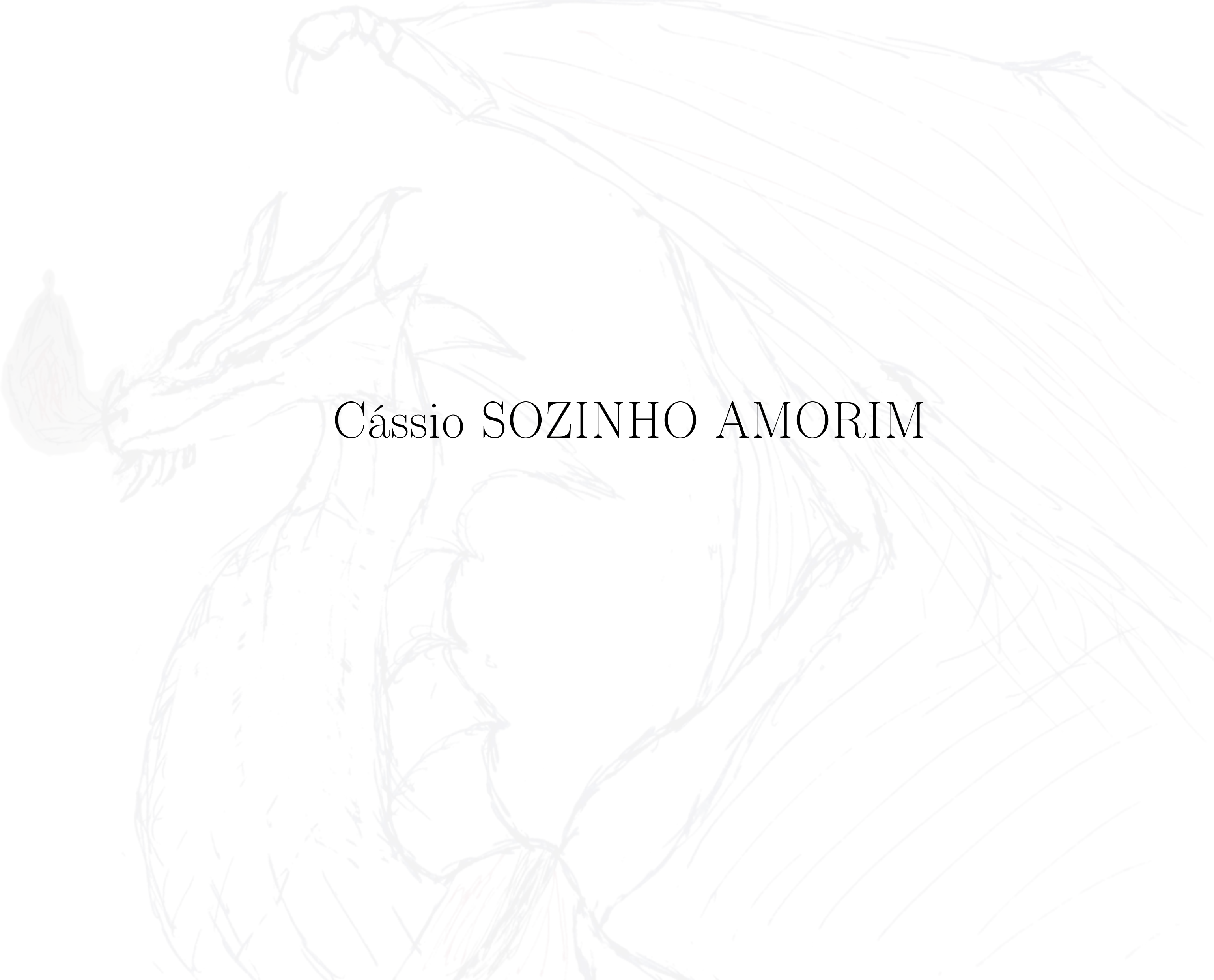
New approaches to quantum correlations: Majorana braiding dynamics and information indistinguishability

- A quantum beasts' monomyth -



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Contents

Foreword

The fundamental cause of the trouble is that in the modern world the stupid are cocksure while the intelligent are full of doubt. Even those of the intelligent who believe that they have a nostrum are too individualistic to combine with other intelligent men from whom they differ on minor points. – Bertran Russel, *The Triumph of Stupidity*, 1933.

A quantum mythology

In the beginning, it was the λόγος (logos). Or so tells us Christian Scriptures. And from λόγος was made φύσις (physis) – nature itself. The meaning of all these words changed with time, as did the meaning of many words we now use. Logos — "word" as translated in the Bible but with deep meaning— is now intuitively linked to knowledge and study, and physis — a change in moods if we take Aristotle — to concrete mechanisms in the world. Many fields of study are designated by the suffix "-logy" as in physiology, neurology, geology, topology – these are the "words" or "tellings" in an abstract sense of these matters, an authoritative telling of the body, the nerves, the Earth, and the places (or spaces in a somewhat concrete but abstract fashion), respectively. However, it is not the case for *physics*.

Like other disciplines, such as mathematics and economics, or some of its fields like optics and mechanics, physics is not bound to a writing objective like geography, nor to authoritative descriptions like geology; physics is an unbounded field of knowledge, an open pursue for the unknown φύσις. This leads to many facets of distinct purposes, or even no purpose at all. Some may be willing to obtain an authoritative description as a true λόγος of the universe. Some might be rather interested in the control and engineering of the φύσις, writing clear maps and plans to navigate chosen regions of the cosmos. Some might be rather fueled by the aesthetic appeal that the world of ideas has, or even just the beauty of the appearances, trying to save them in the best platonic effort (σώζειν τὰ φαινομένα). Despite the objective, the philosopher or thinker (φιλοσόσος) is the only one who truly knows what she pursues.

