Observations on the Sphere Spectrum

球面スペクトラムの観察

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Contents

1 Introduction

As its name suggests, the sphere spectrum $\mathbb{S}$ stems from geometry. It was originally defined as a certain sequence of spheres and homeomorphisms, and the theory of spectra began with it. In the eighties, Waldhausen realized that $\mathbb{S}$ should be considered as a ring deeper than $\mathbb{Z}$, that spectra should be viewed as generalized abelian groups, and that ring spectra should be viewed as generalized rings. It is fair to say that that was the starting point for the theory of so-called brave new rings. Over the last couples of decades, many topologists have been studying these (for instance, [GoHo], [HHR], [Lurie], [Goodwillie2], [Waldhausen]). This thesis joins the sequence of such quests.

In the nineties, nice symmetric monoidal products of spectra were found ([EKMM], [HSS], [Lydakis]). With respect to the symmetric monoidal products, the Eilenberg-MacLane functor sends rings to monoid objects in spectra, which are called ring spectra. Therefore ring spectra literally turned to be viewed as generalized rings. The focus of this thesis will be on various homology theories of those generalized rings such as topological Hochschild homology THH, topological cyclic homology TC, and periodic topological cyclic homology TP. They can be studied for instance in relation to algebraic K-theory with trace methods via the following diagram ([NS], [DGM]),

$$\begin{array}{ccccc} K & \longrightarrow & \mathrm{TC} & \longrightarrow & \mathrm{TP} \\ & & & \searrow & \\ & & & & \mathrm{THH}\,. \end{array}$$

It is safe to say that mathematicians have thought that algebraic K-theory deserves to be considered as a significant object in many branches of mathematics. However, usually it is very difficult to compute, and in some cases calculations of the homology theories mentioned above are comparatively easier. Moreover, in some cases, relative and birelative K and TC coincide. In this sense, such theories have contributed to our understanding of algebraic K-theory. One of our main results needs this coincidence of K and TC.

On the other hand, they are important in their own right especially from the view point of p-adic Hodge theory. By Connes' theory of cyclic objects, THH has a canonical action by the circle group $\mathbb{T}$, which is necessary for the theory of cyclotomic spectra. Hesselholt and Madsen have proven that, using the theory of cyclotomic spectra (which has been reinterpreted in [NS] recently via $(\infty, 1)$-category theory), the 0-th stable homotopy group (which is actually a ring in this case) of fixed points of the subgroup $C_{p^n} \subset \mathbb{T}$ of order p^n of a commutative ring A is isomorphic to the ring of p-typical n-length Witt vectors of the commutative ring,

$$\pi_0(\mathrm{THH}(A)^{C_{p^n}}) \cong W_{p,n}(A).$$

Furthermore, this ring isomorphism is compatible with structure maps, Verschiebung, Frobenius and restriction, which we will review in section 4. It is fair to say that this theorem triggered the growing theory of the connection between p-adic Hodge theory and stable homotopy theory ([BMS2], [NS], [Hesselholt2], [HM4]).

In the spirit of this connection, we show the following two results as our main theorems in section 5. The first result shows that periodic topological cyclic homology TP is not nil-invariant. That is, there is a ring R and a nilpotent ideal I such that the canonical map $R \to R/I$ does not induce an equivalence on TP. We actually show that the map

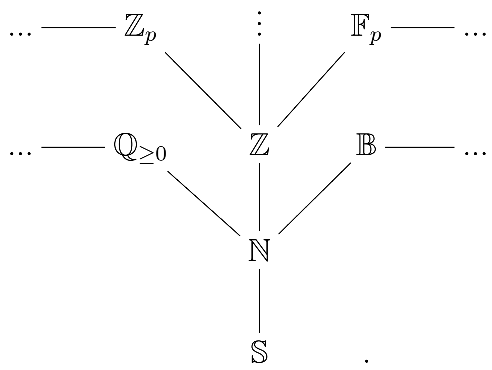
$$\mathrm{TP}(\mathbb{F}_p[x]/(x^k)) \to \mathrm{TP}(\mathbb{F}_p)$$

induced by the canonical projection is not an isomorphism for any prime number p and any natural number k greater than 1, even after inverting p. We remark that [BlMa] and [AMN] have shown that TP behaves very well on dg categories smooth and proper over a perfect field of positive characteristic. More precisely, it satisfies the Künneth formula on those objects. We also note that Hesselholt gives an interpretation of certain zeta functions by TP in [Hesselholt2]. In this way TP originating from stable homotopy theory contributes to arithmetic geometry and number theory. Our result shows one of its fundamental properties, which says TP can distinguish points and fat points. The second result evaluates, in terms of Verschiebung maps in THH, the maps of relative algebraic K-groups

$$K_*(A[x]/(x^m), (x)) \to K_*(A[x]/(x^{nm}), (x))$$

induced by the substitution of x^n for x. For A a regular $\mathbb{F}_p$-algebra, the maps can be further expressed in terms of Verschiebung maps of big de Rham-Witt groups using the translation between THH and de Rham-Witt complexes due to Hesselholt. Taking the colimit along maps defined above for A a perfect field of characteristic $p > 0$, we give a calculation of the relative algebraic K-groups of $\mathcal{O}_K/p\mathcal{O}_K$ for various perfectoid fields K, including $K = \mathbb{Q}_p(p^{1/p^\infty})^\wedge$ and $K = \mathbb{Q}_p(\zeta_{p^\infty})^\wedge$. There is no stable homotopy theory in their statements of these results, although it is needed in their proofs. In this way, number theory has been helped by the theory of ring spectra.

This fruitful chemistry that might integrate homotopy theory and arithmetic geometry should be interpreted from a combinatorial or discrete framework. This idea is due to [CC] and [Connes]. As we will see in section 2, $\mathbb{S}$ can be defined as the inclusion functor from (a skeleton of) the category of pointed finite sets and pointed maps to that of pointed sets and pointed maps. In order to define $\mathbb{S}$ in this fashion, we do not use any homotopy theory. We will also see that, using Segal's Γ-sets ([CC], [Segal]) and the Eilenberg-MacLane functor, $\mathbb{S}$ is indeed a deeper base than $\mathbb{N}$. In this way, we shall have the following diagram of numbers in the category of Γ-sets;



Moreover, Borger recently established the theory of Witt vectors for commutative semirings using plethystic algebra ([Borger2], [BW]). It may be reasonably expected that there should be the theory of semiring spectra according to Hesselholt-Madsen's theorem and Connes-Consani's philosophy, which has not yet been well studied. If we could extend the theorem to commutative semirings, it would be the trigger for a new geometry. That is to say, $\mathbb{S}$ has been playing an important role for a certain geometry of generalized $\mathbb{Z}$-algebras and is probably ready for a new geometry of generalized $\mathbb{N}$-algebras, which the final section of this thesis is about. This is the reason why our observations, which are hopefully not laden by the usual homotopy theory pretty much, are made.

This thesis is organized as follows. Sections 2, 3 and 4 are preliminary parts. In those sections, we review some basic facts to explain how our contributions appear in the sequence of studies mentioned above. We do not give their proofs. In section 2, we will review Γ-sets following Connes-Consani's paper ([CC]). We fix a skeleton Γ^{op} of the category of pointed finite sets and pointed maps and define a Γ-set to be a pointed functor from Γ^{op} to the category Set_* of pointed sets and pointed maps. The category $\mathrm{Mod}_{\mathbb{S}}$ of Γ-sets has a symmetric monoidal structure defined by a Day convolution whose unit is $\mathbb{S}$ and admits a fully faithful functor H, called the Eilenberg-MacLane functor, from the category $\mathrm{Mod}_{\mathbb{N}}$ of commutative monoids. We will especially focus on how the symmetric monoidal structures and H are related to get the diagram of numbers above.

In section 3, we will review the theory of Γ-spaces (i.e. simplicial objects in Γ-sets) and the theory of spectra. The category Γ-Sp of Γ-spaces still has a symmetric monoidal structure defined as a Day convolution again. Furthermore it admits a model structure which is compatible with said symmetric monoidal structure ([Lydakis]). In effect, Γ-Sp gives a model of connective spectra which are equivalent to localized symmetric monoidal categories in the sense of Thomason [Thomason]. After reviewing connective spectra, we will also review (non-connective) spectra by symmetric spectra. Some facts on stable homotopy theory are mentioned as well in order to define the homology theories used for our main results.

In section 4, we will review topological Hochschild homology THH and some variants thereof. In order to show the meanings of our main theorems, some famous results will be explained, such as the relation proved by Hesselholt between THH and the de Rham Witt complexes. Although the theory of cyclotomic spectra is essential, we do not step into the deep theory for simplicity.

In section 5, our main two results explained above will be shown. We remark that we focus on Γ-sets because that category is where $\mathbb{S}$ lives. We do need non-connective spectra to study TP for instance.

In the final section, there will be some observations on commutative semirings and homotopy theory as suggestions for future work. We do not have theorems, but pose some questions.

1.1 Acknowledgements

First of all, I would like to express my sincerely gratitude to my adviser Lars Hesselholt for his patient support and constant encouragements for several years. Without his help and guidance, it would not be possible to complete this thesis. I remember that day, in front of the entrance Nagoya University mathematics department building, he told me he was going to Copenhagen and invited me there. I also sincerely appreciate him bringing me to Copenhagen where I encounter a beautiful three years.

I owe a great debt to both of Shien-shitsu staffs in Nagoya and administrative staffs in Copenhagen. It would be more complicated than usual to care double degree students, but they are always helpful and kind.

My special thanks go to my colleagues for anything concerning mathematics; Alexander, Alexandra, Amalie, Benjamin, Christopher, Clarisson, Clemens, David, Dieter, Dustin, Eduardo, Espen, Francesco, Gergely, Gijs, Isabelle, Irakli, Jens, Joshua, Kevin, Kristian, Manuel, Markus, Martin, Massimiliano, Matthias, Mikala, Niek, Rune, Simon, Tobias, Tomasz, Valerio, Vibeke. I thank them for everything fun in the three years as well. I am also indebted to Inger and Hans Ole for their warm hospitality, so that I live in their house comfortably.

Finally, I would take the liberty of using my mother tongue for my parents. なおざりに扱われた覚えがないということに助けられたこともあった気がします。かたじけなく思っています。

2 Modules and algebras over $\mathbb{S}$

We start with the combinatorial framework for modules and algebras over the sphere spectrum given by [CC]. In this section, we collect basic notions about it. Throughout this thesis the set of natural numbers contains 0.

2.1 The sphere spectrum and smash product

In this section, we recall Connes-Consani's study ([CC]) of the sphere spectrum, which is the central object for this thesis. They pointed out that the category of Γ-sets, which are discrete spectra in a certain sense, is an appropriate category to study algebras. Although they also study hyper algebras, we focus on $\mathbb{N}$-algebras via the Eilenberg-MacLane functor, which originates in algebraic topology.

Definition 2.1 ([Segal], [CC])**.** Γ^{op} *is the category whose objects are the finite pointed sets* $n_+ := \{0, 1, ..., n\}$ *with the base point* 0 *for every* $n \in \mathbb{N}$ *and whose morphisms are pointed maps.*

The category has been considered by many topologists when studying infinite loop spaces, topological abelian groups, and operations on homology theories. However, as mentioned above, we are concerned only with the discrete ones in this section.

Definition 2.2 ([Segal], [CC])**.** *A* Γ*-set is a functor* X *from* Γ^{op} *to the category* Set_* *of pointed sets and pointed maps such that* $X(0_+)$ *is one-point set. The category of* Γ*-sets and natural transformations is denoted by* $\mathrm{Mod}_{\mathbb{S}}$*.*

This is our main object in this section and we reach the following example.

Example 2.3. The inclusion functor from Γ^{op} to Set_* is called the *sphere spectrum* and denoted by $\mathbb{S}$.

The category $\mathrm{Mod}_{\mathbb{S}}$ has a symmetric monoidal product $\otimes_{\mathbb{S}}$ called *smash product* with $\mathbb{S}$ as unit. To define it, we write $\wedge$ for the smash product of pointed sets. Note that $n_+ \wedge m_+ = nm_+$.

Definition 2.4 ([Day], [Lydakis])**.** *Let X, Y be Γ-sets. The smash product $X \otimes_{\mathbb{S}} Y$ of X and Y is the left Kan extension of the following diagram*

$$\begin{array}{ccc} \Gamma^{\mathrm{op}} & & \\ {\scriptstyle \wedge}\uparrow & \searrow^{X \otimes_{\mathbb{S}} Y} & \\ \Gamma^{\mathrm{op}} \times \Gamma^{\mathrm{op}} & \xrightarrow[X(-)\wedge Y(-)]{} & \mathsf{Set}_*, \end{array}$$

where $X(-) \wedge Y(-)$ denotes the piecewise smash product in Set_*.

Let X, Y be Γ-sets. The internal Hom in Γ-sets, $\Gamma(X, Y)$, is defined by:

$$\Gamma(X, Y)(n_+) := \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{S}}}(X, Y(- \wedge n_+)),$$

where $Y(- \wedge n_+)$ the Γ-set given by

$$Y(- \wedge n_+)(m_+) = Y(mn_+)$$

and $\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{S}}}(X, Y(- \wedge n_+))$ is the set of morphisms of Γ-sets.

Lemma 2.5 ([CC], [Lydakis])**.** *The above constructions $(-) \otimes_{\mathbb{S}} (-)$ and $\Gamma(-,-)$ induce an adjunction on $\mathrm{Mod}_{\mathbb{S}}$.*

For every n_+, we let Γ^n denote the Γ-set represented by n_+, namely, $\Gamma^n(m_+) = \mathrm{Hom}_{\Gamma^{op}}(n_+, m_+)$. So $\mathbb{S} = \Gamma^1$. We get the following.

Lemma 2.6 ([Lydakis], Proposition 2.15)**.** *$\Gamma^n \otimes_{\mathbb{S}} \Gamma^m$ is canonically isomorphic to Γ^{nm}. In particular, $\mathbb{S} \otimes_{\mathbb{S}} \Gamma^n$ is canonically isomorphic to Γ^n for any n_+.*

Using these lemmas, we can define a closed symmetric monoidal structure. Omitting some of the structures, we have the folowing.

Theorem 2.7 ([Lydakis])**.** *The triple* $(\mathrm{Mod}_{\mathbb{S}}, \otimes_{\mathbb{S}}, \mathbb{S})$ *is a closed symmetric monoidal category.*

Therefore, we can talk about monoid objects in $(\mathrm{Mod}_{\mathbb{S}}, \otimes_{\mathbb{S}}, \mathbb{S})$, which we call $\mathbb{S}$-*algebras*, and study them in the next section. The category of $\mathbb{S}$-algebras and $\mathbb{S}$-algebra morphisms is denoted by $\mathrm{Alg}_{\mathbb{S}}$.

From now on, we recall a relation between $\mathrm{Mod}_{\mathbb{S}}$ and symmetric monoidal categories. In order for that, we recall special and very special Γ-sets. We need the following maps of Γ-sets to define them: $s : 2_+ \to 1_+$ with $s^{-1}(1) = \{1, 2\}$ and, for any n_+ and $i \in n_+$, $p_i : n_+ \to 1_+$ with $p_i^{-1}(1) = \{i\}$.

