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ON ENRIQUES SURFACES IN CHARACTERISTIC 2 WITH A FINITE GROUP OF AUTOMORPHISMS

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Abstract

Complex Enriques surfaces with a finite group of automorphisms are classified into seven types. In this paper, we determine which types of such Enriques surfaces exist in characteristic 2. In particular we give a 1-dimensional family of classical and supersingular Enriques surfaces with the automorphism group $\mathrm{Aut}(X)$ isomorphic to the symmetric group $\mathfrak{S}_5$ of degree five.

1. Introduction

We work over an algebraically closed field k of characteristic 2. Complex Enriques surfaces with a finite group of automorphisms are completely classified into seven types. The main purpose of this paper is to determine which types of such Enriques surfaces exist in characteristic 2. Recall that, over the complex numbers, a generic Enriques surface has an infinite group of automorphisms (Barth and Peters [4]). On the other hand, Fano [14] gave an Enriques surface with a finite group of automorphisms. Later Dolgachev [9] gave another example of such Enriques surfaces. Then Nikulin [30] proposed a classification of such Enriques surfaces in terms of the periods. Finally the second author [22] classified all complex Enriques surfaces with a finite group of automorphisms, geometrically. There are seven types $\mathrm{I}, \mathrm{II}, \ldots, \mathrm{VII}$ of such Enriques surfaces. The Enriques surfaces of type I or II form an irreducible 1-dimensional family, and each of the remaining types consists of a unique Enriques surface. The first two types contain exactly twelve nonsingular rational curves, on the other hand, the remaining five types contain exactly twenty nonsingular rational curves. The Enriques surfaces of type I (resp. of type VII) is the example given by Dolgachev (resp. by Fano). We call the dual graphs of all nonsingular rational curves on the Enriques surface of type K the dual graph of type K ($K = \mathrm{I}, \mathrm{II}, ..., \mathrm{VII}$).

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In positive characteristics, the classification of Enriques surfaces with a finite group of automorphisms is still open. Especially the case of characteristic 2 is most interesting. In the paper [5], Bombieri and Mumford classified Enriques surfaces in characteristic 2 into three classes, namely, singular, classical and supersingular Enriques surfaces. As in the case of characteristic 0, an Enriques surface X in characteristic 2 has a canonical double cover $\pi : Y \to X$, which is a separable $\mathbf{Z}/2\mathbf{Z}$-cover, a purely inseparable μ_2- or α_2-cover according to X being singular, classical or supersingular. The surface Y might have singularities and it might even be non-normal, but it is $K3$-like in the sense that its dualizing sheaf is trivial. Bombieri and Mumford gave an explicit example of each type of Enriques surface as a quotient of the intersection of three quadrics in $\mathbf{P}^5$. In particular, they gave an α_2-covering $Y \to X$ such that Y is a supersingular $K3$ surface with 12 rational double points of type A_1. Recently Liedtke [25] showed that every Enriques surface can be realized in the form of the example by Bombieri and Mumford [5], that is, its canonical cover is a complete intersection of three quadrics in $\mathbf{P}^5$. Moreover he showed that the moduli space of Enriques surfaces with a polarization of degree 4 has two 10-dimensional irreducible components. A general member of one component (resp. the other component) consists of singular (resp. classical) Enriques surfaces. The intersection of the two components parametrizes supersingular Enriques surfaces.

In this paper we consider the following problem: *does there exist an Enriques surface in characteristic* 2 *with a finite group of automorphisms whose dual graph of all nonsingular rational curves is of type* I, II, ..., VI *or* VII ? Note that if Enriques surface S in any characteristic has the dual graph of type K ($K = \mathrm{I}, \mathrm{II}, ..., \mathrm{VII}$), then the automorphism group $\mathrm{Aut}(S)$ is finite by Vinberg's criterion (see Proposition 2.3).

We will prove the following Theorem:

Theorem I. *The existence or non-existence of Enriques surfaces in characteristic 2 whose dual graphs of all non-singular rational curves are of type* I, II, *. . . , or* VII *is as in the following Table*1:

Type	I	II	III	IV	V	VI	VII
singular	◯	◯	×	×	×	◯	×
classical	×	×	×	×	×	×	◯
supersingular	×	×	×	×	×	×	◯

Table 1

In Table 1, ◯ *means the existence and* × *means the non-existence of an Enriques surface with the dual graph of type* I, ..., VII.

In case of type I, II, VI, the construction of such Enriques surfaces over the complex numbers works well in characteristic 2 (Theorems 4.1, 4.4, 4.6). The most difficult and interesting case is of type VII.

Theorem II. (cf. Theorems 3.15 and 3.19) *There exists a non-isotrivial and one dimensional family of classical and supersingular Enriques surfaces with a finite group of automorphisms whose dual graph of non-singular rational curves is of type* VII. *The minimal resolutions of the canonical double covers of these Enriques surfaces are the unique supersingular* $K3$ *surface with Artin invariant* 1.

For the non-isotriviality of this family, see Theorem 3.20. We remark that there exists the unique Enriques surface with the dual graph of type VII over the complex number field.

Ekedahl, Hyland and Shepherd-Barron [13] studied classical or supersingular Enriques surfaces whose canonical covers are supersingular $K3$ surfaces with 12 rational double points of type A_1. They showed that the moduli space of such Enriques surfaces is an open piece of a $\mathbf{P}^1$-bundle over the moduli space of supersingular $K3$ surfaces. The canonical double covers of Enriques surfaces in Theorem II have 12 rational double points of type A_1 whose minimal non-singular models are the supersingular $K3$ surface with Artin invariant 1.

Recently the authors [18] gave a 1-dimensional family of classical and supersingular Enriques surfaces which contain a remarkable forty divisors, by using a result of Rudakov and Shafarevich [31] on purely inseparable covers of surfaces. We employ here the same method to construct the above classical and supersingular Enriques surfaces with the dual graph of type VII.

It is known that there exist Enriques surfaces in characteristic 2 with a finite group of automorphisms whose dual graphs of all nonsingular rational curves do not appear in the case of complex surfaces (Ekedahl and Shepherd-Barron[12], Salomonsson[32]). See Remark 4.10. The remaining problem of the classification of Enriques surfaces in characteristic 2 with a finite group of automorphisms is to determine such Enriques surfaces appeared only in characteristic 2.

The plan of this paper is as follows. In section 2, we recall the known results on Rudakov-Shafarevich's theory on derivations, lattices and Enriques surfaces. In section 3, we give a construction of a 1-dimensional family of classical and supersingular Enriques surfaces with the dual graph of type VII. Moreover we show the non-existence of singular Enriques surfaces with the dual graph of type VII (Theorem 3.22). In section 4, we discuss other cases, that is, the existence of singular Enriques surfaces of type $\mathrm{I}, \mathrm{II}, \mathrm{VI}$ and the non-existence of other cases (Theorems 4.1, 4.2, 4.4, 4.5, 4.6, 4.7, 4.9). In appendices A and B, we give two remarks. As appendix A, we show that the covering $K3$ surface of any singular Enriques surface has height 1. As appendix B, we show that for each singular Enriques surface with the dual graph of type I its canonical cover is isomorphic to the Kummer surface of the product of two ordinary elliptic curves.

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2. Preliminaries

Let k be an algebraically closed field of characteristic $p > 0$, and let S be a nonsingular complete algebraic surface defined over k. We denote by K_S a canonical divisor of S. A rational vector field D on S is said to be p-closed if there exists a rational function f on S such that $D^p = fD$. A vector field D for which $D^p = 0$ is called of additive type, while that for which $D^p = D$ is called of multiplicative type. Let $\{U_i = \mathrm{Spec} A_i\}$ be an affine open covering of S. We set $A_i^D = \{D(\alpha) = 0 \mid \alpha \in A_i\}$. Affine varieties $\{U_i^D = \mathrm{Spec} A_i^D\}$ glue together to define a normal quotient surface S^D.

Now, we assume that D is p-closed. Then, the natural morphism $\pi : S \longrightarrow S^D$ is a purely inseparable morphism of degree p. If the affine open covering $\{U_i\}$ of S is fine enough, then taking local coordinates x_i, y_i on U_i, we see that there exsit $g_i, h_i \in A_i$ and a rational function f_i such that the divisors defined by $g_i = 0$ and by $h_i = 0$ have no common divisor, and such that

$$D = f_i \left(g_i \frac{\partial}{\partial x_i} + h_i \frac{\partial}{\partial y_i} \right) \quad \text{on } U_i.$$

By Rudakov and Shafarevich [31] (Section 1), divisors (f_i) on U_i glue to a global divisor (D) on S, and the zero-cycle defined by the ideal (g_i, h_i) on U_i gives rise to a well-defined global zero cycle $\langle D \rangle$ on S. A point contained in the support of $\langle D \rangle$ is called an isolated singular point of D. If D has no isolated singular point, D is said to be divisorial. Rudakov and Shafarevich ([31], Theorem 1, Corollary) showed that S^D is nonsingular if $\langle D \rangle = 0$, i.e.,

D is divisorial. When S^D is nonsingular, they also showed a canonical divisor formula

$$K_S \sim \pi^* K_{S^D} + (p-1)(D), \tag{2.1}$$

where $\sim$ means linear equivalence. As for the Euler number $c_2(S)$ of S, we have a formula

$$c_2(S) = \deg\langle D\rangle - \langle K_S, (D)\rangle - (D)^2 \tag{2.2}$$

(cf. Katsura and Takeda [20], Proposition 2.1).

