Tilting objects over preprojective algebras associated to Coxeter groups

(前射影代数上のコクセター群に付随する傾対象)

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Contents

1 Introduction

The aim of the representation theory of algebras is to study the structure of the module categories of algebras, or the structure of triangulated categories arising from algebras. For instance, the derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$ of an algebra A, or the stable category $\underline{\mathsf{mod}}\,A$ of a finite dimensional self-injective algebra A are very basic triangulated categories in the representation theory and studied by many mathematicians. It is known that such triangulated categories are *algebraic*, that is, equivalent to stable categories of Frobenius categories (see Subsection 3.3 for details). To study these triangulated categories, tilting theory has been developed in recent decades.

Tilting theory is one of the main tools in the study of algebraic triangulated categories. One of the most basic triangulated categories is the homotopy category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ of an algebra A. Tilting theory gives an equivalence between an algebraic triangulated category and the homotopy category of an algebra. In fact, it was shown by Keller [Ke94] (see also Theorem 3.9) that an algebraic triangulated category $\mathcal{T}$ is triangle equivalent to the homotopy category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ of an algebra A if and only if $\mathcal{T}$ has a *tilting object* whose endomorphism algebra is isomorphic to A. Hence it is important to construct a tilting object of an algebraic triangulated category and to study its endomorphism algebra.

Let A be a finite dimensional algebra. A typical example of a tilting object is an algebra A itself in the homotopy category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$. Let T be a tilting A-module. Then a minimal projective resolution of T is a tilting object of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$. Moreover, there are many studies which construct tilting objects, for example [BGG, IO, Lu, MY, MU, Y]. Our results in this thesis are contained in this flow.

In this thesis we report on recent results shown by the author which construct and study tilting objects of certain triangulated categories. This thesis consists of three parts. Part I is based on [Ki14]. In this part, we deal with a triangulated category which is constructed from a preprojective algebra and an element of a Coxeter group. We construct a tilting object in the triangulated category and calculate its endomorphism algebra, when the element of the Coxeter group is c-sortable. Part II is based on [Ki16]. In this part, we deal with the same triangulated category as Part I. Here the element of the Coxeter group is more general than c-sortable, that is, c-starting or c-ending. We show that the category always has a silting object, which is a generalization of a tilting object, and show that if the element of the Coxeter group is c-starting or c-ending, then the silting object is a tilting object. Moreover, we compare the equivalence obtained by the tilting object and the equivalence of Amiot-Reiten-Todorov [ART]. Part III is based on [Ki17]. In this part, we deal with the derived category of modules over the stable category of a hereditary algebra, motivated by the result of Iyama and Oppermann [IO].

Tilting objects associated to c-sortable elements

The preprojective algebra for a quiver Q was introduced by Gelfand-Ponomarev [GP] to study the representation theory of all path algebras of quivers whose underlying graphs coincide with Q. Since the preprojective algebra of Q has all information of such path algebras, its representation theory is very rich and appears in many branches of mathematics. In particular, preprojective algebras play an important role in the additive categorification of Fomin-Zelevinsky's cluster algebras [FZ].

In the context of the categorification of cluster algebras, preprojective algebras were firstly studied by Geiss-Leclerc-Schröer [GLS06, GLS07]. Let Q be a Dynkin quiver, that is, the underlying graph of Q is a simply laced Dynkin graph, and Π be the preprojective algebra of Q. In these papers, they showed that the stable category $\underline{\mathsf{mod}}\,\Pi$ is a 2-Calabi-Yau triangulated category, and has cluster tilting objects which are crucial concept of the categorification.

More generally, by using the preprojective algebra Π of a finite acyclic quiver Q, Buan-Iyama-Reiten-Scott [BIRSc] construct a 2-Calabi-Yau triangulated category with cluster tilting objects as follows. For each vertex $u \in Q_0$, let $I_u := \Pi(1-e_u)\Pi$ be a two-sided ideal of Π, where e_u is an idempotent of Π associated to u. The Coxeter group W_Q of Q is a group generated by the set $\{s_u \mid u \in Q_0\}$ with appropriate Coxeter relations. For each element w of W_Q with a reduced expression $s_{u_1}s_{u_2}\cdots s_{u_l}$, consider the assignment $I(w) := I_{u_1}I_{u_2}\cdots I_{u_l}$. Let $\langle I_u \mid u \in Q_0\rangle$ be a semigroup generated ideals I_u, where the multiplication is given by that of two-sided ideals. Then in [BIRSc], the authors first showed the following.

Theorem 1.1. *[BIRSc, Theorem III. 1.9] The assignment w to $I(w)$ gives an isomorphism $W_Q \simeq \langle I_u \mid u \in Q_0\rangle$ of semigroups.*

They defined an algebra $\Pi(w) := \Pi/I(w)$ for each $w \in W_Q$, which plays a central role in their studies and also in Part I and II of this thesis. They showed that the algebra $\Pi(w)$ is Iwanaga-Gorenstein of dimension at most one. This fact gives that the category $\mathsf{Sub}\,\Pi(w)$ of $\Pi(w)$-submodules of free $\Pi(w)$-modules is a Frobenius category, and the stable category $\underline{\mathsf{Sub}}\,\Pi(w)$ is a triangulated category. Let $\mathrm{D} = \mathrm{Hom}_K(-,K)$ be the standard K-dual, where K is a field. One result of [BIRSc] is the following.

Theorem 1.2. *[BIRSc] For any $w \in W_Q$, we have the followings.*

(a) The stable category $\underline{\mathsf{Sub}}\,\Pi(w)$ is a 2-Calabi-Yau triangulated category, that is, for any objects $X, Y \in \underline{\mathsf{Sub}}\,\Pi(w)$, there exists a bifunctorial isomorphism $\underline{\mathrm{Hom}}_{\Pi(w)}(X,Y) \simeq \mathrm{D}\,\underline{\mathrm{Hom}}_{\Pi(w)}(Y, X[2])$.

(b) For any reduced expression $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ of w, the object

$$T(\boldsymbol{w}) = \bigoplus_{i=1}^{l} \Pi/I(s_{u_1}s_{u_2}\cdots s_{u_i})e_{u_i}$$

is a cluster tilting object of $\underline{\mathsf{Sub}}\,\Pi(w)$, that is,

$$\mathsf{add}\,T(\boldsymbol{w}) = \{\, X \in \underline{\mathsf{Sub}}\,\Pi(w) \mid \mathrm{Ext}^1_{\Pi(w)}(X, T(\boldsymbol{w})) = 0 \,\}$$

holds.

We can see that if Q is a Dynkin quiver and if w is the longest element of the Coxeter group W_Q, then $\Pi(w) = \Pi$ and $\mathsf{Sub}\,\Pi(w) = \mathsf{mod}\,\Pi$ holds. Namely, the above theorem covers the Dynkin cases.

Roughly speaking, our results in Part I and Part II are tilting analog of results of [BIRSc]. The preprojective algebra Π and the factor algebra $\Pi(w)$ have natural structures of $(\mathbb{Z})$-graded algebras, which are determined by the orientation of a quiver Q. Then we can take the category $\mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$ of graded $\Pi(w)$-submodules of graded free $\Pi(w)$-modules. This category is also a Frobenius category and the stable category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$ is a triangulated category. We show that the category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$ has a tilting object. This study has two motivations: one comes from a relationship between $\underline{\mathsf{Sub}}\,\Pi(w)$ and cluster categories, and the other comes from the existence of tilting objects in the stable category of an Iwanaga-Gorenstein algebra.

Cluster tilting objects in 2-Calabi-Yau triangulated categories were introduced by Buan-Marsh-Reineke-Reiten-Todorov in [BMRRT]. Let H be a finite dimensional hereditary algebra, that is, the algebra of global dimension at most one. They construct a *cluster category* of H, which is a 2-Calabi-Yau triangulated category, as the orbit category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,H)/F$ of the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,H)$ modulo appropriate auto-functor F. They showed that any tilting H-module is a cluster tilting object in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,H)/F$, and the converse is also true in some sense. In particular, the algebra H itself is cluster tilting in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,H)/F$.

The construction of cluster categories was generalized by Amiot [A] for a finite dimensional algebra A of global dimension at most two. A *cluster category* $\mathsf{C}(A)$ of A is the triangulated hull of the orbit category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)/F$ in the sense of Keller [Ke05] for an appropriate auto-functor F on $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$. By construction, we have a natural triangle functor $\pi : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A) \to \mathsf{C}(A)$. It was shown that the cluster category $\mathsf{C}(A)$ is a 2-Calabi-Yau triangulated category and that the image of the tilting object A of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$ via π is a cluster tilting object of $\mathsf{C}(A)$.

A relationship between 2-Calabi-Yau triangulated categories $\mathsf{C}(A)$ and $\underline{\mathsf{Sub}}\,\Pi(w)$ was studied by Amiot-Reiten-Todorov [ART]. For any element $w \in W_Q$ and a reduced expression $\boldsymbol{w}$ of w, they constructed a finite dimensional algebra $A(\boldsymbol{w})$ (see Section 11) and they showed that there exists a triangle equivalence

$$\mathsf{C}(A(\boldsymbol{w})) \simeq \underline{\mathsf{Sub}}\,\Pi(w), \tag{1.1}$$

where $\pi(A(\boldsymbol{w}))$ goes to $T(\boldsymbol{w})$.

By forgetting the degree, we have a triangle functor $f : \underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w) \to \underline{\mathsf{Sub}}\,\Pi(w)$. Therefore, it is nature to expect that there exists a tilting object M of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$ such that $f(M) = T(\boldsymbol{w})$ holds. In fact, in a cluster category side, $A(\boldsymbol{w})$ is a tilting object of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A(\boldsymbol{w}))$ and $\pi(A(\boldsymbol{w}))$ is a cluster tilting object of $\mathsf{C}(A(\boldsymbol{w}))$. Moreover, it is also expected that the endomorphism algebra of M is isomorphic to $A(\boldsymbol{w})$.

The other motivation of this study comes from one natural question of Iwanaga-Gorenstein algebras. A finite dimensional algebra A is said to be Iwanaga-Gorenstein of dimension at most n if $\operatorname{injdim}{}_AA \leq n$ and $\operatorname{injdim} A_A \leq n$ hold. We call an A-module M *Cohen-Macaulay* if $\operatorname{Ext}_A^{>0}(M, A) = 0$, and denote by $\mathsf{CM}\,A$ the category of Cohen-Macaulay modules. If A is an Iwanaga-Gorenstein algebra of dimension at most n, then $\mathsf{CM}A$ is a Frobenius category. It is easy to see that $\mathsf{CM}A = \mathsf{mod}\,A$ if $n = 0$ and $\mathsf{CM}A = \mathsf{Sub}\,A$ if $n = 1$. If moreover A is a $\mathbb{Z}$-graded algebra, then we can define graded Cohen-Macaulay modules. We denote by $\mathsf{CM}^{\mathbb{Z}}A$ the category of graded Cohen-Macaulay modules, which is also Frobenius. We can also see that $\mathsf{CM}^{\mathbb{Z}}A = \mathsf{mod}^{\mathbb{Z}}\,A$ if $n = 0$ and $\mathsf{CM}^{\mathbb{Z}}A = \mathsf{Sub}^{\mathbb{Z}}\,A$ if $n = 1$.

Let A be a finite dimensional $\mathbb{Z}$-graded Iwanaga-Gorenstein algebra of dimension at most n. We consider the following question. When does the stable category $\underline{\mathsf{CM}}^{\mathbb{Z}}A$ have tilting objects ? In the case where $n = 0$, then a complete answer to this question was given by Yamaura [Y]. In this thesis, we study this question in the case where $A = \Pi(w)$, which is Iwanaga-Gorenstein of dimension at most one. In Part I and II, we give a sufficient condition such that the stable category $\underline{\mathsf{CM}}^{\mathbb{Z}}\Pi(w) = \underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$ has tilting objects.

In Part I, we show that $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$ has a tilting object when w is a c-sortable element, and calculate its endomorphism algebra. c-sortable elements were introduced by Reading [Re] to study noncrossing partitions associated to a Coxeter group. For a Coxeter group of a quiver Q, it is known by [AIRT] that there exists a closed connection between c-sortable elements of W_Q and tilting modules over the path algebra KQ, see Theorem 4.9. This connection enables us to show the existence of a tilting object and to study its endomorphism algebra in detail. For the definition and notation of c-sortable elements, see Definitions 3.2 and 6.1. Our first result is the following.

Theorem 1.3 (Theorem 5.6)**.** *Let $w \in W_Q$ be a c-sortable element with a c-sortable expression $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$. Then*

$$N(\boldsymbol{w}) := \bigoplus_{i=1}^{l} \Pi/I(s_{u_1}\cdots s_{u_i})e_{u_i}(m_i)$$

is a tilting object of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$.

Next we study the endomorphism algebra $B(\boldsymbol{w}) := \underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(N(\boldsymbol{w}))$ of the tilting object. Let N_0 be the degree zero part of $N := N(\boldsymbol{w})$. By [AIRT], it is known that there exists a tilting KQ-module T such that $\mathsf{Sub}\,T$ has an additive generator N_0. Using this notation, we have the following theorem.

Theorem 1.4 (see Theorems 6.2, 6.3, and 7.1)**.** *We have the followings:*

(a) There exists an isomorphism of algebras $B(\boldsymbol{w}) \simeq \mathrm{End}_{KQ}(N_0)/[T]$.

(b) The global dimension of $B(\boldsymbol{w})$ is at most two.

(c) We have a triangle equivalence

$$\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w) \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,B(\boldsymbol{w})).$$

Where the algebra $\mathrm{End}_{KQ}(N_0)/[T]$ is called a *relative stable Auslander algebra.* By construction, clearly we have $f(N) = T(\boldsymbol{w})$. Although we found a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$ when $\boldsymbol{w}$ is a c-sortable expression, the algebra $B(\boldsymbol{w})$ is not isomorphic to $A(\boldsymbol{w})$, in general.

Tilting objects associated to c-starting and c-ending elements

In Part II, we give a sufficient condition such that the category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$ has a tilting object such that its endomorphism algebra is isomorphic to $A(\boldsymbol{w})$. Firstly, we show that for each reduced expression $\boldsymbol{w}$ of any element $w \in W_Q$, $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$ has a silting object. Where silting objects, which are important objects in the representation theory, are a generalization of tilting objects from the point of mutations of tilting objects [AI].

Theorem 1.5 (Theorem 9.18)**.** *Let $w \in W_Q$. For any reduced expression $\boldsymbol{w} = s_{u_1} s_{u_2} \cdots s_{u_l}$ of w, an object*

$$M(\boldsymbol{w}) := \bigoplus_{i=1}^{l} \Pi/I(s_{u_1} \cdots s_{u_i}) e_{u_i}$$

of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\, \Pi(w)$ is a silting object.

We mention that as the above definitions show, two objects $N(\boldsymbol{w})$ and $M(\boldsymbol{w})$ of Theorems 1.3 and 1.5 are quite different even if $\boldsymbol{w}$ is a c-sortable expression. In fact they have different gradings, and such a difference is crucial when we study $\mathbb{Z}$-graded modules.

Note that our $M(\boldsymbol{w})$ is not a tilting object of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\, \Pi(w)$ in general (see Example 9.19). The second result in Part II gives a sufficient condition on $\boldsymbol{w}$ such that $M(\boldsymbol{w})$ is a tilting object of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\, \Pi(w)$. We introduce c-starting and c-ending elements in Definition 10.2, which are generalization of c-sortable elements. In particular, we have a triangle equivalence between $\underline{\mathsf{Sub}}^{\mathbb{Z}}\, \Pi(w)$ and the derived category of the endomorphism algebra of $M(\boldsymbol{w})$.

Theorem 1.6 (Theorem 10.5)**.** *Let $w \in W_Q$ and $\boldsymbol{w}$ be a reduced expression of w. If $\boldsymbol{w}$ is c-ending on Q_0 or c-starting on Q_0, then we have*

(a) the object $M = M(\boldsymbol{w}) \in \underline{\mathsf{Sub}}^{\mathbb{Z}}\, \Pi(w)$ is a tilting object,

(b) the global dimension of the endomorphism algebra $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)$ of M in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\, \Pi(w)$ is at most two, and

(c) there exists a triangle equivalence $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, \underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)) \simeq \underline{\mathsf{Sub}}^{\mathbb{Z}}\, \Pi(w)$.

The third result of Part II is to compare the equivalence obtained by the tilting object $M(\boldsymbol{w})$ and the equivalence (1.1). We show that if the endomorphism algebra $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M(\boldsymbol{w}))$ of $M(\boldsymbol{w})$ coincides with the algebra $A(\boldsymbol{w})$, then two equivalences commute with canonical functors.

Theorem 1.7 (Theorem 11.4)**.** *Let $w \in W_Q$ and $\boldsymbol{w}$ be a reduced expression of w. If $\boldsymbol{w}$ is c-ending on* Supp(w)*, then $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M(\boldsymbol{w})) = A(\boldsymbol{w})$ holds and we have the following commutative diagram up to isomorphism of functors*

$$\begin{tikzcd}
\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A(\boldsymbol{w})) \arrow[r, "\simeq"] \arrow[d, "\pi"] & \underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w) \arrow[d, "f"] \\
\mathsf{C}(A(\boldsymbol{w})) \arrow[r, "\simeq"] & \underline{\mathsf{Sub}}\, \Pi(w).
\end{tikzcd}$$

Note that both tilting objects $N(\boldsymbol{w})$ and $M(\boldsymbol{w})$ have their own advantages. For example:

- If $\boldsymbol{w}$ is c-sortable, then we can show that the endomorphism algebra of the tilting object $N(\boldsymbol{w})$ is isomorphic to a relative stable Auslander algebra, that is, we can show Theorem 1.4 (a).
- If $\boldsymbol{w}$ is c-ending, then we can compare the equivalence obtained by the tilting object $M(\boldsymbol{w})$ and the equivalence of preceding study [ART], that is, we can show Theorem 1.7.

Stable categories of hereditary algebras and derived categories

We first recall the definition of modules over additive categories. Let $\mathcal{C}$ be an additive category. A *$\mathcal{C}$-module* is a contravariant functor from $\mathcal{C}$ to $\mathcal{A}b$, where $\mathcal{A}b$ is the category of abelian groups. This is an analog of modules over rings when we regard $\mathcal{C}$ as a ring with several objects. A finitely presented $\mathcal{C}$-module is also defined in the same way as defining a finitely presented module over a ring. We denote by $\mathsf{mod}\,\mathcal{C}$ the category of finitely presented $\mathcal{C}$-modules. If $\mathcal{C}$ is triangulated, then it is known that $\mathsf{mod}\ \mathcal{C}$ is Frobenius and abelian, and its stable category $\underline{\mathsf{mod}}\,\mathcal{C}$ is triangulated.

In Part III, we focus on the triangulated category $\underline{\mathsf{mod}}\,\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$, where A is a finite dimensional hereditary algebra. We construct a triangle equivalence between this category and the bounded derived category of some abelian category.

Let k be a field and A be a finite dimensional k-algebra. Recall that an algebra A is *representation finite* if there exist only finitely many isomorphism classes of indecomposable A-modules. This is equivalent to the existence of an additive generator X of $\mathsf{mod}\,A$, that is, each A-module is isomorphic to a direct summand of the direct sum of a finitely many copies of X. In [IO], it was shown that if A is a representation finite hereditary algebra, then there exists a triangle equivalence

$$\underline{\mathsf{mod}}\ \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A) \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\ \Gamma_A), \tag{1.2}$$

where $\Gamma_A := \mathrm{End}_A(X)/[A]$ is the *stable Auslander algebra* of A.

The aim of this part is to extend a triangle equivalence (1.2) to the case when A is a representation infinite hereditary algebra. If A is representation finite, then $\mathsf{mod}(\underline{\mathsf{mod}}\,A) \simeq \mathsf{mod}\,\Gamma_A$ holds. Therefore the role of the stable Auslander algebra Γ_A is played by the stable category of A. Our main result of this part is the following.

Theorem 1.8 (Theorem 14.5). *Let A be a hereditary algebra. We have a triangle equivalence*

$$\underline{\mathsf{mod}}\,\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A) \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}(\underline{\mathsf{mod}}\,A)). \tag{1.3}$$

To prove Theorem 1.8, we need to give general preliminary results on functor categories and repetitive categories. The functor category $\mathsf{mod}(\underline{\mathsf{mod}}\,A)$ is an abelian category with enough projectives and enough injectives, since the category $\underline{\mathsf{mod}}\,A$ forms a dualizing k-variety, which is a distinguished class of k-linear categories introduced by Auslander and Reiten [AR74], see Definition 12.12. A key role is played by the repetitive category $\mathsf{R}(\underline{\mathsf{mod}}\,A)$ of $\underline{\mathsf{mod}}\,A$. The following our first result implies that $\mathsf{R}(\underline{\mathsf{mod}}\,A)$ is a dualizing k-variety.

Theorem 1.9 (Theorem 13.7). *Let $\mathcal{A}$ be a dualizing k-variety. Then $\mathsf{R}\mathcal{A}$ is a dualizing k-variety.*

In particular, we can see that $\mathsf{mod}\,\mathsf{R}\mathcal{A}$ is a Frobenius abelian category for any dualizing k-variety $\mathcal{A}$. We denote by $\underline{\mathsf{mod}}\,\mathsf{R}\mathcal{A}$ the stable category of $\mathsf{mod}\,\mathsf{R}\mathcal{A}$, which is triangulated.

In the case where A is a representation finite hereditary algebra, the following Happel's theorem [Ha88] played an important role in the proof of a triangle equivalence (1.2). Let

A be a finite dimensional k-algebra of finite global dimension. The repetitive algebra $\widehat{A}$ of A is an infinite matrix algebra, without identity

$$\widehat{A} := \begin{pmatrix} \ddots & & & & \text{\huge 0} \\ \ddots & A & & & \\ & \mathrm{D}\,A & A & & \\ & & \mathrm{D}\,A & A & \\ \text{\huge 0} & & & \ddots & \ddots \end{pmatrix}$$

in which matrices have only finitely many non-zero entries and the multiplication is induced from the action of A to the A-bimodule $\mathrm{D}\,A := \mathrm{Hom}_k(A, k)$. We can see that the category of finitely generated $\widehat{A}$-modules $\mathsf{mod}\,\widehat{A}$ is a Frobenius category. Then Happel showed that the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$ is triangle equivalent to the stable category $\underline{\mathsf{mod}}\,\widehat{A}$.

In Section 13, we extend Happel's triangle equivalence to dualizing k-varieties and its module categories. In fact, we deal with the following class of categories including dualizing k-varieties. For a k-linear additive category $\mathcal{A}$, we denote by $\mathsf{proj}\,\mathcal{A}$ the category of finitely generated projective $\mathcal{A}$-modules and by $\mathsf{mod}\,\mathcal{A}$ the category of $\mathcal{A}$-modules having resolutions by $\mathsf{proj}\,\mathcal{A}$. We consider the following conditions:

(IFP) $\mathrm{D}\,\mathcal{A}(X, -)$ is in $\mathsf{mod}\,\mathcal{A}$ for each $X \in \mathcal{A}$, where $\mathrm{D} = \mathrm{Hom}_k(-, k)$.

(G) $\mathrm{D}\,\mathcal{A}(X, -)$ has finite projective dimension over $\mathcal{A}$ for each $X \in \mathcal{A}$.

For example, if $\mathcal{A}$ is a dualizing k-variety, then $\mathcal{A}$ satisfies the condition (IFP). On the other hand, the condition (G) is a categorical version of Gorensteinness. Gorenstein-projective modules (also known as Cohen-Macaulay modules, totally reflexive modules) are important class of modules. We denote by $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ the category of Gorenstein-projective $\mathsf{R}\mathcal{A}$-modules of finite projective dimension as $\mathcal{A}$-modules. We prove the following.

Theorem 1.10 (Corollaries 13.17, 13.18)**.** *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category.*

(a) *Assume that $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP) and (G). Then we have a triangle equivalence*

$$\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A}) \simeq \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A}).$$

(b) *Assume that $\mathcal{A}$ is a dualizing k-variety. If each object of* $\mathsf{mod}\,\mathcal{A}$ *and* $\mathsf{mod}\,\mathcal{A}^{\mathrm{op}}$ *has finite projective dimension, then we have a triangle equivalence*

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\mathcal{A}) \simeq \underline{\mathsf{mod}}\,\mathsf{R}\mathcal{A}.$$

Let A be a finite dimensional k-algebra. Then the category of finitely generated projective modules $\mathsf{proj}A$ is a typical example of a dualizing k-variety. If $\mathcal{A} = \mathsf{proj}A$ and A is of finite global dimension, then Theorem 1.10 gives a Happel's triangle equivalence.

In Section 14, we show the following theorem, which together with Theorem 1.10 implies Theorem 1.8.

Theorem 1.11 (Theorem 14.3)**.** *Let A be a finite dimensional hereditary k-algebra. Then we have an equivalence of additive categories*

$$\mathsf{R}(\underline{\mathsf{mod}}\, A) \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A).$$

We mention that Theorem 1.10 holds if a category $\mathcal{A}$ satisfies assumptions of the theorem. Typical examples are $\mathcal{A} = \mathsf{proj}\, A$ or $\mathcal{A} = \underline{\mathsf{mod}}\, A$ for a finite dimensional algebra A of finite global dimension. On the other hand, Theorem 1.11 holds only in the case when A is hereditary. Otherwise, we can easily find a counter example to Theorem 1.11.

2 Acknowledgement

The author would like to thank his supervisor Osamu Iyama for many supports and helpful comments. He is grateful to Kota Yamaura for helpful comments and discussions. The author thanks Takahide Adachi, Ryo Kanda, and Yuya Mizuno for taking care of him. He expresses his gratitude to professors and friends who gave him helpful advices.

Part I

Tilting objects associated to c-sortable elements

This part is based on the paper [Ki14].

Notation

In Part I and Part II, we use the following notation.

We denote by K an algebraically closed field. All categories are K-categories. All subcategories are full and closed under isomorphisms. All algebras are K-algebras, and all graded algebras are $\mathbb{Z}$-graded K-algebras. We always deal with left modules.

For an algebra A, we denote by $\mathsf{Mod}\,A$ (resp, $\mathsf{mod}\ A$, $\mathsf{fd}\,A$, $\mathsf{proj}\ A$) the category of (resp, finitely generated, finite dimensional, finitely generated projective) A-modules. For a graded algebra A, we denote by $\mathsf{Mod}^{\mathbb{Z}}\,A$ (resp, $\mathsf{mod}^{\mathbb{Z}}\,A$, $\mathsf{fd}^{\mathbb{Z}}A$, $\mathsf{proj}^{\mathbb{Z}}$A) the category of (resp, finitely generated, finite dimensional, finitely generated projective) $\mathbb{Z}$-graded A-modules with degree zero morphisms. For graded A-modules M, N, we denote by $\mathrm{Hom}_A^{\mathbb{Z}}(M, N)$ the set of morphisms from M to N in $\mathsf{Mod}^{\mathbb{Z}}\,A$.

For an additive category $\mathcal{C}$ and $M \in \mathcal{C}$, we denote by $\mathsf{add}(M)$ the additive closure of M in $\mathcal{C}$, that is, the full subcategory of $\mathcal{C}$ consisting of direct summands of the direct sum of finitely many copies of M. The composition of morphisms $f : X \to Y$ and $g : Y \to Z$ is denoted by $fg = g \circ f : X \to Z$. For two algebras A and B, we denote by $A \otimes B$ the tensor algebra of A and B over K. For two arrows α, β of a quiver such that the target of α is the source of β, we denote by $\alpha\beta$ the composition of α and β. We denote by $\mathrm{D} = \mathrm{Hom}_K(-, K)$ the standard K-dual. We always denote by [1] the suspension functor on triangulated categories.

3 Preliminary

In this section, we define some notation which we use throughout this thesis and recall some preliminary results. The notation defined in this section will also be used in Part II.

We fix a finite acyclic quiver $Q = (Q_0, Q_1, s, t)$, where $Q_0 = \{1, \dots, n\}$ is the set of vertices, Q_1 is the set of arrows, and an arrow α goes from $s(\alpha)$ to $t(\alpha)$. Let KQ be the path algebra of Q over K, and for a vertex u of Q, we denote by e_u the corresponding idempotent of KQ.

3.1 Coxeter groups and preprojective algebras

The *Coxeter group* $W = W_Q$ of Q is the group generated by the set $\{s_u \mid u \in Q_0\}$ with relations $s_u^2 = 1$, $s_u s_v = s_v s_u$ if there exist no arrows between u and v, and $s_u s_v s_u = s_v s_u s_v$ if there exists exactly one arrow between u and v.

We call an element of the free group generated by $\{s_u \mid u \in Q_0\}$ a *word*. If a word $\boldsymbol{w}$ represents an element $w \in W_Q$, then we say that $\boldsymbol{w}$ is an *expression* of w.

Definition 3.1. Let $w \in W_Q$ and $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ be an expression of w.

(1) A word $s_{u_{i_1}}s_{u_{i_2}}\cdots s_{u_{i_m}}$ is a *subword* of $\boldsymbol{w}$ if $1 \leq i_1 < i_2 < \cdots < i_m \leq l$ holds.

(2) An expression $\boldsymbol{w}$ of w is *reduced* if l is smallest possible.

(3) Let $\boldsymbol{w}$ be a reduced expression of w, put $\mathrm{Supp}(w) := \{u_1, u_2, \ldots, u_l\} \subset Q_0$. Note that, $\mathrm{Supp}(w)$ is independent of the choice of a reduced expression of w (see [BjBr, Corollary 1.4.8 (ii)]).

(4) An element $c \in W_Q$ is called a *Coxeter element* if there exists an expression $s_{v_1}s_{v_2}\cdots s_{v_n}$ of c such that $\{v_1, v_2, \ldots, v_n\}$ is a permutation of Q_0. In this paper, we only consider a Coxeter element c satisfying $e_{u_j}(KQ)e_{u_i} = 0$ for $i < j$ which is uniquely determined by the orientation of Q.

We recall the definition of c-sortable elements, which were introduced and studied in [Re].

Definition 3.2. [Re] Let c be a Coxeter element of W_Q and $\boldsymbol{c}$ a reduced expression of c. An element $w \in W_Q$ is called a *c-sortable element* if w has a reduced expression $\boldsymbol{w}$ of the form $\boldsymbol{w} = \boldsymbol{c}^{(0)}\boldsymbol{c}^{(1)}\cdots\boldsymbol{c}^{(m)}$, where each $\boldsymbol{c}^{(i)}$ is a subword of $\boldsymbol{c}$ and

$$\mathrm{Supp}(\boldsymbol{c}^{(m)}) \subset \mathrm{Supp}(\boldsymbol{c}^{(m-1)}) \subset \cdots \subset \mathrm{Supp}(\boldsymbol{c}^{(0)}) \subset Q_0.$$

In this case, we say that $\boldsymbol{w} = \boldsymbol{c}^{(0)}\boldsymbol{c}^{(1)}\cdots\boldsymbol{c}^{(m)}$ is a *c-sortable expression* of w.

Note that the definition of c-sortable elements independent of the choice of a reduced expression of c. If there is no danger of confusion, for a c-sortable expression $\boldsymbol{c}^{(0)}\boldsymbol{c}^{(1)}\cdots\boldsymbol{c}^{(m)}$, we denote by $c^{(i)}$ the element of W_Q represented by $\boldsymbol{c}^{(i)}$ for $i = 0, \ldots, m$.

Next we recall the preprojective algebra of Q and introduce factor algebras of the preprojective algebra. The *double quiver* $\overline{Q} = (\overline{Q}_0, \overline{Q}_1, s, t)$ of a quiver Q is defined by $\overline{Q}_0 = Q_0$, $\overline{Q}_1 = Q_1 \sqcup \{\alpha^* : t(\alpha) \to s(\alpha) \mid \alpha \in Q_1\}$. Then we define the *preprojective algebra* Π of Q by

$$\Pi := K\overline{Q}/\langle \sum_{\alpha \in Q_1} \alpha\alpha^* - \alpha^*\alpha \rangle.$$

Let u be a vertex of Q. We define the two-sided ideal I_u of Π by

$$I_u := \Pi(1 - e_u)\Pi.$$

Let $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ be a reduced expression of $w \in W_Q$. We define a two-sided ideal $I(w) = I(s_{u_1}s_{u_2}\cdots s_{u_l})$ of Π by

$$I(w) := I_{u_1}I_{u_2}\cdots I_{u_l}.$$

Note that $I(w)$ is independent of the choice of a reduced expression of w by [BIRSc, Theorem Ⅲ. 1.9]. We define the algebra $\Pi(w) = \Pi(s_{u_1}s_{u_2}\cdots s_{u_l})$ by

$$\Pi(w) := \Pi/I(w).$$

For an algebra A, we denote by $\mathsf{Sub}\, A$ the full subcategory of $\mathsf{mod}\, A$ of submodules of finitely generated free A-modules. A finite dimensional algebra A is said to be *Iwanaga-Gorenstein of dimension at most one* if $\operatorname{injdim}{}_A A \leq 1$ and $\operatorname{injdim} A_A \leq 1$ hold. It is well-known that if A is Iwanaga-Gorenstein of dimension at most one, then $\mathsf{Sub}\, A$ is a Frobenius category and therefore, the stable category $\underline{\mathsf{Sub}}\, A$ is a triangulated category. For Frobenius categories, see Subsection 3.3.

We call a category $\mathcal{C}$ Hom-*finite* if the K-vector space $\operatorname{Hom}_{\mathcal{C}}(X,Y)$ is finite dimensional for any $X, Y \in \mathcal{C}$. For a Hom-finite category $\mathcal{C}$, a *Serre functor* $\mathbb{S}$ is an auto-equivalence of $\mathcal{C}$ such that there exists a bifunctorial isomorphism $\operatorname{Hom}_{\mathcal{C}}(X,Y) \simeq \mathrm{D}\operatorname{Hom}_{\mathcal{C}}(Y, \mathbb{S}(X))$ for any $X, Y \in \mathcal{C}$. Our definition of a Serre functor depends on [RV, Section I]. A triangulated category $\mathcal{C}$ is called 2-*Calabi-Yau* if $\mathcal{C}$ has a Serre functor $\mathbb{S} = [2] = [1] \circ [1]$. Let $\mathcal{C}$ be a 2-Calabi-Yau triangulated category and $C \in \mathcal{C}$. We say that C is a *cluster tilting object* of $\mathcal{C}$ if $\mathsf{add}\, C = \{X \in \mathcal{C} \mid \operatorname{Hom}_{\mathcal{C}}(X, C[1]) = 0\}$ holds.

We say that Q is a Dynkin quiver if the underlying graph of Q is a simply laced Dynkin diagram of type A, D or E. We recall results on the ideal $I(w)$ the algebra $\Pi(w)$.

Proposition 3.3. *[BIRSc] For any $w \in W_Q$, we have the followings.*

(a) If Q is non-Dynkin, then a map $x \mapsto (\cdot x)$ gives an isomorphism of algebras $\Pi \xrightarrow{\sim} \operatorname{End}_{\Pi}(I(w))$.

(b) The algebra $\Pi(w)$ is finite dimensional and Iwanaga-Gorenstein of dimension at most one.

(c) The stable category $\underline{\mathsf{Sub}}\, \Pi(w)$ is a 2-Calabi-Yau triangulated category.

(e) For any reduced expression $\boldsymbol{w} = s_{u_1} s_{u_2} \cdots s_{u_l}$ of w, the object

$$T(\boldsymbol{w}) = \bigoplus_{i=1}^{l} \Pi(s_{u_1} s_{u_2} \cdots s_{u_i})$$

is a cluster tilting object of $\underline{\mathsf{Sub}}\, \Pi(w)$.

3.2 The grading of the preprojective algebra of Q

We introduce the grading of a preprojective algebra. We regard the path algebra $K\overline{Q}$ as a graded algebra by the following grading:

$$\deg \beta = \begin{cases} 1 & \beta = \alpha^*, \alpha \in Q_1 \\ 0 & \beta = \alpha, \alpha \in Q_1. \end{cases}$$

Since the element $\sum_{\alpha \in Q_1} (\alpha\alpha^* - \alpha^*\alpha)$ in $K\overline{Q}$ is homogeneous of degree 1, the grading of $K\overline{Q}$ naturally gives a grading on the preprojective algebra $\Pi = \bigoplus_{i \geq 0} \Pi_i$. A $\mathbb{Z}$-algebra A is said to be *positively graded* if $A_i = 0$ for any $i < 0$. Preprojective algebras are positively graded with respect to the above grading.

Remark 3.4. (a) We have $\Pi_0 = KQ$, since Π_0 is spanned by all paths of degree 0.

(b) For any $w \in W$, the ideal $I(w)$ of Π is a homogeneous ideal of Π since so is each I_u.

(c) In particular, the factor algebra $\Pi(w)$ is a graded algebra.

Let $X = \bigoplus_{i\in\mathbb{Z}} X_i$ be a graded module over a positively graded algebra. For any integer j, we define the shifted graded module $X(j)$ by $(X(j))_i = X_{i+j}$. Moreover, for any integer j, we define a graded submodule $X_{\geq j}$ of X by

$$(X_{\geq j})_i = \begin{cases} X_i & i \geq j \\ 0 & \text{else} \end{cases}$$

and define a graded factor module $X_{\leq j}$ of X by $X_{\leq j} = X/(X_{\geq j+1})$. For $i, j \in \mathbb{Z}$, let $X_{[i,j]} = (X_{\leq j})_{\geq i}$.

Let A be a finite dimensional graded algebra which is Iwanaga-Gorensteion of dimension at most one. We denote by $\mathsf{Sub}^{\mathbb{Z}} A$ the full subcategory of $\mathsf{mod}^{\mathbb{Z}} A$ of submodules of graded free A-modules, that is,

$$\mathsf{Sub}^{\mathbb{Z}} A = \left\{ X \in \mathsf{mod}^{\mathbb{Z}} A \mid X \text{ is a submodule of } \bigoplus_{i=1}^{m} A(j_i),\ m, j_i \in \mathbb{Z}, m \geq 0 \right\}.$$

We have the degree forgetful functor $\rho : \mathsf{mod}^{\mathbb{Z}} A \to \mathsf{mod}\, A$, and have the following equalities.

$$\begin{aligned} \mathsf{Sub}^{\mathbb{Z}} A &= \left\{ X \in \mathsf{mod}^{\mathbb{Z}} A \mid \rho(X) \in \mathsf{Sub}\ A \right\}, & (3.1) \\ &= \left\{ X \in \mathsf{mod}^{\mathbb{Z}} A \mid \operatorname{Ext}_A^{>0}(\rho(X), A) = 0, \forall i > 0 \right\}, \\ &= \left\{ X \in \mathsf{mod}^{\mathbb{Z}} A \mid \operatorname{Ext}_{\mathsf{mod}^{\mathbb{Z}} A}^{>0}(X, A(i)) = 0, \forall i \in \mathbb{Z} \right\}. & (3.2) \end{aligned}$$

Clearly $\mathsf{Sub}^{\mathbb{Z}} A$ has enough projectives and is closed under direct summands. By (3.2), $\mathsf{Sub}^{\mathbb{Z}} A$ is closed under extensions. For any $X \in \mathsf{Sub}^{\mathbb{Z}} A$, there exists a left $(\mathsf{proj}^{\mathbb{Z}} A)$-approximations of X which is monomorphism. Thus $\mathsf{Sub}^{\mathbb{Z}} A$ has enough injectives by (3.2). It is easy to see that the projective objects and the injective objects of $\mathsf{Sub}^{\mathbb{Z}} A$ coincide and equals to $\mathsf{proj}^{\mathbb{Z}} A$. Therefore $\mathsf{Sub}^{\mathbb{Z}} A$ is a Frobenius category. We have a triangulated category $\underline{\mathsf{Sub}}^{\mathbb{Z}} A$. In this paper, we get a tilting object in this category.

We give one example which illustrates grading on the algebra $\Pi(w)$ when w is c-sortable.

Example 3.5. Let Q be a quiver $\begin{tikzcd}[column sep=small] & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[rr] & & 3 \end{tikzcd}$. Then we have a graded algebra $\Pi = \Pi e_1 \oplus \Pi e_2 \oplus \Pi e_3$, and these are represented by their radical filtrations, which correspond to the horizontal layers of simples, as follows:

$$\Pi e_1 = \begin{array}{c} \mathbf{1} \\ 2\ \ 3 \\ 3\ \ 1\ \ 2 \\ 1\ \ 2\ \ 3\ \ 1 \\ 2\ \ 3\ \ 1\ \ 2\ \ 3 \\ 3\ \ 1\ \ 2\ \ 3\ \ 1\ \ 2 \end{array}, \qquad \Pi e_2 = \begin{array}{c} \mathbf{2} \\ 3\ \ \mathbf{1} \\ 1\ \ 2\ \ 3 \\ 2\ \ 3\ \ 1\ \ 2 \\ 3\ \ 1\ \ 2\ \ 3\ \ 1 \\ 1\ \ 2\ \ 3\ \ 1\ \ 2\ \ 3 \end{array}, \qquad \Pi e_3 = \begin{array}{c} \mathbf{3} \\ \mathbf{1}\ \ \mathbf{2} \\ 2\ \ 3\ \ \mathbf{1} \\ 3\ \ 1\ \ 2\ \ 3 \\ 1\ \ 2\ \ 3\ \ 1\ \ 2 \\ 2\ \ 3\ \ 1\ \ 2\ \ 3\ \ 1 \end{array},$$

where numbers connected by solid lines are in the same degree, the tops of the Πe_i are concentrated in degree 0, and the degree zero parts are denoted by bold numbers.