Definition 2.8 ([CC], [Segal])**.** *Let* X *be a* Γ*-set. We say that* X *is special if the map*

$$\prod_i p_i : X(n_+) \to \prod_i X(1_+)$$

is a bijection.

If a Γ-set X is special, then the composite

$$X(1_+) \times X(1_+) \xrightarrow{(p_1,p_2)^{-1}} X(2_+) \xrightarrow{s} X(1_+)$$

defines a commutative monoid structure on $X(1_+)$.

Definition 2.9. *A special* Γ*-set* X *is very special if the commutative monoid structure on* $X(1_+)$ *is a commutative group structure.*

This condition, called the Segal condition, plays a role in stable homotopy theory. As we will see in the next chapter, it is known that objects in the stable homotopy category of Γ-spaces have to be very special in a certain sense ([DGM, Corollary 2.2.1.7]).

Segal constructed a functor called Segal's K-theory or direct sum K-theory. The explicit construction of that is written in for example [DGM, 2.3]. Let $(\mathcal{C}, \otimes, u)$ be a symmetric monoidal category. For $k_+ \in \Gamma^{\mathrm{op}}$, we define a category $\tilde{K}(\mathcal{C})(k_+)$. An object $(a, \alpha) \in \tilde{K}(\mathcal{C})(k_+)$ consists of the following data:

A function $a : \mathcal{P}(\{1, ..., k\}) \to \mathcal{C}$ from the power set of $\{1, ..., k\}$ to $\mathcal{C}$ and a collection α of maps $\alpha_{S,T} : a_S \otimes a_T \to a_{S \sqcup T}$ in $\mathcal{C}$ for pairs $S, T \in \mathcal{P}(\{1, ..., k\})$ such that $a_\emptyset = u$ and $\alpha_{S,\emptyset}$ and $\alpha_{\emptyset,S}$ are the structure maps subject to the evident associativity and commutativity conditions.

A morphism $f : (a, \alpha) \to (b, \beta) \in \tilde{K}(\mathcal{C})(k_+)$ is a family of morphisms $f_S : a_S \to b_S$ such that $f_\emptyset = \mathrm{id}_u$ and $f_{S \sqcup T} \circ \alpha_{S,T} = \beta_{S,T} \circ (f_S \otimes f_T)$. For a map $\theta : k_+ \to l_+ \in \Gamma^{\mathrm{op}}$, the induced map $\theta_* : \tilde{K}(\mathcal{C})(k_+) \to \tilde{K}(\mathcal{C})(l_+)$ is $\theta^{-1} : \mathcal{P}(\{1, ..., l\}) \to \mathcal{P}(\{1, ..., k\})$.

The construction defines a functor

$$\tilde{K} : \mathsf{SymMonCat} \to \mathsf{Special}\Gamma\text{-}\mathsf{Cat},$$

where $\mathsf{Special}\Gamma\text{-}\mathsf{Cat}$ is the category of Γ-objects in the category Cat of categories which is special in the obvious sense. This functor $\tilde{K}$ is used for the famous theorem of Thomason [Thomason] which says the homotopy theories of symmetric monoidal categories and connective spectra are equivalent. We will review it later.

Remark 2.10. *Quillen proved that $\tilde{K}$ of the symmetric monoidal category of finite rank projective modules over a commutative ring R gives a model for the algebraic K-theory of R. See [Mandell].*

One of the motivations of introducing Γ-objects is to manipulate symmetric monoidal objects systematically ([Segal]). In this section we learn that the sphere spectrum $\mathbb{S}$ inhabits Γ-objects in sets and furthermore that it is the unit object of the closed symmetric monoidal category. The author thinks that sets is a fundamental object and it is reasonable to view $\mathbb{S}$ as a fundamental symmetric monoidal object as well. In the next section, we focus on commutative monoids that are discrete symmetric monoidal categories in the usual sense.

2.2 Eilenberg-MacLane spectra

Let us write $\mathrm{Mod}_{\mathbb{N}}$ for the category of abelian monoids and additive morphisms. This category admits the symmetric monoidal structure which we denote by $(\mathrm{Mod}_{\mathbb{N}}, \otimes_{\mathbb{N}}, \mathbb{N})$ ([Borger2]). We now consider the two symmetric monoidal categories $(\mathrm{Mod}_{\mathbb{N}}, \otimes_{\mathbb{N}}, \mathbb{N})$ and $(\mathrm{Mod}_{\mathbb{S}}, \otimes_{\mathbb{S}}, \mathbb{S})$ via the following functor.

Definition 2.11 ([CC], [DGM]). *Eilenberg-MacLane functor $H : \mathrm{Mod}_{\mathbb{N}} \to \mathrm{Mod}_{\mathbb{S}}$ is given by*

$$M \in \mathrm{Mod}_{\mathbb{N}},\ HM(k_+) := M^{\times k},$$

$$f : n_+ \to m_+ \in \Gamma^{op}, HM(f) : HM(m_+) \to HM(n_+),$$

$$HM(f)(\phi)_i = \sum_{j \in f^{-1}(i)} \phi_j,$$

where ϕ_j is the j-th factor of $\phi \in M^{\times n}$.

If a commutative monoid M is viewed as a discrete symmetric monoidal category, then its Eilenberg-MacLane spectrum HM and $\tilde{K}(M)$ coincide, up to canonical isomorphism.

By construction, we have the following.

Proposition 2.12. *For a commutative monoid M, the Eilenberg-MacLane object HM is a special Γ-set. For a commutative group G, the Eilenberg-MacLane object HG is a very special Γ-set.*

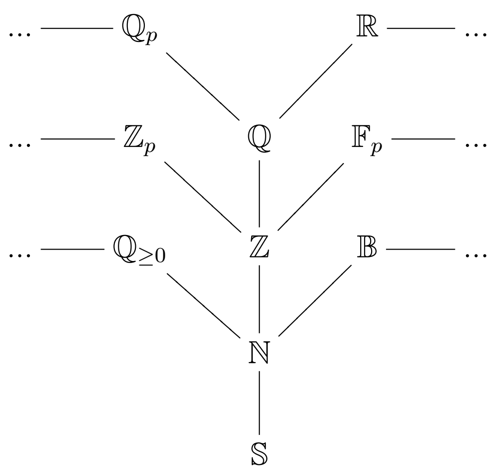
As we will see later, the objects that stable homotopy theory can study are very special. Therefore we are unable to study monoids with stable homotopy theory via Eilenberg-MacLane functor. Also, by definition, we have the following.

Proposition 2.13. *For A and $B \in \mathrm{Mod}_{\mathbb{N}}$, $H(\mathrm{Mod}_{\mathbb{N}}(A, B)) \cong \Gamma(HA, HB)$, where $\mathrm{Mod}_{\mathbb{N}}(A, B)$ denotes the hom-monoid.*

By construction, H is fully faithful and, by the universality of the left Kan extension, it is lax monoidal with respect to above symmetric monidal structures $\otimes_{\mathbb{S}}$ and $\otimes_{\mathbb{N}}$. Moreover, by abstract nonsense, it has a left adjoint, which we denote by $(-) \otimes_{\mathbb{S}} \mathbb{N}$. Connes and Consani have proved the following theorem.

Theorem 2.14 ([CC])**.** *The Eilenberg-MacLane functor induces a fully faithful functor $H : \mathrm{Alg}_{\mathbb{N}} \to \mathrm{Alg}_{\mathbb{S}}$.*

This theorem suggests that it may be reasonable to study $\mathbb{N}$-algebras and commutative monoids via H in $\mathrm{Mod}_{\mathbb{S}}$, since H embeds $\mathbb{N}$-algebras into $\mathbb{S}$-algebras. In other words, the following diagram of Γ-sets will be thought as a diagram of numbers.



where we omit H, since it is a fully faithful embedding. In the next section, we will see how this diagram appears in the stable homotopy theory. Roughly speaking, this diagram is contorted, since stable homotopy theory does not distinguish between $\mathbb{N}$ and $\mathbb{Z}$.

Since every Γ-set can be written by a colimit of representable functors Γ^n, using the lemma 2.6, we can justify to consider the left adjoint $(-) \otimes_{\mathbb{S}} \mathbb{N}$ as a *base change.*

Proposition 2.15. *For Γ-sets X and Y, there is a canonical isomorphism*

$$(X \otimes_{\mathbb{S}} \mathbb{N}) \otimes_{\mathbb{N}} (Y \otimes_{\mathbb{S}} \mathbb{N}) \cong (X \otimes_{\mathbb{S}} Y) \otimes_{\mathbb{S}} \mathbb{N}.$$

Again, since every Γ-set can be written by a colimit of representable functors, we have the following:

Proposition 2.16. *The base change functor $(-)\otimes_{\mathbb{S}}\mathbb{N}$ is symmetric monoidal.*

Using this, we have

Proposition 2.17 ([CC], [Day], [Lydakis])**.** *The adjuction* $((-) \otimes_{\mathbb{S}} \mathbb{N}, H)$

induces an adjunction of the categories of monoid objects,

$$(-) \otimes_{\mathbb{S}} \mathbb{N} : \mathrm{Alg}_{\mathbb{S}} \rightleftarrows \mathrm{Alg}_{\mathbb{N}} : H.$$

In view of [CC] and [Borger1], it should be considered how H and λ-structures are related each other. However, we will not consider that here.

We end this chapter with some calculations of Γ-sets by Connes-Consani. Let $\mathbb{B}$ be the Boolean semifield. As a set, it is $\{0, 1\}$ and the commutative multiplication and the addition are given as follows:

$$0 = 0 + 0 = 0 \cdot 1 = 0 \cdot 0, 1 = 1 + 1 = 1 + 0 = 1 \cdot 1.$$

It is not a ring, but its Eilenberg-MacLane is an $\mathbb{S}$-algebra. We note that $H\mathbb{B}$ is not very special, so that it can not survive in stable homotopy theory.

Definition 2.18 ([CC])**.** *Let k be a natural number.*
(i) A k-relation is a triple $C = (F, G, v)$ where F and G are non-empty finite sets and $v : F \times G \to k_+$ is a map of sets such that no line or column of the corresponding matrix is identically 0.
(ii) A k-relation is reduced if no line and no column is repeated.

Using these notions, Connes and Consani characterized $H\mathbb{B} \otimes_{\mathbb{S}} H\mathbb{B}$ as follows.

Proposition 2.19 ([CC], Theorem 4.9)**.** *Let $\mathbb{B}$ be the Boolean semifield. Then $H\mathbb{B} \otimes_{\mathbb{S}} H\mathbb{B}$ is isomorphic to the Γ-set $\mathcal{R}_+$, where $\mathcal{R}_+(k_+)$ is the pointed set of isomorphism classes of reduced k-relations. In particular it is not isomorphic to $H\mathbb{B}$.*

We note that $\mathbb{B} \otimes_{\mathbb{N}} \mathbb{B} \cong \mathbb{B}$ and $(H\mathbb{B} \otimes_{\mathbb{S}} H\mathbb{B})(1_+)$ is an infinite set, while $H\mathbb{B}(1_+)$ is $\mathbb{B}$. In other words, the product $\otimes_{\mathbb{S}}$ is rather more involved than

$\otimes_{\mathbb{N}}$. The cyclic bar construction of $H\mathbb{B}$ with respect to $\otimes_{\mathbb{S}}$ may have some interesting information, but we do not know this object well so far.

Proposition 2.20 ([CC], Proposition 7.4)**.** *$H\mathbb{Z} \otimes_{\mathbb{S}} H\mathbb{Z}$ is not isomorphic to $H\mathbb{Z}$.*

Using some homotopical replacement in a way, similar statement can be proved ([Kochman, Theorem 3.5]). More precisely, it has been known for a long time that $H\mathbb{Z} \otimes_{\mathbb{S}} H\mathbb{Z}$ and $H\mathbb{Z}$ are not weak equivalent. However the proof in [CC] does not need any homotopical method or rather, there is no notion of homotopy for Γ-sets. This proposition which is one of what $\mathbb{S}$ and $\otimes_{\mathbb{S}}$ are expected to satisfy (compare to Durov's $\mathbb{F}_1$ [Durov]) is proved by a non-homotopical method. It would be possible to speculate that $\mathbb{S}$ is a new class of numbers that is deeper than $\mathbb{N}$.

In order to summarize this section, we emphasize the following again; There is a symmetric monoidal category called $\mathrm{Mod}_{\mathbb{S}}$ which contains any commutative monoids and semirings and whose unit is deeper than the initial commutative semiring $\mathbb{N}$ of natural numbers.

3 Spectra and its homotopy theory

To give some background for our results, in this section, we review some basics of stable homotopy theory which seems to be the most effective theory so far to study the sphere spectrum.

3.1 Γ-spaces

Simplicial objects in Γ-sets are called Γ-spaces. The category of Γ-spaces admits a model structure for connective spectra ([BF]). We recall some basic stable homotopy theory and how the last chapter relates to it. We let $s\mathsf{Set}_*$ denote the category of pointed simplicial sets and pointed morphisms.

Definition 3.1 ([BF], [Segal])**.** *A functor* $X : \Gamma^{\mathrm{op}} \to s\mathsf{Set}_*$ *is a* Γ*-space if* $X(0_+)$ *is a contractible space.*

We let Γ-Sp denote the category of Γ-spaces and natural transformations. For any Γ-set X, we also let X denote the Γ-space $\iota \circ X$ composed with constant inclusion $\iota : \mathsf{Set}_* \to s\mathsf{Set}_*$. This category Γ-$Sp$ has properties similar to $\mathrm{Mod}_{\mathbb{S}}$. We first give it a symmetric monoidal structure as follows, abusing notation.

Definition 3.2 ([Day], [Lydakis])**.** *Let* X, Y *be* Γ*-spaces. Then the smash product* $X \otimes_{\mathbb{S}} Y$ *of* X *and* Y *is the left Kan extension of the following diagram*

$$\begin{array}{ccc} \Gamma^{\mathrm{op}} & & \\ \uparrow{\scriptstyle \wedge} & \overset{X \otimes_{\mathbb{S}} Y}{\searrow} & \\ \Gamma^{\mathrm{op}} \times \Gamma^{\mathrm{op}} & \xrightarrow[X(-) \wedge Y(-)]{} & s\mathsf{Set}_*, \end{array}$$

where $X(-) \wedge Y(-)$ *denotes the degreewise smash product in* $s\mathsf{Set}_*$.

This product is also closed. For Γ-spaces X and Y, we define the mapping Γ-space $\Gamma(X, Y)$ by

$$\Gamma(X,Y)(k_+,[n]) := \mathrm{Hom}_{\Gamma\text{-}Sp}(X \otimes_{\mathbb{S}} (\Delta[n])_+, Y(k_+ \wedge -)),$$

where $X \otimes_{\mathbb{S}} (\Delta[n])_+$ is the Γ-space given by

$$(X \otimes_{\mathbb{S}} (\Delta[n])_+)(k_+) := X(k_+) \wedge \Delta[n]_+,$$

and $Y(k_+ \wedge -)$ is the Γ-space given by $Y(k_+ \wedge -)(l_+) = Y(kl_+)$ for $l_+ \in \Gamma^{\mathrm{op}}$ and $\mathrm{Hom}_{\Gamma\text{-}Sp}(-,-)$ denotes the hom-set.

Theorem 3.3 ([Lydakis]). *Above constructions give rise to a closed symmetric monoidal structure on* Γ*-Sp.*

For short, we let (Γ-Sp, $\otimes_{\mathbb{S}}$, $\mathbb{S}$) denote the symmetric monoidal category. By this theorem, we can talk about monoid objects.

