Classification theory of subcategories around commutative algebra

Hiroki Matsui

Graduate School of Mathematics, Nagoya University
Student ID: 321401248
Advisor: Prof. Ryo Takahashi

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Preface

In the classical representation theory, one of the important problem is to classify all objects of a given category up to isomorphisms. However, this problem is quite difficult and impossible almost always except special cases. Therefore, classifying subcategories which are closed under some operation has been considered so far as the next but most reasonable problem. Namely, instead of classifying objects, we would like to classify subcategories generated by an object in some sense.

Let $\mathcal{C}$ be an essentially small category. Classifying subcategories means for a property $\mathbb{P}$ of full subcategories, finding a one-to-one correspondence

$$\{\text{subcategories of } \mathcal{C} \text{ satisgying } \mathbb{P}\} \rightleftarrows S,$$

between the set of subcategories of $\mathcal{C}$ satisfying $\mathbb{P}$ and some set S. This set S is normally expected to be easier to understand. By using such a correspondence, we would like to understand the set {subcategories of $\mathcal{C}$ satisgying $\mathbb{P}$} and moreover, the structure of $\mathcal{C}$.

For a triangulated category, one of the important classes of subcategories is that of *thick subcategories*. A full subcategory of a triangulated category is called thick if it is triangulated and closed under taking direct summands. It naturally appears as the kernel of some cohomological functor and also as the inverse image of some *tensorial support* (for the definition, we refer to Definitions 1.3 and 2.5).

Classification of subcategories was first considered by Hopkins-Smith [HS] in stable homotopy theory. They classified thick subcategories of p-local finite spectra:

Theorem 1. [HS, Theorem 7] *Let p be a prime number. Then there is a one-to-one correspondence:*

$$\{\textit{thick subcategories of } \mathsf{SH}^{\mathsf{fin}}_{(p)}\} \rightleftarrows \mathbb{Z}_{\geqslant 0} \cup \{\infty\},$$

where $\mathsf{SH}^{\mathsf{fin}}_{(p)}$ denotes the triangulated category of p-local finite spectra.

Indeed, they verified that every thick subcategory of $\mathsf{SH}^{\mathsf{fin}}_{(p)}$ is obtained as the kernel of a cohomological functor given by tensoring with the Morava K-theory spectrum.

Motivated by this classification, Hopkins [Hop] and Neeman [Nee92] proved the following classification theorem for Noetherian affine schemes and Thomason [Tho] generalized it to Noetherian schemes, which classifies thick subcategories of perfect complexes closed under the tensor action by each object:

Theorem 2. [Hop, Nee92, Tho] *Let X be a Noetherian scheme. Then there is a one-to-one correspondence:*

$$\left\{\begin{matrix}\textit{thick subcategories of } \mathsf{D}^{\mathsf{perf}}(X) \\ \textit{closed under } \otimes^{\mathbf{L}}_{\mathcal{O}_X}\end{matrix}\right\} \rightleftarrows \{\textit{specialization closed subsets of } X\},$$

where $\mathsf{D}^{\mathsf{perf}}(X)$ denotes the perfect derived category of X.

In fact, this theorem states that every thick subcategory of $\mathsf{D}^{\mathsf{perf}}(X)$ which is closed under the action via $\otimes^{\mathbf{L}}_{\mathcal{O}_X}$, is obtained as the inverse image $\mathsf{Supp}^{-1} W$ of exactly one specialization closed subset W of X by the cohomological support Supp; see Example 1.4 for details.

On the other hand, in modular representation theory, Benson-Carlson-Rickard [BCR] and Benson-Iyengar-Krause [BIK] classified thick subcategories of the stable module categories of a finite group which are closed under the tensor action by each object:

Theorem 3. [BCR, BIK] *Let k be a field and G a finite group. Then there is a one-to-one correspondence:*

$$\left\{\begin{matrix}\textit{thick subcategories of } \underline{\mathsf{mod}}\, kG \\ \textit{closed under } \otimes_k\end{matrix}\right\} \rightleftarrows \{\textit{specialization closed subsets of } \mathsf{Proj}\,\mathsf{H}^*(G;k)\},$$

where we denote by $\underline{\mathsf{mod}}\, kG$ the stable module category of kG and by $\mathsf{H}^*(G;k)$ the cohomology ring of G with coefficients in k.

As in the case of the above classification, this theorem states that every thick subcategory of $\underline{\mathsf{mod}}\, kG$ which is closed under the action via $\otimes_k$, is obtained as the inverse image $V_G^{-1}(W)$ of exactly one specialization closed subset W of $\mathsf{Proj}\,\mathsf{H}^*(G;k)$ by the support variety V_G; see Example 1.4 for details.

In all of these results, tensor structures of triangulated categories play crucial roles. Indeed, these classifications come from the same framework of *tensor triangular geometry.*

Tensor triangular geometry is a theory established by Balmer at the beginning of this century. Let $(\mathcal{T}, \otimes, \mathbf{1})$ be an (essentially small) *tensor triangulated category*, that is, a triangulated category $\mathcal{T}$ equipped with symmetric tensor product $\otimes$ and unit object $\mathbf{1}$. One can then define the notions of *thick tensor ideals*, *prime thick tensor ideals* and *radical thick tensor ideals* of $\mathcal{T}$, which behave similarly to ideals, prime ideals and radical ideals of a commutative ring. The *Balmer spectrum* $\mathtt{Spec}\,\mathcal{T}$ of $\mathcal{T}$ is defined as the set of prime thick tensor ideals of $\mathcal{T}$. This set has the structure of a topological space. Tensor triangular geometry studies Balmer spectra and develops commutative-algebraic and algebro-geometric observations for them. He accomplished the following monumental work in this direction.

Theorem 4. [Bal05, Theorem 4.10] *Let $\mathcal{T}$ be a tensor triangulated category. Then there is a one-to-one correspondence:*

$$\{\textit{radical thick tensor ideals of } \mathcal{T}\} \rightleftarrows \{\textit{Thomason subsets of } \mathtt{Spec}\,\mathcal{T}\}.$$

Here, a subset of a topological space is said to be *Thomason* if it is the union of closed subsets whose complements are quasi-compact. Thus, the classification of radical thick tensor ideals is interpreted as the study of the topological space $\mathtt{Spec}\,\mathcal{T}$. Balmer [Bal05, Bal10a] also determined the Balmer spectra of $\mathsf{D}^{\mathsf{perf}}(X)$, $\underline{\mathsf{mod}}\, kG$ and the category $\mathsf{SH}^{\mathsf{fin}}$ of finite spectra, by using classification Theorem 1, 2 and 3. Furthermore, these classification theorems are restored from Theorem 4.

The Balmer spectrum is hard to calculate from the definition without classification. To explore $\mathtt{Spec}\,\mathcal{T}$, Balmer [Bal10a] defined a graded commutative ring $\mathrm{R}^\bullet_{\mathcal{T}}$ which is called the *graded central ring* of $\mathcal{T}$ and a continuous map $\rho^\bullet_{\mathcal{T}} : \mathtt{Spec}\,\mathcal{T} \to \mathsf{Spec}^{\mathsf{h}}\,\mathrm{R}^\bullet_{\mathcal{T}}$ ($\mathsf{Spec}^{\mathsf{h}}\,\mathrm{R}^\bullet_{\mathcal{T}}$ stands for the homogeneous prime spectrum of $\mathrm{R}^\bullet_{\mathcal{T}}$). Surjectivity of the comparison map is investigated in [Bal10a] and frequently it becomes surjective (e.g., it does provided $\mathrm{R}^\bullet_{\mathcal{T}}$ is Noetherian). Contrary to this, injectivity of the comparison map is hard to observe and there are only affirmative answers for each individual tensor triangulated category. For example, in the cases of $\mathsf{D}^{\mathsf{perf}}(X)$ for a Noetherian affine scheme X, and $\underline{\mathsf{mod}}\, kG$ for a finite group G with a field k, their comparison maps are injective; see [Bal10a, Propositions 8.1 and 8.5]. Note that these categories are *algebraic triangulated categories*, namely the stable categories of Frobenius exact categories. By contrast, for a *topological tensor triangulated category*, such as $\mathsf{SH}^{\mathsf{fin}}$, the map may not be injective; see [Bal10a, Proposition 9.4]. From this observation, Balmer conjectured the following.

Conjecture 5. [Bal10b, Conjecture 72] *The comparison map*

$$\rho^{\bullet}_{\mathcal{T}} : \mathsf{Spec}\,\mathcal{T} \to \mathsf{Spec}^{\mathsf{h}}\, R^{\bullet}_{\mathcal{T}}$$

is (locally) injective if $\mathcal{T}$ is "algebraic enough".

Here, "algebraic enough" tensor triangulated categories could mean algebraic ones, or derived categories of dg-categories, or ones locally generated by the unit.

Let $\mathcal{A}$ be an (essentially small) additive category. We say that an additive subcategory $\mathcal{X}$ is *additively closed* if it is closed under taking direct summands and *dense* if every object of $\mathcal{A}$ is a direct summand of some object of $\mathcal{X}$. Then one can easily check that every additive subcategory of $\mathcal{A}$ is a dense subcategory of some additively closed subcategory of $\mathcal{A}$. Thus, to classify additive subcategories, it suffices to classify dense ones and additively closed ones. So far, we have considered classifying additively closed subcategories. As classification of dense subcategories, Thomason proved the following theorem.

Theorem 6. [Tho, Theorem 2.1] *Let $\mathcal{T}$ be a triangulated category. Then there is a one-to-one correspondence:*

$$\{\textit{dense triangulated subcategories of } \mathcal{T}\} \rightleftarrows \{\textit{subgroups of } K_0(\mathcal{T})\},$$

where $K_0(\mathcal{T})$ denotes the Grothendieck group of $\mathcal{T}$.

Combining this theorem (classification of dense subcategories) and Theorem 2 (classification of additively closed subcategories) yields a complete classification of the triangulated subcategories of $\mathsf{D}^{\mathsf{perf}}(X)$ for a Noetherian scheme X.

In this thesis, we discuss various problems concerning classification theory of subcategories: tensor triangular geometry for the right bounded derived category $\mathsf{D}^{-}(\mathsf{mod}\, R)$ of a commutative Noetherian ring R, reconstruction problems from classification of subcategories, and classification of dense subcategories of exact categories. We do it mainly for categories appearing in commutative algebra.

This thesis consists of four parts which are based on the papers [MT, Mat17a, Mat17b, Mat17c].

In Part 1, we give a short survey of support theory of triangulated categories and tensor triangular geometry for later use. At the end of this part, we give a geometric criterion for the perfect derived category of a Noetherian scheme to satisfy Balmer's conjecture. Moreover, using this criterion, we prove that Balmer's comparison map $\rho^{\bullet}_{\mathsf{D}^{\mathsf{perf}}(X)}$ is locally injective for the perfect derived categories of Noetherian quasi-affine schemes:

Theorem A (Part 1, Theorem 2.31)**.** *Let X be a Noetherian quasi-affine scheme (i.e., an open subscheme of an affine scheme). Then the comparison map $\rho^{\bullet}_{\mathsf{D}^{\mathsf{perf}}(X)}$ is locally injective.*

Note that if X is quasi-affine, then $\mathsf{D}^{\mathsf{perf}}(X)$ is generated by the unit $\mathcal{O}_X$.

In Part 2, we discuss tensor triangular geometry for the tensor triangulated category $\mathsf{D}^{-}(\mathsf{mod}\, R)$ of right bounded complexes of finitely generated modules over a commutative Noetherian ring R. Tensor triangular geometry for tensor triangulated categories which are *rigid* (small in some sense), have been studied by several authors so far [Bal07, BF, SS] and various results are known. For example, $\mathsf{SH}^{\mathsf{fin}}$, $\mathsf{D}^{\mathsf{perf}}(X)$, and $\underline{\mathsf{mod}}\, kG$ are rigid. However, our tensor triangulated category $\mathsf{D}^{-}(\mathsf{mod}\, R)$ is far from rigid and hence we cannot apply general results on rigid tensor triangulated categories.

First, we classify thick tensor ideals of $\mathsf{D}^{-}(\mathsf{mod}\, R)$ generated by bounded complexes:

Theorem B (Part 2, Theorem 5.12)**.** *Let R be a commutative Noetherian ring. Then there is a one-to-one correspondence:*

$$\left\{\begin{array}{l}\textit{thick tensor ideals of } \mathsf{D}^{-}(\mathsf{mod}\, R)\\ \textit{generated by bounded complexes}\end{array}\right\} \rightleftarrows \{\textit{specialization closed subsets of } \mathsf{Spec}\, R\}.$$

This theorem can be considered as a generalization of Theorem 2 in the case of an affine scheme. To prove this theorem, we generalize the smash nilpotence theorem for perfect complexes due to Hopkins [Hop, Theorem 10] and Neeman [Nee92, Theorem 1.1] to unbounded complexes.

Next, we relate the Balmer spectrum $\mathtt{Spec}\, \mathsf{D}^{-}(\mathsf{mod}\, R)$ to the Zariski spectrum $\mathsf{Spec}\, R$. As stated above, classification of subcategories gives information on the Balmer spectrum. Therefore, Theorem B gives us a way to investigate the structure of $\mathtt{Spec}\, \mathsf{D}^{-}(\mathsf{mod}\, R)$ by comparing with the Zariski spectrum $\mathsf{Spec}\, R$. More precisely, we introduce two order-reversing maps

$$\mathtt{Spec}\, \mathsf{D}^{-}(\mathsf{mod}\, R) \underset{\mathcal{S}}{\overset{\mathfrak{s}}{\rightleftarrows}} \mathsf{Spec}\, R,$$

and investigate $\mathtt{Spec}\, \mathsf{D}^{-}(\mathsf{mod}\, R)$ through $\mathfrak{s}$ and $\mathcal{S}$. Then the map $\mathfrak{s}$ is continuous and $\mathfrak{s} \circ \mathcal{S} = 1$. Furthermore, these maps connect Balmer's classification theorem and our classification theorem (Theorem 9.20). One of our main results in this direction is the following theorem.

Theorem C (Part 2, Corollary 7.5 and Theorem 10.5)**.** (1) *The following are equivalent:*
 (a) *$\mathcal{S}$ is an immersion.*
 (b) *$\mathsf{Spec}\, R$ is a finite set (i.e., R has Krull dimension at most 1 and semilocal).*

(2) *The following are equivalent:*
 (a) *$\mathcal{S}$ is a homeomorphism.*
 (b) *R is Artinian (i.e., R has Krull dimension 0).*

As a direct consequence, we can classify all the thick tensor ideals of $\mathsf{D}^{-}(\mathsf{mod}\, R)$ via specialization closed subsets of $\mathsf{Spec}\, R$ provided R is Artinian. Conversely, such classification is possible only in the case of Artinian rings.

As Balmer's classification theorem says, topological information on the Balmer spectrum provides significant information to classify radical thick tensor ideals. We have the following results concerning topological properties such as Noetherianity, connectedness, and irreducibility of $\mathtt{Spec}\, \mathsf{D}^{-}(\mathsf{mod}\, R)$.

Theorem D (Part 2, Corollary 8.9)**.** (1) *If $\mathtt{Spec}\, \mathsf{D}^{-}(\mathsf{mod}\, R)$ is Noetherian, then $\mathsf{Spec}\, R$ is a finite set.*

(2) *$\mathtt{Spec}\, \mathsf{D}^{-}(R)$ is connected (resp. irreducible) if and only if $\mathsf{Spec}\, R$ is connected (resp. irreducible).*

For our category $\mathsf{D}^{-}(\mathsf{mod}\, R)$, one has $\mathrm{R}^{\bullet}_{\mathsf{D}^{-}(\mathsf{mod}\, R)} = R$ and $\mathsf{Spec}^{\mathsf{h}}\, \mathrm{R}^{\bullet}_{\mathsf{D}^{-}(\mathsf{mod}\, R)} = \mathsf{Spec}\, R$. Actually, the map $\mathfrak{s}$ is nothing but Balmer's comparison map $\rho^{\bullet}_{\mathsf{D}^{-}(\mathsf{mod}\, R)}$ (Proposition 10.9). Using the properties of $\mathfrak{s}$, we obtain the following result.

Theorem E (Part 2, Corollary 10.10)**.** *Assume that R has positive Krull dimension and that R is either a domain or a local ring. Then the map $\mathfrak{s}$ is not locally injective.*

In view of Conjecture 5, this theorem says that $\mathsf{D}^{-}(\mathsf{mod}\, R)$ is not "algebraic enough"; an algebraic tensor triangulated category is not sufficiently "algebraic enough".

For the remainder of this part, we give some calculations of prime thick tensor ideals of $\mathsf{D}^{-}(\mathsf{mod}\,R)$ for a discrete valuation ring R. In this case, it is known that every complex X in $\mathsf{D}^{-}(\mathsf{mod}\,R)$ is isomorphic to its cohomology complex $\mathsf{H}(X) := \oplus_{n\in\mathbb{Z}}\mathsf{H}^n(X)[-n]$ in $\mathsf{D}^{-}(\mathsf{mod}\,R)$. Thus, we can ignore differentials of complexes. This makes it easy to compute tensor products of complexes. The following theorem would say that the Balmer spectrum $\mathsf{Spec}\,\mathsf{D}^{-}(\mathsf{mod}\,R)$ is quite complicated even in the case of a discrete valuation ring.

Theorem F (Part 2, Propositions 11.7, 11.17 and Theorems 11.11, 11.14)**.** *Let (R, xR) be a discrete valuation ring, and let $n \geqslant 0$ be an integer. Let $\mathcal{P}_n$ be the full subcategory of $\mathsf{D}^{-}(\mathsf{mod}\,R)$ consisting of complexes X with finite length homologies such that there exists an integer $t \geqslant 0$ with $\ell\ell(\mathsf{H}^{-i}X) \leqslant ti^n$ for all $i \gg 0$. Here, $\ell\ell(M)$ denotes the Loewy length $\ell\ell(M) := \inf\{i \mid x^iM = 0\}$. Then:*

(1) *$\mathcal{P}_n$ coincides with the smallest thick tensor ideal of $\mathsf{D}^{-}(R)$ containing the complex*

$$\bigoplus_{i>0}(R/x^{i^n}R)[i] = (\cdots \xrightarrow{0} R/x^{3^n}R \xrightarrow{0} R/x^{2^n}R \xrightarrow{0} R/x^{1^n}R \to 0).$$

(2) *$\mathcal{P}_n$ is a prime thick tensor ideal of $\mathsf{D}^{-}(\mathsf{mod}\,R)$.*

(3) *One has $\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \mathcal{P}_2 \subsetneq \cdots$. Hence $\mathsf{Spec}\,\mathsf{D}^{-}(\mathsf{mod}\,R)$ has infinite Krull dimension.*

In Part 3, as an application of classification theory of subcategories, we consider reconstruction of classifying spaces. Here, by a classifying space, we mean a Noetherian sober space whose specialization closed subsets bijectively correspond to thick subcategories of a given triangulated category via some support; for the precise definition, see Definition 13.5. The first main result of this part is the following reconstruction theorem.

Theorem G (Part 3, Theorem 13.11)**.** *Let $\mathcal{T}$ and $\mathcal{T}'$ be essentially small triangulated categories, X and Y their classifying spaces, respectively. If $\mathcal{T}$ and $\mathcal{T}'$ are triangulated equivalent, then X and Y are homeomorphic.*

By [Bal05, Theorem 5.2], such result can be proven using tensor structures, however, what I would like to emphasize is that we don't need any tensor structures to prove this theorem. Such a reconstruction problem has been dealt with so far in connection with non-commutative algebraic geometry. For example, we refer to [Gab, Ros, Bal05].

Of course, a tensor triangle equivalence $\mathcal{T} \cong \mathcal{T}'$ between tensor triangulated categories implies that $\mathsf{Spec}\,\mathcal{T}$ and $\mathsf{Spec}\,\mathcal{T}'$ are homeomorphic. Applying this theorem, we show that the topology of the Balmer spectrum of $\mathcal{T}$ is reconstructed just from the triangle structure of a certain tensor triangulated category $\mathcal{T}$:

Theorem H (Part 3, Corollary 14.5)**.** *Let $\mathcal{T}$ and $\mathcal{T}'$ be closed tensor triangulated categories such that*

(1) *$\mathsf{Spec}\,\mathcal{T}$ and $\mathsf{Spec}\,\mathcal{T}'$ are Noetherian, and*

(2) *$\mathcal{T}$ and $\mathcal{T}'$ are generated by their unit objects.*

If $\mathcal{T}$ and $\mathcal{T}'$ are triangulated equivalent, then $\mathsf{Spec}\,\mathcal{T}$ and $\mathsf{Spec}\,\mathcal{T}'$ are homeomorphic.

Combining this theorem and Theorems 2 and 3, we obtain the following results.

Theorem I (Part 3, Theorem 14.7)**.** *Let X and Y be Noetherian quasi-affine schemes. If $\mathsf{D}^{\mathsf{perf}}(X)$ and $\mathsf{D}^{\mathsf{perf}}(Y)$ are triangulated equivalent, then X and Y are homeomorphic.*

Theorem J (Part 3, Theorem 14.10, Corollary 14.11)**.** *Let k (resp. l) be a field of characteristic p (resp. q), G (resp. H) a finite p-group (resp. q-group). Then the implications*

$$\begin{aligned}\underline{\mathsf{mod}}\,kG \cong \underline{\mathsf{mod}}\,lH &\Longrightarrow \mathsf{Proj}\,\mathsf{H}^*(G;k) \text{ and } \mathsf{Proj}\,\mathsf{H}^*(H;l) \text{ are homoemorphic}\\ &\Longrightarrow r_p(G) = r_q(H)\end{aligned}$$

hold. Here, we set $r_p(G) := \sup\{r \mid (\mathbb{Z}/p)^r \subseteq G\}$ and call it the p-rank of G.

In the 1980s, Buchweitz [Buc] defined the *stable derived category* of a Noetherian ring R, which is recently called the *singularity category* of R. It is by definition the Verdier quotient

$$\mathsf{D_{sg}}(R) := \mathsf{D^b}(\mathsf{mod}\, R) / \mathsf{K^b}(\mathsf{proj}\, R)$$

of the bounded derived category $\mathsf{D^b}(\mathsf{mod}\, R)$ of finitely generated R-modules by its subcategory $\mathsf{K^b}(\mathsf{proj}\, R)$ of bounded complexes of finitely generated projective R-modules (i.e., perfect complexes over R). Singularity categories have been deeply investigated from algebro-geometric and representation-theoretic motivations [Che, IW, Ste, Tak10] and connected to Kontsevich's Homological Mirror Symmetry Conjecture by Orlov [Orl04]. For two commutative Noetherian rings R and S, we say that they are *singularly equivalent* if their singularity categories $\mathsf{D_{sg}}(R)$ and $\mathsf{D_{sg}}(S)$ are equivalent as triangulated categories. Regarding the singularity category, the following classification was obtained by Takahashi.

Theorem K. [Tak10, Theorem 6.7] *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring which is locally a hypersurface on the punctured spectrum. Then there is a one-to-one correspondence:*

$$\left\{\begin{matrix}\textit{thick subcategories of } \mathsf{D_{sg}}(R)\\ \textit{containing } k\end{matrix}\right\} \rightleftarrows \{\textit{non-empty specialization closed subsets of } \mathsf{Sing}\, R\},$$

where $\mathsf{Sing}\, R$ denotes the singular locus of R.

To apply Theorem G for this classification, we have to check that the condition "containing the residue field" for a thick subcategory is preserved by a singular equivalence. For this, we introduce the notion of a *test object* of a triangulated category which is a categorically defined object and hence preserved by a triangle equivalence. In fact, we prove that a thick subcategory of the singularity category contains a test object if and only if it contains the residue field for a complete intersection local ring (Proposition 15.12). Thus, we obtain the following result.

Theorem L (Part 3, Theorem 15.4)**.** *Let R and S be complete intersection local rings that are locally hypersurfaces on the punctured spectra. If R and S are singularly equivalent, then $\mathsf{Sing}\, R$ and $\mathsf{Sing}\, S$ are homeomorphic.*

In Part 4, we handle classification of dense subcategories of an exact category $\mathcal{E}$. We say that an additive subcategory $\mathcal{X}$ of $\mathcal{E}$ is a *2-out-of-3 subcategory* if it satisfies the 2-out-of-3 property with respect to conflations. By definition, an additively closed 2-out-of-3 subcategory is nothing but a so-called *thick subcategory*. The main theorem of this section is the following.

Theorem M (Part 4, Theorem 17.7)**.** *Let $\mathcal{E}$ be an essentially small exact category admitting either a generator or a cogenerator $\mathcal{G}$ (Definition 17.1). There is a one-to-one correspondence:*

$$\{\textit{dense 2-out-of-3 subcategories of } \mathcal{E} \textit{ containing } \mathcal{G}\} \rightleftarrows \left\{\begin{matrix}\textit{subgroups of } K_0(\mathcal{E})\\ \textit{containing the image of } \mathcal{G}\end{matrix}\right\},$$

where $K_0(\mathcal{E})$ stands for the Grothendieck group of $\mathcal{E}$.

Combining this theorem with Theorem 6, we obtain:

Theorem N (Part 4, Corollary 18.3)**.** *Let $\mathcal{E}$ be an essentially small exact category admitting either a generator or a cogenerator $\mathcal{G}$. Then there are one-to-one correspondences among the sets:*

(1) $\{$*dense 2-out-of-3 subcategories of* $\mathcal{E}$ *containing* $\mathcal{G}\}$,
(2) $\{$*dense triangulated subcategories of* $\mathsf{D}^{\mathsf{b}}(\mathcal{E})$ *containing* $\mathcal{G}\}$, *and*
(3) $\{$*subgroups of* $K_0(\mathcal{E})$ *containing the image of* $\mathcal{G}\}$.

This result can be viewed as a dense version of [KS, Theorem 1] whenever we take $\mathcal{E} = \mathsf{proj}\,\mathcal{E}$.

Acknowledgments

I would like to express my deep gratitude to my supervisor Ryo Takahashi for his guidance, support and encouragement since I was an undergraduate student. I am deeply grateful to Osamu Iyama and Yuji Yoshino for their helpful advice and comments. I would like to thank professors and friends who gave me helpful advice. Without their support, I could not be able to finish writing this thesis. I am partly supported by JSPS Grant-in-Aid for JSPS Fellows 16J01067.

Conventions

For a left Noetherian ring A, we denote by $\mathsf{mod}\, A$ (resp. $\mathsf{proj}\, A$) the category of finitely generated left A-modules (resp. finitely generated projective left A-modules).

For an ordered set P, we denote by $\operatorname{Max} P$ (resp. $\operatorname{Min} P$) the set of maximal (resp. minimal) elements of P.

Let R be a commutative Noetherian ring. We denote by $\mathsf{Spec}\, R$ (resp. $\operatorname{Max} R$, $\operatorname{Min} R$) the set of prime (resp. maximal prime, minimal prime) ideals of R. For an ideal I of R, we denote by $\mathrm{V}(I)$ the set of prime ideals of R containing I, and set $\mathrm{D}(I) = \mathrm{V}(I)^{\complement} = \mathsf{Spec}\, R \setminus \mathrm{V}(I)$. When I is generated by a single element x, we simply write $\mathrm{V}(x)$ and $\mathrm{D}(x)$. For a prime ideal $\mathfrak{p}$ of R, the residue field of $R_{\mathfrak{p}}$ is denoted by $\kappa(\mathfrak{p})$, i.e., $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. We denote by $\operatorname{ht} \mathfrak{p}$ the *height* of $\mathfrak{p}$, that is the supremum of the length of all chains of prime ideals contained in $\mathfrak{p}$. For an ideal $\mathfrak{a}$, we also denote by $\operatorname{ht} \mathfrak{a}$ the *height* of $\mathfrak{a}$, which is the infimum of the heights of prime ideals containing $\mathfrak{a}$. For a sequence $\boldsymbol{x} = x_1, \dots, x_n$ of elements of R, the *Koszul complex* of R with respect to $\boldsymbol{x}$ is denoted by $\mathrm{K}(\boldsymbol{x}, R)$; see [BH] for the definition.

For an additive category $\mathcal{C}$ we denote by $\mathbf{0}$ the *zero subcategory* of $\mathcal{C}$, that is, the full subcategory consisting of objects isomorphic to the zero object. For objects X, Y of $\mathcal{C}$, we mean by $X \lessdot Y$ (or $Y \gtrdot X$) that X is a direct summand of Y in $\mathcal{C}$.

For a triangulated category $\mathcal{T}$, its n-shift functor is denoted by $[n]$.

Throughout this thesis, all categories are assumed to be essentially small (i.e., isomorphism classes form a set) and all subcategories are assumed to be full. We often omit subscripts, superscripts and parentheses, if there is no danger of confusion.

Part 1. Support theory of triangulated categories

In this part, we recall the theory of tensor triangular geometry developed in [Bal05, Bal07, Bal10a, Bal10b]. At the end of this part, we discuss Balmer's conjecture for perfect derived categories of schemes.

1. Preliminaries

This section is a recollection of basic notions from point-set topology and the theory of triangulated categories.

Definition 1.1. Let X be a topological space.

(1) A subspace W of X is said to be *specialization-closed* if for any element x of W, its closure $\overline{\{x\}}$ is contained in W. Note that W is specialization-closed if and only if it is a union of closed subspaces of X. Denote by $\mathbf{Spcl}(X)$ the set of specialization closed subsets of X.
(2) We say that X is *irreducible* if it is non-empty and not the union of two proper closed subspaces. For a subspace Y of X, we say that Y is an *irreducible subspace* of X if it is an irreducible space by induced topology. Moreover, an *irreducible component* of X is a maximal irreducible subspace of X, which is automatically closed since the closure of irreducible subspace is also irreducible.
(3) We say that X is *sober* if every irreducible closed subset of X is the closure of exactly one point.
(4) We say that X is *Noetherian* if every descending chain of closed subspaces stabilizes.
(5) The *(Krull) dimension* of X, denoted by $\dim X$, is defined to be the supremum of integers $n \geqslant 0$ such that there exists a chain $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ of nonempty irreducible closed subsets of X.

Definition 1.2. Let $\mathcal{T}$ be a triangulated category. We say that an additive subcategory $\mathcal{X}$ of $\mathcal{T}$ is *thick* if it satisfies the following conditions:

(i) closed under taking shifts: $\mathcal{X}[1] = \mathcal{X}$.
(ii) closed under taking extensions: for a triangle $L \to M \to N \to L[1]$ in $\mathcal{T}$, if L and N belong to $\mathcal{X}$, then so does M.
(iii) closed under taking direct summands: for two objects L, M of $\mathcal{T}$, if the direct sum $L \oplus M$ belongs to $\mathcal{X}$, then so do L and M.

For a subcategory $\mathcal{X}$ of $\mathcal{T}$, denote by $\mathsf{thick}_{\mathcal{T}}\, \mathcal{X}$ the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{X}$. Denote by $\mathbf{Th}(\mathcal{T})$ the set of thick subcategories of $\mathcal{T}$.

Next, let me introduce the notion of a support data for a triangulated category, which appears in many places of this thesis.

Definition 1.3. Let $\mathcal{T}$ be a triangulated category. A *support data* for $\mathcal{T}$ is a pair (X, σ) where X is a topological space and σ is an assignment which assigns to an object M of $\mathcal{T}$ a closed subset $\sigma(M)$ of X satisfying the following conditions:

(1) $\sigma(0) = \emptyset$.
(2) $\sigma(M[n]) = \sigma(M)$ for any $M \in \mathcal{T}$ and $n \in \mathbb{Z}$.
(3) $\sigma(M \oplus N) = \sigma(M) \cup \sigma(N)$ for any $M, N \in \mathcal{T}$.
(4) $\sigma(M) \subseteq \sigma(L) \cup \sigma(N)$ for any triangle $L \to M \to N \to L[1]$ in $\mathcal{T}$.

Support data for triangulated categories is naturally appears various areas of mathematics. The followings are examples of them.

Example 1.4. (1) Let X be a Noetherian scheme. Denote by $\mathsf{D}^{\mathsf{perf}}(X)$ the category of perfect complexes on X. For $\mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(X)$, we define the *cohomological support* of $\mathcal{F}$ by

$$\mathsf{Supp}_X(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \ncong 0 \text{ in } \mathsf{D}^{\mathsf{perf}}(\mathcal{O}_{X,x})\}.$$

Then, $\mathsf{Supp}_X(\mathcal{F}) = \bigcup_{n\in\mathbb{Z}} \mathsf{Supp}_X(\mathsf{H}^n(\mathcal{F}))$ is a finite union of supports of coherent $\mathcal{O}_X$-modules and hence is a closed subspace of X. Moreover, (X, Supp_X) is a support data for $\mathsf{D}^{\mathsf{perf}}(X)$ because the localization is exact. For details, please see [Tho].

(2) Let k be a field of characteristic $p > 0$ and G a finite group such that p divides the order of G. Recall that the *stable module category* $\underline{\mathsf{mod}}\, kG$ of an group algebra kG is the category whose objects are the same as $\mathsf{mod}\, kG$ and the set of morphisms from M to N is given by

$$\underline{\mathsf{Hom}}_{kG}(M, N) := \mathrm{Hom}_{kG}(M, N)/\mathsf{P}_{kG}(M, N),$$

where $\mathsf{P}_{kG}(M, N)$ consists of all kG-linear maps from M to N factoring through some free kG-module. Then the category $\underline{\mathsf{mod}}\, kG$ has the structure of a triangulated category; see [Hap].

We denote by

$$\mathsf{H}^*(G; k) = \begin{cases} \oplus_{i\in\mathbb{Z}} \mathsf{H}^i(G; k) & p = 2 \\ \oplus_{i\in 2\mathbb{Z}} \mathsf{H}^i(G; k) & p : \text{odd} \end{cases}$$

the direct sum of cohomologies of G with coefficient k. Then $\mathsf{H}^*(G; k)$ has the structure of a graded-commutative Noetherian ring by using the cup product and we call it the *cohomology ring* of G with coefficient k. Thus, we can consider its homogeneous prime spectrum $\mathsf{Proj}\, \mathsf{H}^*(G; k)$ of $\mathsf{H}^*(G; k)$. Denote by $V_G(M)$ the *support variety* for a finitely generated kG-module M which is a closed space of $\mathsf{Proj}\, \mathsf{H}^*(G; k)$. Then the pair $(\mathsf{Proj}\, \mathsf{H}^*(G; k), V_G)$ becomes a support data for $\underline{\mathsf{mod}}\, kG$. For details, please refer to [Ben, Chapter 5].

(3) Let R be a commutative Noetherian ring. For $M \in \mathsf{D}_{\mathsf{sg}}(R)$, we define the *singular support* of M by

$$\mathsf{SSupp}_R(M) := \{\mathfrak{p} \in \mathsf{Sing}\, R \mid M_{\mathfrak{p}} \ncong 0 \text{ in } \mathsf{D}_{\mathsf{sg}}(R_{\mathfrak{p}})\}.$$

Then $(\mathsf{Sing}\, R, \mathsf{SSupp}_R)$ is a support data for $\mathsf{D}_{\mathsf{sg}}(R)$. Indeed, it follows from [AIL, Theorem 1.1] and [BM, Lemma 4.5] that $\mathsf{SSupp}_R(M)$ is a closed subset of $\mathsf{Sing}\, R$. The remained conditions (1)-(4) are clear because the localization functor $\mathsf{D}_{\mathsf{sg}}(R) \to \mathsf{D}_{\mathsf{sg}}(R_{\mathfrak{p}})$ is exact.

Assume that R is Gorenstein. Denote by $\mathsf{CM}(R)$ the category of maximal Cohen-Macaulay R-modules (i.e., modules M satisfying $\mathsf{Ext}^i_R(M, R) = 0$ for all integers $i > 0$). As in the case of a group algebra, we can define the stable category $\underline{\mathsf{CM}}(R)$ and it has the structure of a triangulated category. Moreover, the natural inclusion induces a triangle equivalence $F : \underline{\mathsf{CM}}(R) \xrightarrow{\cong} \mathsf{D}_{\mathsf{sg}}(R)$ by [Buc]. Thus we obtain the support data $(\mathsf{Sing}\, R, \underline{\mathsf{Supp}}_R)$ for $\underline{\mathsf{CM}}(R)$ by using this equivalence. Here,

$$\underline{\mathsf{Supp}}_R(M) := \mathsf{SSupp}_R(F(M)) = \{\mathfrak{p} \in \mathsf{Sing}\, R \mid M_{\mathfrak{p}} \ncong 0 \text{ in } \underline{\mathsf{CM}}(R_p)\}$$

for $M \in \underline{\mathsf{CM}}(R)$.

Remark 1.5. Actually, the above examples of support data satisfy the following stronger condition:

$(1')$ $\sigma(M) = \emptyset$ if and only if $M \cong 0$.

Let (X, σ) be a support data for $\mathcal{T}$, $\mathcal{X}$ a thick subcategory of $\mathcal{T}$, and W a specialization-closed subset of X. Then one can easily check that $f_\sigma(\mathcal{X}) := \sigma(\mathcal{X}) := \bigcup_{M \in \mathcal{X}} \sigma(M)$ is a specialization-closed subset of X and $g_\sigma(W) := \sigma^{-1}(W) := \{M \in \mathcal{T} \mid \sigma(M) \subseteq W\}$ is a thick subcategory of $\mathcal{T}$. Therefore, we obtain two order-preserving maps

$$\mathbf{Th}(\mathcal{T}) \underset{g_\sigma}{\overset{f_\sigma}{\rightleftarrows}} \mathbf{Spcl}(X)$$

with respect to the inclusion relations.

2. **Tensor triangular geometry**

2.1. **Balmer spectra**

In this subsection, we give some basic terminologies from the theory of tensor triangulated categories. To begin with, let us recall the definition of a tensor triangulated category.

Definition 2.1. We say that $(\mathcal{T}, \otimes, \mathbf{1})$ is a *tensor triangulated category* if $\mathcal{T}$ is a triangulated category equipped with a symmetric monoidal structure which is compatible with the triangulated structure of $\mathcal{T}$; see [HPS, Appendix A] for the precise definition. Thus, $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a functor which is exact in each variables and $\mathbf{1}$ is an object of $\mathcal{T}$ which is a unit with respect to $\otimes$.

The followings are examples of tensor triangulated categories. The first two examples have been well examined in [Tho, BCR, BIK]. The last one will be discussed in Part 3 of this thesis.

Example 2.2. (1) Let X be a scheme. Then the perfect derived category $\mathsf{D}^{\mathsf{perf}}(X)$ is a tensor triangulated category with symmetric monoidal tensor product $\otimes^{\mathbf{L}}_{\mathcal{O}_X}$ and unit object $\mathcal{O}_X$.
(2) Let k be a field and G a finite group. Then the stable module category $\underline{\mathsf{mod}}\, kG$ is a tensor triangulated category with symmetric monoidal tensor product $\otimes_k$ and unit object k.
(3) Let R be a commutative Noetherian ring. Then the right bounded derived category $\mathsf{D}^{-}(\mathsf{mod}\, R)$ is a tensor triangulated category with symmetric monoidal tensor product $\otimes^{\mathbf{L}}_R$ and unit object R.

For the rest of this part, let us fix a tensor triangulated category $(\mathcal{T}, \otimes, \mathbf{1})$. Then one can define the notions of *thick tensor ideals* and *prime thick tensor ideals*, which behave similarly to ideals and prime ideals of a commutative ring.

Definition 2.3. (1) A full subcategory $\mathcal{X}$ of $\mathcal{T}$ is called a *thick tensor ideal* if it is a thick subcategory of $\mathcal{T}$ and closed under the action of $\mathcal{T}$ by $\otimes$, namely $M \otimes N \in \mathcal{X}$ for any $M \in \mathcal{X}$ and $N \in \mathcal{T}$. We often abbreviate "tensor ideal" to "$\otimes$-ideal". For a subcategory $\mathcal{X}$ of $\mathcal{T}$, denote by $\mathsf{thick}^{\otimes}_{\mathcal{T}} \mathcal{X}$ the smallest thick $\otimes$-ideal of $\mathcal{T}$ containing $\mathcal{X}$.
(2) A proper thick $\otimes$-ideal $\mathcal{P}$ of $\mathcal{T}$ is called *prime* if it satisfies

$$M \otimes N \in \mathcal{P} \Rightarrow M \in \mathcal{P} \text{ or } N \in \mathcal{P}.$$

Denote by $\mathtt{Spec}\, \mathcal{T}$ the set of all prime thick $\otimes$-ideals of $\mathcal{T}$.
(3) For a family $\mathcal{E}$ of objects of $\mathcal{T}$, we denote by $Z(\mathcal{E})$ the following subset of $\mathtt{Spec}\, \mathcal{T}$:

$$Z(\mathcal{E}) := \{\mathcal{P} \in \mathtt{Spec}\, \mathcal{T} \mid \mathcal{E} \cap \mathcal{P} = \emptyset\}.$$

Clearly, we have $\bigcap_{i\in\mathcal{I}} Z(\mathcal{E}_i) = Z(\bigcup_{i\in I}\mathcal{E}_i)$, $Z(\mathcal{E}_1)\cup Z(\mathcal{E}_2) = Z(\mathcal{E}_1\oplus\mathcal{E}_2)$, $Z(\mathcal{T}) = \emptyset$ and $Z(\emptyset) = \mathtt{Spec}\,\mathcal{T}$. Thus, we can define a topology on $\mathtt{Spec}\,\mathcal{T}$ with the family of closed subsets $\{Z(\mathcal{E}) \mid \mathcal{E}\subseteq\mathcal{T}\}$. We call this topological space $\mathsf{Spec}\,\mathcal{T}$ the *Balmer spectrum* of $\mathcal{T}$. Moreover, denote by $\mathtt{U}(\mathcal{E})$ the complement of $Z(\mathcal{E})$ which is an open subset of the Balmer spectrum.

(4) For $M\in\mathcal{T}$, the *Balmer support* of M is defined as a closed subset

$$\mathsf{BSupp}\,M := Z(\{M\}) = \{\mathcal{P}\in\mathtt{Spec}\,\mathcal{T} \mid M\notin\mathcal{P}\}.$$

Remark 2.4. Note that a family $\{\mathsf{BSupp}\,M \mid M\in\mathcal{T}\}$ of Balmer supports forms a closed basis of $\mathsf{Spec}\,\mathcal{T}$. Therefore, a family $\{\mathtt{U}(M) \mid M\in\mathcal{T}\}$ forms an open basis of $\mathsf{Spec}\,\mathcal{T}$.

Balmer supports define a support data for $\mathcal{T}$ with an additional condition which reflects tensor structure. We call such a support data tensorial:

Definition 2.5. We say that a support data (X,σ) for $\mathcal{T}$ is *tensorial* if it satisfies

$$\sigma(M\otimes N) = \sigma(M)\cap\sigma(N)$$

for any $M, N\in\mathcal{T}$.

Note that tensorial support data are called simply support data in [Bal05].

Example 2.6. Support data given in Example 1.4 (1), (2) are tensorial.

Lemma 2.7. [Bal05, Lemma 2.6] *The pair* $(\mathtt{Spec}\,\mathcal{T}, \mathsf{BSupp})$ *is a tensorial support data for* $\mathcal{T}$*, namely it satisfies the following conditions:*

(1) $\mathsf{BSupp}(0) = \emptyset$.
(2) $\mathsf{BSupp}(M[n]) = \mathsf{BSupp}(M)$ *for any* $M\in\mathcal{T}$ *and* $n\in\mathbb{Z}$.
(3) $\mathsf{BSupp}(M\oplus N) = \mathsf{BSupp}(M)\cup\mathsf{BSupp}(N)$ *for any* $M, N\in\mathcal{T}$.
(4) $\mathsf{BSupp}(M)\subseteq\mathsf{BSupp}(L)\cup\mathsf{BSupp}(N)$ *for any triangle* $L\to M\to N\to L[1]$ *in* $\mathcal{T}$.
(5) $\mathsf{BSupp}(M\otimes N) = \mathsf{BSupp}(M)\cap\mathsf{BSupp}(N)$ *for any* $M, N\in\mathcal{T}$.

Recall that a tensor triangulated category $\mathcal{T}$ is *rigid* if

(1) the functor $M\otimes - : \mathcal{T}\to\mathcal{T}$ has a right adjoint $F(M,-) : \mathcal{T}\to\mathcal{T}$ for each $M\in\mathcal{T}$ and
(2) every object M is *strongly dualizable* (i.e., the natural map $F(M,\mathbf{1})\otimes N \to F(M,N)$ is an isomorphism for each N).

If $\mathcal{T}$ is rigid, then $(\mathtt{Spec}\,\mathcal{T}, \mathsf{BSupp})$ satisfies the stronger condition.

Lemma 2.8. *Assume that* $\mathcal{T}$ *is rigid. Then the support data* $(\mathtt{Spec}\,\mathcal{T}, \mathsf{BSupp})$ *satisfies the condition* $(1')$ *in Remark 1.5.*

Proof. Take an object $M\in\mathcal{T}$ with $\mathsf{BSupp}(M) = \emptyset$. By [Bal05, Corollary 2.4], there is a positive integer n such that $M^{\otimes n}\cong 0$. On the other hand, by [HPS, Lemma A 2.6], M^i belongs to $\mathsf{thick}^{\otimes}_{\mathcal{T}}(M^{2i})$ for any positive integer since every object is strongly dualizable. Therefore, by using induction, we conclude that $M\cong 0$. ∎

The following propositions are analogues of the well known result in commutative ring theory.

Proposition 2.9. [Bal05, Proposition 2.3]

(1) *For any proper thick* $\otimes$*-ideal* $\mathcal{I}$ *of* $\mathcal{T}$*, there is a prime ideal* $\mathcal{P}$ *of* $\mathcal{T}$ *containing* $\mathcal{I}$.
(2) *Every maximal proper thick* $\otimes$*-ideal is prime.*
(3) *The Balmer spectrum* $\mathtt{Spec}\,\mathcal{T}$ *is not empty provided* $\mathcal{T}\not\cong\mathbf{0}$.

Proposition 2.10. [Bal05, Proposition 2.9, Proposition 2.18]

(1) *For any* $\mathcal{P} \in \mathtt{Spec}\,\mathcal{T}$*, one has the equality*

$$\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in \mathtt{Spec}\,\mathcal{T} \mid \mathcal{Q} \subseteq \mathcal{P}\}.$$

(2) *For any closed subset* Z *of* $\mathtt{Spec}\,\mathcal{T}$*,* Z *is irreducible if and only if* $Z = \overline{\{\mathcal{P}\}}$ *holds for some* $\mathcal{P} \in \mathtt{Spec}\,\mathcal{T}$*.*

In particular, $\mathtt{Spec}\,\mathcal{T}$ *is a sober space.*

Proposition 2.11. [Bal05, Proposition 2.14] *Let* U *be an open subset of* $\mathtt{Spec}\,\mathcal{T}$*. Then* U *is quasi-compact if and only if* $U = \mathtt{U}(M)$ *for some* $M \in \mathcal{T}$*. In particular,* $\mathtt{Spec}\,\mathcal{T}$ *is quasi-compact as* $\mathtt{Spec}\,\mathcal{T} = \mathtt{U}(0)$*.*

Remark 2.12. A topological space X is said to be a *spectral space* if it is sober, quasi-compact and quasi-compact open subsets form an open basis of X and closed under taking finite intersections. The previous two propositions show that the Balmer spectrum is spectral for any tensor triangulated category.

Proposition 2.13. *For a tensor triangulated functor* $F : \mathcal{T} \to \mathcal{T}'$*, the map*

$${}^{a}F := \mathtt{Spec}\,F : \mathtt{Spec}\,\mathcal{T}' \to \mathtt{Spec}\,\mathcal{T},\ \mathcal{P} \to F^{-1}(\mathcal{P})$$

is continuous.

2.2. **The classification theorem of Balmer**

Since $(\mathtt{Spec}\,\mathcal{T}, \mathsf{BSupp})$ is a support data by Lemma 2.7, there are two maps

$$\mathbf{Th}(\mathcal{T}) \underset{g_{\mathsf{BSupp}}}{\overset{f_{\mathsf{BSupp}}}{\rightleftarrows}} \mathbf{Spcl}(\mathtt{Spec}\,\mathcal{T}).$$

Of course, images of g_{BSupp} is not just thick subcategories but also have more special condition because $(\mathtt{Spec}\,\mathcal{T}, \mathsf{BSupp})$ is tensorial.

Definition 2.14. For a thick $\otimes$-ideal $\mathcal{X}$ of $\mathcal{T}$, define its *radical* by

$$\sqrt{\mathcal{X}} := \{M \in \mathcal{T} \mid M^{\otimes n} \in \mathcal{X} \text{ for some integer } n\},$$

where $M^{\otimes n}$ is an n-fold tensor product. As the following lemma says, the radical of a thick $\otimes$-ideal is also a thick $\otimes$-ideal.

We say that a thick $\otimes$-ideal $\mathcal{X}$ of $\mathcal{T}$ is *radical* if $\sqrt{\mathcal{X}} = \mathcal{X}$. Denote by $\mathbf{Rad}(\mathcal{T})$ the set of radical thick $\otimes$-ideals of $\mathcal{T}$.

Lemma 2.15. [Bal05, Lemma 4.2] *Let* $\mathcal{X}$ *be a thick* $\otimes$*-ideal. Then one has*

$$\sqrt{\mathcal{X}} = \bigcap_{\mathcal{X} \subseteq \mathcal{P}} \mathcal{P}$$

We can easily check that images of g_{BSupp} are radical thick $\otimes$-ideals. Thus, g_{BSupp} takes values in $\mathbf{Rad}(\mathcal{T})$. On the other hand, the image of f_{BSupp} also becomes smaller than $\mathbf{Spcl}(\mathtt{Spec}\,\mathcal{T})$. It takes values in Thomson subsets.

Definition 2.16. For a topological space X, a subset W is called a *Thomason subset* if it is the union of closed subsets whose complements are quasi-compact. Denote by $\mathbf{Thom}(X)$ the set of Thomason subsets of X. Note that $\mathbf{Thom}(X) \subseteq \mathbf{Spcl}(X)$ holds and if X is Noetherian, it becomes an equality.

Balmer proved the following celebrated result. The first statement gives a classification of radical thick $\otimes$-ideals via Thomason subsets of the Balmer spectrum and the second one says that the Balmer spectrum and the Balmer support is uniquely determined by such a property.

Theorem 2.17. [Bal05, Theorem 4.10, Theorem 5.2]

(1) *There is a one-to-one correspondence:*

$$\mathbf{Rad}(\mathcal{T}) \underset{g_{\mathsf{BSupp}}}{\overset{f_{\mathsf{BSupp}}}{\rightleftarrows}} \mathbf{Thom}(\mathsf{Spec}\,\mathcal{T}).$$

(2) *Let (X, σ) be a tensorial support data for $\mathcal{T}$ satisfying:*
   (a) *X is a Noetherian sober space.*
   (b) *There is a one-to-one correspondence:*

$$\mathbf{Rad}(\mathcal{T}) \underset{g_{\sigma}}{\overset{f_{\sigma}}{\rightleftarrows}} \mathbf{Spcl}(X).$$

Then the map

$$\varphi : X \to \mathsf{Spec}\,\mathcal{T},\ x \mapsto \{M \in \mathcal{T} \mid x \notin \sigma(M)\}.$$

is a homeomorphism. Moreover, this map satisfies $\varphi(\sigma(M)) = \mathsf{BSupp}(M)$ for any $M \in \mathcal{T}$.

The conditions (a) and (b) in the above theorem satisfied in the following cases for instance.

Theorem 2.18. [Tho, Theorem 3.15] *Let X be a Noetherian scheme. Then the support data (X, Supp_X) for $\mathsf{D}^{\mathsf{perf}}(X)$ satisfies the conditions (a), (b) in Theorem 2.17(2).*

Theorem 2.19. [BCR, BIK] *Let k be a field of characteristic $p > 0$ and G a finite group such that p divides the order of G. Then the support data $(\mathsf{Proj}\,\mathsf{H}^*(G; k), V_G)$ for $\underline{\mathsf{mod}}\,kG$ satisfies the conditions (a), (b) in Theorem 2.17(2).*

By using the above theorems, Balmer determined the Balmer spectra for tensor triangulated categories $\mathsf{D}^{\mathsf{perf}}(X)$ and $\underline{\mathsf{mod}}\,kG$.

Corollary 2.20. [Bal05, Theorem 6.3]

(1) *Let X be a Noetherian scheme. Then $\mathsf{Spec}\,\mathsf{D}^{\mathsf{perf}}(X)$ and X are homeomorphic.*
(2) *Let k be a field of characteristic p and G a finite group such that p divides the order of G. Then $\mathsf{Spec}\,\underline{\mathsf{mod}}\,kG$ and $\mathsf{Proj}\,\mathsf{H}^*(G; k)$ are homeomorphic.*

Remark 2.21. In [Bal05], Balmer introduced a locally ringed space structure on $\mathsf{Spec}\,\mathcal{T}$ and proved that the above homeomorphisms are actually isomorphisms of schemes, see [Bal05, Theorem 6.3].

2.3. **Balmer's conjecture**

By Theorem 2.17, we can determine the Balmer spectrum of $\mathcal{T}$ using a classification of radical thick $\otimes$-ideals of $\mathcal{T}$. However, classifying radical thick $\otimes$-ideals is hard problem and hence the structure of the Balmer spectrum is as hard as that. Balmer defined a continuous map from the Balmer spectrum to the Zariski spectrum of some graded-commutative ring to analyze the structure of the Balmer spectrum without classification.

The $\mathbb{Z}$-graded abelian group

$$\mathrm{R}^{\bullet}_{\mathcal{T}} := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(\mathbf{1}, \mathbf{1}[n])$$

becomes an associative graded ring via the multiplication:

$$g \cdot f := g[m] \circ f$$

for $f \in \mathrm{R}^{m}_{\mathcal{T}}$ and $g \in \mathrm{R}^{n}_{\mathcal{T}}$. Actually, this multiplication is graded-commutative.

Proposition 2.22. [Bal10a, Proposition 3.3] $\mathrm{R}^{\bullet}_{\mathcal{T}}$ *is a graded-commutative ring (i.e., $g\cdot f = (-1)^{|g||f|} f \cdot g$). In particular,* $\mathrm{R}^{0}_{\mathcal{T}}$ *is a commutative ring.*

Definition 2.23. We call the graded-commutative ring $\mathrm{R}^{\bullet}_{\mathcal{T}}$ the *graded central ring* of $\mathcal{T}$.

Since $\mathrm{R}^{\bullet}_{\mathcal{T}}$ is graded-commutative, we can define homogeneous prime ideals of $\mathrm{R}^{\bullet}_{\mathcal{T}}$ and Zariski topology on the set $\mathsf{Spec}^{\mathsf{h}}\, R^{\bullet}_{\mathcal{T}}$ of homogeneous prime ideals of $\mathrm{R}^{\bullet}_{\mathcal{T}}$. Balmer defined the following map.

Definition 2.24. [Bal10a, Definition 5.1] Let $\mathcal{P}$ be a prime thick $\otimes$-ideal of $\mathcal{T}$. Define $\rho^{\bullet}_{\mathcal{T}}(\mathcal{P})$ to be the homogeneous ideal of $\mathrm{R}^{\bullet}_{\mathcal{T}}$ generated by homogeneous elements f with $\mathsf{cone}(f) \notin \mathcal{P}$:

$$\rho^{\bullet}_{\mathcal{T}}(\mathcal{P}) := (f \in \mathrm{R}^{\mathsf{hom}}_{\mathcal{T}} \mid \mathsf{cone}(f) \notin \mathcal{P}).$$

Similarly, we define the ideal $\rho^{0}_{\mathcal{T}}(\mathcal{P}) = (f \in \mathrm{R}^{0}_{\mathcal{T}} \mid \mathsf{cone}(f) \notin \mathcal{P})$ of $\mathrm{R}^{0}_{\mathcal{T}}$.

Theorem 2.25. [Bal10a, Theorem 5.3, Corollary 5.6]

(1) $\rho^{\bullet}_{\mathcal{T}}(\mathcal{P})$ *is a homogeneous prime ideal of* $\mathrm{R}^{\bullet}_{\mathcal{T}}$.
(2) $\rho^{0}_{\mathcal{T}}(\mathcal{P})$ *is a prime ideal of* $\mathrm{R}^{0}_{\mathcal{T}}$.
(3) *The maps* $\rho^{\bullet}_{\mathcal{T}} : \mathsf{Spec}\,\mathcal{T} \to \mathsf{Spec}^{\mathsf{h}}\,\mathrm{R}^{\bullet}_{\mathcal{T}}$ *and* $\rho^{0}_{\mathcal{T}} : \mathsf{Spec}\,\mathcal{T} \to \mathsf{Spec}\,\mathrm{R}^{0}_{\mathcal{T}}$ *are continuous.*

These continuous maps $\rho^{\bullet}_{\mathcal{T}}$ and $\rho^{0}_{\mathcal{T}}$ frequently become surjective:

Proposition 2.26. [Bal10a, Theorem 7.3, Theorem 7.13]

(1) *The maps* $\rho^{\bullet}_{\mathcal{T}}$ *and* $\rho^{0}_{\mathcal{T}}$ *are surjective if* $R^{\bullet}_{\mathcal{T}}$ *is Noetherian.*
(2) *The map* $\rho^{0}_{\mathcal{T}}$ *is surjective if* $\mathcal{T}$ *is connective (i.e.,* $\mathrm{R}^{>0}_{\mathcal{T}} = \mathrm{Hom}_{\mathcal{T}}(\mathbf{1}, \mathbf{1}[>0]) = 0$*).*

On the other hand, injectivity of $\rho^{\bullet}_{\mathcal{T}}$ or $\rho^{0}_{\mathcal{T}}$ the comparison map is hard to observe and there are only affirmative answers for each individual tensor triangulated category:

Theorem 2.27. [Bal10a, Propositions 8.1, 8.5 and Corollary 9.5]

(1) *Let R be a commutative Noetherian ring. Then* $\rho^{\bullet}_{\mathcal{T}}$ *and* $\rho^{0}_{\mathcal{T}}$ *are equal and homoemorphisms for* $\mathcal{T} = \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,R)$.
(2) *Let k be a field and G a finite group. Then* $\rho^{\bullet}_{\mathcal{T}}$ *is injective for* $\mathcal{T} = \underline{\mathsf{mod}}\,kG$.
(3) $\rho^{0}_{\mathcal{T}}$ *is not injective for* $\mathcal{T} = \mathsf{SH}^{\mathsf{fin}}$ *the stable homotopy category of finite spectra.*

We say that a triangulated category is algebraic if it is a stable category of a Frobenius exact category. From the above observation, Balmer conjectured:

Conjecture 2.28. [Bal10b, Conjecture 72] Let $\mathcal{T}$ be a tensor triangulated category which is "algebraic enough". Then

$$\rho^{\bullet}_{\mathcal{T}} : \mathsf{Spec}\,\mathcal{T} \to \mathsf{Spec}^{\mathsf{h}}\,\mathrm{R}^{\bullet}_{\mathcal{T}}$$

is locally injective.

Here, "algebraic enough" tensor triangulated categories could mean algebraic ones, or derived categories of dg-categories, or ones locally generated by the unit. Recall that a continuous map $f : X \to Y$ of topological spaces is called *locally injective at* $x \in X$ if there exists a neighborhood N of x such that the restriction $f|_N : N \to Y$ is an injective map. We say that f is *locally injective* if it is locally injective at every point in X.

For the lest of this part, we prove that Balmer's conjecture holds for $\mathsf{D}^{\mathsf{perf}}(X)$ for a Noetherian quasi-affine scheme X. First of all, we give a geometric criterion for the perfect derived category of a Noetherian scheme to satisfy Balmer's conjecture.

Lemma 2.29. *Let X be a Noetherian scheme. Then the following are equivalent:*

(1) $\rho^{\bullet}_{\mathsf{D}^{\mathsf{perf}}(X)}$ *is locally injective.*

(2) $\rho^{0}_{\mathsf{D}^{\mathsf{perf}}(X)}$ *is locally injective.*

(3) *For any $x \in X$, there is an affine open subset U such that the natural map $\mathsf{Spec}\,\Gamma(U, \mathcal{O}_X) \to \mathsf{Spec}\,\Gamma(X, \mathcal{O}_X)$ is injective.*

Proof. By Theorem 2.17(2) and Theorem 2.18, there is a homeomorphism

$$\varphi : X \to \mathsf{Spec}\,\mathsf{D}^{\mathsf{perf}}(X)$$

such that $\varphi(x) := \{\mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(X) \mid \mathcal{F}_x \cong 0\}$.