Now we consider an irreducible curve C on S and we set $C' = \pi(C)$. Take an affine open set U_i as above such that $C \cap U_i$ is non-empty. The curve C is said to be integral with respect to the vector field D if $g_i \frac{\partial}{\partial x_i} + h_i \frac{\partial}{\partial y_i}$ is tangent to C at a general point of $C \cap U_i$. Then, Rudakov-Shafarevich [31] (Proposition 1) showed the following proposition:

Proposition 2.1. (i) *If C is integral, then $C = \pi^{-1}(C')$ and $C^2 = pC'^2$.*
(ii) *If C is not integral, then $pC = \pi^{-1}(C')$ and $pC^2 = C'^2$.*

A lattice is a free abelian group L of finite rank equipped with a non-degenerate symmetric integral bilinear form $\langle ., . \rangle : L \times L \to \mathbf{Z}$. with the bilinear form obtained from the bilinear form of L by multiplication by m. The signature of a lattice is the signature of the real vector space $L \otimes \mathbf{R}$ equipped with the symmetric bilinear form extended from the one on L by linearity. A lattice is called even if $\langle x, x\rangle \in 2\mathbf{Z}$ for all $x \in L$. We denote by U the even unimodular lattice of signature $(1,1)$, and by $A_m,\ D_n$ or E_k the even *negative* definite lattice defined by the Cartan matrix of type $A_m,\ D_n$ or E_k respectively. We denote by $L \oplus M$ the orthogonal direct sum of lattices L and M. Let $\mathrm{O}(L)$ be the orthogonal group of L, that is, the group of isomorphisms of L preserving the bilinear form.

In characteristic 2, a minimal algebaic surface with numerically trivial canonical divisor is called an Enriques surface if the second Betti number is equal to 10. Such surfaces S are divided into three classes (for details, see Bombieri and Mumford [5], Section 3):

(i) K_S is not linearly equivalent to zero and $2K_S \sim 0$. Such an Enriques surface is called a classical Enriques surface.
(ii) $K_S \sim 0$, $\mathrm{H}^1(S, \mathcal{O}_S) \cong k$ and the Frobenius map acts on $\mathrm{H}^1(S, \mathcal{O}_S)$ bijectively. Such an Enriques surface is called a singular Enriques surface.
(iii) $K_S \sim 0$, $\mathrm{H}^1(S, \mathcal{O}_S) \cong k$ and the Frobenius map is the zero map on $\mathrm{H}^1(S, \mathcal{O}_S)$. Such an Enriques surface is called a supersingular Enriques surface.

We denote by $\mathcal{M}_c$ (resp. $\mathcal{M}_s$, resp. $\mathcal{M}_{ss}$) the moduli of classical (resp. singular, resp. supersingular) Enriques surfaces of degree 4. Liedtke showed that both $\mathcal{M}_c$ and $\mathcal{M}_s$ are irreducible, unirational and 10-dimensional, and $\mathcal{M}_{ss}$ coincides with the intersection $\mathcal{M}_c \cap \mathcal{M}_s$ which is irreducible, unirational and 9-dimensional ([25]).

Let S be an Enriques surface and let $\mathrm{Num}(S)$ be the quotient of the Néron-Severi group of S by torsion. Then $\mathrm{Num}(S)$ together with the intersection product is an even unimodular lattice of signature $(1,9)$ (Cossec and Dolgachev [6], Chap. II, Theorem 2.5.1), and hence is isomorphic to $U \oplus E_8$. We denote by $\mathrm{O}(\mathrm{Num}(S))$ the orthogonal group of $\mathrm{Num}(S)$. The set

$$\{x \in \mathrm{Num}(S) \otimes \mathbf{R} \ : \ \langle x, x\rangle > 0\}$$

has two connected components. Denote by $P(S)$ the connected component containing an ample class of S. For $\delta \in \mathrm{Num}(S)$ with $\delta^2 = -2$, we define an isometry s_δ of $\mathrm{Num}(S)$ by

$$s_\delta(x) = x + \langle x, \delta\rangle \delta, \quad x \in \mathrm{Num}(S).$$

The isometry s_δ is called the reflection associated with δ. Let $W(S)$ be the subgroup of $\mathrm{O}(\mathrm{Num}(S))$ generated by reflections associated with all nonsingular rational curves on S. Then $P(S)$ is divided into chambers each of which is a fundamental domain with respect to the action of $W(S)$ on $P(S)$. There exists a unique chamber containing an ample class which is nothing but the closure of the ample cone $D(S)$ of S. It is known that the natural map

$$\rho : \mathrm{Aut}(S) \to \mathrm{O}(\mathrm{Num}(S)) \tag{2.3}$$

has a finite kernel (Dolgachev [10], Theorems 4, 6). Since the image $\mathrm{Im}(\rho)$ preserves the ample cone, we see $\mathrm{Im}(\rho) \cap W(S) = \{1\}$. Therefore $\mathrm{Aut}(S)$ is finite if the index $[\mathrm{O}(\mathrm{Num}(S)) : W(S)]$ is finite. Thus we have the following Proposition (see Dolgachev [9], Proposition 3.2).

Proposition 2.2. *If $W(S)$ is of finite index in $\mathrm{O}(\mathrm{Num}(S))$, then $\mathrm{Aut}(S)$ is finite.*

Over the field of complex numbers, the converse of Proposition 2.2 holds by using the Torelli type theorem for Enriques surfaces (Dolgachev [9], Theorem 3.3).

Now, we recall Vinberg's criterion which guarantees that a group generated by finite number of reflections is of finite index in $\mathrm{O}(\mathrm{Num}(S))$.

Let Δ be a finite set of (-2)-vectors in $\mathrm{Num}(S)$. Let Γ be the graph of Δ, that is, Δ is the set of vertices of Γ and two vertices δ and δ' are joined by m-tuple lines if $\langle \delta, \delta'\rangle = m$. We assume that the cone

$$K(\Gamma) = \{x \in \mathrm{Num}(S) \otimes \mathbf{R} \ : \ \langle x, \delta_i\rangle \geq 0, \ \delta_i \in \Delta\}$$

is a strictly convex cone. Such Γ is called non-degenerate. A connected parabolic subdiagram Γ' in Γ is a Dynkin diagram of type $\tilde{A}_m$, $\tilde{D}_n$ or $\tilde{E}_k$ (see [36], p. 345, Table 2). If the number of vertices of Γ' is $r+1$, then r is called the rank of Γ'. A disjoint union of connected parabolic subdiagrams is called a parabolic subdiagram of Γ. We denote by $\tilde{K}_1 \oplus \tilde{K}_2$ a parabolic subdiagram which is a disjoint union of two connected parabolic subdiagrams of type $\tilde{K}_1$ and $\tilde{K}_2$, where K_i is A_m, D_n or E_k. The rank of a parabolic subdiagram is the sum of the rank of its connected components. Note that the dual graph of reducible fibers of an elliptic fibration on S gives a parabolic subdiagram. For example, a singular fiber of type III, IV or I_{n+1} defines a parabolic subdiagram of type $\tilde{A}_1$, $\tilde{A}_2$ or $\tilde{A}_n$ respectively. We denote by $W(\Gamma)$ the subgroup of $\mathrm{O}(\mathrm{Num}(S))$ generated by reflections associated with $\delta \in \Gamma$.

Proposition 2.3. (Vinberg [36], Theorem 2.3) *Let Δ be a set of (-2)-vectors in Num(S) and let Γ be the graph of Δ. Assume that Δ is a finite set, Γ is non-degenerate and Γ contains no m-tuple lines with $m \geq 3$. Then $W(\Gamma)$ is of finite index in $\mathrm{O}(Num(S))$ if and only if every connected parabolic subdiagram of Γ is a connected component of some parabolic subdiagram in Γ of rank 8 (= the maximal one).*

Finally we recall some facts on elliptic fibrations on Enriques surfaces.

Proposition 2.4. (Dolgachev and Liedtke [11], Theorem 4.8.3)

Let $f : S \to \mathbf{P}^1$ be an elliptic fibration on an Enriques surface S in characteristic 2. Then the following hold.

(i) *If S is classical, then f has two tame multiple fibers, each is either an ordinary elliptic curve or a singular fiber of additive type.*

(ii) *If S is singular, then f has one wild multiple fiber which is a smooth ordinary elliptic curve or a singular fiber of multiplicative type.*

(iii) *If S is supersingular, then f has one wild multiple fiber which is a supersingular elliptic curve or a singular fiber of additive type.*

Proof. As for the number of multiple fibers in each case, it is given in Bombieri and Mumford [5], Proposition 11. Let $2G$ be a multiple fiber of $f : S \longrightarrow \mathbf{P}^1$. If S is classical, then the multiple fiber $2G$ is tame. Therefore, the normal bundle $\mathcal{O}_G(G)$ of G is of order 2 (cf. Katsura and Ueno [21], p. 295, (1.7)). On the other hand, neither the Picard variety $\mathrm{Pic}^0(\mathbf{G}_m)$ of the multiplicative group $\mathbf{G}_m$ nor $\mathrm{Pic}^0(E)$ of the supersingular elliptic curve E has any 2-torsion point. Therefore, G is either an ordinary elliptic curve or a singular fiber of additive type. Now, we consider an exact sequence:

$$0 \longrightarrow \mathcal{O}_S(-G) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_G \longrightarrow 0.$$

Then, we have the long exact sequence

$$\to H^1(S, \mathcal{O}_S) \longrightarrow H^1(G, \mathcal{O}_G) \longrightarrow H^2(S, \mathcal{O}_S(-G)) \longrightarrow H^2(S, \mathcal{O}_S) \to 0.$$

If S is either singular or supersingular, we have $H^1(S, \mathcal{O}_S) \cong H^2(S, \mathcal{O}_S) \cong k$. Note that in our case the canonical divisor K_S is linearly equivalent to 0. Since $2G$ is a multiple fiber, by the Serre duality theorem, we have

$$H^2(S, \mathcal{O}_S(-G)) \cong H^0(S, \mathcal{O}_S(K_S + G)) \cong H^0(S, \mathcal{O}_S(G)) \cong k.$$

Therefore, we see that the natural homomorphism

$$H^1(S, \mathcal{O}_S) \longrightarrow H^1(G, \mathcal{O}_G)$$

is an isomorphism. If S is singular, then the Frobenius map F acts bijectively on $H^1(S, \mathcal{O}_S)$. Hence, F acts on $H^1(G, \mathcal{O}_G)$ bijectively. Therefore, G is either an ordinary elliptic curve or a singular fiber of multiplicative type. If S is supersingular, then the Frobenius map F is the zero map on $H^1(S, \mathcal{O}_S)$. Hence, F is also a zero map on $H^1(G, \mathcal{O}_G)$. Therefore, G is either a supersingular elliptic curve or a singular fiber of additive type. □

Let $f : S \to \mathbf{P}^1$ be an elliptic fibration on an Enriques surface S. We use Kodaira's notation for singular fibers of f:

$$\mathrm{I}_n,\ \mathrm{I}_n^*,\ \mathrm{II},\ \mathrm{II}^*,\ \mathrm{III},\ \mathrm{III}^*,\ \mathrm{IV},\ \mathrm{IV}^*.$$

An elliptic fibration f on an Enriques surface is called extremal if its Jacobian fibration has Mordell-Weil rank 0.