Definition 3.4. *An* $\mathbb{S}$*-algebra is a monoid object in* (Γ*-Sp,* $\otimes_{\mathbb{S}}$*,* $\mathbb{S}$)*.*

We use the name $\mathbb{S}$-algebra again. Monoid objects in $\mathrm{Mod}_{\mathbb{S}}$ are also monoid objects in Γ-Sp via the inclusion $\iota : \mathsf{Set}_* \to s\mathsf{Set}_*$. There are some examples.

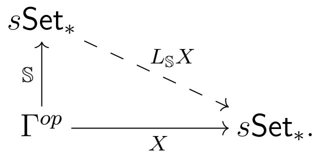
Example 3.5. *(i) The sphere spectrum* $\mathbb{S}$ *with its unique monoid structure is an* $\mathbb{S}$*-algebra,*
(ii) The Eilenberg-MacLane spectra $H(A)$ *for any ring* A *with the canonical monoid structure induced by the universality of the left Kan extension and the monoid structure on* A *is an* $\mathbb{S}$*-algebra*
(iii) For a simplicial monoid M*, the spherical monoid algebra* $\mathbb{S}[M]$ *given by* $\mathbb{S}[M](k_+) = M_+ \otimes_{\mathbb{S}} k_+$ *with the monoid structure given in [DGM, 2.1.4.1]*

Theorem 3.6 ([Lydakis], [DGM]). *The construction above gets the category* Γ-*Sp a category enriched over the symmetric monoidal category* (Γ-*Sp*, $\otimes_{\mathbb{S}}$, $\mathbb{S}$).

Note that $\Gamma(X, Y)(1_+)$ is a pointed simplicial set for Γ-spaces X, Y. We get the following corollary.

Corollary 3.7 ([Lydakis]). Γ-*Sp is an* $sSet_*$*-enriched category via the above constructioin.*

We now recall how Γ-spaces relate to symmetric spectra ([BF], [DGM]). For a Γ-space X, we have an endofunctor on pointed simplicial sets given by the left Kan extension



Then we define a symmetric spectrum $X(\mathbb{S})$ associated to X, whose n-th term is $L_{\mathbb{S}}X(S^n)$, where S^n is the smash product of n copies of the circle S^1. We will abuse notation and write X for $L_{\mathbb{S}}X$. This construction $(-)(\mathbb{S})$ defines a functor from Γ-*Sp* to the category of symmetric spectra ([DGM]). By construction, $\mathbb{S}(\mathbb{S})$ is the sphere spectrum of the standard form, namely, $\mathbb{S}(S^n) = S^n$. In [MMSS], the authors give the category of symmetric spectra a closed symmetric monoidal structure.

Theorem 3.8 ([MMSS]). *The functor* $(-)(\mathbb{S})$ *is symmetric monoidal.*

By this theorem, monoid objects in Γ-*Sp* stay monoid objects in symmetric spectra after sent by $(-)(\mathbb{S})$. In other words, this functor does not break algebra structures.

We now recall a theorem by [Mandell] on Γ-spaces, before stabilization.

Definition 3.9 ([Segal]). *Let X be a Γ-space. We say X is special if the map*

$$\prod_i p_i : X(n_+) \to \prod_i X(1_+)$$

is a weak homotopy equivalence. A special Γ-space X is very special if the induced commutative monoid structure on $\pi_0(X(1_+))$ is a commutative group structure.

We let $\mathsf{SymMonCat}/\sim$ denote the localization of symmetric monoidal categories with respect to weak homotopy equivalences. More precisely, a morphism f of $\mathsf{SymMonCat}$ is a weak homotopy equivalence if $N \circ U(f)$ is weak homotopy equivalence of simplicial sets, where U is the forgetful functor $\mathsf{SymMonCat} \to \mathsf{Cat}$ and N is the nerve functor $\mathsf{Cat} \to s\mathsf{Set}$. We also let $\mathsf{Special}\Gamma\text{-}\mathsf{Sp}/\sim$ denote the localization of special Γ-spaces with respect to objectwise weak homotopy equivalnces.

Theorem 3.10 ([Mandell]). *The functor $N \circ \tilde{K}$ induces an equivalence of categories between $\mathsf{SymMonCat}/\sim$ and $\mathsf{Special}\Gamma\text{-}\mathsf{Sp}/\sim$, where N is the degree-wise nerve functor.*

The category of simplicial sets admits another model structure that models $(\infty, 1)$-categories [Joyal].

Definition 3.11. *Let X be a Γ-space. We say X is quasi-special if the map $\prod_i p_i : X(n_+) \to \prod_i X(1_+)$ is a weak equivalence in the sense of Joyal.*

Example 3.12. *$N \circ \tilde{K}(\mathcal{C})$ is a quasi-special Γ-space for a symmetric monoidal category $\mathcal{C}$.*

We let $\mathsf{q.s.}\Gamma\text{-}\mathsf{Sp}/\!\sim$ denote the localization with respect to objectwise Joyal equivalnces of quasi-special Γ-spaces which are piecewise quasi-cateories.

Proposition 3.13. *The adjunction* $(-)(1_+) \dashv H$ *is an adjoint equivalence* $\mathsf{q.s.}\Gamma\text{-}\mathsf{Sp}/\!\sim \simeq \mathsf{SymMon}(\mathsf{QCat})$.

Remark 3.14. *There is the* (∞, n)*-categorical analogue of the Segal condition for* $n \geq 0$ *and symmetric monoidal* (∞, n)*-categories are defined to be such special* Γ*-objects in* (∞, n)*-categories. See [Barwick, 3.1].*

Next, we will review some basic facts about stable homotopy theory.

3.2 Stable homotopy theory

Stable homotopy theory is the most successful way to analyze $\mathbb{S}$ so far, although there might be a more refined way to approach it. In this section, we recall some basics of stable homotopy theory, mainly using Γ-spaces for simplicity.

Definition 3.15 ([BF], [DGM])**.** *For a* Γ*-space* X *and* $n \in \mathbb{Z}$*, the* n*-th stable homotopy group* $\pi_n(X)$ *is the abelian group* $\operatorname{colim}_{k\to\infty} \pi_{k+n}(X(S^k))$. *A map of* Γ*-spaces is a stable weak equivalence if it induces an isomorphism on stable homotopy groups in each degree.*

This defines the weak equivalences of the following model structure called the stable model structure.

Theorem 3.16 ([BF], [BeMo], [Lydakis])**.** *The symmetric monoidal simplicial category* $(\Gamma\text{-}Sp, \otimes_{\mathbb{S}}, \mathbb{S})$ *admits the following symmetric monoidal simplicial model structure; weak equivalences are stable weak equivalences and cofibrations are generalized Reedy cofibrations.*

The following is a remarkable property of Γ-spaces. We stress again that spectra have negative stable homotopy groups in general.

Proposition 3.17 ([DGM], Lemma 2.2.13)**.** *Let X be a Γ-space. For every negative integer n, $\pi_n(X)$ is 0.*

The Eilenberg-MacLane functor also extends to $H : s\,\mathrm{Mod}_{\mathbb{N}} \to \Gamma\text{-}Sp$ degreewise ([DGM, Example 2.1.2.1]), where $s\,\mathrm{Mod}_{\mathbb{N}}$ is the category of simplicial abelian monoids. Moreover, this functor is lax monoidal [DGM, 2.1.4.1], fully faithful, fully faithful for monoid objects, and has a left adjoint. Therefore, we are able to study simplicial abelian monoids and semirings in $\Gamma\text{-}Sp$ via Eilenberg-MacLane functor H. However, as Lydakis shows and we will see it later, $H(A)$ is not cofibrant in the stable model structure for any nontrivial simplicial monoid A and is not necessarily fibrant. More precisely, the homotopy category can only study very special Γ-spaces and $H(\mathbb{N})$ is not very special.

By the theorem above, cofibrant objects are compartible with $\otimes_{\mathbb{S}}$, since the model structure is monoidal, and by the basic theorem about model categories, every object in the homotopy category can be represented by a fibrant and cofibrant object. Lydakis gave a criterion of cofibracy. Note that $\mathrm{Aut}_{\Gamma^{op}}(n_+)$ is the n-th symmetric group Σ_n.

Lemma 3.18 ([Lydakis], §3)**.** *Let n be a natural number, let X be a Γ-space, and let $X^{(n)}$ denote the n-skeleton. A Γ-space X is cofibrant if and only if the Σ_n-action on $X/X^{(n-1)}(n_+)$ is free for all n.*

Example 3.19 ([Lydakis], Proposition 3.2.)**.** *For every n, Γ^n is cofibrant. In particular, $\mathbb{S}$ is cofibrant.*

The Eilenberg-MacLane spectra, however, are not cofibrant.

Proposition 3.20 ([Lydakis], 3.4)**.** *For any non-trivial simplicial commutative monoid A, $H(A)$ is not cofibrant.*

The bar-construction and the cyclic bar-construction are heavily used in algebraic topology. Our main example of a symmetric monoidal product is the smash product $\otimes_{\mathbb{S}}$. For example, it would be convenient to define topological Hochschild homology, THH, to be the geometric realization of the cyclic bar-construction with respect to $\otimes_{\mathbb{S}}$ ([NS, III 2.3], [PS], [Shipley]). However by the above proposition by Lydakis, to define THH of an Eilenberg-MacLane spectrum in this way, we need to take cofibrant replacement of it. We note that cofibrant replacements cannot be Eilenberg-MacLane spectra. We also note that, in $(\infty, 1)$-categorical language, it does not matter whether we use cofibrant objects or not. Since the cofibrant replacement for our model structure is given by abstract nonsense, it is very difficult to track what happens. In this sense, the proposition is critical.

Fibrant objects are characterized by cofibrations and weak equivalences abstractly. Here is an explicit criterion for fibrancy.

Proposition 3.21 ([Lydakis], 5.7)**.** *Fibrant objects with respect to the model structure are very special.*

Eilenberg-MacLane objects do not behave well in homotopy theory. Especially, Eilenberg-MacLane objects that are not grouplike are not fibrant.

Example 3.22. *The* Γ*-space* $H(\mathbb{N})$ *is special but not very special. The* Γ*-space* $H(\mathbb{Z})$ *is very special.*

Therefore the homotopy category can not detect $H(\mathbb{N})$. As we saw in the last section, the theory of modules over $\mathbb{S}$ potentially covers the theory of modules over $\mathbb{N}$. This fact may suggest that this stable homotopy theory is too coarse to be a theory of numbers. The author does not presently know how to remedy this flaw. We discuss this in the final section.

We have considered Γ-spaces as a model of connective spectra. Finally, we mention the role of connective spectra in stable homotopy theory.

Definition 3.23. *A symmetric spectrum is connective if its homotopy groups in negative degrees are trivial.*

Example 3.24. *(i) For any* Γ*-space* X*, the associated symmetric spectrum* $X(\mathbb{S})$ *is a connective spectrum.*

(ii) The K*-theory spectrum of a symmetric monoidal category is a connective spectrum.*

(iii) Eilenberg-Mac Lane spectra are connective spectra.

(iv) Topological cyclic homology is in general not a connective spectrum.

In [MMSS], the authors give the category of symmetric spectra a model structure.

Theorem 3.25 ([BF], [MMSS])**.** *With respect to above model structures, the adjunction induced by* $(-)(\mathbb{S})$*is a Quillen adjunction. Moreover, the homotopy category of* Γ*-spaces is equivalent to the full subcategory of connective spectra of the stable homotopy category of symmetric spectra via the functor.*

We again note that, for higher algebra, it is not reasonable to consider only Γ-spaces, since not all spectra are connective. To stress the viewpoint of [CC] and to make the proposal in the final section as simple as possible, we focus on connective spectra in this thesis. Similarly, commutative monoid objects in Γ-Sp do not model all commutative monoid objects in symmetric spectra. For instance, $\mathrm{TP}(\mathbb{F}_p)$, which we introduce in the next section and use for our first main theorem, is a non-connective monoid object in symmetric spectra.

In this section we reviewed some basic results on stable homotopy theory to prepare to define homology theories which our theorems use. In the next section, we will define such homology theories.

4 THH and Witt vectors

We study some homology theories in stable homotopy theory with topological Hochschild homology (THH) as the central one. As we see in this section, THH has important relations to p-adic Hodge theory via Witt vectors and de Rham-Witt complexes, which were discovered by Hesselholt and Madsen mainly. Thereby, THH and its relatives recently have been studied in arithmetic geometry as well mainly by Bhatt-Morrow-Scholze (see [BMS1], [BMS2]). Our results concerns this sequence of studies. We state several fundamental theorems in this section after introducing Witt vectors. Although the theory of Witt vectors has numerous applications, especially to number theory, we will just define it.

For a commutative ring A, the (p-adic) ghost map $w : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is defined by $(a_n)_{n\in\mathbb{N}} \mapsto (w_n)_{n\in\mathbb{N}}$ with $w_n = \Sigma_i p^i a_i^{p^{n-i}}$ for a prime number p. Here is a classical theorem.

Definition and Theorem 4.1. *Let A be a commutative ring. The ring $W_p(A)$ of p-typical Witt vectors in A is a commutative ring with the underlying set $A^{\mathbb{N}}$ and the ring structure given by the unique ring structure such that the ω is a natural ring homomorphism, where the target $A^{\mathbb{N}}$ is the product ring.*

$W_p(A)$ possesses three kinds of maps F, V and R which we now define. First, the Frobenius map $F : W_p(A) \to W_p(A)$ is the ring homomorphism that is characterized by making the following diagram

$$\begin{array}{ccc} W_p(A) & \xrightarrow{\omega} & A^{\mathbb{N}} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ W_p(A) & \xrightarrow[\omega]{} & A^{\mathbb{N}}, \end{array}$$

where $F'(w_0, w_1, w_2, ...) = (w_1, w_2, ...)$, commutative.

The *Verschiebung* V is the additive map defined by

$$V : W_p(A) \to W_p(A), (a_0, a_1, a_2, ...) \mapsto (0, a_0, a_1, a_2, ...).$$

The maps similar to these are the main objects for our second results. We write $W_{p,n}(A)$ for $W_p(A)/V^n \, \mathrm{W}_p(A)$.
Finally, the restriction map $R : W_{p,n+1}(A) \to W_{p,n}(A)$ is defined by

$$(a_0, a_1, ..., a_n) \mapsto (a_0, a_1, ..., a_{n-1}).$$

There is also a map called the Teichmüller map $[-] : A \to W_p(A)$, which is defined by $[a] = (a, 0, 0, ...)$. It is multiplicative and makes the diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}} & A \\ {\scriptstyle [-]}\downarrow & & \downarrow{\scriptstyle [-]'} \\ W_p(A) & \xrightarrow[\omega]{} & A^{\mathbb{N}}, \end{array}$$

where $[a]' = (a, a^2, a^3, ...)$, commutative.

We are next going to define topological Hochschild homology, following mainly [HM2], and see it has three kinds of maps as well. Although there are several constructions for THH, we follow the construction in [Shipley], which is due to Bökstedt [Bokstedt2].