Take $f \in \mathrm{R}^n_{\mathsf{D}^{\mathsf{perf}}(X)} := \mathrm{Hom}_{\mathsf{D}^{\mathsf{perf}}(X)}(\mathcal{O}_X, \mathcal{O}_X[n])$ and embed it into a triangle:

$$\mathcal{O}_X \xrightarrow{f} \mathcal{O}_X[n] \to \mathsf{cone}(f) \to \mathcal{O}_X[1].$$

Localize at x, we obtain a triangle

$$\mathcal{O}_{X,x} \xrightarrow{f_x} \mathcal{O}_{X,x}[n] \to \mathsf{cone}(f)_x \to \mathcal{O}_{X,x}[1]$$

in $\mathsf{D}^{\mathsf{perf}}(\mathcal{O}_{X,x})$. If $n \neq 0$, then $\mathsf{cone}(f)_x \gtrdot \mathcal{O}_{X,x}[n] \neq 0$ as $f_x \in \mathsf{Ext}^n_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{X,x}) = 0$. Thus, one has $\mathsf{cone}(f)_x \neq 0$. If $n = 0$, then $\mathsf{cone}(f)_x = 0$ if and only if f_x is an isomorphism. Therefore, $\rho^{\bullet}_{\mathsf{D}^{\mathsf{perf}}(X)}(\varphi(x))$ is a homogeneous ideal of $\mathrm{R}^{\bullet}_{\mathsf{D}^{\mathsf{perf}}(X)}$ generated by $f \in \mathrm{R}^n_{\mathsf{D}^{\mathsf{perf}}(X)}$ for $n > 0$ or f_x is invertible for $n = 0$ and hence

$$\rho^{\bullet}_{\mathsf{D}^{\mathsf{perf}}(X)}(\varphi(x)) = \mathrm{R}^{<0}_{\mathsf{D}^{\mathsf{perf}}(X)} \oplus \rho^{0}_{\mathsf{D}^{\mathsf{perf}}(X)}(\varphi(x)) \oplus \mathrm{R}^{>0}_{\mathsf{D}^{\mathsf{perf}}(X)},$$

where $\rho^{0}_{\mathsf{D}^{\mathsf{perf}}(X)}(\varphi(x)) = \{f \in \mathrm{R}^0_{\mathcal{T}} \mid f_x \text{ is not an isomorphism}\}$. Hence, $\rho^{\bullet}_{\mathsf{D}^{\mathsf{perf}}(X)}$ locally injective if and only if $\rho^{0}_{\mathsf{D}^{\mathsf{perf}}(X)}$ is locally injective.

For an affine open neighborhood U of X, let $r : \mathsf{D}^{\mathsf{perf}}(X) \to \mathsf{D}^{\mathsf{perf}}(U)$ be a restriction functor. Then r is a tensor triangle functor and hence it induces a continuous map

$${}^a r : \mathsf{Spec}\,\mathsf{D}^{\mathsf{perf}}(U) \to \mathsf{Spec}\,\mathsf{D}^{\mathsf{perf}}(X),\ \mathcal{P} \mapsto r^{-1}(\mathcal{P}).$$

by Proposition 2.13. One can easily check that there is a commutative diagram

$$\begin{array}{ccc}
U & \xrightarrow{\ i\ } & X \\
\cong\downarrow & & \downarrow\cong \\
\mathsf{Spec}\,\mathsf{D}^{\mathsf{perf}}(U) & \xrightarrow{\ {}^a r\ } & \mathsf{Spec}\,\mathsf{D}^{\mathsf{perf}}(X) \\
\rho^0_{\mathsf{D}^{\mathsf{perf}}(U)}\cong\downarrow & & \downarrow\rho^0_{\mathsf{D}^{\mathsf{perf}}(X)} \\
\mathsf{Spec}\,\Gamma(U, \mathcal{O}_X) & \longrightarrow & \mathsf{Spec}\,\Gamma(X, \mathcal{O}_X).
\end{array}$$

Here, the bottom map is a continuous map associated to the restriction map $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$. Thus, $\rho^0_{\mathsf{D}^{\mathsf{perf}}(X)} \circ {}^a r$ is injective if and only if $\mathsf{Spec}\,\Gamma(U, \mathcal{O}_X) \to \mathsf{Spec}\,\Gamma(X, \mathcal{O}_X)$ is injective. Hence $\rho^0_{\mathsf{D}^{\mathsf{perf}}(X)}$ is locally injective if and only if the condition (3) is satisfied. ■

Remark 2.30. The comparison map $\rho^{\bullet}_{\mathsf{D}^{\mathsf{perf}}(X)}$ may not be locally injective in general.

Let $n > 1$ be an integer. For the projective space $\mathbb{P}^n_k$ over a field k, one has $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}) = k$. Then the map $\mathbb{P}^n_k \cong \mathsf{Spec}\,\mathsf{D}^{\mathsf{perf}}(\mathbb{P}^n_k) \xrightarrow{\rho^0_{\mathsf{D}^{\mathsf{perf}}(\mathbb{P}^n_k)}} \mathsf{Spec}\,k$ is not locally injective. Therefore, by Lemma 2.29, $\rho^{\bullet}_{\mathsf{D}^{\mathsf{perf}}(\mathbb{P}^n_k)}$ is not locally injective.

Let X be a scheme and $s \in \Gamma(X, \mathcal{O}_X)$. Note that a subset $X_s := \{x \in X \mid s_x = 0 \text{ in } k(x)\}$ is an open subset of X and $s\,|_{X_s}$ is an invertible in $\Gamma(X_s, \mathcal{O}_X)$. Therefore, the restriction map $\Gamma(X, \mathcal{O}_X) \to \Gamma(X_s, \mathcal{O}_X)$ induces a ring map

$$\alpha_s : \Gamma(X, \mathcal{O}_X)_s \to \Gamma(X_s, \mathcal{O}_X).$$

The following theorem is the main result in this part.

Theorem 2.31. *If X is a Noetherian quasi-affine scheme, then $\rho^{\bullet}_{\mathsf{D}^{\mathsf{perf}}(X)}$ is locally injective.*

Proof. By [Iit, Corollary to Theorem 1.15], the map $\alpha_s : \Gamma(X, \mathcal{O}_X)_s \to \Gamma(X_s, \mathcal{O}_X)$ is an isomorphism for any $s \in \Gamma(X, \mathcal{O}_X)$. Consider a commutative diagram:

$$\begin{array}{ccc} \mathsf{Spec}\,\Gamma(X_s, \mathcal{O}_X) & \xrightarrow[\cong]{{}^a\alpha_s} & \mathsf{Spec}\,\Gamma(X, \mathcal{O}_X)_s \\ & \searrow & \downarrow \\ & & \mathsf{Spec}\,\Gamma(X, \mathcal{O}_X). \end{array}$$

Here, the vertical map is injective since it is induced by a localization map. Thus, the natural map $\mathsf{Spec}\,\Gamma(X_s, \mathcal{O}_X) \to \mathsf{Spec}\,\Gamma(X, \mathcal{O}_X)$ induced by restriction is injective.

Since X is quasi-affine, the structure sheaf $\mathcal{O}_X$ is ample, and hence there is a finitely many $s_1, \ldots, s_n \in \Gamma(X, \mathcal{O}_X)$ such that X_{s_i} is affine and $X = \cup_{i=1}^n X_{s_i}$. Consequently, X satisfies the condition (3) in Lemma 2.29. ■

Remark 2.32. If X is quasi-affine scheme, then one has $\mathsf{D}^{\mathsf{perf}}(X) = \mathsf{thick}\,\mathcal{O}_X$. Therefore, this theorem gives an evidence that "algebraic enough" tensor triangulated categories need to be algebraic ones which are locally generated by the unit.

Part 2. Thick tensor ideals of right bounded derived categories

3. Introduction

The contents of this part is based on the author's paper [MT] with R. Takahashi and author's solo work [Mat17b].

Let R be a commutative Noetherian ring. Denote by $\mathsf{D}^{-}(R)$ the right bounded derived category of finitely generated R-modules, namely, the derived category of (cochain) complexes X of finitely generated R-modules such that $\mathsf{H}^i(X) = 0$ for all $i \gg 0$. Then $(\mathsf{D}^{-}(R), \otimes^{\mathbf{L}}_R, R)$ is a tensor triangulated category. The main purpose of this part is to investigate thick tensor ideals of the tensor triangulated category $\mathsf{D}^{-}(R)$ and analyzing the structure of the Balmer spectrum $\mathsf{Spec}\,\mathsf{D}^{-}(R)$ of $\mathsf{D}^{-}(R)$.

From now on, let us explain the main results of this part. First of all, recall that an object M of a triangulated category $\mathcal{T}$ is *compact* (resp. *cocompact*) if the natural morphism

$$\textstyle\bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_{\mathcal{T}}(M, N_\lambda) \to \mathrm{Hom}_{\mathcal{T}}(M, \bigoplus_{\lambda\in\Lambda} N_\lambda)$$
$$\big(\text{resp. } \textstyle\bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_{\mathcal{T}}(N_\lambda, M) \to \mathrm{Hom}_{\mathcal{T}}(\prod_{\lambda\in\Lambda} N_\lambda, M)\big)$$

is an isomorphism for every family $\{N_\lambda\}_{\lambda\in\Lambda}$ of objects of $\mathcal{T}$ with $\bigoplus_{\lambda\in\Lambda} N_\lambda \in \mathcal{T}$ (resp. $\prod_{\lambda\in\Lambda} N_\lambda \in \mathcal{T}$). A thick tensor ideal of $\mathsf{D}^{-}(R)$ is called *compactly generated* (resp. *cocompactly generated*) if it is generated by compact (resp. cocompact) objects of $\mathsf{D}^{-}(R)$ as a thick tensor ideal. For a subcategory $\mathcal{X}$ of $\mathsf{D}^{-}(R)$ we denote by $\mathsf{Supp}\,\mathcal{X}$ the union of the supports of complexes in $\mathcal{X}$, and for a subset S of $\mathsf{Spec}\,R$ we denote by $\langle S\rangle$ the thick tensor ideal of $\mathsf{D}^{-}(R)$ generated by $R/\mathfrak{p}$ with $\mathfrak{p} \in S$. We shall prove the following theorem.

Theorem 3.1 (Proposition 5.1, Theorem 5.12 and Corollary 5.16)**.** (1) *A thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$ is compactly generated if and only if it is cocompactly generated.*
(2) *The assignments $\mathcal{X} \mapsto \mathsf{Supp}\,\mathcal{X}$ and $\langle W\rangle \leftarrow\!\shortmid W$ make mutually inverse bijections*

$$\left\{\begin{matrix}\textit{cocompactly generated}\\ \textit{thick } \otimes\textit{-ideals of } \mathsf{D}^{-}(R)\end{matrix}\right\} \rightleftarrows \{\textit{specialization closed subsets of } \mathsf{Spec}\,R\}.$$

The core of this theorem is constituted by the classification of the *cocompactly* generated thick tensor ideals of $\mathsf{D}^{-}(R)$, which is obtained by establishment of a *generalized smash nilpotence theorem*, extending the classical smash nilpotence theorem due to Hopkins [Hop] and Neeman [Nee92] for the homotopy category of perfect complexes. In view of Theorem 3.1, we may simply call $\mathcal{X}$ *compact* if $\mathcal{X}$ is compactly generated and/or cocompactly generated. We should remark that in general we have

$$\langle W\rangle \neq \mathsf{Supp}^{-1} W,$$

where $\mathsf{Supp}^{-1} W$ consists of the complexes whose supports are contained in W. Thus we call a thick tensor ideal of $\mathsf{D}^{-}(R)$ *tame* if it has the form $\mathsf{Supp}^{-1} W$ for some specialization-closed subset W of $\mathsf{Spec}\,R$.

Next, we relate the Balmer spectrum $\mathsf{Spec}\,\mathsf{D}^{-}(R)$ of $\mathsf{D}^{-}(R)$ to the Zariski spectrum $\mathsf{Spec}\,R$ of R, i.e., the set of prime ideals of R. More precisely, we introduce a pair of order-reversing maps

$$\mathcal{S}: \ \mathsf{Spec}\,R \rightleftarrows \mathsf{Spec}\,\mathsf{D}^{-}(R) \ : \mathfrak{s}$$

and investigate their topological properties. These maps are defined as follows: let $\mathfrak{p} \in \mathsf{Spec}\, R$ and $\mathcal{P} \in \mathtt{Spec}\, \mathsf{D}^-(R)$. Then $\mathcal{S}(\mathfrak{p})$ consists of the complexes $X \in \mathsf{D}^-(R)$ with $X_\mathfrak{p} = 0$, and $\mathfrak{s}(\mathcal{P})$ is the unique maximal element of ideals I of R with $R/I \notin \mathcal{P}$ with respect to the inclusion relation. Our main result in this direction is the following theorem. Denote by ${}^{\mathsf{t}}\mathtt{Spec}\, \mathsf{D}^-(R)$ the set of tame prime thick tensor ideals of $\mathsf{D}^-(R)$, and by $\mathtt{Mx}\, \mathsf{D}^-(R)$ (resp. $\mathtt{Mn}\, \mathsf{D}^-(R)$) the maximal (resp. minimal) elements of $\mathtt{Spec}\, \mathsf{D}^-(R)$ with respect to the inclusion relation. For each full subcategory $\mathcal{X}$ of $\mathsf{D}^-(R)$, let $\mathcal{X}^{\mathsf{tame}}$ stand for the smallest tame thick tensor ideal of $\mathsf{D}^-(R)$ containing $\mathcal{X}$.

Theorem 3.2 (Theorems 6.8, 7.2, 7.4, 7.10, 7.12 and Corollary 6.15)**.** *The following statements hold.*

(1) *One has* $\mathfrak{s} \cdot \mathcal{S} = 1$ *and* $\mathcal{S} \cdot \mathfrak{s} = \mathsf{Supp}^{-1}\, \mathsf{Supp} = ()^{\mathsf{tame}}$. *In particular,* $\dim(\mathtt{Spec}\, \mathsf{D}^-(R)) \geqslant \dim R$.
(2) *The image of* $\mathcal{S}$ *coincides with* ${}^{\mathsf{t}}\mathtt{Spec}\, \mathsf{D}^-(R)$, *and it is dense in* $\mathtt{Spec}\, \mathsf{D}^-(R)$.
(3) *The map* $\mathfrak{s}$ *is continuous, and its restriction* $\mathfrak{s}' : {}^{\mathsf{t}}\mathtt{Spec}\, \mathsf{D}^-(R) \to \mathsf{Spec}\, R$ *is a continuous bijection.*
(4) *The map* $\mathcal{S}' : \mathsf{Spec}\, R \to {}^{\mathsf{t}}\mathtt{Spec}\, \mathsf{D}^-(R)$ *induced by* $\mathcal{S}$ *is an open and closed bijection.*
(5) *The map* $\operatorname{Min} R \to \mathtt{Mx}\, \mathsf{D}^-(R)$ *induced by* $\mathcal{S}$ *is a homeomorphism.*
(6) *The map* $\operatorname{Max} R \to \mathtt{Mn}\, \mathsf{D}^-(R)$ *induced by* $\mathcal{S}$ *is a homeomorphism if* R *is semilocal.*
(7) *One has:* $\mathcal{S}$ *is continuous* $\Leftrightarrow$ $\mathcal{S}'$ *is homeomorphic* $\Leftrightarrow$ $\mathfrak{s}'$ *is homeomorphic* $\Leftrightarrow$ $\mathsf{Spec}\, R$ *is finite.*

The celebrated classification theorem due to Balmer [Bal05] asserts that taking the Balmer support BSupp makes a one-to-one correspondence between the set **Rad** of radical thick tensor ideals of $\mathsf{D}^-(R)$ and the set **Thom** of Thomason subsets of $\mathtt{Spec}\, \mathsf{D}^-(R)$:

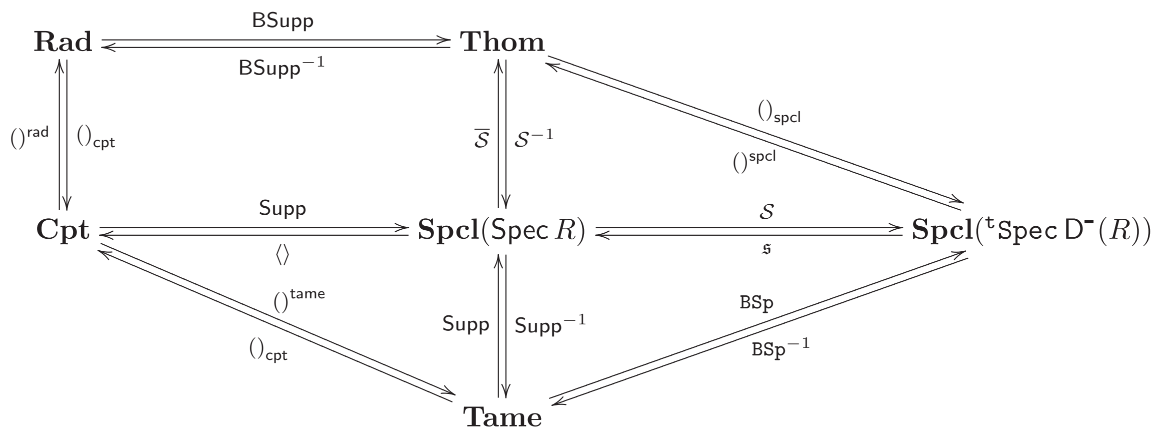
$$\mathsf{BSupp} : \ \mathbf{Rad} \rightleftarrows \mathbf{Thom} \ : \mathsf{BSupp}^{-1}$$

This bijection means that the study of topological structure of $\mathtt{Spec}\, \mathsf{D}^-(R)$ is directly linked to the study of radical thick tensor ideals of $\mathsf{D}^-(R)$. From this motivation, we will investigate some topological properties of $\mathtt{Spec}\, \mathsf{D}^-(R)$. For example, we investigate Noetherianity, connectedness and irreducibility of $\mathtt{Spec}\, \mathsf{D}^-(R)$.

Theorem 3.3 (Theorem 8.1 and Corollary 8.9)**.** (1) *If the Balmer spectrum* $\mathtt{Spec}\, \mathsf{D}^-(R)$ *is a Noetherian topological space, then the Zariski spectrum* $\mathsf{Spec}\, R$ *is a finite set.*
(2) *The Balmer spectrum* $\mathtt{Spec}\, \mathsf{D}^-(R)$ *is connected (resp. irreducible) if and only if the Zariski spectrum* $\mathsf{Spec}\, R$ *is so.*

Our next goal is to complete this one-to-one correspondence to the following commutative diagram, giving complete classifications of compact and tame thick tensor ideals of $\mathsf{D}^-(R)$. Denote by **Cpt** (resp. **Tame**) the set of compact (resp. tame) thick tensor ideals of $\mathsf{D}^-(R)$, and by **Spcl**($\mathsf{Spec}\, R$) (resp. **Spcl**(${}^{\mathsf{t}}\mathtt{Spec}\, \mathsf{D}^-(R)$)) the set of specialization-closed subsets of $\mathsf{Spec}\, R$ (resp. ${}^{\mathsf{t}}\mathtt{Spec}\, \mathsf{D}^-(R)$).

Theorem 3.4 (Theorems 9.13, 9.20)**.** *There is a diagram*



where the pairs of maps $A = (()^{\mathsf{rad}}, ()_{\mathsf{cpt}})$, $B = (\overline{\mathcal{S}}, \mathcal{S}^{-1})$, $C = (()^{\mathsf{spcl}}, ()_{\mathsf{spcl}})$ *are section-retraction pairs (as sets), and all the other pairs consist of mutually inverse bijections. The diagram with the sections (resp. retractions) and bijections is commutative.*

We do not give here the definitions of the maps appearing above (we do this in Section 9); what we want to emphasize now is that those maps are given explicitly.

Moreover, we prove that some/any of the three section-retraction pairs A, B, C in the above theorem are bijections if and only if R is Artinian, which is incorporated into the following theorem.

Theorem 3.5 (Theorem 10.5)**.** *The following are equivalent.*

(1) *R is Artinian.*
(2) *Every thick tensor ideal of* $\mathsf{D}^-(R)$ *is compact, tame and radical.*
(3) *Every radical thick tensor ideal of* $\mathsf{D}^-(R)$ *is tame.*
(4) *The pair of maps* $(\mathcal{S}, \mathfrak{s})$ *consists of mutually inverse homeomorphisms.*
(5) *Some/all of the maps* $\mathcal{S}, \mathfrak{s}$ *are bijective.*
(6) *Some/all of the pairs* A, B, C *consist of mutually inverse bijections.*

This theorem says that in the case of Artinian rings everything is clear. An essential role is played in the proof of this theorem by a certain complex in $\mathsf{D}^-(R)$ constructed from shifted Koszul complexes.

Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a tensor triangulated category. Balmer [Bal10a] constructs a continuous map

$$\rho^\bullet_{\mathcal{T}} : \ \mathsf{Spec}\,\mathcal{T} \longrightarrow \mathrm{Spec^h}\,\mathrm{R}^\bullet_{\mathcal{T}},$$

where $\mathrm{R}^\bullet_{\mathcal{T}} = \mathrm{Hom}_{\mathcal{T}}(\mathbf{1}, \Sigma^\bullet \mathbf{1})$ is a graded-commutative ring. Balmer [Bal10b] conjectures that the map $\rho^\bullet_{\mathcal{T}}$ is (locally) injective when $\mathcal{T}$ is "algebraic enough". Here, "algebraic enough" tensor triangulated categories could mean algebraic ones, or derived categories of dg-categories, or ones locally generated by the unit. Our $\mathsf{D}^-(R)$ is evidently an algebraic triangulated category, but does not satisfy this conjecture under a quite mild assumption:

Theorem 3.6 (Corollary 10.10)**.** *Assume that* $\dim R > 0$ *and that R is either a domain or a local ring. Then the map* $\rho^\bullet_{\mathsf{D}^-(R)}$ *is not locally injective.*

In fact, the assumption of the theorem gives an element $x \in R$ with $\mathrm{ht}(x) > 0$. Then we can find a non-tame prime thick tensor ideal $\mathcal{P}$ of $\mathsf{D}^-(R)$ associated with x at which $\rho^\bullet_{\mathsf{D}^-(R)}$

is not locally injective. In view of Balmer's conjecture, this theorem says that $\mathsf{D}^-(\mathsf{mod}\, R)$ is not "algebraic enough"; an algebraic tensor triangulated category is not sufficiently "algebraic enough".

Finally, we explore thick tensor ideals of $\mathsf{D}^-(R)$ in the case where R is a discrete valuation ring, because this should be the simplest unclear case, now that everything is clarified by Theorem 3.5 in the case of Artinian rings. We show the following theorem, which says that even if R is such a good ring, the structure of the Balmer spectrum of $\mathsf{D}^-(R)$ is rather complicated. (Here, $\ell\ell(-)$ stands for the Loewy length.)

Theorem 3.7 (Propositions 11.7, 11.17 and Theorems 11.11, 11.14)**.** *Let (R, xR) be a discrete valuation ring, and let $n \geqslant 0$ be an integer. Let $\mathcal{P}_n$ be the full subcategory of $\mathsf{D}^-(R)$ consisting of complexes X with finite length homologies such that there exists an integer $t \geqslant 0$ with $\ell\ell(\mathsf{H}^{-i}X) \leqslant ti^n$ for all $i \gg 0$. Then:*

(1) *$\mathcal{P}_n$ coincides with the smallest thick tensor ideal of $\mathsf{D}^-(R)$ containing the complex*

$$\textstyle\bigoplus_{i>0}(R/x^{i^n}R)[i] = (\cdots \xrightarrow{0} R/x^{3^n}R \xrightarrow{0} R/x^{2^n}R \xrightarrow{0} R/x^{1^n}R \to 0).$$

(2) *$\mathcal{P}_n$ is a prime thick tensor ideal of $\mathsf{D}^-(R)$ which is not tame. If $n \geqslant 1$, then $\mathcal{P}_n$ is not compact.*

(3) *One has $\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \mathcal{P}_2 \subsetneq \cdots$. Hence $\mathsf{Spec}\, \mathsf{D}^-(R)$ has infinite Krull dimension.*

4. **Fundamental materials**

In this section, we give several basic definitions and study fundamental properties, which will be used in later sections. Throughout this part, unless otherwise specified, R is a commutative Noetherian ring, and all subcategories are nonempty and full.

We denote by $\mathsf{D}^-(R)$ (resp. $\mathsf{D}^\mathsf{b}(R)$) the derived category of (cochain) complexes X of finitely generated R-modules with $\mathsf{H}^i(X) = 0$ for all $i \gg 0$ (resp. $|i| \gg 0$). We denote by $\mathsf{D}^-_\mathsf{fl}(R)$ (resp. $\mathsf{D}^\mathsf{b}_\mathsf{fl}(R)$) the subcategory of $\mathsf{D}^-(R)$ (resp. $\mathsf{D}^\mathsf{b}(R)$) consisting of complexes X whose homologies have finite length as R-modules. By $\mathsf{K}^-(R)$ (resp. $\mathsf{K}^\mathsf{b}(\mathsf{proj}\, R)$) we denote the homotopy category of complexes P of finitely generated projective R-modules with $P^i = 0$ for all $i \gg 0$ (resp. $|i| \gg 0$). By $\mathsf{K}^{-,\mathsf{b}}(R)$ the subcategory of $\mathsf{K}^-(R)$ consisting of complexes P with $\mathsf{H}^i(P) = 0$ for all $i \ll 0$. Note that there are chains

$$\mathsf{D}^\mathsf{b}_\mathsf{fl}(R) \subseteq \mathsf{D}^\mathsf{b}(R) \subseteq \mathsf{D}^-(R), \quad \mathsf{D}^\mathsf{b}_\mathsf{fl}(R) \subseteq \mathsf{D}^-_\mathsf{fl}(R) \subseteq \mathsf{D}^-(R), \quad \mathsf{K}^\mathsf{b}(\mathsf{proj}\, R) \subseteq \mathsf{K}^{-,\mathsf{b}}(R) \subseteq \mathsf{K}^-(R)$$

of thick subcategories and triangle equivalences

$$\mathsf{D}^-(R) \cong \mathsf{K}^-(R), \quad \mathsf{D}^\mathsf{b}(R) \cong \mathsf{K}^{-,\mathsf{b}}(R).$$

We will often identify $\mathsf{D}^-(R), \mathsf{D}^\mathsf{b}(R)$ with $\mathsf{K}^-(R), \mathsf{K}^{-,\mathsf{b}}(R)$ respectively, via these equivalences. Note that $(\mathsf{K}^\mathsf{b}(\mathsf{proj}\, R), \otimes_R, R)$ and $(\mathsf{D}^-(R), \otimes^{\mathbf{L}}_R, R)$ are essentially small tensor triangulated categories. (In general, if $\mathcal{C}$ is an essentially small additive category, then so is the category of complexes of objects in $\mathcal{C}$, and so is the homotopy category.)

Remark 4.1. The tensor triangulated category $\mathsf{D}^-(R)$ is never rigid. More strongly, it is never closed. In fact, assume that there is a functor $F : \mathsf{D}^-(R) \times \mathsf{D}^-(R) \to \mathsf{D}^-(R)$ such that $\mathrm{Hom}_{\mathsf{D}^-(R)}(X \otimes^{\mathbf{L}}_R Y, Z) \cong \mathrm{Hom}_{\mathsf{D}^-(R)}(Y, F(X, Z))$ for all $X, Y, Z \in \mathsf{D}^-(R)$. We have $\mathrm{Hom}_{\mathsf{D}^-(R)}(X \otimes^{\mathbf{L}}_R Y, Z) = \mathrm{Hom}_{\mathsf{D}(R)}(X \otimes^{\mathbf{L}}_R Y, Z) \cong \mathrm{Hom}_{\mathsf{D}(R)}(Y, \mathbf{R}\mathrm{Hom}_R(X, Z))$, where $\mathsf{D}(R)$ is the unbounded derived category of R-modules. Letting $Y = R[-i]$ for $i \in \mathbb{Z}$, we obtain $\mathsf{H}^i(F(X, Z)) \cong \mathsf{Ext}^i_R(X, Z)$. Since $F(X, Z)$ is in $\mathsf{D}^-(R)$, we have $\mathsf{H}^i(F(X, Z)) = 0$ for $i \gg 0$. Hence $\mathsf{Ext}^{\gg 0}_R(X, Z) = 0$ for all $X, Z \in \mathsf{D}^-(R)$. This is a contradiction.

Here we compute some thick closures and thick tensor ideal closures.

Proposition 4.2. *There are equalities:*

(1) $\mathsf{thick}^{\otimes}_{\mathsf{D}^-(R)} R = \mathsf{D}^-(R)$.

(2) $\mathsf{thick}_{\mathsf{D}^-(R)} R = \mathsf{thick}_{\mathsf{D}^b(R)} R = \mathsf{thick}_{\mathsf{K}^b(\mathsf{proj}\, R)} R = \mathsf{thick}^{\otimes}_{\mathsf{K}^b(\mathsf{proj}\, R)} R = \mathsf{K}^b(\mathsf{proj}\, R)$.

(3) $\mathsf{thick}_{\mathsf{D}^-(R)} k = \mathsf{thick}_{\mathsf{D}^b(R)} k = \mathsf{D}^b_{\mathsf{fl}}(R)$, *if R is local with residue field k.*

Proof. The following hold in general, which are easy to check.

(a) Let $\mathcal{T}$ be a triangulated category, $\mathcal{U}$ a thick subcategory and $U \in \mathcal{U}$. Then $\mathsf{thick}_{\mathcal{U}} U = \mathsf{thick}_{\mathcal{T}} U$.

(b) Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a tensor triangulated category. Then $\mathsf{thick}^{\otimes} \mathbf{1} = \mathcal{T}$.

The assertion is shown by these two statements. ■

From now on, we deal with the supports of objects and subcategories of $\mathsf{D}^-(R)$. Recall that the *support* of an R-module M is defined as the set of prime ideals $\mathfrak{p}$ of R such that the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is nonzero, which is denoted by $\mathsf{Supp}_R M$.

Proposition 4.3. *Let X be a complex in $\mathsf{D}^-(R)$. Then the following three sets are equal.*

(1) $\bigcup_{i \in \mathbb{Z}} \mathsf{Supp}_R \mathsf{H}^i(X)$,

(2) $\{\mathfrak{p} \in \mathsf{Spec}\, R \mid X_{\mathfrak{p}} \ncong 0 \text{ in } \mathsf{D}^-(R_{\mathfrak{p}})\}$,

(3) $\{\mathfrak{p} \in \mathsf{Spec}\, R \mid \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R X \ncong 0 \text{ in } \mathsf{D}^-(R_{\mathfrak{p}})\}$.

Proof. It is clear that the first and second sets coincide. For a prime ideal $\mathfrak{p}$ of R one has $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R X \cong \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} X_{\mathfrak{p}}$. It is seen by [Chr, Corollary (A.4.16)] that the second and third sets coincide. ■

Definition 4.4. The set in Proposition 4.3 is called the *support* of X and denoted by $\mathsf{Supp}_R X$. For a subcategory $\mathcal{C}$ of $\mathsf{D}^-(R)$, we set $\mathsf{Supp}\, \mathcal{C} = \bigcup_{C \in \mathcal{C}} \mathsf{Supp}\, C$, and call this the *support* of $\mathcal{C}$. For a subset S of $\mathsf{Spec}\, R$, we denote by $\mathsf{Supp}^{-1} S$ the subcategory of $\mathsf{D}^-(R)$ consisting of complexes whose supports are contained in S.

Remark 4.5. The fact that the second and third sets in Proposition 4.3 coincide will often play an important role in this part. Note that these two sets are different if X is a complex outside $\mathsf{D}^-(R)$. For example, let $(R, \mathfrak{m}, k)$ be a local ring of positive Krull dimension. Take any nonmaximal prime ideal $\mathfrak{p}$, and let X be the injective hull $\mathrm{E}(R/\mathfrak{p})$ of the R-module $R/\mathfrak{p}$. Then $k \otimes^{\mathbf{L}}_R X = 0$, while $X_{\mathfrak{m}} \neq 0$.

Remark 4.6. For $X \in \mathsf{D}^-(R)$ one has $\mathsf{Supp}\, X = \emptyset$ if and only if $X = 0$. In other words, it holds that $\mathsf{Supp}^{-1} \emptyset = \mathbf{0}$. (If we define the support of X as the third set in Proposition 4.3, then the assumption that X belongs to $\mathsf{D}^-(R)$ is essential, as the example given in Remark 4.5 shows.)

In the following lemma and proposition, we state several basic properties of Supp and Supp^{-1} defined above. Both results will often be used later. First lemma says that the pair $(\mathsf{Spec}\, R, \mathsf{Supp})$ satisfies the condition of a support data except that the support Supp takes values in not only closed subsets but also specialization closed subsets.

Lemma 4.7. *The following statements holds:*

(1) $\mathsf{Supp}(0) = \emptyset$.

(2) $\mathsf{Supp}(M[n]) = \mathsf{Supp}(M)$ *for any $M \in \mathcal{T}$ and $n \in \mathbb{Z}$.*

(3) $\mathsf{Supp}(M \oplus N) = \mathsf{Supp}(M) \cup \mathsf{Supp}(N)$ *for any $M, N \in \mathcal{T}$.*

(4) $\mathsf{Supp}(M) \subseteq \sigma(L) \cup \mathsf{Supp}(N)$ *for any triangle* $L \to M \to N \to L[1]$ *in* $\mathcal{T}$.
(5) $\mathsf{Supp}(M \otimes N) = \mathsf{Supp}(M) \cap \mathsf{Supp}(N)$ *for any* $M, N \in \mathcal{T}$.

Proof. The assertions (1), (2), (3) and (4) are straightforward by definition. For each prime ideal $\mathfrak{p}$ of R there is an isomorphism $(X \otimes_R^{\mathbf{L}} Y)_{\mathfrak{p}} \cong X_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}}^{\mathbf{L}} Y_{\mathfrak{p}}$. Hence $(X \otimes_R^{\mathbf{L}} Y)_{\mathfrak{p}} = 0$ if and only if either $X_{\mathfrak{p}} = 0$ or $Y_{\mathfrak{p}} = 0$ by [Chr, Corollary (A.4.16)]. This shows the assertion (5). ■

Let X be a topological space. For a subset A of X, denote by A_{spcl} the largest specialization-closed subset of X contained in A, which is the union of $\overline{\{a\}}$ with $\overline{\{a\}} \subseteq A$. It will be called the spcl-*closure* of A in Section 9.

Proposition 4.8. (1) *Let S be a subset of* $\mathsf{Spec}\, R$. *Then there are equalities* $\mathsf{Supp}^{-1} S = \mathsf{Supp}^{-1}(S_{\mathsf{spcl}})$ *and* $\mathsf{Supp}(\mathsf{Supp}^{-1} S) = S_{\mathsf{spcl}}$. *Moreover,* $\mathsf{Supp}^{-1} S$ *is a thick* $\otimes$*-ideal of* $\mathsf{D}^-(R)$.
(2) *Let $\mathcal{X}$ be any subcategory of* $\mathsf{D}^-(R)$. *Then* $\mathsf{Supp}\,\mathcal{X}$ *is a specialization-closed subset of* $\mathsf{Spec}\, R$, *and one has* $\mathsf{Supp}\,\mathcal{X} = \mathsf{Supp}(\mathsf{thick}^{\otimes}\,\mathcal{X})$.
(3) *It holds that* $\mathsf{D}^-_{\mathsf{fl}}(R) = \mathsf{Supp}^{-1}(\operatorname{Max} R)$. *In particular,* $\mathsf{D}^-_{\mathsf{fl}}(R)$ *is a thick* $\otimes$*-ideal of* $\mathsf{D}^-(R)$.

Proof. (1) We put $W = S_{\mathsf{spcl}}$. Let X be a complex in $\mathsf{D}^-(R)$. Since $\mathsf{Supp}\, X$ is specialization-closed, it is contained in S if and only if it is contained in W. Hence $\mathsf{Supp}^{-1} S = \mathsf{Supp}^{-1} W$. Evidently, W contains $\mathsf{Supp}(\mathsf{Supp}^{-1} W)$, while we have $\mathfrak{p} \in \mathsf{Supp}\, R/\mathfrak{p} = \mathrm{V}(\mathfrak{p}) \subseteq W$ for $\mathfrak{p} \in W$. Hence $\mathsf{Supp}(\mathsf{Supp}^{-1} W) = W$, and thus $\mathsf{Supp}(\mathsf{Supp}^{-1} S) = W$. It is seen from Lemma 4.7 that $\mathsf{Supp}^{-1} S$ is a thick $\otimes$-ideal of $\mathsf{D}^-(R)$.

(2) We have $\mathsf{Supp}\,\mathcal{X} = \bigcup_{X \in \mathcal{X}} \mathsf{Supp}\, X = \bigcup_{X \in \mathcal{X}} \bigcup_{i \in \mathbb{Z}} \mathsf{Supp}\, \mathsf{H}^i X$ by Proposition 4.3. Since $\mathsf{H}^i X$ is a finitely generated R-module, $\mathsf{Supp}\, \mathsf{H}^i X$ is closed. Hence $\mathsf{Supp}\,\mathcal{X}$ is specialization-closed. A prime ideal $\mathfrak{p}$ of R is not in $\mathsf{Supp}\,\mathcal{X}$ if and only if $\mathcal{X}$ is contained in $\mathsf{Supp}^{-1}(\{\mathfrak{p}\}^{\complement})$, if and only if $\mathsf{thick}^{\otimes}\,\mathcal{X}$ is contained in $\mathsf{Supp}^{-1}(\{\mathfrak{p}\}^{\complement})$, if and only if $\mathfrak{p}$ does not belong to $\mathsf{Supp}(\mathsf{thick}^{\otimes}\,\mathcal{X})$. It follows from (1) that $\mathsf{Supp}^{-1}(\{\mathfrak{p}\}^{\complement})$ is a thick $\otimes$-ideal of $\mathsf{D}^-(R)$, which shows the second equivalence. The other two equivalences are obvious.

(3) The equality is straightforward, and the last assertion is shown by (1). ■

5. Classification of compact thick tensor ideals

In this section, we prove a generalized version of the smash nilpotence theorem due to Hopkins [Hop] and Neeman [Nee92], and using this we give a complete classification of cocompact thick tensor ideals of $\mathsf{D}^-(R)$.

We begin with recalling the definitions of compact and cocompact objects. Let $\mathcal{T}$ be a triangulated category. We say that an object $M \in \mathcal{T}$ is *compact* (resp. *cocompact*) if the natural morphism

$$\bigoplus_{\lambda \in \Lambda} \operatorname{Hom}_{\mathcal{T}}(M, N_\lambda) \to \operatorname{Hom}_{\mathcal{T}}(M, \bigoplus_{\lambda \in \Lambda} N_\lambda)$$
$$(\text{resp. } \bigoplus_{\lambda \in \Lambda} \operatorname{Hom}_{\mathcal{T}}(N_\lambda, M) \to \operatorname{Hom}_{\mathcal{T}}(\prod_{\lambda \in \Lambda} N_\lambda, M))$$

is an isomorphism for every family $\{N_\lambda\}_{\lambda \in \Lambda}$ of objects of $\mathcal{T}$ with $\bigoplus_{\lambda \in \Lambda} N_\lambda \in \mathcal{T}$ (resp. $\prod_{\lambda \in \Lambda} N_\lambda \in \mathcal{T}$). We denote by $\mathcal{T}^{\mathsf{c}}$ (resp. $\mathcal{T}^{\mathsf{cc}}$) the subcategory of $\mathcal{T}$ consisting of compact (resp. cocompact) objects. For $\mathcal{T} = \mathsf{D}^-(R)$ we have explicit descriptions of the compact objects and cocompact objects:

Proposition 5.1. *One has* $\mathsf{D}^-(R)^{\mathsf{c}} = \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$ *and* $\mathsf{D}^-(R)^{\mathsf{cc}} = \mathsf{D}^{\mathsf{b}}(R)$.

Proof. The second statement follows from [OS, Theorem 18]. The first one can be shown in the same way as the proof of the fact that the compact objects of the unbounded derived category of all R-modules coincides with $\mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)$. For the convenience of the reader, we give a proof.

First of all, R is compact since each homology functor H^i commutes with direct sums. Since the compact objects form a thick subcategory, one has $\mathsf{K}^{\mathsf{b}}(\operatorname{proj} R) \subseteq \mathsf{D}^{-}(R)^{\mathsf{c}}$. Next, let $X \in \mathsf{D}^{-}(R)$ be a compact object. Replacing X with its projective resolution, we may assume $X \in \mathsf{K}^{-}(R)$. Consider the chain map

$$\begin{array}{ccc} X & = & (\cdots \longrightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \longrightarrow \cdots) \\ \downarrow{\scriptstyle f_n} & & \qquad\quad \downarrow \qquad\quad \downarrow{\scriptstyle f_n^n} \qquad \downarrow \\ C^n[-n] & = & (\cdots \longrightarrow 0 \longrightarrow C^n \longrightarrow 0 \longrightarrow \cdots), \end{array}$$

where C^n is the cokernel of d^{n-1}, and $f_n^n : X^n \to C^n$ is a natural surjection. Put $Y = \bigoplus_{n\in\mathbb{Z}} C^n[-n]$. A chain map $f : X \to Y$ is induced by $\{f_n\}_{n\in\mathbb{Z}}$. As $X \in \mathsf{K}^{-}(R)$ is compact in $\mathsf{D}^{-}(R)$, we have isomorphisms

$$\begin{aligned} \operatorname{Hom}_K(X,Y) \cong \operatorname{Hom}_{\mathsf{D}^{-}(R)}(X,Y) &\cong \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathsf{D}^{-}(R)}(X, C^n[-n]) \\ &\cong \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_K(X, C^n[-n]), \end{aligned}$$

where K is the homotopy category of R-modules. The composition of these isomorphisms sends f to $(f_n)_{n\in\mathbb{Z}}$, which implies that there exists $t \in \mathbb{Z}$ such that $f_n = 0$ in K for all $n \leqslant t$. Hence, there is an R-linear map $g : X^{n+1} \to C^n$ such that $g \circ d^n = f_n^n$. Let $\overline{d^n} : C^n \to X^{n+1}$ be the map induced by d^n. We have $g\overline{d^n} f_n^n = gd^n = f_n^n$, and obtain $g\overline{d^n} = 1$ as f_n^n is a surjection. Thus, C^n is a direct summand of X^{n+1}, and thereby projective. Also, $\mathsf{H}^n X$ is isomorphic to the kernel of $\overline{d^n}$, which vanishes since $\overline{d^n}$ is a split monomorphism. Consequently, the truncated complex $X' := (0 \to C^t \xrightarrow{\overline{d^t}} X^{t+1} \xrightarrow{d^{t+1}} X^{t+2} \xrightarrow{d^{t+2}} \cdots)$, which is quasi-isomorphic to X, is in $\mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)$. We now conclude that X belongs to $\mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)$. ∎

Next, we make the definitions of the annihilators of morphisms and objects in $\mathsf{D}^{-}(R)$.

Definition 5.2. (1) Let $f : X \to Y$ be a morphism in $\mathsf{D}^{-}(R)$. We define the *annihilator* of f as the set of elements $a \in R$ such that $af = 0$ in $\mathsf{D}^{-}(R)$, and denote it by $\operatorname{Ann}_R(f)$. This is an ideal of R.

(2) The *annihilator* of an object $X \in \mathsf{D}^{-}(R)$ is defined as the annihilator of the identity morphism X, and denoted by $\operatorname{Ann}_R(X)$. This is the set of elements $a \in R$ such that $(X \xrightarrow{a} X) = 0$ in $\mathsf{D}^{-}(R)$.

Here are some properties of annihilators.

Proposition 5.3. (1) *Let $f : X \to Y$ be a morphism in $\mathsf{D}^{-}(R)$ and $\mathfrak{p}$ a prime ideal of R.*

(a) *The ideal $\operatorname{Ann}_R(f)$ is the kernel of the map $\eta_f : R \to \operatorname{Hom}_{\mathsf{D}^{-}(R)}(X,Y)$ given by $a \mapsto af$.*

(b) *If the natural map $\tau_{X,Y,\mathfrak{p}} : \operatorname{Hom}_{\mathsf{D}^{-}(R)}(X,Y)_{\mathfrak{p}} \to \operatorname{Hom}_{\mathsf{D}^{-}(R_{\mathfrak{p}})}(X_{\mathfrak{p}}, Y_{\mathfrak{p}})$ is an isomorphism, then there is an equality $\operatorname{Ann}_R(f)_{\mathfrak{p}} = \operatorname{Ann}_{R_{\mathfrak{p}}}(f_{\mathfrak{p}})$.*

(2) *For any* $X \in \mathsf{D}^{-}(R)$ *one has* $\mathrm{V}(\mathrm{Ann}\, X) \supseteq \mathsf{Supp}\, X$*. The equality holds if* $\tau_{X,X,\mathfrak{p}}$ *is an isomorphism for all* $\mathfrak{p} \in \mathsf{Spec}\, R$*. In particular, for* $X \in \mathsf{D}^{\mathsf{b}}(R)$ *one has* $\mathrm{V}(\mathrm{Ann}\, X) = \mathsf{Supp}\, X$*.*

(3) *Let* $\boldsymbol{x} = x_1, \dots, x_n$ *be a sequence of elements of* R*. Then it holds that* $\mathrm{Ann}\, \mathrm{K}(\boldsymbol{x}, R) = \boldsymbol{x}R$*. In particular, there is an equality* $\mathsf{Supp}\, \mathrm{K}(\boldsymbol{x}, R) = \mathrm{V}(\boldsymbol{x})$*, and* $\mathrm{K}(\boldsymbol{x}, R)$ *belongs to* $\mathsf{Supp}^{-1} \mathrm{V}(\boldsymbol{x})$*.*

Proof. (1) The assertion (a) is obvious, while (b) follows from (a) and the commutative diagram

$$\begin{array}{ccc} R_{\mathfrak{p}} & \xrightarrow{(\eta_f)_{\mathfrak{p}}} & \mathrm{Hom}_{\mathsf{D}^{-}(R)}(X, Y)_{\mathfrak{p}} \\ \| & & \cong \downarrow \tau_{X,Y,\mathfrak{p}} \\ R_{\mathfrak{p}} & \xrightarrow{\eta_{f_{\mathfrak{p}}}} & \mathrm{Hom}_{\mathsf{D}^{-}(R_{\mathfrak{p}})}(X_{\mathfrak{p}}, Y_{\mathfrak{p}}). \end{array}$$

(2) The first assertion is easy to show. Suppose that $\tau_{X,X,\mathfrak{p}}$ is an isomorphism for all $\mathfrak{p} \in \mathsf{Spec}\, R$. By (1) one has $(\mathrm{Ann}_R X)_{\mathfrak{p}} = \mathrm{Ann}_{R_{\mathfrak{p}}} X_{\mathfrak{p}}$. We have $X_{\mathfrak{p}} \neq 0$ if and only if $(\mathrm{Ann}_R X)_{\mathfrak{p}} \neq R_{\mathfrak{p}}$, if and only if $\mathfrak{p} \in \mathrm{V}(\mathrm{Ann}_R X)$. This shows $\mathrm{V}(\mathrm{Ann}_R X) = \mathsf{Supp}_R X$. As for the last assertion, use [AF, Lemma 5.2(b)].

(3) The second statement follows from the first one and (2). Therefore it suffices to show the equality $\mathrm{Ann}\, \mathrm{K}(\boldsymbol{x}, R) = \boldsymbol{x}R$. It follows from [BH, Proposition 1.6.5] that $\mathrm{Ann}\, \mathrm{K}(\boldsymbol{x}, R)$ contains $\boldsymbol{x}R$. Conversely, pick $a \in \mathrm{Ann}\, \mathrm{K}(\boldsymbol{x}, R)$. Then the multiplication map $a : \mathrm{K}(\boldsymbol{x}, R) \to \mathrm{K}(\boldsymbol{x}, R)$ is null-homotopic, and there is a homotopy $\{s_i : \mathrm{K}_{i-1}(\boldsymbol{x}, R) \to \mathrm{K}_i(\boldsymbol{x}, R)\}$ from a to 0. In particular, we have $a = d_1 s_1$, where d_1 is the first differential of $\mathrm{K}(\boldsymbol{x}, R)$. Writing $d_1 = (x_1, \dots, x_n) : R^n \to R$ and $s_1 = {}^t(a_1, \dots, a_n) : R \to R^n$, we get $a = (x_1, \dots, x_n)\, {}^t(a_1, \dots, a_n) = a_1 x_1 + \cdots + a_n x_n \in \boldsymbol{x}R$. Consequently, we obtain $\mathrm{Ann}\, \mathrm{K}(\boldsymbol{x}, R) = \boldsymbol{x}R$. ■

To state our next results, we need to introduce some notation.

Definition 5.4. Let $\mathcal{T}$ be a triangulated category.

(1) For two subcategories $\mathcal{C}_1, \mathcal{C}_2$ of $\mathcal{T}$, we denote by $\mathcal{C}_1 * \mathcal{C}_2$ the subcategory of $\mathcal{T}$ consisting of objects M such that there is an exact triangle $C_1 \to M \to C_2 \rightsquigarrow$ with $C_i \in \mathcal{C}_i$ for $i = 1, 2$.

(2) For a subcategory $\mathcal{C}$ of $\mathcal{T}$, we denote by $\mathsf{add}^{\Sigma}\, \mathcal{C}$ the smallest subcategory of $\mathcal{T}$ that contains $\mathcal{C}$ and is closed under finite direct sums, direct summands and shifts. Inductively we define $\mathsf{thick}^1_{\mathcal{T}}(\mathcal{C}) = \mathsf{add}^{\Sigma}\, \mathcal{C}$ and $\mathsf{thick}^r_{\mathcal{T}}(\mathcal{C}) = \mathsf{add}^{\Sigma}(\mathsf{thick}^{r-1}_{\mathcal{T}}(\mathcal{C}) * \mathsf{add}^{\Sigma}\, \mathcal{C})$ for $r > 1$. This is sometimes called the r-th *thickening* of $\mathcal{C}$. When $\mathcal{C}$ consists of a single object X, we simply denote it by $\mathsf{thick}^r_{\mathcal{T}}(X)$.

(3) For a morphism $f : X \to Y$ in $\mathcal{T}$ and an integer $n \geqslant 1$, we denote by $f^{\otimes n}$ the n-fold tensor product $\underbrace{f \otimes \cdots \otimes f}_{n}$. Note that for $\mathcal{T} = \mathsf{D}^{-}(R)$ we mean by $f^{\otimes n}$ the morphism $\underbrace{f \otimes^{\mathbf{L}}_R \cdots \otimes^{\mathbf{L}}_R f}_{n}$.

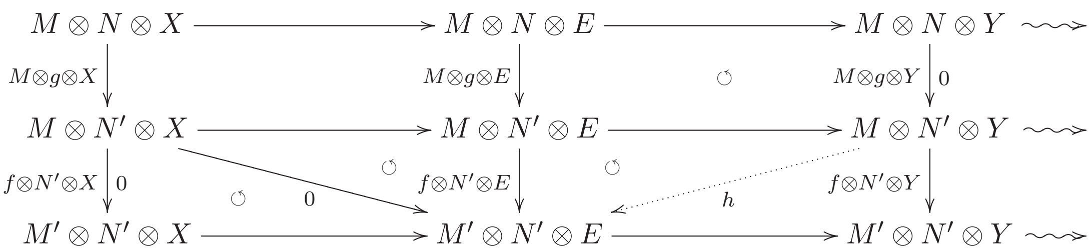
We establish two lemmas, which will be used to show the generalized smash nilpotence theorem. The first one concerns general tensor triangulated categories, while the second one is specific to our $\mathsf{D}^{-}(R)$.

Lemma 5.5. *Let* $\mathcal{T}$ *be a tensor triangulated category.*

(1) *Let $\mathcal{X}, \mathcal{Y}$ be subcategories of $\mathcal{T}$. Let $f : M \to M'$ and $g : N \to N'$ be morphisms in $\mathcal{T}$. If $f \otimes \mathcal{X} = 0$ and $g \otimes \mathcal{Y} = 0$, then $f \otimes g \otimes (\mathcal{X} * \mathcal{Y}) = 0$.*

(2) *Let $\phi : A \to B$ be a morphism in $\mathcal{T}$, and let C be an object of $\mathcal{T}$. If $\phi \otimes C = 0$, then $\phi^{\otimes n} \otimes \mathsf{thick}^n_{\mathcal{T}}(C) = 0$ for all integers $n > 0$.*

Proof. As (2) is shown by induction on n and (1), so let us show (1). Let $X \to E \to Y \rightsquigarrow$ be an exact triangle in $\mathcal{T}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Then $f \otimes X = 0$ and $g \otimes Y = 0$ by assumption. There is a diagram



in $\mathcal{T}$ whose rows are exact triangles, and we obtain a morphism h as in it. It is observed from this diagram that $f \otimes g \otimes E = (f \otimes N' \otimes E) \circ (M \otimes g \otimes E)$ is a zero morphism. ■

Lemma 5.6. (1) *Let $f : X \to Y$ be a morphism in $\mathsf{D}^-(R)$. Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence of elements of R. If $f \otimes^{\mathbf{L}}_R R/(\boldsymbol{x}) = 0$ in $\mathsf{D}^-(R)$, then $f^{\otimes 2^n} \otimes^{\mathbf{L}}_R \mathrm{K}(\boldsymbol{x}, R) = 0$ in $\mathsf{D}^-(R)$.*

(2) *Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence of elements of R, and let $e > 0$ be an integer. Then $\mathrm{K}(\boldsymbol{x}^e, R)$ belongs to $\mathsf{thick}^{ne}_{\mathsf{K}^-(R)}(\mathrm{K}(\boldsymbol{x}, R))$, where $\boldsymbol{x}^e = x_1^e, \dots, x_n^e$.*

Proof. (1) We use induction on n. Let $n = 1$ and set $x = x_1$. There are exact sequences $0 \to (0 : x) \to R \to (x) \to 0$ and $0 \to (x) \to R \to R/(x) \to 0$. Applying the octahedral axiom to $(R \to (x) \to R) = (R \xrightarrow{x} R)$ gives an exact triangle $(0 : x)[1] \to \mathrm{K}(x, R) \to R/(x) \rightsquigarrow$ in $\mathsf{D}^-(R)$. We have $f \otimes^{\mathbf{L}}_R R/(x) = 0$, and $f \otimes^{\mathbf{L}}_R (0 : x)[1] = (f \otimes^{\mathbf{L}}_R R/(x)) \otimes^{\mathbf{L}}_{R/(x)} (0 : x)[1] = 0$. Lemma 5.5(1) yields $f^{\otimes 2} \otimes^{\mathbf{L}}_R \mathrm{K}(x, R) = 0$.

Let $n \geqslant 2$. We have $0 = f \otimes^{\mathbf{L}}_R R/(\boldsymbol{x}) = (f \otimes^{\mathbf{L}}_R R/(x_1)) \otimes^{\mathbf{L}}_{R/(x_1)} R/(\boldsymbol{x})$. The induction hypothesis gives

$$\begin{aligned} 0 &= (f \otimes^{\mathbf{L}}_R R/(x_1))^{\otimes 2^{n-1}} \otimes^{\mathbf{L}}_{R/(x_1)} \mathrm{K}(x_2, \dots, x_n, R/(x_1)) \\ &= (f^{\otimes 2^{n-1}} \otimes^{\mathbf{L}}_R \mathrm{K}(x_2, \dots, x_n, R)) \otimes^{\mathbf{L}}_R R/(x_1). \end{aligned}$$

The induction basis shows $0 = (f^{\otimes 2^{n-1}} \otimes^{\mathbf{L}}_R \mathrm{K}(x_2, \dots, x_n, R))^{\otimes 2} \otimes^{\mathbf{L}}_R \mathrm{K}(x_1, R) = f^{\otimes 2^n} \otimes^{\mathbf{L}}_R \mathrm{K}(x_2, \dots, x_n, \boldsymbol{x}, R)$. Note that $\mathrm{K}(\boldsymbol{x}, R)$ is a direct summand of $\mathrm{K}(x_2, \dots, x_n, \boldsymbol{x}, R)$; see [BH, Proposition 1.6.21]. We thus obtain the desired equality $f^{\otimes 2^n} \otimes^{\mathbf{L}}_R \mathrm{K}(\boldsymbol{x}, R) = 0$.

(2) Again, we use induction on n. Consider the case $n = 1$. Put $x = x_1$. Applying the octahedral axiom to $(R \xrightarrow{x^{e-1}} R \xrightarrow{x} R) = (R \xrightarrow{x^e} R)$, we get an exact triangle $\mathrm{K}(x^{e-1}, R) \to \mathrm{K}(x^e, R) \to \mathrm{K}(x, R) \rightsquigarrow$. Induction on e shows $\mathrm{K}(x^e, R) \in \mathsf{thick}^e\, \mathrm{K}(x, R)$. Let $n \geqslant 2$. By the induction hypothesis, $\mathrm{K}(x_1^e, \dots, x_{n-1}^e, R)$ belongs to $\mathsf{thick}^{(n-1)e}\, \mathrm{K}(x_1, \dots, x_{n-1}, R)$. Applying the exact functor $- \otimes \mathrm{K}(x_n^e, R)$, we see that $\mathrm{K}(\boldsymbol{x}^e, R)$ belongs to $\mathsf{thick}^{(n-1)e}\, \mathrm{K}(x_1, \dots, x_{n-1}, x_n^e, R)$. Applying the exact functor $\mathrm{K}(x_1, \dots, x_{n-1}, R) \otimes -$ to the containment $\mathrm{K}(x_n^e, R) \in \mathsf{thick}^e\, \mathrm{K}(x_n, R)$ gives rise to $\mathrm{K}(x_1, \dots, x_{n-1}, x_n^e, R) \in \mathsf{thick}^e\, \mathrm{K}(\boldsymbol{x}, R)$. Therefore $\mathrm{K}(\boldsymbol{x}^e, R)$ belongs to $\mathsf{thick}^{ne}\, \mathrm{K}(\boldsymbol{x}, R)$. ■

We now achieve the goal of generalizing the Hopkins–Neeman smash nilpotence theorem.

Theorem 5.7 (Generalized Smash Nilpotence)**.** *Let* $f : X \to Y$ *be a morphism in* $\mathsf{K}^-(R)$ *with* $Y \in \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$*. Suppose that* $f \otimes \kappa(\mathfrak{p}) = 0$ *for all* $\mathfrak{p} \in \mathsf{Spec}\, R$*. Then* $f^{\otimes t} = 0$ *for some* $t > 0$*.*

Proof. We have an ascending chain $\mathrm{Ann}_R(f) \subseteq \mathrm{Ann}_R(f^{\otimes 2}) \subseteq \mathrm{Ann}_R(f^{\otimes 3}) \subseteq \cdots$ of ideals of R. Since R is Noetherian, there is an integer c such that $\mathrm{Ann}_R(f^{\otimes c}) = \mathrm{Ann}_R(f^{\otimes i})$ for all $i > c$. Replacing f by $f^{\otimes c}$, we may assume that $\mathrm{Ann}_R(f) = \mathrm{Ann}_R(f^{\otimes i})$ for all $i > 0$. Note that $\mathrm{Ann}_R(f) = R$ if and only if $f = 0$.

We assume $\mathrm{Ann}_R(f) \neq R$, and shall derive a contradiction. Take a minimal prime ideal $\mathfrak{p}$ of $\mathrm{Ann}_R(f)$. Then localization at $\mathfrak{p}$ reduces to the following situation:

> $(R, \mathfrak{m}, k)$ is a local ring, $\mathrm{Ann}_R(f)$ is an $\mathfrak{m}$-primary ideal, $f \otimes_R k = 0$ and $\mathrm{Ann}_R(f) = \mathrm{Ann}_R(f^{\otimes i})$ for all $i > 0$.