Let w be an element of W_Q with a reduced expression $\boldsymbol{w} = s_1s_2s_3s_1s_2s_1$. This w is a c-sortable element by this reduced expression, where $\boldsymbol{c} = \boldsymbol{c}^{(0)} = s_1s_2s_3$, $\boldsymbol{c}^{(1)} = s_1s_2$, and $\boldsymbol{c}^{(3)} = s_1$. Then we have a graded algebra, $\Pi(w) = \Pi(w)e_1 \oplus \Pi(w)e_2 \oplus \Pi(w)e_3$, where

$$\Pi(w)e_1 = \begin{array}{ccccccc} &&&\mathbf{1}&&& \\ &&2&&3&& \\ &3&&1&&2& \\ 1&&&&&&1 \end{array}, \qquad \Pi(w)e_2 = \begin{array}{ccccccc} &&\mathbf{2}&&&& \\ &3&&\mathbf{1}&&& \\ 1&&2&&3&& \\ &&&1&&2& \\ &&&&&&1 \end{array}, \qquad \Pi(w)e_3 = \begin{array}{cccc} &\mathbf{3}&& \\ \mathbf{1}&&\mathbf{2}& \\ &&&\mathbf{1} \end{array}.$$

3.3 Silting and tilting objects of triangulated categories

In this subsection, we recall the definitions of Frobenius categories, silting and tilting objects and recall tilting theorem for algebraic triangulated categories which was shown by Keller.

Let $\mathcal{A}$ be an abelian category. A full subcategory $\mathcal{B}$ of $\mathcal{A}$ is called *extension closed* if for any exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ with $X, Z \in \mathcal{B}$, we have $Y \in \mathcal{B}$. Assume that $\mathcal{B}$ is extension closed subcategory of $\mathcal{A}$. $X \in \mathcal{B}$ is called a *relative-projective* object if $\mathrm{Ext}^1_{\mathcal{A}}(X, B) = 0$ for any $B \in \mathcal{B}$. Dually, we define *relative-injective* objects.

Definition 3.6. [Ha88, He] Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ a full subcategory of $\mathcal{A}$ which is extension closed.

(1) We say that $\mathcal{B}$ has *enough projectives* (resp. *enough injectives*) if for each $X \in \mathcal{B}$, there exists an exact sequences $0 \to Y \to P \to X \to 0$ (resp. $0 \to X \to I \to Y \to 0$) in $\mathcal{A}$ such that $P \in \mathcal{B}$ is relative-projective (resp. $I \in \mathcal{B}$ is relative-injective).

(2) $\mathcal{B}$ is said to be *Frobenius* if the following conditions are satisfied:

  (i) An object in $\mathcal{B}$ is relative-projective if and only if it is relative-injective.

  (ii) $\mathcal{B}$ has enough projectives and enough injectives.

(2) For a Frobenius category $\mathcal{B}$, we define the *stable category* $\underline{\mathcal{B}}$ as follows: The objects of $\underline{\mathcal{B}}$ are the same as $\mathcal{B}$, and the morphism space is given by

$$\underline{\mathrm{Hom}}_{\mathcal{B}}(X, Y) := \mathrm{Hom}_{\mathcal{B}}(X, Y)/P(X, Y)$$

for any $X, Y \in \mathcal{B}$, where $P(X, Y)$ is the submodule of $\mathrm{Hom}_{\mathcal{B}}(X, Y)$ consisting of morphisms which factor through relative-projective objects in $\mathcal{B}$.

Frobenius categories gives triangulated categories, which is shown by Happel.

Definition-Theorem 3.7. *[Ha88] Let $\mathcal{B}$ be a Frobenius category. Then the stable category $\underline{\mathcal{B}}$ has a structure of a triangulated category. Such a triangulated category is called* algebraic*.*

Next we recall the definition of silting and tilting objects. Let $\mathcal{T}$ be a triangulated category. For an object X of $\mathcal{T}$, we denote by $\mathsf{thick}_{\mathcal{T}} X$ the smallest triangulated full subcategory of $\mathcal{T}$ containing X and closed under direct summands.

Definition 3.8. Let $\mathcal{T}$ be a triangulated category.

(1) An object X of $\mathcal{T}$ is called a *silting object* if $\mathrm{Hom}_{\mathcal{T}}(X, X[i]) = 0$ for any $i > 0$ and $\mathsf{thick}\, X = \mathcal{T}$.

(2) An object X of $\mathcal{T}$ is called a *tilting object* if X is a silting object of $\mathcal{T}$ and $\mathrm{Hom}_{\mathcal{T}}(X, X[i]) = 0$ for any $i < 0$.

For example, let A be a finite dimensional algebra. Then A is a tilting object of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$.

Let $\mathcal{C}, \mathcal{C}'$ be additive categories and $X \in \mathcal{C}$. A morphism $e : X \to X$ in $\mathcal{C}$ is called an *idempotent* if $e^2 = e$. We call $\mathcal{C}$ *idempotent complete* if each idempotent of $\mathcal{C}$ has a kernel. An additive functor $F : \mathcal{C} \to \mathcal{C}'$ is called an *equivalence up to direct summands* if it is fully faithful and any object $X \in \mathcal{C}'$ is isomorphic to a direct summand of FY for some $Y \in \mathcal{C}$. It is easy to see that if $\mathcal{C}$ is idempotent complete, then F is an equivalence.

For an algebra A, we denote by $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ the homotopy category of bounded complexes of finitely generated projective A-modules. We have the following theorem for algebraic triangulated categories [Ke94, (4.3)] (see also [IT, Theorem 2.2]).

Theorem 3.9. *Let $\mathcal{T}$ be an algebraic triangulated category with a tilting object X. Then the following statements hold.*

(a) There exists a triangle equivalence $F : \mathcal{T} \to \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, \mathrm{End}_{\mathcal{T}}(X))$ up to direct summands.

(b) If $\mathcal{T}$ is idempotent complete, then F is a triangle equivalence.

In Section 11, we use the following basic lemma of a triangle functor.

Lemma 3.10. *Let $\mathcal{T}, \mathcal{U}$ be triangulated categories and $F : \mathcal{T} \to \mathcal{U}$ be a triangle functor. Moreover, let X be a tilting object of $\mathcal{T}$. Assume that $\mathcal{T}$ is idempotent complete, $F(X)$ is a tilting object of $\mathcal{U}$ and $F_{X,X}$ induces an isomorphism $\mathrm{Hom}_{\mathcal{T}}(X, X) \simeq \mathrm{Hom}_{\mathcal{U}}(F(X), F(X))$. Then the functor F is an equivalence.*

Finally, we recall the definition of Krull-Schmidt categories. An additive category $\mathcal{C}$ is called *Krull-Schmidt* if each object of $\mathcal{C}$ is a finite direct sum of objects such that their endomorphism algebras are local. Note that a Krull-Schmidt category is idempotent complete. For instance, our triangulated categories $\underline{\mathsf{Sub}}\, \Pi(w)$ and $\underline{\mathsf{Sub}}^{\mathbb{Z}}\, \Pi(w)$ are Krull-Schimdt.

4 Graded structure of $I(w)$ and $\Pi(w)$

In this section, we prove some basic properties of gradings of $I(w)$ and $\Pi(w)$. The main result in this section is Proposition 4.5. We also recall some results from [AIRT] which will be used later. Throughout this section, let $c \in W_Q$ be a Coxeter element and $\boldsymbol{c}$ an expression satisfying the statement in Definition 3.1 (4).

Lemma 4.1. *[AIRT, Lemma 2.1] Let Q' be a full subquiver of Q and w an element in $W_{Q'} \subset W_Q$. Then we have $\Pi/I(w) = \Pi'/I'(w)$ as graded algebras, where Π' is a preprojective algebra of Q' and $I'(w)$ is the ideal of Π' associated with w.*

We first calculate the ideal $I(w)$ and the algebra $\Pi(w)$ when w has a reduced expression which is a subword of $\boldsymbol{c}$.

Lemma 4.2. *Let $w \in W_Q$ and assume that w has a reduced expression which is a subword of $\boldsymbol{c}$. Let Q' the full subquiver of Q whose set of vertices is $\mathrm{Supp}(w)$. We denote by Π' the preprojective algebra of Q' and $I'(w)$ the ideal of Π' associated with w. Then the following holds.*

(a) We have $\Pi(w) = \Pi'(w) = KQ'$.

(b) $I(w)_{\geq 1} = \Pi_{\geq 1}$.

(c) $I(w)_0$ is the ideal of KQ generated by idempotents $\{e_u \mid u \in Q_0 \setminus \mathrm{Supp}(w)\}$.

Proof. (a) By Lemma 4.1, we have $\Pi(w) = \Pi'(w)$. By assumption, w is a Coxeter element of $W_{Q'}$. Then, by [BIRSc, Proposition III. 3.2], we have $\Pi'(w) = KQ'$.

(b) By (a), we have $\Pi(w)_0 = \Pi'(w)_0 = KQ'$. This means that $I(w)_{\geq 1} = \Pi_{\geq 1}$.

(c) Since $KQ' = \Pi(w)_0 = \Pi_0/(w)_0 = KQ/I(w)_0$ holds, $I(w)_0$ the ideal generated by the vertices in $Q_0 \setminus \mathrm{Supp}(w)$. $\square$

Then we describe the grading of $I(w)$ for a c-sortable element w.

Lemma 4.3. *Let $w \in W_Q$ be a c-sortable element and $\boldsymbol{w} = \boldsymbol{c}^{(0)}\boldsymbol{c}^{(1)} \cdots \boldsymbol{c}^{(m)}$ a c-sortable expression of w. Then we have $I(c^{(i)}c^{(i+1)})_0 = I(c^{(i)})_0$ for all $0 \leq i \leq m-1$.*

Proof. Since $\Pi(w)$ is positively graded, we have $I(c^{(i)}c^{(i+1)})_0 = I(c^{(i)})_0 I(c^{(i+1)})_0$. By Lemma 4.2, $I(c^{(i)})_0$ and $I(c^{(i+1)})_0$ are generated by idempotents $\{e_v \mid v \in Q_0 \setminus \mathrm{Supp}(c^{(i)})\}$ and $\{e_v \mid v \in Q_0 \setminus \mathrm{Supp}(c^{(i+1)})\}$, respectively. Since w is a c-sortable element, we have $\mathrm{Supp}(c^{(i+1)}) \subset \mathrm{Supp}(c^{(i)})$. Therefore we have $I(c^{(i)})_0 I(c^{(i+1)})_0 = I(c^{(i)})_0$. $\square$

Lemma 4.4. *Let $w \in W_Q$ be a c-sortable element and $\boldsymbol{w} = \boldsymbol{c}^{(0)}\boldsymbol{c}^{(1)} \cdots \boldsymbol{c}^{(m)}$ a c-sortable expression of w. Then we have*

$$I(w)_i = \begin{cases} I(c^{(0)}c^{(1)} \cdots c^{(i)})_i & 0 \leq i \leq m. \\ \Pi_i & m+1 \leq i. \end{cases}$$

In particular, we have $\Pi(w)_{\geq m+1} = 0$.

Proof. We first show that $I(w)_{\geq m+1} = \Pi_{\geq m+1}$. Since Π is generated by Π_1 as a Π_0-algebra, we have $\Pi_{\geq m+1} = \prod_{j=0}^{m}(\Pi_{\geq 1})$. By Lemma 4.2 (b), the equation $\Pi_{\geq 1} = I(c^{(j)})_{\geq 1}$ holds for any $0 \leq j \leq m$. Thus we have

$$I(w)_{\geq m+1} \subset \Pi_{\geq m+1} = \prod_{j=0}^{m} \Pi_{\geq 1} = \prod_{j=0}^{m} I(c^{(j)})_{\geq 1} \subset I(w)_{\geq m+1}.$$

Therefore we have $I(w)_{\geq m+1} = \Pi_{\geq m+1}$.

Assume that $0 \leq i \leq m-1$. We show that $I(w)_i = I(c^{(0)}c^{(1)} \cdots c^{(m-1)})_i$. Since $I(w) \subset I(c^{(0)}c^{(1)} \cdots c^{(m-1)})$, we have $I(w)_i \subset I(c^{(0)}c^{(1)} \cdots c^{(m-1)})_i$. Conversely, we show that

$$I(c^{(0)}c^{(1)} \cdots c^{(m-1)})_i \subset I(w)_i.$$

In general, we have

$$I(c^{(0)}c^{(1)}\cdots c^{(m-1)})_i = \sum_{b_0+b_1+\cdots+b_{m-1}=i} I(c^{(0)})_{b_0} I(c^{(1)})_{b_1}\cdots I(c^{(m-1)})_{b_{m-1}}. \tag{4.1}$$

Since $I(w)_i = 0$ for any $i<0$ and (4.1), it is enough to show that

$$I(c^{(0)})_{a_0} I(c^{(1)})_{a_1}\cdots I(c^{(m-1)})_{a_{m-1}} \subset I(w)_i,$$

for any non-negative integers $a_0, a_1, \ldots, a_{m-1}$ satisfying $\sum_{j=0}^{m-1} a_j = i$. Since $a_0, \ldots, a_{m-1}$ are non-negative and $i \leq m-1$, at least one of them must be zero. Let j be the largest integer satisfying $a_j = 0$. Then we have

$$\begin{aligned}
&I(c^{(0)})_{a_0}\cdots I(c^{(j)})_{a_j} I(c^{(j+1)})_{a_{j+1}}\cdots I(c^{(m-1)})_{a_{m-1}}\\
&= I(c^{(0)})_{a_0}\cdots I(c^{(j)})_{a_j}(\Pi_{a_{j+1}})\cdots(\Pi_{a_{m-1}})\\
&= I(c^{(0)})_{a_0}\cdots I(c^{(j)})_{a_j} I(c^{(j+1)})_0(\Pi_{a_{j+1}})\cdots(\Pi_{a_{m-1}})\\
&= I(c^{(0)})_{a_0}\cdots I(c^{(j)})_{a_j} I(c^{(j+1)})_0 I(c^{(j+2)})_{a_{j+1}}\cdots I(c^{(m)})_{a_{m-1}}\\
&\subset I(w)_i,
\end{aligned}$$

where the first and the third equations come form Lemma 4.2 (b), and the second equation comes from Lemma 4.3. Therefore we have $I(c^{(0)}c^{(1)}\cdots c^{(m-1)})_i \subset I(w)_i$ for $0 \leq i \leq m-1$. By using this equation repeatedly, we have the assertion. □

Now we describe the grading of $\Pi(w)$ for a c-sortable element w. For an element w in W_Q, let $Q^{(1)}$ be the full subquiver of Q whose set of vertices is $\mathrm{Supp}(w)$.

Proposition 4.5. *Let $w \in W_Q$ be a c-sortable element and $\boldsymbol{w} = \boldsymbol{c}^{(0)}\boldsymbol{c}^{(1)}\cdots\boldsymbol{c}^{(m)}$ a c-sortable expression of w. For each $i \leq m$, we have $\Pi(w)_{\leq i} = \Pi(c^{(0)}c^{(1)}\cdots c^{(i)})_{\leq i} = \Pi(c^{(0)}c^{(1)}\cdots c^{(i)})$. In particular, we have $\Pi(w)_0 = \Pi(c^{(0)}) = KQ^{(1)}$.*

Proof. By Lemma 4.4, we have the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & I(w)_{\leq i} \arrow[r] \arrow[d, equal] & \Pi_{\leq i} \arrow[r] \arrow[d, equal] & \Pi(w)_{\leq i} \arrow[r] \arrow[d, "\simeq"] & 0\\
0 \arrow[r] & I(c^{(0)}c^{(1)}\cdots c^{(i)})_{\leq i} \arrow[r] & \Pi_{\leq i} \arrow[r] & \Pi(c^{(0)}c^{(1)}\cdots c^{(i)})_{\leq i} \arrow[r] & 0.
\end{tikzcd}$$

Therefore we have an equality $\Pi(w)_{\leq i} = (\Pi_{c^{(0)}c^{(1)}\cdots c^{(i)}})_{\leq i}$. The equality $\Pi(c^{(0)}c^{(1)}\cdots c^{(i)})_{\leq i} = \Pi(c^{(0)}c^{(1)}\cdots c^{(i)})$ comes from Lemma 4.4. If $i = 0$, then we have $\Pi(w)_0 = \Pi(c^{(0)}) = KQ^{(1)}$, where the second equality comes from Lemma 4.2 (a). □

The following proposition is important to show Theorem 5.6.

Proposition 4.6. *Let $w \in W_Q$ be a c-sortable element and $\boldsymbol{w} = s_{u_1}\cdots s_{u_l} = \boldsymbol{c}^{(0)}\boldsymbol{c}^{(1)}\cdots\boldsymbol{c}^{(m)}$ a c-sortable expression of w. For any integer i and $X \in \mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$, we have $X_{\geq i}, X_{\leq i}, X_i \in \mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$.*

Proof. Since $X_{\geq i}$ is a submodule of X, we have $X_{\geq i} \in \mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$.

By Proposition 3.3 (e), we have $\Pi(u_1 \cdots u_j) \in \mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$ for any $1 \leq j \leq l$. Therefore, by Proposition 4.5, we have $\Pi(w)_{\leq i} \in \mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$ for any integer i. Clearly, the functor $X \mapsto X_{\leq i}$ preserves injective morphisms. Therefore we have $X_{\leq i} \in \mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$. Since $X_{\geq i} \in \mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$, $X_i = (X_{\geq i})_{\leq i} \in \mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$ holds. $\square$

Next we recall the result of [AIRT]. For a reduced expression $\boldsymbol{w} = s_{u_1} \cdots s_{u_l}$ of $w \in W_Q$ and $1 \leq i \leq l$, we define a $\Pi(w)$-module $L^i_{\boldsymbol{w}}$ by $L^1_{\boldsymbol{w}} := \Pi/I_{u_1}$ and

$$L^i_{\boldsymbol{w}} := \frac{I(s_{u_1} \cdots s_{u_{i-1}})}{I(s_{u_1} \cdots s_{u_i})},$$

for $i \geq 2$.

Proposition 4.7. *[AIRT, Proposition 1.3] We have equalities*

$$L^i_{\boldsymbol{w}} = L^i_{\boldsymbol{w}} e_{u_i} = \frac{I(s_{u_1} \cdots s_{u_j})}{I(s_{u_1} \cdots s_{u_i})} e_{u_i},$$

where j is the largest integer satisfying $j < i$ and $u_j = u_i$. If such an integer j does not appear in $1, \ldots, i-1$, then $L^i_{\boldsymbol{w}} = (\Pi/I(s_{u_1} \cdots s_{u_i}))e_{u_i}$.

We use the following notation. Let $\boldsymbol{w} = s_{u_1} \cdots s_{u_l}$ be a reduced expression of $w \in W_Q$ For any $u \in \mathrm{Supp}(w)$, let

$$p_u = \max\{1 \leq j \leq l \mid u_j = u\}.$$

For $1 \leq i \leq l$, let

$$m_i = \sharp\{1 \leq j \leq i-1 \mid u_j = u_i\}.$$

Note that, if $\boldsymbol{w} = s_{u_1} \cdots s_{u_l} = \boldsymbol{c}^{(0)}\boldsymbol{c}^{(1)} \cdots \boldsymbol{c}^{(m)}$ is a c-sortable expression, then we have $m_{p_u} = \max\{j \mid u \in \mathrm{Supp}(c^{(j)})\}$ for any $u \in \mathrm{Supp}(w)$. Using $L^i_{\boldsymbol{w}}$, we have the following information on $\Pi(w)e_u$.

Lemma 4.8. *Let $\boldsymbol{w} = s_{u_1} \cdots s_{u_l} = \boldsymbol{c}^{(0)}\boldsymbol{c}^{(1)} \cdots \boldsymbol{c}^{(m)}$ be a c-sortable expression of $w \in W_Q$. Then, for any $u \in \mathrm{Supp}(w)$ and any integer $i \geq m_{p_u}$, we have*

$$(\Pi(w)e_u)_i = \begin{cases} L^{p_u}_{\boldsymbol{w}} & i = m_{p_u}, \\ 0 & m_{p_u} + 1 \leq i. \end{cases}$$

Proof. Since $I(w)e_u = I(c^{(0)}c^{(1)} \cdots c^{(m_{p_u})})e_u$, we have $\Pi(w)e_u = \Pi(c^{(0)}c^{(1)} \cdots c^{(m_{p_u})})e_u$. Thus, by Lemma 4.4, we have $(\Pi(w)e_u)_i = 0$ for $m_{p_u} + 1 \leq i$.

If $i = m_{p_u}$, we have

$$\begin{aligned}
(\Pi(w)e_u)_i &= \mathrm{Ker}\big((\Pi(w)e_u)_{\leq i} \to (\Pi(w)e_u)_{\leq i-1}\big) \\
&= \mathrm{Ker}\left(\frac{\Pi}{I(c^{(0)}c^{(1)} \cdots c^{(i)})} e_u \to \frac{\Pi}{I(c^{(0)}c^{(1)} \cdots c^{(i-1)})} e_u\right) \\
&= \frac{I(c^{(0)}c^{(1)} \cdots c^{(i-1)})}{I(c^{(0)}c^{(1)} \cdots c^{(i)})} e_u,
\end{aligned}$$

where the second equality comes from Proposition 4.5. Since $I(c^{(0)}c^{(1)} \cdots c^{(i)})e_u = I(s_{u_1} \cdots s_{u_{p_u}})e_u$, we have the desired equality. $\square$

The next theorem is one of the main results in [AIRT], and important in this paper. We use Theorem 4.9 to prove Proposition 5.5. For an element w in W, let $Q^{(1)}$ be the full subquiver of Q whose set of vertices is $\mathrm{Supp}(w)$.

Theorem 4.9. *Let $w \in W_Q$ be a c-sortable element and $\boldsymbol{w} = s_{u_1} \cdots s_{u_l} = \boldsymbol{c}^{(0)}\boldsymbol{c}^{(1)} \cdots \boldsymbol{c}^{(m)}$ a c-sortable expression of w. Then*

$$T = \bigoplus_{u \in Q_0^{(1)}} L_{\boldsymbol{w}}^{p_u} = \bigoplus_{u \in Q_0^{(1)}} (\Pi(w)e_u(m_{p_u}))_0$$

is a tilting $KQ^{(1)}$-module.

Proof. $T = \bigoplus_{u \in Q_0^{(1)}} L_{\boldsymbol{w}}^{p_u}$ is a tilting $KQ^{(1)}$-module by [AIRT, Theorem 3.11]. Moreover $T = \bigoplus_{u \in Q_0^{(1)}} (\Pi(w)e_u(m_{p_u}))_0$ holds by Lemma 4.8. $\square$

We give one example which illustrates the tilting module of Theorem 4.9.

Example 4.10. Let Q be a quiver
$$\begin{tikzcd}[column sep=small] & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[rr] & & 3 \end{tikzcd}$$
and w be an element of W_Q with a reduced expression $\boldsymbol{w} = s_1s_2s_3s_1s_2s_1$. This is a c-sortable expression. Then we have a graded algebra $\Pi(w) = \Pi(w)e_1 \oplus \Pi(w)e_2 \oplus \Pi(w)e_3$,

$$\begin{smallmatrix} & & & \mathbf{1} & & & \\ & & 2 & & 3 & & \\ & 3 & & 1 & & 2 & \\ 1 & & & & & & 1 \end{smallmatrix} \qquad \begin{smallmatrix} & & \mathbf{2} & & & \\ & 3 & & \mathbf{1} & & \\ 1 & & 2 & & 3 & \\ & & & 1 & & 2 \\ & & & & & & 1 \end{smallmatrix} \qquad \begin{smallmatrix} & \mathbf{3} & & \\ \mathbf{1} & & \mathbf{2} & \\ & & & \mathbf{1} \end{smallmatrix}.$$

We have

$$L_{\boldsymbol{w}}^1 = 1, \qquad L_{\boldsymbol{w}}^2 = \begin{smallmatrix} 2 & \\ & 1 \end{smallmatrix}, \qquad L_{\boldsymbol{w}}^3 = \begin{smallmatrix} & 3 & \\ 1 & & 2 \\ & & & 1 \end{smallmatrix},$$

$$L_{\boldsymbol{w}}^4 = \begin{smallmatrix} 2 & & 3 & & \\ & 1 & & 2 & \\ & & & & 1 \end{smallmatrix}, \qquad L_{\boldsymbol{w}}^5 = \begin{smallmatrix} & 3 & & & & \\ 1 & & 2 & & 3 & \\ & & & 1 & & 2 \\ & & & & & & 1 \end{smallmatrix}, \qquad L_{\boldsymbol{w}}^6 = \begin{smallmatrix} & 3 \\ 1 & \end{smallmatrix}.$$

By Theorem 4.9, $L_{\boldsymbol{w}}^3 \oplus L_{\boldsymbol{w}}^5 \oplus L_{\boldsymbol{w}}^6$ is a tilting KQ-module.

5 A tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ for a c-sortable element w

In this section, we construct a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ when w is a c-sortable element. A triangle equivalence induced from tilting objects is given in Section 6.

Definition 5.1. Let $w \in W_Q$ be a c-sortable element and $\boldsymbol{w} = s_{u_1} \cdots s_{u_l} = \boldsymbol{c}^{(0)}\boldsymbol{c}^{(1)} \cdots \boldsymbol{c}^{(m)}$ a c-sortable expression of w. Put

$$M = M(\boldsymbol{w}) := \bigoplus_{i=0}^{m} \left(\Pi(c^{(0)} \cdots c^{(i)})\right)(i).$$

Throughout this section, let $w \in W_Q$ be a c-sortable element and $\boldsymbol{w} = s_{u_1} \cdots s_{u_l} = \boldsymbol{c}^{(0)}\boldsymbol{c}^{(1)} \cdots \boldsymbol{c}^{(m)}$ a c-sortable expression of w. and M be a module as in Definition 5.1. This M belongs to $\mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$ by Proposition 3.3 (e) and (3.1).

Example 5.2. Let Q be a quiver $\begin{tikzcd}[column sep=small] & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[rr] & & 3 \end{tikzcd}$. Let w be an element of W_Q with a reduced expression $\boldsymbol{w} = s_1 s_2 s_3 s_1 s_2 s_1$. This is a c-sortable element. Then we have a graded algebra $\Pi(w) = \Pi(w)e_1 \oplus \Pi(w)e_2 \oplus \Pi(w)e_3$,

$$\begin{array}{ccccccc} &&&\mathbf{1}&&& \\ &&2&&3&& \\ &&&\diagdown\diagup&&\diagdown& \\ &3&&1&&2& \\ &\diagup&&&&&\diagdown \\ 1&&&&&&1 \end{array} \qquad \begin{array}{cccccc} &&\mathbf{2}&&& \\ &&&\diagdown&& \\ &3&&\mathbf{1}&& \\ \diagup&&\diagdown&&& \\ 1&&2&&3& \\ &&&\diagdown\diagup&&\diagdown \\ &&&1&&2 \\ &&&&&\diagdown \\ &&&&&1 \end{array} \qquad \begin{array}{ccc} &\mathbf{3}& \\ \diagup&&\diagdown \\ \mathbf{1}&&\mathbf{2} \\ &&\diagdown \\ &&\mathbf{1} \end{array}.$$

and

$$M = \mathbf{1} \oplus \begin{array}{cc} \mathbf{2}& \\ \diagdown& \\ &\mathbf{1} \end{array} \oplus \left(\begin{array}{ccccc} &1&&& \\ \mathbf{2}&&\mathbf{3}&& \\ \diagdown&\diagup&\diagdown&& \\ &\mathbf{1}&&\mathbf{2}& \\ &&&\diagdown& \\ &&&&\mathbf{1} \end{array} \right)$$

in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$, where the graded projective $\Pi(w)$-modules are removed, and the degree zero parts are denoted by bold numbers.

The following proposition follows from Proposition 4.5.

Proposition 5.3. $M = M_{\leq 0}$.

Proof. We have $M = \bigoplus_{i=0}^m \Pi(c^{(0)} \cdots c^{(i)})_{\leq i}(i) = \bigoplus_{i=0}^m \Pi(c^{(0)} \cdots c^{(i)})(i)_{\leq 0} = M_{\leq 0}$. $\square$

By the following two propositions, we show that this M satisfies the axioms of tilting objects. Note that, by Lemma 4.4, $\Pi(w)_{\leq i} = \Pi(w)$ holds for $i \geq m$, and therefore, we have

$$M = \bigoplus_{i=0}^{m} \Pi(c^{(0)} \cdots c^{(i)})_{\leq i}(i) = \bigoplus_{i \geq 0} \Pi(w)_{\leq i}(i) = \bigoplus_{i \geq 0} \Pi(w)(i)_{\leq 0}$$

in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ by Proposition 4.5.

Proposition 5.4. *We have* $\underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M, M[j]) = 0$ *for any* $j \neq 0$.

Proof. For any $0 \leq i$, we have a short exact sequence,

$$0 \to \Pi(w)(i)_{\geq 1} \to \Pi(w)(i) \to \Pi(w)(i)_{\leq 0} \to 0.$$

Since $(\Pi(w)(i)_{\geq 1})_{\leq 0} = 0$, we have

$$(\Omega M)_{\leq 0} = \bigoplus_{i \geq 0} \big(\Omega\,(\Pi(w)(i)_{\leq 0})\big)_{\leq 0} = \bigoplus_{i \geq 0} \big(\Pi(w)(i)_{\geq 1}\big)_{\leq 0} = 0.$$

Since $\Pi(w)$ is positively graded, we have $\big(\Omega^j(M)\big)_{\leq 0} = 0$ for $j \geq 1$. Therefore

$$\mathrm{Hom}^{\mathbb{Z}}_{\Pi(w)}(M, \Omega^j(M)) = 0 \text{ and } \mathrm{Hom}^{\mathbb{Z}}_{\Pi(w)}(\Omega^j(M), M) = 0$$

hold for any $j \geq 1$ by Proposition 5.3. The first equality implies $\underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M, M[-j]) = 0$ for $j \geq 1$, and the second equality implies $\underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M, M[j]) = 0$ for $j \geq 1$. $\square$

Next we prove that M satisfies the second axiom of tilting objects. Since $\Pi(w)_0 = KQ^{(1)}$ by Proposition 4.5, we regard a $KQ^{(1)}$-module X as a graded $\Pi(w)$-module concentrated in degree 0. For an integer i, let $\mathsf{mod}^{\leq i}\,\Pi(w)$ be the full subcategory of $\mathsf{mod}^{\mathbb{Z}}\,\Pi(w)$ of modules X satisfying $X = X_{\leq i}$.

Proposition 5.5. *We have* $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w) = \mathsf{thick}\,M$.

Proof. Let$X \in \mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$. We show that $X \in \mathsf{thick}\,M$. By Proposition 4.6, we have $X_i \in \mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$ for any $i \in \mathbb{Z}$. Since X has a finite filtration $\{X_{\geq j} \mid j \in \mathbb{Z}\}$, it is enough to show that $X_i \in \mathsf{thick}\,M$ for any $i \in \mathbb{Z}$. Since each X_i is a $KQ^{(1)}$-module and the global dimension of $KQ^{(1)}$ is at most one, it is enough to show that $KQ^{(1)}(i) \in \mathsf{thick}\,M$ for any $i \in \mathbb{Z}$.

Firstly, we show $KQ^{(1)}(i) \in \mathsf{thick}\,M$ for any $i \geq 0$ by induction on i. Since M has a direct summand $\Pi(w)_0 = KQ^{(1)}$, we have $KQ^{(1)} \in \mathsf{thick}\,M$. Assume $KQ^{(1)}(j) \in \mathsf{thick}\,M$ for $0 \leq j \leq i-1$. Consider a short exact sequence

$$0 \to \Pi(w)_{[1,i]}(i) \to \Pi(w)_{\leq i}(i) \to \Pi(w)_0(i) \to 0. \tag{5.1}$$

By taking a finite filtration of $\Pi(w)_{[1,i]}(i)$ and the inductive hypothesis, we conclude that $\Pi(w)_{[1,i]}(i) \in \mathsf{thick}\,M$. Since $\Pi(w)_{\leq i}(i)$ is a direct summand of M or a graded projective $\Pi(w)$-module, we have $KQ^{(1)}(i) = \Pi(w)_0(i) \in \mathsf{thick}\,M$ by (5.1). Consequently, we have that $X \in \mathsf{thick}\,M$ for any $X \in \mathsf{mod}^{\leq 0}\,\Pi(w) \cap \mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$.

Secondly, we show that $KQ^{(1)}(-i) \in \mathsf{thick}\,M$ for any $i \geq 0$ by induction on i. Assume $KQ^{(1)}(-j) \in \mathsf{thick}\,M$ for $0 \leq j \leq i-1$. Thus we have $X \in \mathsf{thick}\,M$ for any $X \in \mathsf{mod}^{\leq i-1}\,\Pi(w) \cap \mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$. By Theorem 4.9, $T = \bigoplus_{u \in Q_0^{(1)}} (\Pi(w)e_u(m_{p_u}))_0$ is a tilting $KQ^{(1)}$-module. There exists a short exact sequence

$$0 \to KQ^{(1)} \to T_0 \to T_1 \to 0,$$

where $T_0, T_1 \in \mathsf{add}\,T$. Therefore it is enough to show that $T(-i) \in \mathsf{thick}\,M$. For each $u \in Q_0^{(1)}$, take a short exact sequence

$$0 \to Te_u(-i) \to \Pi(w)e_u(m_{p_u})(-i) \to \Pi(w)e_u(m_{p_u})_{\leq -1}(-i) \to 0. \tag{5.2}$$

The second term is a graded projective $\Pi(w)$-module. The third term belongs to $\mathsf{thick}\,M$ since $\Pi(w)e_u(m_{p_u})_{\leq -1}(-i)$ is in $\mathsf{mod}^{\leq i-1}\,\Pi(w)$. Consequently, we have $T(-i) \in \mathsf{thick}\,M$ by (5.2). $\square$

Then we have the main theorem of this section.

Theorem 5.6. *Let $w \in W_Q$ be a c-sortable element and $\boldsymbol{w} = s_{u_1} \cdots s_{u_l} = \boldsymbol{c}^{(0)}\boldsymbol{c}^{(1)} \cdots \boldsymbol{c}^{(m)}$ a c-sortable expression of w. Put*

$$M = \bigoplus_{i=0}^{m} \left(\Pi(c^{(0)} \cdots c^{(i)})\right)(i).$$

Then M is a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$.

Proof. By Propositions 5.4, and 5.5, M is a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$. $\square$

Remark 5.7. It was shown by Yamaura [Y] that, for a finite dimensional self-injective positively graded algebra A, the stable category $\underline{\mathsf{mod}}^{\mathbb{Z}}A$ has a tilting object $\bigoplus_{i\geq 0} A(i)_{\leq 0}$ if A_0 has finite global dimension. Our tilting object M in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ is an analog of this since $M = \bigoplus_{i\geq 0}\Pi(w)_{\leq i}(i) = \bigoplus_{i\geq 0}\Pi(w)(i)_{\leq 0}$ holds.

6 The endomorphism algebra of the tilting object

In this section, we calculate the endomorphism algebra of the tilting object which was constructed in Definition 5.1. The aim of this section is to prove Theorems 6.2 and 6.3. Throughout this section, let Q be a finite acyclic quiver.

6.1 A morphism from $\mathrm{End}^{\mathbb{Z}}_{\Pi(w)}(M)$ to $\mathrm{End}_{KQ^{(1)}}(M_0)$

Firstly, we give another description of the tilting object which was constructed in Definition 5.1. Throughout this section, we use the following notation.

Definition 6.1. Let $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ be a reduced expression of $w \in W_Q$. We use the same notation as after Proposition 4.7, that is,

$$p_u = \max\{1 \leq j \leq l \mid u_j = u\}, \qquad \text{for } u \in \mathrm{Supp}(w),$$
$$m_i = \sharp\{1 \leq j \leq i-1 \mid u_j = u_i\}, \qquad \text{for } 1 \leq i \leq l.$$

Moreover, for $1 \leq i \leq l$, put

$$M^i := (\Pi/I(s_{u_1}\cdots s_{u_i}))e_{u_i}(m_i), \qquad M = \bigoplus_{i=1}^{l} M^i,$$
$$P = \bigoplus_{u\in\mathrm{Supp}(w)} M^{p_u}, \qquad T = P_0.$$

Note that $P \in \mathsf{proj}^{\mathbb{Z}}\Pi(w)$ holds since $\Pi(w) = \bigoplus_{u\in\mathrm{Supp}(w)} M^{p_u}(-m_{p_u})$. If $\boldsymbol{w} = s_{u_1}\cdots s_{u_l} = \boldsymbol{c}^{(0)}\boldsymbol{c}^{(1)}\cdots\boldsymbol{c}^{(m)}$ is a c-sortable expression of w, then we have an isomorphism

$$\bigoplus_{i=1}^{l} M^i \simeq \bigoplus_{i=0}^{m} \Pi(c^{(0)}\cdots c^{(i)})(i) \tag{6.1}$$

in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$. In fact, for any $1 \leq i \leq l$, $M^i = (\Pi/I(c^{(0)}\cdots c^{(m_i)}))e_{u_i}(m_i)$ holds by Proposition 4.7, and for any $0 \leq j \leq m$, if $u \in Q_0 \setminus \mathrm{Supp}(c^{(j)})$, then $(\Pi/I(c^{(0)}\cdots c^{(j)}))e_u = \Pi(w)e_u$ holds, which is projective. Therefore we have an isomorphism (6.1). As we have shown in Theorem 5.6, $M = \bigoplus_{i=1}^{l} M^i$ is a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$.

Before starting the calculating of the endomorphism algebra $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)$, we state a triangle equivalence induced from a tilting object. We show that the global dimension of $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)$ is finite and we have the following theorem.

Theorem 6.2. *Let $w \in W_Q$ be a c-sortable element and $\boldsymbol{w} = s_{u_1} \cdots s_{u_l} = \boldsymbol{c}^{(0)}\boldsymbol{c}^{(1)} \cdots \boldsymbol{c}^{(m)}$ a c-sortable expression of w. Let $M = \bigoplus_{i=1}^{l} M^i$ be a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$. Then the global dimension of $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)$ is finite and we have a triangle equivalence*

$$\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w) \simeq \mathsf{D}^{\mathrm{b}}(\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)).$$

Proof. By Proposition 6.14, the global dimension of $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)$ is finite. By Theorems 5.6 and 3.9, we have the assertion. $\square$

We state another theorem of this section. Looking at the degree zero part of graded modules, we have the following functor

$$\mathbb{F} := (-)_0 : \mathsf{mod}^{\mathbb{Z}}\,\Pi \to \mathsf{mod}\,KQ.$$

The functor $\mathbb{F}$ induces the following morphism of algebras

$$F := \mathbb{F}_{M,M} : \mathrm{End}^{\mathbb{Z}}_{\Pi(w)}(M) \to \mathrm{End}_{KQ}(M_0)$$

given by $F(f) = f|_{M_0}$. Then we claim the following.

Theorem 6.3. *Let w be a c-sortable element. The morphism F induces an isomorphism of algebras $\underline{F} : \underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M) \xrightarrow{\sim} \mathrm{End}_{KQ}(M_0)/[T]$, which makes the following diagram commutative*

$$\begin{tikzcd}
\mathrm{End}^{\mathbb{Z}}_{\Pi(w)}(M) \arrow[r, "F"] \arrow[d] & \mathrm{End}_{KQ}(M_0) \arrow[d] \\
\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M) \arrow[r, "\underline{F}"] & \mathrm{End}_{KQ}(M_0)/[T],
\end{tikzcd}$$

where $[T]$ is an ideal of $\mathrm{End}_{KQ}(M_0)$ consisting of morphisms factoring through objects in $\mathsf{add}\,T$, and vertical morphisms are canonical surjections.

Proof. In Proposition 6.15, we show that F actually induces a morphism $\underline{F}$. $\underline{F}$ is surjective by Proposition 6.29. In Proposition 6.31, we show that $\underline{F}$ is injective. $\square$

In Subsection 6.2, we show one theorem which we will use to prove Proposition 6.29.