Yet, physicists pursue universality in the vast range of phenomena studied. And to avoid problematic frictions, philosophy is cast aside of discussions, only upholding logic and its universal language – mathematics – as meaningful ways to universality. This, in turn, begs the question: is it possible to separate physics and philosophy? Does philosophy not depend on φύσις? And can physics stray away from philosophy? My sincere opinion is that it can not happen completely. As long as philosophers pursue wisdom, they cannot completely ignore the φύσις surrounding us and must drink from physics whenever in this pursue, assuring at least a one way dependence. But assuming this is the only direction of dependence is certainly naïve. For as long as physicists decide to unveil the mysteries of nature, as long as they rely on logic, as long as they question the essence of nature and observed phenomena, and as long as they wonder about reality and knowledge, as long as they braid unveiled elegance in the φύσις, it is nothing less than the purest form of philosophy.

On the other hand, physicists and other scientists in general have led a pursue for knowledge that has had more impacts in our lives than just "understanding the universe," a fair but not clear enough claim in my opinion. Scientists have also made stories. They are people as human as any one else, who became heroes and have perpetrated new myths to guide our lives. Their discoveries, sold under the angelic name of *science*, spark dreams in children and adults alike, and nurture souls everywhere. And if no myth is complete without evil and temptation, we also have our parcel of demons and evil dukes to be opposed by saints and princes, that would certainly make a case for Joseph Campbell's *monomyth* analogy[1]. Following Campbell's *Hero of a thousand faces*, a monomyth is that single (*mono-*) myth that pervades all of human collective and individual psych, the myth of each individual's journey through the adventure called life that repeats itself in many situations. It is marked by the call to adventure, a road of trials, and the overcome of an enemy or a shadow, representing an internal conflict within the self. The monomyth is therefore a reflection, in the form of a story, thought and told to take lessons of ethics and life behavior. And if we seek for a reflection of our unconscious in the life of scientists and their achievements, as Campbell would suggest, why not then give such reflection directly to the abstract scientific knowledge, instead of the people that surround it? Perhaps that is a way to create as much interest in the ποίησις behind the knowledge as in the τέχνη that inspire sci-fi dreams. If many people already replace the myths of religion by science, should we avoid giving them what they are looking for anyway, i.e. a scientific myth? With proper care to not be too fallacious, I do not see a problem. If the minstrels can sing about love and dragons, they certainly could sing correlations and entropy. If a mathematician can write stories with references to nursery rhymes that sing about eggs falling from walls[2], a physicist certainly can write stories about electrons pairing up to make superconductors.

In this spirit, I decided to make this thesis as a mythic story, with dragons and fairies flying around, but I cannot sacrifice scientific rigor. Although I am convinced to be possible to make the whole thing a single big story (Oh, how I wanted to!), I decided to amuse where I can, and give the technical (mathematical) analysis apart. This way, I hope to keep the text lighter to every audience. But since every story has a purpose, knowingly or not, and given the underlying philosophy I acknowledge the existence, I also decided to add some short philosophical opinions on everything written in this work. Every chapter shall include in its ending a section that will aspire to give some philosophical taste of the physics discussed there. In general, it may not be a clear conclusion, perhaps leaving more questions than answers, and perhaps disturbing the human psyche more than soothing. But given the contents to be covered here – single body, many body, symmetry, ordering, information, locality, causality, contextuality, indistinguishability... – it would be almost irresponsible to run away from such matters knowing their importance. Western history has given us a legion of names who lit ideas around our known φύσις, like Democritus, Epicurus, Plato, Aristotle, Ptolemy, Bacon, Descartes, Galileo, Newton, Berkeley, Hegel, Kant, Maxwell, Einstein, Heisenberg, Russel, Wittgenstein, von Neuman, Bell, Feynman, Popescu, to name a few (even if my dear reader may question fitting these names together). They did not tackle necessarily the same problems, nor did they proceed the same manner regarding the same problems, but each one has given much insight to their fields, and I will try to put some of them side-by-side.

I do not mean to make a rigorous and comprehensive analysis of the evolution of our understanding of the universe nor will I formally define φύσις, for this would be a titanic quest in its own merit. My mission is much more humble: organize the time flow in the world of ideas regarding the φύσις, from the point of view of a modern physicist in early twenty-first century, amidst the newly merging buzzword *quantum technology*. In this era of information and technology, I shall recall what is information, quantum, τέχνη, as well as λόγος. I shall try to remember how we arrived here, and perhaps dream about where we are heading to. But most importantly, given the eldritch silhouette of our current knowledge of quantum physics, I find it worthwhile to consider where do we stand now in more mundane words, reporting to people what goes on in the world and how (not only why) it matters. To completely fulfill this task is an extremely ambitious goal, so I beg the reader to read this as a candid attempt and perhaps a single step towards that direction, not a complete treatise from λόγος to λόγος in all corners of the φύσις. This said, we may proceed to our story.

[1]Recently, we have seen how a scientist's story can make a glorious myth with National Geographic's series *Genius*, for instance.

[2]Humpty Dumpty sat on a wall,/ Humpty Dumpty had a big fall...

Comments on notation

Throughout this work, some words will be used interchangeably in English and Greek. For example, physis/φύσις and logos/λόγος, as was already done in the previous section. Whenever this happens, these words essentially mean the same things, however the Greek form should stand for its original meaning or something close to it, while the Latin spelling of the word should expose a rather modern meaning. For instance, "physis" may in general be safely replaced by "nature" but its Greek counterpart φύσις reserve a more abstract, uncertain, and questioning aura around it when referring to the universe. Similarly, logos should be presenting modern understanding of the word related to knowledge, whilst λόγος should keep it's original sense, including the one found in the Bible. Also, while it is usual to presenting Latin in italics, I will not abide to this rule. Hence, words like "et al." or "a priori" shall be, a priori, in normal font. They *may* be italicized for emphasis though.