Let $\mathcal{I}$ be the category of finite sets and inclusions. For $j \in \mathbb{N}$ and a commutative ring spectrum R, we have a symmetric spectrum $\mathrm{THH}_j(R) = \mathrm{hocolim}_{\mathcal{I}^{j+1}}(\mathcal{D}^j \tilde{R}(n_0, ..., n_j))$, where $\mathcal{D}^j \tilde{R}(n_0, ..., n_j) = \Omega^{n_0+...+n_j} \mathrm{L}\,\mathrm{F}_0(R_{n_0} \wedge ... \wedge R_{n_j})$. This construction defines a functor $\mathrm{THH}.(R) : \Lambda^{op} \to \mathit{SymmSpctr}$, i.e. a cyclic object in symmetric spectra.

Definition 4.2 ([Shipley]). *Let R be a commutative ring spectrum. Its topological Hochschild homology $\mathrm{THH}(R)$ is the spectrum defined as the geometric realization of the cyclic object $\mathrm{THH}.(R)$ in symmetric spectra.*

We recall from [HM2, 2.2, 3.3] the maps F, V and R maps on THH which correspond to those on W_p. By Connes' theory of cyclic objects $\mathrm{THH}(R)$ has a $\mathbb{T}$-action, so that the fixed points $\mathrm{THH}(R)^{C_n}$ makes sense for any natural number n, where C_n denotes the nth cyclic group. The inclusion of fixed points $F_n : \mathrm{THH}(R)^{C_{mn}} \to \mathrm{THH}(R)^{C_m}$ is called nth Frobenius map. The projection $\mathbb{T}/C_m \to \mathbb{T}/C_{mn}$ also induces $V_n : \mathrm{THH}(R)^{C_m} \to \mathrm{THH}(R)^{C_{nm}}$, which is called nth Verschiebung ([HM2, 3.3]). The cyclotomic structure on $\mathrm{THH}(R)$ gives the nth restriction map $R_n : \mathrm{THH}(R)^{C_{mn}} \to \mathrm{THH}(R)^{C_m}$ ([HM2, 2.2]). Here is a pivotal theorem of higher-algebraic arithmetic geometry.

Theorem 4.3 ([HM2], Theorem 3.3). *Let A be a commutative ring. Then there is a natural ring isomorphism*

$$\pi_0(\mathrm{THH}(A)^{C_{p^n}}) \cong W_{p,n}(A),$$

which is compatible with R, F and V for arbitrary prime number p and natural number n.

It would be reasonable to say that this theorem is a hub of the rising theory studying the new relation between stable homotopy theory and p-adic Hodge theory ([BMS2], [NS], [AMN]). The left hand side in the theorem comes from stable homotopy theory and the right hand side from p-adic Hodge theory.

We now recall the basic periodicity theorem on THH proved by Bökstedt [Bokstedt1]. This periodicity induces the periodicity on periodic topological cyclic homology TP which we will review later.

Theorem 4.4 ([HM3], Bökstedt Periodicity). *For any prime number p,*

$$\pi_*(\mathrm{THH}(\mathbb{F}_p)) = \mathbb{F}_p[x],$$

where deg(x)*=2.*

Below we recall several results which relate to our results. A subset $S \subset \mathbb{N}$ is a truncation set if, for any element $n \in S$, every divisor of n is also in S. For a truncation set S and a natural number n, we define $S/n := \{e \in \mathbb{N} | ne \in S\}$. The big Witt vectors functor $\mathbb{W}_{(-)}$ are defined for truncation sets [Hesselholt3]. For the truncated set $\{1, p, p^2, p^3, ...\}$, $\mathbb{W}_{\{1,p,p^2,p^3,...\}}$ and W_p coincide.

Definition 4.5 ([Hesselholt3], Definition 4.1). *Let A be a commutative ring. A Witt complex over A is a contravariant functor from the category of truncation sets to anti-commutative graded rings, $S \mapsto E_S^\bullet$, with natural ring maps*

$$\eta_S : \mathbb{W}_S(A) \to E_S^0,$$

and the following maps of graded abelian groups

$$\mathrm{d} : E_S^q \to E_S^{q+1}, \mathrm{F_n} : E_S^q \to E_{S/n}^q, \mathrm{V_n} : E_{S/n}^q \to E_S^q,$$

subject to the following axioms (i)-(v);

(i) For $x \in E^q_S$ *and* $y \in E^r_S$, $\mathrm{d}(xy) = \mathrm{d}(x)y + (-1)^q x\,\mathrm{d}(y)$ *and* $\mathrm{d}\,\mathrm{d}(x) = \mathrm{d}\log\eta_S([-1]_S)\,\mathrm{d}(x)$*, where* $\mathrm{d}\log\eta_S([-1]_S) = \eta_S([-1])^{-1}\,\mathrm{d}\,\eta_S([-1]_S)$*.*

(ii) For positive natural numbers m *and* n, $\mathrm{F_1} = \mathrm{V_1}$, $\mathrm{F_m}\,\mathrm{F_n} = \mathrm{F_{mn}}$, $\mathrm{V_m}\,\mathrm{V_n} = \mathrm{V_{mn}}$, $\mathrm{F_n}\,\mathrm{V_n} = n \cdot \mathrm{id}$, $\mathrm{F_m}\,\eta_S = \eta_{S/m}\,\mathrm{F_m}$*, and* $\eta_S\,\mathrm{V_m} = \mathrm{V_n}\,\eta_{S/m}$*. If* $(m,n) = 1$, $\mathrm{F_m}\,\mathrm{V_n} = \mathrm{V_n}\,\mathrm{F_m}$*.*

(iii) For positive natural numbers n, $\mathrm{F_n}$ *is a ring map. For* $x \in E^q_S$ *and* $y \in E^r_{S/n}$, $x\,\mathrm{V_n}(y) = \mathrm{V_n}(\mathrm{F_n}(x)y)$*.*

(iv) For all positive natural numbers n *and* $x \in E^q_{S/n}$, $\mathrm{F_n}\,\mathrm{d}\,\mathrm{V_n}(y) = \mathrm{d}(y) + (n-1)\,\mathrm{d}\log\eta_{S/n}([-1]_{S/n})y$*.*

(v) For all positive natural number n *and elements* $a \in A$, $\mathrm{F_n}\,\mathrm{d}\,\eta_S([a]_S) = \eta_{S/n}([a]^{n-1}_{S/n})\,\mathrm{d}\,\eta_{S/n}([a]_{S/n})$*.*

After introducing Witt complexes, we can talk about de Rham-Witt complex which gives another relation between stable homotopy theory and arithmetic geometry. Our second contribution is related to it.

Definition and Theorem 4.6 ([Hesselholt3], Definition 4.7)**.** *Let* A *be a commutative ring. Then, the category of Witt complexes over* A *has the initial object which is called the de Rham-Witt complex* $\mathbb{W}_{.}\Omega^{\cdot}_A$*.*

Remark 4.7 ([Hesselholt3])**.** *For* A *an* $\mathbb{F}_p$*-algebra and* $S = \{1, p, ..., p^{n-1}\}$, $\mathbb{W}_S\Omega^{\cdot}_A := \mathbb{W}_{.}\Omega^{\cdot}_A(S)$ *is the classical p-typical de Rham-Witt complex* $W_{p,n}\Omega^{\cdot}_A$ *in the sense of Bloch-Deligne-Illusie ([Illusie]).*

Hesselholt and Madsen have proven that THH gives the higher-algebraic de Rham-Witt complex in the following sense.

Theorem 4.8 ([HM3], Theorem 2.2.2)**.** *Let* k *be a perfect field of positive* *characteristic and A a smooth k-algebra. Then there is a natural isomorphism*

$$\bigoplus_{m\geq 0} \mathbb{W}_{(m+1)n}\Omega_A^{*-2m} \to \pi_* \operatorname{holim}_R \operatorname{THH}(A)^{C_s}_{2\lfloor (s-1)/n \rfloor},$$

where $\lfloor x \rfloor$ denotes the integer part of x.

Using this translation between THH and de Rham-Witt complex, Hesselholt constructs the following diagram.

Theorem 4.9 ([Hesselholt1], Theorem A)**.** *Let A be a regular noetherian ring and an $\mathbb{F}_p$-algebra. Then the canonical prjection $f : A[x]/(x^m) \to A[x]/(x^n)$ induces a map of long exact sequences*

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\bigoplus_{l\geq 0} \mathbb{W}_{l+1}\Omega_A^{q-2l} & \longrightarrow & \bigoplus_{l\geq 0} \mathbb{W}_{l+1}\Omega_A^{q-2l} \\
\downarrow V_m & & \downarrow V_n \\
\bigoplus_{l\geq 0} \mathbb{W}_{m(l+1)}\Omega_A^{q-2l} & \longrightarrow & \bigoplus_{l\geq 0} \mathbb{W}_{n(l+1)}\Omega_A^{q-2l} \\
\downarrow & & \downarrow \\
K_{q+1}(A[x]/(x^m),(x)) & \longrightarrow & K_{q+1}(A[x]/(x^n),(x)) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array},$$

where the lowest horizontal map is the map of relative K-groups induced by the canonical projection, where the middle horizontal map takes the lth summand of the domain to the lth summand of the target by the composition of the restriction map and the multiplication by a certain element, where the top horizontal map is zero and where V denotes Verschiebung.

Stable homotopy theory does not show up explicitly in this theorem. However, as is mentioned, it lies behind. Our second contribution is akin to this theorem. In stead of the projection, we evaluate the map of K-groups

$$K(A[x]/(x^k),(x)) \to K(A[x]/(x^{nk}),(x)),$$

induced by $x \mapsto x^n$ with big de Rham-Witt forms. Moreover, for A a perfect field of characteristic $p > 0$, we give a calculation of the relative algebraic K-groups of $\mathcal{O}_K/p\mathcal{O}_K$ for various perfectoid fields K at the end of section 5.2.

We next recall another new homology theory TP called topological periodic cyclic homology from [Hesselholt2]. Our first result studies the question of nil-invariance. Since it is defined by the Tate construction, we first recall the construction.

Let E be a contractible $\mathbb{T}$-CW-complex with free $\mathbb{T}$ action. This is well-defined, up to unique equivariant homotopy equivalence. Then we consider the following cofibration sequence

$$E_+ \to S^0 \to \tilde{E},$$

here E_+ is E with the base point ∞, and where the lefthand map sends ∞ to the base point $\infty \in S^0$ and other points to $0 \in S^0$.

Let M be a $\mathbb{T}$-spectrum. Smashing the internal hom $[E_+, M]$ with the above diagram and taking fixed points of a subgroup $C \subset \mathbb{T}$, we have the following sequence called the Tate cofibration sequence

$$(E_+ \otimes_{\mathbb{S}} [E_+, M])^C \to ([E_+, M])^C \to (\tilde{E} \otimes_{\mathbb{S}} [E_+, M])^C.$$

We write $\hat{\mathrm{H}}^{\cdot}(C, M) := (\tilde{E} \otimes_{\mathbb{S}} [E_+, M])^C$.

In [GeHe], the authors define THH for schemes that also has a certain $\mathbb{T}$-action. Using this, Hesselholt defines the following.

Definition 4.10 ([Hesselholt2])**.** *Let X be a scheme. The periodic topological cyclic homology of X is the spectrum given by*

$$\mathrm{TP}_{\cdot}(X) = \hat{\mathrm{H}}^{\cdot}(\mathbb{T}, \mathrm{THH}(X)).$$

Hesselholt has proved that TP gives an interpretation of Hasse-Weil zeta function.

Theorem 4.11 ([Hesselholt2], Theorem A)**.** *Let k be a finite field with order $q = p^r$ and W be its ring of p-typical Witt vectors, and $\sigma : \mathrm{W} \to \mathbb{C}$ be a choice of embedding. If X is a scheme smooth and proper over $\mathrm{Spec}(k)$, then as meromorphic functions on $\mathbb{C}$,*

$$\zeta(X, s) = \frac{\det_{\infty}(s \cdot \mathrm{id} - \Theta | \, \mathrm{TP}_{\mathrm{od}}(X) \otimes_{\mathrm{W},\sigma} \mathbb{C})}{\det_{\infty}(s \cdot \mathrm{id} - \Theta | \, \mathrm{TP}_{\mathrm{ev}}(X) \otimes_{\mathrm{W},\sigma} \mathbb{C})},$$

where Θ is a $\mathbb{C}$-linear graded derivation such that $q^{\Theta} = \mathrm{Fr}_q^*$, where Fr is the geometric Frobenius, and $\Theta(v) = \frac{2\pi i}{\log q} \cdot v$ with $v \in \mathrm{TP}_{-2}(k)$ is the generator given in [Hesselholt2, §4].

Unfortunately, the author does not have well enough understanding on this topic to give an explanation of this theorem. It should still be worth understanding the properties of TP. Our first result in the next section shows that TP does not have nil-invariance in general. To motivate our result further, we mention [AMN] and [BlMa] show that TP satisfies the Künneth formula in the following sense.

Theorem 4.12 ([AMN], Theorem 1.1). *Let k be a perfect field of characteristic $p > 0$. If $\mathcal{C}$ and $\mathcal{D}$ are smooth and proper k-linear dg categories, then the natural map*

$$\mathrm{TP}(\mathcal{C}) \otimes_{\mathrm{TP}(k)} \mathrm{TP}(\mathcal{D}) \to \mathrm{TP}(\mathcal{C} \otimes_k \mathcal{D})$$

is an equivalence.

According to them, TP can be defined for dg categories. However, unfortunately again, the author does not understand that yet well. We use this theorem as a motivation of our result. That is to say, since it is expected that TP is useful to analyze smooth and proper schemes of positive characteristic, our theorem 5.2 shall give some geometric understanding on such objects.

As the conclusion of this section, we have recalled some homology theories which originate in stable homotopy theory, and seen some relations between them and arithmetics. Based on these celebrated results, we are showing our contributions in the next section.

5 Contributions

In this section, we show our main contributions, taking the previous sections to be preparations for them. The first one studies a property so-called nil-invaliancy of TP, and the other one is about certain maps of algebraic K-theory of truncated polynomial algebras. We will calculate the K-groups of a certain ring as a consequence.

5.1 The non-nil-invariance of TP

5.1.1 The nil-invariance of HP

As we saw in the last section, Hesselholt defined a spectrum $\mathrm{TP}(X)$ for a scheme X using THH and Tate construction, which is the higher-algebraic analogue of Connes's periodic cyclic homology HP defined by Hochschild homology and Tate construction. Goodwillie has proven the following which says, for algebras of characteristic 0, HP has the nil-invariancy.

Theorem 5.1 ([Goodwillie1], Theorem II.5.1)**.** *Let R be an algebra over a field of characteristic 0 and I a nilpotent ideal of R, then the quotient map $R \to R/I$ induces an isomorphim on* HP*.*

By the nature of higher-algebra, the analogous statement for positive characteristic algebras will be asked. We show that such analogous result for TP does not hold, that is, there are an algebra of positive characteristic and a nilpotent ideal such that the quotient map does not induce an isomorphism on TP. Recently in [BlMa] and [AMN], It is shown that TP satisfies the Künneth formula for smooth and proper dg categories over a field of positive characteristic. Our main theorem should contribute to such study.

5.1.2 Main theorem

Our main result is the following

Theorem 5.2. *Let p be a prime number and $k \geq 2$ a natural number. Then the canonical map $\mathrm{TP}_*(\mathbb{F}_p[x]/(x^k)) \to \mathrm{TP}_*(\mathbb{F}_p)$ is not an isomorphism up to p-inverted.*

Before proving our main result, we recall from [HM2] and [Hesselholt2] some calculations concerning $\mathrm{THH}(\mathbb{F}_p[x]/(x^k))$.