Indeed, since Y is in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$, it follows from [AF, Lemma 5.2(b)] that the map $\tau_{X,Y,\mathfrak{p}}$ is an isomorphism, and Proposition 5.3(1) yields $\mathrm{Ann}_{R_\mathfrak{p}}(f_\mathfrak{p}) = \mathrm{Ann}_R(f)_\mathfrak{p}$, which is a $\mathfrak{p}R_\mathfrak{p}$-primary ideal of $R_\mathfrak{p}$. Also, we have $\mathrm{Ann}_{R_\mathfrak{p}}(f_\mathfrak{p}) = \mathrm{Ann}_R(f)_\mathfrak{p} = \mathrm{Ann}_R(f^{\otimes i})_\mathfrak{p} = \mathrm{Ann}_{R_\mathfrak{p}}((f^{\otimes i})_\mathfrak{p}) = \mathrm{Ann}_{R_\mathfrak{p}}((f_\mathfrak{p})^{\otimes i})$ for all $i > 0$. Furthermore, it holds that $f_\mathfrak{p} \otimes_{R_\mathfrak{p}} \kappa(\mathfrak{p}) = f \otimes_R \kappa(\mathfrak{p}) = 0$ by the assumption of the theorem.

For each nonnegative integer n, consider the following two statements.

- $F(n)$: Let $(R, \mathfrak{m}, k)$ be a reduced local ring with $\dim R \leqslant n$. Let $f : X \to Y$ be a morphism in $\mathsf{K}^-(R)$ with $Y \in \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$. If $\mathrm{Ann}_R(f)$ is $\mathfrak{m}$-primary and $f \otimes_R k = 0$, then $f^{\otimes t} = 0$ for some $t > 0$.
- $G(n)$: Let $(R, \mathfrak{m}, k)$ be a local ring with $\dim R \leqslant n$. Let $f : X \to Y$ be a morphism in $\mathsf{K}^-(R)$ with $Y \in \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$. If $\mathrm{Ann}_R(f)$ is $\mathfrak{m}$-primary and $f \otimes_R k = 0$, then $f^{\otimes t} = 0$ for some $t > 0$.

If the statement $G(n)$ holds true for all $n \geqslant 0$, we have $\mathrm{Ann}_R(f) = \mathrm{Ann}_R(f^{\otimes t}) = R$, which gives a desired contradiction. Note that the statement $F(0)$ always holds true since a 0-dimensional reduced local ring is a field. It is thus enough to show the implications $F(n) \Rightarrow G(n) \Rightarrow F(n+1)$.

$F(n) \Rightarrow G(n)$: We consider the reduced ring $R_{\mathrm{red}} = R/\operatorname{nil} R$, where $\operatorname{nil} R$ stands for the nilradical of R. The ideal $\mathrm{Ann}_{R_{\mathrm{red}}}(f \otimes_R R_{\mathrm{red}})$ of R_{red} is $\mathfrak{m}R_{\mathrm{red}}$-primary since it contains $(\mathrm{Ann}_R f)R_{\mathrm{red}}$. We have $(f \otimes_R R_{\mathrm{red}}) \otimes_{R_{\mathrm{red}}} k = f \otimes_R k = 0$. Thus R_{red} and $f \otimes_R R_{\mathrm{red}}$ satisfy the assumption $F(n)$, and we find an integer $t > 0$ such that $f^{\otimes t} \otimes_R R_{\mathrm{red}} = (f \otimes_R R_{\mathrm{red}})^{\otimes t} = 0$. Using Lemma 5.6(1), we get $f^{\otimes tu} \otimes_R \mathrm{K}(\boldsymbol{x}, R) = 0$, where $\boldsymbol{x} = x_1, \cdots, x_n$ is a system of generators of $\operatorname{nil} R$ and $u = 2^n$. Choose an integer $e > 0$ such that $x_i^e = 0$ for all $1 \leqslant i \leqslant n$. Then R is a direct summand of $\mathrm{K}(\boldsymbol{x}^e, R)$ by [BH, Proposition 1.6.21], whence R is in $\mathsf{thick}^{ne} K(\boldsymbol{x}, R)$ by Lemma 5.6(2). Finally, Lemma 5.5(2) gives rise to the equality $f^{\otimes netu} = 0$.

$G(n) \Rightarrow F(n+1)$: We may assume $\dim R = n+1 > 0$. Since R is reduced and $\mathrm{Ann}_R(f)$ is $\mathfrak{m}$-primary, we can choose an R-regular element $x \in \mathrm{Ann}_R(f)$. Then the local ring $R/(x)$ has dimension n, the ideal $\mathrm{Ann}_{R/(x)}(f \otimes_R R/(x))$ of $R/(x)$ is $\mathfrak{m}/(x)$-primary and $(f \otimes_R R/(x)) \otimes_{R/(x)} k = 0$. Hence $R/(x)$ and $f \otimes_R R/(x)$ satisfy the assumption of $G(n)$, and there is an integer $t > 0$ such that $(f \otimes_R R/(x))^{\otimes t} = 0$. The short exact sequence $0 \to R \xrightarrow{x} R \to R/(x) \to 0$ induces an exact triangle $R/(x)[-1] \to R \xrightarrow{x} R \rightsquigarrow$ in $\mathsf{D}^-(R)$. Tensoring Y with this gives an exact triangle $Y \otimes_R R/(x)[-1] \xrightarrow{g} Y \xrightarrow{x} Y \rightsquigarrow$ in $\mathsf{D}^-(R)$. As $xf = 0$, there is a morphism $h : X \to Y \otimes_R R/(x)[-1]$ with $f = gh$. Now $f^{\otimes t+1}$ is decomposed as follows:

$$X^{\otimes t+1} \xrightarrow{h\otimes X^{\otimes t}} (Y \otimes_R R/(x)[-1]) \otimes_R X^{\otimes t} \xrightarrow{(Y\otimes R/(x)[-1])\otimes f^{\otimes t}}$$
$$(Y \otimes_R R/(x)[-1]) \otimes_R Y^{\otimes t} \xrightarrow{g\otimes Y^{\otimes t}} Y^{\otimes t+1}.$$

The middle morphism is identified with $Y[-1] \otimes_R (f \otimes_R R/(x))^{\otimes t}$, which is zero. Thus, $f^{\otimes t+1} = 0$. ■

Remark 5.8. (1) Theorem 5.7 extends the smash nilpotence theorem due to Hopkins [Hop, Theorem 10] and Neeman [Nee92, Theorem 1.1], where X is also assumed to belong to $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$, so that $f : X \to Y$ is a morphism in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$. Under this assumption one can reduce to the case where $X = R$, which plays a key role in the proof of the original Hopkins–Neeman smash nilpotence theorem.

(2) The proof of Theorem 5.7 has a similar frame to that of the original Hopkins–Neeman smash nilpotence theorem, but we should notice that various delicate modifications are actually made there. Indeed, Proposition 5.3, Lemmas 5.5 and 5.6 are all established to prove Theorem 5.7, which are not necessary to prove the original smash nilpotence theorem.

(3) The assumption in Theorem 5.7 that Y belongs to $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$ is used only to have $\mathrm{Ann}_{R_\mathfrak{p}}(f_\mathfrak{p}) = \mathrm{Ann}_R(f)_\mathfrak{p}$.

Our next goal is to classify cocompactly generated thick tensor ideals of $\mathsf{D}^-(R)$. To this end, we begin with deducing the following proposition concerning generation of thick tensor ideals of $\mathsf{D}^-(R)$, which will play an essential role throughout the rest of the part.

Proposition 5.9. *Let X be an object of $\mathsf{D}^-(R)$, and let $\mathcal{Y}$ be a subcategory of $\mathsf{D}^-(R)$. If $\mathrm{V}(\mathrm{Ann}\, X) \subseteq \mathsf{Supp}\, \mathcal{Y}$, then $X \in \mathsf{thick}^{\otimes} \mathcal{Y}$.*

Proof. Clearly, we may assume $X \neq 0$. We prove the proposition by replacing $\mathsf{D}^-(R)$ with $\mathsf{K}^-(R)$. There are a finite number of prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ of R such that $\mathrm{V}(\mathrm{Ann}\, X) = \bigcup_{i=1}^n \mathrm{V}(\mathfrak{p}_i)$. Since each $\mathfrak{p}_i$ is in the support of $\mathcal{Y}$, we find an object $Y_i \in \mathcal{Y}$ with $\mathfrak{p}_i \in \mathsf{Supp}\, Y_i$. All $\mathfrak{p}_i$ are in the support of $Y := Y_1 \oplus \cdots \oplus Y_n \in \mathsf{K}^-(R)$. Choose an integer t with $\mathfrak{p}_1, \cdots, \mathfrak{p}_n \in \bigcup_{i>t} \mathsf{Supp}\, \mathsf{H}^i(Y)$, and let $Y' = (\cdots \to 0 \to 0 \to Y^t \to Y^{t+1} \to \cdots) \in \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$ be the truncated complex of Y. Then $\mathrm{V}(\mathrm{Ann}\, X)$ is contained in $\mathsf{Supp}\, Y'$. Let $f : Y' \to Y$ be the natural morphism, and let $\phi : R \to \mathrm{Hom}_R(Y', Y)$ be the composition of the homothety morphism $R \to \mathrm{Hom}_R(Y, Y)$ and $\mathrm{Hom}_R(f, Y) : \mathrm{Hom}_R(Y, Y) \to \mathrm{Hom}_R(Y', Y)$. There is an exact triangle $Z \xrightarrow{\psi} R \xrightarrow{\phi} \mathrm{Hom}_R(Y', Y) \rightsquigarrow$ in $\mathsf{K}^-(R)$. We establish two claims.

Claim 1. Let $\Phi : R \to C$ be a nonzero morphism in $\mathsf{K}^-(R)$. If R is a field, then Φ is a split monomorphism.

Proof. Since C is isomorphic to $\mathsf{H}(C)$ in $\mathsf{K}^-(R)$, we may assume that the differentials of C are zero. As $z := \Phi^0(1)$ is nonzero, we can construct a chain map $\Psi : C \to R$ with $\Psi^0(z) = 1$ and $\Psi^i = 0$ for all $i \neq 0$. It then holds that $\Psi\Phi = 1$. □

Claim 2. The morphism $\phi \otimes_R \kappa(\mathfrak{p})$ in $\mathsf{K}^-(\kappa(\mathfrak{p}))$ is a split monomorphism for each $\mathfrak{p} \in \mathrm{V}(\mathrm{Ann}\, X)$.

Proof of Claim. Set $S = \bigcup_{i>t} \mathsf{Supp}\, \mathsf{H}^i(Y)$; note that this contains $\mathrm{V}(\mathrm{Ann}\, X)$. We prove the stronger statement that $\phi \otimes \kappa(\mathfrak{p})$ is a split monomorphism for each $\mathfrak{p} \in S$. Since Y' is a perfect complex, there are natural isomorphisms $\mathrm{Hom}_R(Y', Y) \otimes \kappa(\mathfrak{p}) \cong \mathrm{Hom}_R(Y', Y \otimes \kappa(\mathfrak{p})) \cong \mathrm{Hom}_{\kappa(\mathfrak{p})}(Y' \otimes \kappa(\mathfrak{p}), Y \otimes \kappa(\mathfrak{p}))$, which says that $\phi \otimes \kappa(\mathfrak{p})$ is identified with the natural morphism $\kappa(\mathfrak{p}) \to \mathrm{Hom}_{\kappa(\mathfrak{p})}(Y' \otimes \kappa(\mathfrak{p}), Y \otimes \kappa(\mathfrak{p}))$. This induces a map $\mathsf{H}^0(\phi \otimes \kappa(\mathfrak{p})) : \kappa(\mathfrak{p}) \to \mathrm{Hom}_{\mathsf{K}^-(\kappa(\mathfrak{p}))}(Y' \otimes \kappa(\mathfrak{p}), Y \otimes \kappa(\mathfrak{p}))$, sending 1 to $f \otimes \kappa(\mathfrak{p})$. If $f \otimes \kappa(\mathfrak{p}) = 0$ in $\mathsf{K}^-(\kappa(\mathfrak{p}))$, then we see that $\mathsf{H}^{>t}(Y \otimes \kappa(\mathfrak{p})) = 0$, contradicting the fact that $\mathfrak{p} \in S$. Thus $\mathsf{H}^0(\phi \otimes \kappa(\mathfrak{p}))$ is nonzero, and so is $\phi \otimes \kappa(\mathfrak{p})$. Applying Claim 1 completes the proof. □

Claim 2 implies $\psi \otimes_R \kappa(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \mathrm{V}(\mathrm{Ann}\, X)$. Using Theorem 5.7 for the morphism $\psi \otimes_R (R/\mathrm{Ann}\, X)$ in $\mathsf{K}^-(R/\mathrm{Ann}\, X)$, we have $\psi^{\otimes m} \otimes_R (R/\mathrm{Ann}\, X) = 0$ for some $m > 0$. Lemma 5.6(1) shows

$$0 = \psi^{\otimes u} \otimes_R \mathrm{K}(\boldsymbol{x}, R) : Z^{\otimes u} \otimes \mathrm{K}(\boldsymbol{x}, R) \to \mathrm{K}(\boldsymbol{x}, R), \tag{5.9.1}$$

where $\boldsymbol{x} = x_1, \ldots, x_r$ is a system of generators of the ideal $\mathrm{Ann}\, X$, and $u = 2^r m$.

For each $i > 0$, let W_i be the cone of the morphism $\psi^{\otimes i} : Z^{\otimes i} \to R$. Applying the octahedral axiom to the composition $\psi \circ (\psi^{\otimes i} \otimes Z) = \psi^{\otimes i+1}$, we get an exact triangle $W_i \otimes Z \to W_{i+1} \to W_1 \rightsquigarrow$ in $\mathsf{K}^-(R)$. As $W_1 \cong \mathrm{Hom}_R(Y', Y)$ and $Y' \in \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$, we see that W_1 is in $\mathsf{thick}\, Y$. Using the triangle, we inductively observe that W_i belongs to $\mathsf{thick}^{\otimes} Y$ for all $i > 0$, and so does $W_u \otimes \mathrm{K}(\boldsymbol{x}, R)$. It follows from (5.9.1) that $\mathrm{K}(\boldsymbol{x}, R)$ is a direct summand of $W_u \otimes \mathrm{K}(\boldsymbol{x}, R)$, and therefore $\mathrm{K}(\boldsymbol{x}, R)$ belongs to $\mathsf{thick}^{\otimes} Y$.

There is an exact triangle $R \xrightarrow{x_i} R \to \mathrm{K}(x_i, R) \rightsquigarrow$ in $\mathsf{K}^-(R)$ for each $1 \leqslant i \leqslant r$. Tensoring X with this and using the fact that each x_i kills X, we see that X is a direct summand of $X \otimes \mathrm{K}(\boldsymbol{x}, R)$. Consequently, X belongs to $\mathsf{thick}^{\otimes} Y$. By construction Y is in $\mathsf{thick}\, \mathcal{Y}$, and hence X belongs to $\mathsf{thick}^{\otimes} \mathcal{Y}$. ■

Remark 5.10. (1) Proposition 5.9 extends Neeman's result [Nee92, Lemma 1.2], where both X and $\mathcal{Y}$ are contained in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$ (and $\mathcal{Y}$ is assumed to consist of a single object).
(2) Proposition 5.9 is no longer true if we replace $\mathrm{V}(\mathrm{Ann}\, X)$ with $\mathsf{Supp}\, X$, or if we replace $\mathsf{Supp}\, \mathcal{Y}$ with $\mathrm{V}(\mathrm{Ann}\, \mathcal{Y})$. This will be explained in Remarks 10.7(1) and 11.15.

The following results are consequences of Proposition 5.9, which will often be used later.

Corollary 5.11. *Let $\mathcal{X}$ be a thick $\otimes$-ideal of $\mathsf{D}^-(R)$. Let I be an ideal of R and $\boldsymbol{x} = x_1, \ldots, x_n$ a system of generators of I. Then there are equivalences:*

$$\mathrm{V}(I) \subseteq \mathsf{Supp}\, \mathcal{X} \;\Leftrightarrow\; R/I \in \mathcal{X} \;\Leftrightarrow\; \mathrm{K}(\boldsymbol{x}, R) \in \mathcal{X}.$$

Proof. Proposition 5.3(3) implies $\mathsf{Supp}\, R/I = \mathrm{V}(\mathrm{Ann}\, R/I) = \mathrm{V}(I) = \mathrm{V}(\mathrm{Ann}\, \mathrm{K}(\boldsymbol{x}, R)) = \mathsf{Supp}\, \mathrm{K}(\boldsymbol{x}, R)$. The assertion is shown by combining this with Proposition 5.9. ■

Now we can give a complete classification of the cocompactly generated thick tensor ideals of $\mathsf{D}^-(R)$, using Proposition 5.9. For each subset S of $\mathsf{Spec}\, R$, we set $\langle S \rangle = \mathsf{thick}^{\otimes}\{R/\mathfrak{p} \mid \mathfrak{p} \in S\}$.

Theorem 5.12. *The assignments $\mathcal{X} \mapsto \mathsf{Supp}\, \mathcal{X}$ and $\langle W \rangle \leftarrow\!\shortmid W$ make mutually inverse bijections*

$$\left\{\begin{matrix} \textit{cocompactly generated} \\ \textit{thick } \otimes\textit{-ideals of } \mathsf{D}^-(R) \end{matrix}\right\} \rightleftarrows \{\textit{specialization closed subsets of } \mathsf{Spec}\, R\}.$$

Proof. Proposition 4.8(2) shows that the map $\mathcal{X} \mapsto \mathsf{Supp}\, \mathcal{X}$ is well-defined and that for a specialization-closed subset W of $\mathsf{Spec}\, R$ the equality $W = \mathsf{Supp}\langle W \rangle$ holds. It remains to show that for any cocompactly generated thick $\otimes$-ideal $\mathcal{X}$ of $\mathsf{D}^-(R)$ one has $\mathcal{X} = \langle \mathsf{Supp}\, \mathcal{X} \rangle$. Proposition 5.9 implies that $\mathcal{X}$ contains $\langle \mathsf{Supp}\, \mathcal{X} \rangle$. Since $\mathcal{X}$ is cocompactly generated, there is a subcategory $\mathcal{C}$ of $\mathsf{D}^{\mathsf{b}}(R)$ with $\mathcal{X} = \mathsf{thick}^{\otimes} \mathcal{C}$ by Proposition 5.1. Thus, it suffices to prove that each $M \in \mathcal{C}$ belongs to $\langle \mathsf{Supp}\,\mathcal{X} \rangle$. The complex M belongs to $\mathsf{thick}\,\mathsf{H}(M)$ as $M \in \mathsf{D^b}(R)$, and the finitely generated module $\mathsf{H}(M)$ has a finite filtration each of whose subquotients has the form $R/\mathfrak{p}$ with $\mathfrak{p} \in \mathsf{Supp}\,\mathsf{H}(M)$. Hence M is in $\langle \mathsf{Supp}\,M \rangle$, and we are done. ■

Let us give several applications of our Theorem 5.12.

Corollary 5.13. (1) *Let $\mathcal{C}$ be a subcategory of $\mathsf{D^b}(R)$. Then $\mathsf{thick}^{\otimes}_{\mathsf{D^-}(R)}\,\mathcal{C}$ consists of the complexes $X \in \mathsf{D^-}(R)$ with $\mathrm{V}(\mathrm{Ann}\,X) \subseteq \mathsf{Supp}\,\mathcal{C}$. In particular, those complexes X form a thick $\otimes$-ideal of $\mathsf{D^-}(R)$.*

(2) *Let I be an ideal of R. Then $\mathsf{thick}^{\otimes}_{\mathsf{D^-}(R)}(R/I)$ consists of the complexes $X \in \mathsf{D^-}(R)$ with $I \subseteq \sqrt{\mathrm{Ann}\,X}$.*

(3) *Let W be a specialization-closed subset of $\mathsf{Spec}\,R$. Then $\langle W \rangle$ consists of the complexes $X \in \mathsf{D^-}(R)$ such that $\mathrm{V}(\mathrm{Ann}\,X) \subseteq W$.*

(4) *Let $\mathcal{X}, \mathcal{Y}$ be thick subcategories in $\mathsf{D^b}(R)$. Then $\mathsf{thick}^{\otimes}\,\mathcal{X} = \mathsf{thick}^{\otimes}\,\mathcal{Y}$ if and only if $\mathsf{Supp}\,\mathcal{X} = \mathsf{Supp}\,\mathcal{Y}$.*

Proof. (1) Let $\mathcal{X}$ be the subcategory of $\mathsf{D^-}(R)$ consisting of objects $X \in \mathsf{D^-}(R)$ with $\mathrm{V}(\mathrm{Ann}\,X) \subseteq \mathsf{Supp}\,\mathcal{C}$. Proposition 5.9 says that $\mathsf{thick}^{\otimes}\,\mathcal{C}$ contains $\mathcal{X}$. Propositions 4.8(2), 5.1 and Theorem 5.12 yield $\mathsf{thick}^{\otimes}\,\mathcal{C} = \langle \mathsf{Supp}(\mathsf{thick}^{\otimes}\,\mathcal{C}) \rangle = \langle \mathsf{Supp}\,\mathcal{C} \rangle$. For each $\mathfrak{p} \in \mathsf{Supp}\,\mathcal{C}$, the set $\mathrm{V}(\mathrm{Ann}\,R/\mathfrak{p}) = \mathrm{V}(\mathfrak{p})$ is contained in $\mathsf{Supp}\,\mathcal{C}$, whence $R/\mathfrak{p}$ is in $\mathcal{X}$. Hence $\mathsf{thick}^{\otimes}\,\mathcal{C}$ is contained in $\mathcal{X}$, and we get the equality $\mathsf{thick}^{\otimes}\,\mathcal{C} = \mathcal{X}$.

(2) Applying (1) to $\mathcal{C} = \{R/I\}$, we immediately obtain the assertion.

(3) Setting $\mathcal{C} = \{R/\mathfrak{p} \mid \mathfrak{p} \in W\} \subseteq \mathsf{D^b}(R)$, we have $\mathsf{Supp}\,\mathcal{C} = W$. The assertion follows from (1).

(4) Let $\mathcal{C}$ be either $\mathcal{X}$ or $\mathcal{Y}$. By Proposition 5.1 the thick $\otimes$-ideal $\mathsf{thick}^{\otimes}\,\mathcal{C}$ is cocompactly generated, and $\mathsf{Supp}(\mathsf{thick}^{\otimes}\,\mathcal{C}) = \mathsf{Supp}\,\mathcal{C}$ by Proposition 4.8(2). The assertion now follows from Theorem 5.12. ■

We obtain the following one-to-one correspondence by combining our Theorem 5.12 with the celebrated Hopkins–Neeman classification theorem [Nee92, Theorem 1.5].

Corollary 5.14. *The assignments $\mathcal{X} \mapsto \mathcal{X} \cap \mathsf{K^b}(\mathsf{proj}\,R)$ and $\mathsf{thick}^{\otimes}\,\mathcal{Y} \leftarrow\!\shortmid \mathcal{Y}$ make mutually inverse bijections*

$$\{\textit{cocompactly generated thick } \otimes\textit{-ideals of } \mathsf{D^-}(R)\} \rightleftarrows \{\textit{thick subcategories of } \mathsf{K^b}(\mathsf{proj}\,R)\}.$$

In particular, all cocompactly generated thick $\otimes$-ideals of $\mathsf{D^-}(R)$ are compactly generated.

Proof. It is directly verified (resp. follows from Proposition 5.1) that the assignment $\mathcal{X} \mapsto \mathcal{X} \cap \mathsf{K^b}(\mathsf{proj}\,R)$ (resp. $\mathsf{thick}^{\otimes}\,\mathcal{Y} \leftarrow\!\shortmid \mathcal{Y}$) makes a well-defined map. It follows from [Nee92, Theorem 1.5] that

(#) the assignments $\mathcal{X} \mapsto \mathsf{Supp}\,\mathcal{X}$ and $W \mapsto \mathsf{Supp}^{-1}_{\mathsf{K^b}(\mathsf{proj}\,R)}(W) := \mathsf{Supp}^{-1}\,W \cap \mathsf{K^b}(\mathsf{proj}\,R)$ make mutually inverse bijections between the thick subcategories of $\mathsf{K^b}(\mathsf{proj}\,R)$ and the specialization-closed subsets of $\mathsf{Spec}\,R$.

In view of Theorem 5.12 and (#), we have only to show that

(a) $\mathsf{Supp}^{-1}_{\mathsf{K^b}(\mathsf{proj}\,R)}(\mathsf{Supp}\,\mathcal{X}) = \mathcal{X} \cap \mathsf{K^b}(\mathsf{proj}\,R)$ for any cocompactly generated thick $\otimes$-ideal $\mathcal{X}$ of $\mathsf{D^-}(R)$, and

(b) $\langle \mathsf{Supp}\,\mathcal{Y} \rangle = \mathsf{thick}^{\otimes}\,\mathcal{Y}$ for any thick subcategory $\mathcal{Y}$ of $\mathsf{K^b}(\mathsf{proj}\,R)$.

Using Propositions 5.1 and 4.8(2), we see that $\langle \mathsf{Supp}\,\mathcal{Y}\rangle$ and $\mathsf{thick}^{\otimes}\,\mathcal{Y}$ are cocompactly generated thick $\otimes$-ideals of $\mathsf{D}^{-}(R)$ whose supports are equal to $\mathsf{Supp}\,\mathcal{Y}$. Now Theorem 5.12 shows the statement (b).

Clearly, $\mathsf{Supp}(\mathcal{X}\cap\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,R))$ is contained in $\mathsf{Supp}\,\mathcal{X}$. Take a prime ideal $\mathfrak{p}\in\mathsf{Supp}\,\mathcal{X}$, and let $\boldsymbol{x}$ be a system of generators of $\mathfrak{p}$. Then $\mathrm{V}(\mathrm{Ann}\,\mathrm{K}(\boldsymbol{x},R))=\mathsf{Supp}\,\mathrm{K}(\boldsymbol{x},R)=\mathrm{V}(\mathfrak{p})\subseteq\mathsf{Supp}\,\mathcal{X}$ by Proposition 5.3(3), and $\mathrm{K}(\boldsymbol{x},R)\in\mathcal{X}\cap\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,R)$ by Proposition 5.9. It follows that $\mathfrak{p}\in\mathsf{Supp}\,\mathrm{K}(\boldsymbol{x},R)\subseteq\mathsf{Supp}(\mathcal{X}\cap\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,R))$. Thus we get $\mathsf{Supp}(\mathcal{X}\cap\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,R))=\mathsf{Supp}\,\mathcal{X}$, and obtain $\mathsf{Supp}^{-1}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,R)}(\mathsf{Supp}\,\mathcal{X})=\mathsf{Supp}^{-1}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,R)}(\mathsf{Supp}(\mathcal{X}\cap\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,R)))=\mathcal{X}\cap\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,R)$, where the last equality is shown by (#). Now the statement (a) is proved. ■

Remark 5.15. Corollary 5.14 in particular gives a classification of the *compactly* generated thick $\otimes$-ideals of $\mathsf{D}^{-}(R)$. This itself can also be deduced as follows: Let $\mathcal{X},\mathcal{Y}$ be thick subcategories of $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,R)$ with $\mathsf{Supp}(\mathsf{thick}^{\otimes}\,\mathcal{X})=\mathsf{Supp}(\mathsf{thick}^{\otimes}\,\mathcal{Y})$. Then $\mathsf{Supp}\,\mathcal{X}=\mathsf{Supp}\,\mathcal{Y}$ by Proposition 4.8(2), and the Hopkins–Neeman theorem yields $\mathcal{X}=\mathcal{Y}$. Hence $\mathsf{thick}^{\otimes}\,\mathcal{X}=\mathsf{thick}^{\otimes}\,\mathcal{Y}$.

The essential benefit that Corollary 5.14 produces is the classification of the *cocompactly* generated thick $\otimes$-ideals of $\mathsf{D}^{-}(R)$. This should not follow from the Hopkins–Neeman theorem or other known results, but require the arguments established in this section so far (especially, the Generalized Smash Nilpotence Theorem 5.7). A *compactly* generated thick tensor ideal of $\mathsf{D}^{-}(R)$ is clearly *cocompactly* generated by Proposition 5.1, but the converse (shown in Corollary 5.14) should be rather non-trivial.

In view of Corollary 5.14 and Proposition 5.1, we obtain the following result and definition.

Corollary 5.16. *The following four conditions are equivalent for a thick $\otimes$-ideal $\mathcal{X}$ of $\mathsf{D}^{-}(R)$.*

- *$\mathcal{X}$ is compactly generated.*
- *$\mathcal{X}$ is generated by objects in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,R)$.*
- *$\mathcal{X}$ is cocompactly generated.*
- *$\mathcal{X}$ is generated by objects in $\mathsf{D}^{\mathsf{b}}(R)$.*

Definition 5.17. Let $\mathcal{X}$ be a thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$. We say that $\mathcal{X}$ is *compact* if it satisfies one (hence all) of the equivalent conditions in Corollary 7.17.

Next, for two thick $\otimes$-ideals $\mathcal{X},\mathcal{Y}$ of $\mathsf{D}^{-}(R)$ we define the thick $\otimes$-ideals $\mathcal{X}\wedge\mathcal{Y}$ and $\mathcal{X}\vee\mathcal{Y}$ by:

$$\mathcal{X}\wedge\mathcal{Y}=\mathsf{thick}^{\otimes}\{X\otimes^{\mathbf{L}}_{R}Y\mid X\in\mathcal{X},\,Y\in\mathcal{Y}\},\qquad \mathcal{X}\vee\mathcal{Y}=\mathsf{thick}^{\otimes}(\mathcal{X}\cup\mathcal{Y}).$$

These two operations yield a lattice structure in the compact thick $\otimes$-ideals of $\mathsf{D}^{-}(R)$:

Proposition 5.18. (1) *Let A and B be specialization-closed subsets of $\mathsf{Spec}\,R$. One then has equalities*

$$\langle A\rangle\wedge\langle B\rangle=\langle A\cap B\rangle,\qquad \langle A\rangle\vee\langle B\rangle=\langle A\cup B\rangle.$$

(2) *The set of compact thick $\otimes$-ideals of $\mathsf{D}^{-}(R)$ is a lattice with meet $\wedge$ and join $\vee$.*

Proof. (1) It is evident that the second equality holds. Let us show the first one.

We claim that for two subcategories $\mathcal{M},\mathcal{N}$ of $\mathsf{D}^{-}(R)$ it holds that

$$(\mathsf{thick}^{\otimes}\,\mathcal{M})\wedge(\mathsf{thick}^{\otimes}\,\mathcal{N})=\mathsf{thick}^{\otimes}\{M\otimes^{\mathbf{L}}_{R}N\mid M\in\mathcal{M},\,N\in\mathcal{N}\}.$$

In fact, clearly $(\mathsf{thick}^{\otimes}\,\mathcal{M})\wedge(\mathsf{thick}^{\otimes}\,\mathcal{N})$ contains $\mathcal{C}:=\mathsf{thick}^{\otimes}\{M\otimes^{\mathbf{L}}_{R}N\mid M\in\mathcal{M},\,N\in\mathcal{N}\}$. For each $N\in\mathcal{N}$, the subcategory of $\mathsf{D}^{-}(R)$ consisting of objects X with $X\otimes^{\mathbf{L}}_{R}N\in\mathcal{C}$ is a thick $\otimes$-ideal containing $\mathcal{M}$, so contains $\mathsf{thick}^{\otimes}\,\mathcal{M}$. Let X be an object in $\mathsf{thick}^{\otimes}\,\mathcal{M}$. Then

$X \otimes_R^{\mathbf{L}} N$ belongs to $\mathcal{C}$ for all $N \in \mathcal{N}$. The subcategory of $\mathsf{D}^-(R)$ consisting of objects Y with $X \otimes_R^{\mathbf{L}} Y \in \mathcal{C}$ is a thick $\otimes$-ideal containing $\mathcal{N}$, so contains $\mathsf{thick}^{\otimes} \mathcal{N}$. Hence $X \otimes_R^{\mathbf{L}} Y$ is in $\mathcal{C}$ for all $X \in \mathsf{thick}^{\otimes} \mathcal{M}$ and $Y \in \mathsf{thick}^{\otimes} \mathcal{N}$, and the claim follows.

Using the claim, we see that $\langle A\rangle \wedge \langle B\rangle = \mathsf{thick}^{\otimes}\{R/\mathfrak{p} \otimes_R^{\mathbf{L}} R/\mathfrak{q} \mid \mathfrak{p} \in A,\ \mathfrak{q} \in B\}$. Therefore

$$\begin{aligned}\mathsf{Supp}(\langle A\rangle \wedge \langle B\rangle) &= \mathsf{Supp}\{R/\mathfrak{p} \otimes_R^{\mathbf{L}} R/\mathfrak{q} \mid \mathfrak{p} \in A,\ \mathfrak{q} \in B\}\\ &= \textstyle\bigcup_{\mathfrak{p} \in A,\, \mathfrak{q} \in B} \mathsf{Supp}(R/\mathfrak{p} \otimes_R^{\mathbf{L}} R/\mathfrak{q})\\ &= \textstyle\bigcup_{\mathfrak{p} \in A,\, \mathfrak{q} \in B} (\mathrm{V}(\mathfrak{p}) \cap \mathrm{V}(\mathfrak{q})) = A \cap B = \mathsf{Supp}\langle A \cap B\rangle\end{aligned}$$

by Proposition 4.8(2), Lemma 4.7(4) and the assumption that A, B are specialization-closed. Theorem 5.12 implies that $\langle A\rangle \wedge \langle B\rangle = \langle A \cap B\rangle$.

(2) Let $\mathcal{X}, \mathcal{Y}$ be compact thick $\otimes$-ideals of $\mathsf{D}^-(R)$. Theorem 5.12 implies that $\mathcal{X} = \langle \mathsf{Supp}\,\mathcal{X}\rangle$ and $\mathcal{Y} = \langle \mathsf{Supp}\,\mathcal{Y}\rangle$, and $\mathsf{Supp}\,\mathcal{X}$ and $\mathsf{Supp}\,\mathcal{Y}$ are specialization-closed. It follows from (1) that $\mathcal{X} \wedge \mathcal{Y} = \langle \mathsf{Supp}\,\mathcal{X} \cap \mathsf{Supp}\,\mathcal{Y}\rangle$ and $\mathcal{X} \vee \mathcal{Y} = \langle \mathsf{Supp}\,\mathcal{X} \cup \mathsf{Supp}\,\mathcal{Y}\rangle$, which are compact. It is seen by definition that any thick $\otimes$-ideal containing both $\mathcal{X}$ and $\mathcal{Y}$ contains $\mathcal{X} \vee \mathcal{Y}$. Let $\mathcal{Z}$ be a compact thick $\otimes$-ideal contained in both $\mathcal{X}$ and $\mathcal{Y}$. By Theorem 5.12 again we get $\mathcal{Z} = \langle \mathsf{Supp}\,\mathcal{Z}\rangle$. Since $\mathsf{Supp}\,\mathcal{Z}$ is contained in $\mathsf{Supp}\,\mathcal{X} \cap \mathsf{Supp}\,\mathcal{Y}$, we have that $\mathcal{Z}$ is contained in $\mathcal{X} \wedge \mathcal{Y}$. These arguments prove the assertion. ■

Note that the specialization-closed subsets of $\mathsf{Spec}\,R$ form a lattice with meet $\cap$ and join $\cup$. As an immediate consequence of this fact and Proposition 5.18(2), we obtain a refinement of Theorem 5.12:

Theorem 5.19. *The assignments $\mathcal{X} \mapsto \mathsf{Supp}\,\mathcal{X}$ and $\langle W\rangle \leftarrow\!\shortmid W$ induce a lattice isomorphism*

$$\{\textit{compact thick } \otimes\textit{-ideals of } \mathsf{D}^-(R)\} \cong \{\textit{specialization-closed subsets of } \mathsf{Spec}\,R\}.$$

Restricting to the Artinian case, we get a complete classification of thick tensor ideals of $\mathsf{D}^-(R)$.

Corollary 5.20. *Let R be an Artinian ring. Then the following statements are true.*

(1) *All the thick $\otimes$-ideals of $\mathsf{D}^-(R)$ are compact.*
(2) *The assignments $\mathcal{X} \mapsto \mathsf{Supp}\,\mathcal{X}$ and $\langle S\rangle \leftarrow\!\shortmid S$ induce a lattice isomorphism*

$$\{\textit{thick } \otimes\textit{-ideals of } \mathsf{D}^-(R)\} \cong \{\textit{subsets of } \mathsf{Spec}\,R\}.$$

Proof. (1) Take any thick $\otimes$-ideal $\mathcal{X}$ of $\mathsf{D}^-(R)$. We want to show $\mathcal{X} = \langle \mathsf{Supp}\,\mathcal{X}\rangle$. Corollary 5.11 implies that $\mathcal{X}$ contains $\langle \mathsf{Supp}\,\mathcal{X}\rangle$. To show the opposite inclusion, we may assume that $\mathcal{X}$ consists of a single object X. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_s, \mathfrak{m}_{s+1}, \dots, \mathfrak{m}_n$ be the maximal ideals of R with $\mathsf{Supp}\,X = \{\mathfrak{m}_1, \dots, \mathfrak{m}_s\}$. Find an integer $t > 0$ with $(\mathfrak{m}_1 \cdots \mathfrak{m}_n)^t = 0$. The Chinese remainder theorem yields an isomorphism $R \cong R/\mathfrak{m}_1^t \oplus \cdots \oplus R/\mathfrak{m}_n^t$ of R-modules. Tensoring X, we obtain an isomorphism $X \cong (X \otimes_R^{\mathbf{L}} R/\mathfrak{m}_1^t) \oplus \cdots \oplus (X \otimes_R^{\mathbf{L}} R/\mathfrak{m}_n^t)$. Lemma 4.7(4) gives $\mathsf{Supp}(X \otimes_R^{\mathbf{L}} R/\mathfrak{m}_i^t) = \mathsf{Supp}\,X \cap \{\mathfrak{m}_i\}$, which is an empty set for $s+1 \leqslant i \leqslant n$. For such an i we have $X \otimes_R^{\mathbf{L}} R/\mathfrak{m}_i^t = 0$ by Remark 4.6, and get $X \cong (X \otimes_R^{\mathbf{L}} R/\mathfrak{m}_1^t) \oplus \cdots \oplus (X \otimes_R^{\mathbf{L}} R/\mathfrak{m}_s^t)$. It follows that X is in $\mathsf{thick}^{\otimes}\{R/\mathfrak{m}_1^t, \dots, R/\mathfrak{m}_s^t\}$, which is the same as $\langle \mathsf{Supp}\,X\rangle$ by Corollary 5.13.

(2) Since all prime ideals of R are maximal, every subset of $\mathsf{Spec}\,R$ is specialization-closed. (A more general statement will be given in Lemma 7.3.) The assertion follows from (1) and Theorem 5.19. ■

6. **Correspondence between the Balmer and Zariski spectra**

In this section, we construct a pair of maps between the Balmer spectrum $\mathsf{Spec}\,\mathsf{D}^-(R)$ and the Zariski spectrum $\mathsf{Spec}\,R$, which will play a crucial role in later sections. We begin with the following proposition which will be used later.

Proposition 6.1. *For each complex $X \in \mathsf{D}^-(R)$ it holds that*

$$\mathsf{Supp}\,X = \mathsf{Spec}\,R \;\Leftrightarrow\; \mathsf{thick}^{\otimes} X = \mathsf{D}^-(R) \;\Leftrightarrow\; \mathsf{BSupp}\,X = \mathsf{Spec}\,\mathsf{D}^-(R).$$

Proof. The second equivalence follows from [Bal05, Corollary 2.5]. Let us prove the first equivalence. Proposition 4.8(2) implies $\mathsf{Supp}\,X = \mathsf{Supp}(\mathsf{thick}^{\otimes} X)$, which shows ($\Leftarrow$). As for ($\Rightarrow$), for every $M \in \mathsf{D}^-(R)$ we have $\mathrm{V}(\mathrm{Ann}\,M) \subseteq \mathsf{Spec}\,R = \mathsf{Supp}\,X$, by which and Proposition 5.9 we get $M \in \mathsf{thick}^{\otimes} X$. ■

Let us introduce the following notation.

Notation 6.2. For a prime ideal $\mathfrak{p}$ of R, we denote by $\mathcal{S}(\mathfrak{p})$ the subcategory of $\mathsf{D}^-(R)$ consisting of complexes X with $X_\mathfrak{p} \cong 0$ in $\mathsf{D}^-(R_\mathfrak{p})$.

The subcategory $\mathcal{S}(\mathfrak{p})$ is always a prime thick tensor ideal:

Proposition 6.3. *Let $\mathfrak{p}$ be a prime ideal of R. Then $\mathcal{S}(\mathfrak{p})$ is a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$ satisfying*

$$\mathsf{Supp}\,\mathcal{S}(\mathfrak{p}) = \{\mathfrak{q} \in \mathsf{Spec}\,R \mid \mathfrak{q} \nsubseteq \mathfrak{p}\}.$$

Proof. Since $\mathcal{S}(\mathfrak{p})$ does not contain R, it is not equal to $\mathsf{D}^-(R)$. Note that $\mathcal{S}(\mathfrak{p}) = \mathsf{Supp}^{-1}(\{\mathfrak{p}\}^\complement)$. Using Lemma 4.7(4) and Proposition 4.8(1), we observe that $\mathcal{S}(\mathfrak{p})$ is a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$.

Fix a prime ideal $\mathfrak{q}$ of R. If $\mathfrak{q}$ is in $\mathsf{Supp}\,\mathcal{S}(\mathfrak{p})$, then there is a complex $X \in \mathcal{S}(\mathfrak{p})$ with $\mathfrak{q} \in \mathsf{Supp}\,X$, and it follows that $X_\mathfrak{p} = 0 \neq X_\mathfrak{q}$. If $\mathfrak{q}$ is contained in $\mathfrak{p}$, then we have $X_\mathfrak{q} = (X_\mathfrak{p})_\mathfrak{q}$ and get a contradiction. Therefore $\mathfrak{q}$ is not contained in $\mathfrak{p}$. Conversely, assume this. Take a system of generators $\boldsymbol{x} = x_1, \dots, x_n$ of $\mathfrak{q}$, and put $K = \mathrm{K}(\boldsymbol{x}, R)$. Then we have $K_\mathfrak{q} \neq 0 = K_\mathfrak{p}$ by Proposition 5.3(3). Hence K belongs to $\mathcal{S}(\mathfrak{p})$ and $\mathfrak{q}$ is in $\mathsf{Supp}\,K$, which implies $\mathfrak{q} \in \mathsf{Supp}\,\mathcal{S}(\mathfrak{p})$. We thus obtain the equality in the proposition. ■

As an easy consequence of the above proposition, we get the following.

Corollary 6.4. *Let R be an integral domain of dimension one. It then holds that $\mathsf{D}^-_{\mathsf{fl}}(R) = \mathcal{S}((0))$, where (0) stands for the zero ideal of R. Hence $\mathsf{D}^-_{\mathsf{fl}}(R)$ is a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$.*

Proof. For a complex $X \in \mathsf{D}^-(R)$ it holds that

$$X \in \mathsf{D}^-_{\mathsf{fl}}(R) \;\Leftrightarrow\; \ell(\mathsf{H}^i X) < \infty \text{ for all } i \;\Leftrightarrow\; \mathsf{H}^i X_{(0)} = 0 \text{ for all } i \;\Leftrightarrow\; X_{(0)} = 0 \;\Leftrightarrow\; X \in \mathcal{S}((0)),$$

where the second equivalence follows from the fact that $\mathsf{Spec}\,R = \{(0)\} \cup \mathrm{Max}\,R$. This shows $\mathsf{D}^-_{\mathsf{fl}}(R) = \mathcal{S}((0))$. Proposition 6.3 implies that $\mathcal{S}((0))$ is prime, which gives the last statement of the corollary. ■

Remark 6.5. Corollary 6.4 is no longer valid if we remove the assumption that R is an integral domain. More precisely, the assertion of the corollary is not true even if R is reduced. In fact, consider the ring $R = k[[x, y]]/(xy)$, where k is a field. Then R is a 1-dimensional reduced local ring. It is observed by Proposition 5.3(3) that the Koszul complexes $\mathrm{K}(x, R), \mathrm{K}(y, R)$ are outside $\mathsf{D}^-_{\mathsf{fl}}(R)$, while the complex $\mathrm{K}(x, R) \otimes^{\mathbf{L}}_R \mathrm{K}(y, R) = \mathrm{K}(x, y, R)$ is in $\mathsf{D}^-_{\mathsf{fl}}(R)$. This shows that $\mathsf{D}^-_{\mathsf{fl}}(R)$ is not prime.

We have constructed from each prime ideal $\mathfrak{p}$ of R the prime thick tensor ideal $\mathcal{S}(\mathfrak{p})$ of $\mathsf{D}^{-}(R)$. Now we are concerned with the opposite direction, that is, we construct from a prime thick tensor ideal of $\mathsf{D}^{-}(R)$ a prime ideal of R, which is done in the following proposition.

Proposition 6.6. *Let $\mathcal{P}$ be a prime thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$. Let K be the set of ideals I of R such that $\mathrm{V}(I)$ is not contained in $\mathsf{Supp}\,\mathcal{P}$. Then K has the maximum element P with respect to the inclusion relation, and P is a prime ideal of R.*

Proof. We claim that for ideals I, J of R, if $\mathsf{Supp}\,\mathcal{P}$ contains $\mathrm{V}(I+J)$, then it contains either $\mathrm{V}(I)$ or $\mathrm{V}(J)$. Indeed, let $\boldsymbol{x} = x_1, \dots, x_a$ and $\boldsymbol{y} = y_1, \dots, y_b$ be systems of generators of I and J, respectively. Corollary 5.11 yields that $\mathrm{K}(\boldsymbol{x}, \boldsymbol{y}, R)$ is in $\mathcal{P}$. There is an isomorphism $\mathrm{K}(\boldsymbol{x}, R) \otimes_R^{\mathbf{L}} \mathrm{K}(\boldsymbol{y}, R) \cong \mathrm{K}(\boldsymbol{x}, \boldsymbol{y}, R)$ of complexes, whence $\mathrm{K}(\boldsymbol{x}, R) \otimes_R^{\mathbf{L}} \mathrm{K}(\boldsymbol{y}, R)$ belongs to $\mathcal{P}$. Since $\mathcal{P}$ is prime, it contains either $\mathrm{K}(\boldsymbol{x}, R)$ or $\mathrm{K}(\boldsymbol{y}, R)$. Thus $\mathsf{Supp}\,\mathcal{P}$ contains either $\mathrm{V}(I)$ or $\mathrm{V}(J)$ by Corollary 5.11 again.

The claim says that K is closed under sums of ideals of R. Taking into account that R is Noetherian, we see that K has the maximum element P with respect to the inclusion relation. There is a filtration $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_t = R/P$ of submodules of the R-module R/P such that for every $1 \leqslant i \leqslant t$ one has $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ with some $\mathfrak{p}_i \in \mathsf{Supp}_R\, R/P$, whence each $\mathfrak{p}_i$ contains P. Suppose that P is not a prime ideal of R. Then the $\mathfrak{p}_i$ strictly contain P, and the maximality of P shows that $\mathsf{Supp}\,\mathcal{P}$ contains $\mathrm{V}(\mathfrak{p}_i)$. There is an equality $\mathsf{Supp}_R\, R/P = \bigcup_{i=1}^{t} \mathsf{Supp}\, R/\mathfrak{p}_i$, or equivalently, $\mathrm{V}(P) = \bigcup_{i=1}^{t} \mathrm{V}(\mathfrak{p}_i)$. It follows that $\mathsf{Supp}\,\mathcal{P}$ contains $\mathrm{V}(P)$, which is a contradiction. Consequently, P is a prime ideal of R. ■

Thus we have got two maps in the mutually inverse directions, between $\mathsf{Spec}\,R$ and $\mathtt{Spec}\,\mathsf{D}^{-}(R)$:

Notation 6.7. Let $\mathcal{P}$ be a prime thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$. With the notation of Proposition 6.6, we set $\mathbb{I}(\mathcal{P}) = K$ and $\mathfrak{s}(\mathcal{P}) = P$. In view of Proposition 6.3, we obtain a pair of maps

$$\mathcal{S} : \mathsf{Spec}\,R \rightleftarrows \mathtt{Spec}\,\mathsf{D}^{-}(R) : \mathfrak{s}$$

given by $\mathfrak{p} \mapsto \mathcal{S}(\mathfrak{p})$ and $\mathcal{P} \mapsto \mathfrak{s}(\mathcal{P})$ for $\mathfrak{p} \in \mathsf{Spec}\,R$ and $\mathcal{P} \in \mathtt{Spec}\,\mathsf{D}^{-}(R)$.

Now we compare the maps $\mathcal{S}, \mathfrak{s}$, and for this recall two basic definitions from set theory. Let $f : A \to B$ be a map of partially ordered sets. We say that f is *order-reversing* if $x \leqslant y$ implies $f(x) \geqslant f(y)$ for all $x, y \in A$. Also, we call f an *order anti-embedding* if $x \leqslant y$ is equivalent to $f(x) \geqslant f(y)$ for all $x, y \in A$. Note that any order anti-embedding is an injection. We regard $\mathsf{Spec}\,R$ and $\mathtt{Spec}\,\mathsf{D}^{-}(R)$ as partially ordered sets with respect to the inclusion relations. The following theorem is the main result of this section.

Theorem 6.8. *The maps $\mathcal{S} : \mathsf{Spec}\,R \rightleftarrows \mathtt{Spec}\,\mathsf{D}^{-}(R) : \mathfrak{s}$ are order-reversing, and satisfy*

$$\mathfrak{s} \cdot \mathcal{S} = 1, \qquad \mathcal{S} \cdot \mathfrak{s} = \mathsf{Supp}^{-1}\,\mathsf{Supp}.$$

Hence, $\mathcal{S}$ is an order anti-embedding.

Proof. Let $\mathfrak{p}, \mathfrak{q}$ be prime ideals of R with $\mathfrak{q} \subseteq \mathfrak{p}$. Then Proposition 6.3 shows that $\mathfrak{q}$ is not in $\mathsf{Supp}\,\mathcal{S}(\mathfrak{p})$. Hence $X_{\mathfrak{q}} = 0$ for all $X \in \mathcal{S}(\mathfrak{p})$, which means that $\mathcal{S}(\mathfrak{p})$ is contained in $\mathcal{S}(\mathfrak{q})$. On the other hand, let $\mathcal{P}, \mathcal{Q}$ be prime thick $\otimes$-ideals of $\mathsf{D}^{-}(R)$ with $\mathcal{P} \subseteq \mathcal{Q}$. Then $\mathsf{Supp}\,\mathcal{P}$ is contained in $\mathsf{Supp}\,\mathcal{Q}$, and we see from the definition of $\mathfrak{s}$ that $\mathfrak{s}(\mathcal{P})$ contains $\mathfrak{s}(\mathcal{Q})$. Therefore, the maps $\mathcal{S}, \mathfrak{s}$ are order-reversing.

Fix a prime ideal $\mathfrak{p}$ of R. Then $\mathfrak{s}(\mathcal{S}(\mathfrak{p}))$ is the maximum element of $\mathbb{I}(\mathcal{S}(\mathfrak{p}))$, which consists of ideals I with $\mathrm{V}(I) \nsubseteq \mathsf{Supp}\,\mathcal{S}(\mathfrak{p})$. This is equivalent to saying that $I \subseteq \mathfrak{p}$ by Proposition 6.3. Hence $\mathfrak{s}(\mathcal{S}(\mathfrak{p})) = \mathfrak{p}$.

Let $\mathcal{P}$ be a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$. Note that a prime ideal $\mathfrak{p}$ of R belongs to $\mathbb{I}(\mathcal{P})$ if and only if $\mathfrak{p}$ is not in $\mathsf{Supp}\,\mathcal{P}$. Let $X \in \mathsf{D}^-(R)$ be a complex with $X_{\mathfrak{s}(\mathcal{P})} = 0$. If $\mathfrak{p}$ is a prime ideal of R with $X_{\mathfrak{p}} \neq 0$, then $\mathfrak{p}$ is not contained in $\mathfrak{s}(\mathcal{P})$, and $\mathfrak{p}$ must not belong to $\mathbb{I}(\mathcal{P})$, which means $\mathfrak{p} \in \mathsf{Supp}\,\mathcal{P}$. Therefore $\mathsf{Supp}\,X$ is contained in $\mathsf{Supp}\,\mathcal{P}$, and we obtain $\mathcal{S}(\mathfrak{s}(\mathcal{P})) \subseteq \mathsf{Supp}^{-1}\,\mathsf{Supp}\,\mathcal{P}$. Conversely, let $X \in \mathsf{D}^-(R)$ be a complex with $\mathsf{Supp}\,X \subseteq \mathsf{Supp}\,\mathcal{P}$. Since $\mathfrak{s}(\mathcal{P})$ is in $\mathbb{I}(\mathcal{P})$, it does not belong to $\mathsf{Supp}\,\mathcal{P}$. Hence $\mathfrak{s}(\mathcal{P})$ is not in $\mathsf{Supp}\,X$, which means $X_{\mathfrak{s}(\mathcal{P})} = 0$. We thus conclude that $\mathcal{S}(\mathfrak{s}(\mathcal{P})) = \mathsf{Supp}^{-1}\,\mathsf{Supp}\,\mathcal{P}$.

The last assertion is shown by using the equality $\mathfrak{p} = \mathfrak{s}(\mathcal{S}(\mathfrak{p}))$ for all prime ideals $\mathfrak{p}$ of R. ■

From this theorem, we obtain the following evaluation of the (Krull) dimension of $\mathtt{Spec}\,\mathsf{D}^-(R)$.

Proposition 6.9. (1) *Let $\mathcal{T}$ be an essentially small $\otimes$-triangulated category. The dimension of $\mathtt{Spec}\,\mathcal{T}$ is equal to the supremum of integers $n \geqslant 0$ such that there is a chain $\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \cdots \subsetneq \mathcal{P}_n$ in $\mathtt{Spec}\,\mathcal{T}$.*

(2) *There is an inequality*

$$\dim(\mathtt{Spec}\,\mathsf{D}^-(R)) \geqslant \dim R.$$

Proof. Applying Proposition 2.10 shows (1), while (2) follows from (1) and Theorem 6.8. ■

Remark 6.10. We will see that the inequality in Proposition 6.9(2) sometimes becomes equality, and sometimes becomes strict inequality. See Corollaries 7.14, 11.13 and Theorem 11.11.

The above theorem also gives rise to several corollaries, which will often be used later. The rest of this section is devoted to stating and proving them.

Corollary 6.11. *Let $\mathfrak{p}$ be a prime ideal of R, and let $\mathcal{P}$ a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$. It holds that:*

$$\mathfrak{p} \subseteq \mathfrak{s}(\mathcal{P}) \;\Leftrightarrow\; R/\mathfrak{p} \notin \mathcal{P} \;\Leftrightarrow\; \mathfrak{p} \notin \mathsf{Supp}\,\mathcal{P} \;\Leftrightarrow\; \mathcal{P} \subseteq \mathcal{S}(\mathfrak{p}).$$

In particular, $\mathfrak{s}(\mathcal{P})$ is the maximum element of $(\mathsf{Supp}\,\mathcal{P})^{\complement}$ with respect to the inclusion relation.

Proof. The second equivalence follows from Corollary 5.11, while the third one is trivial. If $\mathfrak{p} \notin \mathsf{Supp}\,\mathcal{P}$, then $\mathfrak{p} \subseteq \mathfrak{s}(\mathcal{P})$. If this is the case, then $\mathcal{S}(\mathfrak{p}) \supseteq \mathcal{S}(\mathfrak{s}(\mathcal{P})) = \mathsf{Supp}^{-1}\,\mathsf{Supp}\,\mathcal{P} \supseteq \mathcal{P}$ by Theorem 6.8. ■

Corollary 6.12. *For two prime thick $\otimes$-ideals $\mathcal{P}, \mathcal{Q}$ of $\mathsf{D}^-(R)$ one has:*

$$\mathfrak{s}(\mathcal{P}) \subseteq \mathfrak{s}(\mathcal{Q}) \;\Leftrightarrow\; \mathsf{Supp}\,\mathcal{P} \supseteq \mathsf{Supp}\,\mathcal{Q}, \qquad \mathfrak{s}(\mathcal{P}) = \mathfrak{s}(\mathcal{Q}) \;\Leftrightarrow\; \mathsf{Supp}\,\mathcal{P} = \mathsf{Supp}\,\mathcal{Q}.$$

Proof. Theorem 6.8 and Proposition 4.8(1) yield the first equivalence, which implies the second one. ■

Here we introduce a new class of thick tensor ideals, which will play main roles in the rest of this part.

Definition 6.13. A thick $\otimes$-ideal $\mathcal{X}$ of $\mathsf{D}^-(R)$ is called *tame* if one can write $\mathcal{X} = \mathsf{Supp}^{-1}\,S$ for some subset S of $\mathsf{Spec}\,R$. The set of tame prime thick $\otimes$-ideals of $\mathsf{D}^-(R)$ is denoted by ${}^{\mathsf{t}}\mathtt{Spec}\,\mathsf{D}^-(R)$.

Remark 6.14. For each subcategory $\mathcal{X}$ of $\mathsf{D}^{-}(R)$ the following are equivalent.

(1) $\mathcal{X}$ is a tame thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$.
(2) $\mathcal{X} = \mathsf{Supp}^{-1} S$ for some subset S of $\mathsf{Spec}\, R$.
(3) $\mathcal{X} = \mathsf{Supp}^{-1} W$ for some specialization-closed subset W of $\mathsf{Spec}\, R$.

This is a direct consequence of Proposition 4.8(1).

The following corollary of Theorem 6.8 gives an explicit description of tame prime thick tensor ideals.

Corollary 6.15. *It holds that*

$$^{\mathsf{t}}\mathsf{Spec}\, \mathsf{D}^{-}(R) = \mathsf{Im}\, \mathcal{S} = \{\mathcal{S}(\mathfrak{p}) \mid \mathfrak{p} \in \mathsf{Spec}\, R\}.$$

Proof. For a prime ideal $\mathfrak{p}$ of R, we have $\mathcal{S}(\mathfrak{p}) = \mathcal{S}\mathfrak{s}\mathcal{S}(\mathfrak{p}) = \mathsf{Supp}^{-1}(\mathsf{Supp}\, \mathcal{S}(\mathfrak{p}))$ by Theorem 6.8, which shows that the prime thick $\otimes$-ideal $\mathcal{S}(\mathfrak{p})$ of $\mathsf{D}^{-}(R)$ is tame. On the other hand, let $\mathcal{P}$ be a tame prime thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$. Using Theorem 6.8 and Proposition 4.8, we get $\mathcal{S}(\mathfrak{s}(\mathcal{P})) = \mathsf{Supp}^{-1}(\mathsf{Supp}\, \mathcal{P}) = \mathcal{P}$. ■

Here is one more application of Theorem 6.8, giving a criterion for a thick tensor ideal to be prime.

Corollary 6.16. *Let W be a specialization-closed subset of $\mathsf{Spec}\, R$. The following are equivalent.*

(1) *The tame thick $\otimes$-ideal $\mathsf{Supp}^{-1} W$ of $\mathsf{D}^{-}(R)$ is prime.*
(2) *There exists a prime ideal $\mathfrak{p}$ of R such that $W = \mathsf{Supp}\, \mathcal{S}(\mathfrak{p})$.*
(3) *There exists a prime thick $\otimes$-ideal $\mathcal{P}$ of $\mathsf{D}^{-}(R)$ such that $W = \mathsf{Supp}\, \mathcal{P}$.*
(4) *The set $W^{\complement}$ has a unique maximal element with respect to the inclusion relation.*

Proof. (1) $\Rightarrow$ (2): By Corollary 6.11, the complement of $W = \mathsf{Supp}(\mathsf{Supp}^{-1} W)$ (see Proposition 4.8(2)) has the maximum element $\mathfrak{p} := \mathfrak{s}(\mathsf{Supp}^{-1} W)$. Using Theorem 6.8, we obtain $W = \mathsf{Supp}\, \mathcal{S}(\mathfrak{p})$.

(2) $\Rightarrow$ (3): Take $\mathcal{P} = \mathcal{S}(\mathfrak{p})$, which is a prime thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$ by Proposition 6.3.

(3) $\Rightarrow$ (4): This implication follows from Corollary 6.11.