Another observation is that many physical concepts will receive different but equivalent names. For instance, fermions will be called dragons and bosons will be called fey[3]. What's the point? – One might ask. The point is that, the more new concepts we have to learn and bind together under new chunks, the more cognitive effort it takes. Hence, to attenuate this crescendo of new concepts, I'm substituting for them a familiar but still imaginary (concrete or abstract) counterpart. Understand: it is very complicated, and sometimes impossible to give a simple, classical, and concrete analogy to quantum phenomena. In such times, I'm convinced that human creativity and imagination has already presented us with enough folklore to extract reasonable metaphors, or create new ones. If you doubt, let me tell you that physicists do use "supernumbers" that are divided into *body* and *soul* when dealing with supersymmetry. You might not know what supersymmetry is, and saying that numbers bear body and soul might sound nonsensical, but probably the body/soul concept, despite vague, is the only one you can grasp without considerable effort. To my knowledge, not even Pythagoras, son of Apollo and creator of the famous theorem after his name who believed the world was made of numbers[4], went so far. Oh, if hearing about souls makes you wonder, yes, there are also ghosts (in fact, ghost and anti-ghost numbers) in physical theories; add dark matter and dark energy and it becomes a creepy subject to study (perhaps we will get vampires, imps, werewolves, and so on anytime).

Following common place in physics community, formulae shall appear as part of the text, even if they have a line and number just for it. It means that some formulae will have a period, a comma, or some sort of punctuation after it, not to be thought as having some specific mathematical meaning. Also, we shall adopt natural units with constants $\hbar = c = 1$.

Comments on organization

In total, this thesis comprehends 9 (nine) chapters of varied length, called "levels" instead of the most usual naming. Each level presents us a different central topic of background or research report, binding new ideas to new levels, what represents levels of abstraction that one might compare to floors in a building, levels in a dungeon, or levels in a role playing game, as the reader wishes. Level 1 is our starting point, introducing single-particle quantum mechanics. Level 2 goes one step further to consider two-particle quantum mechanics. Level 3 discusses many-body problems. Both levels 2 and 3 connect to level 1, i.e., single-particle mapping of more complex problems. Up to this point, only fermions are taken into account, so in level 4 fermions, bosons, and statistics are topics discussed. Level 5 introduces us to some concepts in topology, dimensionality, and fractionalization of the discussed issues so far. In level 6, all previous levels come together, and we present the usage and research results regarding Majorana bound states to topological quantum computation (and related matters beyond). Level 7 offers a new perspective on information. Level 8 introduces and discusses the idea of information indistinguishability

[3]Note that the word *fey* will be used both as an adjective *and* as a noun. In other words, it becomes the counterpart of both *dragon* and *draconic*. Both the plural and singular forms of the noun will also be the same word *fey*. Nothing more bosonic than that!

[4]Though "numbers" for him were also shapes, as regular polygons.

and its implications. Level 9 offers a short conclusion on the whole physics discussed throughout this thesis. Levels 6 and 8 can be qualified as the main and original results of this thesis, though section x.3 in every level also bears a certain amount of originality (with possible exception of section 7.3).

A message to reader: Before reading, take an exorcism

My name is Legion, for we are many. — Mark 5:9

We all have the *Classical demons* inside us, that disrupt our notion and ability for understanding through many illusions and fake perceptions. They cloud our eyes and make us believe in what we should not. So, dear reader, before venturing into reading these pages herein, please, go to an exorcism session. Well, you don't need to *go* anywhere, but make sure you can at least exorcise yourself of the following classical demons, if not the whole legion inside you. For if you do not get rid of such pesky thoughts, it will be hard to step into the world that will arise from these letters.

- **Predictability:** Demon with power over illusionary determinism.
- **Locality:** Demoness that craves for binding objects, properties and phenomena to restricted regions of space, always suggesting hidden pillars sustaining physics where none is to be.
- **Certainty:** Demoness specialized in whispering lies on our ears to provide a false sense of security, assuring what cannot be assured at all.
- **Embodiment:** Traitor demon that rejects its own kind, denying any souls their free existence and always implying whole sets of properties and objects are bound to be one single and indivisible cluster.
- **Primal-cause:** (Aristotelian) Arch-demon that lures mortals to hopeless fates, attributing a single and precise cause to whatever the event, in a single simple chain of events in the universe, whether visible or invisible. It is the protector of all other demons, grasping with its claws every single attempt of freedom.
- **Analogy:** Arch-demoness that seduces people into comfort by making them believe one thing can be understood by looking at other unrelated ones. It gains our trust by misleading us into correct comparisons and later substituting them for hallucinating ones.

These demons dwell deep inside our consciousness and only a thorough exorcism that spurge their power over us can allow one to savor the fine delicate tastes of quantum interactions. They are responsible for luring people into peaceful and quiet traps that collapse over their preys as paradoxes; perhaps the most famous being the Einstein-Podolsky-Rosen paradox, described by Einstein as a "spooky action at a distance" that seems to transfer information faster than light. Such paradoxes are nothing but illusions acting on false beliefs, fueled by the demons already mentioned. Therefore I urge you to rebuke them with all your willpower! It may be impossible to loose all the chains they cast upon us, but even realizing the chains are there may be of great usefulness in this discussion. For who said every unseen event must occur like those we see? Who said every small occurrence must have a simple cause? Who said objects in the small scale are also singled out as object in the larger scale we know from experience? Such are the poisons these demons use to fill our souls. Therefore, beware! They will do whatever they can to divert you from this path. But don't worry, we will try to guide you through arcane ways forsaken by the legion of unworthy fiends. Stay firm, and be brave: they will not conquer you, pal!