We give the pointed finite set Π_k=$\{0, 1, x, \ldots, x^{k-1}\}$ with the base point 0 the pointed commutative monoid structure, where 1 is the unit, $0 \cdot 1 = 0 \cdot x^i = 0$, $x^i \cdot x^j = x^{i+j}$, $x^k = 0$. We denote the cyclic bar construction of Π_k by $\mathrm{N}^{\mathrm{cy}}_\bullet(\Pi_k)$. More precisely, the set of l-simplicies is

$$\mathrm{N}^{\mathrm{cy}}_l(\Pi_k) = \Pi_k \wedge \cdots \wedge \Pi_k,$$

where there are $l + 1$ smash factors and the structure maps are given by

$d_i(x_0 \wedge \cdots \wedge x_l) = x_0 \wedge \cdots \wedge x_i x_{i+1} \wedge \cdots \wedge x_l,\ 0 \leq i < l,$

$d_l(x_0 \wedge \cdots \wedge x_l) = x_l x_0 \wedge x_1 \wedge \cdots \wedge x_{k-1},$

$s_i(x_0 \wedge \cdots \wedge x_l) = x_0 \wedge \cdots \wedge x_i \wedge 1 \wedge x_{i+1} \wedge \cdots \wedge x_l,\ 0 \leq i \leq l,$

$t_l(x_0 \wedge \cdots \wedge x_l) = x_l \wedge x_0 \wedge x_1 \wedge \cdots \wedge x_{l-1}.$

We let $\mathrm{N}^{\mathrm{cy}}(\Pi_k)$ denote the geometric realization of $\mathrm{N}^{\mathrm{cy}}_\bullet(\Pi_k)$.

In [HM1, Theorem 7.1], it is proved that there is a natural equivalence of cyclotomic spectra

$$\mathrm{THH}(\mathbb{F}_p[x]/(x^k))) \simeq \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{N}^{\mathrm{cy}}(\Pi_k). \tag{a}$$

For each positive integer i, we also have the cyclic subset

$$\mathrm{N}^{\mathrm{cy}}_{\bullet}(\Pi_k, i) \subset \mathrm{N}^{\mathrm{cy}}_{\bullet}(\Pi_k)$$

generated by the $(i-1)$-simplex $x \wedge \cdots \wedge x$ (i factors), and denote the geometric realization by $\mathrm{N}^{\mathrm{cy}}(\Pi_k, i)$. We also have the cyclic subset $\mathrm{N}^{\mathrm{cy}}_{\bullet}(\Pi_k, 0)$ generated by the 0-simplex 1 with the geometric realization $\mathrm{N}^{\mathrm{cy}}(\Pi_k, 0)$. Thus we obtain the following wedge decomposition

$$\bigvee_{i \geq 0} \mathrm{N}^{\mathrm{cy}}(\Pi_k, i) = \mathrm{N}^{\mathrm{cy}}(\Pi_k).$$

We consider the complex $\mathbb{T}$-representation, where $d = \lfloor (i-1)/k \rfloor$ is the integer part of $(i-1)/k$ for $i \geq 1$,

$$\lambda_d = \mathbb{C}(1) \oplus \mathbb{C}(2) \oplus \cdots \oplus \mathbb{C}(d),$$

where $\mathbb{C}(i) = \mathbb{C}$ with the $\mathbb{T}$ action;

$$\mathbb{T} \times \mathbb{C}(i) \to \mathbb{C}(i)$$

defined by $(z, w) \mapsto z^i w$. Then we have the following by [HM2, theorem B], for $i \geq 1$ such that $i \notin k\mathbb{N}$, there is an equivalence

$$\mathrm{N}^{cy}(\Pi_k, i) \simeq S^{\lambda_d} \wedge (\mathbb{T}/C_i)_+,$$

where C_i is the i-th cyclic group.

Let $\mathrm{THH}(\mathbb{F}_p[x]/(x^k), (x))$ denote the fiber of the canonical map

$$\mathrm{THH}(\mathbb{F}_p[x]/(x^k)) \to \mathrm{THH}(\mathbb{F}_p),$$

and we write

$$\mathrm{TP}(\mathbb{F}_p[x]/(x^k),(x)) = \hat{\mathrm{H}}^{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p[x]/(x^k),(x)))$$

The triviality of $\mathrm{TP}(\mathbb{F}_p[x]/(x^k),(x))[1/p]$ shall imply that TP is not nil-invariant up to p-inverted. In order to obtain the triviality, we use the following decomposition.

Lemma 5.3. *There is a canonical equivalence*

$$\mathrm{TP}(\mathbb{F}_p[x]/(x^k),(x)) \simeq \prod_{i\geq 1} \hat{\mathrm{H}}^{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{N}^{cy}(\Pi_k, i)).$$

Proof. By (a) and the wedge decomposition, we have

$$\Sigma\mathrm{H}.(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p[x]/(x^k),(x))) \simeq \bigvee_{i\geq 1} \Sigma\mathrm{H}.(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{N}^{cy}(\Pi_k, i)),$$

since $\mathrm{H}.(\mathbb{T}, -)$ preserves all homotopy colimits.
Since the connectivity of $\Sigma\mathrm{H}.(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{N}^{cy}(\Pi_k, i))$ goes to ∞ as i goes to ∞, we have

$$\bigvee_{i\geq 1} \Sigma\mathrm{H}.(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{N}^{cy}(\Pi_k, i)) \simeq \prod_{i\geq 1} \Sigma\mathrm{H}.(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{N}^{cy}(\Pi_k, i)).$$

Similarly, since $\mathrm{H}^{\cdot}(\mathbb{T}, -)$ preserves all homotopy limits, we have

$$\mathrm{H}^{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p[x]/(x^k),(x))) \simeq \prod_{i\geq 1} \mathrm{H}^{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{N}^{cy}(\Pi_k, i)).$$

Since $\mathrm{TP}(\mathbb{F}_p[x]/(x^k),(x))$ is the cofiber of

$$\Sigma \mathrm{H}_\cdot(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p[x]/(x^k),(x))) \to \mathrm{H}^\cdot(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p[x]/(x^k),(x))),$$

we get the desired equivalence. □

It is known that, for a $\mathbb{T}$-spectrum X, there is a $\mathbb{T}$-equivalence

$$X \otimes (\mathbb{T}/C_i)_+ \simeq \Sigma[(\mathbb{T}/C_i)_+, X],$$

see for example [HM1, 8.1]. Hence, we have

$$\begin{aligned}
\hat{\mathrm{H}}^\cdot(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p) \otimes (\mathbb{T}/C_i)_+) &= (\tilde{E} \otimes [E_+, \mathrm{THH}(\mathbb{F}_p) \otimes (\mathbb{T}/C_i)_+])^{\mathbb{T}} \\
&\simeq \Sigma(\tilde{E} \otimes [E_+, [(\mathbb{T}/C_i)_+, \mathrm{THH}(\mathbb{F}_p)]])^{\mathbb{T}} \\
&\simeq \Sigma(\tilde{E} \otimes [(\mathbb{T}/C_i)_+, [E_+, \mathrm{THH}(\mathbb{F}_p)]])^{\mathbb{T}} \\
&\simeq (\tilde{E} \otimes (\mathbb{T}/C_i)_+ \otimes [E_+, \mathrm{THH}(\mathbb{F}_p)]])^{\mathbb{T}} \\
&\simeq \Sigma([(\mathbb{T}/C_i)_+, \tilde{E} \otimes [E_+, \mathrm{THH}(\mathbb{F}_p)]])^{\mathbb{T}} \\
&\simeq \Sigma(\tilde{E} \otimes_{\mathbb{S}} [E_+, \mathrm{THH}(\mathbb{F}_p)])^{C_i} \\
&= \Sigma\hat{\mathrm{H}}^\cdot(C_i, \mathrm{THH}(\mathbb{F}_p)).
\end{aligned}$$

Furthermore, by [HM1, 3.2], we have an equivalence of spectra

$$\hat{\mathrm{H}}^\cdot(C_i, \mathrm{THH}(\mathbb{F}_p) \otimes S^{\lambda_d}) \simeq \hat{\mathrm{H}}^\cdot(C_{p^{v_p(i)}}, \mathrm{THH}(\mathbb{F}_p) \otimes S^{\lambda_d}),$$

where v_p is the p-adic valuation.

Hesselholt and Madsen have calculated the homotopy groups of the above spectra [HM1, §9],

$$\pi_* \hat{\mathrm{H}}^{\cdot}(C_{p^n}, \mathrm{THH}(\mathbb{F}_p) \otimes S^{\lambda_d}) \cong S_{\mathbb{Z}/p^n\mathbb{Z}}\{t, t^{-1}\},$$

where α is the divided Bott element. More precisely, $\pi_* \hat{\mathrm{H}}^{\cdot}(C_{p^n}, \mathrm{THH}(\mathbb{F}_p) \otimes S^{\lambda_d})$ is a free module of rank 1 over $\mathbb{Z}/p^n\mathbb{Z}[t, t^{-1}]$ on a generator of degree $2d$. Combining these, we obtain for $i \notin k\mathbb{N}$

$$\pi_j \hat{\mathrm{H}}^{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{N}^{\mathrm{cy}}(\Pi_k, i)) \cong \begin{cases} \mathbb{Z}/p^{v_p(i)}\mathbb{Z}, & j - \lambda_d + 1 : \textit{even} \\ 0, & j - \lambda_d + 1 : \textit{odd}, \end{cases}$$

and by definition $-\lambda_d + 1$ is always odd. They have also calculated for $i \in k\mathbb{N}$,

$$\pi_j \hat{\mathrm{H}}^{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{N}^{\mathrm{cy}}(\Pi_k, i)) \cong \begin{cases} \mathbb{Z}/p^{v_p(k)}\mathbb{Z}, & j : \textit{odd} \\ 0, & j : \textit{even}. \end{cases}$$

Due to this, we have

Theorem 5.4. *If j is an odd integer, then there is a canonical isomorphism*

$$\mathrm{TP}_j(\mathbb{F}_p[x]/(x^k), (x)) \cong \prod_{i \geq 1, i \in k\mathbb{N}} \mathbb{Z}/p^{v_p(k)}\mathbb{Z} \times \prod_{i \geq 1, i \notin k\mathbb{N}} \mathbb{Z}/p^{v_p(i)}\mathbb{Z}.$$

If j is an even integer, then

$$\mathrm{TP}_j(\mathbb{F}_p[x]/(x^k), (x)) = 0.$$

Therefore, we get our main result by this theorem. In addition, by [NS, Corollary 1.5] and [HM1], we get the following as well.

Corollary 5.5. *Topological negative cyclic homology is not nil-invariant up* *to p-inverted.*

5.2 Verschiebung maps among K-groups of truncated polynomial algebras

In [Hesselholt1], Hesselholt constructs the tower of long exact sequences, for a certain class of algebras, calculates the maps of algebraic K-theory of the canonical projections $A[x]/(x^m) \to A[x]/(x^n)$ for $m \geq n$. We apply those methods to the power maps $A[x]/(x^k) \to A[x]/(x^{kn})$, $x \mapsto x^n$. As a consequence, we give a calculation of the relative K-groups of $\mathcal{O}_K/p\mathcal{O}_K$ for certain perfectoid fields K.

5.2.1 Pointed commutative monoids and truncated polynomial algebras

In [HM1], the homotopy classes of the following maps of pointed C_i-spaces are defined; $\theta_d : \Delta^{i-1}/C_i \cdot \Delta^{i-k} \to S^{\lambda_d}$ for $kd < i < k(d+1)$, and $\theta_d : \Delta^{i-1}/C_i \cdot \Delta^{i-k} \to (S^0 * C_k) \wedge S^{\lambda_d}$ for $i = k(d+1)$, where C_m is the m-th cyclic group and S^{λ_d} is the one point compactification of λ_d. They play a key role in this section. In op. cit., for any positive integer i, the following cofibration sequence are constructed using θ;

$$\mathbb{T}_+ \wedge_{C_{i/k}} S^{\lambda_{d_i}} \xrightarrow{\text{pr}} \mathbb{T}_+ \wedge_{C_i} S^{\lambda_{d_i}} \longrightarrow \mathrm{N}^{\mathrm{cy}}(\Pi_k, i) \longrightarrow \Sigma\mathbb{T}_+ \wedge_{C_{i/k}} S^{\lambda_{d_i}},$$

where $\mathbb{T}_+ \wedge_{C_{i/k}} S^{\lambda_{d_i}}$ is trivial when k does not divide i and $d_i = \lfloor (i-1)/k \rfloor$ is the largest natural number less than or equal to $(i-1)/k$. We briefly recall the construction.

For the i-th cyclic group C_i, $\mathbb{R}[C_i]$ denotes the regular representation and $\Delta^{i-1} \subset \mathbb{R}[C_i]$ the convex hull of the generators of C_i. By permutation C_i acts on $\mathbb{R}[C_i]$ and the action restricts on Δ^{i-1}. Let ξ_i denote the generator of

C_i and Δ^{i-m} the convex hull of $1, \xi_i, ..., \xi_i^{i-m}$. The canonical decomposition of $\mathbb{R}[C_i]$ induces the projection map ([Hesselholt1, p.11])

$$\pi_d : \mathbb{R}[C_i] \to \lambda_d,$$

if $2d < i$. We first consider the case $md < i < m(d+1)$. In [HM1], it is proved that $0 \notin \pi_d(C_i \cdot \Delta^{i-m}) \subset \lambda_d$. Composing $\pi_d|_{\Delta^{i-1}}$ and the radial projection, we get a C_i-equivariant map

$$\theta_d : \Delta^{i-1}/C_i \cdot \Delta^{i-m} \to S^{\lambda_d}.$$

We next consider the case $i = m(d+1)$. It is also proved that in [HM1] $0 \notin \pi_{d+1}(C_i \cdot \Delta^{i-m}) \subset \lambda_{d+1}$. Furthermore, that proves

$$\pi_{d+1}(C_i \cdot \Delta^{i-m}) \cap \lambda_d^{\perp} = C'_m,$$

where $\lambda_d^{\perp}$ is the orthogonal completion of the image of the canonical inclusion $\lambda_d \xrightarrow{\iota} \lambda_{d+1}$ and C'_m is the preimage of C_m by the isomorphism $\lambda_d^{\perp} \to \mathbb{C}(d+1)$ induced by ι. Picking a small ball $B \subset \lambda_{d+1} \backslash C'_m$ around a point in the sphere $S(\lambda_d^{\perp})$, we define $U := (C_i \cdot B) \cap S(\lambda_{d+1})$. If B is small enough, the projection π_{d+1} and radial projection define a C_i-equivariant map $\theta'_d : \Delta^{i-1}/C_i \cdot \Delta^{i-m} \to D(\lambda_{d+1})/(S(\lambda_{d+1}) \backslash U)$, where $D(\lambda_{d+1})$ denotes the disk in λ_{d+1}. [Hesselholt1] shows that there is a strong deformation retract of C_i-spaces

$$(S^0 * C_m) \wedge S^{\lambda_d} \to D(\lambda_{d+1})/(S^{\lambda_{d+1}} \backslash U).$$

Therefore we get a homotopy class of C_i-equivariant maps

$$\theta_d : \Delta^{i-1}/C_i \cdot \Delta^{i-m} \to (S^0 * C_m) \wedge S^{\lambda_d}.$$

We also recall some well-known theorems. For A a commutative ring, Hesselholt-Madsen shows that, in [HM2], there is an equivalence

$$\mathrm{THH}(A[x]/(x^k)) \simeq \mathrm{THH}(A) \otimes \mathrm{N}^{\mathrm{cy}}(\Pi_k). \tag{a}$$

This equivalence gives rise to

$$\mathrm{THH}(A[x]/(x^k), (x)) \simeq \bigvee_{i>0} \mathrm{THH}(A) \otimes \mathrm{N}^{\mathrm{cy}}(\Pi_k, i).$$

Here is a corollary of the famous theorem by Dundas-Goodwillie-McCarthy. For A a commutative ring, after p-completion for any prime number p, we have an equivalence

$$K(A[x]/(x^k), (x)) \simeq \mathrm{TC}(A[x]/(x^k), (x)). \tag{b}$$

In the present section, using above theorems, we study the map

$$K(A[x]/(x^k), (x)) \to K(A[x]/(x^{nk}), (x)),$$

induced by $x \mapsto x^n$.