(4) $\Rightarrow$ (1): Let $\mathfrak{p}$ be a unique maximal element of $W^{\complement}$. We claim that there is an equality $W = \mathsf{Supp}\, \mathcal{S}(\mathfrak{p})$. Indeed, $\mathsf{Supp}\, \mathcal{S}(\mathfrak{p})$ consists of the prime ideals $\mathfrak{q}$ of R not contained in $\mathfrak{p}$ by Proposition 6.3. Now fix a prime ideal $\mathfrak{q}$ of R. Suppose that $\mathfrak{q}$ is in W. If $\mathfrak{q}$ is contained in $\mathfrak{p}$, then $\mathfrak{p}$ belongs to W as W is specialization-closed. This contradicts the choice of $\mathfrak{p}$, whence $\mathfrak{q}$ belongs to $\mathsf{Supp}\, \mathcal{S}(\mathfrak{p})$. Conversely, if $\mathfrak{q}$ is not in W, then $\mathfrak{q}$ is in $W^{\complement}$, and the choice of $\mathfrak{p}$ shows that $\mathfrak{q}$ is contained in $\mathfrak{p}$. Thus the claim follows. Applying Theorem 6.8, we obtain $\mathsf{Supp}^{-1} W = \mathcal{S}(\mathfrak{p})$ and this is a prime thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$. ■

7. More on the maps $\mathfrak{s}$ and $\mathcal{S}$

In this section, we study various topological properties of the maps $\mathcal{S}, \mathfrak{s}$ defined in the previous section. We first consider fibers of the map $\mathfrak{s} : \mathsf{Spec}\, \mathsf{D}^{-}(R) \to \mathsf{Spec}\, R$.

Proposition 7.1. *There is a direct sum decomposition of sets*

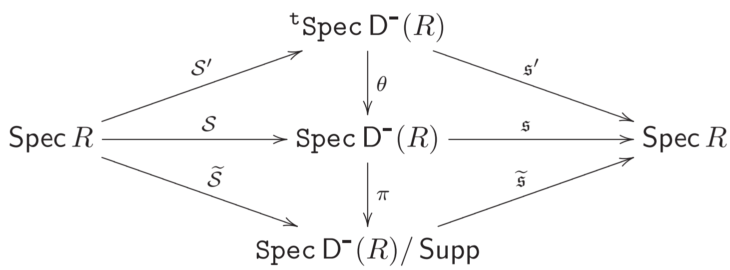
$$\mathsf{Spec}\, \mathsf{D}^{-}(R) = \textstyle\coprod_{\mathfrak{p} \in \mathsf{Spec}\, R} \mathfrak{s}^{-1}(\mathfrak{p}),$$

where $\mathfrak{s}^{-1}(\mathfrak{p}) := \{\mathcal{P} \in \mathsf{Spec}\, \mathsf{D}^{-}(R) \mid \mathfrak{s}(\mathcal{P}) = \mathfrak{p}\} = \{\mathcal{P} \in \mathsf{Spec}\, \mathsf{D}^{-}(R) \mid \mathsf{Supp}\, \mathcal{P} = \{\mathfrak{q} \in \mathsf{Spec}\, R \mid \mathfrak{q} \nsubseteq \mathfrak{p}\}\}$

Proof. Theorem 6.8 says that the map $\mathfrak{s}$ is surjective. Using this, we easily get the direct sum decomposition. Applying Theorem 6.8, Corollary 6.12 and Proposition 6.3, we observe that for any $\mathfrak{p} \in \mathsf{Spec}\, R$ and $\mathcal{P} \in \mathtt{Spec}\, \mathsf{D}^-(R)$ one has $\mathfrak{s}(\mathcal{P}) = \mathfrak{p}$ if and only if $\mathsf{Supp}\, \mathcal{P} = \{\mathfrak{q} \in \mathsf{Spec}\, R \mid \mathfrak{q} \nsubseteq \mathfrak{p}\}$. ■

Let $\mathcal{P}, \mathcal{Q}$ be prime thick $\otimes$-ideals of $\mathsf{D}^-(R)$. We write $\mathcal{P} \sim \mathcal{Q}$ if $\mathsf{Supp}\, \mathcal{P} = \mathsf{Supp}\, \mathcal{Q}$. Then $\sim$ defines an equivalence relation on $\mathtt{Spec}\, \mathsf{D}^-(R)$. We denote by $\mathtt{Spec}\, \mathsf{D}^-(R)/\,\mathsf{Supp}$ the quotient topological space of $\mathtt{Spec}\, \mathsf{D}^-(R)$ by the equivalence relation $\sim$, so that a subset U of $\mathtt{Spec}\, \mathsf{D}^-(R)/\,\mathsf{Supp}$ is open if and only if $\pi^{-1}(U)$ is open in $\mathtt{Spec}\, \mathsf{D}^-(R)$, where $\pi : \mathtt{Spec}\, \mathsf{D}^-(R) \to \mathtt{Spec}\, \mathsf{D}^-(R)/\,\mathsf{Supp}$ stands for the canonical surjection. By definition, π is a continuous map. Denote by $\theta : {}^{\mathtt{t}}\mathtt{Spec}\, \mathsf{D}^-(R) \to \mathtt{Spec}\, \mathsf{D}^-(R)$ the inclusion map, which is continuous. Now we can state our first main result in this section.

Theorem 7.2. (1) *The set* ${}^{\mathtt{t}}\mathtt{Spec}\, \mathsf{D}^-(R)$ *is dense in* $\mathtt{Spec}\, \mathsf{D}^-(R)$.
(2) *The composition* $\pi\theta$ *is a continuous bijection.*
(3) *The maps* $\mathcal{S}, \mathfrak{s}$ *induce the bijections* $\mathcal{S}', \widetilde{\mathcal{S}}, \mathfrak{s}', \widetilde{\mathfrak{s}}$ *which make the diagram below commute.*



In particular, one has $\mathfrak{s}\mathcal{S} = \mathfrak{s}'\mathcal{S}' = \widetilde{\mathfrak{s}}\widetilde{\mathcal{S}} = 1$.
(4) *The maps* $\mathfrak{s}, \mathfrak{s}', \widetilde{\mathfrak{s}}$ *are continuous. The maps* $\mathcal{S}', \widetilde{\mathcal{S}}$ *are open and closed.*

Proof. First of all, recall from Corollary 6.15 that the image of $\mathcal{S}$ coincides with ${}^{\mathtt{t}}\mathtt{Spec}\, \mathsf{D}^-(R)$.

(1) Let X be a complex in $\mathsf{D}^-(R)$, and suppose that $U := \mathtt{U}(X)$ is nonempty. Then U contains a prime thick $\otimes$-ideal $\mathcal{P}$ of $\mathsf{D}^-(R)$, and X is in $\mathcal{P}$. It is seen from Theorem 6.8 that $\mathcal{P}$ is contained in $\mathcal{S}(\mathfrak{s}(\mathcal{P}))$, and hence X is in $\mathcal{S}(\mathfrak{s}(\mathcal{P}))$. Therefore $\mathcal{S}(\mathfrak{s}(\mathcal{P}))$ belongs to the intersection $U \cap {}^{\mathtt{t}}\mathtt{Spec}\, \mathsf{D}^-(R)$, and we have $U \cap {}^{\mathtt{t}}\mathtt{Spec}\, \mathsf{D}^-(R) \neq \emptyset$. This shows that any nonempty open subset of $\mathtt{Spec}\, \mathsf{D}^-(R)$ meets ${}^{\mathtt{t}}\mathtt{Spec}\, \mathsf{D}^-(R)$.

(2) Since π and θ are continuous, so is $\pi\theta$. Let $\mathcal{P}, \mathcal{Q}$ be tame prime thick $\otimes$-ideals of $\mathsf{D}^-(R)$. Then $\mathcal{P} = \mathcal{S}\mathfrak{s}(\mathcal{P})$ and $\mathcal{Q} = \mathcal{S}\mathfrak{s}(\mathcal{Q})$ by Theorem 6.8. One has $\mathcal{P} \sim \mathcal{Q}$ if and only if $\mathfrak{s}(\mathcal{P}) = \mathfrak{s}(\mathcal{Q})$ by Corollary 6.12, if and only if $\mathcal{P} = \mathcal{Q}$ by Theorem 6.8 again. This shows that the map $\pi\theta$ is well-defined and injective. To show the surjectivity, pick a prime thick $\otimes$-ideal $\mathcal{R}$ of $\mathsf{D}^-(R)$. It is seen from Proposition 4.8(1) that $\mathcal{R} \sim \mathsf{Supp}^{-1}\, \mathsf{Supp}\, \mathcal{R}$, and the latter thick $\otimes$-ideal is tame. Consequently, $\pi\theta$ is a bijection.

(3) Using Theorem 6.8, we obtain the bijection $\mathcal{S}'$ satisfying $\theta\mathcal{S}' = \mathcal{S}$. Set $\widetilde{\mathcal{S}} = \pi\mathcal{S}$ and $\mathfrak{s}' = \mathfrak{s}\theta$. Define the map $\widetilde{\mathfrak{s}} : \mathtt{Spec}\, \mathsf{D}^-(R)/\,\mathsf{Supp} \to \mathsf{Spec}\, R$ by $\widetilde{\mathfrak{s}}([\mathcal{P}]) = \mathfrak{s}(\mathcal{P})$ for $\mathcal{P} \in \mathtt{Spec}\, \mathsf{D}^-(R)$. Corollary 6.12 guarantees that this is well-defined, and by definition we have $\widetilde{\mathfrak{s}}\pi = \mathfrak{s}$. Thus the commutative diagram in the assertion is obtained, which and Theorem 6.8 yield $1 = \mathfrak{s}\mathcal{S} = \mathfrak{s}'\mathcal{S}' = \widetilde{\mathfrak{s}}\widetilde{\mathcal{S}}$. It follows that the map $\mathcal{S}'$ is bijective, and so is $\mathfrak{s}'$. We have $\widetilde{\mathcal{S}} = (\pi\theta)\mathcal{S}'$, which is bijective by (2), and so is $\widetilde{\mathfrak{s}}$.

(4) Let $\mathcal{P} \in \mathtt{Spec}\, \mathsf{D}^-(R)$. An ideal I of R is contained in $\mathfrak{s}(\mathcal{P})$ if and only if $\mathrm{V}(I)$ is not contained in $\mathsf{Supp}\, \mathcal{P}$, if and only if R/I does not belong to $\mathcal{P}$ by Corollary 5.11. We

obtain an equality

$$\mathfrak{s}^{-1}(\mathrm{V}(I)) = \mathsf{BSupp}\, R/I,$$

which shows that $\mathfrak{s}$ is a continuous map. Since the map θ is continuous, so is the composition $\mathfrak{s}' = \mathfrak{s}\theta$. The equality $\mathfrak{s}' = \mathcal{S}'^{-1}$ from (3) and the continuity of $\mathfrak{s}'$ imply that the map $\mathcal{S}'$ is open and closed.

Fix an ideal I of R. A prime ideal $\mathfrak{p}$ of R is in $\mathrm{D}(I)$ if and only if $\mathcal{S}(\mathfrak{p})$ is in $\mathtt{U}(R/I)$. This shows $\mathcal{S}(\mathrm{D}(I)) = \mathtt{U}(R/I) \cap {}^{\mathsf{t}}\mathsf{Spec}\,\mathsf{D}^{-}(R)$, and we get $\pi^{-1}\widetilde{\mathcal{S}}(\mathrm{D}(I)) = \pi^{-1}\pi\mathcal{S}(\mathrm{D}(I)) = \pi^{-1}\pi(\mathtt{U}(R/I) \cap {}^{\mathsf{t}}\mathsf{Spec}\,\mathsf{D}^{-}(R))$. Let $\mathcal{P} \in \mathsf{Spec}\,\mathsf{D}^{-}(R)$ and $\mathcal{Q} \in {}^{\mathsf{t}}\mathsf{Spec}\,\mathsf{D}^{-}(R)$. One has $\pi(\mathcal{P}) = \pi(\mathcal{Q})$ if and only if $\mathsf{Supp}\,\mathcal{P} = \mathsf{Supp}\,\mathcal{Q}$, if and only if $\mathsf{Supp}^{-1}\,\mathsf{Supp}\,\mathcal{P} = \mathcal{Q}$ since $\mathsf{Supp}^{-1}\,\mathsf{Supp}\,\mathcal{Q} = \mathcal{Q}$ by Proposition 4.8. Hence $\mathcal{P}$ is in $\pi^{-1}\pi(\mathtt{U}(R/I) \cap {}^{\mathsf{t}}\mathsf{Spec}\,\mathsf{D}^{-}(R))$ if and only if $\mathsf{Supp}^{-1}\,\mathsf{Supp}\,\mathcal{P}$ contains R/I (note here that $\mathsf{Supp}^{-1}\,\mathsf{Supp}\,\mathcal{P}$ is in ${}^{\mathsf{t}}\mathsf{Spec}\,\mathsf{D}^{-}(R)$ by Theorem 6.8), if and only if $\mathsf{Supp}\,\mathcal{P}$ contains $\mathrm{V}(I)$, if and only if R/I belongs to $\mathcal{P}$ by Corollary 5.11. Thus we obtain $\pi^{-1}\widetilde{\mathcal{S}}(\mathrm{D}(I)) = \mathtt{U}(R/I)$, which shows that $\widetilde{\mathcal{S}}(\mathrm{D}(I))$ is an open subset of $\mathsf{Spec}\,\mathsf{D}^{-}(R)/\,\mathsf{Supp}$. Therefore $\widetilde{\mathcal{S}}$ is an open map. This map is also closed since it is bijective. Combining the equality $\widetilde{\mathfrak{s}} = \widetilde{\mathcal{S}}^{-1}$ from (3) and the openness of $\widetilde{\mathcal{S}}$, we observe that $\widetilde{\mathfrak{s}}$ is a continuous map. ■

The assertions of the above theorem naturally lead us to ask when the maps in the diagram in the theorem are homeomorphisms. We start by establishing a lemma.

Lemma 7.3. *The following are equivalent.*

(1) *The set* $\mathsf{Spec}\, R$ *is finite.*
(2) *There are only finitely many specialization-closed subsets of* $\mathsf{Spec}\, R$*.*
(3) *There are only finitely many closed subsets of* $\mathsf{Spec}\, R$*.*
(4) *Every specialization-closed subset of* $\mathsf{Spec}\, R$ *is closed.*
(5) *There are no countable antichains of prime ideals of* R*.*

Proof. (1) ⇒ (2): If $\mathsf{Spec}\, R$ is finite, then there are only finitely many subsets of $\mathsf{Spec}\, R$.

(2) ⇒ (3): This implication follows from the fact that any closed subset is specialization-closed.

(3) ⇒ (4): Every specialization-closed subset is a union of closed subsets. This is a finite union by assumption, and hence it is closed.

(4) ⇒ (5): Since $\mathrm{Max}\, R$ is specialization-closed, it is closed by our assumption. Hence $\mathrm{Max}\, R$ possesses only finitely many minimal elements with respect to the inclusion relation, which means that it is a finite set. Therefore the ring R is semilocal. In particular, it has finite Krull dimension, say d.

Suppose that there is a countable antichain S of prime ideals of R. Then there exists an integer $0 \leqslant n \leqslant d$ such that the set $S_n := \{\mathfrak{p} \in S \mid \mathrm{ht}\,\mathfrak{p} = n\}$ is infinite. Then the specialization-closed subset $W = \bigcup_{\mathfrak{p} \in S_n} \mathrm{V}(\mathfrak{p})$ is not closed because S_n consists of the minimal elements of W, which is an infinite set. This provides a contradiction, and consequently, there are no countable antichains of prime ideals of R.

(5) ⇒ (1): Assume that $\mathsf{Spec}\, R$ is infinite. As every subset of $\mathrm{Max}\, R$ forms an antichain of prime ideals, we may assume that R has only finitely many maximal ideals. Then R has finite Krull dimension d. Since R has infinitely many prime ideals, there is a non-negative integer $0 \leqslant n \leqslant d$ such that the set $\{\mathfrak{p} \in \mathsf{Spec}\, R \mid \mathrm{ht}\,\mathfrak{p} = n\}$ has infinitely many elements. Then a countable subset of this set is a countable antichain of prime ideals of R, a contradiction. ■

Now we can prove the following theorem, which answers the question stated just before the lemma.

Theorem 7.4. *Consider the following seven conditions.*
(1) $\mathcal{S}$ *is continuous.* (2) $\mathcal{S}'$ *is homeomorphic.* (3) $\mathfrak{s}'$ *is homeomorphic.* (4) $\widetilde{\mathcal{S}}$ *is homeomorphic.*
(5) $\widetilde{\mathfrak{s}}$ *is homeomorphic.* (6) $\pi\theta$ *is homeomorphic.* (7) $\mathsf{Spec}\, R$ *is finite.*
Then the following implications hold:

$$\begin{array}{ccccccccc} (1) & \Longleftrightarrow & (2) & \Longleftrightarrow & (3) & \Longleftrightarrow & (5+6) & \Longleftrightarrow & (7) \\ & & & & & \swarrow & & \searrow & \\ & & (4) & \Longleftrightarrow & (5) & & & & (6) \end{array}$$

Proof. In this proof we tacitly use Theorem 7.2.

(2) $\Leftrightarrow$ (3): Note that $\mathcal{S}'$ and $\mathfrak{s}'$ are mutually inverse bijections. The equivalence follows from this.

(4) $\Leftrightarrow$ (5): As $\widetilde{\mathcal{S}}, \widetilde{\mathfrak{s}}$ are mutually inverse bijections, we have the equivalence.

(7) $\Rightarrow$ (2): For each $X \in \mathsf{D}^-(R)$ we have $\mathcal{S}'^{-1}(\mathsf{BSupp}\, X \cap {}^{\mathsf{t}}\mathsf{Spec}\, \mathsf{D}^-(R)) = \{\mathfrak{p} \in \mathsf{Spec}\, R \mid \mathcal{S}(\mathfrak{p}) \in \mathsf{BSupp}\, X\} = \mathsf{Supp}\, X$. As $\mathsf{Supp}\, X$ is specialization-closed, it is closed by Lemma 7.3. Hence the map $\mathcal{S}'$ is continuous.

(2) $\Rightarrow$ (1): This follows from the fact that $\mathcal{S}$ is the composition of the continuous maps $\mathcal{S}'$ and θ.

(1) $\Rightarrow$ (7): It is easy to observe that for any complex $X \in \mathsf{D}^-(R)$ one has

$$\mathcal{S}^{-1}(\mathsf{BSupp}\, X) = \mathsf{Supp}\, X. \tag{7.4.1}$$

Since $\mathcal{S}$ is continuous, $\mathsf{Supp}\, X$ is closed in $\mathsf{Spec}\, R$ for all $X \in \mathsf{D}^-(R)$ by (7.4.1). Suppose that $\mathsf{Spec}\, R$ is an infinite set. Then by Lemma 7.3 there is a non-closed specialization-closed subset W of $\mathsf{Spec}\, R$. There are infinitely many minimal elements of W with respect to the inclusion relation, and we can choose countably many pairwise distinct minimal elements $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \dots$ of W. Consider the complex $X = \bigoplus_{i=1}^{\infty}(R/\mathfrak{p}_i)[i] \in \mathsf{D}^-(R)$. Then $\mathsf{Supp}\, X = \bigcup_{i=1}^{\infty} \mathrm{V}(\mathfrak{p}_i)$ is not closed since it has infinitely many minimal elements. This contradiction shows that $\mathsf{Spec}\, R$ is a finite set.

(2) $\Rightarrow$ (4+6): Since $\pi, \theta, \mathcal{S}'$ are all continuous, so is $\widetilde{\mathcal{S}} = \pi\theta\mathcal{S}'$. Combining this with the fact that $\widetilde{\mathcal{S}}$ is bijective and open, we see that $\widetilde{\mathcal{S}}$ is a homeomorphism. As $\mathcal{S}'$ is homeomorphic, so is $\pi\theta = \widetilde{\mathcal{S}}\mathcal{S}'^{-1}$.

(4+6) $\Rightarrow$ (2): We have $\mathcal{S}' = (\pi\theta)^{-1}\widetilde{\mathcal{S}}$. Since $\pi\theta$ and $\widetilde{\mathcal{S}}$ are homeomorphisms, so is $\mathcal{S}'$. ■

This theorem contains the following corollary.

Corollary 7.5. *The following are equivalent:*
(1) $\mathcal{S} : \mathsf{Spec}\, R \to \mathsf{Spec}\, \mathsf{D}^-(R)$ *is an immersion.*
(2) $\mathfrak{s} : \mathsf{Spec}\, \mathsf{D}^-(R) \to \mathsf{Spec}\, R$ *is a quotient map.*
(3) $\mathsf{Spec}\, R$ *is a finite set.*

Next we consider the maximal and minimal elements of $\mathsf{Spec}\, \mathsf{D}^-(R)$ with respect to the inclusion relation.

Definition 7.6. Let $\mathcal{T}$ be an essentially small tensor triangulated category.

(1) A thick $\otimes$-ideal $\mathcal{M}$ of $\mathcal{T}$ is said to be *maximal* if $\mathcal{M} \neq \mathcal{T}$ and there is no thick $\otimes$-ideal $\mathcal{X}$ of $\mathcal{T}$ with $\mathcal{M} \subsetneq \mathcal{X} \subsetneq \mathcal{T}$. We denote the set of maximal thick $\otimes$-ideals of $\mathcal{T}$ by $\mathtt{Mx}\,\mathcal{T}$. According to Proposition 2.9(2), any maximal thick $\otimes$-ideal is prime, or in other words, it holds that $\operatorname{Max}\mathcal{T} \subseteq \mathtt{Spec}\,\mathcal{T}$.
(2) A prime thick $\otimes$-ideal $\mathcal{P}$ of $\mathcal{T}$ is said to be *minimal* if it is minimal in $\mathtt{Spec}\,\mathcal{T}$ with respect to the inclusion relation. We denote the set of minimal prime thick $\otimes$-ideals of $\mathcal{T}$ by $\operatorname{Min}\mathcal{T}$.

By Proposition 7.1 the Balmer spectrum of $\mathsf{D}^{-}(R)$ is decomposed into the fibers by $\mathfrak{s} : \mathsf{Spec}\,\mathsf{D}^{-}(R) \to \mathsf{Spec}\,R$ as a set. Concerning the fibers of maximal ideals and minimal primes of R, we have the following.

Proposition 7.7. *Let $\mathfrak{m} \in \operatorname{Max} R$ and $\mathfrak{p} \in \operatorname{Min} R$. Then*

$$\operatorname{Min}\mathfrak{s}^{-1}(\mathfrak{m}) \subseteq \operatorname{Min}\mathsf{D}^{-}(R), \qquad \operatorname{Max}\mathfrak{s}^{-1}(\mathfrak{p}) \subseteq \operatorname{Max}\mathsf{D}^{-}(R).$$

Proof. Take $\mathcal{P} \in \operatorname{Min}\mathfrak{s}^{-1}(\mathfrak{m})$, and let $\mathcal{Q}$ be a prime thick $\otimes$-ideal contained in $\mathcal{P}$. Then $\mathfrak{m} = \mathfrak{s}(\mathcal{P}) \subseteq \mathfrak{s}(\mathcal{Q})$ by Theorem 6.8. Since $\mathfrak{m}$ is a maximal ideal, we get $\mathfrak{m} = \mathfrak{s}(\mathcal{P}) = \mathfrak{s}(\mathcal{Q})$, and $\mathcal{Q} \in \mathfrak{s}^{-1}(\mathfrak{m})$. The minimality of $\mathcal{P}$ implies $\mathcal{P} = \mathcal{Q}$. Thus the first inclusion follows. The second inclusion is obtained similarly. ■

Next, we compare maximal elements of the Balmer support $\mathsf{BSupp}\,C$ with minimal elements of the usual support $\mathsf{Supp}\,C$ for a bounded complex $C \in \mathsf{D}^{\mathsf{b}}(R)$. By Theorem 5.12, $C \in \mathcal{X}$ if and only if $\mathsf{Supp}\,C \subseteq \mathsf{Supp}\,\mathcal{X}$ for a thick $\otimes$-ideal $\mathcal{X}$. By combining this with Theorem 6.8, one has $\mathcal{P} \in \mathsf{BSupp}\,C$ if and only if $\mathcal{S}\mathfrak{s}(\mathcal{P}) \in \mathsf{BSupp}\,C$ for a prime thick tensor ideal $\mathcal{P}$ of $\mathsf{D}^{-}(R)$.

Lemma 7.8. (1) $\mathfrak{s}(\mathsf{BSupp}\,C) = \mathsf{Supp}\,C$.
(2) $\mathcal{S}(\mathsf{Supp}\,C) \subseteq \mathsf{BSupp}\,C$.

Proof. (1) For a prime thick tensor ideal $\mathcal{P}$ in $\mathsf{BSupp}\,C$, we have the following equivalences.

$$\begin{aligned}\mathcal{P} \in \mathfrak{s}^{-1}(\mathsf{Supp}\,C) &\Leftrightarrow \mathfrak{s}(\mathcal{P}) \in \mathsf{Supp}\,C\\ &\Leftrightarrow \mathsf{Supp}\,C \not\subseteq \mathsf{Supp}\,\mathcal{P} = \{\mathfrak{p} \in \mathsf{Spec}\,R \mid \mathfrak{p} \not\subseteq \mathfrak{s}(\mathcal{P})\}\\ &\Leftrightarrow C \notin \mathcal{P}\\ &\Leftrightarrow \mathcal{P} \in \mathsf{BSupp}\,C.\end{aligned}$$

Here, the first and the last equivalences are clear. Since $\{\mathfrak{p} \in \mathsf{Spec}\,R \mid \mathfrak{p} \not\subseteq \mathfrak{s}(\mathcal{P})\}$ is the largest specialization closed subset of $\mathsf{Spec}\,R$ not containing $\mathfrak{s}(\mathcal{P})$, the second equivalence holds. The third one follows from the above discussion. As a result, we obtain $\mathsf{Supp}\,C = \mathfrak{s}(\mathfrak{s}^{-1}(\mathsf{Supp}\,C)) = \mathfrak{s}(\mathsf{BSupp}\,C)$ since $\mathfrak{s}$ is surjective.

(2) For an element $\mathfrak{p} \in \mathsf{Supp}\,C$, since $C \notin \mathcal{S}(\mathfrak{p})$ $\mathsf{Supp}\,C \not\subseteq \mathsf{Supp}\,\mathcal{S}(\mathfrak{p}) = \{\mathfrak{q} \in \mathsf{Spec}\,R \mid \mathfrak{q} \not\subseteq \mathfrak{s}\mathcal{S}(\mathfrak{p}) = \mathfrak{p}\}$ shows $C \notin \mathcal{S}(\mathfrak{p})$. Thus, we obtain $\mathcal{S}(\mathfrak{p}) \in \mathsf{BSupp}\,C$. ■

From this lemma, the maps

$$\mathfrak{s} : \mathsf{Spec}\,\mathsf{D}^{-}(R) \rightleftarrows \mathsf{Spec}\,R : \mathcal{S}$$

restrict to maps

$$\mathfrak{s} : \mathsf{BSupp}\,C \rightleftarrows \mathsf{Supp}\,C : \mathcal{S}.$$

Now we can prove the following theorem.

Theorem 7.9. *Let C be an object of $\mathsf{D}^{\mathsf{b}}(R)$. The above pair of maps induce mutually inverse bijections*

$$\mathfrak{s} : \operatorname{Max}\mathsf{BSupp}\,C \rightleftarrows \operatorname{Min}\mathsf{Supp}\,C : \mathcal{S}.$$

Proof. Because $\mathcal{S} : \mathsf{Spec}\, R \to \mathsf{Spec}\, \mathsf{D}^{-}(R)$ is injective, we have only to check that the map $\mathcal{S} : \mathrm{Min}\, \mathsf{Supp}\, C \to \mathrm{Max}\, \mathsf{BSupp}\, C$ is well-defined and surjective.

Let $\mathfrak{p}$ be a minimal element of $\mathsf{Supp}\, C$. We show that $\mathcal{S}(\mathfrak{p})$ is a maximal element of $\mathsf{BSupp}\, C$. Take a prime thick tensor ideal $\mathcal{P}$ in $\mathsf{BSupp}\, C$ containing $\mathcal{S}(\mathfrak{p})$. Then $\mathfrak{s}(\mathcal{P}) \subseteq \mathfrak{s}\mathcal{S}(\mathfrak{p}) = \mathfrak{p}$ by Theorem 6.8. Since both $\mathfrak{p}$ and $\mathfrak{s}(\mathcal{P})$ belong to $\mathsf{Supp}\, C$ by Lemma 7.8, the minimality of $\mathfrak{p}$ shows the equality $\mathfrak{p} = \mathfrak{s}(\mathcal{P})$. Hence, we have

$$\mathsf{Supp}\, \mathcal{P} = \{\mathfrak{q} \in \mathsf{Spec}\, R \mid \mathfrak{q} \not\subseteq \mathfrak{s}(\mathcal{P}) = \mathfrak{p} = \mathfrak{s}\mathcal{S}(\mathfrak{p})\} = \mathsf{Supp}\, \mathcal{S}(\mathfrak{p}).$$

This shows that $\mathcal{P} \subseteq \mathcal{S}(\mathfrak{p})$ and thus $\mathcal{S}(\mathfrak{p})$ is a maximal element in $\mathsf{BSupp}\, C$. For this reason, the map $\mathcal{S} : \mathrm{Min}\, \mathsf{Supp}\, C \to \mathrm{Max}\, \mathsf{BSupp}\, C$ is well-defined.

Next we check the surjectivity of the map $\mathcal{S} : \mathrm{Min}\, \mathsf{Supp}\, C \to \mathrm{Max}\, \mathsf{BSupp}\, C$. Let $\mathcal{P}$ be a maximal element of $\mathsf{BSupp}\, C$. It follows from Lemma 7.8 that $\mathcal{S}\mathfrak{s}(\mathcal{P})$ is also an element in $\mathsf{BSupp}\, C$. On the other hand, $\mathcal{S}\mathfrak{s}(\mathcal{P})$ contains $\mathcal{P}$ by Theorem 6.8. Thus, we get $\mathcal{P} = \mathcal{S}\mathfrak{s}(\mathcal{P})$ from the maximality of $\mathcal{P}$. Let $\mathfrak{p}$ be an element of $\mathsf{Supp}\, C$ with $\mathfrak{p} \subseteq \mathfrak{s}(\mathcal{P})$. Then $\mathcal{P} = \mathcal{S}\mathfrak{s}(\mathcal{P}) \subseteq \mathcal{S}(\mathfrak{p})$. Since $\mathcal{P}$ is maximal in $\mathsf{BSupp}\, C$, one has $\mathcal{P} = \mathcal{S}(\mathfrak{p})$. Hence, $\mathfrak{p} = \mathfrak{s}\mathcal{S}(\mathfrak{p}) = \mathfrak{s}(\mathcal{P})$ and this shows that $\mathfrak{s}(\mathcal{P})$ is a minimal element of $\mathsf{Supp}\, C$. As a result, one has $\mathcal{S}(\mathfrak{p}) = \mathcal{S}\mathfrak{s}(\mathcal{P}) = \mathcal{P}$ and this shows that $\mathcal{S} : \mathrm{Min}\, \mathsf{Supp}\, C \to \mathrm{Max}\, \mathsf{BSupp}\, C$ is surjective. ■

Taking $C = R$, we get the following corollary. This especially says that $\mathsf{D}^{-}(R)$ is "semilocal" in the sense that $\mathsf{D}^{-}(R)$ admits only a finite number of maximal thick tensor ideals. If R is an integral domain, then $\mathsf{D}^{-}(R)$ is "local" in the sense that $\mathsf{D}^{-}(R)$ has a unique maximal thick tensor ideal.

Corollary 7.10. *There is a mutually inverse bijections*

$$\mathfrak{s} : \mathrm{Max}\, \mathsf{D}^{-}(R) \rightleftarrows \mathrm{Min}\, R : \mathcal{S}.$$

The following theorem is opposite to Theorem 7.9.

Theorem 7.11. *Let C be an object of $\mathsf{D}^{\mathsf{b}}(R)$.*

(1) *The map $\mathcal{S} : \mathsf{Supp}\, C \to \mathsf{BSupp}\, C$ restricts to an injective map*

$$\mathcal{S} : \mathrm{Max}\, \mathsf{Supp}\, C \hookrightarrow \mathrm{Min}\, \mathsf{BSupp}\, C.$$

(2) *Assume that $\mathrm{Max}\, \mathsf{Supp}\, C$ consists of finitely many maximal ideals of R. Then the maps $\mathfrak{s}$ and $\mathcal{S}$ restrict to mutually inverse bijections*

$$\mathfrak{s} : \mathrm{Max}\, \mathsf{BSupp}\, C \rightleftarrows \mathrm{Min}\, \mathsf{Supp}\, C : \mathcal{S}.$$

Proof. (1) Fix a maximal element $\mathfrak{m}$ of $\mathsf{Supp}\, C$ and show that $\mathcal{S}(\mathfrak{m})$ is minimal in $\mathsf{BSupp}\, C$. Note that $\mathfrak{m}$ is a maximal ideal. Take $\mathcal{P} \in \mathsf{BSupp}\, C$ with $\mathcal{P} \subseteq \mathcal{S}(\mathfrak{m})$. Then for any $X \in \mathcal{S}(\mathfrak{m})$, $\mathsf{Supp}(X \otimes^{\mathbf{L}}_R R/\mathfrak{m}) = \mathsf{Supp}\, X \cap \{\mathfrak{m}\} = \emptyset$. Remark 4.6 shows $X \otimes^{\mathbf{L}}_R R/\mathfrak{m} = 0$, which belongs to $\mathcal{P}$. As $\mathcal{P}$ is prime, either X or $R/\mathfrak{m}$ is in $\mathcal{P}$. Since $\mathcal{S}(\mathfrak{m})$ does not contain $R/\mathfrak{m}$, neither does $\mathcal{P}$. Therefore X must be in $\mathcal{P}$, and we obtain $\mathcal{P} = \mathcal{S}(\mathfrak{m})$. This shows that the prime thick $\otimes$-ideal $\mathcal{S}(\mathfrak{m})$ is minimal. Thus, $\mathcal{S}$ induces a map $\mathrm{Max}\, \mathsf{Supp}\, C \to \mathrm{Min}\, \mathsf{BSupp}\, C$. The injectivity follows from Theorem 6.8.

(2) We have already shown that the map $\mathcal{S} : \mathrm{Max}\, \mathsf{Supp}\, C \to \mathrm{Min}\, \mathsf{BSupp}\, C$ is injective. Hence it suffices to show that this map is surjective.

Write $\mathrm{Max}\, \mathsf{Supp}\, C = \{\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_r\}$. Suppose that $\mathcal{P}$ is a minimal element of $\mathsf{BSupp}\, C$ with $\mathcal{P} \neq \mathcal{S}(\mathfrak{m}_i)$ for any i. For each $1 \leqslant i \leqslant r$ we find an object X_i of $\mathsf{D}^{-}(R)$ with $X_i \in \mathcal{S}(\mathfrak{m}_i)$ and $X_i \notin \mathcal{P}$. Then $C \otimes^{\mathbf{L}}_R X_1 \otimes^{\mathbf{L}}_R X_2 \otimes^{\mathbf{L}}_R \cdots \otimes^{\mathbf{L}}_R X_r$ has the empty support and hence isomorphic to 0 by Remark 4.6. As $\mathcal{P}$ is prime, it contains some X_i, which is a contradiction.

Consequently, $\mathcal{P}$ must contain $\mathcal{S}(\mathfrak{m})$ for some $\mathfrak{m} \in \operatorname{Max} \mathsf{Supp}\, C$. The minimality of $\mathcal{P}$ shows that $\mathcal{P} = \mathcal{S}(\mathfrak{m})$. We conclude that the map $\mathcal{S} : \operatorname{Max} \mathsf{Supp}\, C \hookrightarrow \operatorname{Min} \mathsf{BSupp}\, C$ is surjective, whence it is bijective. ■

Again taking $C = R$, we obtain the following opposite result of Corollary 7.10. The third assertion says that if R is local, then $\mathsf{D}^-(R)$ is an "integral domain" in the sense that $\mathbf{0}$ is a (unique) minimal prime thick tensor ideal of $\mathsf{D}^-(R)$.

Corollary 7.12. (1) *For every maximal ideal* $\mathfrak{m}$ *of* R*, the subcategory* $\mathcal{S}(\mathfrak{m})$ *is a minimal prime thick* $\otimes$*-ideal of* $\mathsf{D}^-(R)$*, or in other words, the restriction of* $\mathcal{S}$ *to* $\operatorname{Max} R$ *induces an injection*

(7.12.1) $$\mathcal{S} : \operatorname{Max} R \hookrightarrow \mathtt{Mn}\, \mathsf{D}^-(R).$$

(2) *The ring* R *is semilocal if and only if* $\mathsf{D}^-(R)$ *has only finitely many minimal prime thick* $\otimes$*-ideals. When this is the case, the map* (7.12.1) *is a bijection.*

(3) *If* $(R, \mathfrak{m})$ *is a local ring, then* $\mathcal{S}(\mathfrak{m}) = \mathbf{0}$ *is a unique minimal prime thick* $\otimes$*-ideal of* $\mathsf{D}^-(R)$*.*

Question 7.13. Is the map (7.12.1) bijective even if R is not semilocal?

Applying the above two theorems to the Artinian case gives rise to the following result.

Corollary 7.14. *Let* R *be an Artinian ring. Then the map* $\mathcal{S} : \mathsf{Spec}\, R \to \mathtt{Spec}\, \mathsf{D}^-(R)$ *is a homeomorphism. Hence the topological space* $\mathtt{Spec}\, \mathsf{D}^-(R)$ *is Noetherian, and one has* $\dim(\mathtt{Spec}\, \mathsf{D}^-(R)) = \dim R = 0 < \infty$*.*

Proof. Since $\mathsf{Spec}\, R = \operatorname{Min} R = \operatorname{Max} R$, the assertion is deduced from Corollaries 7.10 and 7.12(2). ■

From here we consider when $\mathsf{D}^-(R)$ is a local tensor triangulated category. Let us recall the definition.

Definition 7.15. (1) A topological space X is called *local* if for any open cover $X = \bigcup_{i \in I} U_i$ of X there exists $t \in I$ such that $X = U_t$. In particular, any local topological space is quasi-compact.
(2) An essentially small tensor triangulated category $\mathcal{T}$ is called *local* if $\mathtt{Spec}\, \mathcal{T}$ is a local topological space.

Remark 7.16. It is clear that the topological space $\mathsf{Spec}\, R$ is local if and only if the ring R is local.

For an essentially small $\otimes$-triangulated category $\mathcal{T}$ the following are equivalent ([Bal10a, Proposition 4.2]).

(i) $\mathcal{T}$ is local.
(ii) $\mathcal{T}$ has a unique minimal prime thick $\otimes$-ideal.
(iii) The radical thick $\otimes$-ideal $\sqrt{\mathbf{0}}$ of $\mathcal{T}$ is prime.

If moreover $\mathcal{T}$ is *rigid*, then the above three conditions are equivalent to:

(iv) The zero subcategory $\mathbf{0}$ of $\mathcal{T}$ is a prime thick $\otimes$-ideal.

Also, it follows from [Bal10a, Example 4.4] that $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$ is local if and only if so is R.

The following result says that the same statements hold for $\mathsf{D}^-(R)$. Also, we emphasize that it contains the equivalent condition (4), even though $\mathsf{D}^-(R)$ is not rigid; see Remark 4.1.

Corollary 7.17. *The following are equivalent.*

(1) *The $\otimes$-triangulated category $\mathsf{D}^{-}(R)$ is local.*
(2) *There is a unique minimal thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$.*
(3) *The radical thick $\otimes$-ideal $\sqrt{\mathbf{0}}$ of $\mathsf{D}^{-}(R)$ is prime.*
(4) *The zero subcategory $\mathbf{0}$ of $\mathsf{D}^{-}(R)$ is a prime thick $\otimes$-ideal.*
(5) *The ring R is local.*

Proof. Combining Theorem 7.12 with the result given just before the corollary, we observe that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) $\Rightarrow$ (4) hold. If $\mathbf{0}$ is prime, then it is easy to see that $\sqrt{\mathbf{0}} = \mathbf{0}$. Thus (4) implies (3). ■

One can indeed obtain more precise information on the structure of $\mathtt{Spec}\,\mathsf{D}^{-}(R)$ than Corollary 7.17:

Proposition 7.18. *One has*

$$\mathtt{Spec}\,\mathsf{D}^{-}(R) = \begin{cases} \mathtt{U}(R/\mathfrak{m}) \sqcup \{\mathbf{0}\} & \text{if } (R,\mathfrak{m}) \text{ is local,} \\ \bigcup_{\mathfrak{m}\in \operatorname{Max} R} \mathtt{U}(R/\mathfrak{m}) & \text{if } R \text{ is non-local.} \end{cases}$$

If $\mathfrak{m}, \mathfrak{n}$ are distinct maximal ideals of R, then $\mathtt{Spec}\,\mathsf{D}^{-}(R) = \mathtt{U}(R/\mathfrak{m}) \cup \mathtt{U}(R/\mathfrak{n})$.

Proof. Suppose that $(R,\mathfrak{m},k)$ is a local ring. Corollary 7.17 implies that $\mathbf{0}$ is prime, and $\mathtt{Spec}\,\mathsf{D}^{-}(R)$ contains $\mathtt{U}(k) \cup \{\mathbf{0}\}$. Let $\mathcal{P}$ be a nonzero prime thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$. Then there exists an object $X \neq 0$ in $\mathcal{P}$. By Remark 4.6 the support of X is nonempty and specialization-closed, whence contains $\mathfrak{m}$. Using Lemma 4.7(4), we have $\mathsf{Supp}(X \otimes_R^{\mathbf{L}} k) = \mathsf{Supp}\, X \cap \mathsf{Supp}\, k = \{\mathfrak{m}\} \neq \emptyset$. Hence $X \otimes_R^{\mathbf{L}} k$ is nonzero by Remark 4.6 again. Since $X \otimes_R^{\mathbf{L}} k$ is isomorphic to a direct sum of shifts of k-vector spaces, it contains $k[n]$ as a direct summand for some $n \in \mathbb{Z}$. As $X \otimes_R^{\mathbf{L}} k$ is in $\mathcal{P}$, so is k. Therefore $\mathcal{P}$ is in $\mathtt{U}(k)$, and we obtain $\mathtt{Spec}\,\mathsf{D}^{-}(R) = \mathtt{U}(k) \cup \{\mathbf{0}\}$. It is obvious that $\mathtt{U}(k) \cap \{\mathbf{0}\} = \emptyset$. We conclude that $\mathtt{Spec}\,\mathsf{D}^{-}(R) = \mathtt{U}(k) \sqcup \{\mathbf{0}\}$.

Now, let $\mathfrak{m}$ and $\mathfrak{n}$ be distinct maximal ideals of R. Applying Lemma 4.7(4), we have $\mathsf{Supp}(R/\mathfrak{m} \otimes_R^{\mathbf{L}} R/\mathfrak{n}) = \{\mathfrak{m}\} \cap \{\mathfrak{n}\} = \emptyset$, and hence $R/\mathfrak{m} \otimes_R^{\mathbf{L}} R/\mathfrak{n} = 0$ by Remark 4.6. Therefore we obtain $\mathtt{U}(R/\mathfrak{m}) \cup \mathtt{U}(R/\mathfrak{n}) = \mathtt{U}(R/\mathfrak{m} \otimes_R^{\mathbf{L}} R/\mathfrak{n}) = \mathtt{U}(0) = \mathtt{Spec}\,\mathsf{D}^{-}(R)$, where the first equality follows from Lemma 2.7. Thus the last assertion of the proposition follows, which shows the first assertion in the non-local case. ■

8. **Connectedness of the Balmer spectrum**

In this section, we discuss Noetherianity, connectedness and irreducibility of the Balmer spectrum $\mathtt{Spec}\,\mathsf{D}^{-}(R)$. Besides, we prove that the following theorem which gives a necessary condition for the Balmer spectrum $\mathtt{Spec}\,\mathsf{D}^{-}(R)$ to be Noetherian.

Theorem 8.1. *If the Balmer spectrum $\mathtt{Spec}\,\mathsf{D}^{-}(R)$ is a Noetherian topological space, then $\mathsf{Spec}\, R$ is a finite set (i.e., semilocal ring with Krull dimension at most 1).*

Proof. Assume that $\mathtt{Spec}\,\mathsf{D}^{-}(R)$ is Noetherian. Then for any chain of the form $\mathsf{BSupp}\, M_1 \supseteq \mathsf{BSupp}\, M_2 \supseteq \mathsf{BSupp}\, M_3 \cdots$ with $M_i \in \mathsf{D}^{-}(R)$ stabilizes. Thus, by Theorem 2.17, every descending chain $\sqrt{\mathsf{thick}^{\otimes} M_1} \supseteq \sqrt{\mathsf{thick}^{\otimes} M_2} \supseteq \sqrt{\mathsf{thick}^{\otimes} M_3} \supseteq \cdots$ stabilizes.

Assume furthermore that R has infinitely many prime ideals. From Lemma 7.3, we can take a countable antichain $\{\mathfrak{p}_n\}_{n\geqslant 1}$ of prime ideals. Set $M_n := \bigoplus_{i\geqslant n} R/\mathfrak{p}_i[i]$ to be the complex

$$M_n := (\cdots \xrightarrow{0} R/\mathfrak{p}_{n+2} \xrightarrow{0} R/\mathfrak{p}_{n+1} \xrightarrow{0} R/\mathfrak{p}_n \to 0 \cdots).$$

Here, $R/\mathfrak{p}_i$ fit into the $(-i)$-th component. Then M_n belongs to $\mathsf{D}^{-}(R)$, and M_{n+1} is a direct summand of M_n for each integer $n \geqslant 1$. Therefore, we have a descending chain $\sqrt{\mathsf{thick}^{\otimes} M_1} \supseteq \sqrt{\mathsf{thick}^{\otimes} M_2} \supseteq \sqrt{\mathsf{thick}^{\otimes} M_3} \supseteq \cdots$ of radical thick tensor ideals. From the above argument, we get an equality $\sqrt{\mathsf{thick}^{\otimes} M_n} = \sqrt{\mathsf{thick}^{\otimes} M_{n+1}}$ for some integer $n \geqslant 1$. Taking Supp, we obtain

$$\bigcup_{i \geqslant n} \mathrm{V}(\mathfrak{p}_i) = \mathsf{Supp}\sqrt{\mathsf{thick}^{\otimes} M_n} = \mathsf{Supp}\sqrt{\mathsf{thick}^{\otimes} M_{n+1}} = \bigcup_{i \geqslant n+1} \mathrm{V}(\mathfrak{p}_i).$$

Hence, there is an integer $m \geqslant n+1$ such that $\mathfrak{p}_m \subseteq \mathfrak{p}_n$. This gives a contradiction. ■

Next, we discuss connectedness and irreducibility of $\mathtt{Spec}\,\mathsf{D}^{-}(R)$. To begin with, let us recall some notions from point-set topology.

Definition 8.2. Let X be a topological space.

(1) We say that a subset of X is *clopen* if it is both open and closed.
(2) A subspace W of X is said to be *generalization closed* if for any $x \in W$ and $y \in X$, $x \in \overline{\{y\}}$ implies $y \in W$.
(3) We say that X is *connected* if it contains no non-trivial clopen subset. For a subspace Y of X, we say that Y is a *connected subspace* of X if it is a connected space by induced topology. Moreover, a *connected component* of X is a maximal connected subspace of X. Besides, we show the following theorem which gives a sufficient condition for Noetherianity of the Balmer spectrum $\mathtt{Spec}\,\mathsf{D}^{-}(R)$.

Remark 8.3. (1) A subspace is generalization closed if and only if its complement is specialization closed. In particular, every open subset of X is generalization closed.
(2) Let $X \supseteq Y \supseteq Z$ be subspaces. If Y is specialization closed in X and Z is specialization closed in Y, then Z is specialization closed in X.
(3) Let W be a subspace of $\mathsf{Spec}\,R$. Then W is specialization closed (resp. generalization closed) in $\mathsf{Spec}\,R$ if and only if

$$\mathfrak{p} \in W,\ \mathfrak{p} \subseteq \mathfrak{q} \Longrightarrow \mathfrak{q} \in W.$$
$$(\text{resp. } \mathfrak{q} \in W,\ \mathfrak{p} \subseteq \mathfrak{q} \Longrightarrow \mathfrak{p} \in W.)$$

(4) [Bal05, Proposition 2.9] Let $\mathcal{T}$ be an essentially small tensor triangulated category and W a subspace of $\mathtt{Spec}\,\mathcal{T}$. Then W is specialization closed (resp. generalization closed) in $\mathtt{Spec}\,\mathcal{T}$ if and only if

$$\mathcal{P} \in W,\ \mathcal{P} \supseteq \mathcal{Q} \Longrightarrow \mathcal{Q} \in W.$$
$$(\text{resp. } \mathcal{Q} \in W,\ \mathcal{P} \supseteq \mathcal{Q} \Longrightarrow \mathcal{P} \in W.)$$

The following theorem is the second main result of this section.

Theorem 8.4. *Let $C \in \mathsf{D}^{\mathsf{b}}(R)$ be a bounded complex.*

(1) *There is a one-to-one correspondence*

$$\left\{\textit{connected components of } \mathsf{BSupp}\,C\right\} \underset{\mathfrak{s}^{-1}}{\overset{\mathfrak{s}}{\rightleftarrows}} \left\{\textit{connected components of } \mathsf{Supp}\,C\right\}.$$

(2) *There is a one-to-one correspondence*

$$\left\{\textit{irreducible components of } \mathsf{BSupp}\,C\right\} \underset{\mathfrak{s}^{-1}}{\overset{\mathfrak{s}}{\rightleftarrows}} \left\{\textit{irreducible components of } \mathsf{Supp}\,C\right\}.$$

The proof of this theorem is divided into several lemmata. Let us start the following general fact about connected components of a topological space. For a while, we fix a bounded complex $C \in \mathsf{D}^{\mathsf{b}}(R)$. The first one is a remark about general topological spaces.

Lemma 8.5. *Let X be a topological space. Then every connected component of X is both specialization closed and generalization closed.*

Proof. Fix a connected component O of X. For $x \in O$, $\overline{\{x\}}$ is irreducible and in particular connected. Since $O \cap \overline{\{x\}}$ is non-empty, $O \cup \overline{\{x\}}$ is connected. Thus, $O \cup \overline{\{x\}}$ must be equal to O, and hence $\overline{\{x\}} \subseteq O$. This shows that O is specialization closed in X.

For $x \notin O$, assume that there exists $y \in \overline{\{x\}}$ with $y \in O$. Then $\overline{\{x\}} \cap O$ is non-empty as it contains y. Therefore, the same argument as above shows that $\overline{\{x\}} \subseteq O$. This gives a contradiction to $x \notin O$. Thus, $X \setminus O$ is specialization closed in X and hence O is generalization closed in X. ∎

The following result gives an easier way to check whether a given subspace is clopen for our topological spaces $\mathsf{Supp}\, C$ or $\mathsf{BSupp}\, C$.

Lemma 8.6. *Let X be either $\mathsf{Supp}\, C$ or $\mathsf{BSupp}\, C$ and W a subset of X. If W is both specialization closed and generalization closed, then W is clopen.*

Proof. We show this statement only for $X = \mathsf{BSupp}\, C$ because a similar argument works also for $X = \mathsf{Supp}\, C$. By symmetry, we need to check that W is closed.

Claim. $W = \bigcup_{\mathcal{P} \in \mathrm{Max}\, \mathsf{BSupp}\, C \cap W} \overline{\{\mathcal{P}\}}$.

Proof of claim. Since W is specialization closed, $W \supseteq \bigcup_{\mathcal{P} \in \max \mathsf{BSupp}\, C \cap W} \overline{\{\mathcal{P}\}}$ holds. Let $\mathcal{P}$ be an element of W. Take a minimal element $\mathfrak{p}$ in $\mathsf{Supp}\, C$ contained in $\mathfrak{s}(\mathcal{P})$. We can take such a $\mathfrak{p}$ since $\mathsf{Supp}\, C$ is a closed subset of $\mathsf{Spec}\, R$. Then

$$\mathcal{P} \subseteq \mathcal{S}\mathfrak{s}(\mathcal{P}) \subseteq \mathcal{S}(\mathfrak{p}).$$

By Theorem 7.9, $\mathcal{S}(\mathfrak{p})$ is a maximal element of $\mathsf{BSupp}\, C$. Moreover, $\mathcal{S}(\mathfrak{p})$ belongs to W since W is generalization closed and $\mathcal{P} \in W$. These show that $\mathcal{S}(\mathfrak{p})$ is a maximal element of $\mathsf{BSupp}\, C$. Accordingly, we obtain $\mathcal{P} \in \overline{\{\mathcal{S}(\mathfrak{p})\}}$ with $\mathcal{S}(\mathfrak{p}) \in \max \mathsf{BSupp}\, C$ and hence the converse inclusion holds true. □

Note that $\mathsf{Supp}\, C$ is closed and thus contains only finitely many minimal elements. By using the one-to-one correspondence in Theorem 7.9, $\mathrm{Max}\, \mathsf{BSupp}\, C$ is also a finite set. Consequently, W is a finite union of closed subsets, and hence is closed. ∎

Lemma 8.7. *Let U be a clopen subset of $\mathsf{BSupp}\, C$. Then*

(1) *$\mathfrak{p} \in \mathfrak{s}(U)$ if and only if $\mathcal{S}(\mathfrak{p}) \in U$, and*

(2) *$\mathfrak{s}(U)$ is a clopen subset in $\mathsf{Supp}\, C$.*

Proof. (1) The 'if' part is from Theorem 6.8. Let $\mathfrak{p}$ be an element of $\mathfrak{s}(U) \subseteq \mathfrak{s}(\mathsf{BSupp}\, C) = \mathsf{Supp}\, C$. Then there is a prime thick tensor ideal $\mathcal{P} \in U$ such that $\mathfrak{s}(\mathcal{P}) = \mathfrak{p}$. Then $\mathcal{S}(\mathfrak{p})$ belongs to U because $\mathcal{P} \subseteq \mathcal{S}\mathfrak{s}(\mathcal{P}) = \mathcal{S}(\mathfrak{p})$ and U is generalization closed in $\mathsf{BSupp}\, C$.

(2) By Lemma 8.6, we have only to check that $\mathfrak{s}(U)$ and $\mathsf{Supp}\, C \setminus \mathfrak{s}(U)$ are specialization closed in $\mathsf{Supp}\, C$.

Take $\mathfrak{p} \in \mathfrak{s}(U)$ and $\mathfrak{q} \in \mathrm{V}(\mathfrak{p})$. Then $\mathcal{S}(\mathfrak{q}) \subseteq \mathcal{S}(\mathfrak{p})$. From (1), one has $\mathcal{S}(\mathfrak{p}) \in U$. Since U is specialization closed, we get $\mathcal{S}(\mathfrak{q}) \in U$. Thus, $\mathfrak{q} = \mathfrak{s}\mathcal{S}(\mathfrak{q})$ belongs to $\mathfrak{s}(U)$. This shows that $\mathfrak{s}(U)$ is specialization closed in $\mathsf{Supp}\, C$.

Take $\mathfrak{p} \in \mathsf{Supp}\, C \setminus \mathfrak{s}(U)$ and $\mathfrak{q} \in \mathrm{V}(\mathfrak{p})$. Then $\mathcal{S}(\mathfrak{q}) \subseteq \mathcal{S}(\mathfrak{p})$. From (1), one has $\mathcal{S}(\mathfrak{p}) \notin U$. Assume that $\mathcal{S}(\mathfrak{q})$ belongs to U. Since U is generalization closed, $\mathcal{S}(\mathfrak{p})$ belongs to U, a contradiction. Thus, $\mathcal{S}(\mathfrak{q}) \notin U$ and hence $\mathfrak{q} \notin \mathfrak{s}(U)$ by (1). This shows that $\mathsf{Supp}\, C \setminus \mathfrak{s}(U)$ is specialization closed in $\mathsf{Supp}\, C$. ■

Lemma 8.8. *Let U be a clopen subset of $\mathsf{BSupp}\, C$. Then $\mathfrak{s}^{-1}\mathfrak{s}(U) = U$.*

Proof. The inclusion $U \subseteq \mathfrak{s}^{-1}\mathfrak{s}(U)$ is trivial. For a prime thick tensor ideal $\mathcal{P} \in \mathfrak{s}^{-1}\mathfrak{s}(U)$, one has $\mathfrak{s}(\mathcal{P}) \in \mathfrak{s}(U)$. By Lemma 8.7(1), we obtain $\mathcal{S}\mathfrak{s}(\mathcal{P}) \in U$. Since U is specialization closed in $\mathsf{BSupp}\, C$ and $\mathcal{P} \subseteq \mathcal{S}\mathfrak{s}(\mathcal{P})$, we conclude that $\mathcal{P}$ belongs U. ■

Now, we are ready to prove Theorem 8.4.

(Proof of Theorem 8.4). (1) By Lemma 8.7(2), we obtain a well-defined map

$$\{\text{clopen subsets of } \mathsf{BSupp}\, C\} \to \{\text{clopen subsets of } \mathsf{Supp}\, C\},\ U \mapsto \mathfrak{s}(U).$$

This map is injective by Lemma 8.8 and surjective since $\mathfrak{s} : \mathsf{BSupp}\, C \to \mathsf{Supp}\, C$ is continuous and surjecive. Thus, this map is an order-preserving one-to-one correspondence.

Our topological spaces $\mathsf{BSupp}\, C$ and $\mathsf{Supp}\, C$ have only finitely many connected components by Theorem 7.9 and Lemma 8.5, and the proof of Lemma 8.6. Thus, connected components are nothing but minimal non-empty clopen subsets. Therefore, the statement (1) follows from the above order-isomorphism.

(2) By Proposition 2.10, every irreducible closed subset of $\mathsf{BSupp}\, C$ is of the form

$$\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in \mathsf{Spec}\, \mathsf{D}^{-}(R) \mid \mathcal{Q} \subseteq \mathcal{P}\}$$

for a unique prime thick tensor ideal $\mathcal{P} \in \mathsf{BSupp}\, C$. Since an irreducible component is by definition a maximal irreducible closed subset, every irreducible component of $\mathsf{BSupp}\, C$ is of the form $\overline{\{\mathcal{P}\}}$ for a unique maximal element $\mathcal{P}$ of $\mathsf{BSupp}\, C$. Thus, $\mathcal{P} = \mathcal{S}(\mathfrak{p})$ for some minimal element $\mathfrak{p}$ of $\mathsf{Supp}\, C$ by Theorem 7.9. Similarly, every irreducible component of $\mathsf{Supp}\, C$ is of the form $\overline{\{\mathfrak{p}\}}$ for a unique minimal element $\mathfrak{p}$ of $\mathsf{Supp}\, C$. Therefore, there is a maximal element $\mathcal{P}$ of $\mathsf{BSupp}\, C$ such that $\mathfrak{p} = \mathfrak{s}(\mathcal{P})$ by Theorem 7.9. Altogether, the one-to-one correspondence of Theorem 7.9 gives a one-to-one correspondence what we want. ■

The following connectedness and irreducibility result is a direct consequence of Theorem 8.4.

Corollary 8.9. *For a bounded complex $C \in \mathsf{D}^{\mathsf{b}}(R)$, $\mathsf{BSupp}\, C$ is connected (resp. irreducible) if and only if $\mathsf{Supp}\, C$ is connected (resp. irreducible). In particular, $\mathsf{Spec}\, \mathsf{D}^{-}(R)$ is connected (resp. irreducible) if and only if $\mathsf{Spec}\, R$ is connected (resp. irreducible).*

As an application of Theorem 8.4, we obtain the following corollary.

Corollary 8.10. *Let $C \in \mathsf{D}^{\mathsf{b}}(R)$ be a bounded complex. If C is indecomposable in $\mathsf{D}^{-}(R)$, then $\mathsf{BSupp}\, C$ is connected.*

Proof. By Theorem 8.4, it is enough to show that $\mathsf{Supp}\, C$ is connected if C is indecomposable.

Take an ideal I with $\mathsf{Supp}\, C = \mathrm{V}(I)$. It follows from [Orl11, Lemma 2.1] that there exists a bounded complex B such that

(i) B is isomorphic to C in $\mathsf{D}^{\mathsf{b}}(R)$,
(ii) $\mathsf{Supp}\, B^i \subseteq \mathrm{V}(I)$.

By (ii), we can take an integer $n > 0$ with $I^n B_i = 0$ for each i. Thus by (i), we may assume that $\mathsf{Supp}\, C = \mathrm{V}(I)$ and $IC^i = 0$ for each i.