Proposition 2.5. *Let $f : S \to \mathbf{P}^1$ be an extremal elliptic fibration on an Enriques surface S in characteristic* 2. *Then the type of singular fibers is one of the following*:

$$(\mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_3),\ (\mathrm{I}_5, \mathrm{I}_5),\ (\mathrm{I}_9),\ (\mathrm{I}_4^*),\ (\mathrm{II}^*),\ (\mathrm{II}^*, \mathrm{I}_1),\ (\mathrm{III}, \mathrm{I}_8),$$

$$(\mathrm{I}_1^*, \mathrm{I}_4),\ (\mathrm{III}^*, \mathrm{I}_2),\ (\mathrm{IV}, \mathrm{IV}^*),\ (\mathrm{IV}, \mathrm{I}_2, \mathrm{I}_6),\ (\mathrm{IV}^*, \mathrm{I}_3).$$

Proof. Consider the Jacobian fibration $J(f) : R \to \mathbf{P}^1$ of f which is a rational elliptic surface. It is known that the type of singular fibers of f coincides with that of $J(f)$ (cf. Liu-Lorenzini-Raynaud [26], Theorem 6.6). Now the assertion follows from the classification of singular fibers of extremal rational elliptic surfaces in characteristic 2 due to Lang [23], [24] (also see Ito [15]). □

3. Enriques surfaces with the dual graph of type VII

In this section, we construct Enriques surfaces in characteristic 2 whose dual graph of all nonsingular rational curves is of type VII. The method to construct them is similar to the one in Katsura and Kondo [18], §4.

We consider the nonsingular complete model of the supersingular elliptic curve E defined by

$$y^2 + y = x^3 + x^2.$$

For $(x_1, y_1), (x_2, y_2) \in E$, the addition of this elliptic curve is given by,

$$x_3 = x_1 + x_2 + \left(\frac{y_2+y_1}{x_2+x_1}\right)^2 + 1$$
$$y_3 = y_1 + y_2 + \left(\frac{y_2+y_1}{x_2+x_1}\right)^3 + \left(\frac{y_2+y_1}{x_2+x_1}\right) + \frac{x_1y_2+x_2y_1}{x_2+x_1} + 1.$$

The following lemma is easily proved.

Lemma 3.1. *The* $\mathbf{F}_4$*-rational points of* E *are given by*

$$P_0 = \infty, P_1 = (1, 0), P_2 = (0, 0), P_3 = (0, 1), P_4 = (1, 1).$$

The point P_0 *is the zero point of* E*, and these points generate the cyclic group of order five :*

$$P_i = iP_1 \quad (i = 2, 3, 4),\ P_0 = 5P_1$$

Now we consider the relatively minimal nonsingular complete elliptic surface $\psi : R \longrightarrow \mathbf{P}^1$ defined by

$$y^2 + sxy + y = x^3 + x^2 + s$$

with a parameter s of an affine line $\mathbf{A}^1$ in $\mathbf{P}^1$. This surface is a rational elliptic surface with two singular fibers of type I_5 over the points given by $s = 1, \infty$, and two singular fibers of type I_1 over the points given by $t = \omega, \omega^2$. Here, ω is a primitive cube root of unity. We consider the base change of $\psi : R \longrightarrow \mathbf{P}^1$ by $s = t^2$. Then, we have the elliptic surface defined by

$$(*) \qquad y^2 + t^2xy + y = x^3 + x^2 + t^2.$$

We consider the relatively minimal nonsingular complete model of this elliptic surface :

$$(3.1) \qquad f : Y \longrightarrow \mathbf{P}^1.$$

The surface Y is an elliptic $K3$ surface.

Lemma 3.2. Y *is a supersingular K3 surface, i.e. the Picard number* $\rho(Y)$ *is equal to the second Betti number* $b_2(Y)$*.*

Proof. From Y to R, there exists a generically surjective purely inseparable rational map. We denote by $R^{(\frac{1}{2})}$ the algebraic surface whose coefficients of the defining equations are the square roots of those of R. Then, $R^{(\frac{1}{2})}$ is also a rational surface, and we have the Frobenius morphism $F : R^{(\frac{1}{2})} \longrightarrow R$. F factors through a generically surjective purely inseparable rational map from $R^{(\frac{1}{2})}$ to Y. By the fact that $R^{(\frac{1}{2})}$ is rational we see that Y is unirational. Hence, Y is a supersingular $K3$ surface (cf. Shioda [34], p.235, Corollary 1). □

Lemma 3.3. *The types of singular fibers of the elliptic surface* $f : Y \longrightarrow \mathbf{P}^1$ *are given by the following table*:

t	1	ω	ω^2	∞
type	I_{10}	I_2	I_2	I_{10}

At the point defined by $t = 0$*, there exists precisely one fiber with good and supersingular reduction.*

Proof. The discriminant of the elliptic surface $f : Y \longrightarrow \mathbf{P}^1$ is given by

$$\Delta = (t+1)^{10}(t^2+t+1)^2$$

and the j-invariant is given by

$$j = t^{24}/(t+1)^{10}(t^2+t+1)^2.$$

Therefore, on the elliptic surface $f : Y \longrightarrow \mathbf{P}^1$, there exist two singular fibers of type I_{10} over the points given by $t = 1, \infty$, and two singular fibers of type I_2 over the points given by $t = \omega, \omega^2$. The regular fiber over the point defined by $t = 0$ is the supersingular elliptic curve E. □

By direct calculations, we have the following proposition and corollary.

Proposition 3.4. *The elliptic* $K3$ *surface* $f : Y \longrightarrow \mathbf{P}^1$ *has ten sections* s_i, m_i $(i = 0, 1, 2, 3, 4)$ *given as follows*:

s_0 : *the zero section*	*passing through* P_0 *on* E
$s_1 : x = 1, y = t^2$	*passing through* P_1 *on* E
$s_2 : x = t^2, y = t^2$	*passing through* P_2 *on* E
$s_3 : x = t^2, y = t^4 + t^2 + 1$	*passing through* P_3 *on* E
$s_4 : x = 1, y = 1$	*passing through* P_4 *on* E
$m_0 : x = \frac{1}{t^2}, y = \frac{1}{t^3} + \frac{1}{t^2} + t$	*passing through* P_0 *on* E
$m_1 : x = t^3 + t + 1,\ y = t^4 + t^3 + t$	*passing through* P_1 *on* E
$m_2 : x = t,\ y = t^3$	*passing through* P_2 *on* E
$m_3 : x = t,\ y = 1$	*passing through* P_3 *on* E
$m_4 : x = t^3 + t + 1,\ y = t^5 + t^4 + t^2 + t + 1$	*passing through* P_4 *on* E.

These ten sections generate the cyclic group of order 10*, and the group structure is given by*

$$s_i = is_1,\ m_i = m_0 + s_i\ (i = 0, 1, 2, 3, 4),\ 2m_0 = s_0$$

with s_0*, the zero section.*

Corollary 3.5. *The images of* s_i *(resp.* m_i*)* $(i = 0, 1, 2, 3, 4)$ *on* R *give sections (resp. multi-sections) of* $\psi : R \longrightarrow \mathbf{P}^1$*. The intersection numbers of the sections* s_i, m_i $(i = 0, 1, 2, 3, 4)$ *are given by*

$$\langle s_i, s_j\rangle = -2\delta_{ij},\ \langle m_i, m_j\rangle = -2\delta_{ij},\ \langle s_i, m_j\rangle = \delta_{ij}, \tag{3.2}$$

where δ_{ij} *is Kronecker's delta.*

Proposition 3.6. *The surface Y is a supersingular $K3$ surface with Artin invariant* 1.

Proof. The elliptic fibration (3.1) has two singular fibers of type I_{10}, two singular fibers of type I_2 and ten sections. Since $\sigma \geq 1$, the assertion follows from the Shioda-Tate formula (cf. Shioda [33], Corollary 1.7). □

Incidentally, by the Shioda-Tate formula, we also see that the order of the group of the sections of $f : Y \longrightarrow \mathbf{P}^1$ is equal to 10 and so the Mordell-Weil group is isomorphic to $\mathbf{Z}/10\mathbf{Z}$. We also remark that the elliptic $K3$ surface $f : Y \longrightarrow \mathbf{P}^1$ is the most special case in the list of the classification of elliptic $K3$ surfaces with 2-torsion sections (Ito, Liedtke [16]).

On the singular elliptic surface $(*)$, we denote by F_1 the fiber over the point defined by $t = 1$. F_1 is an irreducible curve and on F_1 the surface $(*)$ has only one singular point P. The surface Y is a surface obtained by the minimal resolution of singularities of $(*)$. We denote the proper transform of F_1 on Y again by F_1, if confusion doesn't occur. We have nine exceptional curves $E_{1,i}$ $(i = 1, 2, \ldots, 9)$ over the point P, and as a singular fiber of type I_{10} of the elliptic surface $f : Y \longrightarrow \mathbf{P}^1$, F_1 and these nine exceptional curves form a decagon $F_1E_{1,1}E_{1,2} \ldots E_{1,9}$ numbered clockwisely. The blowing-up at the singular point P gives two exceptional curves $E_{1,1}$ and $E_{1,9}$, and they intersect each other at a singular point. The blowing-up at the singular point again gives two exceptional curves $E_{1,2}$ and $E_{1,8}$. The exceptional curve $E_{1,2}$ (resp. $E_{1,8}$) intersects $E_{1,1}$ (resp. $E_{1,9}$) transversely. Exceptional curves $E_{1,2}$ and $E_{1,8}$ intersect each other at a singular point, and so on. By successive blowing-ups, the exceptional curve $E_{1,5}$ finally appears to complete the resolution of singularity at the point P, and it intersects $E_{1,4}$ and $E_{1,6}$ transversely. Summarizing these results, we see that F_1 intersects $E_{1,1}$ and $E_{1,9}$ transversely, and that $E_{1,i}$ intersects $E_{1,i+1}$ $(i = 1, 2, \ldots, 8)$ transversely. We choose $E_{1,1}$ as the component which intersects the section m_2. By this resolution of singularities, we know how the 10 sections intersect these 10 curves F_1 and E_{1i} $(i = 1, 2, \ldots, 9)$.