Example 6.4. Let Q be a quiver $\begin{tikzcd}[column sep=small, row sep=small] & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[rr] & & 3 \end{tikzcd}$. Let $\boldsymbol{w} = s_1s_2s_3s_1s_2s_1$ be a reduced expression of $w \in W_Q$. This is a c-sortable element. In Example 5.2, we have

$$\begin{aligned}
M &= M^1 \oplus M^2 \oplus M^3 \oplus M^4 \oplus M^5 \oplus M^6 \\
&= \mathbf{1} \oplus \begin{smallmatrix} \mathbf{2} & & \\ & \diagdown & \\ & & \mathbf{1} \end{smallmatrix} \oplus \begin{smallmatrix} & & \mathbf{3} & & & & \\ & \diagup & & \diagdown & & & \\ \mathbf{1} & & & & \mathbf{2} & & \\ & & & & & \diagdown & \\ & & & & & & \mathbf{1} \end{smallmatrix} \oplus \left(\begin{smallmatrix} & & 1 & & & & & & \\ & & & & & & & & \\ \mathbf{2} & & & & \mathbf{3} & & & & \\ & \diagdown & & \diagup & & \diagdown & & & \\ & & \mathbf{1} & & & & \mathbf{2} & & \\ & & & & & & & \diagdown & \\ & & & & & & & & \mathbf{1} \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} & & & & 2 & & & & & & \\ & & & & & \diagdown & & & & & \\ & & \mathbf{3} & & & & 1 & & & & \\ & \diagup & & \diagdown & & & & & & & \\ \mathbf{1} & & & & \mathbf{2} & & \mathbf{3} & & & & \\ & & & & & \diagdown \diagup & & \diagdown & & & \\ & & & & & \mathbf{1} & & & \mathbf{2} & & \\ & & & & & & & & & \diagdown & \\ & & & & & & & & & & \mathbf{1} \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} & & & & 1 & & & & \\ & & & & & & & & \\ & & 2 & & & & 3 & & \\ & & & \diagdown & & \diagup & & \diagdown & \\ & \mathbf{3} & & & 1 & & & & 2 \\ \diagup & & & & & & & & & \diagdown \\ \mathbf{1} & & & & & & & & & & 1 \end{smallmatrix}\right)
\end{aligned}$$

in $\mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$, where the degree zero parts are denoted by bold numbers. Therefore, we have $P = M^3 \oplus M^5 \oplus M^6$ and

$$T = P_0 = M_0^3 \oplus M_0^5 \oplus M_0^6 = \begin{smallmatrix} & & 3 & & & & \\ & \diagup & & \diagdown & & & \\ 1 & & & & 2 & & \\ & & & & & \diagdown & \\ & & & & & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & & 3 & & & & & & & & & & \\ & \diagup & & \diagdown & & & & & & & & & \\ 1 & & & & 2 & & & & 3 & & & & \\ & & & & & \diagdown & & \diagup & & \diagdown & & & \\ & & & & & & 1 & & & & 2 & & \\ & & & & & & & & & & & \diagdown & \\ & & & & & & & & & & & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & & 3 \\ & \diagup & \\ 1 & & \end{smallmatrix},$$

$$M_0 = M_0^1 \oplus M_0^2 \oplus M_0^4 \oplus T = 1 \oplus \begin{smallmatrix} 2 & & \\ & \diagdown & \\ & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 & & & & 3 & & & & \\ & \diagdown & & \diagup & & \diagdown & & & \\ & & 1 & & & & 2 & & \\ & & & & & & & \diagdown & \\ & & & & & & & & 1 \end{smallmatrix} \oplus T.$$

It is easy to see that the algebra $\mathrm{End}_{KQ}(M_0)/[T]$ is given by the following quiver with relations

$$\Delta = \left[\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \right], \quad ab = 0.$$

By Theorem 6.3 or a direct calculation, we can see that the algebra $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)$ is also given by the same quiver with relations.

We can describe the Auslander-Reiten quiver of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$. Let X be the kernel of the canonical epimorphism $\Pi(w)e_2 \to S_2$, where S_2 is a simple module associated with the vertex 2, and let Y be the cokernel of an inclusion $(\Pi(w)e_1)_1 \to \Pi(w)e_2$:

$$X = \begin{smallmatrix} & & 3 & & & & \mathbf{1} & & & & & & \\ & \diagup & & \diagdown & & & & & & & & & \\ 1 & & & & 2 & & & & 3 & & & & \\ & & & & & \diagdown & & \diagup & & \diagdown & & & \\ & & & & & & 1 & & & & 2 & & \\ & & & & & & & & & & & \diagdown & \\ & & & & & & & & & & & & 1 \end{smallmatrix}, \qquad Y = \begin{smallmatrix} & & & & 2 & & \\ & & & & & \diagdown & \\ & & 3 & & & & 1. \\ & \diagup & & & & & \\ 1 & & & & & & \end{smallmatrix}$$

Then the Auslander-Reiten quiver of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ is the following one:

$$\begin{tikzcd}[column sep=tiny]
\cdots \arrow[dr] & & (\Pi(w)e_2)_1 \arrow[dr] & & \boxed{(\Pi(w)e_1)_0} \arrow[dr] & & (\Pi(w)e_1)_{[1,2]}(1) \arrow[dr] & & \boxed{(\Pi(w)e_1)_{[0,1]}(1)} \arrow[dr] & \\
& (\Pi(w)e_1)_1 \arrow[ur] \arrow[dr] & & X \arrow[ur] \arrow[dr] & & Y \arrow[ur] \arrow[dr] & & (\Pi(w)e_1)(1) \arrow[ur] \arrow[dr] & & \cdots, \\
\cdots \arrow[ur] & & (\Pi(w)e_1)_{[0,1]} \arrow[ur] & & (\Pi(w)e_1)_2(1) \arrow[ur] & & \boxed{Y_0} \arrow[ur] & & (\Pi(w)e_2)_1(1) \arrow[ur] &
\end{tikzcd}$$

where $M = (\Pi(w)e_1)_0 \oplus Y_0 \oplus (\Pi(w)e_1)_{[0,1]}(1)$. We see that the shape of the Auslander-Reiten quiver of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ is actually the same as that of $\mathsf{D}^{\mathrm{b}}(\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M))$.

We first describe the quiver of $\mathrm{End}_{\Pi(w)}(M)$. We recall the following definition of a quiver $Q(\boldsymbol{w})$ associated with a reduced expression $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ of $w \in W_Q$. This $Q(\boldsymbol{w})$ was denoted by $Q(u_1, \dots, u_l)$ in [BIRSc, Subsection III. 4].

Definition 6.5. [BIRSc] We define a quiver $Q(\boldsymbol{w})$ associated with a reduced expression $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ as follows:

- vertices: $Q(\boldsymbol{w})_0 = \{1, 2, \dots, l\}$.

  A vertex $1 \leq i \leq l$ in $Q(\boldsymbol{w})$ is said to be *type* $u \in Q_0$ if $u_i = u$.

- arrows:

(a1) For each $u \in \mathrm{Supp}(w)$, draw an arrow from j to i, where i, j are vertices of type u, $i < j$, and there is no vertex of type u between i and j (we call these arrows *going to the left*).

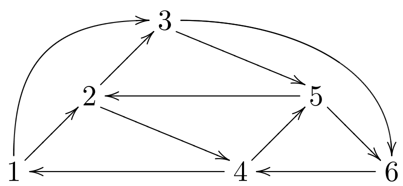
(a2) For each arrow $\alpha : u \to v \in Q_1$, draw an arrow α_i from i to j, where $i < j$, i is a vertex of type u, j is a vertex of type v, there is no vertex of type u between i and j, and j is the biggest vertex of type v before the next vertex of type u (we call these arrows *Q-arrows*).

(a3) For each arrow $\alpha : u \to v \in Q_1$, draw an arrow α_i^* from i to j, where $i < j$, i is a vertex of type v, j is a vertex of type u, there is no vertex of type v between i and j, and j is the biggest vertex of type u before the next vertex of type v (we call these arrows *Q^*-arrows*).

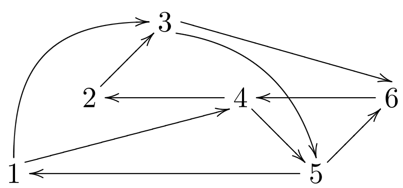
We denote by $\underline{Q}(\boldsymbol{w})$ the full subquiver of $Q(\boldsymbol{w})$ whose the set of vertices is $Q(\boldsymbol{w})_0 \setminus \{p_u \mid u \in \mathrm{Supp}(w)\}$.

Note that the quiver $Q(\boldsymbol{w})$ depends on the choice of a reduced expression of w.

Example 6.6. (a) Let Q be the quiver $\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\gamma"] & \\ 2 \arrow[rr, "\beta"'] & & 3 \end{tikzcd}$, and $w \in W_Q$ with a reduced expression $\boldsymbol{w} = s_{u_1}s_{u_2}s_{u_3}s_{u_4}s_{u_5}s_{u_6} = s_1s_2s_3s_1s_2s_1$. Then we have the quiver $Q(\boldsymbol{w})$ as follows:



(b) Let Q be the same quiver in (a), and $\boldsymbol{w'} = s_{u_1}s_{u_2}s_{u_3}s_{u_4}s_{u_5}s_{u_6} = s_1s_2s_3s_2s_1s_2$ be an another reduced expression of w. Then we have the quiver $Q(\boldsymbol{w'})$ as follows:



It is shown that $Q(\boldsymbol{w})$ gives a quiver of $\mathrm{End}_{\Pi(w)}(M)$ as we see in Theorem 6.8. We define a morphism of algebras $\phi : KQ(\boldsymbol{w}) \to \mathrm{End}_{\Pi(w)}(M)$.

Definition 6.7. Let $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ be a reduced expression of $w \in W_Q$. Then we define a morphism of algebras $\phi : KQ(\boldsymbol{w}) \to \mathrm{End}_{\Pi(w)}(M)$ as

(a0) For a vertex i of $Q(\boldsymbol{w})$, $\phi(e_i)$ is an idempotent of $\mathrm{End}_{\Pi(w)}(M)$ associated with M^i.

(a1) For an arrow $\beta : j \to i$ going to the left, $\phi(\beta)$ is the canonical surjection $M^j \to M^i$.

(a2) For a Q-arrow $\alpha_i : i \to j$ of the arrow $\alpha \in Q_1$, $\phi(\alpha_i)$ is a morphism of $\Pi(w)$-modules from M^i to M^j given by multiplying α from the right.

(a3) For a Q^*-arrow $\alpha_i^* : i \to j$ of the arrow $\alpha \in Q_1$, $\phi(\alpha_i^*)$ is a morphism of $\Pi(w)$-modules from M^i to M^j given by multiplying α^* from the right.

In the following theorem 6.8, we do not consider gradings of $\Pi(w)$ and M^i.

Theorem 6.8. *[BIRSc, Theorem III. 4.1] Let $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ be a reduced expression of $w \in W_Q$. Then the morphism of algebras $\phi : KQ(\boldsymbol{w}) \to \mathrm{End}_{\Pi(w)}(M)$ induces an isomorphism of algebras*

$$\underline{\phi} : KQ(\boldsymbol{w})/I \simeq \mathrm{End}_{\Pi(w)}(M)$$

for an ideal I of $KQ(\boldsymbol{w})$.

Since $\mathrm{End}_{\Pi(w)}(M) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^{\mathbb{Z}}_{\Pi(w)}(M, M(n))$, we regard $\mathrm{End}_{\Pi(w)}(M)$ as a graded algebra by $\mathrm{End}_{\Pi(w)}(M)_n = \mathrm{Hom}^{\mathbb{Z}}_{\Pi(w)}(M, M(n))$. In particular, we have $\mathrm{End}^{\mathbb{Z}}_{\Pi(w)}(M) = \mathrm{End}_{\Pi(w)}(M)_0$. We introduce a grading on $Q(\boldsymbol{w})$, that is, we introduce a map $Q(\boldsymbol{w})_1 \to \mathbb{Z}$.

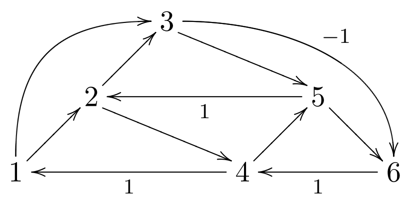
Definition 6.9. Assume that $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ is a reduced expression of $w \in W_Q$. Let $Q(\boldsymbol{w})$ be the quiver of $\mathrm{End}_{\Pi(w)}(M)$ and $Q(\boldsymbol{w})_0 = \{1, \ldots, l\}$. We define a grading on $Q(\boldsymbol{w})$ as follows:

(1) All arrows going to the left are of degree one.

(2) Let $\beta : i \to j$ be a Q-arrow in $Q(\boldsymbol{w})$. Then the degree of β is $m_i - m_j$.

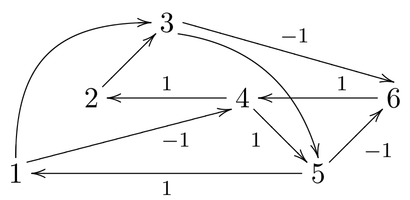
(3) Let $\beta : i \to j$ be a Q^*-arrow in $Q(\boldsymbol{w})$. Then the degree of β is $m_i - m_j + 1$.

Example 6.10. (a) In the quiver of Example 6.6 (a), we have the grading of $Q(\boldsymbol{w})$ as follows:



where non numbered arrows have degree zero.

(b) In the quiver of Example 6.6 (b), we have the grading of $Q(\boldsymbol{w})$ as follows:



where non numbered arrows have degree zero.

We regard $KQ(\boldsymbol{w})$ as a graded algebra by the grading of Definition 6.9. Then the isomorphism in Theorem 6.8 holds as graded algebras.

Proposition 6.11. *The morphism of algebras*

$$\phi : KQ(\boldsymbol{w}) \to \mathrm{End}_{\Pi(w)}(M)$$

is a surjective morphism of graded algebras.

Proof. It is enough to show that the morphism $\phi : KQ(\boldsymbol{w}) \to \mathrm{End}_{\Pi(w)}(M)$ preserves gradings. Since $KQ(\boldsymbol{w})$ is generated by arrows, it is enough to show that ϕ preserves gradings of arrows.

(a1) Let $\beta : j \to i$ be an arrow going to the left. Then $\phi(\beta)$ is given by a surjection

$$(\Pi/I(s_{u_1}s_{u_2}\cdots s_{u_j}))e_{u_j}(m_j) \to (\Pi/I(s_{u_1}s_{u_2}\cdots s_{u_i}))e_{u_i}(m_i).$$

Since there exists no vertex of type $u_i = u_j$ between i and j, we have $m_i + 1 = m_j$. Since $\mathrm{top}\left(M^j\right)$ is concentrated in $-m_j$ and $\mathrm{top}\left(M^i\right)$ is concentrated in $-m_i$, this surjection is degree one.

(a2) Let $\beta = \alpha_i : i \to j$ be a Q-arrow in $Q(\boldsymbol{w})$, where $\alpha \in Q_1$. Then $\phi(\beta)$ is a morphism multiplying α from the right:

$$\phi(\beta) = (\cdot\alpha) : (\Pi/I(s_{u_1}s_{u_2}\cdots s_{u_i}))e_{u_i}(m_i) \to (\Pi/I(s_{u_1}s_{u_2}\cdots s_{u_j}))e_{u_j}(m_j)).$$

This means $\phi(\beta)$ is degree $m_i - m_j$.

(a3) Let $\beta = \alpha_i^* : i \to j$ be a Q^*-arrow in $Q(\boldsymbol{w})$, where $\alpha \in Q_1$. Then $\phi(\beta)$ is a morphism multiplying α^* from the right:

$$\phi(\beta) = (\cdot\alpha^*) : (\Pi/I(s_{u_1}s_{u_2}\cdots s_{u_i}))e_{u_i}(m_i) \to (\Pi/I(s_{u_1}s_{u_2}\cdots s_{u_j}))e_{u_j}(m_j)).$$

This means $\phi(\beta)$ is degree $m_i - m_j + 1$. $\square$

The following lemma is important to show Propositions 6.14 and 6.15.

Lemma 6.12. *Assume that $w = s_{u_1}s_{u_2}\cdots s_{u_l}$ is a c-sortable element. Let $\beta : i \to j$ be an arrow in $Q(\boldsymbol{w})$ which is a Q-arrow or a Q^*-arrow. Then the following holds.*

- *(a) If β has a negative degree, then we have $i = p_{u_i}$ and $j = p_{u_j}$.*
- *(b) If β satisfies $i \neq p_{u_i}$ or $j \neq p_{u_j}$, then β has degree zero.*

Proof. Assume that β is a Q-arrow and i is a vertex of type u and j is a vertex of type v. Then, by the definition of $Q(\boldsymbol{w})$, there exists an arrow $\alpha : u \to v$ in Q which satisfies $\alpha_i = \beta$. Pick up vertices of type u and v from $Q(\boldsymbol{w})_0 = \{1, 2, \ldots, l\}$, then we have the following two cases:

$$1 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_s < b_s < b_{s+1} < \cdots < b_t \leq l, \tag{6.2}$$
$$1 \leq a_1 < b_1 < a_2 < b_2 < \cdots < b_t < a_{t+1} < a_{t+2} < \cdots < a_s \leq l, \tag{6.3}$$

where $a_\bullet$ are vertices of type u and $b_\bullet$ are vertices of type v. By the definition of m_j, we have $m_{a_k} = k - 1$ and $m_{b_k} = k - 1$. Moreover, by the definition of p_u and p_v, we have $p_u = a_s$ and $p_v = b_t$. Let $Q(\boldsymbol{w})(i, j, \alpha)$ be a subquiver of $Q(\boldsymbol{w})$ such that $Q(\boldsymbol{w})(i, j, \alpha)_0 = \{a_1, \cdots, a_s, b_1, \cdots, b_t\}$ and $Q(\boldsymbol{w})(i, j, \alpha)_1$ is the set of all arrows of the form α_k or α_k^* for some $1 \leq k \leq l$ or arrows going to the left.

In the case (6.2), the quiver $Q(\boldsymbol{w})(i, j, \alpha)$ is the following:

$$\begin{tikzcd}
 & b_1 \arrow[dr] & & b_2 \arrow[ll] \arrow[dr] & & \cdots \arrow[ll] \arrow[dr] & & b_s \arrow[ll] & b_{s+1} \arrow[l] & \cdots \arrow[l] & b_t \arrow[l] \\
a_1 \arrow[ur] & & a_2 \arrow[ll] \arrow[ur] & & \cdots \arrow[ll] & & a_s \arrow[ll] \arrow[urrrr, bend right] & & & &
\end{tikzcd}.$$

Since β is a Q-arrow, β is one of the arrows of $a_k \to b_k$ for $1 \le k \le s-1$ or $a_s \to b_t$. For $1 \le k \le s-1$, we have $m_{a_k} - m_{b_k} = 0$. Therefore, in the case (6.2), (a) and (b) hold.

In the case (6.3), the quiver $Q(\boldsymbol{w})(i,j,\alpha)$ is the following:

$$\begin{tikzcd}[column sep=small]
 & b_1 \arrow[dr] & & b_2 \arrow[ll] \arrow[dr] & & \cdots \arrow[ll] & & b_t \arrow[ll] \arrow[rrrd, bend left] & & & \\
a_1 \arrow[ur] & & a_2 \arrow[ll] \arrow[ur] & & \cdots \arrow[ll] & & a_t \arrow[ll] \arrow[ur] & & a_{t+1} \arrow[ll] & \cdots \arrow[l] & a_s \arrow[l]
\end{tikzcd}.$$

Since β is a Q-arrow, β is one of the arrows of $a_k \to b_k$ for $1 \le k \le t$. For $1 \le k \le t$, we have $m_{a_k} - m_{b_k} = 0$. Therefore, in the case (6.3), (a) and (b) hold.

By the same argument, we can show in the case when β is a Q^*-arrow. $\square$

Lemma 6.13. *Assume that $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ be a c-sortable expression of $w \in W_Q$, then any $f \in \mathrm{Hom}^{\mathbb{Z}}_{\Pi(w)}(M, M(a))$ with $a < 0$ factors through $\mathsf{add}\, P = \mathsf{add}(\bigoplus_{u\in \mathrm{Supp}(w)} M^{p_u})$.*

Proof. We identify $\mathrm{End}_{\Pi(w)}(M)$ with $KQ(\boldsymbol{w})/I$ as graded algebras by Theorem 6.8 and Proposition 9.15. Since f is written as a liner combination of paths in $Q(\boldsymbol{w})$, we can assume that $f = p$ for some path p in $Q(\boldsymbol{w})$. Since f has a negative degree, the degree of p is negative. Thus p contains an arrow of negative degree. By Lemma 6.12, p factors through a vertex p_u for some $u \in \mathrm{Supp}(w)$. Therefore, f factors through $\mathsf{add}\, P = \mathsf{add}(\bigoplus_{u\in \mathrm{Supp}(w)} M^{p_u})$. $\square$

Now we are ready to show the finiteness of the global dimension of $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)$.

Proposition 6.14. *Let $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l} = c^{(0)}c^{(1)}\cdots c^{(m)}$ be a c-sortable expression of $w \in W_Q$ and $M = \bigoplus_{i=1}^{l} M^i$ be a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$. Then the global dimension of $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)$ is finite.*

Proof. By [BIRSm, Theorem 6.6], ϕ induces a surjective morphism of graded algebras $\widetilde{\phi} : K\underline{Q}(\boldsymbol{w}) \to \underline{\mathrm{End}}_{\Pi(w)}(M)$. By Lemma 6.12 (a), $K\underline{Q}(\boldsymbol{w})$ is positively graded and therefore $\underline{\mathrm{End}}_{\Pi(w)}(M)$ is also positively graded. By taking degree zero part of these algebras, we have the following commutative diagram

$$\begin{tikzcd}
K\underline{Q}(\boldsymbol{w}) \arrow[r, "\widetilde{\phi}"] \arrow[d] & \underline{\mathrm{End}}_{\Pi(w)}(M) \arrow[d] \\
K\underline{Q}_0(\boldsymbol{w}) \arrow[r, "\overline{\phi}"] & \underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M),
\end{tikzcd}$$

where we denote by $\underline{Q}_0(\boldsymbol{w})$ a subquiver of $\underline{Q}(\boldsymbol{w})$ such that vertices are same as $\underline{Q}(\boldsymbol{w})$ and arrows are all degree zero arrows of $\underline{Q}(\boldsymbol{w})$. We have a surjection $\overline{\phi}$, since $\widetilde{\phi}$ and vertical morphisms are surjections. Because $\underline{Q}_0(\boldsymbol{w})$ does not contains arrows going to the left, $\underline{Q}_0(\boldsymbol{w})$ is acyclic. Therefore the global dimension of $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)$ is finite. $\square$

In Section 7, we show that the global dimension of $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)$ is at most two.

We show that the morphism F actually induces a morphism $\underline{F}$.

Proposition 6.15. *The morphism F induces a morphism of algebras:*

$$\underline{F} : \underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M) \to \mathrm{End}_{KQ}(M_0)/[T].$$

Proof. We show that if a morphism $f : M \to M$ in $\mathsf{mod}^{\mathbb{Z}}\,\Pi(w)$ factors through graded projective $\Pi(w)$-modules, then f factors through $\mathsf{add}\,P = \mathsf{add}(\bigoplus_{u \in \mathrm{Supp}(w)} M^{p_u})$. Without loss of generality, we may assume that $f = h \circ g$ for $g : M \to M^{p_u}(a)$ and $h : M^{p_u}(a) \to M$, where $u \in \mathrm{Supp}(w)$ and $a \in \mathbb{Z}$. We divide into three cases:

- If $a > 0$, then $M^{p_u}(a)_0 = M^{p_u}_a = 0$, since $M^{p_u} = M^{p_u}_{\leq 0}$ by Proposition 4.5. Thus we have $f|_0 = 0$.
- If $a = 0$, then f actually factors through $M^{p_u} \in \mathsf{add}\,P$.
- If $a < 0$, then g factors through $\mathsf{add}\,P$ by Lemma 6.13. Thus f also factors through $\mathsf{add}\,P$.

□

In the rest of this subsection, we give some examples of tilting objects M and its endomorphism algebras.

Example 6.16. If Q is not Dynkin, then $\boldsymbol{w} = \boldsymbol{c}^2 = s_{u_1}s_{u_2}\cdots s_{u_n}s_{u_1}s_{u_2}\cdots s_{u_n}$ is a reduced expression by [BIRSc, Proposition III. 3.1]. Thus we have $\Pi(w) = \Pi_{\leq 1}$ by Proposition 4.5. Since $M = \Pi(c) \oplus \Pi(c^2)(1) \simeq KQ$ in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ and KQ is concentrated in degree 0, we have

$$\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M) = \underline{\mathrm{End}}_{\Pi(w)}(KQ).$$

By [BIRSc, Proposition III. 3.2], we have an isomorphism $\underline{\mathrm{End}}_{\Pi(w)}(KQ) \simeq KQ$. Therefore, we have $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M) \simeq KQ$, and a triangulated equivalence

$$\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w) \simeq \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,KQ) \simeq \mathsf{D}^{\mathrm{b}}(KQ).$$

Example 6.17. Let Q be a quiver $1 \rightrightarrows 2$. Then we have a graded algebra $\Pi = \Pi e_1 \oplus \Pi e_2$, and these are represented by their radical filtrations as follows:

$$\begin{array}{ccccccccccc} & & & & & \mathbf{1} & & & & & \\ & & & & 2 & & 2 & & & & \\ & & & 1 & & 1 & & 1 & & & \\ & & 2 & & 2 & & 2 & & 2 & & \\ & 1 & & 1 & & 1 & & 1 & & 1 & \\ 2 & & 2 & & 2 & & 2 & & 2 & & 2 \end{array} \qquad \begin{array}{ccccccccccc} & & & & & \mathbf{2} & & & & & \\ & & & & \mathbf{1} & & \mathbf{1} & & & & \\ & & & 2 & & 2 & & 2 & & & \\ & & 1 & & 1 & & 1 & & 1 & & \\ & 2 & & 2 & & 2 & & 2 & & 2 & \\ 1 & & 1 & & 1 & & 1 & & 1 & & 1 \end{array},$$

where the degree zero parts are denoted by bold numbers. Let $c = s_1s_2$. This is a Coxeter element. Let $\boldsymbol{w} = \boldsymbol{c}^{n+1} = s_1s_2s_1\cdots s_1s_2$. This is a c-sortable element. We have $(\Pi/I(c^i))e_1 = (\Pi/J^{2i-1})e_1$, and $(\Pi/I(c^i))e_2 = (\Pi/J^{2i})e_2$, where J is the Jacobson radical of Π. By Theorem 5.6, $M = \bigoplus_{i=1}^{n}(\Pi/I(c^i))(i-1)$ is a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$, where graded projective $\Pi(w)$-modules are removed. The endomorphism algebra $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M) \simeq \mathrm{End}_{KQ}(M_0)/[T]$ is given by the following quiver with relations

$$\Delta = \left[\, 1 \overset{a}{\underset{b}{\rightrightarrows}} 2 \overset{a}{\underset{b}{\rightrightarrows}} 3 \overset{a}{\underset{b}{\rightrightarrows}} \quad \cdots \quad \overset{a}{\underset{b}{\rightrightarrows}} 2n-1 \overset{a}{\underset{b}{\rightrightarrows}} 2n \,\right], \quad aa = bb.$$

The algebra $K\Delta/\langle aa - bb\rangle$ has global dimension two.

6.2 Relationship between endomorphism algebras associated with w and w'

In this subsection, we prove Theorem 6.25 which is used to prove Proposition 6.29. Throughout this subsection, we use the notation in Definition 6.1.

Assume that v is a source in Q. Let $Q' = \mu_v(Q)$ be the quiver obtained by reversing all arrows starting at v. Although the preprojective algebras Π and Π' of Q and Q', respectively, are the same as ungraded algebras, they have different gradings.

We first construct a functor from $\mathsf{mod}^{\mathbb{Z}}\,\Pi$ to $\mathsf{mod}^{\mathbb{Z}}\,\Pi'$. Let $\beta_1, \beta_2, \dots, \beta_r$ be the arrows in Q starting at v, and

$$Q'_1 = \big(Q_1 \setminus \{\beta_1, \beta_2, \dots, \beta_r\}\big) \sqcup \{\gamma_1, \dots, \gamma_r\},$$

where $t(\gamma_i) = v$, $t(\beta_i) = s(\gamma_i)$. We have an isomorphism of algebras $\rho : K\overline{Q} \to K\overline{Q'}$ given by $\rho(\beta_i) = \gamma_i^*, \rho(\beta_i^*) = -\gamma_i$, and $\rho(\alpha) = \alpha$ for other arrows. Then ρ induces an isomorphism of the preprojective algebras, we also denote it by ρ:

$$\rho : \Pi \xrightarrow{\sim} \Pi'. \tag{6.4}$$

By calculating the grading of paths of $K\overline{Q}$ and $K\overline{Q'}$, we have the following lemma, where $\delta_{u,v} = 1$ if $u = v$ and 0 otherwise for $u, v \in Q_0$.

Lemma 6.18. *For $u, u' \in Q_0$ and $i \in \mathbb{Z}$, by identifying KQ with KQ' by ρ, we have*

$$e_u(K\overline{Q})_i e_{u'} = e_u(K\overline{Q'})_{i+\delta_{u,v}-\delta_{u',v}} e_{u'}.$$

Moreover, the equation also holds for Π and Π', that is,

$$e_u \Pi_i e_{u'} = e_u \Pi'_{i+\delta_{u,v}-\delta_{u',v}} e_{u'}.$$

For a finitely generated graded Π'-module N, we regard $\mathrm{End}_\Pi(N)$ as a graded algebra by $\mathrm{End}_\Pi(N)_i = \mathrm{Hom}_\Pi^{\mathbb{Z}}(N, N(i))$. The graded preprojective algebras Π and Π' are related as follows.

Lemma 6.19. *We have an isomorphism of graded algebras*

$$\Pi' \to \mathrm{End}_\Pi(\Pi e_v(1) \oplus \Pi(1 - e_v)), \quad x \mapsto (\cdot \rho^{-1}(x)).$$

Proof. It is enough to show that the morphism preserves gradings. This follows from Lemma 6.18. □

Then we construct a functor $\mathbb{G}$ from $\mathsf{mod}^{\mathbb{Z}}\,\Pi$ to $\mathsf{mod}^{\mathbb{Z}}\,\Pi'$. We need the following Lemma.

Lemma 6.20. *We have a surjective morphism of algebras $\Pi \to \mathrm{End}_\Pi(I_v), x \mapsto (\cdot x)$.*

Proof. If Q is a non-Dynkin quiver, then the assertion follows from Proposition 3.3 (a). If Q is a Dynkin quiver, then the assertion follows from [M, Lemma 2.7]. □

More precisely, we have the following surjective morphism of graded algebras.

Lemma 6.21. *Let $v \in Q_0$ be a source and $U := I_v e_v(1) \oplus \Pi(1-e_v) \in \mathsf{mod}^{\mathbb{Z}}\,\Pi$. Then we have a surjective morphism of graded algebras*

$$\Pi' \to \mathrm{End}_\Pi(U), \qquad x \mapsto (\cdot \rho^{-1}(x)).$$

Moreover, we have the following surjective morphism of graded algebras

$$\Pi' \to \mathrm{End}_\Pi(\Pi/I_v), \qquad x \mapsto (\cdot \pi(\rho^{-1}(x))),$$

where $\pi : \Pi \to \Pi/I_v$ is the canonical surjection.

Proof. The morphism is surjective since ρ is an isomorphism and by Lemma 6.20. We have to show that the composite is a morphism of graded algebras.

By Lemma 6.18, for $u, u' \in Q_0$, we have

$$e_u(\Pi'_i)e_{u'} = e_u(\Pi_{i+\delta_{u',v}-\delta_{u,v}})e_{u'}.$$

Moreover, for $u, u' \in Q_0$ and $j \in \mathbb{Z}$, we have

$$\begin{aligned} U_j e_u \cdot e_u(\Pi_{i+\delta_{u',v}-\delta_{u,v}})e_{u'} &= \begin{cases} (I_v)_{j+1}e_u \cdot e_u\Pi_i e_{u'} & u = u' = v \\ (I_v)_{j+1}e_u \cdot e_u\Pi_{i-1} e_{u'} & u = v, u' \neq v \\ \Pi_j e_u \cdot e_u\Pi_{i+1} e_{u'} & u \neq v, u' = v \\ \Pi_j e_u \cdot e_u\Pi_i e_{u'} & u \neq v, u' \neq v \end{cases} \\ &\subset U(i)_j e_{u'}. \end{aligned}$$

Thus, the morphism $\Pi' \to \mathrm{End}_\Pi(U)$ is a morphism of graded algebras. The other follows from a similar calculation. $\square$

By Lemma 6.21, we have a functor

$$\mathbb{G} := \mathrm{Hom}_\Pi(U, -) : \mathsf{mod}^{\mathbb{Z}}\,\Pi \to \mathsf{mod}^{\mathbb{Z}}\,\Pi'.$$

where the grading on the Π'-module $\mathbb{G}(X)$ is given by $\mathbb{G}(X)_i := \mathrm{Hom}_\Pi^{\mathbb{Z}}(U, X(i))$. This functor satisfies $\mathbb{G} \circ (i) \simeq (i) \circ \mathbb{G}$ for any $i \in \mathbb{Z}$.

To show Proposition 6.23, we recall the following proposition. For a reduced expression $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$, let $I_{k,m} = I(s_{u_k}\cdots s_{u_m})$ if $k \leq m$ and $I_{k,m} = \Pi$ if $m < k$.

Proposition 6.22. *[BIRSc, Lemma III. 1.14] Assume that $s_{u_1}s_{u_2}\cdots s_{u_l}$ is a reduced expression. Then we have $I_{k+1,m}/I_{1,m} \simeq \mathrm{Hom}_\Pi(\Pi/I(s_{u_1}\ldots s_{u_k}), \Pi/I(s_{u_1}\ldots s_{u_m}))$ by $x \mapsto (\cdot x)$.*

Proof. If Q is a non-Dynkin quiver, then the assertion holds by [BIRSc, Lemma III. 1.14]. The assertion also holds when Q is a Dynkin quiver by Lemma 4.1. $\square$

We apply the same construction as Definition 6.1 to the reduced expression $\boldsymbol{w}' := s_{u_2}s_{u_3}\cdots s_{u_l}$. Put

$$\begin{aligned} p'_u &= \max\{2 \leq j \leq l \mid u_j = u\} - 1, & &\text{for } u \in \mathrm{Supp}(w'), \\ m'_i &= \sharp\{2 \leq j \leq i-1 \mid u_j = u_i\}, & &\text{for } 2 \leq i \leq l. \end{aligned}$$

Moreover, for $2 \leq i \leq l$, put

$$M'^{i-1} := (\Pi'/I'(s_{u_2} \cdots s_{u_i}))e_{u_i}(m'_i), \qquad P' = \bigoplus_{u \in \mathrm{Supp}(w')} M'^{p'_u}.$$

We have $\Pi'(w') = \bigoplus_{u \in \mathrm{Supp}(w')} M'^{p'_u}(-m'_{p'_u})$. Put $M' = \bigoplus_{i=2}^{l} M'^{i-1}$.

Proposition 6.23. *Assume that $\boldsymbol{w} = s_{u_1} s_{u_2} \cdots s_{u_l}$ is a reduced expression of $w \in W_Q$ and $u_1 = v$ is a source of Q, $l \geq 2$. Let $\boldsymbol{w}' = s_{u_2} \cdots s_{u_l}$. Then*

(a) $\mathbb{G}(M^1) = 0$.

(b) For $2 \leq j \leq l$, we have an isomorphism $\psi_j : \mathbb{G}(M^j) \xrightarrow{\sim} M'^{j-1}$ in $\mathsf{mod}^{\mathbb{Z}}\,\Pi'$, that is,

$$\psi_j : \mathbb{G}\left((\Pi/I_{1,j})e_{u_j}\right)(m_j) \xrightarrow{\sim} \left((\Pi'/I'_{2,j})e_{u_j}\right)(m'_j).$$

(c) We have $\psi = \bigoplus_{j=1}^{l} \psi_j : \mathbb{G}(M) = \mathbb{G}(M/M^1) \xrightarrow{\sim} M'$ in $\mathsf{mod}^{\mathbb{Z}}\,\Pi'$.

Proof. (a) Since a simple module associated with $u_1 = v$ does not appear in $\mathrm{top}(U)$, we have $\mathrm{Hom}_{\Pi}(U, M^1) = 0$.

(b) Since $m'_j = m_j - \delta_{v,u_j}$ holds, we show that

$$\mathbb{G}\left((\Pi/I_{1,j})e_{u_j}\right) \simeq \left((\Pi'/I'_{2,j})e_{u_j}\right)(-\delta_{v,u_j}).$$

By a similar calculation of the proof of Lemma 6.21, we have the following morphism of graded Π'-modules

$$(I'_{2,j}/I'_{1,j})(-\delta_{v,u_j}) \to \mathrm{Hom}_{\Pi}((\Pi/I_v)(1), \Pi/I_{1,j}),$$
$$(\Pi'/I'_{1,j})(-\delta_{v,u_j}) \to \mathrm{Hom}_{\Pi}(\Pi e_v(1) \oplus \Pi(1-e_v), \Pi/I_{1,j}),$$

where both of them are defined by $x \mapsto (\cdot \rho^{-1}(x))$. These morphisms are isomorphisms by Proposition 6.22. By Proposition 3.3 (e), $\mathrm{Ext}^1_{\Pi}(\Pi/I_v, \Pi/I_{1,j}) = 0$ holds. Applying the functor $\mathrm{Hom}_{\Pi}(-, \Pi/I_{1,j})$ to the exact sequence

$$0 \to U \to \Pi e_v(1) \oplus \Pi(1-e_v) \to (\Pi/I_v)(1) \to 0,$$

we have the following commutative diagram of exact sequence in $\mathsf{mod}^{\mathbb{Z}}\,\Pi'$;

$$\begin{tikzcd}
0 \arrow[r] & (I'_{2,j}/I'_{1,j})(-\delta_{v,u_j}) \arrow[r] \arrow[d, "\simeq"] & (\Pi'/I'_{1,j})(-\delta_{v,u_j}) \arrow[r] \arrow[d, "\simeq"] & (\Pi'/I'_{2,j})(-\delta_{v,u_j}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {}_{\Pi}((\Pi/I_v)(1), \Pi/I_{1,j}) \arrow[r] & {}_{\Pi}(\Pi e_v(1) \oplus \Pi(1-e_v), \Pi/I_{1,j}) \arrow[r] & {}_{\Pi}(U, \Pi/I_{1,j}) \arrow[r] & 0.
\end{tikzcd}$$

Therefore we have the assertion.

(c) This comes from (a) and (b). ☐

The following lemma is used later.

Lemma 6.24. *Under the setting in Proposition 6.23, for the functor $\mathbb{G} : \mathsf{mod}^{\mathbb{Z}}\,\Pi \to \mathsf{mod}^{\mathbb{Z}}\,\Pi'$, we have*

(a) $\mathbb{G}$ restricts to a dense functor $\mathsf{proj}^{\mathbb{Z}}\,\Pi(w)$ to $\mathsf{proj}^{\mathbb{Z}}\,\Pi'_{w'}$.

(b) For $i \in \mathbb{Z}$, the map $\mathbb{G}_{M,M(i)}$ is surjective.

Proof. (a) This comes from $\Pi(w) = \bigoplus_{u\in \mathrm{Supp}(w)} M^{p_u}(-m_{p_u})$, $\Pi'_{w'} = \bigoplus_{u\in \mathrm{Supp}(w')} M'^{p'_u}(-m'_{p'_u})$, and Proposition 6.23.

(b) It is enough to show that the map $\mathbb{G}_{M^j,M^k(i)}$ is surjective for $2 \le j,k \le l$. By Lemma 6.22 (b), we have

$$\mathrm{Hom}^{\mathbb{Z}}_{\Pi}(M^j, M^k(i)) = \left(e_{u_j}\frac{I_{j+1,k}}{I_{1,k}}e_{u_k}\right)_{m_k-m_j+i},$$
$$\mathrm{Hom}^{\mathbb{Z}}_{\Pi'}(M'^{j-1}, M'^{k-1}(i)) = \left(e_{u_j}\frac{I'_{j+1,k}}{I'_{2,k}}e_{u_k}\right)_{m'_k-m'_j+i}.$$

For $2 \le j,k \le l$, an equation $m'_k - m'_j + i = m_k - m_j + \delta_{u_j,v} - \delta_{u_k,v} + i$ holds. Thus $\rho : \Pi \to \Pi'$ maps $\left(e_{u_j}(I_{j+1,k}/I_{1,k})e_{u_k}\right)_{m_k-m_j+i}$ to $\left(e_{u_j}(I'_{j+1,k}/I'_{2,k})e_{u_k}\right)_{m'_k-m'_j+i}$ by Lemma 6.18. We have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^{\mathbb{Z}}_{\Pi}(M^j, M^k(i)) \arrow[r, "\mathbb{G}_{M^j,M^k(i)}"] \arrow[d, "\simeq"] & \mathrm{Hom}^{\mathbb{Z}}_{\Pi'}(\mathbb{G}(M^j), \mathbb{G}(M^k)(i)) \arrow[r, "\alpha", "\sim"'] & \mathrm{Hom}^{\mathbb{Z}}_{\Pi'}(M'^{j-1}, M'^{k-1}(i)) \arrow[d, "\simeq"] \\
\left(e_{u_j}\frac{I_{j+1,k}}{I_{1,k}}e_{u_k}\right)_{m_k-m_j+i} \arrow[rr] & & \left(e_{u_j}\frac{I'_{j+1,k}}{I'_{2,k}}e_{u_k}\right)_{m'_k-m'_j+i},
\end{tikzcd} \tag{6.5}$$

where the lower map is induced by $\rho : \Pi \to \Pi'$, and α is defined by $\alpha(f) = \psi_k(i) \circ f \circ \psi_j^{-1}$. Since the lower map is surjective and α is an isomorphism by Proposition 6.23 (b), we have that $\mathbb{G}_{M^j,M^k(i)}$ is surjective. $\square$

The following theorem is a graded version of [IR, Theorem 3.1, (ii)] and the main theorem of this subsection.

Theorem 6.25. *Under the setting in Proposition 6.23, we have an isomorphism of algebras*

$$\underline{G} : \mathrm{End}^{\mathbb{Z}}_{\Pi(w)}(M)/[M^1(i) \mid 0 \le i \le p_{u_1}] \xrightarrow{\sim} \mathrm{End}^{\mathbb{Z}}_{\Pi'_{w'}}(M'),$$

where $G(-) = \psi \circ \mathbb{G}_{M,M}(-) \circ \psi^{-1}$ and $[M^1(i) \mid 0 \le i \le p_{u_1}]$ is an ideal of $\mathrm{End}^{\mathbb{Z}}_{\Pi(w)}(M)$ consisting of morphisms factoring through objects in $\mathsf{add}\{M^1(i) \mid 0 \le i \le p_{u_1}\}$.