5.2.2 The geometric Verschiebung map

In order to study the map $K(A[x]/(x^k), (x)) \to K(A[x]/(x^{nk}), (x))$, we use the two pointed commutative monoids Π_k and Π_{nk} and their realizations of cyclic bar constructions, and construct a map between corresponding cofibration sequences.

In [HM2, 7.2], Hesselholt and Madsen defined an isomorphism between the geometric realization $|\Lambda[n]|$ of standard cyclic set and the product topological space $\mathbb{T} \times \Delta^n$ of the circle and the standard n-simplex as follows;
In [Jones, Theorem 3.4], Jones constructed a homeomorphism between $|\Lambda[n]|$ and $\mathbb{T} \times \Delta^n$ and defined an action of C_{n+1} on $\mathbb{T} \times \Delta^n$ by

$$\tau_n \cdot (x; u_0, ..., u_n) := (x - u_0; u_1, ..., u_n, u_0).$$

However, Hesselholt and Madsen consider a different action of C_{n+1} on $\mathbb{T}\times\Delta^n$ given by

$$\tau_n * (x; u_0, ..., u_n) := (x - 1/(n+1); u_1, ..., u_n, u_0),$$

and defined an $\mathbb{T} \times C_{n+1}$-equivariant homeomorphism $F_n : \mathbb{T} \times \Delta^n \to \mathbb{T} \times \Delta^n$ by

$$F_n(x; u_0, ..., u_n) := (x - f_n(u_0, ..., u_n); u_0, ..., u_n)$$

with an affine map $f_n : \Delta^n \to \mathbb{R}$

$$f_n(u_1, ..., u_n, u_0) - f_n(u_0, ..., u_n) = 1/(n+1) - u_0,$$

and

$$f_n(1, 0, ..., 0) = 0.$$

By construction, the restriction $F_n|_{\Delta^n}$ is the identity map. We identify $|\Lambda[n]|$ with $\mathbb{T} \times \Delta^n$ via this isomorphism. We define a map $\mathrm{e_{i,n}} : \Delta^{i-1} \to \Delta^{in-1}$, which sends the vertex $(0, ..., 0, 1, 0, ..., 0)$, where the (m+1)th coordinates is 1, to the vertex $(0, ..., 0, 1, 0, ..., 0)$, where the (mn+1)th coordinate is 1, for every $m \in \{0, ..., i-1\}$. In other words, $\mathrm{e_{i,n}}(\xi_i^j) = \xi_{in}^{nj}$ for $0 \leq j \leq i-1$, where ξ_i, respectively ξ_{in}, is the generator of C_i, respectively C_{in}.

Lemma 5.6. *The map* $\mathrm{e_{i,n}}$ *induces the map* $\mathrm{g_{i,n}} : \mathrm{N^{cy}}(\Pi_k, i) \to \mathrm{N^{cy}}(\Pi_{nk}, in)$, $a \mapsto b^n$*, via the isomorphism, where* Π_k*, respectively* Π_{nk}*, is generated by* a*, respectively* b*.*

Proof. By [HM1, Lemma 2.2.6], the map $\Lambda[i-1] \xrightarrow{\alpha} \mathrm{N^{cy}}(\Pi_k, i)[-]$ representing the $i-1$-simplex $a \wedge ... \wedge a$ (i factors) induces a $\mathbb{T}$-equivariant homeomorphism after the geometric realization. We write $\Lambda[in-1] \xrightarrow{\beta} \mathrm{N^{cy}}(\Pi_{kn}, in)[-]$ for the map representing the $in-1$-simplex $b \wedge ... \wedge b$ (in factors). Then we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{N^{cy}}(\Pi_k, i)[-] & \xrightarrow{g'_{i,n}} & \mathrm{N^{cy}}(\Pi_{kn}, in)[-] \\ \alpha \uparrow & & \uparrow \beta \\ \Lambda[i-1] & \xrightarrow{\Psi} & \Lambda[in-1], \end{array}$$

where the map Ψ of cyclic sets is the one induced by the composition map $d_{in-1}d_{in-2}...d_2d_1$ except d_{nj} for all $j \in \{1, ..., i-1\}$ and $g'_{i,n}$ is the map of cyclic sets that is given by $a \mapsto b^n$ and induces $g_{i,n}$ via the geometric realization by definition. The geometric realization of Ψ with Hesselholt-Madsen's isomorphism mentioned above is given by

$$\mathbb{T} \times \Delta^{i-1} \to \mathbb{T} \times \Delta^{in-1},$$

$$(t, (u_0, u_1, ..., u_{i-1})) \mapsto (t, (u_0, 0, 0, ..., 0, u_1, 0, ..., 0, u_{i-1}, 0, 0, ..., 0)),$$

where there are $n-1$ zeros between u_{s-1} and u_s. By definition, it is the map

$\mathrm{id}_{\mathbb{T}} \times \mathrm{e_{i,n}}$. In other words, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{T} \times \Delta^{i-1} & \xrightarrow{\mathrm{id}_{\mathbb{T}} \times \mathrm{e_{i,n}}} & \mathbb{T} \times \Delta^{in-1} \\ \cong \uparrow & & \cong \uparrow \\ |\Lambda[i-1]| & \xrightarrow{|\Psi|} & |\Lambda[in-1]|. \end{array}$$

By the definition of the cyclic bar construction and the commutativity of our monoids,

$$\begin{aligned} \beta_{i-1}(d^1 d^2 ... d^{in-2} d^{in-1}) &= \beta_{i-1} \circ D(id_{[in-1]}) \\ &= D_* \circ \beta_{in-1}(id_{[in-1]}) \\ &= D_*(b \wedge ... \wedge b) = b^n \wedge ... \wedge b^n, \end{aligned}$$

$$\begin{array}{ccc} \Lambda[in-1][in-1] & \xrightarrow{\beta_{in-1}} & \mathrm{N^{cy}}(\Pi_{kn}, in)[in-1] \\ \downarrow D & & \downarrow D_* \\ \Lambda[in-1][i-1] & \xrightarrow{\beta_{i-1}} & \mathrm{N^{cy}}(\Pi_{kn}, in)[i-1], \end{array}$$

where D is the image of the map $d_{in-1} d_{in-2} ... d_2 d_1$ except d_{nj} for all $j \in \{1, ..., i-1\}$ by the contravariant functor $\Lambda[in-1][-]$ and D_* is the image of the map $d_{in-1} d_{in-2} ... d_2 d_1$ except d_{nj} for all $j \in \{1, ..., i-1\}$ by the contravariant functor $\mathrm{N^{cy}}(\Pi_{kn}, in)$. □

We now study the relation between the map $\mathrm{g}_{i,n}$ and the cofibration sequences above. More precisely, we have two cofibration sequences for every $i > 0$

$$\mathbb{T}_+ \wedge_{C_{i/k}} S^{\lambda_{d_i}} \xrightarrow{\mathrm{pr}} \mathbb{T}_+ \wedge_{C_i} S^{\lambda_{d_i}} \longrightarrow \mathrm{N^{cy}}(\Pi_k, i),$$

$$\mathbb{T}_+ \wedge_{C_{in/kn}} S^{\lambda_{d_i}} \xrightarrow{\mathrm{pr}} \mathbb{T}_+ \wedge_{C_{in}} S^{\lambda_{d_i}} \longrightarrow \mathrm{N^{cy}}(\Pi_{kn}, in),$$

and are comparing them using $g_{i,n}$.

Proposition 5.7. *(i)For $kd < i < k(d+1)$, the following diagram commutes up to homotopy*

$$\begin{array}{ccc} \Delta^{i-1}/C_i \cdot \Delta^{i-k} & \xrightarrow{\theta_{i,k}} & S^{\lambda_d} \\ \downarrow{\scriptstyle \mathrm{e_{i,n}}_*} & & \downarrow{\scriptstyle \mathrm{id}} \\ \Delta^{in-1}/C_{in} \cdot \Delta^{n(i-k)} & \xrightarrow{\theta_{in,kn}} & S^{\lambda_d}. \end{array}$$

(ii)For $i = k(d+1)$, the following diagram commutes up to homotopy

$$\begin{array}{ccc} \Delta^{i-1}/C_i \cdot \Delta^{i-k} & \xrightarrow{\theta_{i,k}} & (S^0 * C_k) \wedge S^{\lambda_d} \\ \downarrow{\scriptstyle \mathrm{e_{i,n}}_*} & & \downarrow \\ \Delta^{in-1}/C_{in} \cdot \Delta^{n(i-k)} & \xrightarrow{\theta_{in,kn}} & (S^0 * C_{nk}) \wedge S^{\lambda_d}, \end{array}$$

where the right hand side vertical map is induced by the inclusion $C_k \to C_{kn}$, $\xi_k \mapsto \xi_{nk}^k$ and the identity map on S^{λ_d}.

Proof. We prove (i). The same argument holds for (ii). By the construction of θ ([Hesselholt1, §3]), $\theta_{i,k}(\xi_i^j) = [\xi_i^j, \xi_i^{2j}, ..., \xi_i^d]$, where ξ_i is the generator of C_i. Likewise, $\theta_{in,kn}(\xi_{in}^j) = [\xi_{in}^j, \xi_{in}^{2j}, ..., \xi_{in}^d]$, where ξ_{in} is the generator of C_{in}. By the definition of e, we have $\mathrm{e}(\xi_i) = \xi_{in}^n$. In the complex numbers plane $\mathbb{C}$, $\xi_i^j = \xi_{in}^{nj}$. □

By this proposition, we get the following map of cofiber sequences.

Corollary 5.8. *There is a homotopy commutative diagram of cofibration* *sequences*

$$\begin{array}{ccccc}
\mathbb{T}_+ \wedge_{C_{i/k}} S^{\lambda_d} & \xrightarrow{\text{pr}} & \mathbb{T}_+ \wedge_{C_i} S^{\lambda_d} & \longrightarrow & \mathrm{N}^{\mathrm{cy}}(\Pi_k, i) \\
\Big\downarrow{\scriptstyle \text{id}} & & \Big\downarrow{\scriptstyle \text{pr}} & & \Big\downarrow{\scriptstyle g_{i,n}} \\
\mathbb{T}_+ \wedge_{C_{in/nk}} S^{\lambda_d} & \xrightarrow{\text{pr}} & \mathbb{T}_+ \wedge_{C_{in}} S^{\lambda_d} & \longrightarrow & \mathrm{N}^{\mathrm{cy}}(\Pi_{nk}, ni),
\end{array}$$

where $\mathbb{T}_+ \wedge_{C_{i/k}} S^{\lambda_d}$ *and* $\mathbb{T}_+ \wedge_{C_{in/kn}} S^{\lambda_d}$ *are trivial when* k *does not divide* i *and* $d = \lfloor (i-1)/k \rfloor$.

Proof. Again by [HM1, (3.1.1)], the map $\Lambda[j-1] \to \mathrm{N}^{\mathrm{cy}}(\Pi_m, j)[-]$ representing $y \wedge y \wedge ... \wedge y$ (j factors) with the generator y of Π_m induces,

$$\mathrm{N}^{\mathrm{cy}}(\Pi_m, j) \cong \mathbb{T}_+ \wedge_{C_j} (\Delta^{j-1}/C_j \cdot \Delta^{j-m}).$$

We can get two cofibration sequences

$$\mathbb{T}_+ \wedge_{C_{i/k}} S^{\lambda_d} \xrightarrow{\text{pr}} \mathbb{T}_+ \wedge_{C_i} S^{\lambda_d} \longrightarrow \mathrm{N}^{\mathrm{cy}}(\Pi_k, i),$$

$$\mathbb{T}_+ \wedge_{C_{in/kn}} S^{\lambda_d} \xrightarrow{\text{pr}} \mathbb{T}_+ \wedge_{C_{in}} S^{\lambda_d} \longrightarrow \mathrm{N}^{\mathrm{cy}}(\Pi_{kn}, in),$$

applying $\mathbb{T}_+ \wedge_{C_i} (-)$ and $\mathbb{T}_+ \wedge_{C_{in}} (-)$ respectively to diagrams in 5.7. The inclusion map $C_i \to C_{ni}$, $\xi_i^j \mapsto \xi_{in}^{nj}$, induces the maps id, pr and $g_{i,n}$ which make the diagram commutative. □

5.2.3 Proof of theorems

Using the above diagram, we get a map of long exact sequences to study commutative rings.

Theorem 5.9. *Let* A *be a commutative ring and* k *a positive integer. There* *is a map of long exact sequences*

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\prod_{i\geq 1} \mathrm{TR}^{i/k}_{q-\lambda_{\lfloor (i-1)/k \rfloor}}(A) & \xrightarrow{\mathrm{id}} & \prod_{i\geq 1} \mathrm{TR}^{i/kn}_{q-\lambda_{\lfloor (i-1)/kn \rfloor}}(A) \\
\downarrow{\scriptstyle V_{k*}} & & \downarrow{\scriptstyle V_{kn*}} \\
\prod_{i\geq 1} \mathrm{TR}^{i}_{q-\lambda_{\lfloor (i-1)/k \rfloor}}(A) & \xrightarrow{V_{n*}} & \prod_{i\geq 1} \mathrm{TR}^{i}_{q-\lambda_{\lfloor (i-1)/kn \rfloor}}(A) \\
\downarrow & & \downarrow \\
\mathrm{TF}_{q+1}(A[x]/(x^k),(x)) & \longrightarrow & \mathrm{TF}_{q+1}(A[x]/(x^{nk}),(x)) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

where the lower left vertical map is induced by $A[x]/(x^k) \to A[x]/(x^{nk})$, $x \mapsto x^n$, *V maps are given by Verschiebung maps,* $e_i = \lfloor (i-1)/kn \rfloor$, *and* $\mathrm{TR}^{i/l}_s(A)$ *is trivial when* $i/l \notin \mathbb{N}$.