Lemma 3.7. *The* 10 *sections above intersect these* 10 *curves* F_1 *and* E_{1i} $(i = 1, 2, \ldots, 9)$ *transversely as follows*:

sections	s_0	s_1	s_2	s_3	s_4	m_0	m_1	m_2	m_3	m_4
componets	F_1	$E_{1,8}$	$E_{1,6}$	$E_{1,4}$	$E_{1,2}$	$E_{1,5}$	$E_{1,3}$	$E_{1,1}$	$E_{1,9}$	$E_{1,7}$

Here, the table means that the section s_0 intersects the singular fiber over the point defined by $t = 1$ with the component F_1, for example.

The surface Y has an automorphism σ defined by

$$(t, x, y) \mapsto \left(\frac{t}{t+1}, \frac{x + t^4 + t^2 + 1}{(t+1)^4}, \frac{x + y + s^6 + s^2}{(s+1)^6}\right).$$

Lemma 3.8. *The automorphism σ is of order 4 and replaces the fiber over the point $t = 1$ with the one over the point $t = \infty$, and also interchanges the fiber over the point $t = \omega$ with the one over the point $t = \omega^2$. The automorphism σ acts on the ten sections above as follows*:

sections	s_0	s_1	s_2	s_3	s_4	m_0	m_1	m_2	m_3	m_4
σ^**(sections)*	s_0	s_2	s_4	s_1	s_3	m_0	m_2	m_4	m_1	m_3

Using the automorphism σ, to construct the resolution of singularity on the fiber over the point P_∞ defined by $t = \infty$, we use the resolution of singularity on the fiber over the point P_1 defined by $t = 1$. We attach names to the irreducible components of the fiber over P_∞ in the same way as above. Namely, on the singular elliptic surface $(*)$, we denote by F_∞ the fiber over the point defined by $t = \infty$. We also denote the proper transform of F_∞ on Y by F_∞. We have 9 exceptional curves $E_{\infty,i}$ $(i = 1, 2, \ldots, 9)$ over the point P_∞, and as a singular fiber of type I_{10} of the elliptic surface $f : Y \longrightarrow \mathbf{P}^1$, F_∞ and these 9 exceptional curves make a decagon $F_\infty E_{\infty,1} E_{\infty,2} \ldots E_{\infty,9}$ numbered clockwisely. F_∞ intersects $E_{\infty,1}$ and $E_{\infty,9}$ transversely, and that $E_{\infty,i}$ intersects $E_{\infty,i+1}$ $(i = 1, 2, \ldots, 8)$ transversely.

The singular fiber of $f : Y \longrightarrow \mathbf{P}^1$ over the point defined by $t = \omega$ (resp. $t = \omega^2$) consists of two irreducible components F_ω and E_ω (resp. F_{ω^2} and E_{ω^2}), where F_ω (resp. F_{ω^2}) is the proper transform of the fiber over the point P_ω (resp. P_{ω^2}) in $(*)$. Summarizing these results, we have the following proposition.

Proposition 3.9. *The* 10 *sections above intersect singular fibers of elliptic surface $f : Y \longrightarrow \mathbf{P}^1$ as follows*:

sections	s_0	s_1	s_2	s_3	s_4	m_0	m_1	m_2	m_3	m_4
$t = 1$	F_1	$E_{1,8}$	$E_{1,6}$	$E_{1,4}$	$E_{1,2}$	$E_{1,5}$	$E_{1,3}$	$E_{1,1}$	$E_{1,9}$	$E_{1,7}$
$t = \infty$	F_∞	$E_{\infty,6}$	$E_{\infty,2}$	$E_{\infty,8}$	$E_{\infty,4}$	$E_{\infty,5}$	$E_{\infty,1}$	$E_{\infty,7}$	$E_{\infty,3}$	$E_{\infty,9}$
$t = \omega$	F_ω	F_ω	F_ω	F_ω	F_ω	E_ω	E_ω	E_ω	E_ω	E_ω
$t = \omega^2$	F_{ω^2}	F_{ω^2}	F_{ω^2}	F_{ω^2}	F_{ω^2}	E_{ω^2}	E_{ω^2}	E_{ω^2}	E_{ω^2}	E_{ω^2}

TABLE 2

Now, we consider a rational vector field

$$D_{a,b} = (t-a)(t-b)\frac{\partial}{\partial t} + \frac{(1+t^2x)}{t-1}\frac{\partial}{\partial x}$$

with $a, b \in k,\ a + b = ab,\ a^3 \neq 1$.

Lemma 3.10. *Under the notation above, we have*

$$D_{a,b}^2 = abD_{a,b},$$

that is, $D_{a,b}$ is 2-closed and $D_{a,b}$ is of additive type if $a = b = 0$ and of multiplicative type otherwise. Moreover, we have

$$(3.3)\quad \begin{aligned}(D_{a,b}) &= -(F_1 + E_{1,2} + E_{1,4} + E_{1,6} + E_{1,8} + F_\infty + E_{\infty,2} + E_{\infty,4}\\ &\quad + E_{\infty,6} + E_{\infty,8} + E_\omega + E_{\omega^2}).\end{aligned}$$

Proof. This follows from the definition of $D_{a,b}$ and direct calculations. □

From here until Theorem 3.15, the argument is parallel to the one in Katsura and Kondo [18], §4, and so we give just a brief sketch of the proofs for the readers' convenience. We set $D = D_{a,b}$ for the sake of simplicity.

Lemma 3.11. *The quotient surface Y^D is nonsingular.*

Proof. Since Y is a $K3$ surface, we have $c_2(Y) = 24$. Using $(D)^2 = -24$ and the equation (2.2), we have

$$24 = c_2(Y) = \deg\langle D\rangle - \langle K_Y, (D)\rangle - (D)^2 = \deg\langle D\rangle + 24.$$

Therefore, we have $\deg\langle D\rangle = 0$. This means that D is divisorial, and that Y^D is nonsingular. □

By the result on the canonical divisor formula of Rudakov and Shafarevich (see the equation (2.1)), we have

$$K_Y = \pi^* K_{Y^D} + (D).$$

Lemma 3.12. *Let C be an irreducible curve contained in the support of the divisor (D), and set $C' = \pi(C)$. Then, C' is an exceptional curve of the first kind.*

Proof. By direct calculation, C is integral with respect to D. Therefore, we have $C = \pi^{-1}(C')$ by Proposition 2.1. By the equation $2C'^2 = (\pi^{-1}(C'))^2 = C^2 = -2$, we have $C'^2 = -1$. Since Y is a $K3$ surface, K_Y is linearly equivalent to zero. Therefore, we have

$$2\langle K_{Y^D}, C'\rangle = \langle \pi^* K_{Y^D}, \pi^*(C')\rangle = \langle K_Y - (D), C\rangle = C^2 = -2.$$

Therefore, we have $\langle K_{Y^D}, C'\rangle = -1$ and the arithmetic genus of C' is equal to 0. Hence, C' is an exceptional curve of the first kind. □

We denote these 12 exceptional curves on Y^D by E'_i ($i = 1, 2, \ldots, 12$), which are the images of irreducible components of $-(D)$ by π. Let

$$\varphi : Y^D \to X_{a,b}$$

be the blowing-downs of E'_i $(i = 1, 2, \ldots, 12)$. For simplicity, we denote $X_{a,b}$ by X. Now we have the following commutative diagram:

$$\begin{array}{ccc} Y^D & \overset{\pi}{\longleftarrow} & Y \\ \varphi \downarrow & & \downarrow f \\ X = X_{a,b} & & \mathbf{P}^1 \\ g \downarrow & \swarrow_F & \\ \mathbf{P}^1 & & \end{array}$$

Here F is the Frobenius base change. Then, we have

$$K_{Y^D} = \varphi^*(K_X) + \sum_{i=1}^{12} E'_i.$$

Lemma 3.13. *The canonical divisor K_X of X is numerically equivalent to* 0.

Proof. As mentioned in the proof of Lemma 3.12, all irreducible curves which appear in the divisor (D) are integral with respect to the vector field D. For an irreducible component C of (D), we denote by C' the image $\pi(C)$ of C. Then, we have $C = \pi^{-1}(C')$ by Proposition 2.1. Therefore, we have

$$(D) = -\pi^*(\sum_{i=1}^{12} E'_i).$$

Since Y is a $K3$ surface, by the formula (2.1),

$$0 \sim K_Y = \pi^* K_{Y^D} + (D) = \pi^*(\varphi^*(K_X) + \sum_{i=1}^{12} E'_i) + (D) = \pi^*(\varphi^*(K_X))$$

Therefore, K_X is numerically equivalent to zero. □

Lemma 3.14. *The surface X has $b_2(X) = 10$ and $c_2(X) = 12$.*

Proof. Since $\pi : Y \longrightarrow Y^D$ is finite and purely inseparable, the étale cohomology of Y is isomorphic to the étale cohomology of Y^D. Therefore, we have $b_1(Y^D) = b_1(Y) = 0$, $b_3(Y^D) = b_3(Y) = 0$ and $b_2(Y^D) = b_2(Y) = 22$. Since φ is the blowing-downs of 12 exceptional curves of the first kind, we see $b_0(X) = b_4(X) = 1$, $b_1(X) = b_3(X) = 0$ and $b_2(X) = 10$. Therefore, we have

$$c_2(X) = b_0(X) - b_1(X) + b_2(X) - b_3(X) + b_4(X) = 12.$$

□

Theorem 3.15. *Under the notation above, the following statements hold.*

(i) *The surface $X = X_{a,b}$ is a supersingular Enriques surface if $a = b = 0$.*

(ii) *The surface $X = X_{a,b}$ is a classical Enriques surface if $a+b = ab$ and $a \notin \mathbf{F}_4$.*

Proof. Since K_X is numerically trivial, X is minimal and the Kodaira dimension $\kappa(X)$ is equal to 0. Since $b_2(X) = 10$, X is an Enriques surface. Since Y is a supersingular $K3$ surface, X is either supersingular or classical (e.g., [18], Lemma 4.1).