Final disclaimer

It is my real will to write this whole thesis as a single story, a single novel, a single romance. Nevertheless, limited by very strict time constraints, I can not devote even one quarter of the time I wished to this enterprise. Hence, the story falls short from my intention. The contents fall short from my intention. The format, editing, and general style also fall short from my intention. Everything indeed falls short of my will. Please, forgive my lack of further development. But also, within reasonable fairness, I believe I may claim the production herein, specially of every first and last section of chapters, to be mine and therefore express my views on the topic exposed. There is certainly a lot of study, research, and influence from many people and sources to all of it, whose direct and indirect references I hope not to forget.

This thesis deals mainly with the research in refs. [1, 2]. Also, while the relevant bibliography comes in the text, I would like to point that refs. [3–6] lay the foundation of this thesis, together with many more exposed in the bibliography..

Acknowledgements

I am morally obligated to acknowledge the great support of my supervisor Yuki Kawaguchi in the completion of this thesis. Her support during these years of doctoral course has been more than I would expect from most supervisors in the world. Also, the leader of our group, prof. Yukio Tanaka, has also many times offered his support whenever necessary. Prof. Masatoshi Sato, my supervisor during Masters, has also taught me a lot in our communications after that, which ultimately led to this thesis. I also acknowledge the limited but grateful advice for completion of this thesis by other evaluators of this thesis, notably prof. Hideo Kishida and prof. Shogo Tanimura.

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Likewise, I must thank all the friends with whom I had contact the past few years for various support. The list is (fortunately) too long to even consider fitting here, so I shall acknowledge there support collectively. All the members of Tanaka lab; Brazilians and other international students I met in Nagoya University and Nagoya Institute of Technology; members of Academic Jungle in Nagoya University; Catholic friends in the Diocese of Nagoya; colleagues that I met and had great discussions in the many conferences and schools I have attended; friends since long that I can only contact through Facebook now but still receive important support; and the many other scattered encounters I had through the world in these years: to all of you, my sincerest thanks. And as science is closer to art than many would recognize, the many singers and bands I enjoyed the performance in these years also played an important role in pushing me forward.

And certainly, to my parents Maria de Belém Sozinho and Carlos Amorim, who never really explained me the value of study, but always taught me through actions and word to place utmost value in knowledge, I can only say that this work exists thanks to your support (even if sometimes too "noisy"). Equally, they have also taught me to trust in God, and in God we trust, so my deepest thank you is directed to The Holy Father above us.

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Level 1
Lonely Dragon

– Où sont les hommes? reprit enfin le petit prince. On est un peu seul dans le désert...
– On est seul aussi chez les hommes, dit le serpent.[1]

— Antoine de Saint-Éxupery, *Le Petit Prince*, 1943.

But nathelees, whil I have tyme and space,
Er that I ferther in this tale pace,
Me thynketh it acordaunt to resoun
To telle yow al the condicioun
Of ech of hem, so as it semed me,
And whiche they weren and of what degree,
And eek in what array that they were inne;
And at a Knyght than wol I first bigynne.

— Geoffrey Chaucer*, Canterbury Tales, General Prologue, 1387.*

1.1 Alone in the darkness

MANY people do not understand what is to be a lonely dragon, or a lonely fey. They don't seem to understand that, for us, it means nothing, just nothing. When we are alone, we are no longer dragons nor fey, we are just ourselves in our truest nature. You humans might not understand, you seem to call us with one single word: particles. And at this point you are partially correct, for we do lose our identity, our distinct souls, and become just ordinary, just another piece of the universe without contact or interaction with any other corner of it. But don't be silly, there's no such thing as true loneliness, we just look as if we were really lonely, but in fact it is merely an illusion. Still, we should leave such details to be discussed later and just meditate on this cozy wilderness of being alone.

My apologies, I did not introduce myself to you, for the loneliness made me forget of your existence. And, to say the truth, it made me forget of mine as well, so I do not know what to say. For now, you may just see me as a simple dragon. I am told that humans think of dragons as wild but wise, fearsome but respectable, greed and selfish but powerful beasts that need little to survive and sleep all the time. It is clear that you just blame us for every piece of your own contradiction. The only universal characteristic of us that you get right is this: we do indeed like our own territory and protect it fiercely. A dragon never invades another dragon's terrain. For now, it will do; you need to know naught beyond. This is enough for you to understand our nature, and will allow you to see me alone for a while. I believe a human

[1]– Where are the men? Returns the little prince. We're a bit lonely in the desert...
– We're lonely amidst men too, said the serpent.

called Fermi noticed how defensive we are, what made your people label us *fermions*, though you call our territory *states*[i] — I guess this is a way you have to call large political territories ruled by one of your kind, which I reckon is the origin of this word usage. However, perhaps it was another human named Pauli who first realized we ought to be occupying each territory alone, something he called *exclusion principle*, but he was not sure of what was going on.

Anyway, you can just call me dragon for now, or fermion if you prefer. But as I said, I'm alone here, and when we are solo, there is no territory to dispute; every state occupied by a single beast can only have one ruler anyhow. This is why I'll let you see my single life first, to avoid needless intricacies. They will come in due time.

1.1.1 ··· *Single flight*

First of all, be not lost when dealing with me. In my experience, your people get confused very easily when dealing with a single of us. Before anything, do not mistake being lonely in a room or a park or field with being alone in the universe. It is very different.

Whenever we say we are alone, some people think there is nothing but us. That is not what we mean. And not what I mean here. See, when I say I am alone, all I mean is that I feel like no one is around — no friend or mate that we can see or that matters, at least — and fly through the valleys, hover over mountains, dive into seas and canyons. Since I'm alone, I can stretch and flap my wings without hitting any one, and control this whole territory by flapping my wings as I please. You may have trouble to see, since I do fly around the whole space, filling all of it. My personal feelings, my own energy is all that I need to care when choosing how to wiggle my wings' membranes. I can live in my own way, my own fashion, my own mode. Indeed, I believe you humans even use the word *eigenmode* to describe it. But saying it is naught. You cannot understand the *sensation* of being; just being.