Proof. Taking the infinite coproduct of the diagram in the corollary, we get the following map of cofibration sequences

$$\begin{array}{ccccc}
\bigvee_{i\geq 0} \mathbb{T}_+ \wedge_{C_{i/k}} S^{\lambda_{d_i}} & \xrightarrow{\mathrm{pr}} & \bigvee_{i\geq 0} \mathbb{T}_+ \wedge_{C_i} S^{\lambda_{d_i}} & \longrightarrow & \mathrm{N}^{\mathrm{cy}}(\Pi_k) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{g_n}} \\
\bigvee_{i\geq 0} \mathbb{T}_+ \wedge_{C_{i/kn}} S^{\lambda_{e_i}} & \xrightarrow{\mathrm{pr}} & \bigvee_{i\geq 0} \mathbb{T}_+ \wedge_{C_i} S^{\lambda_{e_i}} & \longrightarrow & \mathrm{N}^{\mathrm{cy}}(\Pi_{nk}),
\end{array}$$

where $\mathrm{g_n}$ denotes the map induced by $\Pi_k \to \Pi_{nk}$, $a \mapsto b^n$. Applying the functor $\mathrm{THH}(A) \otimes_{\mathbb{S}} (-)$ to the diagram above and taking fixed points and the homotopy limits along with Frobenius maps and homotopy groups of spectra, we get the desired diagram by (a). □

We can deduce another map of another long exact sequences by the cofibration sequence.

Theorem 5.10. *Let A be a ring in which p is nilpotent. There is a map of long exact sequences*

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\lim_R \mathrm{TR}^{i/k}_{q-\lambda_{d_{i,k}}}(A) & \xrightarrow{\mathrm{id}} & \lim_R \mathrm{TR}^{i/kn}_{q-\lambda_{d_{i,kn}}}(A) \\
\downarrow V_{k*} & & \downarrow V_{kn*} \\
\lim_R \mathrm{TR}^{i}_{q-\lambda_{d_{i,k}}}(A) & \xrightarrow{V_{n*}} & \lim_R \mathrm{TR}^{i}_{q-\lambda_{d_{i,kn}}}(A) \\
\downarrow & & \downarrow \\
K_{q+1}(A[x]/(x^k),(x)) & \xrightarrow{v_n} & K_{q+1}(A[x]/(x^{nk}),(x)) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array},$$

where $d_{i,k} = \lfloor (i-1)/k \rfloor$, $d_{i,kn} = \lfloor (i-1)/kn \rfloor$, V denotes the map induced by Verschiebung maps and the maps in the limits are restriction maps and the maps $K_m(A[x]/(x^k),(x)) \to K_m(A[x]/(x^{nk}),(x))$ are induced by $x \mapsto x^n$.

Proof. The same argument in the proof of [Hesselholt1, 2.1] holds. More precisely, we first smash $\mathrm{THH}(A)$ with the cofibration sequences in the proof of the above theorem and use (a). Next we take homotopy limits along with Frobenius maps and homotopy fixed points of restriction maps. Then by (b), we get the desired diagram. □

In [Hesselholt1, §5] Hesselholt gave an explicit translation of topological Hochschild homology and big de Rham-Witt complex $\mathbb{W}_{(-)}\Omega^*_A$ for regular

$\mathbb{F}_p$-algebras. By the translation, we immediately get the following

Corollary 5.11. *Let A be an regular $\mathbb{F}_p$-algebra. There is a map of long exact sequences*

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\bigoplus_{l\geq 0} \mathbb{W}_{l+1}\Omega^{q-2l}_A & \xrightarrow{\mathrm{id}} & \bigoplus_{l\geq 0} \mathbb{W}_{l+1}\Omega^{q-2l}_A \\
\downarrow{\scriptstyle V_{k*}} & & \downarrow{\scriptstyle V_{kn*}} \\
\bigoplus_{l\geq 0} \mathbb{W}_{k(l+1)}\Omega^{q-2l}_A & \xrightarrow{V_{n*}} & \bigoplus_{l\geq 0} \mathbb{W}_{kn(l+1)}\Omega^{q-2l}_A \\
\downarrow & & \downarrow \\
K_{q+1}(A[x]/(x^k),(x)) & \xrightarrow{v_n} & K_{q+1}(A[x]/(x^{nk}),(x)) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array},$$

where I_p denotes the set of positive integers which are not divisible by p, the subscript $m(l+1)$ denotes the truncation set $\{1, 2, ..., m(l+1)\}$.

Taking the colimit of the diagram in the above theorem, we get the following.

Corollary 5.12. *Let A be an $\mathbb{F}_p$-algebra. Then there is a long exact sequence*

$$... \longrightarrow \lim_R \mathrm{TR}^{i/k}_{q-\lambda_{d^i_k}}(A) \longrightarrow \mathrm{colim}_n \lim_R \mathrm{TR}^{i}_{q-\lambda_{d^i_{kn}}}(A) \longrightarrow$$

$$K_{q+1}(\tilde{A}) \longrightarrow ...,$$

where $\tilde{A} := \mathrm{colim}_n(A[x]/(x^{nk}),(x))$.

Proof. The colimit is filtered. Therefore, we get the long exact sequence by taking the colimit of the long exact sequences obtained in the above theorem and $\mathrm{colim}_n K_*(A[x]/(x^{nk}), (x))$ is canonically isomorphic to $K_*(\tilde{A})$. □

For any $\mathbb{Z}_{(p)}$-algebra A, there is a decomposition by [Hesselholt1, §2]

$$\mathrm{TR}^i_{q-\lambda_d}(A) \cong \prod_{j \in i'\mathbb{N}} \mathrm{TR}^u_{q-\lambda_{\lfloor (p^{u-1}j-1)/k \rfloor}}(A;p),$$

where $i = p^{u-1}i'$ with $i'/p \notin \mathbb{N}$ and $d = \lfloor (i - 1/k) \rfloor$. The above isomorphism is induced by the following maps

$$\mathrm{TR}^i_{q-\lambda_d}(A) \to \mathrm{TR}^{i/j}_{q-\lambda_d} \to \mathrm{TR}^{p^u-1}_{q-\lambda_{(p^{u-1}j-1)/k}}(A),$$

where the the first map is Frobenius and the other one is the restriction map. We define $\mathrm{TR}^u_q(A;p) := \mathrm{TR}^{p^{u-1}}_q(A)$.

Let A be a $\mathbb{Z}_{(p)}$-algebra for a prime number p, and i, n and q natural numbers. Write $i = p^{u-1}i'$ with $i'/p \notin \mathbb{N}$ and $n = p^{v-1}n'$ with $n'/p \notin \mathbb{N}$. Then there is a commutative diagram, see [Hesselholt1],

$$\begin{array}{ccc} \mathrm{TR}^i_{q-\lambda}(A) & \longrightarrow & \prod_{j \in i'\mathbb{N}} \mathrm{TR}^u_{q-\lambda'}(A;p) \\ \downarrow & & \downarrow \\ \mathrm{TR}^{in}_{q-\lambda}(A) & \longrightarrow & \prod_{j \in i'n'\mathbb{N}} \mathrm{TR}^{u+v-1}_{q-\lambda'}(A;p), \end{array}$$

where the left vertical map is n-th Verschiebung map V^n and the right vertical map acts on the j-factor as $n'V^{v-1}$ which lands on jn'-factor and the horizontal maps are isomorphisms defined above. We use the stability lemma [Hesselholt1, Lemma 2.6].

Lemma 5.13. *Let p be a prime number and A a $\mathbb{Z}_{(p)}$-algebra and $i = p^{u-1}i'$,*

$n = p^v n'$ and q natural numbers and $j \in I_p$. Then there is a natural number u' such that the following diagram is commutative

$$\begin{array}{ccc}
\lim_R \mathrm{TR}^u_{q-\lambda_{\lfloor (p^{u-1}j-1)/k \rfloor}}(A;p) & \xrightarrow{V_*^{p^v}} & \lim_R \mathrm{TR}^u_{q-\lambda_{\lfloor (p^{u-1}j-1)/k \rfloor}}(A;p) \\
\downarrow \mathrm{pr} & & \downarrow \mathrm{pr} \\
\mathrm{TR}^{u'}_{q-\lambda_{\lfloor (p^{u'-1}j-1)/k \rfloor}}(A;p) & \xrightarrow{V_*^{p^v}} & \mathrm{TR}^{u'+v}_{q-\lambda_{\lfloor (p^{u'-1}j-1)/k \rfloor}}(A;p),
\end{array}$$

and vertical maps are isomorphisms and $q < 2\lfloor (p^{u'}j-1)/k \rfloor$.

Proof. We first note that $\lfloor (p^{u'+v-1}jn'-1)/kn \rfloor = \lfloor (p^{u'-1}j-1)/k \rfloor$. Therefore, the diagram is commutative. Moreover, by Lemma 2.6 in [Hesselholt1], the vertical maps are isomorphisms for $q < 2\lfloor (p^{u'}j-1)/k \rfloor$. □

Taking the limit on the decomposition, we get the isomorphism

$$\lim_R \mathrm{TR}^i_{q-\lambda_d} \cong \prod_{j \in I_p} \lim_R \mathrm{TR}^u_{q-\lambda_{(p^{u-1}j-1)/k}}(A;p),$$

where I_p is the set of natural numbers which are not divided by p.

Corollary 5.14. *Let A be a $\mathbb{Z}_{(p)}$-algebra. Then we can chose $\tilde{u}$ such that the following is a long exact sequence*

$$\begin{array}{l}
\ldots \longrightarrow \bigoplus_{j \in I_p} \lim_R \mathrm{TR}^{\tilde{u}}_{q-\lambda_{d^{k,p}_{u,j}}}(A;p) \\
\longrightarrow \bigoplus_{j \in I_p} \mathrm{colim}_v \lim_R \mathrm{TR}^{\tilde{u}+v+s}_{q-\lambda_{d^{k,p}_{u,j}}}(A;p) \longrightarrow \\
K_{q+1}(\tilde{A}) \longrightarrow \ldots,
\end{array}$$

where $\tilde{A} := \operatorname{colim}_n(A[x]/(x^{nk}),(x))$ *and* $k = p^s k'$ *and* $n = p^v n'$ *and* $d^{k,p}_{u,j} = \lfloor (p^{u-1}j-1)/k \rfloor$.

Proof. By 5.13, for any $j \in I_p$, it is able to chose large enough $\tilde{u}$ such that the following commutes and vertical maps are isomorphisms

$$\begin{array}{ccc} \lim_R \mathrm{TR}^u_{q-\lambda_{\lfloor (p^{u-1}j-1)/k \rfloor}}(A;p) & \xrightarrow{V_*^{p^s}} & \lim_R \mathrm{TR}^{u+s}_{q-\lambda_{\lfloor (p^{u-1}j-1)/k \rfloor}}(A;p) \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}} \\ \mathrm{TR}^{\tilde{u}}_{q-\lambda_{\lfloor (p^{u'-1}j-1)/k \rfloor}}(A;p) & \xrightarrow{V_*^{p^s}} & \mathrm{TR}^{\tilde{u}+s}_{q-\lambda_{\lfloor (p^{u'-1}j-1)/k \rfloor}}(A;p), \end{array}$$

where $k = p^s k'$. We also have the decomposition for $\mathbb{Z}_{(p)}$-algebra. Therefore, we get the desired long exact sequence from Cor 5.12. □

By [HM1, Theorem A] and our result Corollary 5.11, we have the following commutative diagram of short exact sequences for any j, m, n and any perfect field k of positive characteristic

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{W}_j(k) & \xrightarrow{V_m} & \mathbb{W}_{jm}(k) & \longrightarrow & K_{2j-1}(k[x]/(x^m),(x)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle V_n} & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{W}_j(k) & \xrightarrow{V_{mn}} & \mathbb{W}_{jmn}(k) & \longrightarrow & K_{2j-1}(k[x]/(x^{nm}),(x)) & \longrightarrow & 0, \end{array}$$

where $\mathbb{W}$ denotes big Witt vectors, and the right vertical map is induced by the power map $x \mapsto x^n$. In the rest of this section, we consider applications of this diagram.

Let k be a perfect field with characteristic $p > 0$ and let

$$W(k)[p^{1/p^\infty}] := \operatorname{colim}_n W(k)[p^{1/p^n}],$$

where the structure maps are given by $p^{1/p^n} \mapsto (p^{1/p^{n+1}})^p$. We consider the completion $\mathcal{O}_K := W(k)[p^{1/p^\infty}]^\wedge$ with quotient field $K := \mathcal{O}_K[1/p]$ and residue field k. For example, if $k = \mathbb{F}_p$ then $\mathcal{O}_K = \mathbb{Z}_p[p^{1/p^\infty}]^\wedge$ with quotient field $\mathcal{O}_K[1/p] = \mathbb{Q}_p(p^{1/p^\infty})^\wedge$. Using $W(k)/p = k$ and $W(k)[x]/(x^{p^n} - p) = W(k)[p^{1/p^n}]$, we have

$$\mathcal{O}_K/p\mathcal{O}_K = \operatorname{colim}_n k[x]/(x^{p^n})$$

with structure maps given by $x \mapsto x^p$. We let $\mathfrak{m} \subset \mathcal{O}_K$ denote the maximal ideal. Taking the colimit of the diagram above, we obtain a corollary.

Corollary 5.15. *With the notation above, we have*

$$K_{2j-1}(\mathcal{O}_K/p\mathcal{O}_K, \mathfrak{m}/p\mathcal{O}_K) = \operatorname{colim}_n(\mathbb{W}_{jp^n}(k)/V_{p^n}\mathbb{W}_j(k)),$$

where the colimit is indexed by the category of natural numbers under addition. Moreover, the relative K-groups in even degrees are zero.

Let k again be a perfect field with characteristic $p > 0$ and let

$$W(k)[\zeta_{p^\infty}] := \operatorname{colim}_n W(k)[\zeta_{p^n}],$$

where ζ_{p^n} denotes a primitive p^n-th root of unity and we choose these to satisfy $\zeta_{p^n}^p = \zeta_{p^{n-1}}$. We consider the completion $\mathcal{O}_K := W(k)[\zeta_{p^\infty}]^\wedge$ with quotient field $K = \mathcal{O}_K[1/p]$ and residue field k. For example, if $k = \mathbb{F}_p$ then $\mathcal{O}_K = \mathbb{Z}_p[\zeta_{p^\infty}]^\wedge$ with quotient field $\mathcal{O}_K[1/p] = \mathbb{Q}_p(\zeta_{p^\infty})^\wedge$. Let us write $K_0 = W(k)[1/p]$ and $K_n = K_0(\zeta_{p^n})$. Since $|K_n : K_0| = p^{n-1}(p-1)$ and $\zeta_{p^n} - 1$ is a uniformizer, the map

$$k[x]/(x^{p^{n-1}(p-1)}) \to W(k)(\zeta_{p^n})/p$$

given by $x \mapsto \zeta_{p^n} - 1$ is an isomorphism. Moreover, with these isomorphisms, the following diagram

$$\begin{array}{ccc} W(k)(\zeta_{p^n})/p & \xleftarrow{\cong} & k[x]/(x^{p^{n-1}(p-1)}) \\ \uparrow & & \uparrow \\ W(k)(\zeta_{p^{n-1}})/p & \xleftarrow{\cong} & k[x]/(x^{p^{n-2}(p-1)}) \end{array}$$

commutes, where the left vertical map is given by $\zeta_{p^{n-1}} - 1 \mapsto \zeta_{p^n}^p - 1$ and the right vertical map is given by $x \mapsto x^p$. By this construction, we have

$$\mathcal{O}_K/p\mathcal{O}_K = \operatorname{colim}_n k[x]/(x^{p^{n-1}(p-1)}).$$

We let $\mathfrak{m} \subset \mathcal{O}_K$ denote the maximal ideal. Taking the colimit of the diagram above, we obtain a corollary again.