In the case $a = b = 0$, the elliptic fibration $f : Y \longrightarrow \mathbf{P}^1$ has precisely one fiber that is integral with respect to D, namely the fiber over the point P_0 defined by $t = 0$. Hence $g : X \longrightarrow \mathbf{P}^1$ has only one multiple fiber. Therefore, the multiple fiber is wild, and X is a supersingular Enriques surface (Proposition 2.4). In the case $a \notin \mathbf{F}_4$, the elliptic fibration $f : Y \longrightarrow \mathbf{P}^1$ has precisely two fibers that are integral with respect to D, namely the fibers over the points P_a defined by $t = a$ and P_b defined by $t = b$. Therefore, the multiple fibers are tame, and we conclude that X is a classical Enriques surface (Proposition 2.4). □

Remark 3.16. In the above construction of X, we first take the quotient of Y by the derivation and then contract twelve exceptional curves. The canonical cover of X is the surface obtained by contracting twelve integral curves on Y which has twelve rational double points of type A_1. Thus the Enriques surfaces $X = X_{a,b}$ are the typical examples of Ekedahl, Hyland, Shepherd-Barron's description of the moduli of generic classical or supersingular Enriques surfaces [13].

Remark 3.17. If $a^3 = 1$, $a \neq 1$, then $b = a^2$. In this case, the components F_ω and F_{ω^2} are integral. On the other hand, the exceptional curves E_ω, E_{ω^2} are not integral and appear in the divisor $(D_{a,b})$ as zero divisors with multiplicity 1. The obtained surface $X_{a,b}$ is a rational elliptic surface. It might be interesting to study this surface from the point of view of the moduli space of Enriques surfaces.

Recall that the elliptic fibration $f : Y \to \mathbf{P}^1$ given in (3.1) has two singular fibers of type I_{10}, two singular fibers of type I_2 and ten sections. This fibration induces an elliptic fibration

$$g : X \to \mathbf{P}^1$$

which has two singular fibers of type I_5, two singular fibers of type I_1, and ten 2-sections. Thus we have twenty nonsingular rational curves on X. Denote by $\mathcal{E}$ the set of curves contained in the support of the divisor (D):

$$\mathcal{E} = \{F_1, E_{1,2}, E_{1,4}, E_{1,6}, E_{1,8}, F_\infty, E_{\infty,2}, E_{\infty,4}, E_{\infty,6}, E_{\infty,8}, E_\omega, E_{\omega^2}\}.$$

The singular points of four singular fibers of g consist of twelve points denoted by $\{p_1, ..., p_{12}\}$ which are the images of the twelve curves in $\mathcal{E}$. We may assume that p_{11}, p_{12} are the images of E_ω, E_{ω^2} respectively. Then p_{11}, p_{12} (resp. $p_1, ..., p_{10}$) are the singular points of the singular fibers of g of type I_1 (resp. of type I_5). Each of the twenty nonsingular rational curves passes through two points from $\{p_1, ..., p_{12}\}$ because its preimage on Y meets exactly two curves from twelve curves in $\mathcal{E}$ (see Table 2).

Let $\mathcal{S}_1$ be the set of fifteen nonsingular rational curves which are ten components of two singular fibers of g of type I_5 and five 2-sections which do not pass through p_{11} and p_{12}, that is, the images of $s_0, s_1, ..., s_4$. Then the dual graph of the curves in $\mathcal{S}_1$ is the line graph of the Petersen graph. For the Petersen graph, see Figure 3. Here the line graph $L(G)$ of a graph G is the graph whose vertices correspond to the edges in G bijectively and two vertices in $L(G)$ are joined by an edge iff the corresponding edges meet at a vertex in G. In the following Figure 1, we denote by ten dots the ten points $\{p_1, ..., p_{10}\}$. The fifteen lines denote the fifteen nonsingular rational curves in $\mathcal{S}_1$.

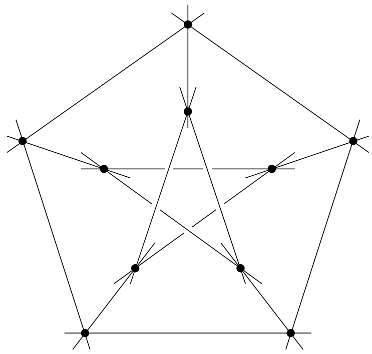

Figure 1

On the other hand, let $\mathcal{S}_2$ be the set of curves which are the images of $m_0, ..., m_4$. Then the dual graph of the curves in $\mathcal{S}_2$ is the complete graph with five vertices in which each pair of the vertices forms the extended Dynkin diagram of type $\tilde{A}_1$ because all of them pass through the two points p_{11} and p_{12}. Each vertex in $\mathcal{S}_1$ meets exactly one vertex in $\mathcal{S}_2$ with multiplicity 2, because any component of the singular fibers of type I_{10} meets exactly one section from $m_0, ..., m_4$ (see Table 2) and s_i meets only m_i $(i = 0, 1, ..., 4)$ (see the equation (3.2)). On the other hand, the vertex in $\mathcal{S}_2$ meets three vertices in $\mathcal{S}_1$ with multiplicity 2, because m_i meets one component of each singular fiber of type I_{10} and s_i. The dual graph Γ of the twenty curves in $\mathcal{S}_1$ and $\mathcal{S}_2$ forms the same dual graph of nonsingular rational curves of the Enriques surfaces of type VII given in Figure 2 (Fig. 7.7 in [22]).

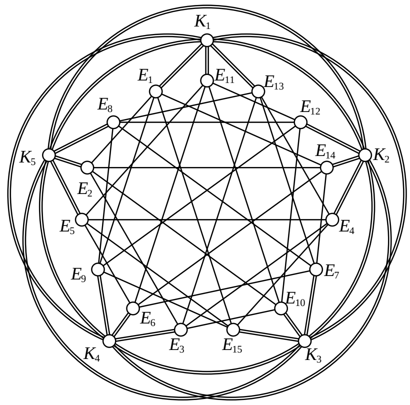

FIGURE 2

The 15 curves in $\mathcal{S}_1$ (resp. five curves in $\mathcal{S}_2$) correspond to $E_1, ..., E_{15}$ (resp. $K_1, ..., K_5$) in Figure 2. It is easy to see that the maximal parabolic subdiagrams in Γ are

$$\tilde{A}_8,\ \tilde{A}_4 \oplus \tilde{A}_4,\ \tilde{A}_5 \oplus \tilde{A}_2 \oplus \tilde{A}_1,\ \tilde{A}_7 \oplus \tilde{A}_1$$

which correspond to elliptic fibrations of type

$$(\mathrm{I}_9),\ (\mathrm{I}_5, \mathrm{I}_5),\ (\mathrm{I}_6, \mathrm{IV}, \mathrm{I}_2),\ (\mathrm{I}_8, \mathrm{III}),$$

respectively. It follows from Vinberg's criterion (Proposition 2.3) that $W(X)$ is of finite index in $\mathrm{O}(\mathrm{Num}(X))$. The same argument in [22], (3.7) implies that X contains exactly twenty nonsingular rational curves in $\mathcal{S}_1, \mathcal{S}_2$.

Lemma 3.18. *The map* $\rho : \mathrm{Aut}(X) \to \mathrm{O}(\mathrm{Num}(X))$ *is injective.*

Proof. Let $\varphi \in \mathrm{Ker}(\rho)$. Then φ preserves each nonsingular rational curve on X. Since each nonsingular rational curve meets the other curves in at least three points in total, φ fixes all 20 nonsingular rational curves pointwisely. Now consider the elliptic fibration $g : X \to \mathbf{P}^1$. Since this fibration has ten 2-sections, φ fixes a general fiber of g and hence φ is the identity. □

By Proposition 2.2, we now have the following theorem.

Theorem 3.19. *The automorphism group* $\mathrm{Aut}(X)$ *is isomorphic to the symmetric group* $\mathfrak{S}_5$ *of degree five and* X *contains exactly twenty nonsingular rational curves whose dual graph is of type* VII.

Proof. We have already shown that $\mathrm{Aut}(X)$ is finite and that X contains exactly twenty nonsigular rational curves whose dual graph Γ is of type VII. It follows from Lemma 3.18 that $\mathrm{Aut}(X)$ is a subgroup of $\mathrm{Aut}(\Gamma) \cong \mathfrak{S}_5$. Then by the same argument as in [22], (3.7), we see that $\mathrm{Aut}(\Gamma)$ is represented by automorphisms of X. □

Theorem 3.20. *The one dimensional family $\{X_{a,b}\}$ is non-isotrivial.*

Proof. Denote by Γ the dual graph of all nonsingular rational curves on X which is given in Figure 2. Γ contains only finitely many extended Dynkin diagrams (= the disjoint union of $\tilde{A}_m, \tilde{D}_n, \tilde{E}_k$), that is, $\tilde{A}_8, \tilde{A}_7 \oplus \tilde{A}_1, \tilde{A}_4 \oplus \tilde{A}_4, \tilde{A}_5 \oplus \tilde{A}_2 \oplus \tilde{A}_1$ (see also Kondo [22], page 274, Table 2). Note that the elliptic fibrations on X bijectively correspond to the extended Dynkin diagrams in Γ. This implies that X has only finitely many elliptic fibrations. The j-invariant of the elliptic curve which appears as the fiber E_a defined by $t = a$ of the elliptic fibration $f : Y \longrightarrow \mathbf{P}^1$ is equal to $a^{24}/(a+1)^{10}(a^2+a+1)^2$ (cf. section 3). Consider the multiple fiber $2E'_a$ on the elliptic fibration on the Enriques surface X which is the image of E_a. Since we have a purely inseparable morphism of degree 2 from E_a to E'_a, we see that the j-invariant of E'_a is equal to $a^{48}/(a+1)^{20}(a^2+a+1)^4$. This implies the infiniteness of the number of elliptic curves which appear as the multiple fibers of the elliptic fibration on an Enriques surface in our family of Enriques surfaces with parameter a. Therefore, in our family of Enriques surfaces there are infinitely many non-isomorphic ones (see also Katsura-Kondō [18], Remark 4.9). □

Remark 3.21. The pullback of an elliptic fibration $\pi : X \to \mathbf{P}^1$ to the covering $K3$ surface Y gives an elliptic fibration $\tilde{\pi} : Y \to \mathbf{P}^1$. The type of reducible singular fibers of $\tilde{\pi}$ is $(\mathrm{I}_{10}, \mathrm{I}_{10}, \mathrm{I}_2, \mathrm{I}_2)$ if π is of type $\tilde{A}_4 \oplus \tilde{A}_4$, $(\mathrm{I}_{16}, \mathrm{I}_1^*)$ if π is of type $\tilde{A}_7 \oplus \tilde{A}_1$, $(\mathrm{I}_{12}, \mathrm{IV}^*, \mathrm{I}_4)$ if π is of type $\tilde{A}_5 \oplus \tilde{A}_2 \oplus \tilde{A}_1$, and type $(\mathrm{I}_{18}, \mathrm{I}_2, \mathrm{I}_2, \mathrm{I}_2)$ if π is of type $\tilde{A}_8$, respectively.