Proof. We show that G is surjective and $\mathrm{Ker}(G) = [M^1(i) \mid 0 \le i \le p_{u_1}]$.

(i) By Lemma 6.24 (b), G is surjective.

(ii) Since ψ is an isomorphism, we have $\mathrm{Ker}(G) = \mathrm{Ker}(\mathbb{G}_{M,M})$. We show that $\mathrm{Ker}(\mathbb{G}_{M,M}) = [M^1(i) \mid 0 \le i \le p_{u_1}]$. By Proposition 6.23 (a), we have $[M^1(i) \mid 0 \le i \le p_{u_1}] \subset \mathrm{Ker}\left(\mathbb{G}_{M^j,M^k}\right)$. Conversely, we show that $\mathrm{Ker}\left(\mathbb{G}_{M^j,M^k}\right) \subset [M^1(i) \mid 0 \le i \le p_{u_1}]$ for $2 \le j,k \le l$. By the commutative diagram (6.5), we have

$$\mathrm{Ker}\left(\mathbb{G}_{M^j,M^k}\right) = \left(e_{u_j}\frac{I_{2,k}}{I_{1,k}}e_{u_k}\right)_{m_k-m_j}.$$

If $u_j \neq u_1$, then $e_{u_j} I_{1,k} = e_{u_j} I_{2,k}$ and we have $\operatorname{Ker}\left(\mathbb{G}_{M^j,M^k}\right) = 0$. If $u_j = u_1$, then we have

$$\begin{aligned}
\operatorname{Ker}\left(\mathbb{G}_{M^j,M^k}\right) &= \left(e_{u_j} \frac{I_{2,k}}{I_{1,k}} e_{u_k}\right)_{m_k - m_j} \\
&= \left(e_{u_j} \frac{\Pi}{I_{u_1}} e_{u_1}\right)\left(e_{u_1} \frac{I_{2,k}}{I_{1,k}} e_{u_k}\right)_{m_k - m_j} \\
&= \operatorname{Hom}^{\mathbb{Z}}_{\Pi}(M^1(m_j), M^k) \circ \operatorname{Hom}^{\mathbb{Z}}_{\Pi}(M^j, M^1(m_j)).
\end{aligned}$$

In particular, we have $\operatorname{Ker}\left(\mathbb{G}_{M^j,M^k}\right) \subset [M^1(i) \mid 0 \leq i \leq p_{u_1}]$. $\square$

We end this subsection by showing the following lemma which is used later to show Lemma 6.28. For a source $v \in Q_0$ and $Q' = \mu_v(Q)$, we have the reflection functor

$$\mathsf{mod}\ KQ \xrightarrow{R_v^+} \mathsf{mod}\ KQ'.$$

Note that U is generated by U_0 as a left Π-module. In fact, $I_v e_u = \Pi e_u$ is generated by e_u for $u \neq v$ and $I_v e_v$ is generated by all arrows in $\overline{Q}$ starting at v.We denote by $\mathbb{F}'$ the degree zero functor on $\mathsf{mod}^{\mathbb{Z}}\,\Pi'$:

$$\mathbb{F}' = (-)_0 : \mathsf{mod}^{\mathbb{Z}}\,\Pi' \to \mathsf{mod}\,KQ'.$$

Lemma 6.26. *Let v be a source of Q and $Q' = \mu_v(Q)$.*

(a) We have a morphism of functors $\phi : \mathbb{F}' \circ \mathbb{G} \to R^+_{u_1} \circ \mathbb{F}$.

(b) For any $X \in \mathsf{mod}^{\leq 0}\,\Pi$, $\phi_X : \mathbb{G}(X)_0 \to R_v^+(X_0)$ is an isomorphism of KQ'-modules, that is, the following diagram of functors is commutative on $\mathsf{mod}^{\leq 0}\,\Pi$:

$$\begin{tikzcd}
\mathsf{mod}^{\mathbb{Z}}\,\Pi \arrow[r, "\mathbb{G}"] \arrow[d, "\mathbb{F}"] & \mathsf{mod}^{\mathbb{Z}}\,\Pi' \arrow[d, "\mathbb{F}'"] \\
\mathsf{mod}\ KQ \arrow[r, "R_v^+"] & \mathsf{mod}\ KQ'.
\end{tikzcd}$$

Proof. By the definition of the functor $\mathbb{G}$, we have $\mathbb{G}(X)_0 = \operatorname{Hom}^{\mathbb{Z}}_{\Pi}(U, X)$. Since $\Pi_i \simeq \tau^{-i}(KQ)$ as KQ-modules, $U_0 = \tau^-(KQe_v) \oplus KQ(1 - e_v)$ holds and this is an APR-tilting KQ-module associated with v. Therefore we have a morphism of KQ'-modules

$$\phi_X : \mathbb{G}(X)_0 = \operatorname{Hom}^{\mathbb{Z}}_{\Pi}(U, X) \to \operatorname{Hom}_{KQ}(U_0, X_0) = R_v^+(X_0),$$

given by $\phi_X(f) = f|_{U_0}$. Clearly this gives a morphism $\phi : \mathbb{F}' \circ \mathbb{G} \to R^+_{u_1} \circ \mathbb{F}$ of functors. Since U is generated by U_0 as a graded Π-module, a morphism $f \in \operatorname{Hom}^{\mathbb{Z}}_{\Pi}(U, X)$ is determined by $\phi_X(f)$. This implies that ϕ_X is injective.

We show that ϕ_X is surjective when X is in $\mathsf{mod}^{\leq 0}\,\Pi$. Let $g \in \operatorname{Hom}_{KQ}(U_0, X_0)$. We define a morphism $f : U \to X$ of KQ-modules by $f|_{U_0} = g$ and $f|_{U_{\geq 1}} = 0$. Then f gives a morphism in $\mathsf{mod}^{\mathbb{Z}}\,\Pi$, since $X \in \mathsf{mod}^{\leq 0}\,\Pi$ and Π is positively graded. $\square$

6.3 $\underline{\mathbf{F}}$ is surjective

We use the notation in Subsections 6.1 and 6.2. For a quiver Q, we denote by W_Q the Coxeter group of Q. Assume that $\boldsymbol{w} = \boldsymbol{c}^{(0)}\boldsymbol{c}^{(1)} \cdots \boldsymbol{c}^{(m)} = s_{u_1}s_{u_2} \cdots s_{u_l}$ is a c-sortable expression of $w \in W_Q$. Without loss of generality by Lemma 4.1, we assume that $Q_0 = \mathrm{Supp}(w)$. Let $Q' = \mu_{u_1}(Q)$. We show that the morphism $\underline{F} : \underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M) \to \mathrm{End}_{KQ}(M_0)/[T]$ is surjective. We first prove the following lemma.

Lemma 6.27. *An element w' with a reduced expression $\boldsymbol{w}' = s_{u_2} \cdots s_{u_l}$ is a $(s_{u_1}\boldsymbol{c}s_{u_1})$-sortable element in $W_{Q'}$.*

Proof. It is clear that $s_{u_1}\boldsymbol{c}s_{u_1}$ is a Coxeter element of $W_{Q'}$ admissible with respect to the orientation of Q'. Let $a = \max\{k \mid u_1 \in \mathrm{Supp}(c^{(k)})\}$. Put

$$\boldsymbol{c}'^{(k)} = \begin{cases} s_{u_1}\boldsymbol{c}^{(k)}s_{u_1} & 0 \le k \le a-1 \\ s_{u_1}\boldsymbol{c}^{(k)} & k = a \\ \boldsymbol{c}^{(k)} & a+1 \le k \le m. \end{cases}$$

Then we have a reduced expression $\boldsymbol{w}' = \boldsymbol{c}'^{(0)}\boldsymbol{c}'^{(1)} \cdots \boldsymbol{c}'^{(m')}$, where $m' = m-1$ if $\mathrm{Supp}(c^{(m)}) = \{u_1\}$, and $m' = m$ if otherwise. Since each $\boldsymbol{c}'^{(k)}$ is a subword of $s_{u_1}\boldsymbol{c}s_{u_1}$, w' is a $(s_{u_1}\boldsymbol{c}s_{u_1})$-sortable element. □

Let $\boldsymbol{w}' = s_{u_2} \cdots s_{u_l}$. By Proposition 6.23 (c), there exists the isomorphism of graded Π'-modules

$$\psi : \mathbb{G}(M/M^1) \xrightarrow{\sim} M'.$$

By using ψ, we have an isomorphism of algebras

$$\alpha : \mathrm{End}^{\mathbb{Z}}_{\Pi'}(\mathbb{G}(M/M^1)) \to \mathrm{End}^{\mathbb{Z}}_{\Pi'}(M')$$

defined by $\alpha(f) = \psi \circ f \circ \psi^{-1}$. Moreover we have an isomorphism of algebras

$$\alpha_0 : \mathrm{End}_{KQ'}(\mathbb{G}(M/M^1)_0) \to \mathrm{End}_{KQ'}(M'_0)$$

defined by $\alpha_0(f) = \psi_0 \circ f \circ \psi_0^{-1}$, where $\psi_0 = \psi|_{\mathbb{G}(M/M^1)_0}$. Let

$$F_{>1} := \mathbb{F}_{M/M^1, M/M^1} : \mathrm{End}^{\mathbb{Z}}_{\Pi}(M/M^1) \to \mathrm{End}_{KQ}\left((M/M^1)_0\right).$$

Lemma 6.28. *The following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{End}^{\mathbb{Z}}_{\Pi}(M/M^1) \arrow[r, "G_{>1}"] \arrow[d, "F_{>1}"] & \mathrm{End}^{\mathbb{Z}}_{\Pi'}(\mathbb{G}(M/M^1)) \arrow[r, "\alpha", "\sim"'] \arrow[d, "\overline{F}'"] & \mathrm{End}^{\mathbb{Z}}_{\Pi'}(M') \arrow[d, "F'"] \\
\mathrm{End}_{KQ}((M/M^1)_0) \arrow[r, "R", "\sim"'] & \mathrm{End}_{KQ'}(\mathbb{G}(M/M^1)_0) \arrow[r, "\alpha_0", "\sim"'] & \mathrm{End}_{KQ'}(M'_0),
\end{tikzcd} \tag{6.6}$$

where $G_{>1} = \mathbb{G}_{M/M^1, M/M^1}$, $\overline{F}' = \mathbb{F}'_{\mathbb{G}(M/M^1), \mathbb{G}(M/M^1)}$, and R is defined by $R(f) = (\phi_{M/M^1})^{-1} \circ R^+_{u_1}(f) \circ \phi_{M/M^1}$.

Proof. The commutativity of the left square comes from the functoriality of ϕ of Lemma 6.26. The commutativity of the right square is clear. □

Proposition 6.29. *Assume that* $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l} = \boldsymbol{c}^{(0)}\boldsymbol{c}^{(1)}\cdots\boldsymbol{c}^{(m)}$ *is a c-sortable expression of* $w \in W_Q$*. Then we have*

(a) The morphism $F : \operatorname{End}^{\mathbb{Z}}_{\Pi(w)}(M) \to \operatorname{End}_{KQ}(M_0)$, $f \mapsto f|_{M_0}$ *is surjective.*

(b) The morphism $\underline{F} : \underline{\operatorname{End}}^{\mathbb{Z}}_{\Pi(w)}(M) \to \operatorname{End}_{KQ}(M_0)/[T]$ *is surjective.*

Proof. (a) We show the assertion by induction on l. Assume that $l = 1$. Then we have $M = M^1 = M^1_0$ and $\Pi(w) = KQ$. Thus we have $\operatorname{End}^{\mathbb{Z}}_{\Pi(w)}(M) = \operatorname{End}_{KQ}(M_0)$. The assertion holds. Assume that $l \geq 2$. We show that two maps

$$F_1 := \mathbb{F}_{M^1,M} : \operatorname{Hom}^{\mathbb{Z}}_{\Pi(w)}(M^1, M) \to \operatorname{Hom}_{KQ}(M^1_0, M_0),$$
$$\mathbb{F}_{M/M^1,M} : \operatorname{Hom}^{\mathbb{Z}}_{\Pi(w)}(M/M^1, M) \to \operatorname{Hom}_{KQ}\big((M/M^1)_0, M_0\big)$$

are surjective. Since $M^1 = M^1_0$, M is in $\mathsf{mod}^{\leq 0}\,\Pi(w)$, and Π_w is positively graded, we can regard any $g \in \operatorname{Hom}_{KQ}(M^1_0, M_0)$ as a morphism in $\mathsf{mod}^{\mathbb{Z}}\,\Pi_w$. Therefore, F_1 is surjective.

By [AIRT, Corollary 3.10], we have $\operatorname{Hom}_{KQ}(M^j_0, M^i_0) = 0$ for $i < j$. Thus we have $\operatorname{Hom}_{KQ}\big((M/M^1)_0, M_0\big) = \operatorname{End}_{KQ}\big((M/M^1)_0\big)$. Therefore it is enough to show that the map

$$F_{>1} := \mathbb{F}_{M/M^1,M/M^1} : \operatorname{End}^{\mathbb{Z}}_{\Pi(w)}(M/M^1) \to \operatorname{End}_{KQ}\big((M/M^1)_0\big)$$

is surjective. We show that $F_{>1}$ is surjective by using the diagram (6.6). Let $\boldsymbol{w}' = s_{u_2}\cdots s_{u_l}$. By Lemma 6.27 (c), w' is a $(s_{u_1}\boldsymbol{c}s_{u_1})$-sortable element in $W_{Q'}$. Thus, by the inductive hypothesis, F' in the diagram (6.6) is surjective. By Theorem 6.25, $G_{>1}$ is surjective. Since α, α_0, and R are isomorphism, $F_{>1}$ is surjective.

(b) We have the following commutative diagram

$$\begin{tikzcd} \operatorname{End}^{\mathbb{Z}}_{\Pi(w)}(M) \arrow[r, "\pi"] \arrow[d, "F"] & \underline{\operatorname{End}}^{\mathbb{Z}}_{\Pi(w)}(M) \arrow[d, "\underline{F}"] \\ \operatorname{End}_{KQ}(M_0) \arrow[r, "\pi'"] & \operatorname{End}_{KQ}(M_0)/[T]. \end{tikzcd} \tag{6.7}$$

Since the bottom and the left morphisms are surjective, the right morphism is surjective. □

6.4 $\underline{\mathbf{F}}$ is injective

We show that the morphism $\underline{F}$ is injective. Let $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ be a c-sortable expression and $\boldsymbol{w}' = s_{u_2}\cdots s_{u_l}$. Without loss of generality by Lemma 4.1, we assume that $Q_0 = \operatorname{Supp}(w)$. Since $\mathbb{G}(M^1) = 0$ and by Lemma 6.28, we have the following commutative diagram:

$$\begin{tikzcd} \operatorname{End}^{\mathbb{Z}}_{\Pi(w)}(M) \arrow[r, "G"] \arrow[d, "F"] & \operatorname{End}^{\mathbb{Z}}_{\Pi'(w')}(M') \arrow[d, "F'"] \\ \operatorname{End}_{KQ}(M_0) \arrow[r, "\overline{R}"] & \operatorname{End}_{KQ'}(M'_0), \end{tikzcd} \tag{6.8}$$

where $\overline{R} = \alpha_0 \circ R$.

Lemma 6.30. *Let $f \in \mathrm{End}^{\mathbb{Z}}_{\Pi_w}(M)$. Assume that $G(f)$ factors through* $\mathsf{add}\, P'$. *Then we have*

(a) f factors through $\mathsf{add}(P \oplus M^1)$.

(b) If $F(f) = 0$, then f factors through $\mathsf{add}(P)$.

Proof. (a) By Proposition 6.23 (d), we have $\mathbb{G}(P) = P'$. Since $G(f)$ factors through $\mathsf{add}\, P'$ and by Theorem 6.25 and Lemma 6.24, there exist morphisms $f_1, g \in \mathrm{End}^{\mathbb{Z}}_{\Pi(w)}(M)$ such that $f = f_1 + g$, f_1 factors through $\mathsf{add}\, P$, and g factors through $\mathsf{add}\{M^1(i) \mid i \geq 0\}$. Thus g is the sum of morphisms $g_1, g_2 \in \mathrm{End}^{\mathbb{Z}}_{\Pi(w)}(M)$ such that g_1 factors through $\mathsf{add}\, M^1$ and g_2 factors through $\mathsf{add}\{M^1(i) \mid i \geq 1\}$. By Lemma 6.13, g_2 factors through $\mathsf{add}\, P$.

(b) By (a), there exists $g \in \mathrm{End}^{\mathbb{Z}}_{\Pi(w)}(M)$ such that g factors through $\mathsf{add}\, M^1$ and $f - g$ factors through $\mathsf{add}\, P$. We show that g factors through $\mathsf{add}\, P$. Since $\mathrm{Hom}^{\mathbb{Z}}_{\Pi(w)}(M/M^1, M^1) = 0$, we have $g|_{M/M^1} = 0$. Therefore we may regard g as a morphism from M^1 to M. Since $F(f) = 0$, $F(g - f) = F(g) : M^1_0 \to M_0$ factors through $\mathsf{add}\, P_0$. By Proposition 6.29 (a), there exists $h \in \mathrm{Hom}^{\mathbb{Z}}_{\Pi(w)}(M^1, M)$ such that h factors through $\mathsf{add}\, P$ and $F(g) = F(h)$. Because $M^1 = M^1_0$, we have $g = h$. $\square$

Proposition 6.31. *The morphism $\underline{F} : \underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M) \to \mathrm{End}_{KQ}(M_0)/[T]$ is injective.*

Proof. We show the assertion by induction on l. If $l = 1$, then we have $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M) = \mathrm{End}_{KQ}(M_0)/[T] = 0$. Thus the claim is clear.

Assume that $l \geq 2$. Let f be a morphism in $\mathrm{End}^{\mathbb{Z}}_{\Pi(w)}(M)$ satisfying $\underline{F}(\pi(f)) = 0$. We show $\pi(f) = 0$. By the commutative diagram (6.7), we have $\pi'(F(f)) = 0$. Since $\mathrm{Ker}\, \pi' = [T]$, $F(f)$ factors through $\mathsf{add}\, T$. By Proposition 6.29 (a) and $\mathbb{F}(P) = T$, there exists $g \in \mathrm{End}^{\mathbb{Z}}_{\Pi(w)}(M)$ such that g factors through $\mathsf{add}\, P$ and $F(f) = F(g)$. Put $h := f - g \in \mathrm{End}^{\mathbb{Z}}_{\Pi(w)}(M)$. We have $\pi(f) = \pi(h)$. Therefore it is enough to show $\pi(h) = 0$.

Consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{End}^{\mathbb{Z}}_{\Pi(w)}(M) \arrow[r, "G"] \arrow[d, "F"] & \mathrm{End}^{\mathbb{Z}}_{\Pi'(w')}(M') \arrow[r, "\eta"] \arrow[d, "F'"] & \underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi'(w')}(M') \arrow[d, "\underline{F'}"] \\
\mathrm{End}_{KQ}(M_0) \arrow[r, "\overline{R}"] & \mathrm{End}_{KQ'}(M'_0) \arrow[r, "\eta'"] & \mathrm{End}_{KQ'}(M'_0)/[T'],
\end{tikzcd}$$

where η and η' are canonical surjections. We have

$$\underline{F'}(\eta(G(h))) = \eta'(F'(G(h))) = \eta'(R(F(h))) = 0,$$

since $F(h) = F(f) - F(g) = 0$. By the inductive hypothesis, $\underline{F'}$ is injective. Thus $\eta(G(h)) = 0$ and $G(h)$ factors through a graded projective $\Pi'(w')$-module. By the proof of Proposition 6.15, $G(h)$ factors through $\mathsf{add}\, P'$. Thus, by Lemma 6.30 (b), h factors through $\mathsf{add}\, P$. Therefore, we have $\pi(f) = \pi(h) = 0$. $\square$

7 The global dimension of the endomorphism algebra

Throughout this section, let A be a finite dimensional algebra and T a *cotilting* A-module of finite injective dimension, that is, T satisfies $\operatorname{injdim} T < \infty$, $\operatorname{Ext}^i_A(T,T) = 0$ for any $i > 0$, and there exists an exact sequence $0 \to T_r \to \cdots \to T_1 \to T_0 \to \mathrm{D}\,A \to 0$ where $T_i \in \mathsf{add}\,T$. We denote by ${}^{\perp_{>0}}T$ the full subcategory consisting of $\mathsf{mod}\,A$ of modules X satisfying $\operatorname{Ext}^i_A(X,T) = 0$ for any $i > 0$. The aim of this section is to show the following theorem.

Theorem 7.1. *Assume that the global dimension of A is at most n and that ${}^{\perp_{>0}}T$ has an additive generator M. Then the global dimension of $\operatorname{End}_A(M)/[T]$ is at most $3n-1$.*

Note that $\operatorname{End}_A(M)$ and $\operatorname{End}_A(M)/[T]$ are relative version of Auslander algebras and stable Auslander algebras. It is known that Auslander algebras have global dimension at most two [ARS], and that stable Auslander algebras have global dimension at most $3(\operatorname{gldim} A) - 1$ [AR74, Proposition 10.2]. We apply Theorem 7.1 to our endomorphism algebra in Theorem 6.3. We denote by $\mathsf{Sub}\,T$ the full subcategory of $\mathsf{mod}\,A$ consisting of submodules of finite direct sums of T.

Corollary 7.2. *Under the setting in Theorem 6.3, the global dimension of $\operatorname{End}_{KQ}(M_0)/[T]$ is at most two.*

Proof. Let $Q^{(1)}$ be the full subquiver of Q whose the set of vertices is $\operatorname{Supp}(w)$. We have $\operatorname{End}_{KQ}(M_0)/[T] = \operatorname{End}_{KQ^{(1)}}(M_0)/[T]$. Moreover, by Theorem 4.9, T is a tilting $KQ^{(1)}$-module. By [AIRT, Theorem 3.11], we have $\mathsf{Sub}\,T = \mathsf{add}\{M_0^1, M_0^2, \dots, M_0^l\}$. By Bongartz's lemma [ASS, Chapter VI, 2.4. Lemma], tilting modules over a hereditary algebra coincide with cotilting modules. Since $KQ^{(1)}$ is hereditary, $\mathsf{Sub}\,T = {}^{\perp_{>0}}T$ holds. Therefore, by applying Theorem 7.1, the global dimension of $\operatorname{End}_{KQ^{(1)}}(M_0)/[T]$ is at most two. $\square$

To show Theorem 7.1, we use cotilting theory. We recall some properties of cotilting modules.

Proposition 7.3. *[AR91, Theorem 5.4, Proposition 5.11] Let T be a cotilting A-module. Then*

(a) For any $X \in {}^{\perp_{>0}}T$, there exists an injective left $(\mathsf{add}\,T)$-approximation of X.

(b) Let $X \in {}^{\perp_{>0}}T$. Then $X \in \mathsf{add}\,T$ if and only if $\operatorname{Ext}^1_A(Y,X) = 0$ for any $Y \in {}^{\perp_{>0}}T$.

In the following lemma and proposition, we construct an important long exact sequence. For $X, Y \in \mathsf{mod}\,A$, we denote by $\overline{\operatorname{Hom}}^T_A(X,Y)$ the quotient of $\operatorname{Hom}_A(X,Y)$ by the subspace consisting of morphisms factoring through $\mathsf{add}\,T$, that is, $\overline{\operatorname{Hom}}^T_A(X,Y) = \operatorname{Hom}_A(X,Y)/[T]$.

Lemma 7.4. *For an exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in ${}^{\perp_{>0}}T$ and any A-module N, we have the following exact sequence*

$$\overline{\operatorname{Hom}}^T_A(Z,N) \xrightarrow{-\circ g} \overline{\operatorname{Hom}}^T_A(Y,N) \xrightarrow{-\circ f} \overline{\operatorname{Hom}}^T_A(X,N).$$

Proof. It is enough to show that $\mathrm{Ker}(-\circ f) \subset \mathrm{Im}(-\circ g)$. Assume that $\alpha \in \mathrm{Hom}_A(Y,N)$ satisfies $f\alpha = 0 \in \mathrm{Hom}_A(X,N)/[T]$. There exists a module $T' \in \mathsf{add}\, T$ and morphisms $h_1 : X \to T'$, $h_2 : T' \to N$ such that $f\alpha = h_1 h_2$. Since $\mathrm{Ext}^1_A(Z,T) = 0$, there exists a morphism $\beta : Y \to T'$ such that $f\beta = h_1$. Since $f(\alpha - \beta h_2) = f\alpha - f\beta h_2 = f\alpha - h_1 h_2 = 0$, there exists a morphism $\gamma : Z \to N$ such that $g\gamma = \alpha - \beta h_2$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h_1"'] & Y \arrow[r, "g"] \arrow[d, "\alpha"] \arrow[dl, dashed, "\beta"'] & Z \arrow[dl, dashed, "\gamma"] \\ T' \arrow[r, "h_2"'] & N. & \end{tikzcd}$$

$\square$

Let $X \in {}^{\perp_{>0}}T$. By Proposition 7.3 (b), there exists an injective left $(\mathsf{add}\, T)$-approximation $f : X \to T'$. We have $\mathrm{Cok}\, f \in {}^{\perp_{>0}}T$. We denote by $\Omega_T^-(X)$ a cokernel of f. Note that $\Omega_T^-(X)$ is uniquely determined by X up to direct summands in $\mathsf{add}\, T$. Let $\Omega_T^{-n}(X) = \Omega_T^-(\Omega_T^{-(n-1)}(X))$ for $n > 1$.

Proposition 7.5. *Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an exact sequence in ${}^{\perp_{>0}}T$. Then*

(a) We have an exact sequence $0 \to Y \to Z \oplus T' \to \Omega_T^-(X) \to 0$, where $T' \in \mathsf{add}\, T$.

(b) For any A-module N, we have the following long exact sequence

$$\cdots \to \overline{\mathrm{Hom}}_A^T(\Omega_T^{-n}(Z), N) \to \overline{\mathrm{Hom}}_A^T(\Omega_T^{-n}(Y), N) \to \overline{\mathrm{Hom}}_A^T(\Omega_T^{-n}(X), N) \to \cdots$$
$$\cdots \to \overline{\mathrm{Hom}}_A^T(\Omega_T^{-}(X), N) \to \overline{\mathrm{Hom}}_A^T(Z, N) \to \overline{\mathrm{Hom}}_A^T(Y, N) \to \overline{\mathrm{Hom}}_A^T(X, N).$$

Proof. (a) Let $h : X \to T'$ be an injective left $(\mathsf{add}\, T)$-approximation of X. Since $\mathrm{Ext}^1_A(Z,T) = 0$, h factors through f, and therefore we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X \arrow[r] & T' \arrow[r] & \Omega_T^-(X) \arrow[r] & 0. \end{tikzcd}$$

Thus we have an exact sequence $0 \to Y \to Z \oplus T' \to \Omega_T^-(X) \to 0$.

(b) By applying (a) and Lemma 7.4 inductively, we have the assertion. $\square$

In the following two propositions, we assume that the global dimension of A is at most n.

Proposition 7.6. *Let $X \in {}^{\perp_{>0}}T$. If the global dimension of A is at most n, then we have $\Omega_T^{-n}(X) \in \mathsf{add}\, T$.*

Proof. By Proposition 7.3 (b), it is enough to show that $\mathrm{Ext}^1_A(Y, \Omega_T^{-n}(X)) = 0$ for any $Y \in {}^{\perp_{>0}}T$. Let $Y \in {}^{\perp_{>0}}T$. By using Proposition 7.3 (a), we have the following exact sequence

$$0 \to X \to T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} \cdots \to T_{n-1} \xrightarrow{f_{n-1}} \Omega_T^{-n}(X) \to 0,$$

where $T_i \in \mathsf{add}\, T$ and $\operatorname{Im} f_i = \Omega_T^{-(i+1)}(X)$. By applying $\operatorname{Hom}_A(Y, -)$ to this exact sequence, we have the following isomorphisms

$$\begin{aligned}\operatorname{Ext}_A^1(Y, \Omega_T^{-n}(X)) &\simeq \operatorname{Ext}_A^2(Y, \Omega_T^{-(n-1)}(X)) \\ &\simeq \operatorname{Ext}_A^3(Y, \Omega_T^{-(n-2)}(X)) \\ &\cdots \\ &\simeq \operatorname{Ext}_A^{n+1}(Y, X) = 0,\end{aligned}$$

where the last equation follows from $\operatorname{gldim} A \leq n$. □

Proposition 7.7. *Assume that the global dimension of A is at most n. For an exact sequence $0 \to X \to Y \to Z \to 0$ in ${}^{\perp_{>0}}T$ and any A-module N, we have the following exact sequence*

$$\begin{aligned}0 \to \overline{\operatorname{Hom}}_A^T(\Omega_T^{-(n-1)}(Z), N) \to \overline{\operatorname{Hom}}_A^T(\Omega_T^{-(n-1)}(Y), N) \to \overline{\operatorname{Hom}}_A^T(\Omega_T^{-(n-1)}(X), N) \to \cdots \\ \cdots \to \overline{\operatorname{Hom}}_A^T(\Omega_T^{-}(X), N) \to \overline{\operatorname{Hom}}_A^T(Z, N) \to \overline{\operatorname{Hom}}_A^T(Y, N) \to \overline{\operatorname{Hom}}_A^T(X, N).\end{aligned}$$

Proof. By Proposition 7.6, we have $\overline{\operatorname{Hom}}_A^T(\Omega_T^{-n}(X), N) = 0$. Therefore, we have a desired exact sequence by Proposition 7.5 (b). □

Then we prove Theorem 7.1.

Proof of Theorem 7.1. We show that projective dimensions of all right $\operatorname{End}_A(M)/[T]$-modules are at most $3n - 1$. Let N be a right $\operatorname{End}_A(M)/[T]$-module. There exist $X, Y \in {}^{\perp_{>0}}T$ and a homomorphism of A-modules $f : X \to Y$ which induce a minimal projective presentation of N,

$$\overline{\operatorname{Hom}}_A^T(Y, M) \xrightarrow{-\circ f} \overline{\operatorname{Hom}}_A^T(X, M) \to N \to 0.$$

Let $g : X \to T'$ be an injective left $(\mathsf{add}\, T)$-approximation of X. We have an injective morphism $h = f \oplus g : X \to Y \oplus T'$. Since g is a left $(\mathsf{add}\, T)$-approximation of X, we have $\operatorname{Cok} h \in {}^{\perp_{>0}}T$. Let $Z = \operatorname{Cok} h$. We have an exact sequence $0 \to X \xrightarrow{h} Y \oplus T' \to Z \to 0$. By Proposition 7.7, we have the following exact sequence

$$\begin{aligned}0 \to \overline{\operatorname{Hom}}_A^T(\Omega_T^{-(n-1)}(Z), N) \to \overline{\operatorname{Hom}}_A^T(\Omega_T^{-(n-1)}(Y), N) \to \overline{\operatorname{Hom}}_A^T(\Omega_T^{-(n-1)}(X), N) \to \cdots \\ \cdots \to \overline{\operatorname{Hom}}_A^T(\Omega_T^{-}(X), N) \to \overline{\operatorname{Hom}}_A^T(Z, N) \to \overline{\operatorname{Hom}}_A^T(Y, N) \xrightarrow{-\circ f} \overline{\operatorname{Hom}}_A^T(X, N).\end{aligned}$$

Therefore the projective dimension of N is at most $3n - 1$. □

Part II
Tilting objects associated to c-starting and c-ending elements

This part is based on the paper [Ki16]. Throughout this part, we use the notation introduced in Section 3.

8 Preliminary

In this section, we recall basic facts and show basic lemmas on graded algebras which we will use.

8.1 Graded algebras

In this subsection, we observe some properties of graded algebras. Recall that a graded algebra $A = \bigoplus_{i\in\mathbb{Z}} A_i$ is said to be *positively graded* if $A_i = 0$ for any $i < 0$. The following lemma is well-known and we omit the proof.

Lemma 8.1. *Let A be a finite dimensional graded algebra and let M, N be finitely generated indecomposable graded A-modules. If M is isomorphic to N in $\mathsf{mod}\, A$, then there exists an integer i such that $M(i)$ is isomorphic to N in $\mathsf{mod}^{\mathbb{Z}} A$.*

We need the following lemma later.

Lemma 8.2. *Let A be a finite dimensional positively graded algebra such that the global dimension of A_0 is at most m. Let $M \in \mathsf{mod}^{\geq 0} A$ and*

$$\cdots \to P^2 \xrightarrow{f^2} P^1 \xrightarrow{f^1} P^0 \xrightarrow{f^0} M \to 0 \tag{8.1}$$

be a minimal projective resolution of M in $\mathsf{mod}^{\mathbb{Z}} A$. Then we have $\mathrm{Ker}(f^m)_0 = 0$.

Proof. We show that, by taking the degree zero part of (8.1), we have a minimal projective resolution of M_0 in $\mathsf{mod}\ A_0$. Since A is positively graded and $M \in \mathsf{mod}^{\geq 0} A$, $P^i \in \mathsf{mod}^{\geq 0} A$ holds for each $i \geq 0$. Thus $(P^i)_0$ is either a projective A_0-module or a zero module for any $i \geq 0$. Therefore the degree zero part of (8.1) gives a projective resolution of M_0 in $\mathsf{mod}\, A_0$. Next we show a minimality, that is, for each $i \geq 0$, $\mathrm{Ker}(f^i)_0$ is a superfluous A_0-submodule of $(P^i)_0$. Let L be an A_0-submodule of $(P^i)_0$ satisfying $L + \mathrm{Ker}(f^i)_0 = (P^i)_0$. There exists an exact sequence $0 \to (P^i)_{\geq 1} \to P^i \to (P^i)_0 \to 0$. By taking a pull-back diagram of $P^i \to (P^i)_0 \hookleftarrow L$, we have an A-module N which is a submodule of P^i and satisfies $N_0 = L$ and $N_{\geq 1} = (P^i)_{\geq 1}$. This implies $N + \mathrm{Ker}(f^i) = P^i$. Therefore we have $N = P^i$ and $L = (P^i)_0$.

Since the global dimension of A_0 is at most m and the degree zero part of (8.1) is a minimal projective resolution of M_0 in $\mathsf{mod}\, A_0$, we have $\mathrm{Ker}(f^m)_0 = 0$. $\square$

We use the following definition in Section 11.

Definition 8.3. Let A, B, C be graded algebras.

(1) We define a grading on the tensor algebra $A \otimes B$ as follows:

$$(A \otimes B)_i = \left\{ \sum a \otimes b \mid a \in A_j,\, b \in B_k,\, j + k = i \right\},$$

for any $i \in \mathbb{Z}$.

(2) Let X be a graded $A \otimes B^{\mathrm{op}}$-module and Y a graded $B \otimes C^{\mathrm{op}}$-module. We define a grading on the $A \otimes C^{\mathrm{op}}$-module $X \otimes_B Y$ as follows:

$$(X \otimes_B Y)_i = \left\{ \sum x \otimes y \mid x \in X_j,\, y \in Y_k,\, j + k = i \right\},$$

for any $i \in \mathbb{Z}$.

8.2 Some results on $\Pi(w)$ and $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$

Let $w \in W_Q$. In this subsection, we show some results on $\Pi(w)$ and the category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$. In particular, we show that $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$ has a Serre functor. The following lemma is an easy observation of the grading on $\Pi(w)$.

Lemma 8.4. *The following holds.*

(a) Let $c \in W_Q$ be the Coxeter element. We have $I(c)_0 = 0$.

(b) Let w be an element of W_Q. If there exists a reduced expression $\boldsymbol{w}$ of w containing an expression of the Coxeter element c as a subword, then we have $I(w)_0 = 0$. In particular, we have $\Pi(w)_0 = KQ$.

(c) Let $\boldsymbol{w} = s_{u_1} s_{u_2} \cdots s_{u_l}$ be a reduced expression of $w \in W_Q$ which is a subword of an expression of the Coxeter element c of W_Q. Then we have $e_{u_1} I(w)_0 e_{u_l} = 0$.

Proof. (a) This comes from [BIRSc, Proposition Ⅲ. 3.2].

(b) By (a), we have $I(w)_0 \subset I(c)_0 = 0$.

(c) We have $e_{u_1} I(w)_0 e_{u_l} \subset e_{u_1} I(c)_0 e_{u_l} = 0$. □

We need the following two observations.

Lemma 8.5. *The category $\mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$ is an extension closed subcategory of $\mathsf{fd}^{\mathbb{Z}}\Pi$.*

Proof. By [BIRSc, Proposition Ⅲ. 2.3] (a), $\mathsf{Sub}\,\Pi(w)$ is an extension closed subcategory of $\mathsf{fd}\,\Pi$. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathsf{fd}^{\mathbb{Z}}\Pi$ and $X, Z \in \mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$. Since $Y \in \mathsf{Sub}\,\Pi(w)$ and $Y \in \mathsf{fd}^{\mathbb{Z}}\Pi$, we have $Y \in \mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$. □

Proposition 8.6. *Let Q be a non-Dynkin quiver and Π be the preprojective algebra of Q. Put $\Pi^e = \Pi \otimes \Pi^{\mathrm{op}}$. Let $\mathcal{D} = \mathsf{D}(\mathsf{Mod}^{\mathbb{Z}}\,\Pi)$ be the derived category of $\mathsf{Mod}^{\mathbb{Z}}\,\Pi$ and X, Y in $\mathcal{D}$. Then the following holds.*

(a) $R\,\mathrm{Hom}_{\Pi^e}(\Pi, \Pi^e) \simeq \Pi[-2](1)$ holds in $\mathsf{D}(\mathsf{Mod}^{\mathbb{Z}}\,\Pi^e)$.

(b) If the homology of X is of finite total dimension, then we have a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) \simeq \mathrm{D}\,\mathrm{Hom}_{\mathcal{D}}(Y, X[2](-1)).$$

Proof. (a) By [GLS07, Section 8], we have a graded Π^e-module resolution of Π:

$$0 \to \bigoplus_{u \in Q_0} (\Pi e_u \otimes e_u \Pi)(-1) \to \bigoplus_{\beta \in \overline{Q}_1} \left(\Pi e_{s(\beta)} \otimes e_{t(\beta)} \Pi\right)(-\deg \beta)$$
$$\to \bigoplus_{u \in Q_0} (\Pi e_u \otimes e_u \Pi) \to \Pi \to 0.$$

This resolution induces the desired isomorphism.

(b) This follows from (a) and [Ke08, Lemma 4.1]. □

Then we have a Serre functor of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$.

Proposition 8.7. *For any $w \in W_Q$, the triangulated category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$ has a Serre functor $[2](-1)$.*

Proof. By Lemma 4.1, we assume that Q is a non-Dynkin quiver. By Lemma 8.5, $\mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$ is an extension closed full subcategory in $\mathsf{fd}^{\mathbb{Z}}\Pi$. Thus we have $\operatorname{Ext}^1_{\mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)}(X,Y) = \operatorname{Ext}^1_{\mathsf{Mod}^{\mathbb{Z}}\,\Pi}(X,Y)$ for $X, Y \in \mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$. Therefore we have

$$\begin{aligned}
\operatorname{Hom}_{\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)}(X, Y[1]) &\simeq \operatorname{Ext}^1_{\mathsf{Mod}^{\mathbb{Z}}\,\Pi}(X,Y)\\
&\simeq \mathrm{D}\operatorname{Ext}^1_{\mathsf{Mod}^{\mathbb{Z}}\,\Pi}(Y, X(-1))\\
&\simeq \mathrm{D}\operatorname{Hom}_{\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)}(Y, X[1](-1)),
\end{aligned}$$

for X, Y in $\mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$, where the second isomorphism comes from Proposition 8.6. This means that $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ has a Serre functor $[2](-1)$. □

We need one result of Iwanaga-Gorenstein algebras. The next theorem is the famous shown in [Bu, Ha91, Ri] and its graded version in the case of injective dimension at most one [IYa]. For a finite dimensional (resp, graded) algebra A, we denote by $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ (resp, $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}^{\mathbb{Z}}\,A)$) the homotopy category of bounded complexes of finitely generated (resp, graded) projective A-modules.

Theorem 8.8. *Let A be an Iwanaga-Gorenstein algebra of dimension at most one. Then the following holds.*

(a) There exists a triangle equivalence

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)/\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A) \xrightarrow{\sim} \underline{\mathsf{Sub}}\,A,$$

where a quasi-inverse of this equivalence is induced from the composite of the canonical functors $\mathsf{Sub}\,A \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)/\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$.

(b) If A is a graded algebra. Then we have the following triangle equivalence

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}}\,A)/\mathsf{K}^{\mathrm{b}}(\mathsf{proj}^{\mathbb{Z}}\,A) \xrightarrow{\sim} \underline{\mathsf{Sub}}^{\mathbb{Z}}A.$$

where a quasi-inverse of this equivalence is induced from the composite of the canonical functors $\mathsf{Sub}^{\mathbb{Z}}\,A \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}}\,A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}}\,A)/\mathsf{K}^{\mathrm{b}}(\mathsf{proj}^{\mathbb{Z}}\,A)$.

Note that categories $\underline{\mathsf{Sub}}\, A$ and $\underline{\mathsf{Sub}}^{\mathbb{Z}}\, A$ for an Iwanaga-Gorenstein algebra A of dimension at most one are often called singularity categories. We denote by ρ_A the composite of triangle functors

$$\rho_A : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\ A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\ A)/\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\ A) \xrightarrow{\sim} \underline{\mathsf{Sub}}A,$$

and denote by $\rho_A^{\mathbb{Z}}$ the graded version of ρ_A if A is a graded algebra.