Consider a decomposition $\mathsf{Supp}\, C = F_1 \sqcup F_2$ with F_1, F_2 closed. Then there are radical ideals I_1 and I_2 such that $F_i = \mathrm{V}(I_i)$ $(i = 1, 2)$, $I_1 + I_2 = R$, and $I_1 \cap I_2 = \sqrt{I}$. Using Chinese remainder theorem, we obtain a direct sum decomposition

$$R/\sqrt{I} \cong R/I_1 \oplus R/I_2.$$

Moreover, from the idempotent lifting theorem (see [Lam, Proposition 21.25]), we obtain the following decomposition

$$R/I \cong R/J_1 \oplus R/J_2.$$

Here, J_1 and J_2 are ideals with $\sqrt{J_i} = I_i$ for $i = 1, 2$. Tensoring with C, we get the following direct sum decomposition:

$$C \cong C \otimes_R R/I \cong (C \otimes_R R/J_1) \oplus (C \otimes_R R/J_2).$$

Since C is indecomposable, $C \otimes_R R/J_1 \cong C$ or $C \otimes_R R/J_2 \cong C$. If $C \otimes_R R/J_1 \cong C$, then we obtain

$$\mathrm{V}(I) = \mathsf{Supp}\, C = \mathsf{Supp}(C \otimes_R R/J_1) \subseteq \mathrm{V}(J_1)$$

and thus $\mathsf{Supp}\, C = \mathrm{V}(I) = \mathrm{V}(I_1) = F_1$. Similarly, if $C \otimes_R R/J_2 \cong C$, then one has $\mathsf{Supp}\, C = F_2$. Thus, we are done. ■

This corollary means that the Balmer support of an indecomposable bounded R-complex is connected. Such a result has been shown by Carlson [Car] for the stable category of finite dimensional representations over a finite group, and more generally, by Balmer [Bal07] for an idempotent complete rigid tensor triangulated category.

For the last of this section, we prove that every clopen subset of $\mathtt{Spec}\, \mathsf{D}^{-}(R)$ is homeomorphic to the Balmer spectrum of the Eilenberg-Moore category of some ring object. Following [Bal11, Bal14], we recall the notion of a ring object and related concepts.

Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a tensor triangulated category. We say that an object $A \in \mathcal{T}$ is a *ring object* of $\mathcal{T}$ if there is a morphisms

$$\mu : A \otimes A \to A,$$
$$\eta : \mathbf{1} \to A$$

satisfying the following commutative diagrams:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{A \otimes \mu} & A \otimes A \\ {\scriptstyle \mu \otimes A}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow[\mu]{} & A \end{array} \qquad \begin{array}{ccccc} \mathbf{1} \otimes A & \xrightarrow{\eta \otimes A} & A \otimes A & \xleftarrow{A \otimes \eta} & A \otimes \mathbf{1} \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \cong} & \\ & & A & & \end{array}$$

We say that a ring object A of $\mathcal{T}$ is *commutative* if $\mu\tau = \mu$ holds, where

$$\tau : A \otimes A \to A \otimes A$$

is the swap of factors. We say that a ring object A of $\mathcal{T}$ is *separable* if there is a morphism

$$\sigma : A \to A \otimes A$$

such that $(A \otimes \mu)(\sigma \otimes A) = \sigma\mu = (\mu \otimes A)(A \otimes \sigma)$.

We say that an object $M \in \mathcal{T}$ is a (left) A-*module* if there is a morphism

$$\lambda : A \otimes M \to M$$

satisfying the following commutative diagrams:

$$\begin{array}{ccc} A\otimes A\otimes M & \xrightarrow{A\otimes\lambda} & A\otimes M \\ {\scriptstyle \mu\otimes M}\downarrow & & \downarrow{\scriptstyle\lambda} \\ A\otimes M & \xrightarrow[\lambda]{} & M \end{array} \qquad \begin{array}{ccc} \mathbf{1}\otimes M & \xrightarrow{\eta\otimes M} & A\otimes M \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle\lambda} \\ & & M \end{array}$$

Denote by $\mathsf{Mod}\, A$ the category of A-modules.

Let me give the following easy observation.

Lemma 8.11. *If R is decomposed into $R = A \times B$ as rings, then A has a unique ring object structure by the natural multiplication $\mu : A \otimes^{\mathbf{L}}_R A \cong A \otimes_R A \cong A$ and the projection $\eta : R \to A$. Moreover, the following holds true.*

(1) *A is a commutative separable ring object in $\mathsf{D}^-(R)$.*
(2) *For any complex $M \in \mathsf{D}^-(R)$, it has an A-module structure if and only if $A\otimes^{\mathbf{L}}_R M \cong M$. This is the case, its A-module structure is uniquely determined by underlying complex.*
(3) *For A-modules M and N, $M \otimes^{\mathbf{L}}_R N$ is an A-module. Hence $\mathsf{Mod}\, A$ is a tensor triangulated category and U_A preserves tensor products.*

Proof. Since A is a projective R-module, the statement (1) means that A is a commutative separable R-algebra in the usual sense and this is clear. Uniqueness of this structure follows from (2).

(2) Let M be an A-module. Consider the following commutative diagram:

$$\begin{array}{ccc} R\otimes^{\mathbf{L}}_R M & \xrightarrow{\eta\otimes^{\mathbf{L}}_R M} & A\otimes^{\mathbf{L}}_R M \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle\lambda} \\ & & M \end{array}$$

In particular, the composition $\mathsf{H}^i(\lambda) \circ \mathsf{H}^i(\eta \otimes^{\mathbf{L}}_R M)$ is an isomorphism for each integer i. Since $\eta \otimes^{\mathbf{L}}_R M$ is a split surjection, $\mathsf{H}^i(\eta \otimes^{\mathbf{L}}_R M)$ is also a split surjection and hence is an isomorphism for each i. This shows that $\eta \otimes^{\mathbf{L}}_R X$ is a quasi-isomorphism. From the above commutative diagram, λ is also a quasi-isomorphism.

Take an object $M \in \mathsf{D}^-(R)$ with $A \otimes^{\mathbf{L}}_R M \cong M$. Then the following morphism gives an A-module structure to M:

$$A \otimes^{\mathbf{L}}_R M \cong A \otimes^{\mathbf{L}}_R A \otimes^{\mathbf{L}}_R M \xrightarrow{\mu\otimes^{\mathbf{L}}_R M} A \otimes^{\mathbf{L}}_R M \cong M.$$

Moreover, the A-module structure λ is uniquely determined as

$$A \otimes^{\mathbf{L}}_R M \xrightarrow{(\eta\otimes^{\mathbf{L}}_R M)^{-1}} R \otimes^{\mathbf{L}}_R M \xrightarrow{\cong} M.$$

The last statement (3) directly follows from the definition of $\otimes_A$ and (2), for details, see [Bal14]. ■

From (2) in the above lemma, we can define a unique A-module structure for a complex $M \in \mathsf{D}^-(R)$ with $A \otimes^{\mathbf{L}}_R M \cong M$. For simplicity, we denote this A-module by M_A. In addition, for a complex $M \in \mathsf{D}^-(R)$, $A \otimes^{\mathbf{L}}_R M$ has an A-module structure and hence we can define a triangulated functor

$$F_A : \mathsf{D}^-(R) \to \mathsf{Mod}\, A,\ M \mapsto A \otimes^{\mathbf{L}}_R M.$$

Corollary 8.12. *For any non-empty clopen subset W of $\mathsf{Spec}\,\mathsf{D}^{-}(R)$, there is a commutative separable ring object A of $\mathsf{D}^{-}(R)$ such that*

$$\varphi_A := {}^aF_A : \mathsf{Spec}(\mathsf{Mod}\,A) \to \mathsf{Spec}\,\mathsf{D}^{-}(R),\ \mathcal{P} \mapsto F_A^{-1}(\mathcal{P})$$

gives a homeomorphism onto W.

Proof. By Lemma 8.7, $\mathfrak{s}(W)$ is a clopen subset of $\mathsf{Spec}\,R$. Therefore, by Corollary 8.10, there is a direct sum decomposition $R = A \times B$ of rings with $\mathfrak{s}(W) = \mathsf{Supp}\,A$. Then Lemma 8.11 shows that A has a commutative separable ring object structure. Since U_A preserves tensor products, one can easily check that the forgetful functor $U_A : \mathsf{Mod}\,A \to \mathsf{D}^{-}(R)$ induces a continuous injective map

$$\psi_A : \mathsf{BSupp}\,A \to \mathsf{Spec}(\mathsf{Mod}\,A), \mathcal{P} \mapsto U_A^{-1}(\mathcal{P}),$$

see Proposition 2.13. Furthermore, the image of φ_A is contained in $\mathsf{BSupp}\,A$ and $\psi_A\varphi_A = 1$ because $F_AU_A \cong 1$. For this reason, we have only to check that the image of φ_A is W.

Let $\mathcal{P}$ be a prime thick tensor ideal of $\mathsf{Mod}\,A$. By definition,

$$\varphi_A(\mathcal{P}) = \{X \in \mathsf{D}^{-}(R) \mid F_A(X) = (A \otimes_R^{\mathbf{L}} X)_A \in \mathcal{P}\}$$

and it contains B because $A \otimes_R^{\mathbf{L}} B = 0$. In particular,

$$\mathsf{Supp}\,B \subseteq \mathsf{Supp}\,\varphi_A(\mathcal{P}) = \{\mathfrak{p} \in \mathsf{Spec}\,R \mid \mathfrak{p} \not\subseteq \mathfrak{s}(\varphi_A(\mathcal{P}))\}$$

and thus $\mathfrak{s}(\varphi_A(\mathcal{P})) \in \mathsf{Spec}\,R \setminus \mathsf{Supp}\,B = \mathsf{Supp}\,A$. Therefore, $\varphi_A(\mathcal{P}) \in W$ by Lemma 8.8. Conversely, take a prime thick tensor ideal $\mathcal{P}$ from W. Then $\mathfrak{s}(\mathcal{P}) \in \mathfrak{s}(W) = \mathsf{Supp}\,A$ implies that $A \notin \mathcal{P}$. Therefore,

$$\varphi_A(\psi_A(\mathcal{P})) = \{X \in \mathsf{D}^{-}(R) \mid A \otimes_R^{\mathbf{L}} X \in \mathcal{P}\} = \mathcal{P}$$

since $A \notin \mathcal{P}$. Thus, we conclude that $\varphi_A(\mathsf{Spec}(\mathsf{Mod}\,A)) = W$. ■

9. **Relationships among thick tensor ideals and specialization-closed subsets**

This section compares compact, tame and radical thick tensor ideals of $\mathsf{D}^{-}(R)$, relating them to specialization closed subsets of $\mathsf{Spec}\,R$, ${}^{\mathsf{t}}\mathsf{Spec}\,\mathsf{D}^{-}(R)$ and Thomason subsets of $\mathsf{Spec}\,\mathsf{D}^{-}(R)$. We start with some notation.

Definition 9.1. (1) Let $\mathcal{T}$ be a tensor triangulated category. Let $\mathbb{P}$ be a property of thick $\otimes$-ideals of $\mathcal{T}$. For a subcategory $\mathcal{X}$ of $\mathcal{C}$ we denote by $\mathcal{X}^{\mathbb{P}}$ (resp. $\mathcal{X}_{\mathbb{P}}$) the $\mathbb{P}$-*closure* (resp. $\mathbb{P}$-*interior*) of $\mathcal{X}$, that is to say, the smallest (resp. largest) thick $\otimes$-ideal of $\mathcal{T}$ which contains (resp. which is contained in) $\mathcal{X}$ and satisfies the property $\mathbb{P}$. We define this only when it exists.
(2) Let X be a topological space. Let $\mathbb{P}$ be a property of subsets of X. For a subset A of X we denote by $A^{\mathbb{P}}$ (resp. $A_{\mathbb{P}}$) the $\mathbb{P}$-*closure* (resp. $\mathbb{P}$-*interior*) of A, namely, the smallest (resp. largest) subset of X that contains (resp. that is contained in) A and satisfies $\mathbb{P}$. We define this only when it exists.

Here is a list of properties $\mathbb{P}$ as in the above definition which we consider:

rad = radical, tame = tame, cpt = compact, spcl = specialization-closed.

Notation 9.2. We denote by **Rad** (resp. **Tame**, **Cpt**) the set of radical (resp. tame, compact) thick $\otimes$-ideals of $\mathsf{D}^{-}(R)$. Also, $\mathbf{Spcl}(\mathsf{Spec})$ (resp. $\mathbf{Spcl}({}^{\mathsf{t}}\mathsf{Spec})$) stands for the set of specialization-closed subsets of the topological space $\mathsf{Spec}\,R$ (resp. ${}^{\mathsf{t}}\mathsf{Spec}\,\mathsf{D}^{-}(R)$).

Our first purpose in this section is to give a certain commutative diagram of bijections. To achieve this purpose, we prepare several propositions. We state here two propositions. The first one is shown by using Proposition 4.8, while the second one is nothing but Theorem 5.19.

Proposition 9.3. *There is a one-to-one correspondence* $\mathsf{Supp} : \mathbf{Tame} \rightleftarrows \mathbf{Spcl}(\mathsf{Spec}) : \mathsf{Supp}^{-1}$.

Proposition 9.4. *There is a one-to-one correspondence* $\mathsf{Supp} : \mathbf{Cpt} \rightleftarrows \mathbf{Spcl}(\mathsf{Spec}) : \langle\rangle$.

Notation 9.5. For an object M of $\mathsf{D}^-(R)$ we denote by $\mathtt{BSp}\,M$ the set of tame prime thick $\otimes$-ideals of $\mathsf{D}^-(R)$ not containing M, i.e., $\mathtt{BSp}\,M = \mathsf{BSupp}\,M \cap {}^{\mathtt{t}}\mathtt{Spec}\,\mathsf{D}^-(R)$. For a subcategory $\mathcal{X}$ of $\mathsf{D}^-(R)$ we set $\mathtt{BSp}\,\mathcal{X} = \bigcup_{M\in\mathcal{X}} \mathtt{BSp}\,M$. For a subset A of $\mathtt{Spec}\,\mathsf{D}^-(R)$ we denote by $\mathtt{BSp}^{-1}\,A$ the subcategory of $\mathsf{D}^-(R)$ consisting of objects M such that $\mathtt{BSp}\,M$ is contained in A.

We make a lemma, whose second assertion is a variant of [Bal05, Lemma 4.8].

Lemma 9.6. (1) *For a subcategory $\mathcal{X}$ of $\mathsf{D}^-(R)$, the subset $\mathtt{BSp}\,\mathcal{X}$ of ${}^{\mathtt{t}}\mathtt{Spec}\,\mathsf{D}^-(R)$ is specialization-closed.*

(2) *For a subset A of ${}^{\mathtt{t}}\mathtt{Spec}\,\mathsf{D}^-(R)$ one has $\mathtt{BSp}^{-1}\,A = \bigcap_{\mathcal{P}\in A^{\complement}} \mathcal{P}$, where $A^{\complement} = {}^{\mathtt{t}}\mathtt{Spec}\,\mathsf{D}^-(R)\setminus A$.*

(3) *Let $\{\mathcal{X}_\lambda\}_{\lambda\in\Lambda}$ be a collection of tame thick $\otimes$-ideals of $\mathsf{D}^-(R)$. Then the intersection $\bigcap_{\lambda\in\Lambda} \mathcal{X}_\lambda$ is also a tame thick $\otimes$-ideal of $\mathsf{D}^-(R)$.*

Proof. (1) We have $\mathtt{BSp}\,\mathcal{X} = \bigcup_{X\in\mathcal{X}} \mathtt{BSp}\,X$, and $\mathtt{BSp}\,X = \mathsf{BSupp}\,X \cap {}^{\mathtt{t}}\mathtt{Spec}\,\mathsf{D}^-(R)$ is closed in ${}^{\mathtt{t}}\mathtt{Spec}\,\mathsf{D}^-(R)$ since $\mathsf{BSupp}\,X$ is closed in $\mathtt{Spec}\,\mathsf{D}^-(R)$. Therefore $\mathtt{BSp}\,\mathcal{X}$ is specialization-closed in ${}^{\mathtt{t}}\mathtt{Spec}\,\mathsf{D}^-(R)$.

(2) An object X of $\mathsf{D}^-(R)$ belongs to $\mathtt{BSp}^{-1}\,A$ if and only if $\mathtt{BSp}\,X$ is contained in A, if and only if $A^{\complement}$ is contained in $(\mathtt{BSp}\,X)^{\complement} = \{\mathcal{P} \in {}^{\mathtt{t}}\mathtt{Spec}\,\mathsf{D}^-(R) \mid X \in \mathcal{P}\}$, if and only if X belongs to $\bigcap_{\mathcal{P}\in A^{\complement}} \mathcal{P}$.

(3) For each $\lambda \in \Lambda$ there is a subset S_λ of $\mathsf{Spec}\,R$ such that $\mathcal{X}_\lambda = \mathsf{Supp}^{-1}\,S_\lambda$. Then it is clear that the equality $\bigcap_{\lambda\in\Lambda} \mathcal{X}_\lambda = \mathsf{Supp}^{-1}(\bigcap_{\lambda\in\Lambda} S_\lambda)$ holds, which shows the assertion. ■

Using the above lemma, we obtain a bijection induced by $\mathtt{BSp}$.

Proposition 9.7. *There is a one-to-one correspondence* $\mathtt{BSp} : \mathbf{Tame} \rightleftarrows \mathbf{Spcl}({}^{\mathtt{t}}\mathtt{Spec}) : \mathtt{BSp}^{-1}$.

Proof. Fix a tame thick $\otimes$-ideal $\mathcal{X}$ of $\mathsf{D}^-(R)$ and a specialization-closed subset U of ${}^{\mathtt{t}}\mathtt{Spec}\,\mathsf{D}^-(R)$. Lemma 9.6(1) implies that $\mathtt{BSp}\,\mathcal{X}$ is specialization-closed in ${}^{\mathtt{t}}\mathtt{Spec}\,\mathsf{D}^-(R)$, that is, $\mathtt{BSp}\,\mathcal{X} \in \mathbf{Spcl}({}^{\mathtt{t}}\mathtt{Spec})$. Lemma 9.6(2) implies that $\mathtt{BSp}^{-1}\,U = \bigcap_{\mathcal{P}\in U^{\complement}} \mathcal{P}$, and each $\mathcal{P} \in U^{\complement}$ is a tame thick $\otimes$-ideal of $\mathsf{D}^-(R)$. Hence $\mathtt{BSp}^{-1}\,U$ is also a tame thick $\otimes$-ideal of $\mathsf{D}^-(R)$ by Lemma 9.6(3), namely, $\mathtt{BSp}^{-1}\,U \in \mathbf{Tame}$.

Let us show that $\mathtt{BSp}(\mathtt{BSp}^{-1}\,U) = U$. It is evident that $\mathtt{BSp}(\mathtt{BSp}^{-1}\,U)$ is contained in U. Pick any $\mathcal{P} \in U$. Corollary 6.15 says $\mathcal{P} = \mathcal{S}(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of R. Since U is specialization-closed in ${}^{\mathtt{t}}\mathtt{Spec}\,\mathsf{D}^-(R)$, the closure C of $\mathcal{S}(\mathfrak{p})$ in ${}^{\mathtt{t}}\mathtt{Spec}\,\mathsf{D}^-(R)$ is contained in U. Using Proposition 2.10, we see that C consists of the prime thick $\otimes$-ideals of the form $\mathcal{S}(\mathfrak{q})$, where $\mathfrak{q}$ is a prime ideal of R with $\mathcal{S}(\mathfrak{q}) \subseteq \mathcal{S}(\mathfrak{p})$. In view of Theorem 6.8, we have $C = \{\mathcal{S}(\mathfrak{q}) \mid \mathfrak{q} \in \mathrm{V}(\mathfrak{p})\}$, and it is easy to observe that this coincides with $\mathtt{BSp}(R/\mathfrak{p})$. Hence $R/\mathfrak{p}$ is in $\mathtt{BSp}^{-1}\,U$, and $\mathcal{P} = \mathcal{S}(\mathfrak{p})$ belongs to $\mathtt{BSp}(\mathtt{BSp}^{-1}\,U)$. Now we obtain $\mathtt{BSp}(\mathtt{BSp}^{-1}\,U) = U$.

It remains to prove that $\mathtt{BSp}^{-1}(\mathtt{BSp}\,\mathcal{X}) = \mathcal{X}$. We have $\mathtt{BSp}^{-1}(\mathtt{BSp}\,\mathcal{X}) = \bigcap_{\mathcal{P}\in(\mathtt{BSp}\,\mathcal{X})^{\complement}} \mathcal{P}$ by Lemma 9.6(2). Fix a prime thick $\otimes$-ideal $\mathcal{P}$ of $\mathsf{D}^{-}(R)$. Then $\mathcal{P}$ is in $(\mathtt{BSp}\,\mathcal{X})^{\complement}$ if and only if $\mathcal{P}$ is tame and $\mathcal{P}$ is not in $\mathtt{BSp}\,\mathcal{X}$. The former statement is equivalent to saying that $\mathcal{P} = \mathcal{S}(\mathfrak{p})$ for some $\mathfrak{p} \in \operatorname{Spec} R$ by Corollary 6.15, while the latter is equivalent to saying that $\mathcal{X}$ is contained in $\mathcal{P}$. Hence $\mathtt{BSp}^{-1}(\mathtt{BSp}\,\mathcal{X}) = \bigcap_{\mathfrak{p}\in\operatorname{Spec} R,\,\mathcal{X}\subseteq\mathcal{S}(\mathfrak{p})} \mathcal{S}(\mathfrak{p})$. Thus an object Y of $\mathsf{D}^{-}(R)$ belongs to $\mathtt{BSp}^{-1}(\mathtt{BSp}\,\mathcal{X})$ if and only if Y belongs to $\mathcal{S}(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathcal{X} \subseteq \mathcal{S}(\mathfrak{p})$, if and only if $Y_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathcal{X}_{\mathfrak{p}} = 0$, if and only if $\operatorname{Supp} Y$ is contained in $\operatorname{Supp}\mathcal{X}$, if and only if Y belongs to $\mathcal{X}$ by Proposition 9.3. ■

Here we consider describing rad-closures, tame-closures and cpt-interiors, and their supports.

Lemma 9.8. *Let $\mathcal{X}$ be a subcategory of $\mathsf{D}^{-}(R)$, and let $\mathcal{Y}$ be a thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$. One has:*

(1) $(\mathsf{thick}^{\otimes}\,\mathcal{X})_{\mathsf{cpt}} = \langle \operatorname{Supp}\mathcal{X}\rangle$, $\quad \mathcal{X}^{\mathsf{rad}} = \sqrt{\mathsf{thick}^{\otimes}\,\mathcal{X}}$, $\quad \mathcal{X}^{\mathsf{tame}} = \operatorname{Supp}^{-1}\operatorname{Supp}\mathcal{X}$,

(2) $\mathcal{Y}_{\mathsf{cpt}} \subseteq \mathcal{Y} \subseteq \mathcal{Y}^{\mathsf{rad}} \subseteq \mathcal{Y}^{\mathsf{tame}}$, $\quad \operatorname{Supp}(\mathcal{Y}_{\mathsf{cpt}}) = \operatorname{Supp}\mathcal{Y} = \operatorname{Supp}(\mathcal{Y}^{\mathsf{rad}}) = \operatorname{Supp}(\mathcal{Y}^{\mathsf{tame}})$,

Proof. (1) It follows from Lemma 2.15 (resp. Remark 6.14) that $\sqrt{\mathsf{thick}^{\otimes}\,\mathcal{X}}$ (resp. $\operatorname{Supp}^{-1}\operatorname{Supp}\mathcal{X}$) is a thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$. It is clear that $\sqrt{\mathsf{thick}^{\otimes}\,\mathcal{X}}$ (resp. $\operatorname{Supp}^{-1}\operatorname{Supp}\mathcal{X}$) is radical (resp. tame) and contains $\mathcal{X}$. If $\mathcal{C}$ is a radical (resp. tame) thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$ containing $\mathcal{X}$, then we have $\sqrt{\mathsf{thick}^{\otimes}\,\mathcal{X}} \subseteq \sqrt{\mathsf{thick}^{\otimes}\,\mathcal{C}} = \sqrt{\mathcal{C}} = \mathcal{C}$ (resp. $\operatorname{Supp}^{-1}\operatorname{Supp}\mathcal{X} \subseteq \operatorname{Supp}^{-1}\operatorname{Supp}\mathcal{C} = \mathcal{C}$ by Proposition 9.3). Thus, we obtain the two equalities $\mathcal{X}^{\mathsf{rad}} = \sqrt{\mathsf{thick}^{\otimes}\,\mathcal{X}}$ and $\mathcal{X}^{\mathsf{tame}} = \operatorname{Supp}^{-1}\operatorname{Supp}\mathcal{X}$. It remains to show the equality $(\mathsf{thick}^{\otimes}\,\mathcal{X})_{\mathsf{cpt}} = \langle\operatorname{Supp}\mathcal{X}\rangle$. Clearly, $\langle\operatorname{Supp}\mathcal{X}\rangle$ is a compact thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$. Applying Corollary 5.11, we observe that $\langle\operatorname{Supp}\mathcal{X}\rangle$ is contained in $\mathsf{thick}^{\otimes}\,\mathcal{X}$. Let $\mathcal{C}$ be a compact thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$ contained in $\mathsf{thick}^{\otimes}\,\mathcal{X}$. Then it follows from Proposition 9.4 that $\mathcal{C} = \langle\operatorname{Supp}\mathcal{C}\rangle$, which is contained in $\langle\operatorname{Supp}(\mathsf{thick}^{\otimes}\,\mathcal{X})\rangle = \langle\operatorname{Supp}\mathcal{X}\rangle$ by Proposition 4.8(2). We now conclude $(\mathsf{thick}^{\otimes}\,\mathcal{X})_{\mathsf{cpt}} = \langle\operatorname{Supp}\mathcal{X}\rangle$.

(2) Fix a prime ideal $\mathfrak{p}$ of R. Proposition 6.3 says that $\mathcal{S}(\mathfrak{p})$ is a prime thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$, whence it is radical. Therefore $\mathcal{Y}_{\mathfrak{p}} = 0$ if and only if $(\sqrt{\mathcal{Y}})_{\mathfrak{p}} = 0$. This shows $\operatorname{Supp}(\sqrt{\mathcal{Y}}) = \operatorname{Supp}\mathcal{Y}$. Hence $\sqrt{\mathcal{Y}}$ is contained in $\operatorname{Supp}^{-1}\operatorname{Supp}\mathcal{Y}$, meaning that $\mathcal{Y}^{\mathsf{rad}}$ is contained in $\mathcal{Y}^{\mathsf{tame}}$ by (1). Thus we get the inclusions $\mathcal{Y}_{\mathsf{cpt}} \subseteq \mathcal{Y} \subseteq \mathcal{Y}^{\mathsf{rad}} \subseteq \mathcal{Y}^{\mathsf{tame}}$, which implies $\operatorname{Supp}(\mathcal{Y}_{\mathsf{cpt}}) \subseteq \operatorname{Supp}\mathcal{Y} \subseteq \operatorname{Supp}(\mathcal{Y}^{\mathsf{rad}}) \subseteq \operatorname{Supp}(\mathcal{Y}^{\mathsf{tame}})$. By (1) and Proposition 4.8 we get $\operatorname{Supp}(\mathcal{Y}^{\mathsf{tame}}) = \operatorname{Supp}\mathcal{Y} = \operatorname{Supp}(\mathcal{Y}_{\mathsf{cpt}})$. The equalities in the assertion follow. ■

The inclusion $\mathcal{Y}^{\mathsf{rad}} \subseteq \mathcal{Y}^{\mathsf{tame}}$ in Lemma 9.8 in particular says:

Corollary 9.9. *Every tame thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$ is radical.*

We now obtain a bijection, using the above lemma.

Proposition 9.10. *There is a one-to-one correspondence* $()^{\mathsf{tame}} : \mathbf{Cpt} \rightleftarrows \mathbf{Tame} : ()_{\mathsf{cpt}}$.

Proof. Fix a compact thick $\otimes$-ideal $\mathcal{X}$, and a tame thick $\otimes$-ideal $\mathcal{Y}$ of $\mathsf{D}^{-}(R)$. We have $(\mathcal{X}^{\mathsf{tame}})_{\mathsf{cpt}} = \langle\operatorname{Supp}(\mathcal{X}^{\mathsf{tame}})\rangle = \langle\operatorname{Supp}\mathcal{X}\rangle = \mathcal{X}$, where the first equality follows from Lemma 9.8(1), the second from Lemma 9.8(2), and the last from Proposition 9.4. Also, it holds that $(\mathcal{Y}_{\mathsf{cpt}})^{\mathsf{tame}} = \operatorname{Supp}^{-1}\operatorname{Supp}(\mathcal{Y}_{\mathsf{cpt}}) = \operatorname{Supp}^{-1}\operatorname{Supp}\mathcal{Y} = \mathcal{Y}$, where the first equality follows from Lemma 9.8(1), the second from Lemma 9.8(2), and the last from Proposition 9.3. Thus we obtain the one-to-one correspondence in the proposition. ■

For each subset A of $\operatorname{Spec} R$, we put $\mathcal{S}(A) = \{\mathcal{S}(\mathfrak{p}) \mid \mathfrak{p} \in A\}$. For each subset B of $\mathtt{Spec}\,\mathsf{D}^{-}(R)$, we put $\mathfrak{s}(B) = \{\mathfrak{s}(\mathcal{P}) \mid \mathcal{P} \in B\}$. We get another bijection.

Proposition 9.11. *There is a one-to-one correspondence* $\mathcal{S} : \mathbf{Spcl}(\mathsf{Spec}) \rightleftarrows \mathbf{Spcl}({}^{\mathsf{t}}\mathtt{Spec}) : \mathfrak{s}$.

Proof. First of all, applying Theorem 6.8 and Corollary 6.15, we observe that
(9.11.1)
$$\mathfrak{s}(\mathcal{S}(\mathfrak{p})) = \mathfrak{p} \text{ for all } \mathfrak{p} \in \mathsf{Spec}\, R \qquad \text{and} \qquad \mathcal{S}(\mathfrak{s}(\mathcal{P})) = \mathcal{P} \text{ for all } \mathcal{P} \in {}^{\mathsf{t}}\mathtt{Spec}\, \mathsf{D}^{-}(R).$$

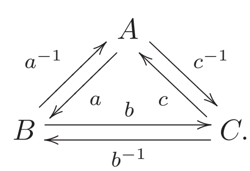
Fix a specialization-closed subset W of $\mathsf{Spec}\, R$ and a specialization-closed subset U of ${}^{\mathsf{t}}\mathtt{Spec}\, \mathsf{D}^{-}(R)$. It follows from (9.11.1) that $\mathfrak{s}(\mathcal{S}(W)) = W$ and $\mathcal{S}(\mathfrak{s}(U)) = U$.

Pick a prime ideal $\mathfrak{p}$ in W. Let X be the closure of $\{\mathcal{S}(\mathfrak{p})\}$ in ${}^{\mathsf{t}}\mathtt{Spec}\, \mathsf{D}^{-}(R)$. Then $X = Y \cap {}^{\mathsf{t}}\mathtt{Spec}\, \mathsf{D}^{-}(R)$, where Y is the closure of $\{\mathcal{S}(\mathfrak{p})\}$ in $\mathtt{Spec}\, \mathsf{D}^{-}(R)$, and hence
$$\begin{aligned} X &= \{\mathcal{P} \in {}^{\mathsf{t}}\mathtt{Spec}\, \mathsf{D}^{-}(R) \mid \mathcal{P} \subseteq \mathcal{S}(\mathfrak{p})\} \\ &= \{\mathcal{S}(\mathfrak{q}) \mid \mathfrak{q} \in \mathsf{Spec}\, R,\ \mathcal{S}(\mathfrak{q}) \subseteq \mathcal{S}(\mathfrak{p})\} \\ &= \{\mathcal{S}(\mathfrak{q}) \mid \mathfrak{q} \in \mathrm{V}(\mathfrak{p})\} \subseteq \mathcal{S}(W), \end{aligned}$$
where the first equality follows from Proposition 2.10, the second from Corollary 6.15, and the third from Theorem 6.8. The inclusion holds since W is a specialization-closed subset of $\mathsf{Spec}\, R$. Therefore, $\mathcal{S}(W)$ is a specialization-closed subset of ${}^{\mathsf{t}}\mathtt{Spec}\, \mathsf{D}^{-}(R)$, namely, $\mathcal{S}(W) \in \mathbf{Spcl}({}^{\mathsf{t}}\mathtt{Spec})$.

Pick $\mathcal{P} \in U$. As U is a subset of ${}^{\mathsf{t}}\mathtt{Spec}\, \mathsf{D}^{-}(R)$, the prime thick $\otimes$-ideal $\mathcal{P}$ is tame. Let $\mathfrak{q}$ be a prime ideal of R containing $\mathfrak{s}(\mathcal{P})$. We then get $\mathcal{S}(\mathfrak{q}) \subseteq \mathcal{S}(\mathfrak{s}(\mathcal{P})) = \mathcal{P}$ by Theorem 6.8 and (9.11.1), which says that $\mathcal{S}(\mathfrak{q})$ belongs to the closure of the set $\{\mathcal{P}\}$ in ${}^{\mathsf{t}}\mathtt{Spec}\, \mathsf{D}^{-}(R)$ by Proposition 2.10. The specialization-closed property of U implies that $\mathcal{S}(\mathfrak{q})$ belongs to U. We have $\mathfrak{q} = \mathfrak{s}(\mathcal{S}(\mathfrak{q}))$ by (9.11.1), which belongs to $\mathfrak{s}(U)$. Consequently, the subset $\mathfrak{s}(U)$ of $\mathsf{Spec}\, R$ is specialization-closed, that is, $\mathfrak{s}(U) \in \mathbf{Spcl}(\mathsf{Spec})$. ■

Here we note an elementary fact on commutativity of a diagram of maps.

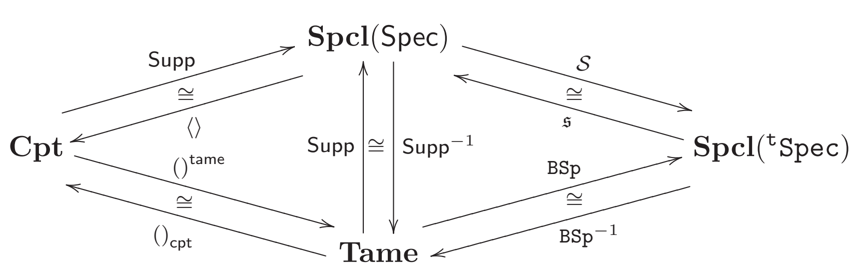
Remark 9.12. Consider the following diagram of bijections



One can choose infinitely many compositions of maps in the diagram, but once one of them is equal to another, this triangle with edges having any directions commutes. To be more explicit, if $c = ba$ for instance, then the set $\{1, a, a^{-1}, b, b^{-1}, c, c^{-1}\}$ is closed under possible compositions.

Now we can state and prove our first main result in this section.

Theorem 9.13. *There is a commutative diagram of mutually inverse bijections:*



Proof. The five one-to-one correspondences in the diagram are shown in Propositions 9.3, 9.4, 9.7, 9.10 and 9.11. It remains to show the commutativity, and for this we take Remark 9.12 into account.

For a thick $\otimes$-ideal $\mathcal{X}$ of $\mathsf{D}^{-}(R)$, we have $\mathsf{Supp}(\mathcal{X}^{\mathsf{tame}}) = \mathsf{Supp}\,\mathcal{X}$ by Lemma 9.8(2), which shows that the left triangle in the diagram commutes. It is easy to observe from Corollary 6.15 that

$$\mathtt{BSp}\,\mathcal{X} = \mathcal{S}(\mathsf{Supp}\,\mathcal{X}) \text{ for any subcategory } \mathcal{X} \text{ of } \mathsf{D}^{-}(R). \tag{9.13.1}$$

The commutativity of the right triangle in the diagram follows from (9.13.1). ■

Remark 9.14. The bijections in the diagram of Theorem 9.13 induce lattice structures in **Tame** and **Spcl**($^{\mathtt{t}}\mathtt{Spec}$), so that the maps are lattice isomorphisms. However, we do not know if there is an explicit way to define lattice structures like the one of **Cpt** given in Proposition 5.18(2).

Let $f : A \to B$ and $g : B \to A$ be maps with $gf = 1$. Then we say that (f, g) is a *section-retraction pair*, and write $f \dashv g$. Our next goal is to construct a certain commutative diagram of section-retraction pairs, and for this we again give several propositions. The first one is a consequence of Theorem 2.17.

Proposition 9.15. *There is a one-to-one correspondence* $\mathsf{BSupp} : \mathbf{Rad} \rightleftarrows \mathbf{Thom} : \mathsf{BSupp}^{-1}$.

Proposition 9.16. *There is a section-retraction pair* $()^{\mathsf{rad}} : \mathbf{Cpt} \rightleftarrows \mathbf{Rad} : ()_{\mathsf{cpt}}$.

Proof. For every $\mathcal{X} \in \mathbf{Cpt}$, we have $(\mathcal{X}^{\mathsf{rad}})_{\mathsf{cpt}} = \langle \mathsf{Supp}(\mathcal{X}^{\mathsf{rad}}) \rangle = \langle \mathsf{Supp}\,\mathcal{X} \rangle = \mathcal{X}_{\mathsf{cpt}} = \mathcal{X}$ by Lemma 9.8. ■

For each subset A of $\mathsf{Spec}\,R$, we set $\overline{\mathcal{S}}(A) = \bigcup_{\mathfrak{p} \in A} \overline{\{\mathcal{S}(\mathfrak{p})\}}$. For each subset B of $\mathtt{Spec}\,\mathsf{D}^{-}(R)$, we set $\mathcal{S}^{-1}(B) = \{\mathfrak{p} \in \mathsf{Spec}\,R \mid \mathcal{S}(\mathfrak{p}) \in B\}$. We obtain another section-retraction pair.

Proposition 9.17. *There is a section-retraction pair* $\overline{\mathcal{S}} : \mathbf{Spcl}(\mathsf{Spec}) \rightleftarrows \mathbf{Thom} : \mathcal{S}^{-1}$.

Proof. Proposition 2.10 and Corollary 6.11 yield

$$\mathsf{BSupp}(R/\mathfrak{p}) = \overline{\{\mathcal{S}(\mathfrak{p})\}} \text{ for any prime ideal } \mathfrak{p} \text{ of } R, \tag{9.17.1}$$

whence $(\overline{\{\mathcal{S}(\mathfrak{p})\}})^{\complement} = \mathtt{U}(R/\mathfrak{p})$, which is quasi-compact by Proposition 2.11. Hence $\overline{\mathcal{S}}(A)$ is a Thomason subset of $\mathtt{Spec}\,\mathsf{D}^{-}(R)$ for any subset A of $\mathsf{Spec}\,R$. In particular, we get a map $\overline{\mathcal{S}} : \mathbf{Spcl}(\mathsf{Spec}) \to \mathbf{Thom}$.

Let T be a Thomason subset of $\mathtt{Spec}\,\mathsf{D}^{-}(R)$. Let $\mathfrak{p}, \mathfrak{q}$ be prime ideals of R with $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathcal{S}(\mathfrak{p}) \in T$. Then $\mathcal{S}(\mathfrak{q})$ belongs to $\overline{\{\mathcal{S}(\mathfrak{p})\}}$ by Proposition 2.10 and Theorem 6.8. Since T is Thomason, it contains $\overline{\{\mathcal{S}(\mathfrak{p})\}}$. Hence $\mathcal{S}(\mathfrak{q})$ belongs to T. Thus the assignment $T \mapsto \mathcal{S}^{-1}(T)$ defines a map $\mathcal{S}^{-1} : \mathbf{Thom} \to \mathbf{Spcl}(\mathsf{Spec})$.

For a specialization-closed subset W of $\mathsf{Spec}\,R$ and a prime ideal $\mathfrak{p}$ of R, one has

$$\begin{aligned} \mathcal{S}(\mathfrak{p}) \in \overline{\{\mathcal{S}(\mathfrak{q})\}} \text{ for some } \mathfrak{q} \in W &\Leftrightarrow \mathcal{S}(\mathfrak{p}) \subseteq \mathcal{S}(\mathfrak{q}) \text{ for some } \mathfrak{q} \in W \\ &\Leftrightarrow \mathfrak{p} \supseteq \mathfrak{q} \text{ for some } \mathfrak{q} \in W \\ &\Leftrightarrow \mathfrak{p} \in W, \end{aligned}$$

where the first and second equivalences follow from Proposition 2.10 and Theorem 6.8, and the last equivalence holds by the fact that W is specialization-closed. This yields $\mathcal{S}^{-1}(\overline{\mathcal{S}}(W)) = W$. ■

Now we consider describing spcl-closures and spcl-interiors.

Proposition 9.18. *Let A be a specialization-closed subset of $\mathtt{Spec}\,\mathsf{D}^{-}(R)$, and let B be a specialization-closed subset of ${}^{\mathsf{t}}\mathtt{Spec}\,\mathsf{D}^{-}(R)$.*

(1) *Let A_{spcl} stand for the spcl-interior of A in ${}^{\mathsf{t}}\mathtt{Spec}\,\mathsf{D}^{-}(R)$. Then*

$$A_{\mathsf{spcl}} = A \cap {}^{\mathsf{t}}\mathtt{Spec}\,\mathsf{D}^{-}(R).$$

(2) *Let B^{spcl} stand for the spcl-closure of B in $\mathtt{Spec}\,\mathsf{D}^{-}(R)$. Then*

$$B^{\mathsf{spcl}} = \{\mathcal{P} \in \mathtt{Spec}\,\mathsf{D}^{-}(R) \mid \mathcal{P}^{\mathsf{tame}} \in B\} = \textstyle\bigcup_{\mathcal{P}\in B^{\mathsf{spcl}}} \mathsf{BSupp}(R/\mathfrak{s}(\mathcal{P})).$$

In particular, B^{spcl} is a Thomason subset of $\mathtt{Spec}\,\mathsf{D}^{-}(R)$.

Proof. (1) We easily observe that $A \cap {}^{\mathsf{t}}\mathtt{Spec}\,\mathsf{D}^{-}(R)$ is a specialization-closed subset of the topological space ${}^{\mathsf{t}}\mathtt{Spec}\,\mathsf{D}^{-}(R)$ contained in A. Also, it is obvious that if X is a specialization-closed subset of ${}^{\mathsf{t}}\mathtt{Spec}\,\mathsf{D}^{-}(R)$ contained in A, then X is contained in $A \cap {}^{\mathsf{t}}\mathtt{Spec}\,\mathsf{D}^{-}(R)$. Hence $A \cap {}^{\mathsf{t}}\mathtt{Spec}\,\mathsf{D}^{-}(R)$ coincides with A_{spcl}.

(2) Let C be the set of prime thick $\otimes$-ideals $\mathcal{P}$ of $\mathsf{D}^{-}(R)$ with $\mathcal{P}^{\mathsf{tame}} \in B$. We proceed step by step.

(a) Each $\mathcal{Q} \in B$ is tame. Hence we have $\mathcal{Q}^{\mathsf{tame}} = \mathcal{Q} \in B$. This shows that C contains B.

(b) Let Y be a specialization-closed subset of $\mathtt{Spec}\,\mathsf{D}^{-}(R)$ containing B. Take any element $\mathcal{P}$ of C. Then $\mathcal{P}^{\mathsf{tame}}$ belongs to B, and hence to Y. Since Y is specialization-closed, $\overline{\{\mathcal{P}^{\mathsf{tame}}\}}$ is contained in Y. Hence $\mathcal{P}$ belongs to Y by Proposition 2.10. It follows that C is contained in Y.

(c) We prove $C = \bigcup_{\mathcal{P}\in C} \mathsf{BSupp}(R/\mathfrak{s}(\mathcal{P}))$. Combining Theorem 6.8, Lemma 9.8(1) and (9.17.1) gives rise to $\mathsf{BSupp}(R/\mathfrak{s}(\mathcal{P})) = \overline{\{\mathcal{P}^{\mathsf{tame}}\}}$, and thus it is enough to verify $C = \bigcup_{\mathcal{P}\in C} \overline{\{\mathcal{P}^{\mathsf{tame}}\}}$. By Proposition2.10 we see that C is contained in $\bigcup_{\mathcal{P}\in C} \overline{\{\mathcal{P}^{\mathsf{tame}}\}}$. Conversely, let $\mathcal{P} \in \mathcal{C}$ and $\mathcal{Q} \in \overline{\{\mathcal{P}^{\mathsf{tame}}\}}$. Then $\mathcal{P}^{\mathsf{tame}}$ belongs to B, and $\mathcal{Q}$ is contained in $\mathcal{P}^{\mathsf{tame}}$ by Proposition 2.10, which shows that $\mathcal{Q}^{\mathsf{tame}}$ is contained in $\mathcal{P}^{\mathsf{tame}}$. Hence $\mathcal{Q}^{\mathsf{tame}}$ is in $\overline{\{\mathcal{P}^{\mathsf{tame}}\}} \cap {}^{\mathsf{t}}\mathtt{Spec}\,\mathsf{D}^{-}(R)$. As B is specialization-closed in ${}^{\mathsf{t}}\mathtt{Spec}\,\mathsf{D}^{-}(R)$, it contains $\overline{\{\mathcal{P}^{\mathsf{tame}}\}} \cap {}^{\mathsf{t}}\mathtt{Spec}\,\mathsf{D}^{-}(R)$, and therefore $\mathcal{Q}^{\mathsf{tame}}$ is in B. Thus $\mathcal{Q}$ belongs to C. We obtain $C = \bigcup_{\mathcal{P}\in C} \overline{\{\mathcal{P}^{\mathsf{tame}}\}}$.

The equality $C = \bigcup_{\mathcal{P}\in C} \mathsf{BSupp}(R/\mathfrak{s}(\mathcal{P}))$ shown in (c) especially says that C is specialization-closed. By this together with (a) and (b) we obtain $C = B^{\mathsf{spcl}}$, and it follows that $C = \bigcup_{\mathcal{P}\in B^{\mathsf{spcl}}} \mathsf{BSupp}(R/\mathfrak{s}(\mathcal{P}))$. ■

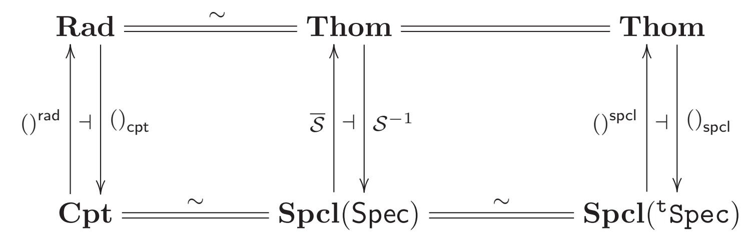
We now obtain another section-retraction pair:

Proposition 9.19. *The operations $()^{\mathsf{spcl}}$ and $()_{\mathsf{spcl}}$ defined in Proposition 9.18 make a section-retraction pair $()^{\mathsf{spcl}} : \mathbf{Spcl}({}^{\mathsf{t}}\mathtt{Spec}) \rightleftarrows \mathbf{Thom} : ()_{\mathsf{spcl}}$.*

Proof. Let U be a specialization-closed subset of ${}^{\mathsf{t}}\mathtt{Spec}\,\mathsf{D}^{-}(R)$. By Proposition 9.18, U^{spcl} is a Thomason subset of $\mathtt{Spec}\,\mathsf{D}^{-}(R)$, and $(U^{\mathsf{spcl}})_{\mathsf{spcl}} = U^{\mathsf{spcl}} \cap {}^{\mathsf{t}}\mathtt{Spec}\,\mathsf{D}^{-}(R) = \{\mathcal{P} \in {}^{\mathsf{t}}\mathtt{Spec}\,\mathsf{D}^{-}(R) \mid \mathcal{P}^{\mathsf{tame}} \in U\} = U$. ■

We can prove our second main result in this section.

Theorem 9.20. *There is a diagram*



where the upper horizontal bijections are the one given in Proposition 9.15 and an equality, and the lower horizontal bijections are the ones appearing in Theorem 9.13. The diagram with vertical arrows from the bottom (resp. top) to the top (resp. bottom) is commutative.

Proof. The three section-retraction pairs are obtained in Propositions 9.16, 9.17 and 9.19.

We claim that for any thick $\otimes$-ideal $\mathcal{X}$ of $\mathsf{D}^-(R)$ one has

$$\mathsf{BSupp}(\mathcal{X}^{\mathsf{rad}}) = \mathsf{BSupp}\,\mathcal{X}. \tag{9.20.1}$$

Indeed, Lemma 9.8(1) shows $\mathcal{X}^{\mathsf{rad}} = \sqrt{\mathcal{X}}$. The inclusion $\mathcal{X} \subseteq \sqrt{\mathcal{X}}$ implies $\mathsf{BSupp}\,\mathcal{X} \subseteq \mathsf{BSupp}\,\sqrt{\mathcal{X}}$. Let $\mathcal{P}$ be a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$. If $\mathcal{X}$ is contained in $\mathcal{P}$, then so is $\sqrt{\mathcal{X}}$ as $\mathcal{P}$ is prime. Therefore we obtain $\mathsf{BSupp}\,\sqrt{\mathcal{X}} = \mathsf{BSupp}\,\mathcal{X}$, and the claim follows.

Fix a thick $\otimes$-ideal $\mathcal{C}$ of $\mathsf{D}^-(R)$. For a prime ideal $\mathfrak{p}$ of R one has $\mathcal{S}(\mathfrak{p}) \in \mathsf{BSupp}\,\mathcal{C}$ if and only if $\mathcal{C} \nsubseteq \mathcal{S}(\mathfrak{p})$, if and only if $\mathcal{C}_\mathfrak{p} \neq 0$, if and only if $\mathfrak{p} \in \mathsf{Supp}\,\mathcal{C}$. This shows $\mathcal{S}^{-1}(\mathsf{BSupp}\,\mathcal{C}) = \mathsf{Supp}\,\mathcal{C}$. Lemma 9.8(2) gives $\mathsf{Supp}(\mathcal{C}_{\mathsf{cpt}}) = \mathcal{S}^{-1}(\mathsf{BSupp}\,\mathcal{C})$. Next, suppose that $\mathcal{C}$ is compact. Lemma 9.8(1), (9.17.1) and (9.20.1) yield

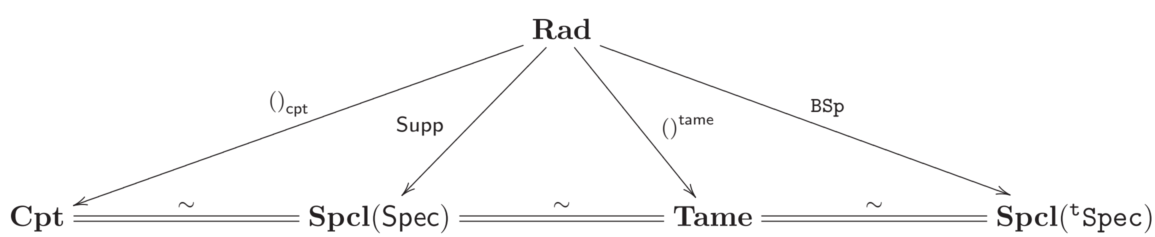
$$\begin{aligned}\mathsf{BSupp}(\mathcal{C}^{\mathsf{rad}}) = \mathsf{BSupp}\,\mathcal{C} = \mathsf{BSupp}(\mathcal{C}_{\mathsf{cpt}}) &= \mathsf{BSupp}(\langle \mathsf{Supp}\,\mathcal{C}\rangle) \\ &= \mathsf{BSupp}\{R/\mathfrak{p} \mid \mathfrak{p} \in \mathsf{Supp}\,\mathcal{C}\} = \overline{\mathcal{S}}(\mathsf{Supp}\,\mathcal{C}).\end{aligned}$$

Thus we obtain the commutativity of the left square of the diagram.

Let A be any subset of $\mathsf{Spec}\,R$. It is clear that $\mathcal{S}(A) = \{\mathcal{S}(\mathfrak{p}) \mid \mathfrak{p} \in A\}$ is contained in $\overline{\mathcal{S}}(A)$. As $\overline{\mathcal{S}}(A)$ is a union of closed subsets of the topological space $\mathtt{Spec}\,\mathsf{D}^-(R)$, it is a specialization-closed subset of $\mathtt{Spec}\,\mathsf{D}^-(R)$. Note that any specialization-closed subset of $\mathtt{Spec}\,\mathsf{D}^-(R)$ containing $\mathcal{S}(A)$ contains $\overline{\mathcal{S}}(A)$. Hence we have $\overline{\mathcal{S}}(A) = (\mathcal{S}(A))^{\mathsf{spcl}}$. Let B be a specialization-closed subset of $\mathtt{Spec}\,\mathsf{D}^-(R)$. Then $\mathcal{S}(\mathcal{S}^{-1}(B)) = \{\mathcal{S}(\mathfrak{p}) \mid \mathfrak{p} \in \mathsf{Spec}\,R,\ \mathcal{S}(\mathfrak{p}) \in B\} = B \cap {}^{\mathsf{t}}\mathtt{Spec}\,\mathsf{D}^-(R) = B_{\mathsf{spcl}}$ by Corollary 6.15 and Proposition 9.18(1). Now it follows that the right square of the diagram commutes. ■

We close this section by producing another commutative diagram, coming from the above theorem.

Corollary 9.21. *There is a commutative diagram:*



Here, the three bijections are the ones appearing in Theorem 9.13, and the other maps are retractions.

Proof. We have the following diagram.

$$\begin{array}{ccccccc} \mathbf{Rad} & & & & & & \\ {\scriptstyle ()^{\mathsf{rad}}}\uparrow \dashv \downarrow{\scriptstyle ()_{\mathsf{cpt}}} & & & & & & \\ \mathbf{Cpt} & \underset{\langle\rangle}{\overset{\mathsf{Supp}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\cong}}}} & \mathbf{Spcl}(\mathsf{Spec}) & \underset{\mathsf{Supp}}{\overset{\mathsf{Supp}^{-1}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\cong}}}} & \mathbf{Tame} & \underset{\mathtt{BSp}^{-1}}{\overset{\mathtt{BSp}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\cong}}}} & \mathbf{Spcl}({}^{\mathsf{t}}\mathtt{Spec}) \end{array}$$

Thus it suffices to verify the equalities of compositions of maps $\mathsf{Supp} \circ ()_{\mathsf{cpt}} = \mathsf{Supp}$, $\mathsf{Supp}^{-1} \circ \mathsf{Supp} = ()^{\mathsf{tame}}$ and $\mathtt{BSp} \circ ()^{\mathsf{tame}} = \mathtt{BSp}$. This is equivalent to showing that the equalities

$$\text{(i) } \mathsf{Supp}(\mathcal{X}_{\mathsf{cpt}}) = \mathsf{Supp}\,\mathcal{X}, \qquad \text{(ii) } \mathsf{Supp}^{-1}\,\mathsf{Supp}\,\mathcal{X} = \mathcal{X}^{\mathsf{tame}}, \qquad \text{(iii) } \mathtt{BSp}(\mathcal{X}^{\mathsf{tame}}) = \mathtt{BSp}\,\mathcal{X}$$

hold for each (radical) thick $\otimes$-ideal $\mathcal{X}$ of $\mathsf{D}^{-}(R)$. The equalities (i) and (ii) immediately follow from Lemma 9.8. We have $\mathtt{BSp}(\mathcal{X}^{\mathsf{tame}}) = \mathtt{BSp}(\mathsf{Supp}^{-1}\,\mathsf{Supp}\,\mathcal{X}) = (\mathtt{BSp} \circ \mathsf{Supp}^{-1})(\mathsf{Supp}\,\mathcal{X}) = \mathcal{S}(\mathsf{Supp}\,\mathcal{X}) = \mathtt{BSp}\,\mathcal{X}$, where the first and last equalities follow from Lemma 9.8(1) and (9.13.1). Proposition 4.8(2) says that $\mathsf{Supp}\,\mathcal{X}$ belongs to $\mathbf{Spcl}(\mathsf{Spec})$, and the third equality above is obtained by Theorem 9.13. Now the assertion (iii) follows, and the proof of the corollary is completed. ■

10. **Distinction between thick tensor ideals, and Balmer's conjecture**

In this section, we consider when the section-retraction pairs in Theorem 9.20 and Corollary 9.21 are one-to-one correspondences, and construct a counterexample to the conjecture of Balmer. We begin with a lemma on the annihilator of an object in the thick $\otimes$-ideal closure.

Lemma 10.1. *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of objects of $\mathsf{D}^{-}(R)$. For $M \in \mathsf{thick}^{\otimes}\{X_\lambda\}_{\lambda\in\Lambda}$ there are (pairwise distinct) indices $\lambda_1,\dots,\lambda_n \in \Lambda$ and integers $e_1,\dots,e_n > 0$ such that $\operatorname{Ann} M$ contains $\prod_{i=1}^{n}(\operatorname{Ann} X_{\lambda_i})^{e_i}$.*

Proof. Let $\mathcal{C}$ be the subcategory of $\mathsf{D}^{-}(R)$ consisting of objects C such that there are $\lambda_1,\dots,\lambda_n \in \Lambda$ and $e_1,\dots,e_n > 0$ such that $\operatorname{Ann} C$ contains $\prod_{i=1}^{n}(\operatorname{Ann} X_{\lambda_i})^{e_i}$. The following statements hold in general.

- If A is an object of $\mathsf{D}^{-}(R)$ and B is a direct summand of A, then $\operatorname{Ann} A \subseteq \operatorname{Ann} B$.
- For each object $A \in \mathsf{D}^{-}(R)$ one has $\operatorname{Ann}(A[\pm 1]) = \operatorname{Ann} A$.
- If $A \to B \to C \to A[1]$ is an exact triangle in $\mathsf{D}^{-}(R)$, then $\operatorname{Ann} B$ contains $\operatorname{Ann} A \cdot \operatorname{Ann} C$.
- For any objects A, B of $\mathsf{D}^{-}(R)$ one has $\operatorname{Ann}(A \otimes_R^{\mathbf{L}} B) \supseteq \operatorname{Ann} A$.

It follows from these that $\mathcal{C}$ is a thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$. Since X_λ is in $\mathcal{C}$ for all $\lambda \in \Lambda$, it holds that $\mathcal{C}$ contains $\mathsf{thick}^{\otimes}\{X_\lambda\}_{\lambda\in\Lambda}$. The assertion of the lemma now follows. ■

The proposition below says in particular that in the case where R is a local ring $\mathsf{D}^{-}(R)$ has a compact prime thick tensor ideal. On the other hand, in the nonlocal case it is often that $\mathsf{D}^{-}(R)$ has no such one.

Proposition 10.2. (1) *If R is a local ring with maximal ideal $\mathfrak{m}$, then $\mathbf{Cpt} \cap \mathfrak{s}^{-1}(\mathfrak{m}) = \{\mathbf{0}\} \neq \emptyset$.*

(2) *Let R be a nonlocal semilocal domain. Then there exists no compact prime thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$. In particular, one has $\mathcal{P}_{\mathsf{cpt}} \subsetneq \mathcal{P} = \mathcal{P}^{\mathsf{rad}}$ for all $\mathcal{P} \in \mathsf{Spec}\,\mathsf{D}^{-}(R)$.*

Proof. (1) Let $\mathcal{P}$ be in $\mathsf{Spec}\,\mathsf{D}^-(R)$. Then $\mathcal{P}$ is in $\mathfrak{s}^{-1}(\mathfrak{m})$ if and only if $\mathsf{Supp}\,\mathcal{P} = \{\mathfrak{p} \in \mathsf{Spec}\,R \mid \mathfrak{p} \not\subseteq \mathfrak{m}\} = \emptyset$ by Proposition 7.1, if and only if $\mathcal{P} = \mathbf{0}$ by Remark 4.6. Since $\mathbf{0}$ is compact, we are done.

(2) Let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be the (pairwise distinct) maximal ideals of R with $n \geqslant 2$. For each $1 \leqslant i \leqslant n$ one finds an element $x_i \in \mathfrak{m}_i$ that does not belong to any other maximal ideals. As R is a domain of positive dimension, x_i is a non-zerodivisor of R. Set $C_i = \bigoplus_{t \geqslant 0} R/x_i^{t+1}[t]$; note that this is an object of $\mathsf{D}^-(R)$. We have $\mathsf{Supp}(C_1 \otimes_R^{\mathbf{L}} \cdots \otimes_R^{\mathbf{L}} C_n) = \bigcap_{i=1}^n \mathsf{Supp}\,C_i = \bigcap_{i=1}^n \mathrm{V}(x_i) = \mathrm{V}(x_1, \dots, x_n) = \emptyset$ by Lemma 4.7(4) and the fact that (x_1, x_2) is a unit ideal of R. Remark 4.6 gives $C_1 \otimes_R^{\mathbf{L}} \cdots \otimes_R^{\mathbf{L}} C_n = 0$.