Being. This is all of it, we just are. At least, I am. But let me try to make you understand. Look at my wings. We, dragons and feys, are always flying. Therefore, we have our wings always flapping, even if very slowly. Some of you may remember sharks slowly moving while sleeping. Well, so does my wings. It is what you call *zero oscillation*. Yes, my wings are always oscillating, always spread in all directions, always alive. And the more energetic I feel, the faster the flapping, and the shorter the waves on my wings.

But my wings are not simply an extension of my will. They allow me to feel my terrain and mold myself to it, to fit in the corners of the fields of my own existence. Yes, my wings are not only meant to fly, but to feel and adjust to my surroundings. Indeed, I can glide on the winds of valleys with my wings, and that is how I feel my own energy growing, that is how I accelerate faster and faster. All this background landscape fills me with or depletes me of energy.

In fact, that is how I feel the outer world. Look at the mountains. See that they may change, move, dance, shift, raise, and fall at any moment. At any moment, a mountain may raise bellow me and raise me, and my wings respond immediately to a faster tempo. But I do not mind. At times, mountains may also appear in my way, and once in a while I just keep flying through it. If all this seems uncommon to you, remember we dragons have our intrinsic magic, which allows us to permeate the surface of anything for a little while, though never too deep. This permits me to eventually cross any barrier, what you usually call *tunneling*, though no real tunnel is carved in the process. All I got to do is to slowly let my wings loose while inside the barriers, and if it is not too painful I can go to the other side.

1.1.2 ··· *Beyond the landscape*

If you are still following me, I must say the world is more than mountains, valleys, caves, and such. Look above and you will see skies. Look bellow and you will see pools with vortices. There are sources of energy in the heavens above, drains in the rifts below. We pay great attention to them, because they may disrupt our flight. Whenever I see them, I always adjust my flight to their influence. Of course, it means changing my flight modes if they move around. But let me show you what I mean in more details.

First, look at that sun. It is a bright source of energy that makes me want not to get close to it. When its energy heats my body, I vigorously flap my wings and spit a jet of fire to cool down. I will tell more about my fire breath later, for its magic is fascinating by itself, but for now it suffices to know I do it. You, humans, would call my energetic mood an *excited state*, and my breath a *spontaneous emission*. Yet, sometimes you seem to call this whole process of exciting me and catching my breath a *measurement*. But there are other measurements too. The vortices in the pools bellow create winds that I avoid. It is easier to avoid the wind going left or right depending on how I flap my wings in that area, fitting and not fighting the wind. According to my wing-flapping and the direction the vortices swirl, I change my flight direction too, towards left or right, and you can follow my path to how my wings move. This is another measurement, one that tells how my wings flap, for my wings do not flap uniquely in one way. Their wavy deformation may, for example, be in what you would call horizontal or vertical direction, to say one example. I believe you would call it the *polarization* of my wings or sorts. And even if I wiggle a bit more up or down, left or right, tilt slightly here or there – for I'm always moving and things may get slightly erratic, – you can still throw me in one defined polarization with such vortices throwing me into such specific paths.

Wait, I hear something! It seems my pleasing solitude is gone, for we have a visitor. I can already hear its draconic call in my veins – I'm alone for not much longer. We shall continue this conversation later! But be warned: if a dispute scales up, you might see some fierce fight!

1.2 Describing the quantum world

Our dragon has given us its perspective of a single particle quantum problem. It accounts for the simplest case, therefore being the beginning and sometimes the end of many discussions. Hence, let us mathematically formalize the concepts it described us.

The lonely dragon represent a *single particle state*. We often write down (pure) quantum states by using a notation introduced by Dirac, known as Dirac notation or *ket* notation. For instance, we would write

$$|\psi\rangle \tag{1.1}$$

to represent a generic (pure) quantum state, single particle or not. It is a generalized form used by physicist to represent vectors in quantum theory. The letter ψ inside the ket is a generic label, and could be literally anything, even emojis if one would like, say |🐲⟩ or |🐈⟩ or |😸⟩ or |👄⟩. In fact, it could also be not one, but a whole collection of labels too. Right now, it does not matter much. These are *ket vectors*, and it is worth to stress the *vector* (since the "ket" is quite obvious when looking at it). Vectors can be thought as merely points in a given linear space, meaning they have (a set of) coordinates to identify them in whatever this linear space is.

Vectors can also be *transformed* into other vectors. For instance, one can *rotate* vectors, or stretch/compress them (what is called *scaling*). We just care to the final result of such changes, not much how the changes were performed, therefore the use of the word "transformation." Hence, by taking quantum states to be mathematically represented by ket vectors, which is nothing but a way of generalizing vectors, we can also represent quantum states by certain coordinates, and transform states into other states. This transformation capability is where the importance of ket states relies.

When time goes by, things change or "evolve" when compared with their previous state, and quantum states are no exception. This change is a transformation in itself, often called *time evolution*. Say, a state we may call $|\text{inital}\rangle$ evolves into a state $|\text{final}\rangle$ after some time, or maybe a state vector $|0\rangle$ goes into $|1\rangle$, or a |😸⟩ becomes a |👄⟩. This is definitely something we are interested in knowing. And we want to know more things, like the energy of such states, their spatial profile, and so on. For this, we need a way to transform such vectors into other new vector, and by an adequate set of transformations, we might get whatever we want. For example, how can we obtain the spatial distribution of a state? For this, we need a certain *basis set*.