9 A silting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$

In this section, we show that the category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$ has a silting object for any $w \in W_Q$. For the definition of silting objects, see Subsection 3.3. In Subsection 9.1, we study a more general triangulated category than $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$.

9.1 Cluster tilting subcategories and thick subcategories

In this subsection, let $\mathcal{T}$ be a Hom-finite, Krull-Schmidt triangulated category with a Serre functor $\mathbb{S}$. Put $\mathbb{S}_2 = \mathbb{S} \circ [-2]$. We denote by $\mathcal{T}/\mathbb{S}_2$ the orbit category of $\mathcal{T}$ associated with $\mathbb{S}_2$. For any object M of $\mathcal{T}$, we regard the endomorphism algebra $\mathrm{End}_{\mathcal{T}/\mathbb{S}_2}(M)$ as a graded algebra by $\mathrm{End}_{\mathcal{T}/\mathbb{S}_2}(M)_i = \mathrm{Hom}_{\mathcal{T}}(M, \mathbb{S}_2^{-i}(M))$. For a subcategory $\mathcal{C}$ of $\mathcal{T}$, put $\mathcal{C}^{\perp} = \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(\mathcal{C}, X) = 0\}$ and ${}^{\perp}\mathcal{C} = \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(X, \mathcal{C}) = 0\}$.

A subcategory $\mathcal{C}$ of $\mathcal{T}$ is called a *contravariantly finite subcategory* of $\mathcal{T}$ if for any $X \in \mathcal{T}$, there exists a morphism $f : Y \to X$ with $Y \in \mathcal{C}$ such that the map $\mathrm{Hom}_{\mathcal{T}}(Z, f) : \mathrm{Hom}_{\mathcal{T}}(Z, Y) \to \mathrm{Hom}_{\mathcal{T}}(Z, X)$ is surjective for any $Z \in \mathcal{C}$. Dually, we define a *covariantly finite subcategory* of $\mathcal{T}$. We call $\mathcal{C}$ a *functorially finite subcategory* of $\mathcal{T}$ if $\mathcal{C}$ is a contravariantly and covariantly finite subcategory of $\mathcal{T}$.

We recall the definition of cluster tilting subcategories.

Definition 9.1. [IYo] Let $\mathcal{C}$ be a subcategory of $\mathcal{T}$. We call $\mathcal{C}$ a *cluster tilting subcategory* of $\mathcal{T}$ if $\mathcal{C}$ is a functorially finite subcategory of $\mathcal{T}$ and

$$\mathcal{C} = \mathcal{C}[-1]^{\perp} = {}^{\perp}\mathcal{C}[1].$$

We recall the following property of cluster tilting subcategories.

Proposition 9.2. *[IYo, Theorem 3.1] If $\mathcal{C}$ is a cluster tilting subcategory of $\mathcal{T}$, then for any object X of $\mathcal{T}$, there exists a triangle $C_0 \to X \to C_1[1] \to C_0[1]$ with $C_0, C_1 \in \mathcal{C}$.*

We recall some definitions. We denote by $J_{\mathcal{T}}$ the *Jacobson radical* of $\mathcal{T}$. We call a morphism $f : X \to Y$ in $\mathcal{T}$ *right minimal* if f does not have a direct summand of the form $X' \to 0$ for some $X' \in \mathcal{T}$. Let $\mathcal{C}$ be a full subcategory of $\mathcal{T}$. A morphism $f : X \to Y$ in $\mathcal{C}$ is called a *right minimal almost split morphism* of Y in $\mathcal{C}$ if the following three conditions are satisfied:

(i) f is not a retraction.

(ii) f induces a surjective map $\mathrm{Hom}_{\mathcal{T}}(Z, X) \to J_{\mathcal{T}}(Z, Y)$ for any $Z \in \mathcal{C}$.

(iii) f is right minimal.

Dually, a *left minimal almost split morphism* is defined.

Note that if there exists a left (resp, right) minimal almost split morphism of Y in $\mathcal{C}$, then it is unique up to isomorphism. We use the following theorem.

Theorem 9.3. *[IYo, Theorem 3.10] Let $\mathcal{C}$ be a cluster tilting subcategory of $\mathcal{T}$ and X be an indecomposable object of $\mathcal{C}$. Then there exist triangles*

$$\mathbb{S}_2(X) \xrightarrow{g} C_1 \to Y \to \mathbb{S}_2(X)[1], \quad Y \to C_0 \xrightarrow{f} X \to Y[1], \tag{9.1}$$

where f is a right minimal almost split morphism in $\mathcal{C}$ and g is a left minimal almost split morphism in $\mathcal{C}$. Dually, there exist triangles

$$X \xrightarrow{g'} C^0 \to Z \to X[1], \quad Z \to C^1 \xrightarrow{f'} \mathbb{S}_2^{-1}(X) \to Z[1], \tag{9.2}$$

where g' is a left minimal almost split in $\mathcal{C}$ and f' is a right minimal almost split in $\mathcal{C}$.

Note that the triangles (9.2) are obtained by applying the functor $\mathbb{S}_2^{-1}$ to the triangles (9.1). In [IYo], the triangles (9.1), regarded as a complex of $\mathcal{T}$, is called an *Auslander-Reiten* 4*-angle* ending at X (AR 4-angle, for short).

Let X be an object of $\mathcal{T}$ and $X \simeq \bigoplus_{i=1}^{l} X_i$ be an indecomposable decomposition of X. We call X a *basic* object if $X_i \not\simeq X_j$ holds for any $i \neq j$. We assume the following condition.

Assumption 9.4. *Let M be a basic object of a triangulated category $\mathcal{T}$.*

(i) We have a cluster tilting subcategory $\mathcal{U}$ of $\mathcal{T}$ given by

$$\mathcal{U} := \mathsf{add}\{\mathbb{S}_2^i(M) \mid i \in \mathbb{Z}\}.$$

(ii) The graded algebra $\mathrm{End}_{\mathcal{T}/\mathbb{S}_2}(M)$ is generated by homogeneous elements of degree zero and one.

The condition (ii) *is equivalent to the following condition:*

(ii)′ *There exists a finite quiver Q with a map $\deg : Q_1 \to \{0,1\}$ such that there exist a surjective morphism $\phi : KQ \to \mathrm{End}_{\mathcal{T}/\mathbb{S}_2}(M)$ of graded algebras and the kernel of ϕ is contained in the ideal of KQ generated by paths of length at least two.*

The following lemma is a fundamental observation on the quiver Q of $\mathrm{End}_{\mathcal{T}/\mathbb{S}_2}(M)$ and on right or left minimal almost split morphisms of M in $\mathcal{U}$.

Lemma 9.5. *Under the Assumption 9.4. For each $j \in Q_0$, let M^j be an indecomposable direct summand of M associated with an idempotent $\phi(e_j)$. For $j \in Q_0$, let*

$$f := (\phi(\alpha)) : \bigoplus_{\alpha \in Q_1,\, t(\alpha)=j} \mathbb{S}_2^{\deg(\alpha)}(M^{s(\alpha)}) \to M^j$$

be a morphism in $\mathcal{T}$. Then f is a right minimal almost split morphism of M^j in $\mathcal{U}$. Dually, let

$$g := (\phi(\alpha)) : M^j \to \bigoplus_{\alpha \in Q_1,\, s(\alpha)=j} \mathbb{S}_2^{-\deg(\alpha)}(M^{t(\alpha)})$$

be a morphism in $\mathcal{T}$. Then g is a left minimal almost split morphism of M^j in $\mathcal{U}$.

Proof. We show that f is a right minimal almost split morphism of M^j in $\mathcal{U}$. Dually, it is shown that g is a left minimal almost split morphism of M^j in $\mathcal{U}$.

By definition, f is right minimal and not a retraction. We denote by X the domain of f and $E := \mathrm{End}_{\mathcal{T}/\mathbb{S}_2}(M)$. Since Q is the quiver of E, f induces a surjective morphism $\mathrm{Hom}_{\mathcal{T}/\mathbb{S}_2}(M, X) \to \mathrm{rad}\, Ee_j$. Since $\mathrm{rad}\, Ee_j = \mathrm{rad}\big(\mathrm{Hom}_{\mathcal{T}/\mathbb{S}_2}(M, M^j)\big) = \bigoplus_{i\in\mathbb{Z}} J_{\mathcal{T}}(\mathbb{S}_2^i(M), M^j)$ holds, we have a surjective map $f^* : \mathrm{Hom}_{\mathcal{T}}(Z, X) \to J_{\mathcal{T}}(Z, M^j)$ for any $Z \in \mathcal{U}$. $\square$

Before stating the main theorem of this subsection, we need the following definition. Let Q be a finite quiver with a map $\deg : Q_1 \to \{0, 1\}$. We define a quiver Q^* by $Q^*_0 = Q_0$ and $Q^*_1 = \{\, \alpha \in Q_1 \mid \deg(\alpha) = 0 \,\} \sqcup \{\, \alpha^* : t(\alpha) \to s(\alpha) \mid \alpha \in Q_1,\, \deg(\alpha) = 1 \,\}$.

Definition-Proposition 9.6. *Let Q be a finite quiver with a map $\deg : Q_1 \to \{0, 1\}$. We call a quiver Q $\deg$-acyclic if one of the following equivalent conditions hold.*

- *(a) The quiver Q^* is acyclic.*
- *(b) There exists an order $\{1, 2, \dots, l\}$ on Q_0 which satisfies the following conditions: for any arrow $\alpha : i \to j$ in Q, if $\deg(\alpha) = 0$, then $j < i$, and if $\deg(\alpha) = 1$, then $i < j$.*

The following is the main theorem of this subsection.

Theorem 9.7. *Under the Assumption 9.4. If the quiver Q is $\deg$-acyclic, then we have $\mathsf{thick}_{\mathcal{T}} M = \mathcal{T}$.*

Proof. Let $\{1, 2, \dots, l\}$ be an order on Q_0 which satisfies the condition of Definition-Proposition 9.6 (b). Let $M = \bigoplus_{j=1}^{l} M^j$ be an indecomposable direct decomposition of M such that each M^j corresponds with a vertex $j \in \{1, 2, \dots, l\} = Q_0$. We show that $\mathbb{S}_2^i(M^j) \in \mathsf{thick}_{\mathcal{T}} M$ by an induction on i and j.

Let $i \geq 1$. Assume that $\mathbb{S}_2^k(M) \in \mathsf{thick}_{\mathcal{T}} M$ for $0 \leq k \leq i-1$ and $\mathbb{S}_2^i(M^k) \in \mathsf{thick}_{\mathcal{T}} M$ for $0 \leq k \leq j-1$, where $M^0 := 0$. We show that $\mathbb{S}_2^i(M^j) \in \mathsf{thick}_{\mathcal{T}} M$. By Theorem 9.3, we have an AR 4-angle ending at M^j

$$\mathbb{S}_2(M^j) \xrightarrow{g} C_1 \to X_1 \to \mathbb{S}_2(M^j)[1], \quad X_1 \to C_0 \xrightarrow{f} M^j \to X_1[1], \tag{9.3}$$

where f is a right minimal almost split of M^j in $\mathcal{U}$ and g is a left minimal almost split of $\mathbb{S}_2(M^j)$ in $\mathcal{U}$. By Lemma 9.5 and a uniqueness of a right (resp, left) minimal almost split morphism, we have

$$C_0 \simeq \bigoplus_{\alpha\in Q_1,\, t(\alpha)=j} \mathbb{S}_2^{\deg(\alpha)}(M^{s(\alpha)}), \quad C_1 \simeq \bigoplus_{\alpha\in Q_1,\, s(\alpha)=j} \mathbb{S}_2^{1-\deg(\alpha)}(M^{t(\alpha)}).$$

By applying $\mathbb{S}_2^{i-1}$ to (9.3), we have an AR 4-angle ending at $\mathbb{S}_2^{i-1}(M^j)$. Since $\{1, 2, \dots, l\} = Q_0$ satisfies the condition of Definition-Proposition 9.6 (b) and by the inductive hypothesis, we have $\mathbb{S}_2^{i-1}(C_0), \mathbb{S}_2^{i-1}(C_1) \in \mathsf{thick}_{\mathcal{T}} M$. Thus we have $\mathbb{S}_2^i(M^j) \in \mathsf{thick}_{\mathcal{T}} M$ and $\mathsf{add}\{\mathbb{S}_2^i(M) \mid i \geq 0\} \subset \mathsf{thick}_{\mathcal{T}} M$ holds. An inclusion $\mathsf{add}\{\mathbb{S}_2^i(M) \mid i \leq 0\} \subset \mathsf{thick}_{\mathcal{T}} M$ follows from the dual property of Theorem 9.3 and a similar argument. Therefore we have $\mathcal{U} \subset \mathsf{thick}_{\mathcal{T}} M$. By Proposition 9.2, we have the assertion. $\square$

We end this subsection with the following proposition which calculates the global dimension of the endomorphism algebra $\mathrm{End}_{\mathcal{T}}(M)$.

Proposition 9.8. *Under the Assumption 9.4, suppose that* $\mathrm{Hom}_{\mathcal{T}}(M, M[-1]) = 0$. *Then the global dimension of* $\mathrm{End}_{\mathcal{T}}(M)$ *is at most two.*

Proof. Let X be an indecomposable direct summand of M. Take an AR 4-angle ending at X

$$\mathbb{S}_2(X) \to C_1 \to Y \to \mathbb{S}_2(X)[1], \quad Y \to C_0 \to X \to Y[1].$$

By applying the functor $\mathrm{Hom}_{\mathcal{T}}(M, -)$ to the first triangle, we have

$$\mathrm{Hom}_{\mathcal{T}}(M, C_1) \simeq \mathrm{Hom}_{\mathcal{T}}(M, Y),$$

since $\mathcal{U}$ is a cluster tilting subcategory and $\mathrm{End}_{\mathcal{T}}(M)$ is positively graded. By applying the functor $\mathrm{Hom}_{\mathcal{T}}(M, -)$ to the second triangle, since $\mathrm{Hom}_{\mathcal{T}}(M, M[-1]) = 0$, we have an exact sequence of $\mathrm{End}_{\mathcal{T}}(M)$-modules

$$0 \to \mathrm{Hom}_{\mathcal{T}}(M, C_1) \to \mathrm{Hom}_{\mathcal{T}}(M, C_0) \to \mathrm{Hom}_{\mathcal{T}}(M, M^j).$$

By Lemma 9.5, we have $C_0, C_1 \in \mathsf{add}\{\, \mathbb{S}_2^i(M) \mid i = 0, 1\,\}$. Since $\mathrm{End}_{\mathcal{T}}(M)$ is positively graded, the $\mathrm{End}_{\mathcal{T}}(M)$-modules $\mathrm{Hom}_{\mathcal{T}}(M, C_0)$ and $\mathrm{Hom}_{\mathcal{T}}(M, C_1)$ are projective $\mathrm{End}_{\mathcal{T}}(M)$-modules. Therefore the projective dimension of the simple $\mathrm{End}_{\mathcal{T}}(M)$-module associated with X is at most two, and we have the assertion. □

9.2 A cluster tilting subcategory of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$

Let $\boldsymbol{w} = s_{u_1} s_{u_2} \cdots s_{u_l}$ be a reduced expression of $w \in W_Q$, and put

$$M(\boldsymbol{w})^i = M^i = (\Pi/I(s_{u_1} s_{u_2} \cdots s_{u_i}))e_{u_i}, \quad M(\boldsymbol{w}) = M = \bigoplus_{i=1}^{l} M(\boldsymbol{w})^i.$$

Remark 9.9. As easily seen, the tilting object of Part I (see Subsection 6.1) and M in this part are quite different even if $\boldsymbol{w}$ is c-sortable. In fact they have different gradings, and such a difference is crucial when we study $\mathbb{Z}$-graded modules.

Whenever there is no danger of confusion, we denote $M(\boldsymbol{w})^i$ and $M(\boldsymbol{w})$ by M^i and M, respectively. In this subsection, we show that the object M of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$ is a silting object. Note that by Proposition 8.7, $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ has a Serre functor $\mathbb{S} = [2] \circ (-1)$, and hence we have $\mathbb{S}_2 = (-1)$. Let

$$\mathcal{U} := \mathsf{add}\{\, M(i) \mid i \in \mathbb{Z}\,\}$$

be the full subcategory of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$.

Lemma 9.10. $\mathcal{U}$ *is a cluster tilting subcategory of* $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$.

Proof. Let $X \in \mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$. Since M and X are finite dimensional, there exists an integer $N > 0$ such that $\mathrm{Hom}^{\mathbb{Z}}_{\Pi(w)}(M, X(i)) = \mathrm{Hom}^{\mathbb{Z}}_{\Pi(w)}(X, M(i)) = 0$ for any $i > |N|$. This means that $\mathcal{U}$ is functorially finite in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$. Since $\mathbb{S} = [2](-1)$ is a Serre functor on $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$, we have

$$\mathcal{U}[-1]^{\perp} = {}^{\perp}\mathcal{U}[1].$$

By Proposition 3.3 (d), $\underline{\mathrm{Hom}}_{\Pi(w)}(M, M[1]) = 0$ holds. Therefore we have an equality $\underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M, M[1](i)) = 0$ for any integer i. This means $\mathcal{U} \subset {}^{\perp}\mathcal{U}[1]$. Let $X \in \mathsf{Sub}^{\mathbb{Z}}\,\Pi(w)$ be an indecomposable object such that $X \in {}^{\perp}\mathcal{U}[1]$ in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$. By forgetting gradings, we have $\underline{\mathrm{Hom}}_{\Pi(w)}(M, X[1]) = 0$. Since M is a cluster tilting object in $\underline{\mathsf{Sub}}\,\Pi(w)$, X is isomorphic to some indecomposable direct summand of M in $\mathsf{mod}\,\Pi(w)$. By Lemma 8.1, we have $X \in \mathcal{U}$. □

For the convenience of the reader, we recall the definition of the quiver $Q(\boldsymbol{w})$ of $\mathrm{End}_{\Pi(w)}(M(\boldsymbol{w}))$, which is already defined in Definition 6.5.

Definition 9.11. [BIRSc] Let w be an element of W. We define a quiver $Q(\boldsymbol{w})$ associated with a reduced expression $\boldsymbol{w} = s_{u_1} s_{u_2} \cdots s_{u_l}$ of w as follows:

- vertices: $Q(\boldsymbol{w})_0 = \{1, 2, \ldots, l\}$.

  A vertex $1 \leq i \leq l$ in $Q(\boldsymbol{w})$ is said to be *type* $u \in Q_0$ if $u_i = u$.

- arrows:

  (a1) For each $u \in \mathrm{Supp}(w)$, draw an arrow from j to i, where i, j are vertices of type u, $i < j$, and there is no vertex of type u between i and j (we call these arrows *going to the left*).

  (a2) For each arrow $\alpha : u \to v \in Q_1$, draw an arrow α_i from i to j, where $i < j$, i is a vertex of type u, j is a vertex of type v, there is no vertex of type u between i and j, and j is the biggest vertex of type v before the next vertex of type u (we call these arrows *Q-arrows*).

  (a3) For each arrow $\alpha : u \to v \in Q_1$, draw an arrow α_i^* from i to j, where $i < j$, i is a vertex of type v, j is a vertex of type u, there is no vertex of type v between i and j, and j is the biggest vertex of type u before the next vertex of type v (we call these arrows *Q^*-arrows*).

We denote by $\underline{Q}(\boldsymbol{w})$ the full subquiver of $Q(\boldsymbol{w})$ whose the set of vertices is $Q(\boldsymbol{w})_0 \setminus \{p_u \mid u \in \mathrm{Supp}(w)\}$, where $p_u = \max\{1 \leq j \leq l \mid u_j = u\}$, for $u \in \mathrm{Supp}(w)$.

Note that the quiver $Q(\boldsymbol{w})$ depends on the choice of a reduced expression of w. We introduce a map $\deg : Q(\boldsymbol{w})_1 \to \{0, 1\}$.

Definition 9.12. We define a map $\deg : Q(\boldsymbol{w})_1 \to \{0, 1\}$ as follows:

- $\deg(\beta) = 1$ if β is a Q^*-arrow.

- $\deg(\beta) = 0$ if β is a Q-arrow or an arrow going to the left.

We define a map deg on $\underline{Q}(\boldsymbol{w})$ as the restriction of $\deg : Q(\boldsymbol{w})_1 \to \{0, 1\}$ to $\underline{Q}(\boldsymbol{w})_1$.

Remark 9.13. The map $\deg : Q(\boldsymbol{w})_1 \to \{0,1\}$ in Definition 9.12 is not equal to that of Definition 6.9, so that our object M in Part II is different from that in Part I.

We give an example of a quiver $Q(\boldsymbol{w})$.

Example 9.14. Let Q be the quiver
$$\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\gamma"] & \\ 2 \arrow[rr, "\beta"'] & & 3 \end{tikzcd}.$$
Let w be an element of W_Q with its expression $\boldsymbol{w} = s_1 s_2 s_3 s_1 s_3 s_2 s_1$. Then we have the quiver $Q(\boldsymbol{w})$ with a map $\deg : Q(\boldsymbol{w})_1 \to \{0,1\}$ as follows:

$$\begin{tikzcd}
 & & 3 \arrow[ddr, "1"] & & 5 \arrow[ll] \arrow[dr, "1"] \arrow[ddrr, bend left, "1"] & & \\
 & 2 \arrow[urrr] \arrow[drr, "1"] & & & & 6 \arrow[llll] \arrow[dr, "1"] & \\
1 \arrow[ur] \arrow[uurr, bend left] & & & 4 \arrow[lll] \arrow[uur] \arrow[urr] & & & 7, \arrow[lll]
\end{tikzcd}$$

where non numbered arrows have degree zero.

We define a morphism of algebras $\phi : KQ(\boldsymbol{w}) \to \operatorname{End}_{\Pi(w)}(M)$ by

(a0) For a vertex i of $Q(\boldsymbol{w})$, $\phi(e_i)$ is an idempotent of $\operatorname{End}_{\Pi(w)}(M)$ associated with M^i.

(a1) For an arrow $\beta : j \to i$ going to the left, $\phi(\beta)$ is the canonical surjection $M^j \to M^i$.

(a2) For a Q-arrow $\alpha_i : i \to j$ of the arrow $\alpha \in Q_1$, $\phi(\alpha_i)$ is a morphism of $\Pi(w)$-modules from M^i to M^j given by multiplying α from the right.

(a3) For a Q^*-arrow $\alpha_i^* : i \to j$ of the arrow $\alpha \in Q_1$, $\phi(\alpha_i^*)$ is a morphism of $\Pi(w)$-modules from M^i to M^j given by multiplying α^* from the right.

We regard the path algebra $KQ(\boldsymbol{w})$ as a graded algebra by the map deg of Definition 9.12. The following proposition gives the quiver of the endomorphism algebra

$$\operatorname{End}_{(\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w))/(-1)}(M) = \bigoplus_{n \in \mathbb{Z}} \underline{\operatorname{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M, M(n)) = \underline{\operatorname{End}}_{\Pi(w)}(M).$$

Lemma 9.15. *The morphism $\phi : KQ(\boldsymbol{w}) \to \operatorname{End}_{\Pi(w)}(M)$ induces a surjective morphism $\underline{\phi} : K\underline{Q}(\boldsymbol{w}) \to \underline{\operatorname{End}}_{\Pi(w)}(M)$ of graded algebras such that the kernel of $\underline{\phi}$ is contained in the ideal of $K\underline{Q}(\boldsymbol{w})$ generated by paths of length at least two.*

Proof. The morphism ϕ is a morphism of graded algebras, since ϕ preserves gradings by the definitions of ϕ and the map deg. The morphism ϕ induces a surjective morphism $\underline{\phi}$ of graded algebras by [BIRSc, Theorem Ⅲ. 4.1]. The kernel of $\underline{\phi}$ is contained in the ideal of $K\underline{Q}(\boldsymbol{w})$ generated by paths of length at least two by [BIRSm, Theorem 6.6]. $\square$

Then we have the following proposition.

Proposition 9.16. *Let $\boldsymbol{w}$ be a reduced expression of $w \in W_Q$. Then the following holds.*

(a) The object M of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$ satisfies Assumption 9.4, where the quiver of $\underline{\operatorname{End}}_{\Pi(w)}(M)$ is $\underline{Q}(\boldsymbol{w})$ and a map deg *is given by Definition 9.12.*

(b) The quiver $Q(\boldsymbol{w})$ is deg*-acyclic. In particular, $\underline{Q}(\boldsymbol{w})$ is* deg*-acyclic.*

Proof. (a) This comes from Lemma 9.10 and Lemma 9.15.

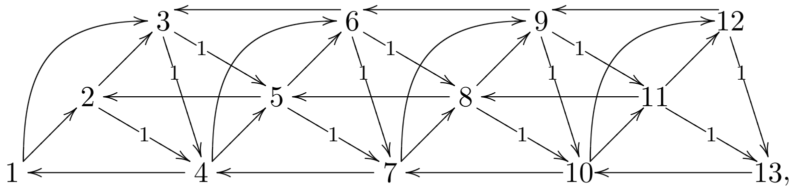
(b) By definition, $(Q(\boldsymbol{w})^*)_1$ is a disjoint union of arrows going to the left, Q-arrows and reversed arrows of Q^*-arrows. We define a map

$$\psi : (Q(\boldsymbol{w})^*)_1 \to Q_0 \sqcup Q_1$$

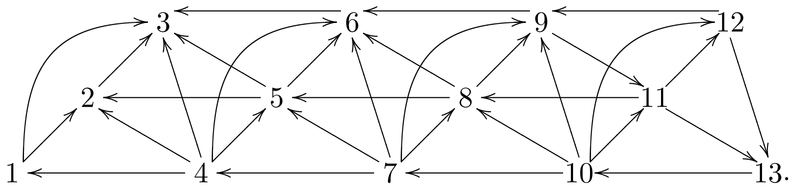
by $\psi(\beta) = u$ if β is an arrow going to the left associated with a vertex $u \in Q_0$ and $\psi(\beta) = \alpha$ if β is a Q-arrow or a reversed arrow of Q^*-arrow associated with an arrow $\alpha \in Q_1$. Then ψ extends to a map from the set of all paths in $Q(\boldsymbol{w})^*$ to the set of all paths in Q. We also denote it by ψ.

If there exists a cycle p in $Q(\boldsymbol{w})^*$, then $\psi(p)$ is a cycle in Q. This is a contradiction. $\square$

Example 9.17. (a) Let Q be the quiver $\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\gamma"] & \\ 2 \arrow[rr, "\beta"'] & & 3 \end{tikzcd}$. Put $c = s_1s_2s_3$. Let $w = c^4s_1 = s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1$. Then we have the quiver $Q(\boldsymbol{w})$ as follows:



where non numbered arrows have degree zero. Then we have the quiver $Q(\boldsymbol{w})^*$ of $Q(\boldsymbol{w})$



As a result, we have the following theorem.

Theorem 9.18. *Let $w \in W_Q$. For any reduced expression $\boldsymbol{w}$ of w, the object $M = M(\boldsymbol{w})$ is a silting object of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$.*

Proof. By Theorem 9.7 and Proposition 9.16, we have $\mathsf{thick}\, M = \underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$.

We show that M satisfies $\underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M, M[j]) = 0$ for any $j > 0$. By Proposition 3.3 (c), $\underline{\mathrm{Hom}}_{\Pi(w)}(M, M[1]) = 0$ holds. Therefore we have $\underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M, M[1]) = 0$. Assume that $j > 1$. By Proposition 8.7, we have

$$\underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M, M[j]) \simeq \mathrm{D}\,\underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M, M[2-j](-1)).$$

Since $2 - j \leq 0$ and $\Pi(w)$ is positively graded, we have $\Omega^{-(2-j)}(M) \in \mathsf{mod}^{\geq 0}\,\Pi(w)$. Therefore $\Omega^{-(2-j)}(M)(-1) \in \mathsf{mod}^{\geq 1}\,\Pi(w)$ holds. Since M is generated by $(M)_0$ as a $\Pi(w)$-module, we have $\mathrm{Hom}^{\mathbb{Z}}_{\Pi(w)}(M, \Omega^{-(2-j)}(M)(-1)) = 0$. This means $\underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M, M[2-j](-1)) = 0$ for $j > 1$. $\square$

Note that $M(\boldsymbol{w})$ is not a tilting object of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$ in general.

Example 9.19. Let Q be a quiver $\begin{tikzcd}[column sep=small] & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[rr] & & 3 \end{tikzcd}$. Then we have a graded algebra $\Pi = \Pi e_1 \oplus \Pi e_2 \oplus \Pi e_3$, and these are represented by their radical filtrations as follows:

$$\Pi e_1 = \begin{array}{cccccc} \mathbf{1} &&&&& \\ 2 & 3 &&&& \\ 3 & 1 & 2 &&& \\ 1 & 2 & 3 & 1 && \\ 2 & 3 & 1 & 2 & 3 & \\ 3 & 1 & 2 & 3 & 1 & 2 \end{array}, \quad \Pi e_2 = \begin{array}{cccccc} \mathbf{2} &&&&& \\ 3 & \mathbf{1} &&&& \\ 1 & 2 & 3 &&& \\ 2 & 3 & 1 & 2 && \\ 3 & 1 & 2 & 3 & 1 & \\ 1 & 2 & 3 & 1 & 2 & 3 \end{array}, \quad \Pi e_3 = \begin{array}{cccccc} \mathbf{3} &&&&& \\ \mathbf{1} & \mathbf{2} &&&& \\ 2 & 3 & \mathbf{1} &&& \\ 3 & 1 & 2 & 3 && \\ 1 & 2 & 3 & 1 & 2 & \\ 2 & 3 & 1 & 2 & 3 & 1 \end{array},$$

where numbers connected by solid lines are concentrated in the same degree, the tops of the Πe_i are concentrated in degree 0, and the degree zero parts are denoted by bold numbers.

Let w be an element of W_Q which has a reduced expression $\boldsymbol{w} = s_3 s_2 s_1 s_2 s_3 s_2$. Then we have a graded algebra, $\Pi(w) = \Pi(w)e_1 \oplus \Pi(w)e_2 \oplus \Pi(w)e_3$, where

$$\Pi(w)e_1 = \begin{array}{cc} \mathbf{1} & \\ 2 & 3 \\ 3 & \end{array}, \qquad \Pi(w)e_2 = \begin{array}{ccc} \mathbf{2} && \\ 3 & \mathbf{1} & \\ 1 & 2 & 3 \\ 2 & 3 & \\ 3 && \end{array}, \qquad \Pi(w)e_3 = \begin{array}{ccc} \mathbf{3} && \\ \mathbf{1} & \mathbf{2} & \\ 2 & 3 & \mathbf{1} \\ 3 && 3 \end{array}.$$

We have a silting object $M = M(\boldsymbol{w})$ of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$ as follows:

$$M = M^1 \oplus M^2 \oplus M^4 = \mathbf{3} \oplus \begin{array}{c} \mathbf{2} \\ 3 \end{array} \oplus \begin{array}{cc} & \mathbf{2} \\ 3 & \mathbf{1} \\ & 3 \end{array}.$$

This $M(\boldsymbol{w})$ is not a tilting object of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$, since we see that $\underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M^4, \Omega(M^1)) \neq 0$. Note that another reduced expression of w gives a tilting object of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$ (see Example 10.9 (a)).

10 A tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$

Let $w \in W_Q$ and $\boldsymbol{w}$ be a reduced expression of w. In this section, we give a sufficient condition on $\boldsymbol{w}$ such that $M = M(\boldsymbol{w})$ is a tilting object of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$. Throughout this section, by Lemma 4.1, without loss of generality assume that $\mathrm{Supp}(w) = Q_0$. We first show the following lemma.

Lemma 10.1. *If one of the following holds, then we have* $\underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M, M[j]) = 0$ *for any* $j < -1$.

(i) There exists a reduced expression of w containing an expression of the Coxeter element of W_Q as a subword.

(ii) The global dimension of $\Pi(w)_0$ is at most one.

Proof. By Lemma 8.4 (b), we assume that (ii) holds. By the definition of M, M is in $\mathsf{mod}^{\geq 0}\,\Pi(w)$. Therefore by Lemma 8.2, we have $\Omega^j(M)_0 = 0$ for any $j > 1$. Since M is generated by $(M)_0$ as a $\Pi(w)$-module, we have $\mathrm{Hom}^{\mathbb{Z}}_{\Pi(w)}(M, \Omega^j(M)) = 0$ for any $j > 1$. This means $\underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M, M[j]) = 0$ for any $j < -1$. □

Next we observe when $\underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M, M[-1]) = 0$ holds. We define some notation. A full subquiver Q' of Q is said to be *convex* in Q if any path in Q such that its start and target are in Q' is a path in Q'. For any $u, v \in Q_0$, we denote by $Q(u, v)$ the minimal convex full subquiver of Q containing u and v. Let $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ be a reduced expression of w. For any $u \in Q_0$, put

$$p_u = \max\{1 \leq j \leq l \mid u_j = u\}, \quad m_u = \min\{1 \leq j \leq l \mid u_j = u\}.$$

Definition 10.2. Let $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ be a reduced expression of $w \in W_Q$ and S be a subset of Q_0.

(1) An expression $\boldsymbol{w}$ is *c-ending on S* if for any $u, v \in S$, $p_u < p_v$ holds whenever there exists an arrow from u to v in Q.

(2) An expression $\boldsymbol{w}$ is *c-starting on S* if for any $u, v \in S$, $m_u < m_v$ holds whenever there exists an arrow from u to v in Q.

The following lemma is an easy observation.

Lemma 10.3. *Let $w \in W_Q$ and $\boldsymbol{w}$ be a reduced expression of w. If $\boldsymbol{w}$ is c-ending or c-starting on Q_0, then $\boldsymbol{w}$ contains an expression of the Coxeter element of W_Q as a subword, in particular the global dimension of $\Pi(w)_0$ is at most one.*

Recall the following notation. For a reduced expression $s_{u_1}s_{u_2}\cdots s_{u_l}$, let $I_{k,m} = I(s_{u_k}\cdots s_{u_m})$ if $k \leq m$ and $I_{k,m} = \Pi$ if $m < k$. The following proposition is important to show the main theorem of this section.

Proposition 10.4. *Let $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ be a reduced expression of $w \in W_Q$ and $i, j \in \{1, \ldots, l\} \setminus \{p_u \mid u \in Q_0\}$. If an expression $\boldsymbol{w}$ is c-ending on $Q(u_i, u_j)_0$ or c-starting on $Q(u_i, u_j)_0$, then we have $\mathrm{Hom}^{\mathbb{Z}}_{\Pi(w)}(M^i, \Omega(M^j)) = 0$.*

Proof. By Proposition 6.22 and applying the functor $\mathrm{Hom}^{\mathbb{Z}}_{\Pi(w)}(M^i, -)$ to an exact sequence $0 \to \Omega(M^j) \to \Pi(w)e_{u_j} \to M^j \to 0$, we have

$$\mathrm{Hom}^{\mathbb{Z}}_{\Pi(w)}(M^i, \Omega(M^j)) \simeq e_{u_i}\left(\frac{I_{1,j} \cap I_{i+1,l}}{I(w)}\right)_0 e_{u_j}.$$

Therefore it is enough to show that $e_{u_i}(I_{1,j} \cap I_{i+1,l})_0 e_{u_j} = 0$.

Since $(I_{1,j} \cap I_{i+1,l})_0 \subset KQ$, if $e_{u_i}KQe_{u_j} = 0$, then we have $e_{u_i}(I_{1,j} \cap I_{i+1,l})_0 e_{u_j} = 0$. Assume that $e_{u_i}KQe_{u_j} \neq 0$. Let $c_{i,j}$ be the Coxeter element of $W_{Q(u_i,u_j)}$. Since $Q(u_i, u_j)$ is a full subquiver of Q, an expression of $c_{i,j}$ is a subword of an expression of the Coxeter element of W_Q. Since $Q(u_i, u_j)$ is a minimal convex subquiver of Q, u_i is a unique source of $Q(u_i, u_j)$ and u_j is a unique sink of $Q(u_i, u_j)$. Therefore by Lemma 8.4 (c), we have $e_{u_i}I(c_{i,j})_0 e_{u_j} = 0$.

If $\boldsymbol{w}$ is c-ending on $Q(u_i,u_j)_0$, then an expression $s_{u_{i+1}}\cdots s_{u_l}$ contains an expression of $c_{i,j}$ as a subword, and therefore $e_{u_i}(I_{i+1,l})_0e_{u_j}\subset e_{u_i}I(c_{i,j})_0e_{u_j}=0$ holds.

If $\boldsymbol{w}$ is c-starting on $Q(u_i,u_j)_0$, then an expression $s_{u_1}\cdots s_{u_j}$ contains an expression of $c_{i,j}$ as a subword, and therefore $e_{u_i}(I_{1,j})_0e_{u_j}\subset e_{u_i}I(c_{i,j})_0e_{u_j}=0$. We have the assertion. $\square$

Then we show the main theorem of this section.

Theorem 10.5. *Let $w\in W_Q$ and $\boldsymbol{w}=s_{u_1}s_{u_2}\cdots s_{u_l}$ be a reduced expression of w. Put*

$$M^i=(\Pi/I(s_{u_1}s_{u_2}\cdots s_{u_i}))e_{u_i},\quad M=\bigoplus_{i=1}^{l}M^i.$$

If the expression $\boldsymbol{w}$ is c-ending on Q_0 or c-starting on Q_0, then we have the following.

(a) M is a tilting object of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$.

(b) The global dimension of $A=\underline{\mathrm{End}}_{\Pi(w)}^{\mathbb{Z}}(M)$ is at most two.

(c) We have a triangle equivalence $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)\simeq\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$.

Proof. (a) By Theorem 9.18, Lemmas 10.1 and 10.3, we only have to show $\underline{\mathrm{Hom}}_{\Pi(w)}^{\mathbb{Z}}(M,M[-1])=0$. We show that $\mathrm{Hom}_{\Pi(w)}^{\mathbb{Z}}(M^i,\Omega(M^j))=0$ for any $i,j\in\{1,2,\dots,l\}\setminus\{p_u\mid u\in Q_0\}$. Since $\boldsymbol{w}$ is c-ending on Q_0 or c-starting on Q_0, $\boldsymbol{w}$ is c-ending on $Q(u_i,u_j)_0$ or c-starting on $Q(u_i,u_j)_0$. Therefore, we have $\mathrm{Hom}_{\Pi(w)}^{\mathbb{Z}}(M^i,\Omega(M^j))=0$ by Proposition 10.4.

(b) This comes from (a) and Proposition 9.8.

(c) This follows from (a), (b) and Theorem 3.9. $\square$

Remark 10.6. The property (b) of Theorem 10.5 was already shown by [ART] in the case when $\boldsymbol{w}$ is c-ending on $\mathrm{Supp}(w)$ (see Theorem 11.2).

Next we give a more general condition on $\boldsymbol{w}$ such that $M(\boldsymbol{w})$ satisfies $\underline{\mathrm{Hom}}_{\Pi(w)}^{\mathbb{Z}}(M,M[-1])=0$. For a reduced expression $\boldsymbol{w}$, let $S(\boldsymbol{w}):=\{u\in Q_0\mid p_u=m_u\}$.

Definition 10.7. A reduced expression $\boldsymbol{w}$ satisfies ($\diamondsuit$) if for any $u,v\in Q_0\setminus S(\boldsymbol{w})$, $\boldsymbol{w}$ is c-ending on $Q(u,v)_0$ or c-starting on $Q(u,v)_0$.

Put $J=\{1,2,\dots,l\}\setminus\{p_u\mid u\in Q_0\}$. Note that $\{\,u_i\mid i\in J\,\}=Q_0\setminus S(\boldsymbol{w})$ holds. We have the following theorem.

Theorem 10.8. *Let $w\in W_Q$. Assume that the global dimension of $\Pi(w)_0$ is at most one. Let $\boldsymbol{w}=s_{u_1}s_{u_2}\cdots s_{u_l}$ be a reduced expression of w and M be the same object as that in Theorem 10.5. If $\boldsymbol{w}$ satisfies ($\diamondsuit$), then the assertions* (a), (b) *and* (c) *of Theorem 10.5 hold.*

Proof. These are shown by the same argument as that in Theorem 10.5 since $\{\,u_i\mid i\in J\,\}=Q_0\setminus S(\boldsymbol{w})$ holds. $\square$

An example of a reduced expression which satisfies ($\diamondsuit$) but is neither c-ending nor c-starting on Q_0 is given in Example 10.9 (c). We end this section by giving some examples.