Suppose that there exists a compact prime thick $\otimes$-ideal $\mathcal{P}$ of $\mathsf{D}^-(R)$. Then $C_1 \otimes_R^{\mathbf{L}} \cdots \otimes_R^{\mathbf{L}} C_n = 0$ is contained in $\mathcal{P}$, and so is C_ℓ for some $1 \leqslant \ell \leqslant n$. We have $\mathcal{P} = \langle \mathsf{Supp}\,\mathcal{P} \rangle$ by Proposition 9.4, and by Lemma 10.1 there exist prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r \in \mathsf{Supp}\,\mathcal{P}$ and integers $e_1, \dots, e_r > 0$ such that $\mathrm{Ann}\,C_\ell$ contains $(\mathrm{Ann}\,R/\mathfrak{p}_1)^{e_1} \cdots (\mathrm{Ann}\,R/\mathfrak{p}_r)^{e_r} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$. Since R is a domain and x_ℓ is a non-unit of R, we have $\mathrm{Ann}\,C_\ell = \bigcap_{t \geqslant 0} x_\ell^{t+1} R = 0$ by Krull's intersection theorem. Therefore $\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r} = 0$, and $\mathfrak{p}_s = 0$ for some $1 \leqslant s \leqslant r$ as R is a domain. Thus the zero ideal 0 of R belongs to $\mathsf{Supp}\,\mathcal{P}$, which implies $\mathsf{Supp}\,\mathcal{P} = \mathsf{Spec}\,R$. We obtain $\mathcal{P} = \mathsf{D}^-(R)$ by Proposition 6.1, which is a contradiction. ∎

To show a main result of this section, we make two lemmas. The first one concerns the structure of the radical and tame closures, while the second one gives an elementary characterization of Artinian rings.

Lemma 10.3. *Let $\mathcal{X}$ be a subcategory of $\mathsf{D}^-(R)$. One has*

$$\mathcal{X}^{\mathsf{rad}} = \bigcap\nolimits_{\mathcal{X} \subseteq \mathcal{P} \in \mathsf{Spec}\,\mathsf{D}^-(R)} \mathcal{P}, \qquad \mathcal{X}^{\mathsf{tame}} = \bigcap\nolimits_{\mathcal{X} \subseteq \mathcal{P} \in {}^{\mathsf{t}}\mathsf{Spec}\,\mathsf{D}^-(R)} \mathcal{P}.$$

Proof. Lemma 9.8(1) implies $\mathcal{X}^{\mathsf{rad}} = \sqrt{\mathsf{thick}^{\otimes} \mathcal{X}}$, which coincides with the intersection of the prime thick $\otimes$-ideals of $\mathsf{D}^-(R)$ containing $\mathsf{thick}^{\otimes} \mathcal{X}$ by Lemma 2.15. This is equal to the intersection of the prime thick $\otimes$-ideals containing $\mathcal{X}$, and thus the first equality holds. As for the second equality, if $\mathcal{P}$ is a tame thick $\otimes$-ideal containing $\mathcal{X}$, then we have $\mathcal{X}^{\mathsf{tame}} \subseteq \mathcal{P}^{\mathsf{tame}} = \mathcal{P}$, which shows the inclusion ($\subseteq$). Let M be an object of $\mathsf{D}^-(R)$ belonging to all $\mathcal{P} \in {}^{\mathsf{t}}\mathsf{Spec}\,\mathsf{D}^-(R)$ with $\mathcal{X} \subseteq \mathcal{P}$. Corollary 6.15 says that M is in $\mathcal{S}(\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ of R with $\mathcal{X} \subseteq \mathcal{S}(\mathfrak{p})$. This means that $\mathsf{Supp}\,M$ is contained in $\mathsf{Supp}\,\mathcal{X}$. Hence M is in $\mathsf{Supp}^{-1}\,\mathsf{Supp}\,\mathcal{X}$, which coincides with $\mathcal{X}^{\mathsf{tame}}$ by Lemma 9.8(1). Thus the second equality follows. ∎

Lemma 10.4. *The ring R is Artinian if and only if for any sequence $I_1, I_2, \dots$ of ideals of R it holds that* $\mathrm{V}(\bigcap_{n \geqslant 1} I_n) = \bigcup_{n \geqslant 1} \mathrm{V}(I_n)$.

Proof. First of all, note that the inclusion $\mathrm{V}(\bigcap_{n \geqslant 1} I_n) \supseteq \bigcup_{n \geqslant 1} \mathrm{V}(I_n)$ always holds.

If R is Artinian, then there exists an integer $m \geqslant 1$ such that $\bigcap_{n \geqslant 1} I_n = \bigcap_{j=1}^m I_j$. From this we obtain $\mathrm{V}(\bigcap_{n \geqslant 1} I_n) = \mathrm{V}(\bigcap_{j=1}^m I_j) = \bigcup_{j=1}^m \mathrm{V}(I_j) \subseteq \bigcup_{n \geqslant 1} \mathrm{V}(I_n)$. This shows the "only if" part.

Let us prove the "if" part. Assume first that R has infinitely many maximal ideals, and take a sequence $\mathfrak{m}_1, \mathfrak{m}_2, \dots$ of pairwise distinct maximal ideals of R. By assumption, we get $\mathrm{V}(\bigcap_{n \geqslant 1} \mathfrak{m}_n) = \bigcup_{n \geqslant 1} \mathrm{V}(\mathfrak{m}_n)$. Since $\mathrm{V}(\bigcap_{n \geqslant 1} \mathfrak{m}_n)$ is a closed subset of $\mathsf{Spec}\,R$, it has only finitely many minimal elements with respect to the inclusion relation. However, $\bigcup_{n \geqslant 1} \mathrm{V}(\mathfrak{m}_n) = \{\mathfrak{m}_1, \mathfrak{m}_2, \dots\}$ has infinitely many minimal elements, which is a contradiction. Thus, R is a semilocal ring. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_t$ be the maximal ideals of R, and $J = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_t$ the Jacobson radical of R. Applying the assumption to the sequence $\{J^n\}_{n \geqslant 1}$ of ideals gives $\mathrm{V}(\bigcap_{n \geqslant 1} J^n) = \bigcup_{n \geqslant 1} \mathrm{V}(J^n) = \mathrm{V}(J)$. By

Krull's intersection theorem, we obtain $\bigcap_{n\geqslant 1} J^n = 0$, whence $\mathrm{V}(J) = \mathsf{Spec}\, R$. Hence $\mathsf{Spec}\, R = \{\mathfrak{m}_1, \dots, \mathfrak{m}_t\} = \mathrm{Max}\, R$, and we conclude that R is Artinian. ∎

Now we can prove our first main result in this section. Roughly speaking, if our ring R is Artinian, then everything is explicit and behaves well, and vice versa. Note that this result includes Corollary 7.14.

Theorem 10.5. *The following are equivalent.*

(1) *The ring R is Artinian.*
(2) *Every thick $\otimes$-ideal of $\mathsf{D}^-(R)$ is compact, tame and radical.*
(3) *The maps $\mathcal{S} : \mathsf{Spec}\, R \rightleftarrows \mathtt{Spec}\, \mathsf{D}^-(R) : \mathfrak{s}$ are mutually inverse homeomorphisms.*
(4) *The section-retraction pair $\mathcal{S} : \mathsf{Spec}\, R \rightleftarrows \mathtt{Spec}\, \mathsf{D}^-(R) : \mathfrak{s}$ is a one-to-one correspondence.*
(5) *The section-retraction pair $()_{\mathsf{cpt}} : \mathbf{Rad} \rightleftarrows \mathbf{Cpt} : ()^{\mathsf{rad}}$ is a one-to-one correspondence.*
(6) *The section-retraction pair $\mathcal{S}^{-1} : \mathbf{Thom} \rightleftarrows \mathbf{Spcl}(\mathsf{Spec}) : \overline{\mathcal{S}}$ is a one-to-one correspondence.*
(7) *The section-retraction pair $()_{\mathsf{spcl}} : \mathbf{Thom} \rightleftarrows \mathbf{Spcl}({}^{\mathsf{t}}\mathtt{Spec}) : ()^{\mathsf{spcl}}$ is a one-to-one correspondence.*
(8) *The retraction $\mathsf{Supp} : \mathbf{Rad} \to \mathbf{Spcl}(\mathsf{Spec})$ is a bijection.*
(9) *The retraction $()^{\mathsf{tame}} : \mathbf{Rad} \to \mathbf{Tame}$ is a bijection.*
(10) *The retraction $\mathtt{BSp} : \mathbf{Rad} \to \mathbf{Spcl}({}^{\mathsf{t}}\mathtt{Spec})$ is a bijection.*
(11) *The inclusion $\mathbf{Rad} \supseteq \mathbf{Tame}$ is an equality.*

Proof. Theorems 6.8, 9.13, 9.20 and Corollary 9.9 imply that the pairs in (4), (5), (6), (7) are section-retraction pairs, the maps in (8), (9), (10) are retractions, and one has the inclusion in (11).

The equivalences (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) and (5) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9) $\Leftrightarrow$ (10) follow from Theorem 9.20 and Corollary 9.21, respectively. It is trivial that (3) implies (4), while (1) implies (2) by Corollaries 5.20, 9.9 and Proposition 4.8(1). If $\mathtt{Spec}\, \mathsf{D}^-(R) = {}^{\mathsf{t}}\mathtt{Spec}\, \mathsf{D}^-(R)$, then $\mathcal{S} = \mathcal{S}'$ and $\mathfrak{s} = \mathfrak{s}'$. From Theorems 7.2(3) and 7.4 we see that (2) implies (3). Corollary 9.9 says $\mathcal{X}^{\mathsf{tame}} \in \mathbf{Rad}$ for each $\mathcal{X} \in \mathbf{Rad}$. Hence, if $()^{\mathsf{tame}} : \mathbf{Rad} \to \mathbf{Tame}$ is injective, then $\mathcal{X} = \mathcal{X}^{\mathsf{tame}}$ holds. This shows that (9) implies (11). It is easily seen that the converse is also true, and we get the equivalence (9) $\Leftrightarrow$ (11). When $\mathcal{S} : \mathsf{Spec}\, R \to \mathtt{Spec}\, \mathsf{D}^-(R)$ is surjective, we have $\mathtt{Spec}\, \mathsf{D}^-(R) = {}^{\mathsf{t}}\mathtt{Spec}\, \mathsf{D}^-(R)$, and for a radical thick $\otimes$-ideal $\mathcal{X}$ it holds that $\mathcal{X} = \mathcal{X}^{\mathsf{rad}} = \bigcap_{\mathcal{X} \subseteq \mathcal{P} \in \mathtt{Spec}\, \mathsf{D}^-(R)} \mathcal{P} = \bigcap_{\mathcal{X} \subseteq \mathcal{P} \in {}^{\mathsf{t}}\mathtt{Spec}\, \mathsf{D}^-(R)} \mathcal{P} = \mathcal{X}^{\mathsf{tame}}$ by Lemma 10.3, whence $\mathcal{X}$ is tame. Therefore, (4) implies (11).

Now it remains to prove that (11) implies (1). By Lemma 10.4, it suffices to prove that $\mathrm{V}(\bigcap_{n\geqslant 1} I_n)$ is contained in $\bigcup_{n\geqslant 1} \mathrm{V}(I_n)$ for any sequence $I_1, I_2, \dots$ of ideals of R. For each $n \geqslant 1$, fix a system of generators $\boldsymbol{x}(n)$ of I_n. Set $C = \bigoplus_{n\geqslant 1} \mathrm{K}(\boldsymbol{x}(n), R)[n]$; note that this is defined in $\mathsf{D}^-(R)$. Then $\mathsf{Supp}\, C = \bigcup_{n\geqslant 1} \mathsf{Supp}\, \mathrm{K}(\boldsymbol{x}(n), R) = \bigcup_{n\geqslant 1} \mathrm{V}(I_n)$ by Proposition 5.3(3). The radical closure $\mathcal{E}$ of $\langle \bigcup_{n\geqslant 1} \mathrm{V}(I_n) \rangle$ is tame by assumption. Lemma 9.8 implies $\mathsf{Supp}\, \mathcal{E} = \bigcup_{n\geqslant 1} \mathrm{V}(I_n) = \mathsf{Supp}\, C$. Thus C is in $\mathsf{Supp}^{-1} \mathsf{Supp}\, \mathcal{E} = \mathcal{E}$ by Proposition 9.3, and $C^{\otimes r} \in \langle \bigcup_{n\geqslant 1} \mathrm{V}(I_n) \rangle$ for some $r > 0$. Using [BH, Proposition 1.6.21], we have

(10.5.1)
$$\begin{aligned} C^{\otimes r} &= \textstyle\bigoplus_{n\geqslant 1} \big(\bigoplus_{i_1+\cdots+i_r=n} \mathrm{K}(\boldsymbol{x}(i_1), R) \otimes^{\mathbf{L}}_R \cdots \otimes^{\mathbf{L}}_R \mathrm{K}(\boldsymbol{x}(i_r), R)\big)[n] \\ &\gtrdot \textstyle\bigoplus_{n\geqslant 1} \mathrm{K}(\boldsymbol{x}(n), R)^{\otimes r}[nr] = \bigoplus_{n\geqslant 1} \mathrm{K}(\underbrace{\boldsymbol{x}(n), \dots, \boldsymbol{x}(n)}_{r}, R)[nr] \\ &\gtrdot \textstyle\bigoplus_{n\geqslant 1} \mathrm{K}(\boldsymbol{x}(n), R)[nr] =: B. \end{aligned}$$

Thus B is in $\langle\bigcup_{n\geqslant 1}\mathrm{V}(I_n)\rangle$, and Corollary 5.13(3) implies $\mathrm{V}(\operatorname{Ann} B)\subseteq\bigcup_{n\geqslant 1}\mathrm{V}(I_n)$. We have $\operatorname{Ann} B=\bigcap_{n\geqslant 1}\operatorname{Ann}\mathrm{K}(\boldsymbol{x}(n),R)=\bigcap_{n\geqslant 1} I_n$ by Proposition 5.3(3). It follows that $\mathrm{V}(\bigcap_{n\geqslant 1} I_n)\subseteq\bigcup_{n\geqslant 1}\mathrm{V}(I_n)$. ■

Our second main result in this section deals with the difference between the radical and tame closures.

Theorem 10.6. *Let W be a specialization-closed subset of $\operatorname{Spec} R$. Set $\mathcal{X}=\langle W\rangle$ and $\mathcal{Y}=\operatorname{Supp}^{-1} W$.*

(1) *The subcategory $\mathcal{X}$ is compact, and satisfies $\mathcal{X}^{\mathsf{rad}}=\sqrt{\mathcal{X}}$ and $\mathcal{X}^{\mathsf{tame}}=\mathcal{Y}$.*
(2) *The subcategory $\mathcal{X}$ (resp. $\mathcal{Y}$) is the smallest (resp. largest) thick $\otimes$-ideal of $\mathsf{D}^-(R)$ whose support is W. In particular, one has $\mathcal{X}\subseteq\sqrt{\mathcal{X}}\subseteq\mathcal{Y}$.*
(3) *Assume that R is either a domain or a local ring, and that W is nonempty and proper. Then one has $\sqrt{\mathcal{X}}\subsetneq\mathcal{Y}$. Hence $\mathcal{Y}$ is not compact, and $\mathcal{X}^{\mathsf{rad}}\subsetneq\mathcal{X}^{\mathsf{tame}}$.*

Proof. (1) The first statement is evident. The equalities follows from Lemma 9.8 and Proposition 4.8.

(2) Let $\mathcal{Z}$ be a thick $\otimes$-ideal of $\mathsf{D}^-(R)$ whose support is W. Then it is clear that $\mathcal{Z}$ is contained in $\mathcal{Y}$. Proposition 5.9 implies that $R/\mathfrak{p}$ belongs to $\mathcal{Z}$ for each $\mathfrak{p}\in W$, which shows that $\mathcal{Z}$ contains $\mathcal{X}$.

(3) Since W is nonempty, there is a prime ideal $\mathfrak{p}\in W$. Let $\boldsymbol{x}=x_1,\dots,x_r$ be a system of generators of $\mathfrak{p}$, and put $C=\bigoplus_{i\geqslant 0}\mathrm{K}(\boldsymbol{x}^{i+1},R)[i]$, which is an object of $\mathsf{D}^-(R)$. The support of C is equal to $\mathrm{V}(\mathfrak{p})$ by Proposition 5.3(3), which is contained in W as it is specialization-closed. Hence C is in $\operatorname{Supp}^{-1} W=\mathcal{Y}$.

Suppose that $\sqrt{\mathcal{X}}$ coincides with $\mathcal{Y}$, and let us derive a contradiction. There exists an integer $n>0$ such that the n-fold tensor product $D:=C\otimes_R^{\mathbf{L}}\cdots\otimes_R^{\mathbf{L}} C$ belongs to $\mathcal{X}$. An analogous argument to (10.5.1) yields that D contains $E:=\bigoplus_{k\geqslant 0}\mathrm{K}(\boldsymbol{x}^{k+1},R)[nk]$ as a direct summand, whence E belongs to $\mathcal{X}$. We use a similar technique to the one in the latter half of the proof of Proposition 10.2. By Lemma 10.1, there are prime ideals $\mathfrak{p}_1,\dots,\mathfrak{p}_m\in W$ and integers $e_1,\dots,e_m>0$ such that $\operatorname{Ann} E$ contains $\mathfrak{p}_1^{e_1}\cdots\mathfrak{p}_m^{e_m}$. We have

$$\operatorname{Ann} E=\bigcap_{k\geqslant 0}\operatorname{Ann}\mathrm{K}(\boldsymbol{x}^{k+1},R)=\bigcap_{k\geqslant 0}\boldsymbol{x}^{k+1}R=0 \tag{10.6.1}$$

by Proposition 5.3(3) and Krull's intersection theorem. This yields $\mathfrak{p}_1^{e_1}\cdots\mathfrak{p}_m^{e_m}=0$, which says that each prime ideal of R contains $\mathfrak{p}_i$ for some $1\leqslant i\leqslant m$. As W is specialization-closed, we observe that $W=\operatorname{Spec} R$, which is contrary to the assumption. Consequently, $\sqrt{\mathcal{X}}$ is strictly contained in $\mathcal{Y}$.

If $\mathcal{Y}$ is compact, then we have $\mathcal{Y}=\langle\operatorname{Supp}\mathcal{Y}\rangle=\langle W\rangle=\mathcal{X}\subseteq\sqrt{\mathcal{X}}$ by Proposition 9.4 and Proposition 4.8(1), which is a contradiction. Hence $\mathcal{Y}$ is not compact. ■

Remark 10.7. (1) Let $\mathfrak{p},C$ be as in the proof of Theorem 10.6(3). Then

(a) $\operatorname{Supp} C$ is contained in $\operatorname{Supp} R/\mathfrak{p}$, but C does not belong to $\mathsf{thick}^{\otimes} R/\mathfrak{p}$.
(b) $\mathrm{V}(\operatorname{Ann} R)$ is contained in $\mathrm{V}(\operatorname{Ann} C)$, but R does not belong to $\mathsf{thick}^{\otimes} C$.

This guarantees in Proposition 5.9 one cannot replace $\mathrm{V}(\operatorname{Ann} X)$ by $\operatorname{Supp} X$, or $\operatorname{Supp}\mathcal{Y}$ by $\mathrm{V}(\operatorname{Ann}\mathcal{Y})$.

Indeed, we have $\operatorname{Supp} C=\operatorname{Supp} R/\mathfrak{p}=\mathrm{V}(\mathfrak{p})\subseteq W\neq\operatorname{Spec} R$ and $\operatorname{Ann} C=\bigcap_{i\geqslant 0}\boldsymbol{x}^{i+1}R=0$. The former together with Proposition 6.1 shows $R\notin\mathsf{thick}^{\otimes} C$, while the latter implies $\mathrm{V}(\operatorname{Ann} R)=\mathrm{V}(0)=\mathrm{V}(\operatorname{Ann} C)$. Assume C is in $\mathsf{thick}^{\otimes} R/\mathfrak{p}$. Then $\operatorname{Ann} C=0$ contains some power of $\operatorname{Ann} R/\mathfrak{p}=\mathfrak{p}$ by Lemma 10.1. Hence $\mathrm{V}(\mathfrak{p})=\operatorname{Spec} R$, which is a contradiction. Therefore C is not in $\mathsf{thick}^{\otimes} R/\mathfrak{p}$.

(2) The assumption in Theorem 10.6(3) that R is either domain or local is indispensable. In fact, let $R = A \times B$ be a direct product of two commutative Noetherian rings. Then $\mathsf{Spec}\, R = \mathsf{Spec}\, A \sqcup \mathsf{Spec}\, B$ and $\mathsf{D}^{-}(R) \cong \mathsf{D}^{-}(A) \times \mathsf{D}^{-}(B)$, which imply that $\mathsf{Supp}^{-1}_{\mathsf{D}^{-}(R)}(\mathsf{Spec}\, A) = \mathsf{D}^{-}(A) = \langle \mathsf{Spec}\, A \rangle_{\mathsf{D}^{-}(R)}$.

(3) Recall that we have the following first section-retraction pair (Proposition 9.16), while Corollary 9.9 gives rise to the following second section-retraction pair.

$$()^{\mathsf{rad}} : \mathbf{Cpt} \rightleftarrows \mathbf{Rad} : ()_{\mathsf{cpt}}, \qquad \mathsf{inc} : \mathbf{Tame} \rightleftarrows \mathbf{Rad} : ()^{\mathsf{tame}}.$$

Corollary 9.21 implies that the left diagram below commutes. Therefore, it is natural to ask whether the right diagram below also commutes.

This is equivalent to asking if $(\mathcal{X}_{\mathsf{cpt}})^{\mathsf{rad}} = \mathcal{X}$ for all $\mathcal{X} \in \mathbf{Tame}$, and to asking if $\mathcal{Y}^{\mathsf{tame}} = \mathcal{Y}^{\mathsf{rad}}$ for all $\mathcal{Y} \in \mathbf{Cpt}$. Theorem 10.6 gives rise to a negative answer to this question.

Finally, we consider a conjecture of Balmer. Recall that Balmer [Bal10b, Conjecture 72] conjectures the following; see Part 1 for details.

Conjecture 10.8 (Balmer)**.** Let $\mathcal{T}$ be a tensor triangulated category. The map $\rho^{\bullet}_{\mathcal{T}}$ is (locally) injective when $\mathcal{T}$ is "algebraic enough".

Here, "algebraic enough" tensor triangulated categories could mean algebraic ones, or derived categories of dg-categories, or ones locally generated by the unit. Recall that a continuous map $f : X \to Y$ of topological spaces is called *locally injective at* $x \in X$ if there exists a neighborhood N of x such that the restriction $f|_N : N \to Y$ is an injective map. We say that f is *locally injective* if it is locally injective at every point in X. If for any $x \in X$ there exists a neighborhood E of $f(x)$ such that the induced map $f^{-1}(E) \to E$ is injective, then f is locally injective.

Let us consider the above conjecture for our tensor triangulated category $\mathsf{D}^{-}(R)$. It turns out that for $\mathcal{T} = \mathsf{D}^{-}(R)$, Balmer's constructed map $\rho^{\bullet}_{\mathcal{T}}$ coincides with our constructed map $\mathfrak{s} : \mathsf{Spec}\, \mathsf{D}^{-}(R) \to \mathsf{Spec}\, R$.

Proposition 10.9. *Let $\mathcal{P}$ be a prime thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$. One then has the following.*
(1) $\mathfrak{s}(\mathcal{P}) = (a \in R \mid R/a \notin \mathcal{P}) = \{a \in R \mid R/a \notin \mathcal{P}\}$. (2) $\mathfrak{s}(\mathcal{P}) = \rho^{\bullet}_{\mathsf{D}^{-}(R)}(\mathcal{P})$.

Proof. Corollary 5.11 and (1) imply (2). Let us show (1). Set $J = (a \in R \mid R/a \notin \mathcal{P})$. As R is Noetherian, we find a finite number of elements $x_1, \dots, x_n$ with $R/x_1, \dots, R/x_n \notin \mathcal{P}$ and $J = (x_1, \dots, x_n)$. Therefore $\mathrm{K}(x_1, \dots, x_n, R) = \mathrm{K}(x_1, R) \otimes^{\mathbf{L}}_R \cdots \otimes^{\mathbf{L}}_R \mathrm{K}(x_n, R)$ is not in $\mathcal{P}$ by Corollary 5.11 and the fact that $\mathcal{P}$ is prime. Using Corollary 5.11 again shows $J \in \mathbb{I}(\mathcal{P})$, whence J is contained in $\mathfrak{s}(\mathcal{P})$. Next, take any $a \in \mathfrak{s}(\mathcal{P})$. Since $\mathrm{V}(\mathfrak{s}(\mathcal{P}))$ is not contained in $\mathsf{Supp}\, \mathcal{P}$, neither is $\mathrm{V}(a)$. This implies $R/a \notin \mathcal{P}$ by Corollary 5.11. ■

As an application of our Theorem 10.6, we confirm that Conjecture 10.8 is not true in general; our $\mathsf{D}^{-}(R)$ is an algebraic triangulated category, but does not satisfy Conjecture 10.8 under quite mild assumptions:

Corollary 10.10. *Assume that R has positive dimension, and that R is either a domain or a local ring. Then the map $\mathfrak{s} : \mathsf{Spec}\, \mathsf{D}^{-}(R) \to \mathsf{Spec}\, R$ is not locally injective.*

In view of Conjecture 10.8, this theorem says that $\mathsf{D}^-(\operatorname{mod} R)$ is not "algebraic enough"; an algebraic tensor triangulated category is not sufficiently "algebraic enough".

Proof. We can choose a nonunit $x \in R$ such that the ideal xR of R has positive height (hence it has height 1). Put $\mathcal{X} = \langle \mathrm{V}(x) \rangle$. Using Theorem 10.6(3) and Lemma 10.3, we find a prime thick $\otimes$-ideal $\mathcal{P}$ such that $\mathcal{X} \subseteq \mathcal{P} \subsetneq \mathcal{P}^{\mathsf{tame}}$. Suppose that $\mathfrak{s}$ is locally injective at $\mathcal{P}$. Then there exists a complex $M \in \mathsf{D}^-(R)$ with $\mathcal{P} \in \mathtt{U}(M)$ such that the restriction $\mathfrak{s}|_{\mathtt{U}(M)} : \mathtt{U}(M) \to \operatorname{Spec} R$ is injective. Since M is in $\mathcal{P}$, it is also in $\mathcal{P}^{\mathsf{tame}}$. Hence both $\mathcal{P}$ and $\mathcal{P}^{\mathsf{tame}}$ belong to $\mathtt{U}(M)$. However, these two prime thick $\otimes$-ideals are sent by $\mathfrak{s}$ to the same point; see Theorem 6.8. This contradicts the injectivity of $\mathfrak{s}|_{\mathtt{U}(M)}$, and we conclude that $\mathfrak{s}$ is not locally injective at $\mathcal{P}$. The last assertion of the corollary follows from Proposition 10.9(2). ■

Remark 10.11. The reader may think that Corollary 10.10 can also be obtained by showing that the map

$$f : \operatorname{Spec} \mathsf{D}^-(R) \to \operatorname{Spec} \mathsf{K}^{\mathsf{b}}(\operatorname{proj} R), \quad \mathcal{P} \mapsto \mathcal{P} \cap \mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)$$

is not injective. We are not sure whether the non-injectivity of the map f implies Corollary 10.10, but at least showing the non-injectivity of f is equivalent to our approach: Using Proposition 5.9, we see that $\mathcal{P} \cap \mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)$ contains the Koszul complex of a system of generators of each prime ideal belonging to $\operatorname{Supp} \mathcal{P}$. Hence $\operatorname{Supp}(\mathcal{P} \cap \mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)) = \operatorname{Supp} \mathcal{P}$, and the Hopkins–Neeman theorem implies $\mathcal{P} \cap \mathsf{K}^{\mathsf{b}}(\operatorname{proj} R) = \operatorname{Supp}^{-1}_{\mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)} \operatorname{Supp} \mathcal{P}$. Therefore, for $\mathcal{P}, \mathcal{Q} \in \operatorname{Spec} \mathsf{D}^-(R)$ it holds that

$$f(\mathcal{P}) = f(\mathcal{Q}) \iff \operatorname{Supp} \mathcal{P} = \operatorname{Supp} \mathcal{Q},$$

which says that the map f is injective if and only if all the prime thick $\otimes$-ideals of $\mathsf{D}^-(R)$ are tame. In the end, even if we intend to prove Corollary 10.10 by showing the non-injectivity of the map f, we must find a non-tame prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$, which is what we have done in this section.

11. **Thick tensor ideals over discrete valuation rings**

In this section, we concentrate on handling the case where R is a discrete valuation ring. Several properties that are specific to this case are found out in this section. Just for convenience, we write complexes as chain complexes, rather than as cochain complexes. We start by studying complexes with zero differentials.

Proposition 11.1. *Let* $X = \bigoplus_{i \geqslant 0} X_i[i] = (\cdots \xrightarrow{0} X_3 \xrightarrow{0} X_2 \xrightarrow{0} X_1 \xrightarrow{0} X_0 \to 0)$ *be a complex in* $\mathsf{D}^-(R)$*. Then it holds that* $\operatorname{thick}^{\otimes} X = \operatorname{thick}^{\otimes} Y$ *in* $\mathsf{D}^-(R)$*, where*

$$Y = \bigoplus_{i \geqslant 0} (\bigoplus_{j=0}^{i} X_j)[i] = (\cdots \xrightarrow{0} X_3 \oplus X_2 \oplus X_1 \oplus X_0 \xrightarrow{0} X_2 \oplus X_1 \oplus X_0 \xrightarrow{0} X_1 \oplus X_0 \xrightarrow{0} X_0 \to 0).$$

Proof. Putting $F = \bigoplus_{j \geqslant 0} R[j]$, we have $X \otimes^{\mathbf{L}}_R F = (\bigoplus_{i \geqslant 0} X_i[i]) \otimes^{\mathbf{L}}_R (\bigoplus_{j \geqslant 0} R[j]) = \bigoplus_{i,j \geqslant 0} X_i[i+j] = Y$. Hence $\operatorname{thick}^{\otimes} X$ contains $\operatorname{thick}^{\otimes} Y$. The opposite inclusion also holds as X is a direct summand of Y. ■

Proposition 11.2. *Let* $X = \bigoplus_{i \geqslant 0} X_i[i] = (\cdots \xrightarrow{0} X_3 \xrightarrow{0} X_2 \xrightarrow{0} X_1 \xrightarrow{0} X_0 \to 0)$ *be a complex in* $\mathsf{D}^-(R)$*. Then for all integers* $a_i \geqslant 0$*, the thick* $\otimes$*-ideal closure* $\operatorname{thick}^{\otimes} X$ *in* $\mathsf{D}^-(R)$ *contains*

$$\bigoplus_{i \geqslant 0} X_i^{\oplus a_i}[2i] = (\cdots \to X_3^{\oplus a_3} \to 0 \to X_2^{\oplus a_2} \to 0 \to X_1^{\oplus a_1} \to 0 \to X_0^{\oplus a_0} \to 0).$$

Proof. In the category $\mathsf{D}^{-}(R)$ the complex $\bigoplus_{i\geqslant 0} X_i^{\oplus a_i}[2i] = \bigoplus_{i\geqslant 0}(X_i \otimes_R^{\mathbf{L}} R^{\oplus a_i})[2i]$ is a direct summand of $\bigoplus_{i,j\geqslant 0}(X_i \otimes_R^{\mathbf{L}} R^{\oplus a_j})[i+j] = (\bigoplus_{i\geqslant 0} X_i[i]) \otimes_R^{\mathbf{L}} (\bigoplus_{j\geqslant 0} R^{\oplus a_j}[j]) = X \otimes_R^{\mathbf{L}} Y$, where $Y = \bigoplus_{j\geqslant 0} R^{\oplus a_j}[j] = (\cdots \xrightarrow{0} R^{\oplus a_2} \xrightarrow{0} R^{\oplus a_1} \xrightarrow{0} R^{\oplus a_0} \to 0)$ is a complex in $\mathsf{D}^{-}(R)$. Thus the assertion follows. ■

Corollary 11.3. *Let* $X = \bigoplus_{i\geqslant 0} X_i[i] = (\cdots \xrightarrow{0} X_3 \xrightarrow{0} X_2 \xrightarrow{0} X_1 \xrightarrow{0} X_0 \to 0)$ *be a complex in* $\mathsf{D}^{-}(R)$*. Then for any integers* $a_i \geqslant 0$ *the complex*

$$Y = \bigoplus_{i\geqslant 0} X_i^{\oplus a_i}[i] = (\cdots \xrightarrow{0} X_3^{\oplus a_3} \xrightarrow{0} X_2^{\oplus a_2} \xrightarrow{0} X_1^{\oplus a_1} \xrightarrow{0} X_0^{\oplus a_0} \to 0)$$

is in $\mathsf{thick}^{\otimes}\{X_{\mathrm{even}}, X_{\mathrm{odd}}\}$*, where* $X_{\mathrm{even}} = \bigoplus_{i\geqslant 0} X_{2i}[i] = (\cdots \xrightarrow{0} X_6 \xrightarrow{0} X_4 \xrightarrow{0} X_2 \xrightarrow{0} X_0 \to 0)$ *and* $X_{\mathrm{odd}} = \bigoplus_{i\geqslant 0} X_{2i+1}[i] = (\cdots \xrightarrow{0} X_7 \xrightarrow{0} X_5 \xrightarrow{0} X_3 \xrightarrow{0} X_1 \to 0)$.

Proof. The complex Y is the direct sum of $A = (\cdots \to 0 \to X_4^{\oplus a_4} \to 0 \to X_2^{\oplus a_2} \to 0 \to X_0^{\oplus a_0} \to 0)$ and $B = (\cdots \to X_5^{\oplus a_5} \to 0 \to X_3^{\oplus a_3} \to 0 \to X_1^{\oplus a_1} \to 0 \to 0)$. Proposition 11.2 shows that A is in $\mathsf{thick}^{\otimes} X_{\mathrm{even}}$ and B is in $\mathsf{thick}^{\otimes} X_{\mathrm{odd}}$. Therefore Y belongs to $\mathsf{thick}^{\otimes}\{X_{\mathrm{even}}, X_{\mathrm{odd}}\}$. ■

A natural question arises from Proposition 11.2 and Corollary 11.3:

Question 11.4. Does $\mathsf{thick}^{\otimes}(\cdots \to 0 \to X_2 \to 0 \to X_1 \to 0 \to X_0 \to 0)$ contain $(\cdots \xrightarrow{0} X_2 \xrightarrow{0} X_1 \xrightarrow{0} X_0 \to 0)$? Does $\mathsf{thick}^{\otimes}(\cdots \xrightarrow{0} X_1 \xrightarrow{0} X_0 \to 0)$ contain $(\cdots \xrightarrow{0} X_1^{\oplus a_1} \xrightarrow{0} X_0^{\oplus a_0} \to 0)$ for all integers $a_i \geqslant 0$?

We do not know the general answer to this question. The following example gives an affirmative answer.

Example 11.5. Let (R, xR) be a discrete valuation ring. Then

$$\begin{aligned}&\mathsf{thick}^{\otimes}(\cdots \xrightarrow{0} R/x^3 \xrightarrow{0} R/x^2 \xrightarrow{0} R/x \to 0)\\ &\qquad = \mathsf{thick}^{\otimes}(\cdots \to 0 \to R/x^3 \to 0 \to R/x^2 \to 0 \to R/x \to 0).\end{aligned}$$

Proof. In fact, the inclusion ($\supseteq$) follows from Proposition 11.2. To check the inclusion ($\subseteq$), set $A = (\cdots \xrightarrow{0} R/x^3 \xrightarrow{0} R/x^2 \xrightarrow{0} R/x \to 0)$ and $B = (\cdots \to 0 \to R/x^3 \to 0 \to R/x^2 \to 0 \to R/x \to 0)$. Note that for each integer $n \geqslant 0$ there is an exact sequence $0 \to R/x^n \xrightarrow{x^{n+1}} R/x^{2n+1} \to R/x^{n+1} \to 0$ of R-modules. This induces an exact sequence $0 \to C \to A \to B \to 0$ of complexes of R-modules, where

$$\begin{aligned}C = (\cdots \xrightarrow{0} R/x^n \xrightarrow{0} R/x^{2n} \xrightarrow{0} R/x^{n-1} \xrightarrow{0} R/x^{2(n-1)} \xrightarrow{0} \cdots\\ \xrightarrow{0} R/x^2 \xrightarrow{0} R/x^4 \xrightarrow{0} R/x \xrightarrow{0} R/x^2 \to 0).\end{aligned}$$

We see that $C = B[2] \oplus D$, where $D = (\cdots \to 0 \to R/x^{2n} \to 0 \to \cdots \to 0 \to R/x^4 \to 0 \to R/x^2 \to 0)$, and have an exact sequence $0 \to B[1] \to D \to B[1] \to 0$ of complexes. The assertion now follows. ■

The *Loewy length* of a finitely generated R-module M, denoted by $\ell\ell_R(M)$, is by definition the infimum of integers i such that the ideal $(\operatorname{rad} R)^i$ kills M. Let us consider thick $\otimes$-ideals defined by Loewy lengths.

Notation 11.6. Let R be a local ring with maximal ideal $\mathfrak{m}$. Let $c \geqslant 0$ be an integer.

(1) Let $\mathcal{L}_c$ be the subcategory of $\mathsf{D}^{-}_{\mathsf{fl}}(R)$ consisting of complexes X such that there exists an integer $t \geqslant 0$ with $\ell\ell(\mathsf{H}_i X) \leqslant t i^{c-1}$ for all $i \gg 0$.

(2) When $c \geqslant 1$, let G_c be the complex

$$\bigoplus_{i>0}(R/\mathfrak{m}^{i^{c-1}})[i] = (\cdots \xrightarrow{0} R/\mathfrak{m}^{3^{c-1}} \xrightarrow{0} R/\mathfrak{m}^{2^{c-1}} \xrightarrow{0} R/\mathfrak{m} \to 0).$$

Proposition 11.7. *Let $(R,\mathfrak{m},k)$ be local. One has $\mathcal{L}_0 \subsetneq \mathcal{L}_1 \subsetneq \mathcal{L}_2 \subsetneq \cdots$ and $\mathcal{L}_0 = \mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R) = \mathsf{thick}_{\mathsf{D}^-(R)}\, k$.*

Proof. Fix an integer $n \geqslant 0$. It is clear that $\mathcal{L}_n$ is contained in $\mathcal{L}_{n+1}$. We have $\ell\ell(\mathsf{H}_i G_{n+1}) = i^n$ for each $i \geqslant 0$, which shows $\mathcal{L}_n \neq \mathcal{L}_{n+1}$. Hence the chain $\mathcal{L}_0 \subsetneq \mathcal{L}_1 \subsetneq \mathcal{L}_2 \subsetneq \cdots$ is obtained. Let X be a complex in $\mathsf{D}^-(R)$. Suppose that there exists an integer $t \geqslant 0$ such that $\ell\ell(\mathsf{H}_i X) \leqslant ti^{-1}$ for $i \gg 0$. Then we have to have $\ell\ell(\mathsf{H}_i X) = 0$ for $i \gg 0$, which says that $\mathsf{H}_j X = 0$ for $j \gg 0$. Thus we obtain $\mathcal{L}_0 = \mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R) = \mathsf{thick}_{\mathsf{D}^-(R)}\, k$, where the second equality is shown in Proposition 4.2. ■

Recall that an abelian category $\mathcal{A}$ is called *hereditary* if it has global dimension at most one, that is, if $\mathsf{Ext}^2_{\mathcal{A}}(\mathcal{A},\mathcal{A}) = 0$. Recall also that a ring R is called *hereditary* if R has global dimension at most one.

From now on, we study thick $\otimes$-ideals of $\mathsf{D}^-(R)$ when R is local and hereditary. In this case, R is either a field or a discrete valuation ring. If R is a field, then by Corollary 5.20 there are only trivial thick $\otimes$-ideals. So, we mainly consider the case of a discrete valuation ring. First, we mention a well-known fact, saying that each complex in the derived category of a hereditary abelian category has zero differentials.

Lemma 11.8. [Kra, 1.6] *Let $\mathcal{A}$ be a hereditary abelian category. Then for each object $M \in \mathsf{D}(\mathcal{A})$ there exists an isomorphism $M \cong \mathsf{H}(M) = \bigoplus_{i\in\mathbb{Z}} \mathsf{H}_i(M)[i]$ in $\mathsf{D}(\mathcal{A})$.*

The lemma below is part of our first main result in this section.

Lemma 11.9. *Let R be a discrete valuation ring. Then $\mathcal{L}_c$ is a thick $\otimes$-ideal of $\mathsf{D}^-(R)$ for every $c \geqslant 1$.*

Proof. By Proposition 4.8(3), it suffices to show $\mathcal{L}_c$ is a thick $\otimes$-ideal of $\mathsf{D}^-_{\mathsf{fl}}(R)$. We do this step by step.

(1) Take any complex X in $\mathcal{L}_c$. There exist integers $t, u \geqslant 0$ such that $\ell\ell(\mathsf{H}_i X) \leqslant ti^{c-1}$ for all $i \geqslant u$. Let Y be a direct summand of X in $\mathsf{D}^-_{\mathsf{fl}}(R)$. Then $\mathsf{H}_i Y$ is a direct summand of $\mathsf{H}_i X$, and we have $\ell\ell(\mathsf{H}_i Y) \leqslant \ell\ell(\mathsf{H}_i X) \leqslant ti^{c-1}$ for all $i \geqslant u$. Hence Y belongs to $\mathcal{L}_c$.

(2) Let $X \to Y \to Z \rightsquigarrow$ be an exact triangle in $\mathsf{D}^-_{\mathsf{fl}}(R)$. Suppose that both X and Z belong to $\mathcal{L}_c$. Then there exist integers $t, u, a, b \geqslant 0$ such that $\ell\ell(\mathsf{H}_i X) \leqslant ti^{c-1}$ and $\ell\ell(\mathsf{H}_j Z) \leqslant uj^{c-1}$ for all $i \geqslant a$ and $j \geqslant b$. An exact sequence $\cdots \to \mathsf{H}_k X \to \mathsf{H}_k Y \to \mathsf{H}_k Z \to \cdots$ is induced, and from this we see that $\ell\ell(\mathsf{H}_k Y) \leqslant \ell\ell(\mathsf{H}_k X) + \ell\ell(\mathsf{H}_k Z) \leqslant (t+u)k^{c-1}$ for all $k \geqslant \max\{a, b\}$. Therefore, Y belongs to $\mathcal{L}_c$.

(3) Let X be a complex in $\mathcal{L}_c$. Then there exist integers $t, u \geqslant 0$ such that $\ell\ell(\mathsf{H}_i X) \leqslant ti^{c-1}$ for all $i \geqslant u$. It holds that $\ell\ell(\mathsf{H}_i(X[1])) = \ell\ell(\mathsf{H}_{i-1} X) \leqslant t(i-1)^{c-1} \leqslant ti^{c-1}$ for all $i \geqslant u+1$ for all $i \geqslant u+1$, where the second inequality holds as $c \geqslant 1$. Also, $\ell\ell(\mathsf{H}_i(X[-1])) = \ell\ell(\mathsf{H}_{i+1} X) \leqslant t(i+1)^{c-1} \leqslant t(i+i)^{c-1} = (2^{c-1}t) \cdot i^{c-1}$ for all $i \geqslant \max\{1, u-1\}$, where the first inequality holds as $i \geqslant u-1$, and the second one holds since $i \geqslant 1$ and $c \geqslant 1$. Thus the complexes $X[1]$ and $X[-1]$ belong to $\mathcal{L}_c$.

(4) Let X, Y be complexes in $\mathsf{D}^-_{\mathsf{fl}}(R)$. Suppose that X belongs to $\mathcal{L}_c$. We want to show that $X \otimes^{\mathbf{L}}_R Y$ also belongs to $\mathcal{L}_c$. Taking into account (3) and Lemma 11.8, we may assume that $X = \bigoplus_{i\geqslant 1} X_i[i]$ and $Y = \bigoplus_{j\geqslant 0} Y_j[j]$ with X_i, Y_j being R-modules, and that there exist $s \geqslant 1, t \geqslant 0$ such that $\ell\ell(X_i) \leqslant ti^{c-1}$ for all $i \geqslant s$. Set $u = \max\{\ell\ell(X_i) \mid$

$1 \leqslant i \leqslant s-1\}$; note that each X_i has finite length, whence has finite Loewy length. We have $X \otimes_R^{\mathbf{L}} Y = \bigoplus_{i\geqslant 1,\, j\geqslant 0}(X_i \otimes_R^{\mathbf{L}} Y_j)[i+j]$, and from this we get $\mathsf{H}_k(X \otimes_R^{\mathbf{L}} Y) = \bigoplus_{i\geqslant 1,\, j\geqslant 0,\, i+j\leqslant k} \mathsf{Tor}_{k-i-j}^R(X_i, Y_j)$ for all integers k. Note here that $\mathsf{Tor}_{k-i-j}^R(X_i, Y_j) = 0$ for $i+j > k$.

We claim that $\ell\ell(X_i) \leqslant (t+u)i^{c-1}$ for all $i \geqslant 1$. In fact, recall $c \geqslant 1$ and $t, u \geqslant 0$. If $i \geqslant s$, then $\ell\ell(X_i) \leqslant ti^{c-1} \leqslant (t+u)i^{c-1}$. If $1 \leqslant i \leqslant s-1$, then $\ell\ell(X_i) \leqslant u \leqslant t+u \leqslant (t+u)i^{c-1}$. The claim follows.

Fix three integers i, j, k with $i \geqslant 1$, $j \geqslant 0$ and $i+j \leqslant k$. Then $(t+u)k^{c-1} \geqslant (t+u)i^{c-1}$ since $k \geqslant i$ and $c \geqslant 1$. The claim shows that X_i is killed by $\mathfrak{m}^{(t+u)k^{c-1}}$, and so is $\mathsf{Tor}_{k-i-j}^R(X_i, Y_j)$, where $\mathfrak{m}$ stands for the maximal ideal of R. Hence $\ell\ell(\mathsf{H}_k(X \otimes_R^{\mathbf{L}} Y)) \leqslant (t+u)k^{c-1}$ for all $k \in \mathbb{Z}$, which implies $X \otimes_R^{\mathbf{L}} Y \in \mathcal{L}_c$.

It follows from the above arguments (1)–(4) that $\mathcal{L}_c$ is a thick $\otimes$-ideal of $\mathsf{D}^-_{\mathsf{fl}}(R)$. ■

Remark 11.10. Let $(R, \mathfrak{m}, k)$ be a local ring. When $c = 0$, the subcategory $\mathcal{L}_c$ is never a thick $\otimes$-ideal of $\mathsf{D}^-(R)$. Indeed, by Proposition 11.7 we have $\mathcal{L}_0 = \mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R)$. The module k is in $\mathcal{L}_0$, but the complex $(\cdots \xrightarrow{0} k \xrightarrow{0} k \to 0) = k \otimes_R^{\mathbf{L}} (\cdots \xrightarrow{0} R \xrightarrow{0} R \to 0)$ is not in $\mathcal{L}_0$.

Now we have our first theorem concerning the subcategories $\mathcal{L}_c$ of $\mathsf{D}^-(R)$ for a discrete valuation ring R. This especially says that the equality of Proposition 6.9(2) does not necessarily hold.

Theorem 11.11. *Let R be a discrete valuation ring. Then $\mathcal{L}_c$ is a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$ for all integers $c \geqslant 1$. In particular, one has*

$$\dim(\mathsf{Spec}\,\mathsf{D}^-(R)) = \infty > 1 = \dim R.$$

Proof. Lemma 11.9 says that $\mathcal{L}_c$ is a thick $\otimes$-ideal of $\mathsf{D}^-(R)$. Proposition 11.7 especially says $\mathcal{L}_c \neq \mathsf{D}^-(R)$. Let X, Y be complexes in $\mathsf{D}^-(R)$ with $X \otimes_R^{\mathbf{L}} Y \in \mathcal{L}_c$, and we shall prove that either X or Y is in $\mathcal{L}_c$. Applying Lemma 11.8 and taking shifts if necessary, we may assume $X = \bigoplus_{i\geqslant 0} X_i[i]$ and $Y = \bigoplus_{j\geqslant 0} Y_j[j]$, where X_i, Y_j are finitely generated R-modules. Assume that X is not in $\mathsf{D}^-_{\mathsf{fl}}(R)$. Then X_a has infinite length for some $a \geqslant 0$. As R is a discrete valuation ring, X_a has a nonzero free direct summand. Hence $R[a]$ is a direct summand of X, and $Y[a] = R[a] \otimes_R^{\mathbf{L}} Y$ is a direct summand of $X \otimes_R^{\mathbf{L}} Y$. As $X \otimes_R^{\mathbf{L}} Y$ is in $\mathcal{L}_c$, so is Y. Similarly, if $Y \notin \mathsf{D}^-_{\mathsf{fl}}(R)$, then $X \in \mathcal{L}_c$. This argument shows that we may assume that both X and Y belong to $\mathsf{D}^-_{\mathsf{fl}}(R)$, or equivalently, that all X_i and Y_j have finite length as R-modules. Since $X \otimes_R^{\mathbf{L}} Y$ belongs to $\mathcal{L}_c$, there exist integers $t, u \geqslant 0$ such that $\mathsf{H}_n(X \otimes_R^{\mathbf{L}} Y)$ has Loewy length at most tn^{c-1} for all $n \geqslant u$. Assume that X is not in $\mathcal{L}_c$. Then we can find an integer $e \geqslant u$ such that $\ell\ell(X_e) > te^{c-1}$. We have $X \otimes_R^{\mathbf{L}} Y = \bigoplus_{i,j\geqslant 0}(X_i \otimes_R^{\mathbf{L}} Y_j)[i+j]$, which gives rise to $\mathsf{H}_n(X \otimes_R^{\mathbf{L}} Y) = \bigoplus_{i,j\geqslant 0} \mathsf{Tor}_{n-i-j}(X_i, Y_j)$ for all integers n. Setting $a_i = \ell\ell(X_i)$ and $b_j = \ell\ell(Y_j)$ for $i, j \geqslant 0$, we obtain for every integer $n \geqslant e$:

$$\mathsf{H}_n(X \otimes_R^{\mathbf{L}} Y) \gtrdot \mathsf{Tor}_{n-e-(n-e)}(X_e, Y_{n-e}) = X_e \otimes_R Y_{n-e} \gtrdot R/x^{a_e} \otimes_R R/x^{b_{n-e}} = R/x^{\min\{a_e, b_{n-e}\}}$$

It is seen that $\min\{a_e, b_{n-e}\} \leqslant tn^{c-1}$ for all $n \geqslant e$. As $a_e > te^{c-1}$, we must have $a_e > b_{n-e}$, and $b_{n-e} \leqslant tn^{c-1}$ for all $n \geqslant e$. Hence $\ell\ell(\mathsf{H}_n(Y[e])) = \ell\ell(Y_{n-e}) = b_{n-e} \leqslant tn^{c-1}$ for $n \geqslant e$, which implies that $Y[e]$ is in $\mathcal{L}_c$, and so is Y. Similarly, if Y is not in $\mathcal{L}_c$, then X is in $\mathcal{L}_c$. Thus $\mathcal{L}_c$ is a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$. Now $\mathcal{L}_1 \subsetneq \mathcal{L}_2 \subsetneq \mathcal{L}_3 \subsetneq \cdots$ from Lemma 11.9 is an ascending chain of prime thick $\otimes$-ideals with infinite length, which shows the inequality in the proposition; see Proposition 6.9(1). ■

To make an application of the above theorem, we state and prove a lemma.

Lemma 11.12. *For each prime ideal* $\mathfrak{p}$ *of* R*, one has* $\dim \operatorname{Spec} \mathsf{D}^-(R_\mathfrak{p}) \leqslant \dim \operatorname{Spec} \mathsf{D}^-(R)$.

Proof. We first show that the localization functor $L : \mathsf{D}^-(R) \to \mathsf{D}^-(R_\mathfrak{p})$ is an essentially surjective. Let $X = (\cdots \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \to 0)$ be a complex in $\mathsf{D}^-(R_\mathfrak{p})$. What we want is a complex $Y \in \mathsf{D}^-(R)$ such that $X \cong L(Y)$. For each integer $i \geqslant 0$, choose a finitely generated R-module Y_i with $(Y_i)_\mathfrak{p} = X_i$, and R-linear maps $d_i^Y : Y_i \to Y_{i-1}$ and $s_i \in R \setminus \mathfrak{p}$ such that $d_i^X = \frac{d_i^Y}{s_i}$ in $\operatorname{Hom}_{R_\mathfrak{p}}(X_i, X_{i-1}) = \operatorname{Hom}(Y_i, Y_{i-1})_\mathfrak{p}$. Then $\frac{d_{i-1}^Y d_i^Y}{s_{i-1}s_i} = d_{i-1}^X d_i^X = 0$, and there is an element $t_i \in R \setminus \mathfrak{p}$ such that $t_i d_{i-1}^Y d_i^Y = 0$. Define a complex $Y = (\cdots \xrightarrow{t_{i+1}d_{i-1}^Y} Y_i \xrightarrow{t_i d_i^Y} \cdots \xrightarrow{t_2 d_2^Y} Y_1 \xrightarrow{t_1 d_1^Y} Y_0 \to 0)$ in $\mathsf{D}^-(R)$. Then there is an isomorphism

$$\begin{array}{ccccccccccccc}
Y_\mathfrak{p} & = & (\cdots \to & (Y_i)_\mathfrak{p} & \xrightarrow{\frac{t_i d_i^Y}{1}} & (Y_{i-1})_\mathfrak{p} & \to \cdots \to & (Y_2)_\mathfrak{p} & \xrightarrow{\frac{t_2 d_2^Y}{1}} & (Y_1)_\mathfrak{p} & \xrightarrow{\frac{t_1 d_1^Y}{1}} & (Y_0)_\mathfrak{p} & \to 0) \\
\downarrow & & & \cong \downarrow u_i & & \cong \downarrow u_{i-1} & & \cong \downarrow u_2 & & \cong \downarrow u_1 & & \| & \\
X & = & (\cdots \to & X_i & \xrightarrow{d_i^X} & X_{i-1} & \to \cdots \to & X_2 & \xrightarrow{d_2^X} & X_1 & \xrightarrow{d_1^X} & X_0 & \to 0),
\end{array}$$

of complexes, where $u_i := t_1 \cdots t_i s_1 \cdots s_i$. Thus, we obtain $L(Y) = Y_\mathfrak{p} \cong X$.

The essentially surjective tensor triangulated functor L induces an injective continuous map $\operatorname{Spec} L : \operatorname{Spec} \mathsf{D}^-(R_\mathfrak{p}) \to \operatorname{Spec} \mathsf{D}^-(R)$ given by $\mathcal{P} \mapsto L^{-1}(\mathcal{P})$; see [Bal05, Corollary 3.8]. This map sends a chain $\mathcal{P}_0 \subsetneq \cdots \subsetneq \mathcal{P}_n$ of prime thick $\otimes$-ideals of $\mathsf{D}^-(R_\mathfrak{p})$ to the chain $L^{-1}(\mathcal{P}_0) \subsetneq \cdots \subsetneq L^{-1}(\mathcal{P}_n)$ of prime thick $\otimes$-ideals of $\mathsf{D}^-(R)$. The lemma now follows. ∎

The following corollary of Theorem 11.11 provides a class of rings R such that the Balmer spectrum of $\mathsf{D}^-(R)$ has infinite Krull dimension. This class includes normal local domains for instance.

Corollary 11.13. *If* $R_\mathfrak{p}$ *is regular for some* $\mathfrak{p}$ *with* $\operatorname{ht} \mathfrak{p} > 0$*, then* $\dim \operatorname{Spec} \mathsf{D}^-(R) = \infty$.

Proof. We may assume $\operatorname{ht} \mathfrak{p} = 1$. We have $\dim \operatorname{Spec} \mathsf{D}^-(R) \geqslant \dim \operatorname{Spec} \mathsf{D}^-(R_\mathfrak{p}) = \infty$, where the inequality follows from Lemma 11.12, and the equality is shown in Theorem 11.11. ∎

Next we study generation of the thick $\otimes$-ideals $\mathcal{L}_c$. In fact each of them possesses a single generator.

Theorem 11.14. *Let* (R, xR, k) *be a discrete valuation ring, and let* $c \geqslant 1$ *be an integer. It then holds that* $\mathcal{L}_c = \mathsf{thick}^{\otimes}_{\mathsf{D}^-(R)} G_c$. *In particular, one has* $\mathcal{L}_1 = \mathsf{thick}^{\otimes}_{\mathsf{D}^-(R)} k$.

Proof. Clearly, G_c is in $\mathcal{L}_c$. Lemma 11.9 implies that $\mathsf{thick}^{\otimes} G_c$ is contained in $\mathcal{L}_c$. We establish a claim.

Claim. Let $0 \leqslant n \leqslant c-1$ be an integer. Let $X \in \mathsf{D}^-_{\mathsf{fl}}(R)$ be a complex. Suppose that there exists an integer $t \geqslant 0$ such that $\ell\ell(\mathsf{H}_i X) \leqslant t i^n$ for all $i \gg 0$. Then X belongs to $\mathsf{thick}^{\otimes} G_c$.

Once we show this claim, it will follow that $\mathcal{L}_c$ is contained in $\mathsf{thick}^{\otimes} G_c$, and we will be done.

First of all, note that k is a direct summand of G_c. Combining this with Proposition 4.2, we have

$$\mathsf{thick}^{\otimes} G_c \supseteq \mathsf{thick}^{\otimes} k \supseteq \mathsf{thick}\, k = \mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R). \tag{11.14.1}$$

Let X be a complex as in the claim. Using Lemma 11.8, we may assume $X = \bigoplus_{i \geqslant s} X_i[i]$ for some integer s and R-modules X_i of finite length. There is an integer $u \geqslant s$ with

$\ell\ell(X_i) \leqslant ti^n$ for all $i \geqslant u$. We have $X = (\bigoplus_{i\geqslant u} X_i[i]) \oplus (\bigoplus_{i=s}^{u-1} X_i[i])$, whose latter summand is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R)$. In view of (11.14.1), replacing X with the former summand, we may assume $u = s$. When $s \geqslant 0$, we set $X_i = 0$ for $0 \leqslant i \leqslant s-1$. When $s < 0$, we have $X = (\bigoplus_{i\geqslant 0} X_i[i]) \oplus (\bigoplus_{i=s}^{-1} X_i[i])$, whose latter summand is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R)$. By similar replacement as above, we may assume $s = 0$. Thus, $X = \bigoplus_{i\geqslant 0} X_i[i]$ and $\ell\ell(X_i) \leqslant ti^n$ for all $i \geqslant 0$.

Since R is a discrete valuation ring with maximal ideal xR, for every $i \geqslant 1$ there is an integer $a_{ij} \geqslant 0$ such that X_i is isomorphic to $\bigoplus_{j=1}^{ti^n}(R/x^j)^{\oplus a_{ij}}$. Therefore it holds that

$$\begin{aligned} X &\cong \textstyle\bigoplus_{i\geqslant 0}(\bigoplus_{j=1}^{ti^n}(R/x^j)^{\oplus a_{ij}})[i] \lessdot \bigoplus_{i\geqslant 0}(\bigoplus_{j=1}^{ti^n} R/x^j)^{\oplus a_i}[i] \\ &\in \mathsf{thick}^{\otimes}\left\{\textstyle\bigoplus_{i\geqslant 0}(\bigoplus_{j=1}^{t(2i)^n} R/x^j)[i],\ \bigoplus_{i\geqslant 0}(\bigoplus_{j=1}^{t(2i+1)^n} R/x^j)[i]\right\} \\ &= \mathsf{thick}^{\otimes}\left\{A_1, A_2 \oplus (\bigoplus_{j=1}^{t} R/x^j)\right\}, \end{aligned}$$

where $a_i := \max\{a_{ij} \mid 1 \leqslant j \leqslant ti^n\}$ and $A_l := \bigoplus_{i\geqslant 1}(\bigoplus_{j=t(2i-l)^n+1}^{t(2i-l+2)^n} R/x^j)[i]$ for $l = 1,2$. The relations "$\in$" and "$=$" follow from Corollary 11.3 and Proposition 11.1, respectively. Since $\bigoplus_{j=1}^{t} R/x^j$ is in $\mathsf{thick}^{\otimes} G_c$ by (11.14.1), it suffices to show that A_l belongs to $\mathsf{thick}^{\otimes} G_c$ for $l = 1,2$.