A basis is nothing but a set of ket vectors that span our vector space. In other words, a set of vectors that, together, allow us to write a certain state by their (weighted) summation. For example, if we had only two places where a state can reside, we could have two bases, say $|0\rangle$ and $|1\rangle$. If there is no overlap between them, they are said to be *orthogonal*, and we will assume them to be from this point on. To know where a certain state $|\psi\rangle$ resides, we need a way to transform it into each basis, $|0\rangle$ and $|1\rangle$, and know how much of the state is still there; i.e. we need to keep track somehow of how much change occurred during such transformation. So, if by transforming state $|\psi\rangle$ into $|0\rangle$ no change at all happened, but it changes completely when brought to $|1\rangle$, we know it resides in $|0\rangle$. On the other hand, suppose we know that only "half" of the state remains when it is transformed into $|0\rangle$ or $|1\rangle$, regardless of which. In this case, we know we are in a superposition of both basis, and we would write $|\psi\rangle = |0\rangle + |1\rangle$.

It is worth to make an observation here. In general, vectors have length. But in quantum mechanics, such length has limited meaning, so we often just set their length to unity, what is called *normalizing* vectors. To do it, we need a way to calculate such vector's length, which also allows us to calculate "how much of the (former) vector is still there" after a transformation. In other words, it gives us a way to calculate distance. For this, we use ket vectors' sibling called *bra vector* and written

$$\langle\psi|. \tag{1.2}$$

With this, the names bra and ket must be self-evident. This bra vector is said dual to a ket vector. Just like a ket vector, it is spanned by a basis set of bra vectors, summed with a certain weight. For instance, an arbitrary state, following our two orthogonal bases, can be written $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, with the greek letters being coefficients indication the weight of each basis. A similar rule holds for their dual bra. However, one observation must be made: the coefficients α and β may actually be complex numbers, so we use their complex conjugate for the dual bra. Explicitly, we would write $\langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1|$.

To calculate a bra or ket vector's length, we take its product with its conjugate. If our bases are orthogonal (actually, orthonormal, both orthogonal and normalized to unity length), their product will equal unity only if they coincide, vanishing to zero otherwise. That is,

$$\begin{aligned}\langle\psi|\psi\rangle &= \alpha\alpha^*\langle 0|0\rangle + \beta\beta^*\langle 1|1\rangle + \alpha\beta^*\underbrace{\langle 1|0\rangle}_{=0} + \beta\alpha^*\underbrace{\langle 0|1\rangle}_{=0} \\ &= |\alpha|^2 + |\beta|^2, \end{aligned} \tag{1.3}$$

and it gives us vector $|\psi\rangle$ length. By forcing it to be unity, $|\alpha|^2 + |\beta|^2 = 1$, we can normalize vector $|\psi\rangle$. One can see that, if it were evenly distributed between our two bases, we would write $|\psi\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big)$. Such states are quite common, and we shall abbreviate them for later use as $|\pm\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle \pm |1\rangle\big)$.

If we understand our vectors and know that they have a length that we may use to keep track of them (by checking how much they shrink, for instance), we need an explicit way to perform the transformations so far mentioned. This is done with *operators*. Operators, often written as capital letters with a hat (though not always and font may vary widely), like $\hat{O}$, act on ket vectors from the left to produce new ket vectors, like

$$|\psi'\rangle = \hat{O}|\psi\rangle. \tag{1.4}$$

This new state $|\psi'\rangle$ can, in general, be represented as a weighted sum of other states (including a "sum" of a single term with a factor). It may also happen that a certain operator will "change" the state to itself, that is,

$$\hat{O}|\psi\rangle = \lambda|\psi\rangle, \quad \lambda \in \mathbb{C}. \tag{1.5}$$

In such a case, $|\psi\rangle$ is called an *eigenstate* of the operator $\hat{O}$ and λ its *eigenvalue*. For bra vector, all the same idea can be used, applying operators to the right side of bras. And as bras are dual to ket, there are also dual forms of the operators given by their *Hermite conjugate*, represented by a dagger,

$$\langle\psi|\hat{O}^\dagger \leftrightarrow \hat{O}|\psi\rangle. \tag{1.6}$$

A particular kind of operators is the so called *unitary* operators, and are the ones that will interest us. They obey $\hat{O}\hat{O}^\dagger = \hat{O}^\dagger\hat{O} = 1$, meaning that their Hermite conjugate is their *inverse* operator, i.e.

$\hat{O}^{-1} = \hat{O}^{\dagger}$ (the inverse equals the hermitian conjugate). In quantum theory, this has the important meaning of preserving the total probability of the system equal to unity, i.e. there is no dissipation, loss, or gain. This is what we look at in quantum theory, the probability distribution of a particle, normalized to unity and conserved, meaning that the total probability of finding the particle in *some* state adds to 100% and it does not disappear nor pops out of nothing. Therefore, quantum theory is *unitary.* Note that, with caution, this condition can be relaxed, but we shall not consider such complex situations (yet). Hence, any transformations in quantum theory will be represented by *unitary operators.*

Also, an important type of operators follow the relation $\hat{O} = \hat{O}^{\dagger}$ and are called *hermitian.* All physical quantities measurable in experiments are given by hermitian operators. For example, the hamiltonian, often written as $\hat{H}$ is the operator that gives us the energy of a system, and is arguably the most important operator in quantum mechanics. Eigenstates of the the hamiltonian are often called energy eigenstates and often taken as bases to discuss various phenomena. And the eigenvalues of the hamiltonian stand for energy eigenvalues, making up the *energy spectrum*[ii] of a system, another important physical information.