Example 10.9. (a) Let Q be a quiver $\begin{tikzcd}[column sep=small, row sep=small] & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[rr] & & 3 \end{tikzcd}$. Let w be an element of W_Q which has a reduced expression $\boldsymbol{w} = s_3s_2s_1s_3s_2s_3$. Note that this w is the same element as that in Example 9.19. The expression $\boldsymbol{w}$ is c-ending on Q_0. Then we have a graded algebra, $\Pi(w) = \Pi(w)e_1 \oplus \Pi(w)e_2 \oplus \Pi(w)e_3$, where

$$\Pi(w)e_1 = \begin{smallmatrix} & & \mathbf{1} & \\ & 2 & & 3 \\ 3 & & & \end{smallmatrix}, \qquad \Pi(w)e_2 = \begin{smallmatrix} & & & & \mathbf{2} & & \\ & & & 3 & & \mathbf{1} & \\ & & 1 & & 2 & & 3 \\ & 2 & & 3 & & & \\ 3 & & & & & & \end{smallmatrix}, \qquad \Pi(w)e_3 = \begin{smallmatrix} & & & \mathbf{3} & & & \\ & & \mathbf{1} & & \mathbf{2} & & \\ & 2 & & 3 & & \mathbf{1} & \\ 3 & & & & & & 3 \end{smallmatrix}.$$

We have a tilting object $M = M(\boldsymbol{w})$ of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$ as follows:

$$M = \mathbf{3} \oplus \begin{smallmatrix} & \mathbf{2} \\ 3 & \end{smallmatrix} \oplus \begin{smallmatrix} & & & \mathbf{3} & \\ & & \mathbf{1} & & \mathbf{2} \\ & 2 & & 3 & \\ 3 & & & & \end{smallmatrix}.$$

The endomorphism algebra $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)$ is given by the following quiver with relations

$$\Delta = \left[\begin{tikzcd} \bullet \arrow[r, "a"] & \bullet \arrow[r, "b"] & \bullet \end{tikzcd} \right], \quad ab = 0.$$

(b) Let Q be the same quiver as that in (a) and w be an element of W_Q with its expression $\boldsymbol{w} = s_1s_2s_3s_1s_3s_2s_1$. This expression $\boldsymbol{w}$ is a reduced expression and c-starting on Q_0. Then we have

$$\Pi(w)e_1 = \begin{smallmatrix} & & & & \mathbf{1} & & & \\ & & & 2 & & 3 & & \\ & & 3 & & 1 & & 2 & \\ & 1 & & 2 & & & & 1 \\ 2 & & & & & & & \end{smallmatrix}, \qquad \Pi(w)e_2 = \begin{smallmatrix} & & & \mathbf{2} & & & & \\ & & 3 & & \mathbf{1} & & & \\ & 1 & & 2 & & 3 & & \\ 2 & & & & 1 & & 2 & \\ & & & & & & & 1 \end{smallmatrix}, \qquad \Pi(w)e_3 = \begin{smallmatrix} & & \mathbf{3} & & \\ & \mathbf{1} & & \mathbf{2} & \\ 2 & & & & \mathbf{1} \end{smallmatrix}.$$

A tilting object $M = M(\boldsymbol{w})$ of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$ is described as follows:

$$M = \mathbf{1} \oplus \begin{smallmatrix} \mathbf{2} & \\ & \mathbf{1} \end{smallmatrix} \oplus \begin{smallmatrix} & \mathbf{3} & & \\ \mathbf{1} & & \mathbf{2} & \\ & & & \mathbf{1} \end{smallmatrix} \oplus \begin{smallmatrix} & \mathbf{1} & & & \\ 2 & & 3 & & \\ & 1 & & 2 & \\ & & & & 1 \end{smallmatrix}.$$

The endomorphism algebra $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)$ is given by the following quiver with relations

$$\Delta = \left[\begin{tikzcd} & & 3 \\ & 2 & \\ 1 \arrow[uurr, bend left, "c"] \arrow[ur, "b"] & & 4 \arrow[ll, "a"] \end{tikzcd} \right], \quad ab = ac = 0.$$

It is easy to see that the algebra $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)$ is derived equivalent to the path algebra of Dynkin quiver of type D_4.

(c) Let Q be a quiver

$$\begin{tikzcd}[row sep=small] & & 3 \\ 1 \arrow[r, Rightarrow] & 2 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\ & & 4 \end{tikzcd}$$

and w be an element of W_Q with its reduced expression $\boldsymbol{w} = s_4 s_1 s_2 s_3 s_2 s_3 s_1 s_2 s_4$. An expression $s_1 s_2 s_3 s_4$ is an expression of the Coxeter element of W_Q. The expression $\boldsymbol{w}$ contains $s_1 s_2 s_3 s_4$ as a subword, and hence the global dimension of $\Pi(w)_0$ is at most one. We can see that $\boldsymbol{w}$ satisfies ($\diamondsuit$). Thus $M = M(\boldsymbol{w})$ is a tilting object of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$. The endomorphism algebra $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)$ is given by the following quiver with relations:

$$\Delta = \left[\bullet \qquad \bullet \Longrightarrow \bullet \xrightarrow{a} \bullet \overset{b}{\underset{c}{\Longrightarrow}} \bullet\right], \quad ab = ac = 0.$$

Note that $\boldsymbol{w}$ is neither c-ending on Q_0 nor c-starting on Q_0.

There exist examples such that a reduced expression $\boldsymbol{w}$ does not satisfies ($\diamondsuit$), but $M = M(\boldsymbol{w})$ is a tilting object. In fact, in the following example, $\mathrm{Hom}^{\mathbb{Z}}_{\Pi(w)}(M, \Omega(M)) \neq 0$, but $\underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M, M[-1]) = 0$ holds.

Example 10.10. Let Q be the same quiver as in Example 10.9 (a) and w be an element of W_Q with its reduced expression $\boldsymbol{w} = s_3 s_1 s_2 s_3 s_1 s_3$. Note that $\boldsymbol{w}$ does not satisfies ($\diamondsuit$). We have

$$M^1 = \mathbf{3}, \quad M^2 = \begin{smallmatrix} \mathbf{1} \\ 3 \end{smallmatrix}, \quad M^3 = \Pi(w)e_2 = \begin{smallmatrix} & \mathbf{2} & \\ 3 & & \mathbf{1} \\ & & 3 \end{smallmatrix},$$

$$M^4 = \begin{smallmatrix} & \mathbf{3} & & \\ \mathbf{1} & & \mathbf{2} & \\ & 3 & & \mathbf{1} \\ & & & 3 \end{smallmatrix}, \quad M^5 = \Pi(w)e_1 = \begin{smallmatrix} & & \mathbf{1} & & \\ & 2 & & 3 & \\ 3 & & 1 & & 2 \\ & & & 3 & & 1 \\ & & & & 3 & \end{smallmatrix}, \quad M^6 = \Pi(w)e_3 = \begin{smallmatrix} & & \mathbf{3} & & \\ & \mathbf{1} & & \mathbf{2} & \\ 2 & & 3 & & \mathbf{1} \\ 3 & & & & 3 \end{smallmatrix}.$$

It is easy to see that $\mathrm{Hom}^{\mathbb{Z}}_{\Pi(w)}(M^2, \Omega(M^1)) \neq 0$ and $\underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M^2, \Omega(M^1)) = 0$. Moreover, we see that $\underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M, M[-1]) = 0$. The expression $\boldsymbol{w}$ contains an expression of the Coxeter element of W_Q. Therefore, $M = M(\boldsymbol{w})$ is a tilting object of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w)$.

11 The relationship with the result of Amiot-Reiten-Todorov

Before describing the result of [ART], we recall the definition of cluster categories which are introduced by Amiot [A]. Let A be a finite dimensional algebra of global dimension at most two. We denote by $\mathbb{S} = \mathrm{D}\,A \otimes^{\mathbf{L}}_A (-)$ a Serre functor on $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$. Put $\mathbb{S}_2 = \mathbb{S} \circ [-2]$. A *cluster category* $\mathsf{C}(A)$ of A is the triangulated hull of the orbit category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)/\mathbb{S}_2$ in the sense of Keller [Ke05]. We have the composition of functors

$$\pi_A : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)/\mathbb{S}_2 \to \mathsf{C}(A).$$

Note that π_A is a triangle functor. Let $w \in W_Q$. For a reduced expression $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ of w, let

$$M(\boldsymbol{w})^i = M^i = (\Pi/I(s_{u_1}s_{u_2}\cdots s_{u_i}))\, e_{u_i}, \quad M(\boldsymbol{w}) = M = \bigoplus_{i=1}^{l} M(\boldsymbol{w})^i,$$
$$A(\boldsymbol{w}) = A = \mathrm{End}^{\mathbb{Z}}_{\Pi(w)}(M(\boldsymbol{w})).$$

We denote by e_i the idempotent of A associated with M^i for each $1 \le i \le l$. Let $e_F = \sum_{j\in F} e_j$, where $F = \{p_u \mid u \in \mathrm{Supp}(w)\}$. Put

$$\underline{A} = A/Ae_FA.$$

By definition, we have an exact sequence

$$0 \to Ae_FA \to A \to \underline{A} \to 0. \tag{11.1}$$

Note that, by the definition, M is a right A-module and we have $Me_F = \Pi(w)$ as left $\Pi(w)$-modules.

We see that the algebra $\underline{A}$ coincides with the our endomorphism algebra $\underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)$.

Lemma 11.1. *We have $Ae_FA = \mathcal{P}(M,M)$. In particular, we have $\underline{A} = \underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)$.*

Proof. Clearly $Ae_FA \subset \mathcal{P}(M,M)$ holds. Let $f \in \mathcal{P}(M,M)$. We can assume that f factors through $(\Pi(w))(j) = Me_F(j)$ for some $j \in \mathbb{Z}$. Then we have a morphism $g : M \to Me_F$ of degree j and $h : Me_F \to M$ of degree $-j$ such that $f = gh$. Since $\mathrm{End}_{\Pi(w)}(M)$ is positively graded by Lemma 9.15, we have $j = 0$. This means $f \in Ae_FA$. $\square$

Next we recall the result of [ART]. We denote by $\rho_{\Pi(w)}$ the composite of triangle functors

$$\rho_{\Pi(w)} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\ \Pi(w)) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\ \Pi(w))/\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\ \Pi(w)) \xrightarrow{\sim} \underline{\mathsf{Sub}}\Pi(w).$$

Amiot-Reiten-Todorov showed the following theorem.

Theorem 11.2. *[ART, Theorem 3.1, Theorem 4.4] Let $w \in W_Q$ and $\boldsymbol{w}$ be a reduced expression of w. Put $N := M \otimes^{\mathbf{L}}_A \underline{A} \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}(\Pi(w) \otimes \underline{A}^{\mathrm{op}}))$. If $\boldsymbol{w}$ is c-ending on* Supp(w)*, then we have the following.*

(a) The global dimension of $\underline{A}$ is at most two.

(b) There exists a triangle equivalence $G : \mathsf{C}(\underline{A}) \to \underline{\mathsf{Sub}}\,\Pi(w)$ which makes the following diagram commutative up to isomorphism of functors

$$\begin{tikzcd} \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\underline{A}) \arrow[r, "N\otimes^{\mathbf{L}}_{\underline{A}}-"] \arrow[d, "\pi_{\underline{A}}"] & \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Pi(w)) \arrow[d, "\rho_{\Pi(w)}"] \\ \mathsf{C}(\underline{A}) \arrow[r, dashed, "G"] & \underline{\mathsf{Sub}}\,\Pi(w). \end{tikzcd}$$

Remark 11.3. For any reduced expression $\boldsymbol{w}$ of $w \in W_Q$, since Q is acyclic, there exists a quiver Q' such that whose underlying graph coincides with that of Q and $\boldsymbol{w}$ is c-ending on $\mathrm{Supp}(w)$ as an element of $W_{Q'}$. Since $\underline{\mathsf{Sub}}\,\Pi(w)$ is independent of an orientation of Q, we have an equivalence (1.1) by Theorem 11.2.

We construct a functor $\Phi : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\ \underline{A}) \to \underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ as follows. By Definition 8.3, the algebra $\Pi(w) \otimes A^{\mathrm{op}}$ is a graded algebra and M is a graded $\Pi(w) \otimes A^{\mathrm{op}}$-module. Therefore $N = M \otimes^{\mathbf{L}}_A \underline{A}$ is an object of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}}(\Pi(w) \otimes \underline{A}^{\mathrm{op}}))$ and we have a derived functor

$$N \otimes^{\mathbf{L}}_{\underline{A}} - : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\ \underline{A}) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}}\,\Pi(w)).$$

We denote by $\rho^{\mathbb{Z}}_{\Pi(w)}$ the graded version of $\rho_{\Pi(w)}$, that is,

$$\rho^{\mathbb{Z}}_{\Pi(w)} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}}\,\Pi(w)) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}}\,\Pi(w))/\mathsf{K}^{\mathrm{b}}(\mathsf{proj}^{\mathbb{Z}}\,\Pi(w)) \xrightarrow{\sim} \underline{\mathsf{Sub}}^{\mathbb{Z}}\,\Pi(w).$$

By composing $N \otimes^{\mathbf{L}}_{\underline{A}} -$ and $\rho^{\mathbb{Z}}_{\Pi(w)}$, we have a triangle functor

$$\Phi = \rho^{\mathbb{Z}}_{\Pi(w)} \circ N \otimes^{\mathbf{L}}_{\underline{A}} - : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\ \underline{A}) \to \underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w).$$

In this section, we show the following theorem which is a graded version of Theorem 11.2.

Theorem 11.4. *Let $w \in W_Q$ and $\boldsymbol{w}$ be a reduced expression of w. If $\boldsymbol{w}$ is c-ending on* $\mathrm{Supp}(w)$*, then we have the following.*

(a) The triangle functor $\Phi = \rho^{\mathbb{Z}}_{\Pi(w)} \circ N \otimes^{\mathbf{L}}_{\underline{A}} - : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\ \underline{A}) \to \underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$ is an equivalence.

(b) We have the following commutative diagram up to isomorphism of functors

$$\begin{tikzcd}
\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\ \underline{A}) \arrow[r, "\Phi"] \arrow[d, "\pi_{\underline{A}}"] & \underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w) \arrow[d, "\mathrm{Forget}"] \\
\mathsf{C}(\underline{A}) \arrow[r, "G"] & \underline{\mathsf{Sub}}\,\Pi(w).
\end{tikzcd}$$

We begin with the following lemma.

Lemma 11.5. *[ART, Lemma 3.2] If a reduced expression $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ of w is c-ending on* $\mathrm{Supp}(w)$*, then we have a projective resolution $0 \to P^1 \to P^0 \to Ae_i \to \underline{A}e_i \to 0$ of A-module $\underline{A}e_i$, where $i \in \{1 \le j \le l\} \setminus F$ and $P^0, P^1 \in \mathsf{add}(Ae_F)$.*

Proof. Since $\boldsymbol{w}$ is c-ending on $\mathrm{Supp}(w)$ and by [ART, Lemma 4.3], the conditions $(H1) \sim (H4)$ in [ART] are satisfied. Then the assertion follows immediately from [ART, Lemma 3.2]. $\square$

We need the following lemma.

Lemma 11.6. *If a reduced expression $\boldsymbol{w} = s_{u_1}s_{u_2}\cdots s_{u_l}$ of w is c-ending on* $\mathrm{Supp}(w)$*, then we have the following.*

(a) $Ae_FAe_i = Ae_i$ for any $i \in F$.

(b) *We have a projective resolution of Ae_FA as an A-module*

$$0 \to P^1 \to P^0 \to Ae_FA \to 0, \tag{11.2}$$

where $P^0, P^1 \in \mathsf{add}(Ae_F)$.

(c) *We have $M \otimes^{\mathbf{L}}_A (Ae_FA) \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}^{\mathbb{Z}}\, \Pi(w))$.*

Proof. (a) Since e_i is an idempotent, this is clear.

(b) We have an exact sequence (11.1). Thus the assertion follows from (a) and Lemma 11.5.

(c) By (b), $Ae_FA \in \mathsf{thick}\, Ae_F$ holds. Thus we have $M \otimes^{\mathbf{L}}_A (Ae_FA) \in \mathsf{thick}(M \otimes^{\mathbf{L}}_A Ae_F) = \mathsf{K}^{\mathrm{b}}(\mathsf{proj}^{\mathbb{Z}}\, \Pi(w))$, where the last equality follows from $M \otimes^{\mathbf{L}}_A Ae_F = Me_F = \Pi(w)$. □

Then we are ready to show the main theorem.

Proof of Theorem 11.4. (a) We will apply Lemma 3.10 for the triangle functor Φ. We first show that $\Phi(\underline{A}) = \rho^{\mathbb{Z}}_{\Pi(w)}(N \otimes^{\mathbf{L}}_{\underline{A}} \underline{A}) \simeq M$ in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$. Recall that $N := M \otimes^{\mathbf{L}}_A \underline{A}$. By applying $M \otimes^{\mathbf{L}}_A -$ to the sequence (11.1), we have the following triangle in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}}\, \Pi(w))$

$$M \otimes^{\mathbf{L}}_A (Ae_FA) \to M \otimes^{\mathbf{L}}_A A \to M \otimes^{\mathbf{L}}_A \underline{A} \to M \otimes^{\mathbf{L}}_A (Ae_FA)[1].$$

By Lemma 11.6 (c) and this triangle, M is isomorphic to $\rho^{\mathbb{Z}}_{\Pi(w)}(N \otimes^{\mathbf{L}}_{\underline{A}} \underline{A})$ in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$.

By Theorem 10.5, M is a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w)$. Since the global dimension of $\underline{A}$ is at most two, $\underline{A}$ is a tilting object of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, \underline{A})$.

We next show that $\Phi_{\underline{A},\underline{A}}$ induces an isomorphism $\mathrm{Hom}_{\underline{A}}(\underline{A}, \underline{A}) \simeq \underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M, M)$. We use the following notations:

$$\rho^{\mathbb{Z}}_{\Pi(w)} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}}\, \Pi(w)) \xrightarrow{\pi} \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}}\, \Pi(w))/\mathsf{K}^{\mathrm{b}}(\mathsf{proj}^{\mathbb{Z}}\, \Pi(w)) \xrightarrow{\rho} \underline{\mathsf{Sub}}^{\mathbb{Z}}\, \Pi(w),$$

where π is a canonical triangle functor and ρ is an triangle equivalence of Theorem 8.8 (b). For any $a \in A = \mathrm{End}^{\mathbb{Z}}_{\Pi(w)}(M)$, we denote by $\underline{a} \in \underline{A} = \underline{\mathrm{End}}^{\mathbb{Z}}_{\Pi(w)}(M)$ the element represented by a. We denote by $\cdot a$ the image of a by the usual isomorphism $A \simeq \mathrm{End}_A(A, A)$, and we use the same notation for $\cdot\underline{a}$. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & Ae_FA \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\cdot a"] & \underline{A} \arrow[r] \arrow[d, "\cdot \underline{a}"] & 0 \\
0 \arrow[r] & Ae_FA \arrow[r] & A \arrow[r] & \underline{A} \arrow[r] & 0.
\end{tikzcd}$$

By applying $M \otimes^{\mathbb{L}}_A -$ to this diagram, we have the following commutative diagram:

$$\begin{tikzcd}
M \otimes^{\mathbf{L}}_A (Ae_FA) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "a"] & M \otimes^{\mathbf{L}}_A \underline{A} \arrow[r] \arrow[d, "\mathrm{id}_M \otimes^{\mathbf{L}}_A (\cdot\underline{a})"] & M \otimes^{\mathbf{L}}_A (Ae_FA)[1] \arrow[d] \\
M \otimes^{\mathbf{L}}_A (Ae_FA) \arrow[r] & M \arrow[r] & M \otimes^{\mathbf{L}}_A \underline{A} \arrow[r] & M \otimes^{\mathbf{L}}_A (Ae_FA)[1].
\end{tikzcd}$$

This means that the morphism $(\pi \circ N \otimes^{\mathbb{L}}_{\underline{A}} -)_{\underline{A},\underline{A}} : \mathrm{Hom}_{\underline{A}}(\underline{A}, \underline{A}) \to \mathrm{Hom}_{\mathcal{U}}(\pi(M), \pi(M))$ sends $\cdot\underline{a}$ to $\pi_{M,M}(a)$, where $\mathcal{U} := \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}}\, \Pi(w))/\mathsf{K}^{\mathrm{b}}(\mathsf{proj}^{\mathbb{Z}}\, \Pi(w))$. By Theorem 8.8 (b), the composition $\rho_{M,M} \circ \pi_{M,M} : \mathrm{Hom}^{\mathbb{Z}}_{\Pi(w)}(M,M) \to \underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M,M)$ corresponds to a canonical morphism. Therefore, $\Phi_{\underline{A},\underline{A}} = (\rho \circ \pi \circ N \otimes^{\mathbb{L}}_{\underline{A}} -)_{\underline{A},\underline{A}}$ sends $\cdot\underline{a}$ to $\underline{a}$. This means that $\Phi_{\underline{A},\underline{A}}$ induces an isomorphism $\mathrm{Hom}_{\underline{A}}(\underline{A},\underline{A}) \simeq \underline{\mathrm{Hom}}^{\mathbb{Z}}_{\Pi(w)}(M,M)$.

By Lemma 3.10, the functor $\Phi = \rho^{\mathbb{Z}}_{\Pi(w)} \circ (N \otimes^{\mathbf{L}}_{\underline{A}} -)$ is an equivalence.

(b) We have the following commutative diagram up to isomorphism of functors

$$\begin{tikzcd}
\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\ \underline{A}) \arrow[rr, "N\otimes^{\mathbf{L}}_{\underline{A}}-"] \arrow[dr, "N\otimes^{\mathbf{L}}_{\underline{A}}-"'] \arrow[d, "\pi_{\underline{A}}"] & & \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}}\,\Pi(w)) \arrow[dl] \arrow[d, "\rho^{\mathbb{Z}}_{\Pi(w)}"] \\
\mathsf{C}(\underline{A}) \arrow[dr, "G"] & \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\ \Pi(w)) \arrow[d, "\rho_{\Pi(w)}"] & \underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w) \arrow[dl] \\
 & \underline{\mathsf{Sub}}\,\Pi(w) &
\end{tikzcd}$$

where $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}}\,\Pi(w)) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Pi(w))$ and $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi(w) \to \underline{\mathsf{Sub}}\,\Pi(w)$ are degree forgetful functors. In particular, we obtain the desired diagram. $\square$

Part III
Stable categories of hereditary algebras and derived categories

This part is based on the paper [Ki17].

Notation

In this part, we denote by k a field. All subcategories are full and closed under isomorphisms. Let $\mathcal{C}$ be an additive category and $\mathcal{S}$ be a subclass of objects of $\mathcal{C}$ or a subcategory of $\mathcal{C}$. We denote by $\mathsf{add}\,\mathcal{S}$ the subcategory of $\mathcal{C}$ whose objects are direct summands of finite direct sums of objects in $\mathcal{S}$. For subcategories $\mathcal{C}_i$ $(i \in I)$ of $\mathcal{C}$, we denote by $\bigvee_{i \in I} \mathcal{C}_i$ the smallest additive subcategory of $\mathcal{C}$ containing all $\mathcal{C}_i$ and closed under direct summands. For objects $X, Y \in \mathcal{C}$, we denote by $\mathcal{C}(X, Y)$ the set of morphisms from X to Y in $\mathcal{C}$. We call a category *skeletally small* if the class of isomorphism class of objects is a set. We assume that all categories in this paper are skeletally small.

12 Preliminaries

12.1 Functor categories

In this subsection, we recall the definition of modules over categories. Let $\mathcal{A}$ be an additive category. An $\mathcal{A}$*-module* is a contravariant additive functor from $\mathcal{A}$ to $\mathcal{A}b$, where $\mathcal{A}b$ is the category of abelian groups. We denote by $\mathsf{Mod}\,\mathcal{A}$ the category of $\mathcal{A}$-modules, where morphisms of $\mathsf{Mod}\,\mathcal{A}$ are morphisms of functors. Since $\mathcal{A}$ is skeletally small, $\mathsf{Mod}\,\mathcal{A}$ is a category. It is well known that $\mathsf{Mod}\,\mathcal{A}$ is abelian.

For two morphisms $f : L \to M$ and $g : M \to N$ of $\mathsf{Mod}\,\mathcal{A}$, the sequence $L \to M \to N$ is exact in $\mathsf{Mod}\,\mathcal{A}$ if and only if the induced sequence $L(X) \to M(X) \to N(X)$ is exact in $\mathcal{A}b$ for any $X \in \mathcal{A}$.

Example 12.1. For each $X \in \mathcal{A}$, we have an $\mathcal{A}$-module $\mathcal{A}(-, X)$. By Yoneda's lemma, $\mathcal{A}(-, X)$ is projective in $\mathsf{Mod}\,\mathcal{A}$.

The following notation is basic and used throughout this paper. We call an $\mathcal{A}$-module M *finitely generated* if there exists an epimorphism $\mathcal{A}(-, X) \to M$ in $\mathsf{Mod}\,\mathcal{A}$ for some $X \in \mathcal{A}$. We denote by $\mathsf{proj}\,\mathcal{A}$ the subcategory of $\mathsf{Mod}\,\mathcal{A}$ consisting of all finitely generated projective $\mathcal{A}$-modules. Note that finitely generated projective modules are precisely direct summands of representable functors. We need the following notation which is called FP_n in some literatures (e.g. [BGI, Br]).

Definition 12.2. Let $\mathcal{A}$ be an additive category and $n \geq 0$ be an integer.

(1) We denote by $\mathsf{mod}_n\,\mathcal{A}$ the subcategory of $\mathsf{Mod}\,\mathcal{A}$ consisting of all $\mathcal{A}$-modules M such that there exists an exact sequence

$$P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

in $\mathsf{Mod}\,\mathcal{A}$, where P_i is in $\mathsf{proj}\,\mathcal{A}$ for each $0 \leq i \leq n$.

(2) We denote by $\mathsf{mod}\,\mathcal{A}$ the subcategory of $\mathsf{Mod}\,\mathcal{A}$ consisting of all $\mathcal{A}$-modules M such that there exists an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

in $\mathsf{Mod}\,\mathcal{A}$, where P_i is in $\mathsf{proj}\,\mathcal{A}$ for each $i \geq 0$.

The following lemma is a basic observation on $\mathsf{mod}_n\,\mathcal{A}$.

Lemma 12.3. *The following statements hold for an additive category $\mathcal{A}$.*

(a) *Let $M \in \mathsf{mod}_n\,\mathcal{A}$. Assume that there exists an exact sequence $P_l \to P_{l-1} \to \cdots \to P_0 \to M \to 0$ with $P_i \in \mathsf{proj}\,\mathcal{A}$ and $l \leq n$. Then there exist $P_{l+1}, \ldots, P_n \in \mathsf{proj}\,\mathcal{A}$ and an exact sequence $P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$.*

(b) *Let $M \in \mathsf{Mod}\,\mathcal{A}$. Assume that there exist the following two exact sequences*

$$0 \to K \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0,$$
$$0 \to L \to Q_n \to Q_{n-1} \to \cdots \to Q_0 \to M \to 0,$$

where $P_i, Q_i \in \mathsf{proj}\,\mathcal{A}$ for each $i \geq 0$. Then there exist $P, Q \in \mathsf{proj}\,\mathcal{A}$ such that $K \oplus P \simeq L \oplus Q$.

Proof. (a) This follows from (b).

(b) The case where $n = 0$ is well known as Schanuel's Lemma. The case where $n > 0$ is shown by an induction on n and by using the case where $n = 0$. □

The following lemma gives a sufficient condition when an $\mathcal{A}$-module is in $\mathsf{mod}_n\,\mathcal{A}$. For simplicity, we use the notation $\mathsf{mod}_{-1}\,\mathcal{A} := \mathsf{Mod}\,\mathcal{A}$, $\mathsf{mod}_{\infty}\,\mathcal{A} := \mathsf{mod}\,\mathcal{A}$ and $\infty - 1 := \infty$.

Lemma 12.4. *Let $\mathcal{A}$ be an additive category and M be an $\mathcal{A}$-module. Then we have the following properties.*

(a) *Let $n \geq 0$ be an integer. If there exists an exact sequence $X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0$ in $\mathsf{Mod}\,\mathcal{A}$ with $X_i \in \mathsf{mod}_{n-i}\,\mathcal{A}$ for any $0 \leq i \leq n$, then we have $M \in \mathsf{mod}_n\,\mathcal{A}$.*

(b) *If there exists an exact sequence $\cdots \to X_2 \to X_1 \to X_0 \to M \to 0$ in $\mathsf{Mod}\,\mathcal{A}$ with $X_i \in \mathsf{mod}\,\mathcal{A}$ for any $i \geq 0$, then we have $M \in \mathsf{mod}\,\mathcal{A}$.*

(c) *Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. For an exact sequence $0 \to L \to M \to N \to 0$ in $\mathsf{Mod}\,\mathcal{A}$ with $L \in \mathsf{mod}_{n-1}\,\mathcal{A}$ and $M \in \mathsf{mod}_n\,\mathcal{A}$, we have $N \in \mathsf{mod}_n\,\mathcal{A}$.*

Proof. (a) We have the following commutative diagram

$$\begin{tikzcd}
X_n \arrow[r] & X_{n-1} \arrow[r] & \cdots \arrow[r] & X_0 \arrow[r] & M \arrow[r] & 0 \\
P_{n,0} \arrow[u] \arrow[r] & P_{n-1,0} \arrow[u] \arrow[r] & \cdots \arrow[r] & P_{0,0} \arrow[u] & & \\
& P_{n-1,1} \arrow[u] \arrow[r] & \cdots \arrow[r] & P_{0,1} \arrow[u] & & \\
& & & \cdots \arrow[u] & & \\
& & & P_{0,n} \arrow[u] & &
\end{tikzcd}$$

in $\mathsf{Mod}\,\mathcal{A}$, where each $P_{i,0} \to X_i$ is epimorphism for $0 \le i \le n$, each vertical sequence is exact and each $P_{i,j}$ is in $\mathsf{proj}\,\mathcal{A}$. Thus we have an exact sequence

$$\overline{P}_n \to \cdots \to \overline{P}_1 \to \overline{P}_0 \to M \to 0$$

in $\mathsf{Mod}\,\mathcal{A}$, where $\overline{P}_i = \bigoplus_{j=0}^{i} P_{j,i-j}$ for $0 \le i \le n$. Since $\overline{P}_i$ is in $\mathsf{proj}\,\mathcal{A}$ for $0 \le i \le n$, M is an object of $\mathsf{mod}_n\,\mathcal{A}$.

(b) This comes from the same argument as (a).

(c) This follows from (a) for $n \in \mathbb{Z}_{\geq 0}$ and (b) for $n = \infty$. $\square$

Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ be a subcategory of $\mathcal{A}$. We say that $\mathcal{B}$ is a *thick* subcategory of $\mathcal{A}$ if $\mathcal{B}$ is closed under direct summands and for any exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, if two of X, Y, Z are in $\mathcal{A}$, then so is the third. We have the following observation of the categories $\mathsf{mod}_n\,\mathcal{A}$.

Lemma 12.5. *Let $\mathcal{A}$ be an additive category. Then we have the following statements.*

(a) $\mathsf{mod}_n\,\mathcal{A}$ *is closed under extensions and direct summands in* $\mathsf{Mod}\,\mathcal{A}$ *for each* $n \ge 0$.

(b) $\mathsf{mod}\,\mathcal{A} = \bigcap_{n \ge 0} \mathsf{mod}_n\,\mathcal{A}$ *holds.*

(c) (e.g. [E, Proposition 2.6]) $\mathsf{mod}\,\mathcal{A}$ *is a thick subcategory of* $\mathsf{Mod}\,\mathcal{A}$.

Proof. (a) By Horseshoe Lemma, $\mathsf{mod}_n\,\mathcal{A}$ is closed under extensions in $\mathsf{Mod}\,\mathcal{A}$. Let $X \oplus Y \in \mathsf{mod}_n\,\mathcal{A}$. We show that $X, Y \in \mathsf{mod}_n\,\mathcal{A}$ by an induction on n. If $n = 0$, then the claim is clear. Assume $n > 0$. Since $X \oplus Y \in \mathsf{mod}_n\,\mathcal{A} \subset \mathsf{mod}_{n-1}\,\mathcal{A}$ holds, by the inductive hypothesis, we have $X, Y \in \mathsf{mod}_{n-1}\,\mathcal{A}$. Then by Lemma 12.4 (c), we have $X, Y \in \mathsf{mod}_n\,\mathcal{A}$.

(b) In general $\mathsf{mod}\,\mathcal{A} \subset \mathsf{mod}_n\,\mathcal{A}$ holds for each $n \ge 0$. The converse follows from Lemma 12.3 (a).

(c) By (a) and (b), $\mathsf{mod}\,\mathcal{A}$ is closed under extensions and direct summands. Let $0 \to L \to M \to N \to 0$ be an exact sequence in $\mathsf{Mod}\,\mathcal{A}$. By Lemma 12.4 (c), if $L, M \in \mathsf{mod}\,\mathcal{A}$, then $N \in \mathsf{mod}\,\mathcal{A}$ holds. Assume that $M, N \in \mathsf{mod}\,\mathcal{A}$. There exists an exact sequence $0 \to \Omega N \to P \to N \to 0$ such that $P \in \mathsf{proj}\,\mathcal{A}$ and $\Omega N \in \mathsf{mod}\,\mathcal{A}$. By taking a pull-back diagram of $M \to N \leftarrow P$, we have an exact sequence $0 \to \Omega N \to P \oplus L \to M \to 0$. Since $\mathsf{mod}\,\mathcal{A}$ is closed under extensions and direct summands, we have $L \in \mathsf{mod}\,\mathcal{A}$. $\square$

12.2 Gorenstein-projective modules

We define Gorenstein-projective modules. Let $\mathcal{A}$ be an additive category. We first define a contravariant functor

$$(-)^* : \mathsf{Mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathcal{A}^{\mathrm{op}}$$

as follows: for $M \in \mathsf{Mod}\,\mathcal{A}$ and $X \in \mathcal{A}$, let $(M)^*(X) := (\mathsf{Mod}\,\mathcal{A})(M, \mathcal{A}(-, X))$. By the same way, we define a contravariant functor $(-)^* : \mathsf{Mod}\,\mathcal{A}^{\mathrm{op}} \to \mathsf{Mod}\,\mathcal{A}$. Let $P_\bullet := (P_i, d_i : P_i \to P_{i+1})_{i \in \mathbb{Z}}$ be a complex of finitely generated projective $\mathcal{A}$-modules. We say that $P_\bullet$ is *totally acyclic* if complexes $P_\bullet$ and $\cdots \to (P_{i+1})^* \to (P_i)^* \to (P_{i-1})^* \to \cdots$ are acyclic.

Definition 12.6. Let $\mathcal{A}$ be an additive category. An $\mathcal{A}$-module M is said to be *Gorenstein-projective* if there exists a totally acyclic complex $P_\bullet$ such that $\operatorname{Im} d_0$ is isomorphic to M. We denote by $\mathsf{GP}\mathcal{A}$ the full subcategory of $\mathsf{Mod}\,\mathcal{A}$ consisting of all Gorenstein-projective $\mathcal{A}$-modules.

For instance, a finitely generated projective $\mathcal{A}$-module is Gorenstein-projective. In general, $\mathsf{GP}\mathcal{A} \subset \mathsf{mod}\,\mathcal{A}$ holds. We see a fundamental properties of Gorenstein-projective modules.

Let $\mathcal{W}$ be a subcategory of $\mathsf{Mod}\,\mathcal{A}$. We denote by ${}^\perp\mathcal{W}$ the subcategory of $\mathsf{Mod}\,\mathcal{A}$ consisting of $\mathcal{A}$-modules M satisfying $\operatorname{Ext}^i_{\mathsf{Mod}\,\mathcal{A}}(M, W) = 0$ for any $W \in \mathcal{W}$ and any $i > 0$. We denote by $\mathcal{X}_\mathcal{W}$ the subcategory of ${}^\perp\mathcal{W}$ consisting of $\mathcal{A}$-modules M such that there exists an exact sequence $0 \to M \to W_0 \xrightarrow{f_0} W_1 \xrightarrow{f_1} \cdots$ with $W_i \in \mathcal{W}$ and $\operatorname{Im} f_i \in {}^\perp\mathcal{W}$ for any $i \geq 0$. By [AR91, Proposition 5.1], $\mathcal{X}_{\mathsf{proj}\,\mathcal{A}}$ is closed under extensions, direct summands and kernels of epimorphisms in $\mathsf{Mod}\,\mathcal{A}$.

Lemma 12.7. *Let $\mathcal{A}$ be an additive category. Then the following holds.*

(a) *The functor $(-)^* : \mathsf{Mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathcal{A}^{\mathrm{op}}$ induces a duality $(-)^* : \mathsf{GP}\mathcal{A} \to \mathsf{GP}\mathcal{A}^{\mathrm{op}}$.*

(b) *$\mathcal{X}_{\mathsf{proj}\,\mathcal{A}} \cap \mathsf{mod}\,\mathcal{A} = \mathsf{GP}\mathcal{A}$ holds. In particular, $\mathsf{GP}\mathcal{A}$ is closed under extensions, direct summands and kernels of epimorphisms in $\mathsf{Mod}\,\mathcal{A}$.*

Proof. (a) This follows from the definition of $\mathsf{GP}\mathcal{A}$ and the fact that $(-)^*$ induces a duality between $\mathsf{proj}\,\mathcal{A}$ and $\mathsf{proj}\,\mathcal{A}^{\mathrm{op}}$.

(b) In general $\mathcal{X}_{\mathsf{proj}\,\mathcal{A}} \cap \mathsf{mod}\,\mathcal{A} \supset \mathsf{GP}\mathcal{A}$ holds. If $M \in \mathcal{X}_{\mathsf{proj}\,\mathcal{A}} \cap \mathsf{mod}\,\mathcal{A}$, then there exists an exact sequence $P_\bullet = (P_i, d_i : P_i \to P_{i+1})_{i \in \mathbb{Z}}$, where $M \simeq \operatorname{Im} d_0$, $P_i \in \mathsf{proj}\,\mathcal{A}$ for any $i \in \mathbb{Z}$ and $\operatorname{Im} d_i \in {}^\perp(\mathsf{proj}\,\mathcal{A})$ for any $i \geq 1$. Then this sequence is totally acyclic, since $\operatorname{Im} d_i \in {}^\perp(\mathsf{proj}\,\mathcal{A})$ holds for any $i \geq 1$. $\square$

Let $\mathcal{B}$ be an extension closed subcategory of an abelian category $\mathcal{A}$. An exact sequence in $\mathcal{A}$ is called an exact sequence in $\mathcal{B}$ if each term of it is an object of $\mathcal{B}$. We say that an object Z in $\mathcal{B}$ is *relative-projective* if any exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{B}$ splits. Dually, we define *relative-injective* objects. We say that $\mathcal{B}$ has *enough projectives* if for any $X \in \mathcal{B}$, there exists an exact sequence $0 \to Z \to P \to X \to 0$ in $\mathcal{B}$ such that P is relative-projective. Dually, we define a subcategory of $\mathcal{A}$ which has *enough injectives*. An extension closed subcategory $\mathcal{B}$ of $\mathcal{A}$ is said to be *Frobenius* if $\mathcal{B}$ has enough projectives, enough injectives and the relative-projective objects coincide with the relative-injective objects.

The following observation is immediate (cf. [C]).

Proposition 12.8. *Let $\mathcal{A}$ be an additive category. Then $\mathsf{GP}\mathcal{A}$ is a Frobenius category, where the relative-projective objects are precisely finitely generated $\mathcal{A}$-modules.*

Proof. $\mathsf{GP}\mathcal{A}$ is extension closed in $\mathsf{Mod}\,\mathcal{A}$ by Lemma 12.7 (b). By the definition of $\mathsf{GP}\mathcal{A}$ and the duality $(-)^* : \mathsf{GP}\mathcal{A} \to \mathsf{GP}\mathcal{A}^{\mathrm{op}}$, $\mathsf{GP}\mathcal{A}$ has enough projectives and enough injectives. Again by the definition of $\mathsf{GP}\mathcal{A}$, the relative-projective objects coincide with the relative-injective objects, which coincide with finitely generated projective $\mathcal{A}$-modules. $\square$

12.3 Dualizing k-varieties and Serre dualities

In this subsection, we recall the definition of dualizing k-varieties. Let $\mathcal{A}$ be an additive category. We call an object of $\mathsf{mod}_1 \mathcal{A}$ a *finitely presented* $\mathcal{A}$-module.

A morphism $X \to Y$ in $\mathcal{A}$ is a *weak kernel* of a morphism $Y \to Z$ if the induced sequence $\mathcal{A}(-, X) \to \mathcal{A}(-, Y) \to \mathcal{A}(-, Z)$ is exact in $\mathsf{Mod}\,\mathcal{A}$. We say that $\mathcal{A}$ has weak kernels if each morphism in $\mathcal{A}$ has a weak kernel. The following lemma says when an additive category has weak kernels.

Lemma 12.9. *Let $\mathcal{A}$ be an additive category. The following statements are equivalent.*

(i) *$\mathcal{A}$ has weak kernels.*

(ii) *$\mathsf{mod}_1 \mathcal{A}$ is abelian.*

(iii) *$\mathsf{mod}_1 \mathcal{A} = \mathsf{mod}\,\mathcal{A}$ holds.*

Proof. It is well known that the statements (i) and (ii) are equivalent. The statements (i) and (iii) are equivalent by [E, Proposition 2.7]. □

Let $\mathcal{A}$ be an additive category and $X \in \mathcal{A}$. A morphism $e : X \to X$ in $\mathcal{A}$ is called an *idempotent* if $e^2 = e$. We call $\mathcal{A}$ *idempotent complete* if each idempotent of $\mathcal{A}$ has a kernel.

Let k be a field. A *k-linear category* $\mathcal{A}$ is a category such that $\mathcal{A}(X, Y)$ admits a structure of k-modules and the composition of morphisms of $\mathcal{A}$ is k-bilinear. A contravariant functor $F : \mathcal{A} \to \mathcal{B}$ between k-linear categories are called *k-functor* if $F_{X,Y} : \mathcal{A}(X, Y) \to \mathcal{B}(FY, FX)$ is k-linear for any $X, Y \in \mathcal{A}$. If $\mathcal{A}$ is an additive k-linear category, then any $\mathcal{A}$-module can be regarded as a contravariant additive k-functor from $\mathcal{A}$ to $\mathsf{Mod}\,k$, where $\mathsf{Mod}\,k$ is the category of k-modules.