We got the following theorem by the advices of M. Schütt and H. Ito.

Theorem 3.22. *There are no singular Enriques surfaces with the dual graph of type* VII.

Proof. Assume that there exists an Enriques surface S with the dual graph of type VII. In the dual graph of type VII there exists a parabolic subdiagram $\tilde{A}_5 \oplus \tilde{A}_2 \oplus \tilde{A}_1$. By Proposition 2.5, it corresponds to an elliptic fibration on S with singular fibers of type $(\mathrm{IV}, \mathrm{I}_2, \mathrm{I}_6)$. For example, the linear system $|2(E_1 + E_2 + E_{14})|$ defines such a fibration. Moreover the dual graph of type VII tells us that the singular fiber $E_1 + E_2 + E_{14}$ of type IV is a multiple fiber because E_3 is a 2-section of this fibration (see Figure 2). This contradicts to Proposition 2.4, (ii). □

4. Examples of singular $K3$ surfaces with a finite automorphism group

In this section, we show that the examples of complex Enriques surfaces of type I, II and of type VI also work in characteristic 2, and prove the non-existence of the remaining cases. In case of type I and of type II, the following constructions are sometimes called Horikawa representations ([3], Chap. VIII, §18).

4.1. Type I.

Theorem 4.1. *There exists a one dimensional family of singular Enriques surfaces whose dual graph of nonsingular rational curves is of type* I. *The automorphism group* $\mathrm{Aut}(X)$ *is isomorphic to the dihedral group* D_4 *of order* 8.

Proof. Let $[x_0, x_1, x_2, x_3]$ be homogeneous coodinates on $\mathbf{P}^3$. Consider the nonsingular quadric Q in $\mathbf{P}^3$ defined by

$$x_0x_3 + x_1x_2 = 0 \tag{4.1}$$

which is the image of the map

$$\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3, \quad ([u_0, u_1], [v_0, v_1]) \to [u_0v_0, u_0v_1, u_1v_0, u_1v_1].$$

The involution of $\mathbf{P}^1 \times \mathbf{P}^1$

$$([u_0, u_1], [v_0, v_1]) \to ([u_1, u_0], [v_1, v_0])$$

induces an involution

$$\tau : [x_0, x_1, x_2, x_3] \to [x_3, x_2, x_1, x_0] \tag{4.2}$$

of Q whose fixed point set on Q is one point $[1, 1, 1, 1]$. Consider the four lines on Q defined by

$$L_{01} : x_0 = x_1 = 0, \quad L_{02} : x_0 = x_2 = 0,$$

$$L_{13} : x_1 = x_3 = 0, \quad L_{23} : x_2 = x_3 = 0,$$

and a τ-invariant pencil of quadrics

$$C_{\lambda,\mu} : \lambda(x_0 + x_3)(x_1 + x_2) + \mu x_0x_3 = 0$$

passing through the four vertices

$$[1, 0, 0, 0], \quad [0, 1, 0, 0], \quad [0, 0, 1, 0], \quad [0, 0, 0, 1]$$

of the quadrangle $L_{01}, L_{02}, L_{13}, L_{23}$. Note that the two conics

$$Q_1 : x_0 + x_3 = 0, \quad Q_2 : x_1 + x_2 = 0$$

tangent to $C_{\lambda,\mu}$ at two vertices of the quadrangle. Obviously

$$C_{1,0} = Q_1 + Q_2, \quad C_{0,1} = L_{01} + L_{02} + L_{13} + L_{23},$$

and $C_{\lambda,\mu}$ ($\lambda \cdot \mu \neq 0$) is a nonsingular elliptic curve. Thus we have the same configuration of curves given in [22], Figure 1.1 except Q_1 and Q_2 tangent at $[1,1,1,1]$.

Now we fix $(\lambda_0, \mu_0) \in \mathbf{P}^1$ ($\lambda_0 \cdot \mu_0 \neq 0$) and take Artin-Schreier covering $S \to Q$ defined by the triple (L, a, b) where $L = \mathcal{O}_Q(2,2)$, $a \in H^0(Q, L)$ and $b \in H^0(Q, L^{\otimes 2})$ satisfying $Z(a) = C_{0,1}$ and $Z(b) = C_{0,1} + C_{\lambda_0,\mu_0}$. The surface S has four singular points over the four vertices of quadrangle given locally by $z^2 + uvz + uv(u+v) = 0$. In the notation in Artin's list (see [1], §3), it is of type D_4^1. Let Y be the minimal nonsingular model of S. Then the exceptional divisor over a singular point has the dual graph of type D_4. The canonical bundle formula implies that Y is a $K3$ surface. The pencil $\{C_{\lambda,\mu}\}_{(\lambda,\mu)\in\mathbf{P}^1}$ induces an elliptic fibration on Y. The preimage of $L_{01} + L_{02} + L_{13} + L_{23}$ is the singular fiber of type I_{16} and the preimage of $Q_1 + Q_2$ is the union of two singular fibers of type III. Note that the pencil has four sections. Thus we have 24 nodal curves on Y. Note that the dual graph of these 24 nodal curves coincide with the one given in [22], Figure 1.3. The involution τ can be lifted to a fixed point free involution σ of Y because the branch divisor $C_{0,1}$ does not contain the point $[1,1,1,1]$. By taking the quotient of Y by σ, we have a singular Enriques surface $X = Y/\langle\sigma\rangle$. Thus we have the following commutative diagram:

$$\begin{array}{ccc} S & \longleftarrow & Y \\ \downarrow & & \downarrow \\ Q & & X. \end{array}$$

The above elliptic fibration induces an elliptic pencil on X with singular fibers of type I_8 and of type III. Since the ramification divisor of the covering $S \to Q$ is the preimage of $L_{01} + L_{02} + L_{13} + L_{23}$, the multiple fiber of this pencil is the singular fiber of type I_8. By construction, X contains twelve nonsingular rational curves whose dual graph coincides with the one given in [22], Figure 1.4. It follows from Vinberg's criterion (Proposition 2.3) that $W(X)$ is of finite index in $\mathrm{O}(\mathrm{Num}(X))$, and hence the automorphism group $\mathrm{Aut}(X)$ is finite (Proposition 2.2). The same argument as in the proof of [22], Theorem 3.1.1 shows that $\mathrm{Aut}(X)$ is isomorphic to the dihedral group D_4 of order 8. Thus we have the following theorem. □

Theorem 4.2. *There are no classical and supersingular Enriques surfaces with the dual graph of type* I.

Proof. From the dual graph of type I, we can see that such an Enriques surface has an elliptic fibration with a multiple fiber of type I_8. The assertion now follows from Proposition 2.4. □

Remark 4.3. In the above, we consider special quadrics $C_{\lambda,\mu}$ tangent to Q_1, Q_2. If we drop this condition and consider general τ-invariant quadrics through the four vertices of the quadrangle $L_{01}, L_{02}, L_{13}, L_{23}$, we have a two dimensional family of singular Enriques surfaces X. The covering transformation of $Y \to S$ descends to a numerically trivial involution of X, that is, an involution of X acting trivially on $\mathrm{Num}(X)$. In appendix B, we discuss Enriques surfaces with a numerically trivial involution.

4.2. Type II.

Theorem 4.4. *There exists a one dimensional family of singular Enriques surfaces whose dual graph of nonsingular rational curves is of type* II. *The automorphism group* $\mathrm{Aut}(X)$ *is isomorphic to the symmetric group* $\mathfrak{S}_4$ *of degree four.*

Proof. We use the same notation as in 4.1. We consider a τ-invariant pencil of quadrics defined by

$$C_{\lambda,\mu} : \lambda(x_0 + x_1 + x_2 + x_3)^2 + \mu x_0 x_3 = 0$$

which is tangent to the quadrangle $L_{01}, L_{02}, L_{13}, L_{23}$ at $[0, 0, 1, 1]$, $[0, 1, 0, 1]$, $[1, 0, 1, 0]$, $[1, 1, 0, 0]$ respectively. Let

$$L_1 : x_0 + x_1 = x_2 + x_3 = 0, \quad L_2 : x_0 + x_2 = x_1 + x_3 = 0$$

be two lines on Q which passes the tangent points of $C_{\lambda,\mu}$ and the quadrangle $L_{03}, L_{12}, L_{02}, L_{13}$. Note that

$$C_{1,0} = 2L_1 + 2L_2, \quad C_{0,1} = L_{01} + L_{02} + L_{13} + L_{23},$$

and $C_{\lambda,\mu}$ $(\lambda \cdot \mu \neq 0)$ is a nonsingular elliptic curve. Thus we have the same configuration of curves given in [22], Figure 2.1.