One thing worth to notice is that there is nothing extremely special about ket vectors and operators, apart from them being practical mathematical tools. We may discuss quantum effects using functions, or maybe writing kets, or perhaps just talking in terms of operators. Also, ket vectors can be generically written as $|\psi\rangle$, but Dirac also argued that the bar in front of it plays not much role and one could just as well write $\psi\rangle$, and went further to introduce just $\rangle$ as *standard ket*, on which operators apply transformations to create different states. We will not use such reductions here. In fact, we must recognize that, indeed, redundancy in notation may have an important role as much as redundancy in communication, notably error correction, which is why a new ket notation will be introduced in the next level of this work, adding nothing new but the ability to state clearly that we are dealing with a many-body state.

It is also worth to point that not every quantum state can be really represented by kets, an issue we shall revisit in more details in the next level. So far, given the picture presented here of a single particle existing isolated from the rest of the world, a ket state is all we need. And this should be enough to have a glimpse on the limitations here: particles may not be *isolated.* In fact, the dragon showed us a world with a somewhat rich landscape, on the top of which it ventures lonely. This background landscape may represent many things, but can in general be thought of as a *potential* landscape through which the dragon moves. Overall, physical quantities that matter in the quantum world are discussed in terms of "potential" and "gauges" that are all embedded in the background discussed by the dragon, but it may vary considerably in meaning depending in what kind of space we are considering this dragon lives in. For instance, we may think this dragon represents an electron moving through an electromagnetic field in free space. Or it maybe an electron in a solid, like a metal. In the later case, a solid is made of many atoms and electrons, making the problem of an electron's dynamics very far from a single particle problem. Nevertheless, we can approximate it to a single electron moving through a periodic potential, defined by the crystal lattice of the solid. As a result, we obtain *bands* of energy that relate a certain electron momentum with a certain energy. In other words, we can think of electrons as single particles, attached to such bands that do have peaks and valleys. Such bands might become the background landscape in question. We shall return to this issue once we reach level 3.

Another issue is that we have not defined *state*, taking a rather axiomatic approach to it, assuming its existence and building up on top of it. But things may become confusing once we consider *measurements* and how they probe "states." We shall take a superficial look on this matter in the next section, with well defined measurements probing well defined (ket) states and the complete opposite situation too – not so clear "measurements" of not so clear "states," for lack of better terminology.

1.2.1 · · · Inside-out: looks and measurements from outer space, and beyond quantum theory

"What is there? What is beyond?" are some of the questions explorers and children (the greatest explorers of the world) often ask themselves and to others. To answer what is in the bottom is one the the purposes of quantum science. And for this, scientists perform a plethora of experiments we call *measurements.* A detailed coverage of measurement and the field of quantum metrology is way beyond the scope of this thesis, but we need to cover some ground in order to secure our foundations.

First of all, we need to define measurement. Unfortunately, this is a problem in itself. It happens even nowadays that some physicists will propose a certain "measurement" scheme that others will say it is anything but a measurement. Since the discussion of measurements is not our central objective, let us keep a loose definition, calling "measurement" any physical process, experimental or theoretical, that retrieves a finite amount of information from a given system of interest. "Theoretical measurements" are essentially performed only ideally and/or mathematically on a theoretical system, and may not be realizable in practice, but they become, at times, useful tools to abstract and study properties of quantum theories, as we shall descry later. Before it, though, notice we talk about quantum theor**ies** at this point, in plural, and this shall be explained first.

In general, whenever one talks about *quantum theory*, *quantum mechanics*, and alike, they are talking about the field in science that has a set of tools (being described here) used to study microscopic phenomena, i.e., phenomena in a scale small enough that we observe quantum interference, quantum superposition, quantum coherence, and other quantum effects (naturally, *quantization* is behind all that, to begin with). This is what we mean by using the singular term "quantum theory." However, at least in principle, one could conceive different theories that gave the same result. So far, quantum theory looks nice, but there are caveats to it too that may turn out to become the death of the theory itself. For comparison, Newton's mechanics seemed perfect for about three hundred years, and even after Einstein introduced a new relativistic mechanics, men went to the moon about fifty years ago using Newtonian mechanics, for even if not perfect, it was still good enough. So far, all we can say about quantum mechanics is that it has been good enough for everything we thought.

But what if? This is where we come to plural. For now, quantum mechanics has a hard time to deal with more than one particle, as we will discuss from next level onwards. There are ways to do it, but there could be some lost contradiction not yet found. This is where *General Probabilistic Theories* (or GPTs, for short) come in. This is what is meant by "quantum theories," but to avoid confusion, the term GPTs shall be used. First, we must look at quantum theory as a probabilistic theory, one that uses kets and wavefunctions to deliver a probability distribution of observing certain specific *effects* on a system. By "effect" we indicate some transformation from a initial state to a following (final) state, which may be a measurement or some other kind of control. A simple comparison with the traditional probability theory of Laplace indicates a big difference in that Laplace's probability use real numbers, while quantum theory's probability use complex numbers. In other words, one could say that quantum theory is more general. Then why not even more general? Why not use quaternions? Why using kets and algebra? The best answer we can give is because it is working. But it is worth thinking of any other possibilities. These are the possibilities collectively called **GPT**s. Some might be nonsensical, some might be just as good as quantum theory, perhaps even better.

But *why* does it matter? Because this influences the very way we think of measurements. To discuss measurements, let us focus on quantum mechanics alone once again. Perhaps the most common measurements one would think of are *projective measurements.* This is the kind of measurement we are talking about when our dragon is thrown into a polarized state. Translating the dragon to physics, particles have a property called *spin*, which for charged particles generate a magnetic moment, which in turn couples to magnetic field and deflect their trajectory to different directions depending on the direction of this magnetic moment, or, conversely, depending on their spin. The metaphor of magnetic fields being presented as vortices is also very natural in physics, for electric current loops and angular momenta are usually thought as "sources" to magnetic field. All this may sound fancy, but there is a very well known and relatively simple experimental setup working exactly like this: the Stern-Gelarch experiment[5, 7]. Magnets create an uneven magnetic field that deflects (silver) atoms according to their outermost electron spin, dividing an atomic beam into two *spin-polarized* beams, up and down. Hence, with this experiment, one succeeds in measuring a particle's spin while trowing it into a specific up/down state[iii]. This is an example of a projective measurement. Other commonly considered form of measurements are POVMs(positive operator valued measurements)[6].