Corollary 5.16. *With the notation above, we have*

$$K_{2j-1}(\mathcal{O}_K/p\mathcal{O}_K, \mathfrak{m}/p\mathcal{O}_K) = \operatorname{colim}_n(\mathbb{W}_{jp^{n-1}(p-1)}(k)/V_{p^{n-1}(p-1)}\mathbb{W}_j(k)),$$

where the colimit is indexed by the category of natural numbers under addition. Moreover, the relative K-groups in even degrees are zero.

The p-typical decomposition of the right-hand sides in Corollary 5.15 and Corollary 5.16 is explained in [Hesselholt1, p.4-5].

6 Semirings and spectra

This section is about future work. The Witt vector functor has been extended to commutative semirings by Borger ([Borger2]) using the theory of plethystic algebra ([BW]). We recall here Witt vectors for commutative semirings, give some discussion on an obstruction for extending THH to commutative semirings using the usual stable homotopy theory, and discuss a possible way to deal with the problem. In the following, the definition of $\mathbb{N}$-algebra and that of semiring are the same.

As we have seen, the stable homotopy theory of $\mathbb{S}$-algebras can be understood as a theory of numbers and moreover $\mathbb{S}$ is literally deeper than $\mathbb{N}$ in the setting of [CC]. However, as we have seen, $\mathbb{N}$ is an anomaly in the theory of numbers in a sense. The author believes that $\mathbb{N}$ should not be an anomaly in any theory of numbers. Apparently this kind of negligence, which might be ascribed to the structural irreversibility (e.g. the non-existence of inverse in a monoid), is not only for stable homotopy theory but also for other branches of geometry. For instance, the introduction of [Borger2] says, " *There is also a larger purpose to this chapter, which is to show that the formalism of (commutative) semirings—and more broadly, scheme theory over* $\mathbb{N}$*—is a natural and well-behaved formalism, both in general and in its applications to Witt vectors and positivity. It has gotten almost no attention from people working with scheme theory over* $\mathbb{Z}$*, but it deserves to be developed seriously—and independently of any applications, which are inevitable in my view."*. Grandis also studies irreversible worlds in [Grandis].

We naively assume that to study $\mathbb{N}$-algebras as algebras over $\mathbb{S}$ is also inevitable. So, without prudent preparations, we are going to try to observe what can happen.

Let $\Lambda_{\mathbb{N}}$ be the semiring of infinite variable symmetric functions with coefficients in the commutative semiring $\mathbb{N}$. More precisely, let $\Lambda_{\mathbb{Z}}$ be the ring of symmetric functions in infinitely many variables. As an abelian group, it has a basis consisting of the completed elementary functions

$$\overline{e_k}(x_1, x_2, x_3, ...) = \Sigma_{j_1 \geq ... \geq j_k} x_{j_1} x_{j_2} \cdots x_{j_k}.$$

The semiring $\Lambda_{\mathbb{N}}$ is the sub-$\mathbb{N}$-module

$$\Lambda_{\mathbb{N}} \subset \Lambda_{\mathbb{Z}}$$

spanned by this basis. In [Borger2], Borger constructs a plethory structure on $\Lambda_{\mathbb{N}}$ which is actually the restriction of the usual plethory structure on $\Lambda_{\mathbb{Z}}$ (called plethism or composition structure in [Macdonald, p. 135]), so that the set $\mathrm{Alg}_{\mathbb{N}}(\Lambda_{\mathbb{N}}, A)$ of algebra homomorphisms has a natural λ-semiring structure for any commutative semiring A ([Borger2, §4]). To be more precise, there is a monoidal product $\odot$ on the category of bi-$\mathbb{N}$-algebras characterized by

$$\mathrm{Alg}_{\mathbb{N}}(P \odot R, S) = \mathrm{Alg}_{\mathbb{N}}(R, \mathrm{Alg}_{\mathbb{N}}(P, S)).$$

Defining a plethory to be a monoid object in the monoidal structure, he gives an explicit monoidal structure for $\Lambda_{\mathbb{N}}$ ([Borger2, p. 19]). Using it, he reached the following definition in [Borger2, §6.1]

Definition 6.1 ([Borger2])**.** *For any commutative semiring A, the commutative semiring of big Witt vectors in A is*

$$\mathbb{W}(A) := \mathrm{Alg}_{\mathbb{N}}(\Lambda_{\mathbb{N}}, A),$$

where the semiring structure is induced by the bisemiring structure on $\Lambda_{\mathbb{N}}$*.*

We abused the notation $\mathbb{W}$ in the above definition. However, by construction, he proved that this definition was a generalization of the usual one.

Theorem 6.2 ([Borger2])**.** *For any commutative ring A,* $\mathbb{W}(A)$ *is the ring of classical big Witt vectors in A.*

He calculated the Witt vectors of the initial semiring $\mathbb{N}$.

Theorem 6.3 ([Borger2], 7.9)**.** *Let* $\mathcal{O}_{\overline{\mathbb{Q}}}^{tp}$ *be the multiplicative monoid of algebraic numbers which are integral at all finite places and which are real and positive at all infinite places. The Witt vectors* $\mathbb{W}(\mathbb{N})$ *is* $\mathbb{N}[\mathcal{O}_{\overline{\mathbb{Q}}}^{tp}]^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$.

We note that, by this theorem, $\mathbb{W}(\mathbb{N})$ is a countably infinite set, while $\mathbb{W}(\mathbb{Z})$ is known to be uncountably infinite. In particular, there is no bijection between $\mathbb{W}(\mathbb{N})$ and $\mathbb{W}(\mathbb{Z})$. As we saw in section 4, for commutative rings, THH and Witt vectors are related in the sense of Hesselholt-Madsen. In this way, the above theorem may suggest the existence of THH for semirings.

Remark 6.4. *Choosing an embedding* $\mathcal{O}_{\overline{\mathbb{Q}}} \hookrightarrow \mathbb{C}$*,* $\mathbb{W}(\mathbb{N})$ *may admit a conjugation induced by that on* $\mathbb{C}$*. According to Kottwitz [Scholze, Construction 9.3 (iv)], the cohomology theory conjectured at [Scholze, Conjecture 9.5] will have to have a graded antiholomorphic isomorphism when restricted to* $\mathrm{Kt}_{\mathbb{R}}$*. One might hope that it relates to* $\mathbb{W}(\mathbb{N})$*. This is not what we are discussing here mainly, but it seems to be worth mentioning.*

Borger also established p-typical, finite length Witt vectors for commutative semirings in [Borger2, §8] and proved that they were the usual ones for commutative rings. We write $W_{p,n}$ for the p-typical n-length Witt vectors.

Thanks to these results, we may perhaps be able to expect the existence of THH for commutative semirings which may require a new homotopy theory. In addition, from the viewpoint of Connes-Consani which we reviewed in previous sections, it will be fair to say that $\mathbb{N}$-algebras also can be studied as $\mathbb{S}$-algebras. In the rest of this section, reconsidering the notion of space, we will give some observations on (commutative) monoids and higher categories. They are not proved yet, at least by the author.

Our aim is to extend Hesselholt-Madsen's theorem (Therorem 4.3) to commutative semirings in accordance with Borger's theory and we hope that it will give an evidence for the existence of geometry of irreversible objects. In order for our aim, we need to define "$\pi_0(\mathrm{THH}(A)^{C_{p^n}})$" that should not be a commutative ring in general, for a commutative semiring A, but has to be a commutative semiring, since $W_{p,n}(A)$ is a commutative semiring in general. So, at first, we will try to observe what "stable homotopy monoids" should be. We use the letter τ instead of π tentatively.

As we saw in section 3.2, the stable homotopy group of a (nice) spectrum is defined as a certain colimit of homotopy groups of spaces. Therefore, to define "stable homotopy monoids", we may need to define "higher homotopy monoids of spaces". Basically, our idea is to use (∞, ∞)-categories instead of $(\infty, 0)$-categories. We note that, by the homotopy hypothesis, $(\infty, 0)$-categories are the same as spaces. The author personally thinks that objects named spaces should not necessarily have such full reversibility that any groupoid has. Spaces focusing on groups are spaces in the usual sense, namely every path in such a space is invertible. Spaces focusing on monoids may be highly directed things as mentioned below. Perhaps, spaces focusing on general $\mathbb{S}$-modules might be some kind of disconnected or granular one. In other words, every (commutative) group is a one-object groupoid and every (commutative) monoid is a one-object strict (∞, ∞)-category. Thus, every $\mathbb{S}$-module should be a one-object something as well. We could try to see what that should be, but it is too far from the author's understanding by now.

Special Γ-spaces might be another candidate for a model of such homotopy theory. However, commutative *monoids* in ∞-*groupoids* might not be very natural and we may not be able to get higher stable homotopy monoids by special Γ-spaces. Grandis' directed spaces ([Grandis]) should be also related to this topic. However, we may need higher irreversible simplices to study Eilenberg-Maclane spectra of commutative monoids.

It is known that the loop space of an $(\infty, n+1)$-category is an (∞, n)-category ([GeHa, §6.3]) in the sense of Gepner-Haugseng. For example, the loop space ΩX of a quasicategory X is a Kan complex. Thus the homotopy category $\mathrm{h}(\Omega X)$ is a groupoid and the fundamental monoid $\mathrm{End}_{\mathrm{h}(\Omega X)}(*)$ at a point $*$ is a group.

More generally, for an (∞, ∞)-category X, we might be able to define the m-th homotopy monoid $\tau_m(X, *)$ of X at a point $*$ to be $\mathrm{End}_{\mathrm{h}(\Omega^{m-1}X)}(*)$ so that $\tau_m(X, *)$ would be isomorphic to $\tau_{m-1}(\Omega X, *)$ for any $m \geq 1$. We may define $\tau_0(X)$ to be $\mathrm{ob}(\mathrm{h}(X))/\,\mathrm{isom}$ which should be the same as Joyal's τ_0 when X is a quasicategory [Joyal, §2].

Let next X be an (∞, n)-category for some natural number n. If m is sufficiently larger than n, then $\tau_m(X)$ would be a group. For example, if $m = n+1$, $\Omega^{m-1}X$ would be an $(\infty, 0)$-category. So $\tau_m(X)$ would be a group. In other words, we may need full irreversibility for our spaces. Considering this, spaces for our homotopy theory should mean (∞, ∞)-categories, since stable homotopy monoid τ_*^{st} would be defined as a certain colimit. Studying

Γ-objects in (∞, ∞)-categories and their smash products might perhaps give us a homotopy theory of (connective) semiring spectra. We pose some naive questions.

Question 6.5. *(i) Do we need topological alternatives of* (∞, ∞)*-categories?*

(ii) Is the loop space of an (∞, ∞)*-category an* (∞, ∞)*-category as well?*

(iii) Does an (∞, ∞)*-categorical analogue of the Freudenthal suspension theorem hold?*

For the first question, we note that the topological 1-dimensional sphere has finite cyclic groups as its subgroups, however, the simplicial 1-dimensional sphere does not, since it does not have many objects. We might be able to deal with this problem using subdivision techniques. Note that subdivisions may change the homotopy types of simplicial sets in the sense of Joyal. For instance, the edgewise subdivision ([BHM]) of $\Delta[2]$ is not weakly equivalent to $\Delta[2]$ in the sense of Joyal.

For the questions above, Verity's theory of weak complicial sets ([Verity]) will be worth considering. He has defined a model structure on the category of stratified simplicial sets, which models (∞, ∞)-categories. So we may define the generalized Reedy model structure on the category of Γ-objects in stratified simplicial sets ([BeMo]). As we have done for Γ-spaces in section 3, we also get an endofunctor $L_{\mathbb{S}}X \in \mathrm{End}(\mathsf{StrsSet}_*)$ for X a Γ-object in stratified simplicial sets, where $\mathsf{StrsSet}_*$ denotes the category of pointed stratified simplicial sets. Then we may define the stable homotopy monoid $\tau_*^{st}(X)$ of X to be $\mathrm{colim}_{k\to\infty}\, \tau_{k+*}(L_{\mathbb{S}}X(S^k))$ and also localize the generalized Reedy model structure with respect to τ_*^{st}-isomorphisms. Therefore we obtain natural questions.

Question 6.6. *(i) Does the localized model structure exist?*

(ii) Are fibrant objects of the localized model category special and piecewise weak complicial sets?

(iii) Is it possible to construct an adjunction using suspension and loop space for stratified simplicial sets in a reasonable sense?

We now try to give an observation on geometrical differences between (commutative) monoids and (commutative) groups. As is well known, any commutative monoid is homotopic to its groupification. Let M be a (commutative) monoid. Then we may get a strict (∞, ∞)-category as follows; the set of objects is a one point set, 1-morphisms are elements of the underlying set of M, for two 1-morphisms f and g, the set of 2-morphisms from f to g is $\{h \in M | f + h = g\}$, and so on. If the monoid M is a group, then the resulting strict (∞, ∞)-category will be just a strict $(\infty, 0)$-category. More precisely, for 1-morphisms f and g, there exists the unique 2-isomorphism $g - f$ from f to g. If the monoid M is a cancellative monoid, then the resulting strict (∞, ∞)-category will be a strict $(\infty, 1)$-category. More precisely, for 1-morphisms f and g, there is at most one 2-morphism from f to g. The same argument for (strict) symmetric monoidal categories may work. We can also view a commutative monoid M as the strict (∞, ∞)-category such that the object set is the underlying set M. In addition, it might be crucial that, for a (commutative) monoid M, the action on EM by the multiplication of M is not free in general.

Therefore, the Eilenberg-MacLane spectrum of a commutative monoid perhaps should be viewed as a special Γ-object in weak complicial sets. Also we can define smash product as the Day-convolution and consider monoid objects similarly to what we reviewed in section 2 and section 3. Such monoid objects would be called semiring spectra.

Question 6.7. *(i) Is the Eilenberg-MacLane spectrum of a semiring a semiring spectrum?*

(ii) Is the smash product compatible with both of the generalized Reedy model structure and the localized model structure?

(iii) Does $\tau_*^{st}(X)$ have a graded semiring structure for any bi-fibrant monoid object X in Γ-stratified simplicial sets?

For a commutative semiring A, it perhaps might be possible to define the topological Hochschild homology $\mathrm{THH}(A)$ as the geometric realization of the derived cyclic bar-construction of the Eilenberg-MacLane spectrum of A with respect to the Day convolution mentioned above. Note that it is not the entire analogy of Bökstedt's construction for ring spectra in section 4, so that it might be very difficult to analyze (non-homotopy) fixed points of $\mathrm{THH}(A)$. By the construction of the geometric realization ([Drinfeld]), $\mathrm{THH}(A)$ will have a $\mathbb{T}$-action. Then its fixed points may make sense, although we do not know their properties.

Question 6.8. *For a commutative semiring A, are* $\mathrm{THH}(A)$ *and its fixed points* $\mathrm{THH}(A)^{C_{p^n}}$ *commutative semiring spectra?*

Let us think every question is cleared. Here is our expectation.

Expectation 6.9. *There is a natural semiring isomorphism for any prime p, any natural number n and any commutative semiring A,*

$$\tau_0^{st}(\mathrm{THH}(A)^{C_{p^n}}) \cong W_{p,n}(A).$$

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