We prove this by induction on n. When $n = 0$, we have $A_1 = A_2 = 0 \in \mathsf{thick}^{\otimes} G_c$, and are done. Let $n \geqslant 1$. Fix $l = 1,2$. The exact sequences

$$0 \to R/x^{t(2i-l)^n} \xrightarrow{x^j} R/x^{j+t(2i-l)^n} \to R/x^j \to 0 \qquad (i \geqslant 1,\ 1 \leqslant j \leqslant tb_{il})$$

with $b_{il} = (2i-l+2)^n - (2i-l)^n$ induce exact sequences

$$0 \to (R/x^{t(2i-l)^n})^{\oplus tb_{il}} \to \textstyle\bigoplus_{j=t(2i-l)^n+1}^{t(2i-l+2)^n} R/x^j \to \bigoplus_{j=1}^{tb_{il}} R/x^j \to 0 \qquad (i \geqslant 1),$$

which induce an exact triangle $B_l \to A_l \to C_l \rightsquigarrow$ in $\mathsf{D}^{-}_{\mathsf{fl}}(R)$, where we set $B_l = \bigoplus_{i\geqslant 1}(R/x^{t(2i-l)^n})^{\oplus tb_{il}}[i]$ and $C_l = \bigoplus_{i\geqslant 1}(\bigoplus_{j=1}^{tb_{il}} R/x^j)[i]$. Since $\ell\ell(\mathsf{H}_i C_l) = tb_{il}$ has degree at most $n-1$ as a polynomial in i, the induction hypothesis implies that C_l is in $\mathsf{thick}^{\otimes} G_c$. By Corollary 11.3, B_l belongs to

$$\mathsf{thick}^{\otimes}\{\textstyle\bigoplus_{i\geqslant 0}(R/x^{t(4i+r)^n})[i] \mid 0 \leqslant r \leqslant 3\}.$$

Let $f(i)$ be a polynomial in i over $\mathbb{N}$ with leading term ei^n. The exact sequences

$$0 \to R/x^{(t-1)f(i)} \xrightarrow{x^{f(i)}} R/x^{tf(i)} \to R/x^{f(i)} \to 0 \qquad (i \geqslant 0)$$

induce an exact triangle $D_{t-1} \to D_t \to D_1 \rightsquigarrow$ in $\mathsf{D}^{-}_{\mathsf{fl}}(R)$, where we put $D_t = \bigoplus_{i\geqslant 0} R/x^{tf(i)}[i]$. An inductive argument on t shows that D_t belongs to the thick closure of D_1. The exact sequences

$$0 \to R/x^{f(i)-(m+1)i^n} \xrightarrow{x^{i^n}} R/x^{f(i)-mi^n} \to R/x^{i^n} \to 0 \qquad (i \geqslant 0)$$

induce an exact triangle $E_{m+1} \to E_m \to G_c \rightsquigarrow$, where we set $E_m = \bigoplus_{i\geqslant 0}(R/x^{f(i)-mi^n})[i]$ for $0 \leqslant m \leqslant e$. Hence E_0 is in the thick closure of G_c and E_e. Since $\ell\ell(\mathsf{H}_i E_e) = f(i) - ei^n$ has degree at most $n-1$ as a polynomial in i, the induction hypothesis shows that E_e is in $\mathsf{thick}^{\otimes} G_c$. Hence $D_1 = E_0$ is also in $\mathsf{thick}^{\otimes} G_c$, and so is D_t. Therefore B_l is in $\mathsf{thick}^{\otimes} G_c$. Thus A_l belongs to $\mathsf{thick}^{\otimes} G_c$ for $l = 1,2$. ■

Remark 11.15. Let (R, xR, k) be a discrete valuation ring, and let $c \geqslant 2$ be an integer. Then $\mathsf{Supp}\, G_c = \{xR\} = \mathsf{Supp}\, k$. In particular, we have $\mathsf{Supp}\, G_c \neq \mathsf{Spec}\, R$, so that R is not in $\mathsf{thick}^{\otimes} G_c$ by Proposition 6.1. Krull's intersection theorem implies $\operatorname{Ann} G_c = 0 = \operatorname{Ann} R$. Proposition 11.7 and Theorem 11.14 imply that G_c is not in $\mathcal{L}_1 = \mathsf{thick}^{\otimes} k$. In summary:

(1) $\mathsf{Supp}\, G_c$ is contained in $\mathsf{Supp}\, k$, but G_c does not belong to $\mathsf{thick}^{\otimes} k$.
(2) $\mathrm{V}(\operatorname{Ann} R)$ is contained in $\mathrm{V}(\operatorname{Ann} G_c)$, but R does not belong to $\mathsf{thick}^{\otimes} G_c$.

This guarantees that in Proposition 5.9 one cannot replace $\mathrm{V}(\operatorname{Ann} X)$ by $\mathsf{Supp}\, X$, or $\mathsf{Supp}\, \mathcal{Y}$ by $\mathrm{V}(\operatorname{Ann} \mathcal{Y})$.

Example 11.16. Let us deduce the conclusion of Proposition 10.2(1) directly in the case where $(R, \mathfrak{m}, k)$ is a discrete valuation ring. In this case, we have $\mathbf{Spcl}(\mathsf{Spec}) = \{\emptyset, \{\mathfrak{m}\}, \mathsf{Spec}\, R\}$. Using Proposition 9.4, we obtain $\mathbf{Cpt} = \{\mathbf{0}, \mathsf{thick}^{\otimes} k, \mathsf{D}^{-}(R)\}$. Note that $\mathbf{0} = \mathcal{S}(\mathfrak{m})$ is prime and Theorems 11.14, 11.11 say that $\mathsf{thick}^{\otimes} k$ is prime. Thus the compact prime thick $\otimes$-ideals of $\mathsf{D}^{-}(R)$ are $\mathbf{0}$ and $\mathsf{thick}^{\otimes} k$. It follows from Corollary 6.11 that $\mathfrak{s}(\mathsf{thick}^{\otimes} k)$ does not contain $\mathfrak{m}$, which implies $\mathfrak{s}(\mathsf{thick}^{\otimes} k) = 0$. Hence $\mathbf{Cpt} \cap \mathfrak{s}^{-1}(\mathfrak{m}) = \{\mathbf{0}\}$.

Let us consider for a discrete valuation ring R the tameness and compactness of the thick $\otimes$-ideals $\mathcal{L}_c$.

Proposition 11.17. *Let R be a discrete valuation ring, and let $c \geqslant 1$ be an integer. Then $\mathcal{L}_c$ is a non-tame prime thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$. If $c \geqslant 2$, then $\mathcal{L}_c$ is non-compact.*

Proof. It is shown in Theorem 11.11 that $\mathcal{L}_c$ is a prime thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$. Denote by xR the maximal ideal of R. Using Proposition 11.7 and Theorem 11.14, we easily see that $\mathsf{Supp}\, \mathcal{L}_c = \mathrm{V}(x) = \{xR\}$.

Suppose that $\mathcal{L}_c$ is tame. Then $\mathcal{L}_c = \mathsf{Supp}^{-1}\{xR\}$ by Proposition 9.3. For example, consider the complex $E = \bigoplus_{i \geqslant 0} (R/x^{i!})[i]$. We have $\mathsf{Supp}\, E = \{xR\}$, which shows $E \in \mathcal{L}_c$. Hence there exists an integer $t \geqslant 0$ such that $i! = \ell\ell(\mathsf{H}_i E) \leqslant t i^{c-1}$ for all $i \gg 0$. This contradiction shows that $\mathcal{L}_c$ is not tame.

Suppose that $\mathcal{L}_c$ is compact. Then $\mathcal{L}_c = \langle \mathsf{Supp}\, \mathcal{L}_c \rangle = \mathsf{thick}^{\otimes} k = \mathcal{L}_1$ by Proposition 9.4 and Theorem 11.14. This gives a contradiction when $c \geqslant 2$; see Proposition 11.7. Thus $\mathcal{L}_c$ is not compact for all $c \geqslant 2$. ■

Remark 11.18. Theorem 11.14 implies that $\mathcal{L}_c$ is generated by the complex G_c, whose support is the closed subset $\{\mathfrak{m}\}$ of $\mathsf{Spec}\, R$. Corollary 11.17 says that $\mathcal{L}_c$ is not compact for $c \geqslant 2$. This gives an example of a non-compact thick $\otimes$-ideal which is generated by objects with closed supports.

In the proof of Proposition 11.17, a complex defined by using factorials of integers played an essential role. In relation to this, a natural question arises.

Question 11.19. Let (R, xR) be a discrete valuation ring. Consider the complex

$$E = \bigoplus_{i \geqslant 0} (R/x^{i!})[i] = (\cdots \xrightarrow{0} R/x^{120} \xrightarrow{0} R/x^{24} \xrightarrow{0} R/x^{6} \xrightarrow{0} R/x^{2} \xrightarrow{0} R/x \xrightarrow{0} R/x \to 0)$$

in $\mathsf{D}^{-}(R)$. Is it possible to establish a similar result to Theorem 11.14 for $\mathsf{thick}^{\otimes} E$? For example, can one characterize the objects of $\mathsf{thick}^{\otimes} E$ in terms of the Loewy lengths of their homologies?

We have no idea to answer this question. In relation to it, in the next example we will consider complexes defined by using not factorials but polynomials. To do this, we provide a lemma.

Lemma 11.20. *Let x be a non-zerodivisor of R. Then the complex $\bigoplus_{i\geqslant 0}(R/x^{a_i+b_i})[i]$ belongs to the thick closure of $\bigoplus_{i\geqslant 0}(R/x^{a_i})[i]$ and $\bigoplus_{i\geqslant 0}(R/x^{b_i})[i]$ for all integers $a_i, b_i \geqslant 0$. In particular, the complex $\bigoplus_{i\geqslant 0}(R/x^{ca_i})[i]$ is in the thick closure of $\bigoplus_{i\geqslant 0}(R/x^{a_i})[i]$ for all integers $c, a_i \geqslant 0$.*

Proof. For each $i \geqslant 0$ there is an exact sequence $0 \to R/x^{a_i} \xrightarrow{x^{b_i}} R/x^{a_i+b_i} \to R/x^{b_i} \to 0$. From this an exact sequence $0 \to \bigoplus_{i\geqslant 0}(R/x^{a_i})[i] \to \bigoplus_{i\geqslant 0}(R/x^{a_i+b_i})[i] \to \bigoplus_{i\geqslant 0}(R/x^{b_i})[i] \to 0$ is induced. The first assertion follows from this. The second assertion is shown by induction and the first assertion. ■

Example 11.21. Let $x \in R$ be a non-zerodivisor. For integers $a, b, c \geqslant 0$, define a complex

$$X(a,b,c) = \bigoplus_{i\geqslant 0}(R/f_i)[i] = (\cdots \xrightarrow{0} R/f_2 \xrightarrow{0} R/f_1 \xrightarrow{0} R/f_0 \to 0),$$

where $f_i = x^{ai^2+bi+c} \in R$. Then it holds that $\mathsf{thick}^{\otimes}\{X(a,b,c) \mid a,b,c \geqslant 0\} = \mathsf{thick}^{\otimes}\{X(1,0,0)\}$.

Proof. It is obvious that the left-hand side contains the right-hand side. In view of Lemma 11.20, the opposite inclusion will follow if we show that $X(1,0,0), X(0,1,0), X(0,0,1)$ are in $\mathsf{thick}^{\otimes}\{X(1,0,0)\}$, whose first containment is evident. The complex $X(1,0,0)$ has the direct summand $(R/x)[1]$, so the module R/x belongs to $\mathsf{thick}^{\otimes}\{X(1,0,0)\}$. We have $X(0,0,1) = R/x \otimes^{\mathbf{L}}_R (\cdots \xrightarrow{0} R \xrightarrow{0} R \to 0)$, which is in $\mathsf{thick}^{\otimes}\{X(1,0,0)\}$. The exact sequences $0 \to R/x^{i^2} \xrightarrow{x^{2i+1}} R/x^{(i+1)^2} \to R/x^{2i+1} \to 0$ and $0 \to R/x^{2i+1} \xrightarrow{x} R/x^{2i+2} \to R/x \to 0$ with $i > 0$ induce exact sequences $0 \to X(1,0,0) \to X(1,0,0)[-1] \to X(0,2,1) \to 0$ and $0 \to X(0,2,1) \to X(0,2,2) \to X(0,0,1) \to 0$, which shows that $\mathsf{thick}^{\otimes}\{X(1,0,0)\}$ contains $X(0,2,1) = (\cdots \xrightarrow{0} R/x^5 \xrightarrow{0} R/x^3 \xrightarrow{0} R/x \to 0)$ and $X(0,2,2) = (\cdots \xrightarrow{0} R/x^6 \xrightarrow{0} R/x^4 \xrightarrow{0} R/x^2 \to 0)$. Applying Corollary 11.3, we see that $X(0,1,0)$ belongs to $\mathsf{thick}^{\otimes}\{X(1,0,0)\}$. ■

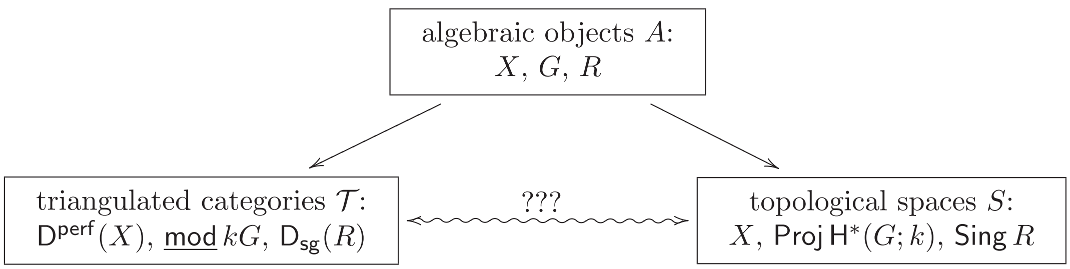
Remark 11.22. One can consider a general statement of Example 11.21 by defining $f_i = x^{a_0 i^d + a_1 i^{d-1} + \cdots + a_d}$, so that it is nothing but the example for $d = 2$. We do not know if it holds for $d \geqslant 3$.

Part 3. Classification of subcategories and reconstruction of classifying spaces

12. Introduction

The contents of this part is based on the author's paper [Mat17c].

As is a common approach in many branches of algebra including algebraic geometry, modular representation theory and commutative ring theory, we assign to an algebraic object A (e.g., a scheme X, a finite group G, a commutative Noetherian ring R) a triangulated category $\mathcal{T}$ (e.g., the perfect derived category $\mathsf{D}^{\mathsf{perf}}(X)$, the stable module category $\underline{\mathsf{mod}}\, kG$, the singularity category $\mathsf{D}_{\mathsf{sg}}(R)$) and a topological space S (e.g., the underlying topological spaces X, $\mathsf{Proj}\,\mathsf{H}^*(G;k)$, $\mathsf{Sing}\,R$). By studying such a triangulated category and a topological space, we aim to grasp the structure of the original algebraic object. From this motivation, it is natural to ask what kind of relationship there exists between $\mathcal{T}$ and S.



In this part, we consider this question, more precisely, the following:

Question 12.1. Let A, A' be algebraic objects, $\mathcal{T}$, $\mathcal{T}'$ corresponding triangulated categories, and S, S' corresponding topological spaces, respectively. Does the implication

$$\mathcal{T} \cong \mathcal{T}' \Longrightarrow S \cong S'$$

hold?

We introduce the notion of a *classifying space* of a triangulated category (see Definition 13.5), and prove the following result, which gives a machinery to answer the above question.

Theorem 12.2 (Theorem 14.10)**.** *Let $\mathcal{T}$, $\mathcal{T}'$ be essentially small triangulated categories and S, S' classifying spaces for $\mathcal{T}$ and $\mathcal{T}'$, respectively. Then the implication*

$$\mathcal{T} \cong \mathcal{T}' \Longrightarrow S \cong S'$$

holds.

The key role to prove this theorem is played by the *support theory* for triangulated categories. For tensor triangulated categories, the support theory has been developed by Balmer [Bal02, Bal05] and is a powerful tool to show such a reconstruction theorem. Since we focus on triangulated categories without tensor structure, we need to invent the *support theory without tensor structure*.

12.1. Algebraic geometry

Let X be a scheme. The derived category of perfect complexes on X is called the *perfect derived category* and denoted by $\mathsf{D}^{\mathsf{perf}}(X)$. The case where $X = \mathsf{Spec}\,R$ is affine, it is well known that the original scheme is reconstructed from $\mathsf{D}^{\mathsf{perf}}(R) := \mathsf{D}^{\mathsf{perf}}(X)$. Indeed, for two commutative rings R and S, if the perfect derived categories of R and S are equivalent, then R is isomorphic to S (see [Ric, Proposition 9.2]), and hence

$$\mathsf{D}^{\mathsf{perf}}(R) \cong \mathsf{D}^{\mathsf{perf}}(S) \implies \mathsf{Spec}\, R \cong \mathsf{Spec}\, S \text{ as topological spaces.} \qquad (*)$$

However, such a result no longer holds for non-affine schemes. In fact, there exist a lot of non-isomorphic schemes X and Y such that $\mathsf{D}^{\mathsf{perf}}(X) \cong \mathsf{D}^{\mathsf{perf}}(Y)$; see [Muk, Orl97]. When there is a triangulated equivalence $\mathsf{D}^{\mathsf{perf}}(X) \cong \mathsf{D}^{\mathsf{perf}}(Y)$, X and Y are said to be *derived equivalent*. In section 3, we shall prove that the underlying topological spaces of a certain class of schemes can be reconstructed from their perfect derived categories:

Theorem 12.3 (Theorem 14.7)**.** *Let X and Y be Noetherian quasi-affine schemes (i.e., open subschemes of affine schemes). Then the implication*

$$\mathsf{D}^{\mathsf{perf}}(X) \cong \mathsf{D}^{\mathsf{perf}}(Y) \Longrightarrow X \cong Y \text{ as topological spaces}$$

holds.

This theorem recovers $(*)$ for Noetherian rings as any affine scheme is quasi-affine. A typical example of a non-affine quasi-affine scheme is the punctured spectrum of a local ring. As an application of this theorem, we obtain that a derived equivalence of X and Y yields the equality of the dimensions of X and Y.

12.2. **Modular representation theory**

In modular representation theory, finite groups are studied in various contexts. From an algebraic viewpoint, a finite group G has been studied through its group algebra kG and *stable module category* $\underline{\mathsf{mod}}\, kG$, where k is a field whose characteristic divides the order of G. Here, $\underline{\mathsf{mod}}\, kG$ is a triangulated category consisting of finitely generated kG-modules modulo projectives. On the other hand, the *cohomology ring* $\mathsf{H}^*(G; k)$ gives an approach to study a finite group G from the topological aspect because it is isomorphic to the cohomology ring of a classifying space BG of G; see [Ben, Chapter 2] for instance. The second main result in section 3 is the following:

Theorem 12.4 (Theorem 14.10)**.** *Let k (resp. l) be a field of characteristic p (resp. q), and let G (resp. H) be a finite p-group (resp. q-group). Then the implication*

$$\underline{\mathsf{mod}}\, kG \cong \underline{\mathsf{mod}}\, lH \Longrightarrow \mathsf{Proj}\, \mathsf{H}^*(G; k) \cong \mathsf{Proj}\, \mathsf{H}^*(H; l) \text{ as topological spaces}$$

holds.

If there exists a triangulated equivalence $\underline{\mathsf{mod}}\, kG \cong \underline{\mathsf{mod}}\, lH$, we say that kG and lH are *stably equivalent*. As an application of this theorem, we have that a stable equivalence of kG and lH yields that the p-rank of G and the q-rank of H are equal.

12.3. **Commutative ring theory**

Let R be a left Noetherian ring. The *singularity category* of R is by definition the Verdier quotient

$$\mathsf{D}_{\mathsf{sg}}(R) := \mathsf{D}^{\mathsf{b}}(\mathsf{mod} R)/\mathsf{D}^{\mathsf{perf}}(R),$$

which has been introduced by Buchweitz [Buc] in 1980s. Here, $\mathsf{mod}\, R$ stands for the category of finitely generated left R-modules and $\mathsf{D}^{\mathsf{b}}(\mathsf{mod} R)$ its bounded derived category. The singularity categories have been deeply investigated from algebro-geometric and representation-theoretic motivations [Che, IW, Ste, Tak10] and connected to the Homological Mirror Symmetry Conjecture by Orlov [Orl04].

One of the important subjects in representation theory of rings is to classify rings up to certain category equivalence. For example, left Noetherian rings R and S are said to be:

- *Morita equivalent* if $\mathsf{mod}\, R \cong \mathsf{mod}\, S$ as abelian categories,
- *derived equivalent* if $\mathsf{D^b}(\mathsf{mod}\, R) \cong \mathsf{D^b}(\mathsf{mod}\, S)$ as triangulated categories,
- *singularly equivalent* if $\mathsf{D_{sg}}(R) \cong \mathsf{D_{sg}}(S)$ as triangulated categories.

It is well known that these equivalences have the following relations:

$$\text{Morita equivalence} \Rightarrow \text{derived equivalence} \Rightarrow \text{singular equivalence}.$$

Complete characterizations of Morita and derived equivalence have already been obtained in [Mor, Ric], while singular equivalence is quite difficult to characterize even in the case of commutative rings. Indeed, only a few examples of singular equivalences of commutative Noetherian rings are known. Furthermore, for all of such known examples, the singular loci of rings are homeomorphic. Thus, it is natural to ask the following question.

Question 12.5. Let R and S be commutative Noetherian rings. Are their singular loci homeomorphic if R and S are singularly equivalent?

In section 4, we show that this question is affirmative for certain classes of commutative Noetherian rings. To be precise, we shall prove the following theorem.

Theorem 12.6 (Theorem 15.4)**.** *Let R and S be commutative Noetherian local rings that are locally hypersurfaces on the punctured spectra. Assume that R and S are either*

(a) *complete intersection rings, or*
(b) *Cohen-Macaulay rings with quasi-decomposable maximal ideal.*

Then the implication

$$\mathsf{D_{sg}}(R) \cong \mathsf{D_{sg}}(S) \Longrightarrow \mathsf{Sing}\, R \cong \mathsf{Sing}\, S \text{ as topological spaces}$$

holds.

Here, we say that an ideal I of a commutative ring R is *quasi-decomposable* if there is an R-regular sequence $\underline{x}$ in I such that $I/(\underline{x})$ is decomposable as an R-module. Moreover, we prove that singular equivalence localizes by using such a homeomorphism.

The organization of this part is as follows. In section 2, we introduce the notions of a classifying support data for a given triangulated category and develop the support theory without tensor structure, and finally prove Theorem 12.2. In section 3, we connect the results obtained in section 2 with the support theory for tensor triangulated categories and study reconstructing the topologies of the Balmer spectra without tensor structure. Using this method, we prove Theorem 12.3 and 12.4. In section 4, we prove Theorem 12.6 and give examples of commutative rings which are not singularly equivalent.

For two triangulated category $\mathcal{T}$, $\mathcal{T}'$ (resp. topological spaces X, X'), the notation $\mathcal{T} \cong \mathcal{T}'$ (resp. $X \cong X'$) means that $\mathcal{T}$ and $\mathcal{T}'$ are equivalent as triangulated categories (resp. X and X' are homeomorphic) unless otherwise specified.

13. **Reconstruction of classifying spaces**

In this section, we discuss the support theory for triangulated categories and reconstruction of classifying spaces.

Definition 13.1. Let $\mathcal{U}$ be a full subcategory of $\mathcal{T}$. We say that $\mathcal{U}$ is a *$\oplus$-ideal* if it satisfies

$$M \in \mathcal{U}, N \in \mathcal{T} \Rightarrow M \oplus N \in \mathcal{U}.$$

Remark 13.2. $\mathcal{U} \subseteq \mathcal{T}$ is a $\oplus$-ideal if and only if $\mathcal{T} \setminus \mathcal{U}$ is closed under taking direct summands.

Example 13.3. (1) The full subcategory $\mathcal{T} \setminus \{0\}$ is a $\oplus$-ideal.
(2) The full subcategory $\mathsf{T}(\mathcal{T})$ of test objects (see Definition 15.8 below) of $\mathcal{T}$ is a $\oplus$-ideal.

Let us fix the following notations:

Notation 13.4. Let $\mathcal{T}$ be a triangulated category, $\mathcal{U} \subseteq \mathcal{T}$ a $\oplus$-ideal, and X a topological space. Then we set:

- $\mathbf{Th}(\mathcal{T}) :=$ {thick subcategories of $\mathcal{T}$},
- $\mathbf{Th}_{\mathcal{U}}(\mathcal{T}) :=$ {thick subcategories of $\mathcal{T}$ containing an object of $\mathcal{U}$},
- $\mathbf{Spcl}(X) :=$ {specialization closed subsets of X},
- $\mathbf{Nesc}(X) :=$ {non-empty specialization-closed subsets of X},
- $\mathbf{Nec}(X) :=$ {non-empty closed subsets of X},
- $\mathbf{Irr}(X) :=$ {irreducible closed subsets of X}.

Let (X, σ) be a support data for $\mathcal{T}$, $\mathcal{X}$ a thick subcategory of $\mathcal{T}$, and W a specialization-closed subset of X. Recall that $f_\sigma(\mathcal{X}) := \bigcup_{M \in \mathcal{X}} \sigma(M)$ is a specialization-closed subset of X and $g_\sigma(W) := \{M \in \mathcal{T} \mid \sigma(M) \subseteq W\}$ is a thick subcategory of $\mathcal{T}$. Therefore, we obtain two order-preserving maps

$$\mathbf{Th}(\mathcal{T}) \underset{g_\sigma}{\overset{f_\sigma}{\rightleftarrows}} \mathbf{Spcl}(X).$$

with respect to the inclusion relations. Now, let us introduce the notions of a classifying support data and a classifying space.

Definition 13.5. Let (X, σ) be a support data for $\mathcal{T}$ and $\mathcal{U} \subseteq \mathcal{T}$ a $\oplus$-ideal. Then we say that (X, σ) is a *classifying support data* for $\mathcal{T}$ with respect to $\mathcal{U}$ if

(i) X is a Noetherian sober space, and
(ii) the above maps f_σ and g_σ restrict to mutually inverse bijections:

$$\mathbf{Th}_{\mathcal{U}}(\mathcal{T}) \underset{g_\sigma}{\overset{f_\sigma}{\rightleftarrows}} \mathbf{Nesc}(X).$$

When this is the case, we say that X is a classifying space of $\mathcal{T}$ with respect to $\mathcal{U}$.

If (X, σ) is a classifying support data for $\mathcal{T}$ with respect to $\mathcal{U} = \mathcal{T} \setminus \{0\}$, then we call it a *classifying support data for* $\mathcal{T}$ and X a *classifying space of* $\mathcal{T}$ for simplicity.

Remark 13.6. A classifying support data (X, σ) for $\mathcal{T}$ classifies all thick subcategories of $\mathcal{T}$ containing $g_\sigma(\emptyset) = \sigma^{-1}(\emptyset)$. Indeed, the map $g_\sigma : \mathbf{Nesc}(X) \to \mathbf{Th}_{\mathcal{U}}(\mathcal{T})$ is injective with image $\{\mathcal{X} \in \mathbf{Th}(\mathcal{T}) \mid \mathcal{X} \supsetneq \sigma^{-1}(\emptyset)\}$. In particular, if (X, σ) satisfies the condition $(1')$ in Remark 1.5, we obtain a one-to-one correspondence:

$$\mathbf{Th}(\mathcal{T}) \underset{g_\sigma}{\overset{f_\sigma}{\rightleftarrows}} \mathbf{Spcl}(X).$$

Every classifying support data automatically satisfies the following realization property.

Lemma 13.7. *Let (X, σ) be a classifying support data for $\mathcal{T}$ with respect to $\mathcal{U}$. Then for any non-empty closed subset Z of X, there is an object M of $\mathcal{U}$, such that $Z = \sigma(M)$.*

Proof. Since X is a Noetherian sober space and $\sigma(M) \cup \sigma(N) = \sigma(M \oplus N)$, we may assume that $Z = \overline{\{x\}}$ for some $x \in X$. From the assumption, one has $Z = f_\sigma g_\sigma(Z) = \bigcup_{M \in g_\sigma(Z)} \sigma(M)$. Hence, there is an element x of $\sigma(M)$ for some $M \in g_\sigma(Z)$. Then we obtain $x \in \sigma(M) \subseteq Z = \overline{\{x\}}$ and this implies that $\sigma(M) = \overline{\{x\}} = Z$.

By definition of a classifying support data with respect to $\mathcal{U}$, $g_\sigma(Z) = \{N \in \mathcal{T} \mid \sigma(N) \subseteq \sigma(M)\}$ contains a object T of $\mathcal{U}$. We conclude that $\sigma(T \oplus M) = \sigma(T) \cup \sigma(M) = \sigma(M) = Z$ for $T \oplus M \in \mathcal{U}$. ■

Let me give two more classes of thick subcategories which play an important role in the proof of first main theorem.

Definition 13.8. Let $\mathcal{U}$ be a $\oplus$-ideal of $\mathcal{T}$.

(1) We say that a thick subcategory $\mathcal{X}$ of $\mathcal{T}$ is *$\mathcal{U}$-principal* if there is an object M of $\mathcal{U}$ such that $\mathcal{X} = \mathsf{thick}_\mathcal{T} M$. Denote by $\mathbf{PTh}_\mathcal{U}(\mathcal{T})$ the set of all $\mathcal{U}$-principal thick subcategories of $\mathcal{T}$.

(2) We say that a $\mathcal{U}$-principal thick subcategory $\mathcal{X}$ of $\mathcal{T}$ is *$\mathcal{U}$-irreducible* if $\mathcal{X} = \mathsf{thick}_\mathcal{T}(\mathcal{X}_1 \cup \mathcal{X}_2)$ $(\mathcal{X}_1, \mathcal{X}_2 \in \mathbf{PTh}_\mathcal{U}(\mathcal{T}))$ implies that $\mathcal{X}_1 = \mathcal{X}$ or $\mathcal{X}_2 = \mathcal{X}$. Denote by $\mathbf{Irr}_\mathcal{U}(\mathcal{T})$ the set of all $\mathcal{U}$-irreducible thick subcategories of $\mathcal{T}$.

The following lemma shows that by using classifying support data with respect to $\mathcal{U}$, we can also classify $\mathcal{U}$-principal thick subcategories and $\mathcal{U}$-irreducible thick subcategories.

Lemma 13.9. *Let (X, σ) be a classifying support data for $\mathcal{T}$ with respect to $\mathcal{U}$, then the one-to-one correspondence*

$$\mathbf{Th}_\mathcal{U}(\mathcal{T}) \underset{g_\sigma}{\overset{f_\sigma}{\rightleftarrows}} \mathbf{Nesc}(X)$$

restricts to one-to-one correspondences

$$\mathbf{PTh}_\mathcal{U}(\mathcal{T}) \underset{g_\sigma}{\overset{f_\sigma}{\rightleftarrows}} \mathbf{Nec}(X),$$

$$\mathbf{Irr}_\mathcal{U}(\mathcal{T}) \underset{g_\sigma}{\overset{f_\sigma}{\rightleftarrows}} \mathbf{Irr}(X).$$

Proof. Note that $f_\sigma(\mathsf{thick}_\mathcal{T} M) = \sigma(M)$ for any $M \in \mathcal{T}$. Therefore, the injective map $f_\sigma : \mathbf{Th}_\mathcal{U}(\mathcal{T}) \to \mathbf{Nesc}(X)$ induces a well defined injective map $f_\sigma : \mathbf{PTh}_\mathcal{U}(\mathcal{T}) \to \mathbf{Nec}(X)$. The surjectivity has already been shown in Lemma 13.7.

Next, we show the second one-to-one correspondence. For $\mathcal{X}_1, \mathcal{X}_2 \in \mathbf{Th}_\mathcal{U}(\mathcal{T})$, one has

$$\begin{aligned} f_\sigma(\mathsf{thick}_\mathcal{T}(\mathcal{X}_1 \cup \mathcal{X}_2)) &= \bigcup_{M \in \mathsf{thick}_\mathcal{T}(\mathcal{X}_1 \cup \mathcal{X}_2)} \sigma(M) \\ &= \bigcup_{M \in \mathcal{X}_1 \cup \mathcal{X}_2} \sigma(M) \\ &= (\bigcup_{M \in \mathcal{X}_1} \sigma(M)) \cup (\bigcup_{M \in \mathcal{X}_2} \sigma(M)) \\ &= f_\sigma(\mathcal{X}_1) \cup f_\sigma(\mathcal{X}_2). \end{aligned} \tag{1}$$

On the other hand, for $Z_1, Z_2 \in \mathbf{Nesc}(X)$, one has

$$\begin{aligned} f_\sigma(\mathsf{thick}_\mathcal{T}(g_\sigma(Z_1) \cup g_\sigma(Z_2))) &= f_\sigma(g_\sigma(Z_1)) \cup f_\sigma(\sigma(Z_2)) \\ &= Z_1 \cup Z_2. \end{aligned}$$

Applying g_σ to this equality, we get

$$\mathsf{thick}_{\mathcal{T}}(g_\sigma(Z_1) \cup g_\sigma(Z_2)) = g_\sigma(Z_1 \cup Z_2). \tag{2}$$

Let W be an irreducible closed subset of X. Assume $g_\sigma(W) = \mathsf{thick}_{\mathcal{T}}(\mathcal{X}_1 \cup \mathcal{X}_2)$ for some $\mathcal{X}_1, \mathcal{X}_2 \in \mathbf{PTh}_{\mathcal{U}}(\mathcal{T})$. Then from the above equality (1), we obtain an equality

$$W = f_\sigma(g_\sigma(W)) = f_\sigma(\mathsf{thick}_{\mathcal{T}}(\mathcal{X}_1 \cup \mathcal{X}_2)) = f_\sigma(\mathcal{X}_1) \cup f_\sigma(\mathcal{X}_2).$$

Since W is irreducible, $f_\sigma(\mathcal{X}_1) = W$ or $f_\sigma(\mathcal{X}_2) = W$ and hence $\mathcal{X}_1 = g_\sigma(f_\sigma(\mathcal{X}_1)) = g_\sigma(W)$ or $\mathcal{X}_2 = g_\sigma(f_\sigma(\mathcal{X}_2)) = g_\sigma(W)$. This shows that $g_\sigma(W)$ is $\mathcal{U}$-irreducible.

Conversely, take a $\mathcal{U}$-irreducible thick subcategory $\mathcal{X}$ of $\mathcal{T}$ and assume $f_\sigma(\mathcal{X}) = Z_1 \cup Z_2$ for some non-empty closed subsets Z_1, Z_2 of X. From the above equality (2), we get

$$\mathcal{X} = g_\sigma(f_\sigma(\mathcal{X})) = g_\sigma(Z_1 \cup Z_2) = \mathsf{thick}_{\mathcal{T}}(g_\sigma(Z_1) \cup g_\sigma(Z_2)).$$

Since $\mathcal{X}$ is $\mathcal{U}$-irreducible, $\mathcal{X} = g_\sigma(Z_1)$ or $\mathcal{X} = g_\sigma(Z_2)$ and therefore, $Z_1 = f_\sigma(g_\sigma(Z_1)) = f_\sigma(\mathcal{X})$ or $Z_2 = f_\sigma(g_\sigma(Z_2)) = f_\sigma(\mathcal{X})$. Thus, $f_\sigma(\mathcal{X})$ is irreducible.

These observations show the second one-to-one correspondence. ■

From this lemma, we can show the following uniqueness result for classifying support data with respect to $\mathcal{U}$.

Proposition 13.10. *Let (X, σ) and (Y, τ) be classifying support data for $\mathcal{T}$ with respect to a $\oplus$-ideal $\mathcal{U}$. Then X and Y are homeomorphic.*

Proof. First note that for a topological space X, the natural map $\iota_X : X \to \mathbf{Irr}(X), x \mapsto \overline{\{x\}}$ is bijective iff X is sober.

Define maps $\varphi : X \to Y$ and $\psi : Y \to X$ to be the composites

$$\varphi : X \xrightarrow{\iota_X} \mathbf{Irr}(X) \xrightarrow{g_\sigma} \mathbf{Irr}_{\mathcal{U}}(\mathcal{T}) \xrightarrow{f_\tau} \mathbf{Irr}(Y) \xrightarrow{\iota_Y^{-1}} Y,$$

$$\psi : Y \xrightarrow{\iota_Y} \mathbf{Irr}(Y) \xrightarrow{g_\tau} \mathbf{Irr}_{\mathcal{U}}(\mathcal{T}) \xrightarrow{f_\sigma} \mathbf{Irr}(X) \xrightarrow{\iota_X^{-1}} X.$$

Then φ and ψ are well defined and mutually inverse bijections by Lemma 13.9.

Fix $x \in X$. For $x' \in \overline{\{x\}}$, one has $\iota_X(x') \subseteq \iota_X(x)$ and hence

$$\overline{\{\varphi(x')\}} = \iota_Y(\varphi(x')) = f_\tau(g_\sigma(\iota_X(x'))) \subseteq \overline{\{\varphi(x)\}} = \iota_Y(\varphi(x)) = f_\tau(g_\sigma(\iota_X(x))).$$

In particular, $\varphi(x')$ belongs to $\overline{\{\varphi(x)\}}$. Therefore, $\varphi(\overline{\{x\}}) \subseteq \overline{\{\varphi(x)\}}$.

Conversely, for $y \in \overline{\{\varphi(x)\}}$, the above argument shows

$$\psi(y) \in \psi(\overline{\{\varphi(x)\}}) \subseteq \overline{\{\psi\varphi(x)\}} = \overline{\{x\}}.$$

Applying φ to this inclusion, we obtain $y \in \varphi(\overline{\{x\}})$ and therefore, $\overline{\{\varphi(x)\}} \subseteq \varphi(\overline{\{x\}})$. Thus, we conclude that $\varphi(\overline{\{x\}}) = \overline{\{\varphi(x)\}}$. Since X is Noetherian, this equation means that φ is a closed map. Similarly, ψ is also a closed map. ■

The following theorem is the main result of this section.

Theorem 13.11. *Consider the following setting:*

- *$\mathcal{T}$ and $\mathcal{T}'$ are triangulated categories.*
- *$\mathcal{U}$ and $\mathcal{U}'$ are $\oplus$-ideals of $\mathcal{T}$ and $\mathcal{T}'$, respectively.*
- *(X, σ) and (Y, τ) are classifying support data for $\mathcal{T}$ and $\mathcal{T}'$ with respect to $\mathcal{U}$ and $\mathcal{U}'$, respectively.*

Suppose that there is a triangle equivalence $F : \mathcal{T} \to \mathcal{T}'$ with $F(\mathcal{U}) = \mathcal{U}'$. Then X and Y are homeomorphic.

Proof. From the assumption, F induces a one-to-one correspondence

$$\tilde{F} : \mathbf{Th}_{\mathcal{U}}(\mathcal{T}) \xrightarrow{\cong} \mathbf{Th}_{\mathcal{U}'}(\mathcal{T}'),\ \mathcal{X} \mapsto \tilde{F}(\mathcal{X}),$$

where $\tilde{F}(\mathcal{X}) := \{N \in \mathcal{T}' \mid \exists M \in \mathcal{X} \text{ such that } N \cong F(M)\}$. For an object M of $\mathcal{T}$, set $\tau^F(M) := \tau(F(M))$. Then we can easily verify that the pair (Y, τ^F) is a support data for $\mathcal{T}$. Furthermore, it becomes a classifying support data for $\mathcal{T}$ with respect to $\mathcal{U}$. Indeed, for $\mathcal{X} \in \mathbf{Th}_{\mathcal{U}}(\mathcal{T})$ and $W \in \mathbf{Nesc}(Y)$, we obtain

$$f_{\tau^F}(\mathcal{X}) = \bigcup_{M \in \mathcal{X}} \tau^F(M) = \bigcup_{M \in \mathcal{X}} \tau(F(M)) = \bigcup_{N \in \tilde{F}(\mathcal{X})} \tau(N) = f_\tau(\tilde{F}(\mathcal{X})),$$

$$\begin{aligned} \tilde{F}(g_{\tau^F}(W)) &= \tilde{F}(\{M \in \mathcal{T} \mid \tau^F(M) \subseteq W\}) \\ &= \{N \in \mathcal{T}' \mid \tau(N) \subseteq W\} = g_\tau(W). \end{aligned}$$

From these equalities, we get equalities $f_{\tau^F} = f_\tau \circ \tilde{F}$ and $\tilde{F} \circ g_{\tau^F} = g_\tau$ and thus f_{τ^F} and g_{τ^F} give mutually inverse bijections between $\mathbf{Th}_{\mathcal{U}}(\mathcal{T})$ and $\mathbf{Nesc}(Y)$. Consequently, we obtain two classifying support data (X, σ) and (Y, τ^F) for $\mathcal{T}$ with respect to $\mathcal{U}$, and hence X and Y are homeomorphic by Proposition 13.10. ■

14. Comparison with tensor triangulated structure

In this section, we discuss relation between the support theory we discussed in the previous section and that for tensor triangulated categories. Throughout this section, fix a tensor triangulated category $(\mathcal{T}, \otimes, \mathbf{1})$.

Recall that a support data (X, σ) for $\mathcal{T}$ is *tensorial* if it satisfies:

$$\sigma(M \otimes N) = \sigma(M) \cap \sigma(N)$$

for any $M, N \in \mathcal{T}$. Then $g_\sigma(W)$ is a radical thick $\otimes$-ideal of $\mathcal{T}$ for every specialization-closed subset W of X. We say that a tensorial support data (X, σ) is *classifying* if X is a Noetherian sober space and there is a one-to-one correspondence:

$$\mathbf{Rad}(\mathcal{T}) \underset{g_\sigma}{\overset{f_\sigma}{\rightleftarrows}} \mathbf{Spcl}(X).$$

Balmer showed the following celebrated result:

Theorem 14.1. [Bal05, Lemma 2.6, Theorem 4.10]

(1) *The pair* $(\mathsf{Spec}\,\mathcal{T}, \mathsf{BSupp})$ *is a tensorial support data for* $\mathcal{T}$.

(2) *There is a one-to-one correspondence:*

$$\{\textit{radical thick } \otimes\textit{-ideals of } \mathcal{T}\} \underset{g_{\mathsf{BSupp}}}{\overset{f_{\mathsf{BSupp}}}{\rightleftarrows}} \mathbf{Thom}(\mathsf{Spec}\,\mathcal{T}).$$

Remark 14.2. The above theorem shows that $(\mathsf{Spec}\,\mathcal{T}, \mathsf{BSupp})$ is a classifying tensorial support data for $\mathcal{T}$ provided $\mathsf{Spec}\,\mathcal{T}$ is Noetherian.

Note that a tensorial classifying support data for $\mathcal{T}$ is a classifying tensorial support data for $\mathcal{T}$. Indeed, for a tensorial classifying support data (X, σ) for $\mathcal{T}$ and $\mathcal{X} \in \mathbf{Th}(\mathcal{T})$, we obtain an equalities

$$\mathcal{X} = g_\sigma(f_\sigma(\mathcal{X})) = g_\sigma(f_\sigma(\sqrt{\mathsf{thick}^\otimes \mathcal{X}})) = \sqrt{\mathsf{thick}^\otimes \mathcal{X}}.$$

The following lemma gives a criterion for the converse implication of this fact.

Lemma 14.3. *Let (X, σ) be a classifying tensorial support data for $\mathcal{T}$. Suppose that $\mathcal{T}$ is rigid. Then the following are equivalent:*

(1) *There is a one-to-one correspondence:*

$$\mathbf{Th}(\mathcal{T}) \underset{g_\sigma}{\overset{f_\sigma}{\rightleftarrows}} \mathbf{Spcl}(X).$$

(2) *(X, σ) is a classifying support data for $\mathcal{T}$.*
(3) *Every thick subcategory of $\mathcal{T}$ is a thick $\otimes$-ideal.*
(4) *$\mathcal{T} = \mathsf{thick}_{\mathcal{T}}\,\mathbf{1}$.*

Proof. By Lemma 2.8 and Theorem 2.17, (X, σ) satisfies the condition $(1')$ in Remark 1.5. Therefore, (1) and (2) means the same conditions from Remark 13.6.

(1) $\Rightarrow$ (3): From the assumption, every thick subcategory $\mathcal{X}$ of $\mathcal{T}$ is of the form $\mathcal{X} = g_\sigma(W)$ for some specialization-closed subset W of X. On the other hand, $g_\sigma(W)$ is a radical thick $\otimes$-ideal as (X, σ) is a tensorial support data.

(3) $\Rightarrow$ (4): By assumption, the thick subcategory $\mathsf{thick}_{\mathcal{T}}\,\mathbf{1}$ is a thick tensor ideal. Thus, for any $M \in \mathcal{T}$, $M \cong M \otimes \mathbf{1}$ belongs to $\mathsf{thick}_{\mathcal{T}}\,\mathbf{1}$.

(4) $\Rightarrow$ (1): Note that $\mathbf{1}$ is strongly dualizable and the family of all strongly dualizable objects forms a thick subcategory of $\mathcal{T}$ by [HPS, Theorem A.2.5 (a)]. Therefore, every object of $\mathcal{T} = \mathsf{thick}_{\mathcal{T}}\,\mathbf{1}$ is strongly dualizable. Thus, for any object $M \in \mathcal{T}$, M belongs to $\mathsf{thick}^{\otimes}_{\mathcal{T}}(M \otimes M)$ by [HPS, Lemma A.2.6]. Then [Bal05, Proposition 4.4] shows that every thick tensor ideal of $\mathcal{T}$ is radical.

On the other hand, for any thick subcategory $\mathcal{X}$ of $\mathcal{Y}$, one can easily verify that the subcategory

$$\mathcal{Y} := \{M \in \mathcal{T} \mid M \otimes \mathcal{X} \subseteq \mathcal{X}\}$$

is a thick $\otimes$-ideal of $\mathcal{T}$ containing $\mathbf{1}$. Thus, we obtain $\mathcal{Y} = \mathsf{thick}_{\mathcal{T}}\,\mathbf{1} = \mathcal{T}$ and hence $\mathcal{X}$ is a thick $\otimes$-ideal.

From these discussion, we conclude that every thick subcategory of $\mathcal{T}$ is a radical thick $\otimes$-ideal and this shows the implication (4) $\Rightarrow$ (1). ■

The following corollaries are direct consequences of this lemma, Proposition 13.10 and Theorem 13.11.

Corollary 14.4. *Let $\mathcal{T}$ be a closed tensor triangulated category. Assume that the Balmer spectrum $\mathsf{Spec}\,\mathcal{T}$ of $\mathcal{T}$ is Noetherian and that $\mathcal{T} = \mathsf{thick}_{\mathcal{T}}\,\mathbf{1}$. Then for any classifying support data (X, σ) for $\mathcal{T}$, X is homeomorphic to $\mathsf{Spec}\,\mathcal{T}$.*

Corollary 14.5. *Let $\mathcal{T}$ and $\mathcal{T}'$ be closed tensor triangulated categories such that*

(1) *$\mathsf{Spec}\,\mathcal{T}$ and $\mathsf{Spec}\,\mathcal{T}'$ are Noetherian, and*
(2) *$\mathcal{T}$ and $\mathcal{T}'$ are generated by their unit objects.*

If $\mathcal{T}$ and $\mathcal{T}'$ are equivalent as triangulated categories, then $\mathsf{Spec}\,\mathcal{T}$ and $\mathsf{Spec}\,\mathcal{T}'$ are homeomorphic.

Next, we consider applications of these corollaries to tensorial support data appeared in Example 1.4.

Thomason showed the following classification theorem of thick tensor ideas of $\mathsf{D}^{\mathsf{perf}}(X)$:

Theorem 14.6. [Tho, Theorem 3.15] *Let X be a Noetherian scheme. Then (X, Supp_X) is a classifying tensorial support data for $\mathsf{D}^{\mathsf{perf}}(X)$.*

As an application of Corollary 14.5, we can reconstruct underlying topological spaces of a certain class of schemes from their perfect derived categories without tensor structure.

Theorem 14.7. *Let X and Y be Noetherian quasi-affine schemes (i.e., open subschemes of affine schemes). If X and Y are derived equivalent, then X and Y are homeomorphic. In particular, topologically determined properties, such as the dimensions and the numbers of irreducible components of quasi-affine Noetherian schemes are preserved by derived equivalences.*

Proof. First, let me remark that the functor $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_X} - : \mathsf{D}^{\mathsf{perf}}(X) \to \mathsf{D}^{\mathsf{perf}}(X)$ has a right adjoint $\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, -) : \mathsf{D}^{\mathsf{perf}}(X) \to \mathsf{D}^{\mathsf{perf}}(X)$ for each $\mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(X)$. Thus, $\mathsf{D}^{\mathsf{perf}}(X)$ is a closed tensor triangulated category.

Note that a scheme X is quasi-affine if and only if its structure sheaf $\mathcal{O}_X$ is ample. Thus, every thick subcategory of $\mathsf{D}^{\mathsf{perf}}(X)$ is thick tensor ideal by [Tho, Proposition 3.11.1]. Applying Corollary 14.5, we obtain the result. ■

Remark 14.8. Let X and Y be Noetherian schemes.

(1) As we have already remarked in the introduction, if X and Y are affine, then a derived equivalence $\mathsf{D}^{\mathsf{perf}}(X) \cong \mathsf{D}^{\mathsf{perf}}(Y)$ implies that X and Y are isomorphic as schemes.
(2) By [Bal02, Theorem 9.7], if $\mathsf{D}^{\mathsf{perf}}(X)$ and $\mathsf{D}^{\mathsf{perf}}(Y)$ are equivalent as tensor triangulated categories, then X and Y are isomorphic as schemes.

Next consider stable module categories over group rings of finite groups. In this case, the following classification theorem is given by Benson-Carlson-Rickard for algebraically closed field k and by Benson-Iyenger-Krause for general k.

Theorem 14.9. [BCR, BIK] *Let k be a field of characteristic $p > 0$ and G a finite group such that p divides the order of G. Then the support data $(\mathsf{Proj}\,\mathsf{H}^*(G;k), V_G)$ is a classifying tensorial support data for $\underline{\mathsf{mod}}\,kG$.*

Applying Corollary 14.5 to this classifying tensorial support data, we obtain the following result:

Theorem 14.10. *Let k (resp. l) be field of characteristic p (resp. q), G (resp. H) be a finite p-group (resp. q-group). If kG and lH are stably equivalent, then $\mathsf{Proj}\,\mathsf{H}^*(G;k)$ and $\mathsf{Proj}\,\mathsf{H}^*(H;l)$ are homeomorphic.*

Proof. For each $M \in \underline{\mathsf{mod}}\,kG$, the functor $M \otimes_k - : \underline{\mathsf{mod}}\,kG \to \underline{\mathsf{mod}}\,kG$ has a right adjoint $\mathrm{Hom}_k(M, -) : \underline{\mathsf{mod}}\,kG \to \underline{\mathsf{mod}}\,kG$. Thus, $\underline{\mathsf{mod}}\,kG$ is a closed tensor triangulated category. Moreover, for a p-group G, kG has only one simple module k. Therefore, we have $\underline{\mathsf{mod}}\,kG = \mathsf{thick}_{\underline{\mathsf{mod}}\,kG}\,k$. Applying Corollary 14.5, we are done. ■

Recall that the *p-rank* of a finite group G is by definition,

$$r_p(G) := \sup\{r \mid (\mathbb{Z}/p)^r \subseteq G\}.$$

Quillen [Qui] showed that the dimension of the cohomology ring $H^*(G;k)$ is equal to the p-rank of G. Thus, the p-rank is an invariant of stable equivalences:

Corollary 14.11. *Let k, l, G, H be as in Theorem 14.10. Assume that there is a stable equivalence between kG and lH, then $r_p(G) = r_q(H)$.*

Remark 14.12. Let G and H be a p-group and k a field of characteristic p.

(1) By [Lin, Corollary 3.6], if there exists a stable equivalence between kG and kH, then $|G| = |H|$.

(2) By [Lin, Corollary 3.2], if there exists a stable equivalence of Morita type between kG and kH, then $G \cong H$.

15. **A necessary condition for singular equivalences**

Recall that commutative Noetherian rings R and S are said to be *singularly equivalent* if their singularity categories are equivalent as triangulated categories. The only known examples of singular equivalences are the following:

Example 15.1. (1) If $R \cong S$, then $\mathsf{D_{sg}}(R) \cong \mathsf{D_{sg}}(S)$.
(2) If R and S are regular, then $\mathsf{D_{sg}}(R) \cong 0 \cong \mathsf{D_{sg}}(S)$.
(3) (Knörrer's periodicity [Yos, Chapter 12]) Let k be an algebraically closed field of characteristic 0. Set $R := k[[x_0, x_1, ..., x_d]]/(f)$ and $S := k[[x_0, x_1, ..., x_d, u, v]]/(f+uv)$. Then $\mathsf{D_{sg}}(R) \cong \mathsf{D_{sg}}(S)$.

Remark 15.2. All of these singular equivalences, the singular loci $\mathsf{Sing}\, R$ and $\mathsf{Sing}\, S$ are homeomorphic. In fact, the cases (1) and (2) are clear. Consider the case of $R := k[[x_0, x_1, ..., x_d]]/(f)$ and $S := k[[x_0, x_1, ..., x_d, u, v]]/(f + uv)$. Then

$$\begin{aligned}
\mathsf{Sing}\, S &= \mathrm{V}(\partial f/\partial x_0, \dots \partial f/\partial x_d, u, v)\\
&\cong \mathsf{Spec}(S/(\partial f/\partial x_0, \dots, \partial f/\partial x_d, u, v))\\
&\cong \mathsf{Spec}(k[[x_0, x_1, ..., x_d, u, v]]/(f + uv, \partial f/\partial x_0, \dots, \mathsf{D} f/\mathsf{D} x_d, u, v))\\
&\cong \mathsf{Spec}(k[[x_0, x_1, ..., x_d]]/(f, \partial f/\partial x_0, \dots, \partial f/\partial x_d)\\
&\cong \mathrm{V}(\partial f/\partial x_0, \dots \partial f/\partial x_d) = \mathsf{Sing}\, R.
\end{aligned}$$

Here, the first and the last equalities are known as the Jacobian criterion.

Let me give some definitions appearing in the statement of the main theorem of this section.

Definition 15.3. Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring.

(1) We say that an ideal I of R is *quasi-decomposable* if there is an R-regular sequence $\underline{x}$ of I such that $I/(\underline{x})$ is decomposable as an R-module.
(2) A local ring R is said to be *complete intersection* if there is a regular local ring S and an S-regular sequence $\underline{x}$ such that the completion $\hat{R}$ of R is isomorphic to $S/(\underline{x})$. We say that R is a *hypersurface* if we can take $\underline{x}$ to be an S-regular sequence of length 1.
(3) A local ring R is said to be *locally a hypersurface on the punctured spectrum* if $R_{\mathfrak{p}}$ is a hypersurface for every non-maximal prime ideal $\mathfrak{p}$.

The following theorem is the main result of this section.

Theorem 15.4. *Let R and S be commutative Noetherian local rings that are locally hypersurfaces on the punctured spectra. Assume that R and S are either*
(a) *complete intersection rings, or*
(b) *Cohen-Macaulay rings with quasi-decomposable maximal ideal.*
If R and S are singularly equivalent, then $\mathsf{Sing}\, R$ *and* $\mathsf{Sing}\, S$ *are homeomorphic.*

For a ring R satisfying the condition (b) in Theorem 15.4, Nasseh-Takahashi [NT, Theorem B] shows that $(\mathsf{Sing}\, R, \mathsf{SSupp}_R)$ is a classifying support data for $\mathsf{D_{sg}}(R)$. Therefore, the statement of Theorem 15.4 follows from Theorem 13.11. Therefore, the problem is the case of (a).

For a ring R satisfying the condition (a) in Theorem 15.4, Takahashi [Tak10] classified thick subcategories of $\mathsf{D_{sg}}(R)$ containing the residue field k of R by using the singular locus $\mathsf{Sing}\, R$ and the singular support SSupp_R. We would like to apply Theorem 13.11 also for this case. The problem is that whether the condition "containing the residue field k" is preserved by stable equivalences. As we will show later, this condition is actually preserved by singular equivalences for local complete intersection rings. To do this, we discuss replacing the residue field k with some categorically defined object.

First of all, let us recall the notion of a test module.

Definition 15.5. Let R be a Noetherian ring. We say that a finitely generated R-module T is a *test module* if for any finitely generated R-module M,

$$\mathsf{Tor}_n^R(T, M) = 0 \text{ for } n \gg 0 \Rightarrow \mathsf{pd}_R M < \infty.$$

Example 15.6. For a Noetherian local ring $(R, \mathfrak{m}, k)$, the syzygy $\Omega^n k$ of its residue field is a test module for each n.

For commutative Noetherian rings admitting dualizing complexes (e.g., Gorenstein rings), there is another characterization for test modules:

Theorem 15.7. [CDT, Theorem 3.2] *Let R be a commutative Noetherian ring admitting a dualizing complex. Then, test modules are nothing but finitely generated R-modules T satisfying the following condition: for any finitely generated R-module M,*

$$\mathsf{Ext}_R^n(T, M) = 0 \textit{ for } n \gg 0 \Rightarrow RM < \infty.$$

Motivated by this theorem, we introduce the following notion.

Definition 15.8. Let $\mathcal{T}$ be a triangulated category. We say that $T \in \mathcal{T}$ is a *test object* if for any object M of $\mathcal{T}$,

$$\mathrm{Hom}_{\mathcal{T}}(T, M[n]) = 0 \text{ for } n \gg 0 \Rightarrow M = 0.$$

Denote by $\mathsf{T}(\mathcal{T})$ the full subcategory of $\mathcal{T}$ consisting of test objects.

The following lemma shows that we can consider the notion of a test object is a generalization of the notion of a test module.

Lemma 15.9. *Let R be a Gorenstein ring. Then one has*

$$\mathsf{T}(\underline{\mathsf{CM}}(R)) = \{T \in \underline{\mathsf{CM}}(R) \mid T \textit{ is a test module}\}.$$

Proof. By Theorem 15.7, we have only to show

$$\mathsf{T}(\underline{\mathsf{CM}}(R)) = \{T \in \underline{\mathsf{CM}}(R) \mid \text{all } N \in \mathsf{mod}\, R \text{ with } \mathsf{Ext}_R^{\gg 0}(M, N) = 0 \text{ satisfy } RN < \infty\}.$$

Fix a maximal Cohen-Macaulay R-module T and a finitely generated R-module M. Since R is Gorenstein and T is maximal Cohen-Macaulay, one has $\mathsf{Ext}_R^1(T, R) = 0$. Therefore, we get isomorphisms

$$\mathsf{Ext}_R^i(T, M) \cong \mathsf{Ext}_R^{i+1}(T, \Omega_R M) \cong \mathsf{Ext}_R^{i+2}(T, \Omega_R^2 M) \cong \cdots$$

for any positive integer i. Therefore, we get isomorphisms

$$\underline{\mathsf{Hom}}_R(T, \Omega_R^d M[d+n]) \cong \mathsf{Ext}_R^{d+n}(T, \Omega_R^d M) \cong \mathsf{Ext}_R^n(T, M)$$

for $n > 0$. Here, d denotes the dimension of R. Thus, we are done since $\Omega_R^d M$ is free if and only if M has finite injective dimension. ∎

Let us recall several classes of subcategories of modules.

Definition 15.10. (1) An additive subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ is called *resolving* if it satisfies the following conditions:

(i) $\mathcal{X}$ is closed under extensions: for an exact sequence $0 \to L \to M \to N \to 0$ in $\mathsf{mod}\, R$, if L and N belong to $\mathcal{X}$, then so does M.
(ii) $\mathcal{X}$ is closed under kernels of epimorphisms: for an exact sequence $0 \to L \to M \to N \to 0$ in $\mathsf{mod}\, R$, if M and N belong to $\mathcal{X}$, then so does L.
(iii) $\mathcal{X}$ contains all projective R-modules.