Now we fix $[\lambda_0, \mu_0] \in \mathbf{P}^1$ $(\lambda_0 \cdot \mu_0 \neq 0)$ and take Artin-Schreier covering $S \to Q$ defined by the triple (L, a, b) where $L = \mathcal{O}_Q(2, 2)$, $a \in H^0(Q, L)$ and $b \in H^0(Q, L^{\otimes 2})$ satisfying $Z(a) = C_{0,1}$ and $Z(b) = C_{0,1} + C_{\lambda_0,\mu_0}$. The surface S has four singular points over the four tangent points of C_{λ_0,μ_0} with the quadrangle and four singular points over the four vertices of the quadrangle. A local equation of each of the first four singular points is given by $z^2 + uz + u(u + v^2) = 0$ and the second one is given by $z^2 + uvz + uv = 0$. In the first case, by the change of coordinates

$$t = z + \omega u + v^2, \quad s = z + \omega^2 u + v^2, \quad v = v$$

$(\omega^3 = 1,\ \omega \neq 1)$, then we have $v^4 + ts = 0$ which gives a rational double point of type A_3. In the second case, obviously, it is a rational double point of type A_1. Let Y be the minimal nonsingular model of S. Then the exceptional divisor over a singular point in the first case has the dual graph of type A_3 and in the second case the dual graph of type A_1. The canonical bundle formula implies that Y is a $K3$ surface. The pencil $\{C_{\lambda,\mu}\}_{(\lambda,\mu)\in\mathbf{P}^1}$ induces an elliptic fibration on Y. The preimage of $L_{01} + L_{02} + L_{13} + L_{23}$ is the singular fiber of type I_8 and the preimage of $C_{1,0}$ is the union of two singular fibers of type I_1^*. Note that the pencil has four sections. Thus we have 24 nodal curves on Y. Note that the dual graph of these 24 nodal curves coincide with the one given in [22], Figure 2.3. The involution τ can be lifted to a fixed point free involution σ of Y because the branch divisor $C_{0,1}$ does not contain the point $[1,1,1,1]$. By taking the quotient of Y by σ, we have a singular Enriques surface $X = Y/\langle\sigma\rangle$. Thus we have the following commutative diagram:

$$\begin{array}{ccc} S & \longleftarrow & Y \\ \downarrow & & \downarrow \\ Q & & X. \end{array}$$

The above elliptic fibration induces an elliptic pencil on X with singular fibers of type I_4 and of type I_1^*. Since the ramification divisor of the covering $S \to Q$ is the preimage of $L_{01} + L_{02} + L_{13} + L_{23}$, the multiple fiber of this pencil is the singular fiber of type I_4. By construction, X contains twelve nonsingular rational curves whose dual graph Γ coincides with the one given in [22], Figure 2.4. The same argument as in the proof of [22], Theorem 3.2.1 shows that $W(X)$ is of finite index in $O(\mathrm{Num}(X))$ and X contains only these twelve nonsingular rational curves. It now follows from Proposition 2.2 that the automorphism group $\mathrm{Aut}(X)$ is finite. By the similar argument as in the proof of Lemma 3.18, we see that the map $\rho : \mathrm{Aut}(X) \to O(\mathrm{Num}(X))$ is injective. Moreover, by the same argument as in the proof of [22], Theorem 3.2.1, $\mathrm{Aut}(X)$ is isomorphic to $\mathrm{Aut}(\Gamma) \cong \mathfrak{S}_4$. □

Theorem 4.5. *There are no classical and supersingular Enriques surfaces with the dual graph of type* II.

Proof. From the dual graph of type II, we can see that such Enriques surface has an elliptic fibration with a multiple fiber of type I_4. The assertion now follows from Proposition 2.4. □

4.3. Type VI. Over the field of complex numbers, the following example was studied by Dardanelli and van Geemen [8], Remark 2.4. This surface X is isomorphic to the Enriques surface of type VI given in [22] (In [8], Remark 2.4, they claimed that X is of type IV, but this is a misprint). Their construction also works in characteristic 2.

Theorem 4.6. *There exists a singular Enriques surface* X *whose dual graph of nonsingular rational curves is of type* VI. *The automorphism group* $Aut(X)$ *is isomorphic to the symmetric group* $\mathfrak{S}_5$ *of degree five.*

Proof. Let $[x_1, \cdots, x_5]$ be homogeneous coodinates on $\mathbf{P}^4$. Consider the surface S in $\mathbf{P}^4$ defined by

$$\sum_{i=1}^{5} x_i = \sum_{i=1}^{5} 1/x_i = 0. \tag{4.3}$$

Let

$$\ell_{ij} : x_i = x_j = 0 \ \ (1 \le i < j \le 5),$$
$$p_{ijk} : x_i = x_j = x_k = 0 \ \ (1 \le i < j < k \le 5).$$

The ten lines ℓ_{ij} and ten points p_{ijk} lie on S. By taking partial derivatives, we see that S has ten nodes at p_{ijk}. Let Y be the minimal nonsingular model of S. Then Y is a $K3$ surface. Denote by L_{ij} the proper transform of ℓ_{ij} and by E_{ijk} the exceptional curve over p_{ijk}. The Cremona transformation

$$[x_i] \to [1/x_i]$$

induces an automorphism σ of order 2 on Y. Note that the fixed point set of the Cremona transformation is exactly one point $[1, 1, 1, 1, 1]$. Hence σ is a fixed point free involution of Y. The quotient surface $X = Y/\langle\sigma\rangle$ is a singular Enriques surface. Obviously the permutation group $\mathfrak{S}_5$ acts on S which commutes with σ. Therefore $\mathfrak{S}_5$ acts on X as automorphisms. The involution σ interchanges L_{ij} and E_{klm}, where $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$. The images of twenty nonsingular rational curves L_{ij}, E_{ijk} give ten nonsingular rational curves on X whose dual graph is given by the following Figure 3. Note that this graph is the well-known Petersen graph (see Figure 3).

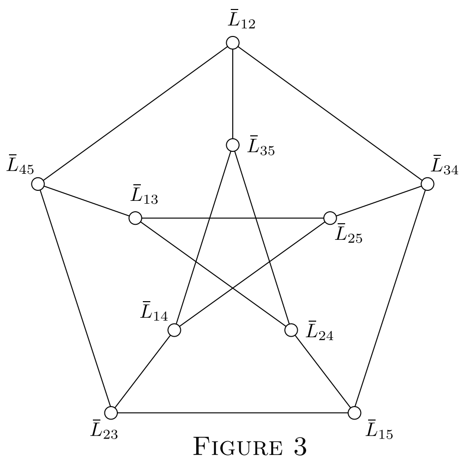

FIGURE 3

Here $\bar{L}_{ij}$ is the image of L_{ij} (and E_{klm}). Note that $\mathfrak{S}_5$ is the automorphism group of the Petersen graph.

The hyperplane section $x_i+x_j=0$ on S is the union of the double line $2\ell_{ij}$ and two lines through p_{klm} defined by $x_kx_l+x_kx_m+x_lx_m=0$. Thus we have additional twenty nodal curves on Y. Note that the Cremona transformation changes two lines defined by $x_kx_l+x_kx_m+x_lx_m=0$. Thus X contains twenty nonsingular rational curves whose dual graph Γ coincides with the one of the Enriques surface of type VI (see Fig.6.4 in [22]). It now follows from Proposition 2.2 that the automorphism group $\mathrm{Aut}(X)$ is finite. The same argument as in the proof of [22], Theorem 3.1.1 shows that X contains only these 20 nonsingular rational curves. By a similar argument to the one in the proof of Lemma 3.18, we see that the map $\rho : \mathrm{Aut}(X) \to \mathrm{O}(\mathrm{Num}(X))$ is injective. Since the classes of twenty nonsingular rational curves generate $\mathrm{Num}(X)\otimes \mathbf{Q}$, $\mathrm{Aut}(X)$ is isomorphic to $\mathrm{Aut}(\Gamma)\cong \mathfrak{S}_5$. □

Theorem 4.7. *There are no classical and supersingular Enriques surfaces with the dual graph of type* VI*.*

Proof. A pentagon in the Figure 3, for example, $|\bar{L}_{12}+\bar{L}_{34}+\bar{L}_{15}+\bar{L}_{24}+\bar{L}_{35}|$, defines an elliptic fibration on X. The multiple fiber of this fibration is nothing but the pentagon, that is, of type I_5. The assertion now follows from Proposition 2.4. □

Remark 4.8. Over the field of complex numbers, Ohashi found that the unique Enriques surface of type VII in [22] is isomorphic to the following surface (see [29], §1.2). Let $[x_1,\cdots,x_5]$ be homogeneous coodinates on $\mathbf{P}^4$. Consider the surface in $\mathbf{P}^4$ defined by

$$\sum_{i<j} x_ix_j = \sum_{i<j<k} x_ix_jx_k = 0 \tag{4.4}$$

which has five nodes at coodinate points and whose minimal resolution is a $K3$ surface Y. The standard Cremona transformation

$$[x_i] \to [1/x_i]$$

induces an automorphism σ of order 2 on Y. In this case, σ is fixed point free, and hence the quotient surface $X=Y/\langle\sigma\rangle$ is a complex Enriques surface. In characteristic 2, the involution σ has a fixed point $[1,1,1,1,1]$ on Y, and hence the quotient is not an Enriques surface.

4.4. Type III, IV, V.

Theorem 4.9. *There are no Enriques surfaces with the same dual graph as in case of type* III, IV *or* V*.*

Proof. In each case of type III, IV, V, from the dual graph (cf. Kondo [22], Figures 3.5, 4.4, 5.5) we can find an elliptic fibration which has two reducible multiples fibers. In fact, the parabolic subdiagram of type $\tilde{A}_3\oplus\tilde{A}_3\oplus\tilde{A}_1\oplus\tilde{A}_1$ in case III (of type $\tilde{A}_3\oplus\tilde{A}_3\oplus\tilde{A}_1\oplus\tilde{A}_1$ in case IV, of type $\tilde{A}_5\oplus\tilde{A}_2\oplus\tilde{A}_1$ in case V) defines such an elliptic fibration (see [22], Table 2, page 274). Hence if an Enriques surface with the same dual graph of nodal curves exists in characteristic 2, then it should be classical (Proposition 2.4). On the other hand, in each case of type III, IV, V, there exists an elliptic fibration which has a reducible multiple fiber of multiplicative type (see [22], Table 2, page 274). However this is impossible because any multiple fiber of an elliptic fibration on a classical Enriques surface is nonsingular or singular of additive type (Proposition 2.4). □

Combining Theorems 3.19, 3.22, 4.1, 4.2, 4.4, 4.5, 4.6, 4.7, 4.9, we have shown Theorem I in the introduction.