Let $\mathcal{A}$ be a k-linear additive category. We call $\mathcal{A}$ *Hom-finite* if $\mathcal{A}(X, Y)$ is finitely generated over k for any $X, Y \in \mathcal{A}$. We recall one proposition about the Krull-Schmidt property of k-linear additive categories.

Proposition 12.10. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category. Then the following properties are equivalent.*

(i) *$\mathcal{A}$ is idempotent complete.*

(ii) *The endomorphism algebra of each indecomposable object in $\mathcal{A}$ is local.*

(iii) *$\mathcal{A}$ is Krull-Schmidt, that is, each object of $\mathcal{A}$ is a finite direct sum of objects whose endomorphism algebras are local.*

Moreover the decomposition of (iii) *is unique up to isomorphism.*

Proposition 12.11. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category. Then* $\mathsf{mod}\,\mathcal{A}$ *is Krull-Schmidt. In particular, each object of* $\mathsf{mod}\,\mathcal{A}$ *has a minimal projective resolution.*

Proof. Since $\mathsf{mod}\,\mathcal{A}$ is closed under direct summands in $\mathsf{Mod}\,\mathcal{A}$, $\mathsf{mod}\,\mathcal{A}$ is idempotent complete. $\mathsf{mod}\,\mathcal{A}$ is Hom-finite, since $\mathcal{A}$ is Hom-finite. □

We recall the definition of dualizing k-varieties. Let $\mathcal{A}$ be a k-linear additive category. We have contravariant functors $\mathrm{D} : \mathsf{Mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathcal{A}^{\mathrm{op}}$ and $\mathrm{D} : \mathsf{Mod}\,\mathcal{A}^{\mathrm{op}} \to \mathsf{Mod}\,\mathcal{A}$ given by $(\mathrm{D}\,M)(X) := \mathrm{D}(M(X))$.

Definition 12.12. Let $\mathcal{A}$ be a k-linear, Hom-finite, idempotent complete additive category. We call $\mathcal{A}$ a *dualizing k-variety* if the functor $\mathrm{D} : \mathsf{Mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathcal{A}^{\mathrm{op}}$ induces a duality between $\mathsf{mod}_1\,\mathcal{A}$ and $\mathsf{mod}_1\,\mathcal{A}^{\mathrm{op}}$.

The following is typical examples of dualizing k-varieties.

Example 12.13. [AR74]

(a) If $\mathcal{A}$ is a dualizing k-variety, then $\mathcal{A}^{\mathrm{op}}$ is a dualizing k-variety.

(b) Let A be a finite dimensional k-algebra and $\mathsf{mod}\,A$ be the category of finitely generated A-modules. Let $\mathsf{proj}\,A$ be the full subcategory of $\mathsf{mod}\,A$ consisting of all finitely generated projective A-modules. Then $\mathsf{mod}\,A$ and $\mathsf{proj}\,A$ are dualizing k-varieties.

We state some properties of dualizing k-varieties.

Lemma 12.14. *[AR74] Let $\mathcal{A}$ be a dualizing k-variety, then we have the following properties.*

(a) *$\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ have weak kernels.*

(b) *$\mathsf{mod}\,\mathcal{A}$ is a dualizing k-variety.*

(c) *Each object in $\mathsf{mod}\,\mathcal{A}$ has a projective cover and an injective hull.*

Let $\mathcal{A}$ be a k-linear, Hom-finite additive category. A *Serre functor* on $\mathcal{A}$ is an auto-equivalence $\mathbb{S} : \mathcal{A} \to \mathcal{A}$ such that there exists a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \simeq \mathrm{D}\,\mathrm{Hom}_{\mathcal{A}}(Y, \mathbb{S}(X))$$

for any $X, Y \in \mathcal{A}$. We denote by $\mathbb{S}^{-1}$ a quasi-inverse of $\mathbb{S}$. It is easy to see that if $\mathcal{A}$ has a Serre functor $\mathbb{S}$, then $\mathcal{A}^{\mathrm{op}}$ has a Serre functor $\mathbb{S}^{-1}$.

If $\mathcal{A}$ has a Serre functor $\mathbb{S}$, then $(-)^*$ is described as in the following lemma. Since $\mathbb{S}$ is an auto-equivalence, we have an equivalence $\mathsf{Mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathcal{A}$ given by $M \mapsto M \circ \mathbb{S}^{-1}$. By composing the functor $\mathrm{D} : \mathsf{Mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathcal{A}^{\mathrm{op}}$, we have a contravariant functor $\mathsf{Mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathcal{A}^{\mathrm{op}}$ given by $M \mapsto \mathrm{D}(M \circ \mathbb{S}^{-1})$. We denote by $\mathsf{Mod}_{\mathrm{fg}}\,\mathcal{A}$ the subcategory of $\mathsf{Mod}\,\mathcal{A}$ consisting of $\mathcal{A}$-modules M such that $M(X)$ is finitely generated over k for any $X \in \mathcal{A}$. Note that D induces a duality $\mathsf{Mod}_{\mathrm{fg}}\,\mathcal{A} \to \mathsf{Mod}_{\mathrm{fg}}\,\mathcal{A}^{\mathrm{op}}$ and the categories $\mathsf{mod}_0\,\mathcal{A}$ and $\mathsf{GP}\mathcal{A}$ are contained in $\mathsf{Mod}_{\mathrm{fg}}\,\mathcal{A}$.

Lemma 12.15. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category with a Serre functor $\mathbb{S}$. Then the following statements hold.*

(a) *We have an isomorphism of functors $(-)^* \simeq \mathrm{D}(- \circ \mathbb{S}^{-1}) : \mathsf{Mod}_{\mathrm{fg}}\,\mathcal{A} \to \mathsf{Mod}_{\mathrm{fg}}\,\mathcal{A}^{\mathrm{op}}$, and this functor is a duality.*

(b) *Let $M \in \mathsf{Mod}\,\mathcal{A}$. The following statements are equivalent.*

  (i) *$M \in \mathsf{GP}\mathcal{A}$.*

  (ii) *$M \in \mathsf{mod}\,\mathcal{A}$ and $M^* \in \mathsf{mod}\,\mathcal{A}^{\mathrm{op}}$.*

Proof. (a) Let $M \in \mathsf{Mod}_{\mathrm{fg}}\,\mathcal{A}$ and $X \in \mathcal{A}$. We have the following equalities.

$$\begin{aligned}(M)^*(X) &= (\mathsf{Mod}\,\mathcal{A})(M, \mathcal{A}(-,X)) \\ &\simeq (\mathsf{Mod}\,\mathcal{A}^{\mathrm{op}})(\mathrm{D}\,\mathcal{A}(-,X), \mathrm{D}\,M) \\ &\simeq (\mathsf{Mod}\,\mathcal{A}^{\mathrm{op}})(\mathcal{A}(\mathbb{S}^{-1}(X),-), \mathrm{D}\,M) \\ &\simeq \mathrm{D}(M(\mathbb{S}^{-1}(X))),\end{aligned}$$

which functorial on X. Thus we have an isomorphism of functors $(-)^* \simeq \mathrm{D}(-\circ\mathbb{S}^{-1})$. This functor is a duality, since D is a duality and $\mathbb{S}$ is an equivalence.

(b) Assume that $M \in \mathsf{GP}\mathcal{A}$. By Lemma 12.7 (a), we have $M^* \in \mathsf{GP}\mathcal{A}^{\mathrm{op}}$. In general $\mathsf{GP}\mathcal{A} \subset \mathsf{mod}\,\mathcal{A}$ holds, thus (i) implies (ii). Assume that (ii) holds. There exists an exact sequence $\cdots \to Q_2 \to Q_1 \to M^* \to 0$, where $Q_i \in \mathsf{proj}\,\mathcal{A}^{\mathrm{op}}$. By (a), $(-)^*$ is an exact functor. Therefore we have an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{d} Q_1^* \to Q_2^* \to \cdots,$$

where $P_i, Q_i^* \in \mathsf{proj}\,\mathcal{A}$ and $\mathrm{Im}\,d \simeq M$. This exact sequence is totally acyclic, since $(-)^*$ is exact. We have $M \in \mathsf{GP}\mathcal{A}$. □

Later we use the following characterization of dualizing k-varieties with Serre functors.

Proposition 12.16. *Let $\mathcal{A}$ be a k-linear, Hom-finite, idempotent complete additive category. Then the following statements are equivalent.*

(i) *$\mathcal{A}$ is a dualizing k-variety and has a Serre functor.*

(ii) *$\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ have weak kernels and $\mathcal{A}$ has a Serre functor.*

(iii) *$\mathsf{GP}\mathcal{A} = \mathsf{mod}_1\,\mathcal{A}$, $\mathsf{GP}\mathcal{A}^{\mathrm{op}} = \mathsf{mod}_1\,\mathcal{A}^{\mathrm{op}}$ hold and $\mathrm{D}\,\mathcal{A}(X,-) \in \mathsf{mod}_1\,\mathcal{A}$, $\mathrm{D}\,\mathcal{A}(-,X) \in \mathsf{mod}_1\,\mathcal{A}^{\mathrm{op}}$ hold for any $X \in \mathcal{A}$.*

Proof. By Lemma 12.14, (i) implies (ii). We show that (ii) implies (i). Let $M \in \mathsf{mod}_1\,\mathcal{A}$. We show that $\mathrm{D}\,M$ is in $\mathsf{mod}_1\,\mathcal{A}^{\mathrm{op}}$. There exists an exact sequence $P_1 \to P_0 \to M \to 0$ for some $P_1, P_0 \in \mathsf{proj}\,\mathcal{A}$. By the functor $\mathrm{D} : \mathsf{Mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathcal{A}^{\mathrm{op}}$, we have an exact sequence $0 \to \mathrm{D}\,M \to \mathrm{D}\,P_0 \to \mathrm{D}\,P_1$ in $\mathsf{Mod}\,\mathcal{A}$. Since $\mathcal{A}$ has a Serre functor, we have $\mathrm{D}\,P_1, \mathrm{D}\,P_0 \in \mathsf{proj}\,\mathcal{A}^{\mathrm{op}}$. Since $\mathcal{A}^{\mathrm{op}}$ has weak kernels, $\mathrm{D}\,M$ is in $\mathsf{mod}_1\,\mathcal{A}^{\mathrm{op}}$. By the dual argument, for any $N \in \mathsf{mod}_1\,\mathcal{A}^{\mathrm{op}}$, we have $\mathrm{D}\,N \in \mathsf{mod}_1\,\mathcal{A}$. Thus $\mathrm{D} : \mathsf{mod}_1\,\mathcal{A} \to \mathsf{mod}_1\,\mathcal{A}^{\mathrm{op}}$ is a duality.

We show that (i) implies (iii). Since $\mathcal{A}$ is a dualizing k-variety, $\mathrm{D}\,\mathcal{A}(X,-) \in \mathsf{mod}_1\,\mathcal{A}$, $\mathrm{D}\,\mathcal{A}(-,X) \in \mathsf{mod}_1\,\mathcal{A}^{\mathrm{op}}$ hold for any $X \in \mathcal{A}$. By Lemma 12.9, we have $\mathsf{mod}\,\mathcal{A} = \mathsf{mod}_1\,\mathcal{A}$ and $\mathsf{mod}\,\mathcal{A}^{\mathrm{op}} = \mathsf{mod}_1\,\mathcal{A}^{\mathrm{op}}$. In general $\mathsf{GP}\mathcal{A} \subset \mathsf{mod}\,\mathcal{A}$ holds. Let $M \in \mathsf{mod}\,\mathcal{A}$. We show that $M \in \mathsf{GP}\mathcal{A}$. Since $\mathcal{A}$ is a dualizing k-variety, $\mathrm{D}\,M \in \mathsf{mod}\,\mathcal{A}^{\mathrm{op}}$ holds. By Lemma 12.15 (a), $M^* \in \mathsf{mod}\,\mathcal{A}^{\mathrm{op}}$ holds. Thus by Lemma 12.15 (b), $M \in \mathsf{GP}\mathcal{A}$ holds.

We show that (iii) implies (ii). In general, $\mathsf{GP}\mathcal{A} \subset \mathsf{mod}\,\mathcal{A} \subset \mathsf{mod}_1\,\mathcal{A}$ holds. Therefore by Lemma 12.9, $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ have weak kernels. Consider the functor $\mathrm{D}\circ(-)^* : \mathsf{Mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathcal{A}$. This functor induces an equivalence $\mathsf{proj}\,\mathcal{A} \xrightarrow{\sim} \mathsf{proj}\,\mathcal{A}$. In fact, if $M \in \mathsf{proj}\,\mathcal{A}$, then $M^* \in \mathsf{proj}\,\mathcal{A}^{\mathrm{op}}$. By the assumption, we have $\mathrm{D}(M^*) \in \mathsf{mod}_1\,\mathcal{A} = \mathsf{GP}\mathcal{A}$. Since $\mathrm{D} : \mathsf{Mod}_{\mathrm{fg}}\,\mathcal{A}^{\mathrm{op}} \to \mathsf{Mod}_{\mathrm{fg}}\,\mathcal{A}$ is a duality, $\mathrm{D}(M^*)$ is an injective object of $\mathsf{Mod}_{\mathrm{fg}}\,\mathcal{A}$. In particular, $\mathrm{D}(M^*)$ is a relative-injective object of $\mathsf{GP}\mathcal{A}$. Since $\mathsf{GP}\mathcal{A}$ is Frobenius, $\mathrm{D}(M^*)$ is an object of $\operatorname{\mathsf{proj}}\mathcal{A}$. Thus we have a functor $\mathrm{D}\circ(-)^* : \operatorname{\mathsf{proj}}\mathcal{A} \to \operatorname{\mathsf{proj}}\mathcal{A}$. This is an equivalence, since its quasi-inverse is given by $(-)^*\circ\mathrm{D}$. Since $\mathcal{A}$ is idempotent complete, the Yoneda embedding $\mathcal{A} \to \operatorname{\mathsf{proj}}\mathcal{A}$, $X \mapsto \mathcal{A}(-,X)$ is equivalence. Thus there exists an equivalence $\mathbb{S} : \mathcal{A} \to \mathcal{A}$ such that the following diagram is commutative:

$$\begin{tikzcd} \operatorname{\mathsf{proj}}\mathcal{A} \arrow[r, "\mathrm{D}\circ(-)^*"] & \operatorname{\mathsf{proj}}\mathcal{A} \\ \mathcal{A} \arrow[u, "\simeq"] \arrow[r, "\mathbb{S}"] & \mathcal{A}. \arrow[u, "\simeq"'] \end{tikzcd}$$

For $X, Y \in \mathcal{A}$, we have the following isomorphisms which are functorial at X, Y:

$$\begin{aligned} \mathcal{A}(Y, \mathbb{S}X) &\simeq \mathrm{D}(\mathcal{A}(-,X)^*)(Y) \\ &\simeq \mathrm{D}(\operatorname{\mathsf{Mod}}\mathcal{A}(\mathcal{A}(-,X), \mathcal{A}(-,Y))) \\ &\simeq \mathrm{D}\,\mathcal{A}(X,Y). \end{aligned}$$

This means that $\mathbb{S}$ is a Serre functor on $\mathcal{A}$. □

12.4 Some observations on triangulated categories

In this subsection, we state some propositions which we use later. We state one theorem for Frobenius categories. Let $\mathcal{A}$ be an additive category and $\mathcal{B}$ be a subcategory of $\mathcal{A}$. For two objects $X, Y \in \mathcal{A}$, we denote by $\mathcal{A}_{\mathcal{B}}(X,Y)$ the subspace of $\mathcal{A}(X,Y)$ consisting of all morphisms which factor through an object of $\mathcal{B}$. We denote by $\mathcal{A}/[\mathcal{B}]$ the category defined as follows: the objects of $\mathcal{A}/[\mathcal{B}]$ are the same as $\mathcal{A}$ and the morphism space is defined by

$$(\mathcal{A}/[\mathcal{B}])(X,Y) := \mathcal{A}(X,Y)/\mathcal{A}_{\mathcal{B}}(X,Y),$$

for $X, Y \in \mathcal{A}$.

Let $\mathcal{F}$ be a Frobenius category, $\mathcal{P}$ the full subcategory of $\mathcal{F}$ consisting of the projective objects in $\mathcal{F}$ and $\underline{\mathcal{F}} := \mathcal{F}/[\mathcal{P}]$. By Happel [Ha88], it is known that $\underline{\mathcal{F}}$ is a triangulated category. Assume that $\mathcal{P}$ is idempotent complete. Let $\mathsf{K}^{\mathrm{b}}(\mathcal{P})$ be the homotopy category of complexes of $\mathcal{P}$. We denote by $\mathsf{K}^{-,\mathrm{b}}(\mathcal{P})$ the full subcategory of $\mathsf{K}(\mathcal{P})$ consisting of complexes $X = (X^i, d^i : X^i \to X^{i+1})$ satisfying the following conditions.

(1) There exists $n_X \in \mathbb{Z}$ such that $X^i = 0$ for any $i > n_X$.

(2) There exist $m_X \in \mathbb{Z}$ and exact sequences $0 \to Y^{i-1} \xrightarrow{a^{i-1}} X^i \xrightarrow{b^i} Y^i \to 0$ in $\mathcal{F}$ for any $i \le m_X$ such that $d^i = a^i b^i$ for any $i < m_X$.

We identify the category $\mathcal{F}$ with the full subcategory of $\mathsf{K}^{-,\mathrm{b}}(\mathcal{P})$ consisting of X satisfying $n_X \le 0 \le m_X$. Then we have the following analogy of the well known equivalence due to [Bu, KV, Ri].

Theorem 12.17. *[IYa] Let $\mathcal{F}$ be a Frobenius category and $\mathcal{P}$ the full subcategory of $\mathcal{F}$ consisting of the projective objects. Assume that $\mathcal{P}$ is idempotent complete. Then the composite $\mathcal{F} \to \mathsf{K}^{-,\mathrm{b}}(\mathcal{P}) \to \mathsf{K}^{-,\mathrm{b}}(\mathcal{P})/\mathsf{K}^{\mathrm{b}}(\mathcal{P})$ induces a triangle equivalence $\underline{\mathcal{F}} \xrightarrow{\sim} \mathsf{K}^{-,\mathrm{b}}(\mathcal{P})/\mathsf{K}^{\mathrm{b}}(\mathcal{P})$.*

Let $\mathcal{U}$ be a triangulated category and $\mathcal{X}$ be a full subcategory of $\mathcal{U}$. We call $\mathcal{X}$ a *thick* subcategory of $\mathcal{U}$ if $\mathcal{X}$ is a triangulated subcategory of $\mathcal{U}$ and closed under direct summands. We denote by $\mathsf{thick}_{\mathcal{U}}\, \mathcal{X}$ the smallest thick subcategory of $\mathcal{U}$ which contains $\mathcal{X}$. Whenever if there is no danger of confusion, let $\mathsf{thick}_{\mathcal{U}}\, \mathcal{X} = \mathsf{thick}\, \mathcal{X}$.

Lemma 12.18. *Let $\mathcal{T}$,$\mathcal{U}$ be triangulated categories and $F : \mathcal{U} \to \mathcal{T}$ a triangle functor. Let $\mathcal{X}$ be a full subcategory of $\mathcal{U}$. Then the following holds.*

- *Assume that a map*

$$F_{M,N[n]} : \mathcal{U}(M,N) \to \mathcal{T}(FM, FN[n])$$

  is an isomorphism for any $M, N \in \mathcal{X}$ and any $n \in \mathbb{Z}$. Then $F : \mathsf{thick}\, \mathcal{X} \to \mathcal{T}$ is fully faithful.

- *If moreover $\mathcal{U}$ is idempotent complete, $\mathsf{thick}\, \mathcal{X} = \mathcal{U}$ and $\mathsf{thick}(\operatorname{Im}(F)) = \mathcal{T}$, then F is an equivalence.*

13 Repetitive categories

13.1 Repetitive categories

We recall the definition of repetitive categories of additive categories. The aim of this subsection is to show Theorem 13.7.

Definition 13.1. Let $\mathcal{A}$ be a k-linear additive category. The *repetitive category* $\mathsf{R}\mathcal{A}$ is the k-linear additive category generated by the following category: the class of objects is $\{(X, i) \mid X \in \mathcal{A}, i \in \mathbb{Z}\}$ and the morphism space is given by

$$\mathsf{R}\mathcal{A}\big((X,i),(Y,j)\big) = \begin{cases} \mathcal{A}(X,Y) & i = j, \\ \mathrm{D}\,\mathcal{A}(Y,X) & j = i+1, \\ 0 & \text{else.} \end{cases}$$

For $f \in \mathsf{R}\mathcal{A}\big((X,i),(Y,j)\big)$ and $g \in \mathsf{R}\mathcal{A}\big((Y,j),(Z,k)\big)$, the composition is given by

$$g \circ f = \begin{cases} g \circ f & i = j = k, \\ \big(\mathrm{D}\,\mathcal{A}(Z,f)\big)(g) & i = j = k-1, \\ \big(\mathrm{D}\,\mathcal{A}(g,X)\big)(f) & i+1 = j = k, \\ 0 & \text{else.} \end{cases}$$

We describe fundamental properties of repetitive categories of Hom-finite categories.

Lemma 13.2. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category. The following statements hold.*

(a) $\mathsf{R}\mathcal{A}$ *is* Hom-*finite.*

(b) $\mathsf{R}\mathcal{A}$ *has a Serre functor $\mathbb{S}$ which is defined by $\mathbb{S}(X,i) := (X, i+1)$.*

(c) *If $\mathcal{A}$ is idempotent complete, then so is $\mathsf{R}\mathcal{A}$.*

Proof. (a) (b) These are clear by the definition.

(c) By the definition, an object of $\mathsf{R}\mathcal{A}$ is indecomposable if and only if it is isomorphic to an object (X,i), where X is an indecomposable object of $\mathcal{A}$ and i is some integer. Let X be an indecomposable object of $\mathcal{A}$ and i be an integer. Since $\mathcal{A}$ is idempotent complete and Proposition 12.10, $\mathrm{End}_{\mathsf{R}\mathcal{A}}(X,i) = \mathrm{End}_{\mathcal{A}}(X)$ is local. Therefore again by Proposition 12.10, $\mathsf{R}\mathcal{A}$ is idempotent complete. □

We see a relation between the categories $\mathsf{mod}\,\mathcal{A}$ and $\mathsf{mod}\,\mathsf{R}\mathcal{A}$ and consequently, we show Theorem 13.7. Let $\mathcal{A}$ be a k-linear additive category and $i \in \mathbb{Z}$. Put the following full subcategory of $\mathsf{R}\mathcal{A}$:

$$\mathcal{A}_i := \mathsf{add}\{\, (X,i) \in \mathsf{R}\mathcal{A} \mid X \in \mathcal{A} \,\}.$$

An inclusion functor $\mathcal{A}_i \to \mathsf{R}\mathcal{A}$ induces an exact functor

$$\rho_i : \mathsf{Mod}\,\mathsf{R}\mathcal{A} \to \mathsf{Mod}\,\mathcal{A}_i.$$

Since a functor $\mathcal{A} \to \mathcal{A}_i$ defined by $X \mapsto (X,i)$ is an equivalence, we denote an object (X,i) of $\mathcal{A}_i$ by X for simplicity.

Since we have a full dense functor $\mathsf{R}\mathcal{A} \to \mathcal{A}_i$ given by $(X,j) \mapsto X$ if $j = i$ and $(X,j) \mapsto 0$ if else, we have a fully faithful functor from $\mathsf{Mod}\,\mathcal{A}_i$ to $\mathsf{Mod}\,\mathsf{R}\mathcal{A}$. Therefore we identify $\mathsf{Mod}\,\mathcal{A}_i$ with the full subcategory of $\mathsf{Mod}\,\mathsf{R}\mathcal{A}$ consisting of $\mathsf{R}\mathcal{A}$-modules M such that $M(X,j) = 0$ for any $j \neq i$ and any $X \in \mathcal{A}$.

Lemma 13.3. *Let $\mathcal{A}$ be an additive category and $i,j \in \mathbb{Z}$.*

(a) *We have $\rho_j|_{\mathsf{Mod}\,\mathcal{A}_i} = \mathrm{id}_{\mathsf{Mod}\,\mathcal{A}_i}$ if $j = i$ and $\rho_j|_{\mathsf{Mod}\,\mathcal{A}_i} = 0$ if else.*

(b) *For any $X \in \mathcal{A}$, we have an exact sequence*

$$0 \to \mathrm{D}\,\mathcal{A}_{i-1}(X,-) \xrightarrow{\beta} \mathsf{R}\mathcal{A}(-,(X,i)) \xrightarrow{\alpha} \mathcal{A}_i(-,X) \to 0 \tag{13.1}$$

in $\mathsf{Mod}\,\mathsf{R}\mathcal{A}$. In particular, we have $\rho_j(P) \in \mathsf{add}\{\mathcal{A}_j(-,X), \mathrm{D}\,\mathcal{A}_j(X,-) \mid X \in \mathcal{A}\}$ for any $P \in \mathsf{proj}\,\mathsf{R}\mathcal{A}$ and $j \in \mathbb{Z}$.

(c) *Each finitely generated $\mathcal{A}_i$-module is a finitely generated $\mathsf{R}\mathcal{A}$-module.*

Proof. (a) The assertions follow from the definition of ρ_j.

(b) We construct morphisms α, β in $\mathsf{Mod}\,\mathsf{R}\mathcal{A}$. For an object (Y,j) of $\mathsf{R}\mathcal{A}$, define

$$\alpha_{(Y,j)} := \begin{cases} \mathrm{id}_{\mathcal{A}(Y,X)} & j = i, \\ 0 & \text{else,} \end{cases} \qquad \beta_{(Y,j)} := \begin{cases} \mathrm{id}_{\mathrm{D}\,\mathcal{A}(X,Y)} & j+1 = i, \\ 0 & \text{else,} \end{cases}$$

and extend α and β on $\mathsf{R}\mathcal{A}$ additively. We can show that α and β are actually morphisms in $\mathsf{Mod}\,\mathsf{R}\mathcal{A}$. By definitions of α and β, for an object (Y,j) of $\mathsf{R}\mathcal{A}$, we have the following exact sequence

$$0 \to \mathrm{D}\,\mathcal{A}_{i-1}(X,(Y,j)) \xrightarrow{\beta_{(Y,j)}} \mathsf{R}\mathcal{A}((Y,j),(X,i)) \xrightarrow{\alpha_{(Y,j)}} \mathcal{A}_i((Y,j),X) \to 0$$

in $\mathsf{Mod}\,k$. Thus we have an exact sequence (13.1). Since ρ_j is exact, by applying ρ_j to the exact sequence (13.1) and by using (a), we have the assertion.

(c) This follows from (b). □

By the following lemma, we construct a filtration of a module over repetitive categories. For $M \in \mathsf{Mod}\,\mathsf{R}\mathcal{A}$, put $\operatorname{Supp} M := \{ i \in \mathbb{Z} \mid \rho_i(M) \neq 0 \}$.

Lemma 13.4. *Let $M \in \mathsf{Mod}\,\mathsf{R}\mathcal{A}$ and $i \in \mathbb{Z}$.*

(a) *If $\rho_{i-1}(M) = 0$, then there exists a short exact sequence*

$$0 \to \rho_i(M) \xrightarrow{\alpha} M \to N \to 0$$

in $\mathsf{Mod}\,\mathsf{R}\mathcal{A}$ such that $\rho_i(N) = 0$ and $\rho_j(N) = \rho_j(M)$ for any $j > i$.

(b) *Assume that $\operatorname{Supp} M$ is a finite set and put $m := \max \operatorname{Supp} M$ and $n := \min \operatorname{Supp} M$. Then there exists a sequence of subobjects of M:*

$$0 = M_{n-1} \subset M_n \subset \cdots \subset M_{m-1} \subset M_m = M$$

such that $M_i/M_{i-1} \simeq \rho_i(M)$ for any $i = n, n+1, \ldots, m$.

Proof. (a) We construct a monomorphism $\alpha : \rho_i(M) \to M$ in $\mathsf{Mod}\,\mathsf{R}\mathcal{A}$. For an object (X, j) of $\mathsf{R}\mathcal{A}$, define

$$\alpha_{(X,j)} := \begin{cases} \mathrm{id}_{M(X,j)} & j = i, \\ 0 & \text{else,} \end{cases}$$

and extend this on $\mathsf{R}\mathcal{A}$ additively. Since $\rho_{i-1}(M) = 0$, α is a morphism of $\mathsf{Mod}\,\mathsf{R}\mathcal{A}$. By the definition, α is mono. Then we have an exact sequence $0 \to \rho_i(M) \to M \to N \to 0$ in $\mathsf{Mod}\,\mathsf{R}\mathcal{A}$, where $N := \operatorname{Cok}(\alpha)$. By Lemma 13.3, we have $\rho_j(\rho_i(M)) = \rho_i(M)$ if $j = i$ and $\rho_j(\rho_i(M)) = 0$ if else. Therefore by applying the functor ρ_j to this exact sequence, we have the assertion.

(b) This follows from (a). $\square$

By the following two lemmas, we see that the functors $\mathsf{Mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathsf{R}\mathcal{A}$ and $\rho_i : \mathsf{Mod}\,\mathsf{R}\mathcal{A} \to \mathsf{Mod}\,\mathcal{A}$ restrict to functors between $\mathsf{mod}\,\mathcal{A}$ and $\mathsf{mod}\,\mathsf{R}\mathcal{A}$ under certain assumptions. For simplicity, we use the notation $\mathsf{mod}_{-1}\,\mathcal{A} := \mathsf{Mod}\,\mathcal{A}$, $\mathsf{mod}_{\infty}\,\mathcal{A} := \mathsf{mod}\,\mathcal{A}$ and $\infty - 1 := \infty$.

Lemma 13.5. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category and $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Assume that $\mathrm{D}\,\mathcal{A}(X, -) \in \mathsf{mod}_{n-1}\,\mathcal{A}$ holds for any $X \in \mathcal{A}$. Then an inclusion functor $\mathsf{Mod}\,\mathcal{A}_i \to \mathsf{Mod}\,\mathsf{R}\mathcal{A}$ restricts to a functor $\mathsf{mod}_n\,\mathcal{A}_i \to \mathsf{mod}_n\,\mathsf{R}\mathcal{A}$ for any $i \in \mathbb{Z}$.*

Proof. Let $n \in \mathbb{Z}_{\geq 0}$. It is sufficient to show that $\mathcal{A}_i(-, X) \in \mathsf{mod}_n\,\mathsf{R}\mathcal{A}$ for any $i \in \mathbb{Z}$. In fact, any $M \in \mathsf{mod}_n\,\mathcal{A}_i$ has an exact sequence $P_n \to \cdots \to P_0 \to M \to 0$ with $P_i \in \mathsf{proj}\,\mathcal{A}_i$ and hence M belongs to $\mathsf{mod}_n\,\mathsf{R}\mathcal{A}$ by Lemma 12.4 (a).

We show $\mathsf{proj}\,\mathcal{A}_i \subset \mathsf{mod}_n\,\mathsf{R}\mathcal{A}$ for any $i \in \mathbb{Z}$ by an induction on n. If $n = 0$, then by Lemma 13.3 (c), we have the assertion. Let $n > 0$, $X \in \mathcal{A}$ and $i \in \mathbb{Z}$. By Lemma 13.3 (b), there exists an exact sequence

$$0 \to \mathrm{D}\,\mathcal{A}_{i-1}(X, -) \to \mathsf{R}\mathcal{A}(-, (X, i)) \to \mathcal{A}_i(-, X) \to 0.$$

By the inductive hypothesis, $\mathrm{D}\,\mathcal{A}_{i-1}(X, -) \in \mathsf{mod}_{n-1}\,\mathsf{R}\mathcal{A}$ holds. Therefore we have $\mathcal{A}_i(-, X) \in \mathsf{mod}_n\,\mathsf{R}\mathcal{A}$ by Lemma 12.4 (c).

By an argument similar to the above, the assertion holds when $n = \infty$. $\square$

Lemma 13.6. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category, $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Assume that $\mathrm{D}\,\mathcal{A}(X,-) \in \mathsf{mod}_n\,\mathcal{A}$ holds for any $X \in \mathcal{A}$. Then the functor $\rho_i : \mathsf{Mod}\,\mathsf{R}\mathcal{A} \to \mathsf{Mod}\,\mathcal{A}_i$ restricts to a functor $\mathsf{mod}_n\,\mathsf{R}\mathcal{A} \to \mathsf{mod}_n\,\mathcal{A}_i$ for any $i \in \mathbb{Z}$.*

Proof. Let $n \in \mathbb{Z}_{\geq 0}$ and $M \in \mathsf{mod}_n\,\mathsf{R}\mathcal{A}$. We have an exact sequence $P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ in $\mathsf{Mod}\,\mathsf{R}\mathcal{A}$, where $P_j \in \mathsf{proj}\,\mathsf{R}\mathcal{A}$ for each $j \geq 0$. Since ρ_i is exact, we have an exact sequence $\rho_i(P_n) \to \cdots \to \rho_i(P_1) \to \rho_i(P_0) \to \rho_i(M) \to 0$ in $\mathsf{Mod}\,\mathcal{A}_i$. By the assumption and Lemma 13.3 (b), $\rho_i(P_j) \in \mathsf{mod}_n\,\mathcal{A}_i$ holds for any $j \geq 0$. Therefore $\rho_i(M) \in \mathsf{mod}_n\,\mathcal{A}_i$ holds by Lemma 12.4 (a).

By an argument similar to the above, the assertion holds when $n = \infty$. $\square$

Note that in general $\mathsf{mod}\,\mathsf{R}\mathcal{A} = \mathsf{mod}_1\,\mathsf{R}\mathcal{A}$ does not hold for a k-linear additive category $\mathcal{A}$. This is the case where $\mathcal{A}$ is a dualizing k-variety by Theorem 13.7 below. Note that there exists an equivalence $(\mathsf{R}\mathcal{A})^{\mathrm{op}} \simeq \mathsf{R}(\mathcal{A}^{\mathrm{op}})$ given by $(X,i) \mapsto (X,-i)$.

Theorem 13.7. *Let $\mathcal{A}$ be a dualizing k-variety. Then the following statements hold.*

(a) $\mathsf{R}\mathcal{A}$ *and* $(\mathsf{R}\mathcal{A})^{\mathrm{op}}$ *have weak kernels.*

(b) $\mathsf{R}\mathcal{A}$ *is a dualizing k-variety.*

Proof. Note that since $\mathcal{A}$ is a dualizing k-variety, $\mathrm{D}\,\mathcal{A}(-,X) \in \mathsf{mod}_1\,\mathcal{A}$ holds for any $X \in \mathcal{A}$ and $\mathsf{mod}_1\,\mathcal{A} = \mathsf{mod}\,\mathcal{A}$ holds.

(a) Let $\mathsf{X},\mathsf{Y} \in \mathsf{R}\mathcal{A}$ and $f : \mathsf{R}\mathcal{A}(-,\mathsf{X}) \to \mathsf{R}\mathcal{A}(-,\mathsf{Y})$ be a morphism of $\mathsf{mod}\,\mathsf{R}\mathcal{A}$. We show that $K := \mathrm{Ker}(f)$ is a finitely generated $\mathsf{R}\mathcal{A}$-module. For any $i \in \mathbb{Z}$, we have an exact sequence $0 \to \rho_i(K) \to \rho_i(\mathsf{R}\mathcal{A}(-,\mathsf{X})) \to \rho_i(\mathsf{R}\mathcal{A}(-,\mathsf{Y}))$ in $\mathsf{Mod}\,\mathcal{A}_i$. By Lemma 13.6, we have $\rho_i(\mathsf{R}\mathcal{A}(-,\mathsf{X})), \rho_i(\mathsf{R}\mathcal{A}(-,\mathsf{Y})) \in \mathsf{mod}\,\mathcal{A}_i$. Therefore $\rho_i(K) \in \mathsf{mod}\,\mathcal{A}_i$ for any $i \in \mathbb{Z}$, since $\mathcal{A}_i \simeq \mathcal{A}$ is a dualizing k-variety. By Lemma 13.5, $\rho_i(K) \in \mathsf{mod}\,\mathsf{R}\mathcal{A}$ for any $i \in \mathbb{Z}$. Since K is a submodule of $\mathsf{R}\mathcal{A}(-,\mathsf{X})$, $\mathrm{Supp}\,K$ is a finite set. Thus by Lemma 13.4 (b), K has a finite filtration by finitely presented $\mathsf{R}\mathcal{A}$-modules $\{\rho_i(K) \mid i \in \mathbb{Z}\}$ and we have $K \in \mathsf{mod}\,\mathsf{R}\mathcal{A}$. In particular, K is finitely generated and $\mathsf{R}\mathcal{A}$ has weak kernels. Since $(\mathsf{R}\mathcal{A})^{\mathrm{op}} \simeq \mathsf{R}(\mathcal{A}^{\mathrm{op}})$ holds and $\mathcal{A}^{\mathrm{op}}$ is a dualizing k-variety, $(\mathsf{R}\mathcal{A})^{\mathrm{op}}$ has weak kernels.

(b) By the definition of dualizing k-varieties, $\mathcal{A}$ is Hom-finite and idempotent complete. By Lemma 13.2, $\mathsf{R}\mathcal{A}$ is Hom-finite and idempotent complete with a Serre functor. Therefore by Proposition 12.16, $\mathsf{R}\mathcal{A}$ is a dualizing k-variety. $\square$

13.2 Tilting subcategories

The aim of this subsection is to show Theorem 13.10. Before stating the main theorem, we need the following definition.

Let $\mathcal{A}$ be a k-linear, Hom-finite additive category. We denote by

$$\rho : \mathsf{Mod}\,\mathsf{R}\mathcal{A} \to \mathsf{Mod}\,\mathcal{A}$$

the forgetful functor, that is, $\rho(M) := \bigoplus_{i \in \mathbb{Z}} \rho_i(M)$ for any $M \in \mathsf{Mod}\,\mathsf{R}\mathcal{A}$, where we regard an $\mathcal{A}_i$-module $\rho_i(M)$ as an $\mathcal{A}$-module by the equivalence $\mathsf{Mod}\,\mathcal{A}_i \simeq \mathsf{Mod}\,\mathcal{A}$. Note that ρ is an exact functor. We denote by $\mathsf{GP}(\mathsf{R}\mathcal{A},\mathcal{A})$ the full subcategory of $\mathsf{GP}(\mathsf{R}\mathcal{A})$ consisting of all objects M such that the projective dimension of $\rho(M)$ over $\mathcal{A}$ is finite, that is,

$$\mathsf{GP}(\mathsf{R}\mathcal{A},\mathcal{A}) := \{\, M \in \mathsf{GP}(\mathsf{R}\mathcal{A}) \mid \mathrm{projdim}_{\mathcal{A}}\,\rho(M) < \infty \,\}.$$

We consider the following condition on $\mathcal{A}$:

(G) : the projective dimension of $\mathrm{D}\,\mathcal{A}(X,-)$ over $\mathcal{A}$ is finite for any $X \in \mathcal{A}$.

Proposition 13.8. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category. Then $\mathcal{A}$ satisfies (G) if and only if $\mathsf{proj}\,\mathsf{R}\mathcal{A} \subset \mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ holds. In this case, the following statements fold.*

(a) *$\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ is a Frobenius category such that the projective objects is the objects of $\mathsf{proj}\,\mathsf{R}\mathcal{A}$.*

(b) *The inclusion functor $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A}) \to \mathsf{GP}(\mathsf{R}\mathcal{A})$ induces a fully faithful triangle functor $\underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A}) \to \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A})$.*

Proof. The first assertion follows from Lemma 13.3 (b). Assume that $\mathcal{A}$ satisfies (G).

(a) By the definition and since ρ is exact, $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ is extension closed subcategory of $\mathsf{Mod}\,\mathsf{RA}$ and has enough projectives and enough injectives. Clearly, an object of $\mathsf{proj}\,\mathsf{R}\mathcal{A}$ is relative projective of $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$. Let Q be a relative projective object of $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$. There exists an exact sequence $0 \to M \to P \to Q \to 0$ in $\mathsf{GP}(\mathsf{R}\mathcal{A})$ with $P \in \mathsf{proj}\,\mathsf{R}\mathcal{A}$. We have $M \in \mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ and therefore this sequence splits. Consequently, the relative projective objects of $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ is the objects of $\mathsf{proj}\,\mathsf{R}\mathcal{A}$.

(b) This follows from (a). □

We regard $\underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A})$ as a thick subcategory of $\underline{\mathsf{GP}}(\mathsf{R}\mathcal{A})$ by Proposition 13.8 (b) if $\mathcal{A}$ satisfies (G). Let $\mathcal{A}$ be a k-linear, Hom-finite additive category. We consider the following condition on $\mathcal{A}$:

$$\text{(IFP)} : \mathrm{D}\,\mathcal{A}(X,-) \in \mathsf{mod}\,\mathcal{A} \text{ holds for any } X \in \mathcal{A}.$$

Note that if $\mathcal{A}$ is a dualizing k-variety, then $\mathcal{A}$ satisfies (IFP). We denote by $\mathcal{M}$ the full subcategory of $\mathsf{Mod}\,\mathsf{R}\mathcal{A}$ given by

$$\mathcal{M} := \mathsf{add}\{\, \mathcal{A}_0(-,X) \mid X \in \mathcal{A}\,\}.$$

We recall the definition of tilting subcategories of a triangulated category.