For a finitely generated R-module M, denote by $\mathsf{res}_R(M)$ the smallest resolving subcategory of $\mathsf{mod}\, R$ containing M.

(2) A non-empty additive subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ is called *thick* if $\mathcal{X}$ satisfies 2-out-of-3 property: for an exact sequence $0 \to L \to M \to N \to 0$ in $\mathsf{mod}\, R$, if 2-out-of $\{L, M, N\}$ belong to $\mathcal{X}$, then so does the third. For a finitely generated R-module M, denote by $\mathsf{thick}_R(M)$ the smallest thick subcategory of $\mathsf{mod}\, R$ containing M.

Lemma 15.11. *Let $\mathcal{T}$ be a triangulated category and T an object of $\mathcal{T}$. If $\mathsf{thick}_{\mathcal{T}}(T)$ contains a test object of $\mathcal{T}$, then T is also a test object.*

Proof. Take an object $M \in \mathcal{T}$ with $\operatorname{Hom}_{\mathcal{T}}(T, M[n]) = 0$ for $n \gg 0$. Set

$$\mathcal{X} := \{N \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(N, M[n]) = 0 \text{ for } n \gg 0\}.$$

Then one can easily verify that $\mathcal{X}$ is a thick subcategory of $\mathcal{T}$. By assumption, $\mathcal{X}$ contains a test object as $\mathcal{X}$ contains T. Thus, M must be zero and hence T is a test object. ∎

The next proposition plays a key role to prove our main theorem.

Proposition 15.12. *Let $(R, \mathfrak{m}, k)$ be a d-dimensional local complete intersection ring and T a finitely generated R-module. Then the following are equivalent:*

(1) *T is a test module.*
(2) $\Omega^d_R k \in \mathsf{res}_R(T)$.
(3) $k \in \mathsf{thick}_R(T \oplus R)$.
(4) $k \in \mathsf{thick}_{\mathsf{D^b}(\mathsf{mod}\, R)}(T \oplus R)$.
(5) $k \in \mathsf{thick}_{\mathsf{D_{sg}}(R)}(T)$.
(6) $\Omega^d_R k \in \mathsf{thick}_{\underline{\mathsf{CM}}(R)}(\Omega^d T)$.

Proof. Notice $\mathsf{res}_R(\Omega^i_R T) \subseteq \mathsf{res}_R(T)$, $\mathsf{thick}_R(T \oplus R) = \mathsf{thick}_R(\Omega^i_R T \oplus R)$, $\mathsf{thick}_{\mathsf{D^b}(\mathsf{mod}\, R)}(T \oplus R) = \mathsf{thick}_{\mathsf{D^b}(\mathsf{mod}\, R)}(\Omega^i_R T \oplus R)$, $\mathsf{thick}_{\mathsf{D_{sg}}(R)}(T) = \mathsf{thick}_{\mathsf{D_{sg}}}(R)(\Omega^i_R T)$ and T is a test module if and only if so is $\Omega_R T$. Hence we may assume that T is maximal Cohen-Macaulay. Then we have

$$\mathsf{res}_R(T) \subseteq \mathsf{thick}_R(T \oplus R) = \mathsf{thick}_{\mathsf{D^b}(\mathsf{mod}\, R)}(T \oplus R) \cap \mathsf{mod}\, R.$$

Here, the first inclusion directly follows from the definition, and the second equality is given by [KS, Theorem 1]. Moreover, the composition functor $\mathsf{D^b}(\mathsf{mod}\, R) \to \mathsf{D_{sg}}(R) \xrightarrow{\cong} \underline{\mathsf{CM}}(R)$ sends k to $\Omega^d_R k[d]$, and the inverse image of $\mathsf{thick}_{\underline{\mathsf{CM}}(R)}(T)$ is $\mathsf{thick}_{\mathsf{D^b}(\mathsf{mod}\, R)}(T \oplus R)$. Therefore, the implications (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) hold true. Furthermore, by using Lemma 15.9 and Lemma 15.11, the implication (5) $\Rightarrow$ (1) follows.

Thus, it remains to show the implication (1) $\Rightarrow$ (2). Assume that T is a test module. Recall that the *complexity* $\mathsf{cx}_R(M)$ of a finitely generated R-module M is the dimension of the *support variety* $V_R(M)$ associated to M; see [AvrBuc] for details. By [CDT, Proposition 2.7], T has maximal complexity, namely $\mathsf{cx}_R(T) = \mathsf{codim}(R) =: c$. Thanks to the prime avoidance lemma, we can take an R-regular sequence $\underline{x}$ of length d from $\mathfrak{m} \setminus \mathfrak{m}^2$. Set $\overline{R} = R/(\underline{x})$ and $\overline{T} = T/(\underline{x})$. Then $\overline{R}$ is an Artinian complete intersection ring and $\mathsf{cx}_{\overline{R}}(\overline{T}) = \mathsf{cx}_R(T) = c = \mathsf{codim} R = \mathsf{codim}(\overline{R})$. Moreover, one has

$$V_{\overline{R}}(\overline{T}) = \mathbb{A}^c_{k^a} = V_{\overline{R}}(k),$$

where k^a denotes the algebraic closure of k. This follows from the fact that $V_{\overline{R}}(\overline{T})$ and $V_{\overline{R}}(k)$ are c-dimensional closed subvarieties of the c-dimensional affine space $\mathbb{A}^c_{k^a}$. Hence, by [CI, Theorem 5.6], k belongs to $\mathsf{thick}_{\mathsf{D}^b(\mathsf{mod}\,\overline{R})}(\overline{T})$. As a result, we get

$$k \in \mathsf{thick}_{\mathsf{D}^b(\mathsf{mod}\,\overline{R})}(\overline{T}) \cap \mathsf{mod}(\overline{R}) \subseteq \mathsf{thick}_{\mathsf{D}^b(\mathsf{mod}\,\overline{R})}(\overline{T} \oplus \overline{R}) \cap \mathsf{mod}(\overline{R}) = \mathsf{thick}_{\overline{R}}(\overline{T} \oplus \overline{R}).$$

Again, the second equality uses [KS, Theorem 1]. Since $\mathsf{thick}_{\overline{R}}(\overline{T} \oplus \overline{R}) = \mathsf{res}_{\overline{R}}(\overline{T})$ by [DT14, Corollary 4.16], we deduce $\Omega^d_R k \in \mathsf{res}_R(T)$ by using [Tak10, Lemma 5.8]. ■

Gathering [Tak10, Theorem 6.7], [NT, Theorem B], Lemma 15.9 and Proposition 15.12, we obtain the following proposition.

Proposition 15.13. *Let R be a Noetherian local ring.*

(1) *If R satisfies the condition (a) in Theorem 15.4, then $(\mathsf{Sing}\, R, \mathsf{SSupp}_R)$ is a classifying support data for $\mathsf{D_{sg}}(R)$ with respect to $\mathsf{T}(\mathsf{D_{sg}}(R))$.*

(2) *If R satisfies the condition (b) in Theorem 15.4, then $(\mathsf{Sing}\, R, \mathsf{SSupp}_R)$ is a classifying support data for $\mathsf{D_{sg}}(R)$.*

Now, the proof of Theorem 15.4 has almost been done.

Proof of Theorem 15.4. Use Proposition 15.13 and Theorem 13.11. Here, let me remark that test objects are preserved by singular equivalences. ■

Remark 15.14. For a hypersurface ring R, the triangulated category $\mathsf{D_{sg}}(R)$ becomes a pseudo tensor triangulated category (i.e., tensor triangulated category without unit). It is shown by Yu implicitly in the paper [Yu] that for two hypersurfaces R and S, if a singular equivalence between R and S preserves tensor products, then $\mathsf{Sing}\, R$ and $\mathsf{Sing}\, S$ are homeomorphic. Indeed, $\mathsf{Sing}\, R$ is reconstructed from $\mathsf{D_{sg}}(R)$ by using its pseudo tensor triangulated structure.

Since Theorem 15.4 gives a necessary condition for singular equivalences, we can generate many pairs of rings which are not singularly equivalent. Let us start with the following lemma.

Lemma 15.15. *Let R be a local complete intersection ring with only an isolated singularity and $r > 1$ an integer. Then the ring $R[[u]]/(u^r)$ is a local complete intersection ring which is locally a hypersurface on the punctured spectrum, and $\mathsf{Sing}(R[[u]]/(u^r))$ is homeomorphic to $\mathsf{Spec}\, R$.*

Proof. Of course $T := R[[u]]/(u^r)$ is a local complete intersection ring.

The natural inclusion $R \to T$ induces a homeomorphism $f : \mathsf{Spec}\, T \xrightarrow{\cong} \mathsf{Spec}\, R$. Then one can easily check that $P = (f(P), u)T$ for any $P \in \mathsf{Spec}\, T$ and $T_P \cong R_{f(P)}[[u]]/(u^r)$. Therefore, T is locally a hypersurface on the punctured spectrum and $\mathsf{Sing}\, T = \mathsf{Spec}\, T$. ■

Corollary 15.16. *Let R and S be local complete intersection rings which have only isolated singularities. Assume that $\mathsf{Spec}\, R$ and $\mathsf{Spec}\, S$ are not homeomorphic. Then for any integers $r, s > 1$, one has*

$$\mathsf{D_{sg}}(R[[u]]/(u^r)) \not\cong \mathsf{D_{sg}}(S[[v]]/(v^s)).$$

In particular, $\mathsf{D_{sg}}(R * R) \ncong \mathsf{D_{sg}}(S * S)$*. Here* $R * R$ *denotes the trivial extension ring of a commutative ring* R*.*

Proof. From the above lemma, we obtain

(1) $R[[u]]/(u^r)$ and $S[[v]]/(v^s)$ satisfies the condition (a) in Theorem 15.4,
(2) $\mathsf{Sing}\, R[[u]]/(u^r) \cong \mathsf{Spec}\, R$ and $\mathsf{Sing}\, S[[u]]/(v^r) \cong \mathsf{Spec}\, S$ are not homeomorphic.

Thus, we conclude $\mathsf{D_{sg}}(R[[u]]/(u^r)) \ncong \mathsf{D_{sg}}(S[[v]]/(v^s))$ by Theorem 15.4.

The second statement follows from an isomorphism $R * R \cong R[[u]]/(u^2)$. ■

The following corollary says that a Knörrer-type equivalence fails over a non-regular ring.

Corollary 15.17. *Let* S *be a regular local ring. Assume that* $S/(f)$ *has an isolated singularity. Then one has*

$$\mathsf{D_{sg}}(S[[u]]/(f, u^2)) \ncong \mathsf{D_{sg}}(S[[u, v, w]]/(f + vw, u^2)).$$

Proof. $\mathsf{Sing}\, S[[u]]/(f, u^2) \cong \mathsf{Spec}\, S/(f)$ and $\mathsf{Sing}\, S[[u, v, w]]/(f+vw, u^2) \cong \mathsf{Spec}\, S[[v, w]]/(f+vw)$ have different dimensions and hence are not homeomorphic. ■

For the last of this part, we will show that singular equivalence localizes.

Lemma 15.18. *Let* R *be a* d*-dimensional Gorenstein local ring and* $\mathfrak{p}$ *a prime ideal of* R*. Then a full subcategory* $\mathcal{X}_\mathfrak{p} := \{M \in \mathsf{D_{sg}}(R) \mid M_\mathfrak{p} \cong 0 \text{ in } \mathsf{D_{sg}}(R_\mathfrak{p})\}$ *is thick and there is a triangle equivalence*

$$\mathsf{D_{sg}}(R)/\mathcal{X}_\mathfrak{p} \cong \mathsf{D_{sg}}(R_\mathfrak{p}).$$

Proof. By using the triangle equivalence $\mathsf{D_{sg}}(R) \cong \underline{\mathsf{CM}}(R)$, we may show the triangle equivalence

$$\underline{\mathsf{CM}}(R)/\mathcal{X}_\mathfrak{p} \cong \underline{\mathsf{CM}}(R_\mathfrak{p}),$$

where $\mathcal{X}_\mathfrak{p} := \{M \in \underline{\mathsf{CM}}(R) \mid M_\mathfrak{p} \cong 0 \text{ in } \underline{\mathsf{CM}}(R_\mathfrak{p})\}$.

Note that the localization functor $L_\mathfrak{p} : \underline{\mathsf{CM}}(R) \to \underline{\mathsf{CM}}(R_\mathfrak{p}), M \mapsto M_\mathfrak{p}$ is triangulated. Since $\mathcal{X}_\mathfrak{p} = \mathsf{Ker}\, L_\mathfrak{p}$, $\mathcal{X}_\mathfrak{p}$ is a thick subcategory of $\underline{\mathsf{CM}}(R)$ and $L_\mathfrak{p}$ induces a triangulated functor $\overline{L}_\mathfrak{p} : \underline{\mathsf{CM}}(R)/\mathcal{X}_\mathfrak{p} \to \underline{\mathsf{CM}}(R_\mathfrak{p})$. Thus, we have only to verify that $\overline{L}_\mathfrak{p}$ is dense and fully faithful.

(i): $\overline{L_\mathfrak{p}}$ is dense.

Let U be an $R_\mathfrak{p}$-module. Take a finite free presentation $R_\mathfrak{p}^n \xrightarrow{\delta} R_\mathfrak{p}^m \to U \to 0$ of U. Then δ can be viewed as an $m \times n$-matrix (α_{ij}) with entries in $R_\mathfrak{p}$. Write $\alpha_{ij} = a_{ij}/s$ for some $a_{ij} \in R$ and $s \in R \setminus \mathfrak{p}$. Then the cokernel $M := \mathsf{Cok}((a_{ij}) : R^n \to R^m)$ is a finitely generated R-module and $M_\mathfrak{p} \cong U$. Since $M_\mathfrak{p}$ is a maximal Cohen-Macaulay $R_\mathfrak{p}$-module, we obtain isomorphisms

$$(\Omega_R^{-d}\Omega_R^d M)_\mathfrak{p} \cong \Omega_{R_\mathfrak{p}}^{-d}\Omega_{R_\mathfrak{p}}^d M_\mathfrak{p} \cong M_\mathfrak{p} \cong U$$

in $\underline{\mathsf{CM}}(R_\mathfrak{p})$. This shows that the functor $\overline{L_\mathfrak{p}}$ is dense.

(ii): $\overline{L_\mathfrak{p}}$ is faithful.

Let $\alpha : M \to N$ be a morphism in $\underline{\mathsf{CM}}(R)/\mathcal{X}_\mathfrak{p}$. Then α is given by a fraction f/s of morphisms $f : M \to Z$ and $s : N \to Z$ in $\underline{\mathsf{CM}}(R)$ such that the mapping cone $C(s)$ of s belongs to $\mathcal{X}_\mathfrak{p}$. Assume $\overline{L}_\mathfrak{p}(\alpha) = L_\mathfrak{p}(s)^{-1}L_\mathfrak{p}(f) = (s_\mathfrak{p})^{-1}f_\mathfrak{p} = 0$. Then $f_\mathfrak{p} = 0$ in $\underline{\mathsf{Hom}}_{R_\mathfrak{p}}(M_\mathfrak{p}, Z_\mathfrak{p})$. From the isomorphism $\underline{\mathsf{Hom}}_R(M, Z)_\mathfrak{p} \cong \underline{\mathsf{Hom}}_{R_\mathfrak{p}}(M_\mathfrak{p}, Z_\mathfrak{p})$, there is $a \in R \setminus \mathfrak{p}$ such that $af = 0$ in $\underline{\mathsf{Hom}}_R(M, Z)$. Since $a : Z_\mathfrak{p} \to Z_\mathfrak{p}$ is isomorphism, the mapping cone of the morphism $a : Z \to Z$ in $\underline{\mathsf{CM}}(R)$ belongs to $\mathcal{X}_{\mathfrak{p}}$. Thus, $\alpha = f/s = (af)/(as) = 0$ in $\underline{\mathsf{CM}}(R)/\mathcal{X}_{\mathfrak{p}}$. This shows that $\overline{L}_{\mathfrak{p}}$ is faithful.

(iii): $\overline{L_{\mathfrak{p}}}$ is full.

Let $g : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ be a morphism in $\underline{\mathsf{CM}}(R_{\mathfrak{p}})$ where $M, N \in \underline{\mathsf{CM}}(R)$. By the isomorphism $\underline{\mathsf{Hom}}_R(M, N)_{\mathfrak{p}} \cong \underline{\mathsf{Hom}}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$, there is a morphism $f : M \to N$ in $\underline{\mathsf{CM}}(R)$ and $a \in R \backslash \mathfrak{p}$ such that $g = f_{\mathfrak{p}}/a$. Since the mapping cone of $a : N \to N$ is in $\mathcal{X}_{\mathfrak{p}}$, we obtain a morphism $f/a : M \to N$ in $\underline{\mathsf{CM}}(R)/\mathcal{X}_{\mathfrak{p}}$ and $\overline{L}_{\mathfrak{p}}(f/a) = f_{\mathfrak{p}}/a = g$. This shows that $\overline{L}_{\mathfrak{p}}$ is full. ■

Corollary 15.19. *Let R and S be complete intersection rings which are locally hypersurfaces on the punctured spectra. If R and S are singularly equivalent, then there is a homeomorphism $\varphi : \mathsf{Sing}\, R \to \mathsf{Sing}\, S$ such that $R_{\mathfrak{p}}$ and $S_{\varphi(\mathfrak{p})}$ are singularly equivalent for any $\mathfrak{p} \in \mathsf{Sing}\, R$.*

Proof. As in Lemma 15.18, we may consider the category $\underline{\mathsf{CM}}(R)$.

Let $F : \underline{\mathsf{CM}}(R) \to \underline{\mathsf{CM}}(S)$ be a triangle equivalence. Take a homeomorphism $\varphi : \mathsf{Sing}\, R \to \mathsf{Sing}\, S$ given in Proposition 13.10 and Theorem 13.11. Then by construction, it satisfies

$$\overline{\{\varphi(\mathfrak{p})\}} = \bigcup_{M \in \underline{\mathsf{CM}}(R),\ \underline{\mathsf{Supp}}_R(M) \subseteq \mathrm{V}(\mathfrak{p})} \underline{\mathsf{Supp}}_S F(M)$$

for each $\mathfrak{p} \in \mathsf{Sing}\, R$. Moreover, the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{Th}_{\mathsf{T}(\underline{\mathsf{CM}}(R))}(\underline{\mathsf{CM}}(R)) & \xrightarrow{\tilde{F}} & \mathbf{Th}_{\mathsf{T}(\underline{\mathsf{CM}}(S))}(\underline{\mathsf{CM}}(S)) \\ {\scriptstyle f_{\underline{\mathsf{Supp}}_R}}\downarrow & & \downarrow{\scriptstyle f_{\underline{\mathsf{Supp}}_S}} \\ \mathbf{Nesc}(\mathsf{Sing}\, R) & \xrightarrow[\tilde{\varphi}]{} & \mathbf{Nesc}(\mathsf{Sing}\, S), \end{array}$$

where the map $\tilde{F}$ and $\tilde{\varphi}$ are defined by $\tilde{F}(\mathcal{X}) := \{N \in \mathcal{T}' \mid \exists M \in \mathcal{X} \text{ such that } N \cong F(M)\}$ and $\tilde{\varphi}(W) := \varphi(W)$, respectively.

Let $\mathfrak{p}$ be an element of $\mathsf{Sing}\, R$. Set $W_{\mathfrak{p}} := \{\mathfrak{q} \in \mathsf{Sing}\, R \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}$ which is a specialization-closed subset of $\mathsf{Sing}\, R$. We establish two claims.

Claim 3. $g_{\underline{\mathsf{Supp}}_R}(W_{\mathfrak{p}}) = \mathcal{X}_{\mathfrak{p}}$.

Proof of Claim 1. Let $M \in \mathcal{X}_{\mathfrak{p}}$. Since $M_{\mathfrak{p}} = 0$ in $\underline{\mathsf{CM}}(R_{\mathfrak{p}})$, one has $\mathfrak{p} \notin \underline{\mathsf{Supp}}_R(M)$. Thus, $\underline{\mathsf{Supp}}_R(M) \subseteq W_{\mathfrak{p}}$ and hence $M \in g_{\underline{\mathsf{Supp}}_R}(W_{\mathfrak{p}})$.

Next, take $M \in g_{\underline{\mathsf{Supp}}_R}(W_{\mathfrak{p}})$. Then $\underline{\mathsf{Supp}}_R(M) \subseteq W_{\mathfrak{p}}$ means that $\mathfrak{p}$ does not belong to $\underline{\mathsf{Supp}}_R(M)$. Therefore, $M_{\mathfrak{p}} = 0$ in $\underline{\mathsf{CM}}(R_{\mathfrak{p}})$ and hence $M \in \mathcal{X}_{\mathfrak{p}}$. ■

Claim 4. $\varphi(W_{\mathfrak{p}}) = W_{\varphi(\mathfrak{p})} := \{\mathfrak{q} \in \mathsf{Sing}\, S \mid \mathfrak{q} \not\subseteq \varphi(\mathfrak{p})\}$.

Proof of Claim 2. One can easily check that φ is order isomorphism with respect to the inclusion relations. Since $\mathsf{Sing}\, R \setminus W_{\mathfrak{p}}$ has a unique maximal element $\mathfrak{p}$, $\varphi(\mathsf{Sing}\, R \setminus W_{\mathfrak{p}}) = \mathsf{Sing}\, S \setminus \varphi(W_{\mathfrak{p}})$ also has a unique maximal element $\varphi(\mathfrak{p})$. This shows $\varphi(W_{\mathfrak{p}}) = W_{\varphi(\mathfrak{p})}$. ■

From the above two claims, we obtain

$$\tilde{F}(\mathcal{X}_{\mathfrak{p}}) = \tilde{F}(g_{\underline{\mathsf{Supp}}_R}(W_{\mathfrak{p}})) = g_{\underline{\mathsf{Supp}}_S}(\tilde{\varphi}(W_{\mathfrak{p}})) = g_{\underline{\mathsf{Supp}}_S}(W_{\varphi(\mathfrak{p})}) = \mathcal{X}_{\varphi(\mathfrak{p})},$$

where the second equality comes from the above commutative diagram and the last equality is shown by the same proof as Claim 1. Consequently, the triangle equivalence F induces triangle equivalences:

$$\underline{\mathsf{CM}}(R_{\mathfrak{p}}) \cong \underline{\mathsf{CM}}(R)/\mathcal{X}_{\mathfrak{p}} \cong \underline{\mathsf{CM}}(S)/\mathcal{X}_{\varphi(\mathfrak{p})} \cong \underline{\mathsf{CM}}(S_{\varphi(\mathfrak{p})}).$$

Part 4. Classification of dense subcategories

16. introduction

The content of this part is based on the authors paper [Mat17a]. In this part, we discuss classifying dense subcategories.

Let $\mathcal{A}$ be an additive category and $\mathcal{X}$ an additive subcategory of $\mathcal{A}$. We say that $\mathcal{X}$ is *additively closed* if it is closed under taking direct summands, and that $\mathcal{X}$ is *dense* if any object in $\mathcal{A}$ is a direct summand of some object of $\mathcal{X}$. We can easily show that $\mathcal{X}$ is additively closed if and only if $\mathcal{X} = \mathsf{add}\,\mathcal{X}$ and that $\mathcal{X}$ is dense if and only if $\mathcal{A} = \mathsf{add}\,\mathcal{X}$. Here, $\mathsf{add}\,\mathcal{X}$ denotes the smallest additive subcategory of $\mathcal{A}$ which is closed under taking direct summands and contains $\mathcal{X}$. Therefore, for any additive subcategory $\mathcal{X}$ of $\mathcal{A}$, $\mathcal{X}$ is a dense subcategory of $\mathsf{add}\,\mathcal{X}$ and $\mathsf{add}\,\mathcal{X}$ is an additively closed subcategory of $\mathcal{A}$. For this reason, to classify additive subcategories, it suffices to classify additively closed ones and dense ones. Thomason classified dense triangulated subcategories:

Theorem 16.1. [Tho, Theorem 2.1] *Let $\mathcal{T}$ be an essentially small triangulated category. Then there is a one-to-one correspondence*

$$\{\textit{dense triangulated subcategories of } \mathcal{T}\} \underset{g}{\overset{f}{\rightleftarrows}} \{\textit{subgroups of } K_0(\mathcal{T})\},$$

where f and g are given by $f(\mathcal{X}) := \langle [X] \mid X \in \mathcal{X}\rangle$ and $g(H) := \{X \in \mathcal{T} \mid [X] \in H\}$, respectively, and $K_0(\mathcal{T})$ stands for the Grothendieck group of $\mathcal{T}$.

Motivated by this theorem, we discuss classifying dense resolving and dense coresolving subcategories of exact categories. The notion of a resolving subcategory has been introduced by Auslander and Bridger [AusBri] and that of a coresolving subcategory is its dual notion. Resolving and coresolving subcategories have been widely studied so far, for example, see [AR, DT, KS, Tak11]. The main theorem of this part is the following.

Theorem 16.2 (Proposition 17.5, Theorem 17.7)**.** *Let $\mathcal{E}$ be an essentially small exact category with either a generator or a cogenerator $\mathcal{G}$.*

(1) *The following subcategories of $\mathcal{E}$ are the same:*
   (i) *dense $\mathcal{G}$-resolving subcategories*
   (ii) *dense $\mathcal{G}$-coresolving subcategories*
   (iii) *dense $\mathcal{G}$-2-out-of-3 subcategories*
(2) *There is a one-to-one correspondence*
   where f and g are given by $f(\mathcal{X}) := \langle [X] \mid X \in \mathcal{X}\rangle$ and $g(H) := \{X \in \mathcal{E} \mid [X] \in H\}$, respectively, and $K_0(\mathcal{E})$ stands for the Grothendieck group of $\mathcal{E}$.

Here, the notion of a $\mathcal{G}$-resolving (resp. $\mathcal{G}$-coresolving) subcategory is a slight generalization of that of a resolving (resp. coresolving) subcategory. Indeed, they coincide when $\mathcal{G}$ consists of the projective (resp. injective) objects. In addition, $\mathcal{G}$-2-out-of-3 subcategory is a subcategory which is both $\mathcal{G}$-resolving and $\mathcal{G}$-coresolving. The precise definitions of these subcategories will be given in Definition 17.3.

17. Classification of dense resolving subcategories

In this section, we show our main result. Throughout this part, let $\mathcal{A}$ be an abelian category, $\mathcal{E}$ an exact category, and $\mathcal{T}$ a triangulated category.

We begin with recalling several notions, which are key notions of this part.

Definition 17.1. Let $\mathcal{G}$ be a family of objects of $\mathcal{E}$. We call $\mathcal{G}$ a *generator* (resp. a *cogenerator*) of $\mathcal{E}$ if for any object $A \in \mathcal{E}$, there is a short exact sequence

$$A' \rightarrowtail G \twoheadrightarrow A \ \text{(resp. } A \rightarrowtail G \twoheadrightarrow A')$$

in $\mathcal{E}$ with $G \in \mathcal{G}$.

Example 17.2. (1) Clearly, $\mathcal{E}$ is both a generator and a cogenerator of $\mathcal{E}$
(2) If $\mathcal{E}$ has enough projective (resp. injective) objects, then the subcategory $\mathsf{proj}\,\mathcal{E}$ (resp. $\mathsf{inj}\,\mathcal{E}$) consisting of projective (resp. injective) objects is a generator (resp. a cogenerator) of $\mathcal{E}$.

Next we give the definitions of $\mathcal{G}$-resolving and $\mathcal{G}$-coresolving subcategories.

Definition 17.3. Let $\mathcal{X}$ be a subcategory of $\mathcal{E}$ and $\mathcal{G}$ a family of objects of $\mathcal{E}$.

(1) We say that $\mathcal{X}$ is a $\mathcal{G}$-*resolving subcategory* of $\mathcal{E}$ if the following three conditions are satisfied:
  (i) $\mathcal{X}$ is closed under extensions: for a short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$, if X and Z are in $\mathcal{X}$, then so is Y.
  (ii) $\mathcal{X}$ is closed under kernels of admissible epimorphisms: for a short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$, if Y and Z are in $\mathcal{X}$, then so is X.
  (iii) $\mathcal{X}$ contains $\mathcal{G}$.
  If $\mathcal{E}$ has enough projective objects, we shall call $\mathcal{X}$ simply *resolving* if it is $\mathsf{proj}\,\mathcal{E}$-resolving.
(2) We say that $\mathcal{X}$ is a $\mathcal{G}$-*coresolving subcategory* of $\mathcal{E}$ if the following three conditions are satisfied:
  (i) $\mathcal{X}$ is closed under extensions: for a short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$, if X and Z are in $\mathcal{X}$, then so is Y.
  (ii) $\mathcal{X}$ is closed under cokernels of admissible monomorphisms: for a short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$, if X and Y are in $\mathcal{X}$, then so is Z.
  (iii) $\mathcal{X}$ contains $\mathcal{G}$.
(3) We say that $\mathcal{X}$ is a $\mathcal{G}$-*2-out-of-3 subcategory* of $\mathcal{E}$ if the following conditions are satisfied:
  (i) $\mathcal{X}$ satisfies 2-out-of-3 property: for a short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$, if 2 out of $\{X, Y, Z\}$ belong to $\mathcal{X}$, then so is the third.
  (ii) $\mathcal{X}$ contains $\mathcal{G}$.

Remark 17.4. Unlike the definition due to Auslander and Bridger [AusBri], we do not assume that resolving subcategories are closed under direct summands. Therefore, our definition is rather close to the definitions in [AR].

The following proposition shows that dense $\mathcal{G}$-resolving, dense $\mathcal{G}$-coresolving, and dense $\mathcal{G}$-2-out-of-3 subcategories are the same thing.

Proposition 17.5. *Let $\mathcal{X}$ be a dense subcategory of $\mathcal{E}$.Then $\mathcal{X}$ is closed under cokernels of admissible monomorphisms if and only if it is closed under kernels of admissible epimorphisms.*

Proof. We have only to show the 'if' part. The 'only if' part is proved by the dual argument.

Let $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$ be a short exact sequence in $\mathcal{E}$ with $X, Y \in \mathcal{X}$. Since $\mathcal{X}$ is dense, we can take $Z' \in \mathcal{E}$ with $Z \oplus Z' \in \mathcal{X}$. Consider a short exact sequence

$$X \oplus Z \overset{\begin{pmatrix} f & 0 \\ 0 & id_Z \\ 0 & 0 \end{pmatrix}}{\rightarrowtail} Y \oplus Z \oplus Z' \overset{\begin{pmatrix} g & 0 & 0 \\ 0 & 0 & id_{Z'} \end{pmatrix}}{\twoheadrightarrow} Z \oplus Z'.$$

Then $X \oplus Z$ is an object of $\mathcal{X}$ because $\mathcal{X}$ is closed under kernels of admissible epimorphisms. From the split short exact sequence $Z \rightarrowtail X \oplus Z \twoheadrightarrow X$, we obtain $Z \in \mathcal{X}$ since $\mathcal{X}$ is closed under kernels of admissible epimorphisms. ∎

Now we recall the definition of the Grothendieck group of an exact category.

Definition 17.6. Let $\mathcal{E}$ be an exact category. Let F be the free abelian group generated by the isomorphism classes of objects of $\mathcal{E}$. Let I be the subgroup of F generated by the elements of the form $[A] - [B] + [C]$ where $A \rightarrowtail B \twoheadrightarrow C$ are short exact sequences in $\mathcal{E}$. Then we define the *Grothendieck group* of $\mathcal{E}$, denoted by $K_0(\mathcal{E})$, as the quotient group F/I.

The following theorem is our main result of this part.

Theorem 17.7. *Let $\mathcal{E}$ be an essentially small exact category with either a generator or a cogenerator $\mathcal{G}$. Then there are one-to-one correspondences among the following sets:*

(1) $\{$*dense $\mathcal{G}$-resolving subcategories of $\mathcal{E}$*$\}$,
(2) $\{$*dense $\mathcal{G}$-coresolving subcategories of $\mathcal{E}$*$\}$,
(3) $\{$*dense $\mathcal{G}$-2-out-of-3 subcategories of $\mathcal{E}$*$\}$, *and*
(4) $\{$*subgroups of $K_0(\mathcal{E})$ containing the image of $\mathcal{G}$*$\}$.

One-to-one correspondences among (1), (2) and (3) have been already shown in Proposition 17.5. Thus, it suffice to show the bijection between (1) and (4). Moreover, we will show this bijection only in the case that $\mathcal{G}$ is a generator because in the cogenerator case, it can be shown by the dual argument. The following lemma is essential in the proof of our theorem.

Lemma 17.8. *Let $\mathcal{G}$ be a generator of $\mathcal{E}$ and $\mathcal{X}$ a dense $\mathcal{G}$-resolving subcategory of $\mathcal{E}$. Then for an object A in $\mathcal{E}$, $A \in \mathcal{X}$ if and only if $[A] \in \langle [X] \mid X \in \mathcal{X} \rangle$.*

Proof. Define an equivalence relation $\sim$ on the isomorphism classes $\mathcal{E}/_{\cong}$ of objects of $\mathcal{E}$, as follows: $A \sim A'$ if there are $X, X' \in \mathcal{X}$ such that $A \oplus X \cong A' \oplus X'$. Set $\langle \mathcal{E} \rangle_{\mathcal{X}} := (\mathcal{E}/_{\cong})/_{\sim}$ and denote by $\langle A \rangle$ the class of A. Then $\langle \mathcal{E} \rangle_{\mathcal{X}}$ is an abelian group with $\langle A \rangle + \langle B \rangle := \langle A \oplus B \rangle$. Indeed, obviously, $+$ is well-defined, commutative, associative, and $\langle 0 \rangle$ is an identity element. Since $\mathcal{X}$ is dense, for any $A \in \mathcal{E}$, there is $A' \in \mathcal{E}$ such that $A \oplus A' \in \mathcal{X}$, and hence $\langle A \rangle + \langle A' \rangle = \langle A \oplus A' \rangle = \langle 0 \rangle$. Therefore, $\langle A' \rangle$ is an inverse element of $\langle A \rangle$.

Let $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$ be a short exact sequence in $\mathcal{E}$. Taking $A', C' \in \mathcal{E}$ with $A \oplus A', C \oplus C' \in \mathcal{X}$ and considering a short exact sequence

$$A \oplus A' \overset{\begin{pmatrix} f & 0 \\ 0 & id_{A'} \\ 0 & 0 \end{pmatrix}}{\rightarrowtail} B \oplus A' \oplus C' \overset{\begin{pmatrix} g & 0 & 0 \\ 0 & 0 & id_{C'} \end{pmatrix}}{\twoheadrightarrow} C \oplus C'.$$

we have $B \oplus A' \oplus C' \in \mathcal{X}$. This shows $\langle B \rangle - \langle A \rangle - \langle C \rangle = \langle B \oplus A' \oplus C' \rangle = \langle 0 \rangle$. Therefore, there is a group homomorphism

$$\varphi : K_0(\mathcal{E}) \to \langle \mathcal{E} \rangle_{\mathcal{X}}, \quad [A] \mapsto \langle A \rangle.$$

Note that $\langle [X] \mid X \in \mathcal{X} \rangle$ is contained in $\mathsf{Ker}\,\varphi$.

From the definition of the Grothendieck group, any element of $K_0(\mathcal{E})$ is denoted by $[A] - [B]$. Moreover, since there is a short exact sequence

$$B' \rightarrowtail G \twoheadrightarrow B$$

in $\mathcal{E}$ with $G \in \mathcal{G}$, $[A] - [B] = [A \oplus B'] - [G]$. Thus, any element of $K_0(\mathcal{E})$ is denoted by $[A] - [G]$ with $G \in \mathcal{G}$.

Let $[A] - [G]$ with $G \in \mathcal{G}$ be an element of $\mathsf{Ker}\,\varphi$. Since $\mathcal{X}$ contains $\mathcal{G}$, $[A] \in \mathsf{Ker}\,\varphi$. This means $\langle A \rangle = \langle 0 \rangle$ and there are $X, X' \in \mathcal{X}$ such that $A \oplus X \cong X'$. Considering the split short exact sequence

$$A \rightarrowtail A \oplus X \twoheadrightarrow X,$$

we obtain $A \in \mathcal{X}$ since $\mathcal{X}$ is closed under kernels of epimorphisms. Thus, $A \in \mathcal{X}$ if and only if $[A] \in \langle [X] \mid X \in \mathcal{X} \rangle$. ■

Proof of Theorem 17.7. By Lemma 17.5, the set (2) is nothing but the set (1). Therefore, we show that there is a one-to-one correspondence between the sets (1) and (3).

For a dense $\mathcal{G}$-resolving subcategory $\mathcal{X}$, define

$$f(\mathcal{X}) := \langle [X] \mid X \in \mathcal{X} \rangle,$$

and for a subgroup H of $K_0(\mathcal{E})$ containing the image of $\mathcal{G}$, define

$$g(H) := \{A \in \mathcal{E} \mid [A] \in H\}.$$

We show that f and g give mutually inverse bijections between (1) and (3).

First note that $g(H) := \{A \in \mathcal{E} \mid [A] \in H\}$ is a dense $\mathcal{G}$-resolving subcategory of $\mathcal{E}$ for a subgroup H of $K_0(\mathcal{E})$ containing the image of $\mathcal{G}$. Indeed, for any object $A \in \mathcal{E}$, take a short exact sequence $A' \rightarrowtail G \twoheadrightarrow A$ in $\mathcal{E}$ with $G \in \mathcal{G}$. Then $[A \oplus A'] = [A] + [A'] = [G] \in H$, and hence $A \oplus A' \in g(H)$. Thus $g(H)$ is dense. Obviously, $g(H)$ contains $\mathcal{G}$. Furthermore, for any short exact sequence $A \rightarrowtail B \twoheadrightarrow C$, the relation $[A] - [B] + [C] = 0$ implies that $g(H)$ is $\mathcal{G}$-resolving. Besides, $f(\mathcal{X})$ is clearly a subgroup of $K_0(\mathcal{E})$ containing the image of $\mathcal{G}$. As a result, f and g are well-defined maps between the sets (1) and (3)

Let H be a subgroup of $K_0(\mathcal{E})$ containing the image of $\mathcal{G}$. Then the inclusion $fg(H) \subset H$ is trivial. For any $[A] - [G] \in H$ with $G \in \mathcal{G}$, $[A] = ([A] - [G]) + [G] \in H$ implies $A \in g(H)$, and thus $[A] - [G] \in fg(H)$. Therefore, $fg(H) = H$.

Let $\mathcal{X}$ be a dense resolving subcategory of $\mathcal{E}$ containing $\mathcal{G}$. Then the inclusion $\mathcal{X} \subset gf(\mathcal{X})$ is trivial. Conversely, for any $A \in gf(\mathcal{X})$, since $[A] \in f(\mathcal{X}) = \langle [X] \mid X \in \mathcal{X} \rangle$, we have $A \in \mathcal{X}$ by Lemma 17.8. Therefore, $gf(\mathcal{X}) = \mathcal{X}$. Consequently, f and g are mutually inverse bijections between (1) and (3). ■

18. **Relations with dense triangulated subcategories**

In this section, we consider some combinations of Theorem 16.1 and Theorem 17.7. Let us start with the definition of the Grothendieck group for a triangulated category.

Definition 18.1. Let $\mathcal{T}$ be a triangulated category. Let F be the free abelian group generated by the isomorphism classes of objects of $\mathcal{T}$. Let I be the subgroup generated by the elements of the form $[A] - [B] + [C]$ where $A \to B \to C \to A[1]$ are exact triangles in $\mathcal{T}$. Then we define the *Grothendieck group* of $\mathcal{T}$, denoted by $K_0(\mathcal{T})$, as the quotient group F/I.

First, we discuss dense subcategories of exact categories and their derived categories. Please refer to [Buh, Nee90] for the definition of the derived category of an exact category.

Lemma 18.2. [Wei, Lemma 9.2.4] *Let $\mathcal{E}$ be an essentially small exact category. Then the canonical functor $\mathcal{E} \to \mathsf{D}^{\mathsf{b}}(\mathcal{E})$ induces an isomorphism $\varphi : K_0(\mathcal{E}) \to K_0(\mathsf{D}^{\mathsf{b}}(\mathcal{E}))$.*

Combining Theorem 16.1, Theorem 17.7 and this lemma, we have the following corollary.

Corollary 18.3. *Let $\mathcal{E}$ be an essentially small exact category with either a generator or a cogenerator $\mathcal{G}$. Then there are one-to-one correspondences among the following sets:*
(1) $\{$*dense $\mathcal{G}$-resolving subcategories of $\mathcal{E}$*$\}$,
(2) $\{$*dense triangulated subcategories of $\mathsf{D}^{\mathsf{b}}(\mathcal{E})$ containing $\mathcal{G}$*$\}$, *and*
(3) $\{$*subgroups of $K_0(\mathcal{E})$ containing the image of $\mathcal{G}$*$\}$.

Taking $\mathcal{G} = \mathsf{proj}\,\mathcal{E}$ in this corollary gives the dense version of the following theorem due to Krause and Stevenson:

Theorem 18.4. [KS, Theorem 1] *Let $\mathcal{E}$ be an exact category with enough projective objects. Then there is one-to-one correspondence between*
(1) $\{$*thick subcategories of $\mathcal{E}$ containing* $\mathsf{proj}\,\mathcal{E}\}$ *and*
(2) $\{$*thick triangulated subcategories of $\mathsf{D}^{\mathsf{b}}(\mathcal{E})$ containing* $\mathsf{proj}\,\mathcal{E}\}$.

Next, we give a more concrete corollary.

Let S be an *Iwanaga-Gorenstein ring* (i.e. S is Noetherian on both sides and S is of finite injective dimension as a left S-module and a right S-module). Let us give several remarks about Iwanaga-Gorenstein rings (cf. [Buc, Yos]).

Remark 18.5. (1) [Buc, Lemma 4.4.2] We say that a finitely generated left S-module X is *maximal Cohen-Macaulay* if $\mathsf{Ext}^i_S(X, S) = 0$ for all integers $i > 0$. $\mathsf{CM}(S)$ denotes the subcategory of $\mathsf{mod}\,S$ consisting of maximal Cohen-Macaulay S-modules. Then it is a Frobenius category, and hence, its stable category $\underline{\mathsf{CM}}(S)$ is triangulated.
(2) Natural inclusions $\mathsf{CM}(S) \hookrightarrow \mathsf{mod}\,S \hookrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,S)$ induce isomorphisms

$$K_0(\mathsf{CM}(S)) \cong K_0(\mathsf{mod}\,S) \cong K_0(\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,S)).$$

Here, the first isomorphism is shown in [Yos, Lemma 13.2] and the second isomorphism is by Lemma 18.2
(3) [Buc, Theorem 4.4.1] Composition of the natural inclusion $\mathsf{CM}(S) \hookrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,S)$ and the quotient functor $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,S) \to \mathsf{D}_{\mathsf{sg}}(S) := \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,S)/\,\mathsf{K}^{\mathsf{b}}(\mathsf{proj}(\mathsf{mod}\,S))$ induces a triangle equivalence

$$\underline{\mathsf{CM}}(S) \cong \mathsf{D}_{\mathsf{sg}}(S).$$

Corollary 18.6. *Let S be an Iwanaga-Gorenstein ring. Then there are one-to-one correspondences among the following sets:*
(1) $\{$*dense resolving subcategories of* $\mathsf{CM}(S)\}$,
(2) $\{$*dense resolving subcategories of* $\mathsf{mod}\,S\}$,
(3) $\{$*dense triangulated subcategories of $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,S)$ containing* $\mathsf{proj}(\mathsf{mod}\,S)\}$,
(4) $\{$*dense triangulated subcategories of* $\underline{\mathsf{CM}}(S) \cong \mathsf{D}_{\mathsf{sg}}(S)\}$,
(5) $\{$*subgroups of $K_0(\mathsf{mod}\,S)$ containing the image of* $\mathsf{proj}(\mathsf{mod}\,S)\}$, *and*
(6) $\{$*subgroups of* $K_0(\underline{\mathsf{CM}}(S))\}$.

Proof. One-to-one correspondences among (1), (2), (3), and (5) follow from the above remarks and Corollary 18.3. The bijection between (4) and (6) follows from Thomason's result, Theorem 16.1. Thus, we show the one-to-one correspondence between (5) and (6).

The localization sequence

$$\mathsf{D}^{\mathsf{b}}(\mathsf{proj}(\mathsf{mod}\, S)) \to \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, S) \to \mathsf{D}_{\mathsf{sg}}(S)$$

yields the exact sequence

$$K_0(\mathsf{D}^{\mathsf{b}}(\mathsf{proj}(\mathsf{mod}\, S))) \to K_0(\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, S)) \to K_0(\mathsf{D}_{\mathsf{sg}}(S)) \to 0.$$

The equivalence $\mathsf{D}_{\mathsf{sg}}(S) \cong \underline{\mathsf{CM}}(S)$ and Lemma 18.2 turns this sequence into the exact sequence

$$K_0(\mathsf{proj}(\mathsf{mod}\, S)) \to K_0(\mathsf{mod}\, S) \to K_0(\underline{\mathsf{CM}}(S)) \to 0.$$

Then, the equivalence is clear. ■

In the last two corollaries, we constructed a triangulated category $\mathsf{D}^{\mathsf{b}}(\mathcal{E})$ from an given exact category $\mathcal{E}$ and discussed their dense subcategories. Next, we consider the opposite direction. More precisely, we construct an abelian category from a given triangulated category, and then we discuss their dense subcategories.

Let us recall the definition and some basic properties of t-structures; for details, see [GM].

Definition 18.7. (1) A *t-structure* on $\mathcal{T}$ is a pair $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ of subcategories in $\mathcal{T}$ satisfying the following conditions:
- (i) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}^{\leqslant -1}, \mathcal{T}^{\geqslant 0}) = 0$.
- (ii) For any object $X \in \mathcal{T}$, there exists an exact triangle $X' \to X \to X'' \to X'[1]$ in $\mathcal{T}$ with $X' \in \mathcal{T}^{\leqslant -1}$ and $X'' \in \mathcal{T}^{\geqslant 0}$.
- (iii) $\mathcal{T}^{\leqslant -1} \subset \mathcal{T}^{\leqslant 0}$ and $\mathcal{T}^{\geqslant 0} \subset \mathcal{T}^{\geqslant -1}$.

Here, $\mathcal{T}^{\leqslant -n} := \mathcal{T}^{\leqslant 0}[n]$ and $\mathcal{T}^{\geqslant -n} := \mathcal{T}^{\geqslant 0}[n]$. Moreover, the intersection $\mathcal{T}^{\leqslant 0} \cap \mathcal{T}^{\geqslant 0}$ has the structure of an abelian category and we call it the *heart* of the t-structure.

(2) A t-structure $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ on $\mathcal{T}$ is called *bounded* if $\mathcal{T} = \bigcup_{i,j \in \mathbb{Z}} \mathcal{T}^{\leqslant i} \cap \mathcal{T}^{\geqslant j}$.

Example 18.8. Let $\mathcal{A}$ be an abelian category and put

$$\mathsf{D}^{\mathsf{b}}(\mathcal{A})^{\leqslant 0} := \{X \in \mathsf{D}^{\mathsf{b}}(\mathcal{A}) \mid \mathsf{H}^i(X) = 0 \ (\forall i > 0)\},$$

$$\mathsf{D}^{\mathsf{b}}(\mathcal{A})^{\geqslant 0} := \{X \in \mathsf{D}^{\mathsf{b}}(\mathcal{A}) \mid \mathsf{H}^i(X) = 0 \ (\forall i < 0)\}.$$

Then $(\mathsf{D}^{\mathsf{b}}(\mathcal{A})^{\leqslant 0}, \mathsf{D}^{\mathsf{b}}(\mathcal{A})^{\geqslant 0})$ defines a bounded t-structure on $\mathsf{D}^{\mathsf{b}}(\mathcal{A})$ and its heart is $\mathcal{A}$.

The next proposition is a variant of Lemma 18.2.

Proposition 18.9. [Nee01] *Let $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ be a bounded t-structure on $\mathcal{T}$ with heart $\mathcal{A}$. Then the inclusion functor induces an isomorphism $K_0(\mathcal{A}) \cong K_0(\mathcal{T})$.*

From this proposition and Theorem 17.7, we have the following corollary.

Corollary 18.10. *Let $\mathcal{T}$ be an essentially small triangulated category, $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ a bounded t-structure on $\mathcal{T}$ with heart $\mathcal{A}$, and either a generator or a cogenerator $\mathcal{G}$ of $\mathcal{A}$. Then there are one-to-one correspondences among the following sets:*

(1) $\{$*dense $\mathcal{G}$-resolving subcategories of $\mathcal{A}$*$\}$,
(2) $\{$*dense triangulated subcategories of $\mathcal{T}$ containing $\mathcal{G}$*$\}$, *and*
(3) $\{$*subgroups of $K_0(\mathcal{T})$ containing the image of $\mathcal{G}$*$\}$.

19. **Examples**

In this section, we give some examples of module categories which have only finitely many dense resolving subcategories.

Let us start with the following remark.

Remark 19.1. Let L be an abelian group. Then there are only finitely many subgroups of L if and only if L is a finite group. Indeed, 'if part' is clear. Suppose that there are only finitely many subgroups of L. Then L is an Noetherian $\mathbb{Z}$-module and in particular, finitely generated. Therefore, there is an isomorphism

$$L \cong \mathbb{Z}^{\oplus r} \oplus \bigoplus_{i=1}^{n} (\mathbb{Z}/n\mathbb{Z})^{\oplus m_i},$$

where r, n and m_i are non-negative integers. We obtain $r = 0$ due to our assumption as $\mathbb{Z}$ has infinitely many subgroups. For this reason, L is isomorphic to a finite direct sum of finite abelian groups, and thus is a finite group.

From this remark and Theorem 17.7, for a left Noetherian ring A, the following two conditions are equivalent:

(1) There are only finitely many dense resolving subcategories of $\mathsf{mod}\, A$.
(2) $K_0(\mathsf{mod}\, A)/\langle [P] \mid P \in \mathsf{proj}(\mathsf{mod}\, A)\rangle$ is a finite group.

19.1. **The case of finite dimensional algebras**

First we consider the case of finite dimensional algebras. Let A be a basic finite dimensional algebra over a field k with a complete set $\{e_1, \dots, e_n\}$ of primitive orthogonal idempotents. Denote $S_i := Ae_i/\operatorname{rad}_A(Ae_i)$ by the simple A-module corresponds to e_i. Then by [ASS, Theorem 3.5], $\{[S_1], \dots, [S_n]\}$ forms a free basis of the Grothendieck group $K_0(\mathsf{mod}\, A)$, and hence there is an isomorphism of abelian groups:

$$K_0(\mathsf{mod}\, A) \cong \mathbb{Z}^{\oplus n}.$$

The Cartan matrix of A is an $n \times n$-matrix $C_A := (\dim_k e_i A e_j)_{i,j=1,\dots,n}$. Then the above isomorphism induces the following isomorphism (see [ASS, Proposition 3.8]).

$$K_0(\mathsf{mod}\, A)/\langle [P] \mid P \in \mathsf{proj}(\mathsf{mod}\, A)\rangle \cong \mathsf{Cok}(\mathbb{Z}^{\oplus n} \xrightarrow{C_A} \mathbb{Z}^{\oplus n}).$$

Therefore if C_A has elementary divisors $(m_1, \cdots, m_r, 0, \cdots, 0)$, then we obtain a decomposition:

$$K_0(\mathsf{mod}\, A)/\langle [P] \mid P \in \mathsf{proj}(\mathsf{mod}\, A)\rangle \cong \mathbb{Z}^{\oplus n-r} \oplus \mathbb{Z}/(m_1) \oplus \cdots \oplus \mathbb{Z}/(m_r),$$

where $m_1, \dots, m_r$ are not zero. Furthermore, one has

$$\mathsf{det}\, C_A = \begin{cases} 0 & (r < n) \\ m_1 \cdot m_2 \cdots m_n & (r = n). \end{cases}$$

As a result, the abelian group $K_0(\mathsf{mod}\, A)/\langle [P] \mid P \in \mathsf{proj}(\mathsf{mod}\, A)\rangle$ is a finite group if and only if the determinant of C_A is not zero.

From this argument, we have the following corollary.

Corollary 19.2. *Let A be a basic finite dimensional algebra over a field k. Then* $\mathsf{mod}\, A$ *has only finitely many dense resolving subcategories if and only if its Cartan matrix has non-zero determinant. This is the case, the number of dense resolving subcategories is* $d(m_1)\cdots d(m_n)$. *Here,* $(m_1, \dots, m_n)$ *are elementary divisors of* C_A *and* $d(l)$ *denotes the number of divisors of* l.

Remark 19.3. For the case of gentle algebras, Holm [Hol] gives a characterization of algebras with non-zero Cartan determinant $\mathsf{det}\, C_A$.

19.2. **The case of simple singularities**

Next we consider the case of simple singularities. Let k be an algebraically closed field of characteristic 0. We say that a commutative Noetherian local ring $R := k[[x, y, z]]/(f)$ has a *simple (surface) singularity* if f is one of the following form:

$$
\begin{array}{ll}
(\mathsf{A}_n) & x^2 + y^{n+1} + z^2 \ (n \geqslant 1), \\
(\mathsf{D}_n) & x^2y + y^{n-1} + z^2 \ (n \geqslant 4), \\
(\mathsf{E}_6) & x^3 + y^4 + z^2, \\
(\mathsf{E}_7) & x^3 + xy^3 + z^2, \\
(\mathsf{E}_8) & x^3 + y^5 + z^2.
\end{array}
$$

In this case, the Grothendieck group of $\mathsf{mod}\, R$ is given as follows (see [Yos, Proposition 13.10]):

	$K_0(\mathsf{mod}\, R)$	$\#\{$ dense resolv. subcat. of $\mathsf{mod}\, R\}$
(A_n)	$\mathbb{Z} \oplus \mathbb{Z}/(n+1)\mathbb{Z}$	the number of divisors of $n+1$
(D_n) ($n =$ even)	$\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	5
(D_n) ($n =$ odd)	$\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	3
(E_6)	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	2
(E_7)	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	2
(E_8)	$\mathbb{Z}$	1

Here, $\mathbb{Z}$ appearing in $K_0(\mathsf{mod}\, R)$ is generated by $[R]$. Owing to Theorem 17.7, there are only finitely many dense resolving subcategories of $\mathsf{mod}\, R$. Hence the following natural question arises.

Question 19.4. Let R be a Gorenstein local ring of dimension two. Then does the condition $\#\{$dense resolving subcategories of $\mathsf{mod}\, R\} < \infty$ imply that R has a simple singularity?

Remark 19.5. 1-dimensional simple singularities may have infinitely many dense resolving subcategories (see [Yos, Proposition 13.10]).

Let R be a Noetherian normal local domain with residue field k. Denote by $\mathsf{Cl}(R)$ the divisor class group of R. Then there is a surjective homomorphism

$$u = \begin{pmatrix} \mathsf{rk} \\ c_1 \end{pmatrix} : K_0(\mathsf{mod}\, R) \to \mathbb{Z} \oplus \mathsf{Cl}(R),$$

where rk is the *rank function* and c_1 is the *first Chern class*. Moreover, $u([R]) = {}^t(1, 0)$ and the kernel of u is the subgroup of $K_0(\mathsf{mod}\, R)$ generated by modules of codimension at least 2; see [Bour]. In particular, if R is a 2-dimensional Noetherian normal local domain with residue field k, we obtain a short exact sequence

$$0 \to \langle [k] \rangle \to K_0(\mathsf{mod}\, R) \xrightarrow{\binom{\mathsf{rk}}{c_1}} \mathbb{Z} \oplus \mathsf{Cl}(R) \to 0$$

of abelian groups. This sequence induces the following short exact sequence since $\mathsf{rk}(R) = 1$ and $c_1(R) = 0$:

$$0 \to \langle [k], [R] \rangle / \langle [R] \rangle \to K_0(\mathsf{mod}\, R)/\langle [R] \rangle \xrightarrow{c_1} \mathsf{Cl}(R) \to 0.$$

Therefore, we have an isomorphism

$$\mathsf{Cl}(R) \cong K_0(\mathsf{mod}\, R)/\langle [k], [R] \rangle$$

and the following result is deduced from Theorem 17.7.

Theorem 19.6. *Let R be a Noetherian normal local domain of dimension two. Then there is a one-to-one correspondence*

$$\left\{ \begin{matrix} \textit{dense resolving subcategories of } \mathsf{mod}\, R \\ \textit{containing } k \end{matrix} \right\} \underset{g}{\overset{f}{\rightleftarrows}} \{\textit{subgroups of } \mathsf{Cl}(R)\}$$

where f and g are given by $f(\mathcal{X}) := \langle c_1(X) \mid X \in \mathcal{X} \rangle$ and $g(H) := \{X \in \mathsf{mod}\, R \mid c_1(X) \in H\}$ respectively.

The following answers Question 19.4 for domains.

Corollary 19.7. *Let R be a 2-dimensional complete non-regular Gorenstein normal local domain with algebraically closed residue field k of characteristic 0. Then the following are equivalent:*

(1) *R has a simple singularity.*
(2) *There are only finitely many dense resolving subcategories of $\mathsf{mod}\, R$.*
(3) *There are only finitely many dense resolving subcategories of $\mathsf{mod}\, R$ containing k.*

Proof. (1) $\Rightarrow$ (2): If R has a simple singularity, then $K_0(\mathsf{mod}\, R)/\langle [R] \rangle$ is a finite group; see [Yos, Proposition 13.10]. Thus, Theorem 17.7 shows that $\mathsf{mod}\, R$ has only finitely many dense resolving subcategories.

(2) $\Rightarrow$ (3): This implication is trivial.

(3) $\Rightarrow$ (1): From Theorem 19.6, $\mathsf{Cl}(R)$ has finitely many subgroups. Therefore, $\mathsf{Cl}(R)$ is a finite group, and thus by [DITV, Corollary 3.3] we have $\Omega\mathsf{CM}(R) = \mathsf{add}\, G$ for some module G, where $\Omega\mathsf{CM}(R)$ stands for the category of first syzygies of maximal Cohen-Macaulay R-modules. Now, since R is Gorenstein, $\mathsf{CM}(R) = \Omega\mathsf{CM}(R)$ has only finitely many indecomposable objects up to isomorphism. Consequently, R has a simple singularity from [Yos, Theorem 8.10]. ■

Example 19.8. Let R be a 2-dimensional simple singularity of type (A_1). Namely, $R = k[[x, y, z]]/(x^2 + y^2 + z^2)$. Then the indecomposable maximal Cohen-Macaulay R-modules are R and the ideal $I = (x + \sqrt{-1}y, z)$ up to isomorphism. Thus, every maximal Cohen-Macaulay module is of the form $R^{\oplus n} \oplus I^{\oplus m}$. Then the dense resolving subcategories of $\mathsf{mod}\, R$ are:

- $\mathsf{mod}\, R$, and
- $\{M \in \mathsf{mod}\, R \mid \Omega^2 M \cong R^{\oplus n} \oplus I^{\oplus 2m}$ for some $m, n \in \mathbb{Z}_{\geqslant 0}\}$.

Proof. Set $G := K_0(\mathsf{mod}\, R)$ and let H be the subgroup generated by $[R]$.

First note that there is a non-split short exact sequence $0 \to I \to R^{\oplus 2} \to I \to 0$, see [Yos, Chapter 10]. Therefore, $[R]$ and $[I]$ satisfy $2[R] = 2[I]$ in G. Moreover, the isomorphism $G \cong K_0(\mathsf{CM}(R))$ shows that G and H are only subgroups of G containing $[R]$.

Using the notation of Theorem 17.7, we know that $g(G) = \mathsf{mod}\, R$. It thus suffices to show that $g(H) = \mathcal{X}$. Let M be an object of $\mathcal{X}$. From the exact sequence $0 \to \Omega^2 M \to R^{\oplus n_1} \to R^{\oplus n_0} \to M \to 0$, one has

$$[M] \equiv [\Omega^2 M] \equiv 0 \bmod H.$$

This shows that $M \in g(H)$. Next, take $M \notin \mathcal{X}$. Then $\Omega^2 M \cong R^{\oplus n} \oplus I^{\oplus(2m+1)}$ for some $n, m \in \mathbb{Z}_{\geqslant 0}$. Using the similar argument, one has

$$[M] \equiv [\Omega^2 M] \equiv (2m+1)[I] \equiv [I] \bmod H.$$

Hence if $[M]$ is in H, then so is $[I]$. This gives a contradiction to $G \neq H$. Therefore, $[M]$ cannot be in H. Thus, we are done. ■

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