Remark 4.10. In characteristic 2, there exist Enriques surfaces with a finite group of automorphisms whose dual graphs of all nonsingular rational curves do not appear in the case of complex surfaces. For example, it is known that there exists an Enriques surface X which has a genus 1 fibration with a multiple singular fiber of type $\tilde{E}_8$ and with a 2-section (Ekedahl and Shepherd-Barron[12], Theorem A, Salomonsson[32], Theorem 1). We have ten nonsingular rational curves on X, that is, nine components of the singular fiber and a 2-section, whose dual graph is given in Figure 4.

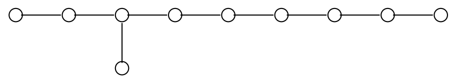

FIGURE 4

It is easy to see that they generate $\mathrm{Num}(X) \cong U \oplus E_8$. Moreover it is known that the reflection subgroup generated by reflections associated with these (-2)-vectors is of finite index in $\mathrm{O}(\mathrm{Num}(X))$ (Vinberg [36], Table 4; also see Proposition 2.3) and hence $\mathrm{Aut}(X)$ is finite (Proposition 2.2).

Added in proof. The authors have classified all supersingular and classical Enriques surfaces with finite automorphism group [19].

Appendix A. The height of the covering $K3$ surfaces of singular Enriques surfaces

In this section we prove the following theorem.

Theorem A.1. *In characteristic* 2*, if a* $K3$ *surface* Y *has a fixed point free involution, then the height* $h(Y)$ *of the formal Brauer group of* Y *is equal to* 1*.*

Corollary A.2. *Let Y be the covering $K3$ surface of a singular Enriques surface. Then the height $h(Y) = 1$.*

Proof. Suppose $h = h(Y) \neq 1$. Since $\mathrm{H}^2(Y, \mathcal{O}_Y)$ is the tangent space of the formal Brauer group of Y (cf. Artin-Mazur [2], Corollary (2.4)), the Frobenius map

$$F : \mathrm{H}^2(Y, \mathcal{O}_Y) \to \mathrm{H}^2(Y, \mathcal{O}_Y)$$

is the zero map. Then, we have an isomorphism as abelian group

$$\mathrm{id} - F : \mathrm{H}^2(Y, \mathcal{O}_Y) \to \mathrm{H}^2(Y, \mathcal{O}_Y).$$

Let $W_i(\mathcal{O}_Y)$ be the sheaf of ring of Witt vectors of length i on Y. Assume $\mathrm{id} - F : \mathrm{H}^2(Y, W_{i-1}(\mathcal{O}_Y)) \longrightarrow \mathrm{H}^2(Y, W_{i-1}(\mathcal{O}_Y))$ is an isomorphism. We have an exact sequence

$$0 \to W_{i-1}(\mathcal{O}_Y) \xrightarrow{V} W_i(\mathcal{O}_Y) \xrightarrow{R} \mathcal{O}_Y \to 0,$$

where V is the Verschiebung and R is the restriction. Then, we have a diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}^2(Y, W_{i-1}(\mathcal{O}_Y)) & \xrightarrow{V} & \mathrm{H}^2(Y, W_i(\mathcal{O}_Y)) & \xrightarrow{R} & \mathrm{H}^2(Y, \mathcal{O}_Y) & \to & 0 \\ & & \mathrm{id} - F \downarrow & & \mathrm{id} - F \downarrow & & \mathrm{id} - F \downarrow & & \\ 0 & \to & \mathrm{H}^2(Y, W_{i-1}(\mathcal{O}_Y)) & \xrightarrow{V} & \mathrm{H}^2(Y, W_i(\mathcal{O}_Y)) & \xrightarrow{R} & \mathrm{H}^2(Y, \mathcal{O}_Y) & \to & 0. \end{array}$$

By the assumption of induction, the first and the third downarrows are isomorphisms. Therefore, by the 5-lemma, we have an isomorphism

$$\mathrm{id} - F : \mathrm{H}^2(Y, W_i(\mathcal{O}_Y)) \cong \mathrm{H}^2(Y, W_i(\mathcal{O}_Y)).$$

Therefore, taking the projective limit, we have an isomorphism

$$\mathrm{id} - F : \mathrm{H}^2(Y, W(\mathcal{O}_Y)) \cong \mathrm{H}^2(Y, W(\mathcal{O}_Y))$$

Therefore, denoting by K the quotient field of the ring of Witt vectors $W(k)$ of infinite length, we have an isomorphism

$$\mathrm{id} - F : \mathrm{H}^2(Y, W(\mathcal{O}_Y)) \otimes K \cong \mathrm{H}^2(Y, W(\mathcal{O}_Y)) \otimes K.$$

Let $\mathrm{H}^2_{et}(Y, \mathbf{Q}_2)$ be the second 2-adic étale cohomology of Y. Then, we have an exact sequence

$$0 \to \mathrm{H}^2_{et}(Y, \mathbf{Q}_2) \to \mathrm{H}^2(Y, W(\mathcal{O}_Y)) \otimes K \xrightarrow{\mathrm{id}-F} \mathrm{H}^2(Y, W(\mathcal{O}_Y)) \otimes K \to 0$$

(cf. Crew [7], (2.1.2) for instance). Therefore, we have $\mathrm{H}^2_{et}(Y, \mathbf{Q}_2) = 0$. On the other hand, we consider the quotient surface X of Y by the fixed point free involution. Then, X is a singular Enriques surface and under the assumption Crew showed $\dim \mathrm{H}^2_{et}(Y, \mathbf{Q}_2) = 1$ for the $K3$ covering Y of X (Crew [7], p41), a contradiction. □

Appendix B. Enriques surfaces associated with Kummer surfaces

In this section we show that the Enriques surfaces given in Remark 4.3 are obtained from Kummer surfaces associated with the product of two ordinary elliptic curves. Let E, E' be two ordinary elliptic curves and let $\iota = \iota_E \times \iota_{E'}$ be the inversion of the abelian surface $E \times E'$. Let $\mathrm{Km}(E \times E')$ be the minimal resolution of the quotient surface $(E \times E')/\langle\iota\rangle$. It is known that $\mathrm{Km}(E \times E')$ is a $K3$ surface called Kummer surface associated with $E \times E'$ (Shioda [35], Proposition 1, see also Katsura [17], Theorem B). The projection from $E \times E'$ to E gives an elliptic fibration which has two singular fibers of type I_4^* and two sections.

Let $p_E \in E$, $p_{E'} \in E'$ be the unique non-zero 2-torsion points on E, E' respectively. Denote by t the translation of $E \times E'$ by the 2-torsion point $(p_E, p_{E'})$. The involution $(\iota_E \times 1_{E'}) \circ t = t \circ (\iota_E \times 1_{E'})$ induces a fixed point free involution σ of $\mathrm{Km}(E \times E')$. Thus we have an Enriques surface $S = \mathrm{Km}(E \times E')/\langle\sigma\rangle$. The involution $\iota_E \times 1_{E'}$ (or t) induces a numerically trivial involution η of S.

Theorem B.1. *The pair (S, η) is isomorphic to an Enriques surface given in Remark* 4.3*.*

Proof. Let E and E' be ordinary elliptic curves over k. Thus, there exist $b, b' \in k$ with $bb' \neq 0$ such that they are given by the equations

$$E : y^2 + xy = x^3 + bx, \quad E' : y'^2 + x'y' = x'^3 + b'x'.$$

The inversion ι_E is then expressed by

$$(x, y) \to (x, y + x)$$

and the translation by the non-zero 2-torsion on E is given by

$$(x, y) \to (b/x, by/x^2 + b/x).$$

Then the function field of $(E \times E')/\langle\iota\rangle$ is given by

$$k((E \times E')/\langle\iota\rangle) = k(x, x', z)$$

with the relation

$$z^2 + xx'z = x^2(x'^3 + b'x') + x'^2(x^3 + bx) \tag{B.1}$$

where $z = xy' + x'y$ (see Shioda [35], the equation (8)). The fixed point free involution σ is expressed by

$$\sigma(x, x', z) = (b/x, b'/x', bb'z/x^2x'^2 + bb'/xx'), \tag{B.2}$$

and the involution induced by $\iota_E \times 1_{E'}$ on $\mathrm{Km}(E \times E')$ is given by

$$(x, x', z) \to (x, x', z + xx'). \tag{B.3}$$

On the other hand, we consider the quadric Q given in (4.1). Instead of τ in (4.2), we consider the involution given by

$$\tau' : (x_0, x_1, x_2, x_3) \to (x_3, b'x_2, bx_1, bb'x_0) \tag{B.4}$$

whose fixed point is $(1, b', b, bb')$. The Artin-Schreier covering is defined by the equation

$$z^2 + x_0x_3z = x_0x_3(x_1x_3 + b'x_0x_2 + x_2x_3 + bx_0x_1)$$

(in the example given in the subsection 4.1, the term $\mu(x_0x_3)^2$ appears in the Artin-Schreier covering. If $\mu \neq 0$, then changing z by $z + ax_0x_3$ where $a^2 + a + \mu = 0$, we can delete this term). Now, by putting here

$$x_0 = u_0v_0,\ x_1 = u_0v_1,\ x_2 = u_1v_0,\ x_3 = u_1v_1$$

and considering an affine locus $u_0 \neq 0, v_0 \neq 0$, we have

$$z^2 + u_1v_1z = u_1v_1(u_1v_1^2 + u_1^2v_1 + bv_1 + b'u_1)$$

which is the same as the equation given in (B.1). Moreover the lifting of τ' and the covering transformation of the Artin-Schreier covering coincide with the ones given in (B.2) and (B.3) respectively. □

Remark B.2. Using appendix A, we see that the height of the formal Brauer group of Kummer surfaces associated with the product of two ordinary elliptic curves is equal to 1.

Remark B.3. All complex Enriques surfaces with cohomologically or numerically trivial automorphisms are classified by Mukai and Namikawa [28], Main theorem (0.1), and Mukai [27], Theorem 3. There are three types: one of them is an Enriques surface associated with $\mathrm{Km}(E \times E')$ and the second one is mentioned in Remark 4.3. For the third one we refer the reader to Mukai [27], Theorem 3. In positive characteristic, Dolgachev ([10], Theorems 4 and 6) determined the order of cohomologically or numerically trivial automorphisms. However, the explicit classification is not known. The above Theorem B.1 implies that two different type of complex Enriques surfaces with a numerically trivial involution coincide in characteristic 2.

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