Definition 13.9. Let $\mathcal{T}$ be a triangulated category. A full subcategory $\mathcal{M}$ of $\mathcal{T}$ is called a *tilting subcategory* of $\mathcal{T}$ if $\mathcal{T}(\mathcal{M}, \mathcal{M}[i]) = 0$ for any $i \neq 0$ and $\mathsf{thick}\,\mathcal{M} = \mathcal{T}$.

We establish the following result.

Theorem 13.10. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category and assume that $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP). Then the following holds.*

(a) *If $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (G), then $\mathcal{M} \subset \mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ holds and $\mathcal{M}$ gives a tilting subcategory of $\underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A})$.*

(b) *If each object of $\mathsf{mod}\,\mathcal{A}$ and $\mathsf{mod}\,\mathcal{A}^{\mathrm{op}}$ has finite projective dimension, then $\mathcal{M} \subset \mathsf{GP}(\mathsf{R}\mathcal{A})$ holds and $\mathcal{M}$ gives a tilting subcategory of $\underline{\mathsf{GP}}(\mathsf{R}\mathcal{A})$.*

In the case where $\mathcal{A}$ is a dualizing k-variety, we have the following corollary.

Corollary 13.11. *Let $\mathcal{A}$ be a dualizing k-variety. If each object of $\mathsf{mod}\,\mathcal{A}$ and $\mathsf{mod}\,\mathcal{A}^{\mathrm{op}}$ has finite projective dimension, then $\mathcal{M}$ is a tilting subcategory of $\underline{\mathsf{mod}}\,\mathsf{R}\mathcal{A}$.*

Before starting the proof of Theorem 13.10, we prepare two lemmas. Let $\mathcal{A}$ be a k-linear additive category and $i \in \mathbb{Z}$. Put the following full subcategories of $\mathsf{R}\mathcal{A}$:

$$\mathcal{A}_{<i} := \bigvee_{j<i} \mathcal{A}_j, \quad \mathcal{A}_{\geq i} := \bigvee_{j\geq i} \mathcal{A}_j.$$

For $M \in \operatorname{\mathsf{Mod}} \mathsf{R}\mathcal{A}$ and $i \in \mathbb{Z}$, let $\rho_{<i}(M) := \bigoplus_{j<i} \rho_j(M)$ and $\rho_{\geq i}(M) := \bigoplus_{j\geq i} \rho_j(M)$.

Lemma 13.12. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category. Let M and N be finitely generated $\mathsf{R}\mathcal{A}$-modules and $i \in \mathbb{Z}$. Assume that $\rho_{\geq i}(M) = 0$ and $\rho_{<i}(N) = 0$.*

(a) *There exist epimorphisms*

$$\mathsf{R}\mathcal{A}(-, \mathsf{X}) \to M, \quad \mathsf{R}\mathcal{A}(-, \mathsf{Y}) \to N,$$

for some $\mathsf{X} \in \mathcal{A}_{<i}$ and $\mathsf{Y} \in \mathcal{A}_{\geq i}$.

(b) *We have $(\operatorname{\mathsf{Mod}} \mathsf{R}\mathcal{A})(M, N) = 0$ and $(\operatorname{\mathsf{Mod}} \mathsf{R}\mathcal{A})(N, M) = 0$.*

(c) *Assume $M \in \operatorname{\mathsf{mod}} \mathsf{R}\mathcal{A}$. Let*

$$\cdots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0 \tag{13.2}$$

be a minimal projective resolution of M in $\operatorname{\mathsf{mod}} \mathsf{R}\mathcal{A}$. Then we have $\rho_{\geq i}(\operatorname{Ker} f_l) = 0$ for $l \geq 0$. Moreover by applying a functor ρ_{i-1}, we have a minimal projective resolution of $\rho_{i-1}(M)$ in $\operatorname{\mathsf{mod}} \mathcal{A}_{i-1}$.

Proof. (a) Since M and N are finitely generated, there exist epimorphisms $\mathsf{R}\mathcal{A}(-, \mathsf{X}) \to M$ and $\mathsf{R}\mathcal{A}(-, \mathsf{Y}) \to N$, where X and Y are in $\mathsf{R}\mathcal{A}$. Let W be an object of $\mathcal{A}_{\geq i}$. By Yoneda's lemma and the assumption, we have $(\operatorname{\mathsf{Mod}} \mathsf{R}\mathcal{A})(\mathsf{R}\mathcal{A}(-, \mathsf{W}), M) \simeq M(\mathsf{W}) = 0$. Therefore we can replace X with an object of $\mathcal{A}_{<i}$. Similarly, we can replace Y with an object of $\mathcal{A}_{\geq i}$.

(b) By (a), there exists an epimorphism $\mathsf{R}\mathcal{A}(-, \mathsf{X}) \to M$, where $\mathsf{X} \in \mathcal{A}_{<i}$. We have a monomorphism $(\operatorname{\mathsf{Mod}} \mathsf{R}\mathcal{A})(M, N) \to (\operatorname{\mathsf{Mod}} \mathsf{R}\mathcal{A})(\mathsf{R}\mathcal{A}(-, \mathsf{X}), N)$. Since $(\operatorname{\mathsf{Mod}} \mathsf{R}\mathcal{A})(\mathsf{R}\mathcal{A}(-, \mathsf{X}), N) \simeq N(\mathsf{X}) = 0$, $(\operatorname{\mathsf{Mod}} \mathsf{R}\mathcal{A})(M, N) = 0$ holds. Similarly, by applying $(\operatorname{\mathsf{Mod}} \mathsf{R}\mathcal{A})(-, M)$ to an epimorphism $\mathsf{R}\mathcal{A}(-, \mathsf{Y}) \to N$, we have $(\operatorname{\mathsf{Mod}} \mathsf{R}\mathcal{A})(N, M) = 0$.

(c) By (a), there exists $\mathsf{X}_0 \in \mathcal{A}_{<i}$ such that P_0 is a direct summands of $\mathsf{R}\mathcal{A}(-, \mathsf{X}_0)$. We have $\rho_{\geq i}(\mathsf{R}\mathcal{A}(-, \mathsf{X}_0)) = 0$. Therefore the submodule $\operatorname{Ker} f_0$ of $\mathsf{R}\mathcal{A}(-, \mathsf{X}_0)$ satisfies $\rho_{\geq i}(\operatorname{Ker} f_0) = 0$. By using this argument inductively, we have that there exist $\mathsf{X}_l \in \mathcal{A}_{<i}$ such that P_l is a direct summands of $\mathsf{R}\mathcal{A}(-, \mathsf{X}_l)$ for any $l \geq 0$. Therefore we have $\rho_{\geq i}(\operatorname{Ker} f_l) = 0$ for $l \geq 0$.

For any $l \geq 0$, by Lemma 13.3, $\rho_{i-1}(P_l)$ is a direct sum of $\mathcal{A}_{i-1}(-, X)$ for some $X \in \mathcal{A}$ and zero objects. Therefore each $\rho_{i-1}(P_l)$ is a projective $\mathcal{A}_{i-1}$-module. Minimality comes from the minimality of the resolution (13.2). $\square$

We see when $\mathsf{GP}(\mathsf{R}\mathcal{A})$ contains the representable functors on $\mathcal{A}$. Note that there exists an equivalence $(\mathsf{R}\mathcal{A})^{\mathrm{op}} \simeq \mathsf{R}(\mathcal{A}^{\mathrm{op}})$ given by $(X, i) \mapsto (X, -i)$. Thus we have a duality

$$\operatorname{\mathsf{Mod}}_{\mathrm{fg}} \mathsf{R}\mathcal{A} \xrightarrow{\mathrm{D}} \operatorname{\mathsf{Mod}}_{\mathrm{fg}} (\mathsf{R}\mathcal{A})^{\mathrm{op}} \xrightarrow{\sim} \operatorname{\mathsf{Mod}}_{\mathrm{fg}} \mathsf{R}(\mathcal{A}^{\mathrm{op}}).$$

By this duality, a full subcategory $\operatorname{\mathsf{mod}} \mathcal{A}_i$ of $\operatorname{\mathsf{mod}} \mathsf{R}\mathcal{A}$ goes to a full subcategory $\operatorname{\mathsf{mod}} (\mathcal{A}^{\mathrm{op}})_{-i}$ of $\operatorname{\mathsf{mod}} \mathsf{R}(\mathcal{A}^{\mathrm{op}})$.

Lemma 13.13. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category.*

(a) *The following statements are equivalent.*

(i) *$\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP).*

(ii) *$\mathcal{A}_i(-,X) \in \mathsf{GP}(\mathsf{R}\mathcal{A})$ and $\mathcal{A}_i(X,-) \in \mathsf{GP}(\mathsf{R}\mathcal{A})^{\mathrm{op}}$ hold for any $X \in \mathcal{A}$ and $i \in \mathbb{Z}$.*

(iii) *$\mathrm{D}\,\mathcal{A}_i(X,-) \in \mathsf{GP}(\mathsf{R}\mathcal{A})$ and $\mathrm{D}\,\mathcal{A}_i(-,X) \in \mathsf{GP}(\mathsf{R}\mathcal{A})^{\mathrm{op}}$ hold for any $X \in \mathcal{A}$ and $i \in \mathbb{Z}$.*

(b) *If $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP), then $\rho_i(M) \in \mathsf{GP}(\mathsf{R}\mathcal{A})$ holds for any $M \in \mathsf{GP}(\mathsf{R}\mathcal{A})$ and $i \in \mathbb{Z}$.*

Proof. Note that by Lemma 13.2, $\mathsf{R}\mathcal{A}$ has a Serre functor $\mathbb{S}$. Thus by Lemma 12.15, we have an isomorphism of functors $(-)^* \simeq \mathrm{D}(- \circ \mathbb{S}^{-1}) : \mathsf{Mod}_{\mathrm{fg}}\,\mathsf{R}\mathcal{A} \to \mathsf{Mod}_{\mathrm{fg}}\,\mathsf{R}(\mathcal{A}^{\mathrm{op}})$. We have

$$(\mathcal{A}_i(-,X))^* \simeq \mathrm{D}(\mathcal{A}^{\mathrm{op}})_{-i-1}(X,-) = \mathrm{D}\,\mathcal{A}_{-i-1}(-,X) \tag{13.3}$$

for any $X \in \mathcal{A}$ and $i \in \mathbb{Z}$. Therefore (ii) and (iii) of (a) are equivalent.

(a) We show that (i) implies (ii). Let $X \in \mathcal{A}$. By Lemma 13.5, $\mathcal{A}_i(-,X) \in \mathsf{mod}\,\mathsf{R}\mathcal{A}$ holds. We have $(\mathcal{A}_i(-,X))^* \in \mathsf{mod}(\mathsf{R}\mathcal{A})^{\mathrm{op}}$, by the equality (13.3) and Lemma 13.5. Therefore by Lemma 12.15 (b), we have $\mathcal{A}_i(-,X) \in \mathsf{GP}(\mathsf{R}\mathcal{A})$. Dually, we have $\mathcal{A}_i(X,-) \in \mathsf{GP}(\mathsf{R}\mathcal{A})^{\mathrm{op}}$.

We show that (ii) implies (i). Let $X \in \mathcal{A}$. Take a minimal projective resolution of $\mathcal{A}_i(-,X)$ in $\mathsf{mod}\,\mathsf{R}\mathcal{A}$:

$$\cdots \to Q_2 \to Q_1 \xrightarrow{d_1} \mathsf{R}\mathcal{A}(-,(X,i)) \to \mathcal{A}_i(-,X) \to 0.$$

By Lemma 13.3 (b), we have $\operatorname{Im} d_1 = \mathrm{D}\,\mathcal{A}_{i-1}(X,-)$. By Lemma 13.12 (c), applying ρ_{i-1}, we have $\mathrm{D}\,\mathcal{A}_{i-1}(X,-) \in \mathsf{mod}\,\mathcal{A}_{i-1}$. This means $\mathrm{D}\,\mathcal{A}(X,-) \in \mathsf{mod}\,\mathcal{A}$. Dually, we have $\mathrm{D}\,\mathcal{A}(-,X) \in \mathsf{mod}\,\mathcal{A}^{\mathrm{op}}$.

(b) By Lemma 13.3 (b), we have $\rho_i(P) \in \mathsf{add}\{\mathcal{A}_i(-,X), \mathrm{D}\,\mathcal{A}_i(X,-) \mid X \in \mathcal{A}\}$ for any $P \in \mathsf{proj}\,\mathsf{R}\mathcal{A}$. Therefore $(\rho_i(P))^* \in \mathsf{mod}(\mathcal{A}^{\mathrm{op}})_{-i-1}$ holds by the equality (13.3) and the assumption. Let $M \in \mathsf{GP}(\mathsf{R}\mathcal{A})$ and $P_\bullet = (P_j, d_j : P_j \to P_{j+1})$ be a totally acyclic complex such that $\operatorname{Im} d_0 = M$, where $P_j \in \mathsf{proj}\,\mathsf{R}\mathcal{A}$. By applying ρ_i, we have an exact sequence $\rho_i(P_\bullet) = (\rho_i(P_j), \rho_i(d_j) : \rho_i(P_j) \to \rho_i(P_{j+1}))$ such that $\operatorname{Im} \rho_i(d_0) = \rho_i(M)$. We have an exact sequence $\cdots \to \rho_i(P_{-1}) \to \rho_i(P_0) \to \rho_i(M) \to 0$. By Lemmas 12.4 (b) and 13.5, $\rho_i(M) \in \mathsf{mod}\,\mathsf{R}\mathcal{A}$ holds. By applying a functor $(-)^*$ to $0 \to \rho_i(M) \to \rho_i(P_1) \to \rho_i(P_2) \to \cdots$, and using Lemma 12.4 (b) to the resulting exact sequence, we have $(\rho_i(M))^* \in \mathsf{mod}(\mathsf{R}\mathcal{A})^{\mathrm{op}}$. Therefore we have $\rho_i(M) \in \mathsf{GP}(\mathsf{R}\mathcal{A})$ by Lemma 12.15 (b). $\square$

By Lemma 13.13, if $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP), then $\mathcal{M} \subset \mathsf{GP}(\mathsf{R}\mathcal{A})$ holds. We also denote by $\mathcal{M}$ the subcategory of $\underline{\mathsf{GP}}(\mathsf{R}\mathcal{A})$ consisting of objects $\mathcal{A}_0(-,X)$ for any $X \in \mathcal{A}$. Then we show Theorem 13.10. We divide the proof into two propositions. Put $\mathcal{T} := \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A})$.

Proposition 13.14. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category and assume that $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP). Then we have $\mathcal{T}(\mathcal{M}, \mathcal{M}[i]) = 0$ for any $i \neq 0$.*

Proof. Let $X \in \mathcal{A}$ and

$$\cdots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} \mathcal{A}_0(-,X) \to 0$$

be a minimal projective resolution in $\mathsf{mod}\,\mathsf{R}\mathcal{A}$. Put $K^i := \operatorname{Ker}(f^{i-1})$ for $i \geq 1$. By Lemmas 13.3 (b) and 13.12 (c), we have $\rho_{\geq 0}(K^i) = 0$ for $i \geq 1$. Let $Y \in \mathcal{A}$. Since $\rho_{<0}(\mathcal{A}_0(-,Y)) = 0$ and Lemma 13.12 (b), we have

$$(\mathsf{Mod}\,\mathsf{R}\mathcal{A})(K^i, \mathcal{A}_0(-,Y)) = 0, \quad (\mathsf{Mod}\,\mathsf{R}\mathcal{A})(\mathcal{A}_0(-,Y), K^i) = 0,$$

for any $i \geq 1$. Therefore we have

$$\mathcal{T}(\mathcal{A}_0(-,Y), \mathcal{A}_0(-,X)[-i]) = \mathcal{T}(\mathcal{A}_0(-,Y), K^i) = 0,$$
$$\mathcal{T}(\mathcal{A}_0(-,X), \mathcal{A}_0(-,Y)[i]) = \mathcal{T}(K^i, \mathcal{A}_0(-,Y)) = 0,$$

for any $i \geq 1$. □

Proposition 13.15. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category and assume that $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP). If $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (G), then we have* $\mathsf{thick}_{\mathcal{T}}\,\mathcal{M} = \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A})$.

Proof. Since $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP), we have $\mathcal{M} \subset \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A})$. Therefore we have $\mathsf{thick}\,\mathcal{M} := \mathsf{thick}_{\mathcal{T}}\,\mathcal{M} \subset \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A})$.

Let $i \in \mathbb{Z}$ and $N \in \mathsf{mod}\,\mathcal{A}_i$. Assume that N has finite projective dimension over $\mathcal{A}_i$. Since the inclusion $\mathsf{mod}\,\mathcal{A}_i \to \mathsf{mod}\,\mathsf{R}\mathcal{A}$ is exact, we have a resolution of N by objects of the form $\mathcal{A}_i(-,X)$, $(X \in \mathcal{A})$ in $\mathsf{mod}\,\mathsf{R}\mathcal{A}$. Therefore if N is an object of $\mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$, then N is in $\mathsf{thick}\,\mathcal{M}$ if $\mathcal{A}_i(-,X)$ is in $\mathsf{thick}\,\mathcal{M}$ for any $X \in \mathcal{A}$.

Let $M \in \mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$. Since M is a factor module of a finitely generated projective $\mathsf{R}\mathcal{A}$-module, $\operatorname{Supp} M$ is a finite set. Thus by Lemma 13.4 (b), M has a finite filtration by $\rho_i(M)$ for $i = n, n+1, \ldots, m$, where $n = \min \operatorname{Supp} M$ and $m = \max \operatorname{Supp} M$. By Lemma 13.13 (b) and since $\rho(M)$ has finite projective dimension over $\mathcal{A}$, $\rho_i(M) \in \mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ for any $i \in \mathbb{Z}$. Therefore M is in $\mathsf{thick}\,\mathcal{M}$ if $\mathcal{A}_i(-,X)$ is in $\mathsf{thick}\,\mathcal{M}$ for any $X \in \mathcal{A}$ and $i = n, n+1, \ldots, m$.

We show that $\mathcal{A}_i(-,X)$ is in $\mathsf{thick}\,\mathcal{M}$ for any $X \in \mathcal{A}$ and $i \in \mathbb{Z}$ by an induction on i. We first show $\mathcal{A}_i(-,X) \in \mathsf{thick}\,\mathcal{M}$ for $i \geq 0$. Since $\mathcal{A}_0(-,X) \in \mathcal{M}$, we have $\mathcal{A}_0(-,X) \in \mathsf{thick}\,\mathcal{M}$. Assume that $\mathcal{A}_j(-,X) \in \mathsf{thick}\,\mathcal{M}$ for $0 \leq j \leq i-1$. By Lemma 13.3, we have an exact sequence in $\mathsf{GP}(\mathsf{R}\mathcal{A})$

$$0 \to \mathrm{D}\,\mathcal{A}_{i-1}(X,-) \to \mathsf{R}\mathcal{A}(-,(X,i)) \to \mathcal{A}_i(-,X) \to 0.$$

Since $\mathrm{D}\,\mathcal{A}_{i-1}(X,-)$ has finite projective dimension over $\mathcal{A}$ and by the inductive hypothesis, we have $\mathrm{D}\,\mathcal{A}_{i-1}(X,-) \in \mathsf{thick}\,\mathcal{M}$. Therefore $\mathcal{A}_i(-,X)$ is in $\mathsf{thick}\,\mathcal{M}$.

Next we show that $\mathcal{A}_{-i}(-,X) \in \mathsf{thick}\,\mathcal{M}$ for $i > 0$. Assume that $\mathcal{A}_{-j}(-,X) \in \mathsf{thick}\,\mathcal{M}$ for $0 \leq j \leq i-1$. Let n be the projective dimension of $\mathrm{D}\,\mathcal{A}_{-i}(-,X) \simeq \mathrm{D}(\mathcal{A}^{\mathrm{op}})_i(X,-)$ in $\mathsf{mod}(\mathcal{A}^{\mathrm{op}})_i$ and

$$Q_n \xrightarrow{f} \cdots \to Q_1 \to Q_0 \to \mathrm{D}\,\mathcal{A}_{-i}(-,X) \to 0$$

be a minimal projective resolution in $\mathsf{mod}(\mathsf{R}\mathcal{A})^{\mathrm{op}} \simeq \mathsf{mod}\,\mathsf{R}(\mathcal{A}^{\mathrm{op}})$. Put $K := \operatorname{Ker} f$. We have $K \in \mathsf{GP}(\mathsf{R}(\mathcal{A}^{\mathrm{op}}))$ by Lemmas 12.7 (b) and 13.13 (a). By applying ρ to this resolution, we have $K \in \mathsf{GP}(\mathsf{R}(\mathcal{A}^{\mathrm{op}}), \mathcal{A}^{\mathrm{op}})$. Since the projective dimension of $\mathrm{D}\,\mathcal{A}_{-i}(-,X)$ in $\mathsf{mod}(\mathcal{A}^{\mathrm{op}})_i$ is n and by Lemma 13.12 (c), we have $\rho_i(K) = 0$. Moreover by Lemma 13.12 (c), we have $\rho_{\geq i+1}(K) = 0$. Therefore a $\mathsf{R}\mathcal{A}$-module $\mathrm{D}\,K$ satisfies $\rho_{<-i+1}(\mathrm{D}\,K) = 0$. Since $\mathrm{D}\,K$ is a finitely generated $\mathsf{R}\mathcal{A}$-module, $\mathrm{Supp}\,\mathrm{D}\,K$ is finite. Thus by Lemma 13.4 (b), $\mathrm{D}\,K$ has a finite filtration by $\rho_j(\mathrm{D}\,K)$ for $-i+1 \leq j \leq m$, where $m = \max \mathrm{Supp}\,\mathrm{D}\,K$. By the inductive hypothesis, $\mathrm{D}\,K \in \mathsf{thick}\,\mathcal{M}$ holds. We have an exact sequence in $\mathsf{GP}(\mathsf{R}\mathcal{A})$

$$0 \to \mathcal{A}_{-i}(-,X) \to \mathrm{D}\,Q_0 \to \mathrm{D}\,Q_1 \to \cdots \to \mathrm{D}\,Q_n \to \mathrm{D}\,K \to 0,$$

where each $\mathrm{D}\,Q_l$ is a projective $\mathsf{R}\mathcal{A}$-module. This means $\mathcal{A}_{-i}(-,X) \simeq (\mathrm{D}\,K)[-n-1]$ in $\underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A})$. Therefore we have $\mathcal{A}_{-i}(-,X) \in \mathsf{thick}\,\mathcal{M}$. ☐

Proof of Theorem 13.10. (a) This follows from Propositions 13.14 and 13.15.

(b) Sine each object of $\mathsf{mod}\,\mathcal{A}$ has finite projective dimension, $\underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A}) = \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A})$ holds. Thus the assertion follows from (a). ☐

Proof of Corollary 13.11. If $\mathcal{A}$ is a dualizing k-variety, then $\mathsf{GP}(\mathsf{R}\mathcal{A}) = \mathsf{mod}\,\mathsf{R}\mathcal{A}$ holds. The assertion directly follows from Theorem 13.10. ☐

13.3 Happel's theorem for functor categories

As an application of Theorem 13.10, we show Happel's theorem for functor categories. We need the following lemma.

Lemma 13.16. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category and assume that $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP). Let $X, Y \in \mathcal{A}$, $\mathcal{T} := \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A})$. We have the following equality:*

$$\mathcal{T}(\mathcal{A}_0(-,X), \mathcal{A}_0(-,Y)[n]) \simeq \begin{cases} \mathcal{A}(X,Y) & n = 0, \\ 0 & \text{else.} \end{cases}$$

Proof. By Proposition 13.14, $\mathcal{T}(\mathcal{A}_0(-,X), \mathcal{A}_0(-,Y)[n \neq 0]) = 0$ holds. Moreover we have

$$\begin{aligned} (\mathsf{Mod}\,\mathsf{R}\mathcal{A})(\mathcal{A}_0(-,X), \mathsf{R}\mathcal{A}(-,(Y,0))) &\simeq (\mathsf{Mod}(\mathsf{R}\mathcal{A})^{\mathrm{op}})(\mathrm{D}\,\mathsf{R}\mathcal{A}(-,(Y,0)), \mathrm{D}\,\mathcal{A}_0(-,X)) \\ &\simeq (\mathsf{Mod}(\mathsf{R}\mathcal{A})^{\mathrm{op}})(\mathsf{R}\mathcal{A}((Y,-1),-), \mathrm{D}\,\mathcal{A}_0(-,X)) \\ &\simeq \mathrm{D}\,\mathcal{A}_0((Y,-1),X) = 0, \end{aligned} \tag{13.4}$$

where we use Lemma 13.2 (b) and Yoneda's lemma. By Lemma 13.3 (b), if a morphism $f : \mathcal{A}_0(-,X) \to \mathcal{A}_0(-,Y)$ in $\mathsf{Mod}\,\mathsf{R}\mathcal{A}$ factors though an object of $\mathsf{proj}\,\mathsf{R}\mathcal{A}$, then f factors though $\mathsf{R}\mathcal{A}(-,(Y,0))$. Thus by the equality (13.4), we have

$$\mathcal{T}(\mathcal{A}_0(-,X), \mathcal{A}_0(-,Y)) = (\mathsf{Mod}\,\mathsf{R}\mathcal{A})(\mathcal{A}_0(-,X), \mathcal{A}_0(-,Y)).$$

By applying the functor $(\mathsf{Mod}\,\mathsf{R}\mathcal{A})(-, \mathcal{A}_0(-,Y))$ to the exact sequence of Lemma 13.3 (b), since $(\mathsf{Mod}\,\mathsf{R}\mathcal{A})(\mathrm{D}\,\mathcal{A}_{-1}(X,-), \mathcal{A}_0(-,Y)) = 0$ holds, we have

$$\begin{aligned} (\mathsf{Mod}\,\mathsf{R}\mathcal{A})(\mathcal{A}_0(-,X), \mathcal{A}_0(-,Y)) &\simeq (\mathsf{Mod}\,\mathsf{R}\mathcal{A})(\mathsf{R}\mathcal{A}(-,(X,0)), \mathcal{A}_0(-,Y)) \\ &\simeq \mathcal{A}_0((X,0),Y) \\ &\simeq \mathcal{A}(X,Y). \end{aligned}$$

☐

We have the following result, which is a functor category version of Happel's theorem.

Corollary 13.17. *Let $\mathcal{A}$ be a k-linear, Hom-finite additive category and assume that $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (IFP).*

(a) *If $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ satisfy (G), then we have a triangle equivalence*

$$\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A}) \simeq \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A}).$$

(b) *If each object of $\mathsf{mod}\,\mathcal{A}$ and $\mathsf{mod}\,\mathcal{A}^{\mathrm{op}}$ has finite projective dimension, then we have a triangle equivalence*

$$\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A}) \simeq \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}).$$

Proof. (a) Let $\mathcal{F} := \mathsf{GP}(\mathsf{R}\mathcal{A}, \mathcal{A})$ and $\mathcal{P} := \mathsf{proj}\,\mathsf{R}\mathcal{A}$. An inclusion functor $\mathsf{proj}\,\mathcal{A} \simeq \mathsf{proj}\,\mathcal{A}_0 \to \mathcal{F}$ induces a triangle functor $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A}) \to \mathsf{K}^{-,\mathrm{b}}(\mathcal{P})$. Then we have the following triangle functors

$$F : \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A}) \to \mathsf{K}^{-,\mathrm{b}}(\mathcal{P}) \to \mathsf{K}^{-,\mathrm{b}}(\mathcal{P})/\mathsf{K}^{\mathrm{b}}(\mathcal{P}) \to \underline{\mathcal{F}},$$

where the third is a quasi-inverse of Theorem 12.17. We denote by F the composite of these functors. We show that F is an equivalence by using Lemma 12.18.

Put $\mathcal{U} := \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A})$ and $\mathcal{T} := \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A}) = \underline{\mathcal{F}}$. Note that $\mathsf{proj}\,\mathcal{A}$ is a subcategory of $\mathcal{U}$. We show that a map

$$F_{M,N[n]} : \mathcal{U}(M, N) \to \mathcal{T}(FM, FN[n])$$

is an isomorphism for any $M, N \in \mathsf{proj}\,\mathcal{A}$ and $n \in \mathbb{Z}$. By Theorem 12.17, a quasi-inverse of $\mathsf{K}^{-,\mathrm{b}}(\mathcal{P})/\mathsf{K}^{\mathrm{b}}(\mathcal{P}) \to \underline{\mathcal{F}}$ is induced from the composite of the canonical functors $\mathcal{F} \to \mathsf{K}^{-,\mathrm{b}}(\mathcal{P}) \to \mathsf{K}^{-,\mathrm{b}}(\mathcal{P})/\mathsf{K}^{\mathrm{b}}(\mathcal{P})$. Therefore we have $F(\mathcal{A}(-, X)) = \mathcal{A}_0(-, X)$ for any $X \in \mathcal{A}$. For any $X, Y \in \mathcal{A}$, we have

$$\mathcal{U}(\mathcal{A}(-, X), \mathcal{A}(-, Y)) = \mathcal{A}(X, Y), \quad \mathcal{U}(\mathcal{A}(-, X), \mathcal{A}(-, Y)[n \neq 0]) = 0.$$

Consequently, by Lemma 13.16, $F_{M,N[n]}$ is an isomorphism for any $M, N \in \mathsf{proj}\,\mathcal{A}$ and $n \in \mathbb{Z}$.

Since $\mathsf{proj}\,\mathcal{A}$ is Hom-finite and idempotent complete, so is $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathcal{A})$. Clearly we have $\mathsf{thick}_{\mathcal{U}}(\mathsf{proj}\,\mathcal{A}) = \mathcal{U}$. Since $\mathrm{Im}(F|_{\mathsf{proj}\,\mathcal{A}}) = \mathcal{M}$ holds, we have $\mathsf{thick}(\mathrm{Im}(F)) = \mathcal{T}$ by Theorem 13.10 (a). Therefore F is an equivalence by Lemma 12.18.

(b) Since each object of $\mathsf{mod}\,\mathcal{A}$ has finite projective dimension, we have $\underline{\mathsf{GP}}(\mathsf{R}\mathcal{A}, \mathcal{A}) \simeq \underline{\mathsf{GP}}(\mathsf{R}\mathcal{A})$. Therefore we have the assertion by (a). □

Corollary 13.18. *Let $\mathcal{A}$ be a dualizing k-variety. If each object of $\mathsf{mod}\,\mathcal{A}$ and $\mathsf{mod}\,\mathcal{A}^{\mathrm{op}}$ has finite projective dimension, then we have the following triangle equivalence*

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\mathcal{A}) \simeq \underline{\mathsf{mod}}\,\mathsf{R}\mathcal{A}.$$

Proof. If $\mathcal{A}$ is a dualizing k-variety, then $\mathsf{GP}(\mathsf{R}\mathcal{A}) = \mathsf{mod}\,\mathsf{R}\mathcal{A}$ holds. The assertion directly follows from Corollary 13.17. □

14 Proof of Theorem 1.8

Throughout this section, let k be an algebraically closed field. Let A be a finite dimensional hereditary k-algebra, that is, $\mathrm{gldim}(A) \leq 1$. In this section, we apply Corollary 13.18 to $\underline{\mathsf{mod}}\, A$ and show Theorem 14.5.

We denote by $\mathsf{mod}\, A$ the category of the finitely generated A-modules and denote by τ and τ^{-1} the Auslander-Reiten translations on $\mathsf{mod}\, A$. We call an indecomposable A-module M *preprojective* (resp. *preinjective*) if there exists an indecomposable projective A-module P (resp. injective A-module I) and an integer i such that $M \simeq \tau^i(P)$ (resp. $M \simeq \tau^i(I)$). We call an indecomposable A-module M *regular* if $\tau^i(M) \neq 0$ for any $i \in \mathbb{Z}$. Put the following subcategories of $\mathsf{mod}\, A$:

$$\begin{aligned}
\mathcal{P} &:= \mathsf{add}\{M \in \mathsf{mod}\, A \mid M \text{ is a preprojective module}\},\\
\mathcal{I} &:= \mathsf{add}\{M \in \mathsf{mod}\, A \mid M \text{ is a preinjective module}\},\\
\mathcal{R} &:= \mathsf{add}\{M \in \mathsf{mod}\, A \mid M \text{ is a regular module}\}.
\end{aligned}$$

We denote by $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ the bounded derived category of $\mathsf{mod}\, A$ and denote by $\mathbb{S}$ a Serre functor of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$. We regard $\mathsf{mod}\, A$ as a full subcategory of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ by the canonical inclusion. Thus for any $X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$, $X \in \mathsf{mod}\, A$ if and only if $\mathrm{H}^i(X) = 0$ for any $i \neq 0$.

The following proposition is well known (see [ASS, Chapter VIII. 2.1. Proposition] [Ha88, Chapter I, 5.2, Lemma]).

Proposition 14.1. *Let A be a representation infinite hereditary algebra. Then we have the following equalities.*

$$\begin{aligned}
\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A) &= \bigvee_{i\in\mathbb{Z}} (\mathsf{mod}\, A)[i],\\
\mathsf{mod}\, A &= \mathcal{P} \vee \mathcal{R} \vee \mathcal{I}.
\end{aligned}$$

We denote by $\mathsf{mod}_{\mathrm{p}}\, A$ the full subcategory of $\mathsf{mod}\, A$ consisting of modules without non-zero projective direct summands. We define an additive functor

$$\Phi : \mathsf{R}(\mathsf{mod}_{\mathrm{p}}\, A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$$

as follows. For $X \in \mathsf{mod}_{\mathrm{p}}\, A$ and $i \in \mathbb{Z}$, let $\Phi(X, i) := \mathbb{S}^i(X)$. For $X, Y \in \mathsf{mod}_{\mathrm{p}}\, A$ and $i, j \in \mathbb{Z}$, since $\mathbb{S}$ is a Serre functor of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$, we have

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)}\big(\mathbb{S}^i(X), \mathbb{S}^j(Y)\big) \simeq \begin{cases} \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)}(X, Y) & i = j,\\ \mathrm{D}\,\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)}(Y, X) & j = i+1,\\ 0 & \text{else}, \end{cases}$$

where the last isomorphism follows from Lemma 14.2. By using these isomorphisms, we define a map

$$\Phi_{(X,i),(Y,j)} : \mathrm{Hom}_{\mathsf{R}(\mathsf{mod}_{\mathrm{p}}\, A)}((X, i), (Y, j)) \to \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)}\big(\mathbb{S}^i(X), \mathbb{S}^j(Y)\big),$$

and we extend Φ on $\mathsf{R}(\mathsf{mod}_{\mathrm{p}}\, A)$ additively. Φ is actually a functor, since a Serre duality is bifunctorial.

Lemma 14.2. *Let A be a representation infinite hereditary algebra. For any $i < 0$ and $j > 1$, we have*

$$\mathbb{S}^i(\mathsf{mod}_{\mathrm{p}} A) \subset \mathsf{add}(A) \vee \bigvee_{l<0} \mathsf{mod}\, A\,[l], \quad \mathbb{S}^j(\mathsf{mod}_{\mathrm{p}} A) \subset \mathsf{add}(\mathrm{D}\, A) \vee \bigvee_{l>1} \mathsf{mod}\, A\,[l].$$

Proof. The assertions come from Proposition 14.1. □

The first theorem of this section is the following. Put $\mathbb{S}_1 := \mathbb{S} \circ [-1]$. Note that $H^0(\mathbb{S}_1(M)) \simeq \tau(M)$ and $H^0(\mathbb{S}_1^{-1}(M)) \simeq \tau^{-1}(M)$ hold for any $M \in \mathsf{mod}\, A$.

Theorem 14.3. *The functor $\Phi : \mathsf{R}(\mathsf{mod}_{\mathrm{p}} A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ is an equivalence of additive categories.*

Proof. By the definition, Φ is fully faithful. We show that Φ is dense. Let X be an indecomposable object of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$. By Proposition 14.1, there exist an indecomposable A-module M and an integer l such that $X \simeq M[l]$.

Assume that M is a preprojective module. There exist an indecomposable projective A-module P and $i \geq 0$ such that $M \simeq \mathbb{S}_1^{-i}(P)$. If $i+l > 0$, then we have $\mathbb{S}_1^{-(i+l)}(P) \in \mathsf{mod}_{\mathrm{p}} A$ and

$$\begin{aligned}\Phi(\mathbb{S}_1^{-(i+l)}(P), -l) &= \mathbb{S}^l(\mathbb{S}_1^{-(i+l)}(P)) \\ &= \mathbb{S}_1^{-i}(P)[l].\end{aligned}$$

If $i + l \leq 0$, then we have $\mathbb{S}_1^{-(i+l)}(\mathbb{S}(P)) \in \mathsf{mod}_{\mathrm{p}} A$ and

$$\begin{aligned}\Phi(\mathbb{S}_1^{-(i+l)}(\mathbb{S}(P)), -l+1) &= \mathbb{S}^{l-1}(\mathbb{S}_1^{-(i+l)}(\mathbb{S}(P))) \\ &= \mathbb{S}_1^{-i}(P)[l].\end{aligned}$$

Next assume that M is a preinjective module. There exist an indecomposable injective A-module I and $i \geq 0$ such that $M \simeq \mathbb{S}_1^{i}(I)$. If $i - l \geq 0$, then we have $\mathbb{S}_1^{i-l}(I) \in \mathsf{mod}_{\mathrm{p}} A$ and

$$\begin{aligned}\Phi(\mathbb{S}_1^{i-l}(I), -l) &= \mathbb{S}^l(\mathbb{S}_1^{i-l}(I)) \\ &= \mathbb{S}_1^{i}(I)[l].\end{aligned}$$

If $i - l < 0$, then we have $\mathbb{S}_1^{i-l}(\mathbb{S}^{-1}(I)) \in \mathsf{mod}_{\mathrm{p}} A$ and

$$\begin{aligned}\Phi(\mathbb{S}_1^{i-l}(\mathbb{S}^{-1}(I)), -l-1) &= \mathbb{S}^{l+1}(\mathbb{S}_1^{i-l}(\mathbb{S}^{-1}(I))) \\ &= \mathbb{S}_1^{i}(I)[l].\end{aligned}$$

Assume that M is a regular module. Then we have $\mathbb{S}_1^{-l}(M) \in \mathcal{R} \subset \mathsf{mod}_{\mathrm{p}} A$ and $\Phi(\mathbb{S}_1^{-l}(M), -l) = \mathbb{S}^l(\mathbb{S}_1^{-l}(M)) = M[l]$ holds. Therefore the functor $\Phi : \mathsf{R}(\mathsf{mod}_{\mathrm{p}} A) \to \mathcal{D}$ is dense. □

Theorem 14.3 is an analog of the well known equivalence $\mathsf{D}^{\mathrm{b}}(\mathcal{H}) \simeq \mathsf{Rep}\, \mathcal{H}$ for a hereditary abelian category $\mathcal{H}$ [Le, Theorem 3.1]. But they are quite different, since the definitions of $\mathsf{Rep}\, \mathcal{H}$ and $\mathsf{R}(\underline{\mathsf{mod}}\, A)$ are quite different.

We recall the following proposition.

Proposition 14.4. *[AR74, Propositions 6.2, 10.2] Let $\mathcal{A}$ be a dualizing k-variety and $\mathcal{B} := \mathsf{mod}\,\mathcal{A}$. Let $\mathcal{P}$ be the full subcategory of $\mathcal{B}$ consisting of the projective modules. Then the following statements hold.*

(a) *$\mathcal{B}/[\mathcal{P}]$ is a dualizing k-variety.*

(b) *Assume that the global dimension of $\mathsf{mod}\,\mathcal{A}$ is at most n, then the global dimension of $\mathsf{mod}(\mathcal{B}/[\mathcal{P}])$ is at most $3n-1$.*

Then we apply Corollary 13.18 to $\underline{\mathsf{mod}}\,A$.

Theorem 14.5. *Let A be a representation infinite hereditary algebra. Then we have the following triangle equivalences*

$$\underline{\mathsf{mod}}\,\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A) \simeq \underline{\mathsf{mod}}\,\mathsf{R}(\underline{\mathsf{mod}}\,A) \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}(\underline{\mathsf{mod}}\,A)).$$

Proof. Since A is hereditary, a canonical functor $\mathsf{mod}_{\mathrm{p}}\,A \to \underline{\mathsf{mod}}\,A$ induces an equivalence $\mathsf{mod}_{\mathrm{p}}\,A \simeq \underline{\mathsf{mod}}\,A$. Therefore the first equivalence comes from Theorem 14.3. By Proposition 14.4, $\underline{\mathsf{mod}}\,A$ is a dualizing k-variety such that the global dimension of $\mathsf{mod}(\underline{\mathsf{mod}}\,A)$ is at most two. Therefore we can apply Corollary 13.18 to the dualizing k-variety $\underline{\mathsf{mod}}\,A$. We have the second equivalence. □

We say that two dualizing k-varieties $\mathcal{A}$ and $\mathcal{A}'$ are derived equivalent if the derived categories of $\mathsf{mod}\,\mathcal{A}$ and $\mathsf{mod}\,\mathcal{A}'$ are triangle equivalent.

Corollary 14.6. *Let A, A' be representation infinite hereditary algebras. If A and A' are derived equivalent, then $\underline{\mathsf{mod}}\,A$ and $\underline{\mathsf{mod}}\,A'$ are derived equivalent.*

Remark 14.7. If A is a representation finite hereditary algebra, then Theorems 14.3, 14.5 and Corollary 14.6 were shown by [IO].

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