Notice that this is a case where we are talking about a *beam*, therefore a large number of atoms made of many particles, so there is no real *single* particle state here. Nevertheless, this is an example of experiment where only a property of one particle (the outermost electron) matters, and whether the experiment is

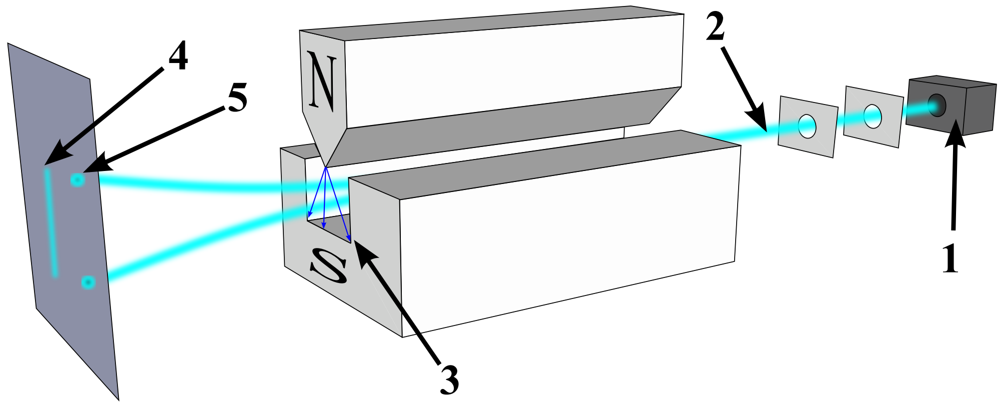


Figure 1.1: Apparatus of a Stern-Gelarch experiment. A silver atom beam (blue beam) is made to travel through an inhomogeneous magnetic field. 1: furnace. 2: beam of silver atoms. 3: inhomogeneous magnetic field. 4: expected result. 5: what was actually observed. © By Tatoute - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=34095239

made with a single electron, a single atom, or a large amount of those makes no difference. This becomes a basic example of when and why single particle quantum mechanics matters; a satisfactory explanation of this experiment relies on the existence of a quantum property (in this case, the inner degree of freedom of the electron, called "spin" even if does not spin) of a single particle. Therefore, even though the particles may not be *really* alone, we can forget the others and treat them as so. Perhaps, if we were to be more rigorous, we would not call it a "lonely dragon" but a "forgotten one," for while it may be surrounded by other particles, there are situation we can simply ignore them all and look at a single one. Nevertheless, other cases require to think of *means* or *averages*, what comes under the so called *mean field theory.* In this case, we do consider the collection of particles we have in hand, but force it down to a single "average" particle. This is an approach to many-body problems, and will be covered later. Just remember that this can be done, for now.

But projective measurements can be said somewhat ideal, since they disregard the relation between disturbance and fluctuations. A detailed coverage of such relations is out of our scope, yet we shoud keep in mind such existence whenever possible. For instance, besides projective measurements, other methods for obtaining information from a physical system are also available. *Weak measurements* provide a way to extract information through pre/post-selection on the states of a system, i.e., both initial and final states are selected, and what happens in between is investigated. Weak measurements rely on disturbances smaller than the system's fluctuation and can be used in different situations to magnify small effects, or to separate properties of quantum states. But some people argue the term "measurement" is not adequate and it should just be called "weak value" instead.

Regardless of considering it a "measurement" or not, we should point to one characteristic of such procedures in particular. The definition of a "state" is clearly looser than the one we have presented so far. For example, the so called *Cheshire cat state* [8] can be realized with this technique, separating a photon or other particle from its spin, an internal degree of freedom, as mentioned. In other words, it separates the particle from its property, like the Cheshire cat in *Alice in Wonderland* that disappears and leaves its grin. However, this process relies on pre-selecting a specific input state, and post-selecting another, different specific output state, analyzing events in between. If we stay by our ket notation, we input $|\text{inital}\rangle$ and obtain $|\text{final}\rangle$ states. The state in between cannot be written as $|\smile\rangle$ in such case. This becomes our first example of a "state" that cannot be satisfactorily represented by a ket. But if we agree that as long as we have an input state being somehow transformed into an output state, we should agree that the state that undergoes the transformation does not cease to exist and is still there, hence the word "state" seems adequate. Surely, it may not be the case and we may have an state disappearing while a completely new one is created, and there is no state at all in the middle. But in quantum mechanics, we abide by one rule: conservation of probability flow, i.e., a fancy way of saying that if a particle were there and didn't become something else, it is still somewhere. Therefore, we ought to accept the term "state" here, and the fact that it cannot be written with a ket. We will come back to this matter in the next level, under a different approach.

Still, this may leave one wondering. What may or may not happen? How can we be sure of what is going on, whether things are there or not, and so on? We cannot address all these questions at once easily, but we may return to GPTs to try addressing a collection of questions, or at least one question that cannot be easily answered. Measurement, for one, can certainly be take under GPT's light. For GPTs, we only consider a probability distribution of measurement outcomes, and their influence on other measurements and probabilities. Later on, we will consider a type of measurement called *sharp measurement* and the GPTs defined by it. It is an idealized form of measurement that is minimally disturbing and repeatable. In other words, it should not change the outcome of measurements compatible with it, and can be confirmed. Naturally, one may think of theories that do not have such kind of ideal measurements possible not even in principle, hence requiring that a theory allows their (theoretical) existence already limits the GPTs we are thinking about. Nevertheless, it turns out that such consideration offers insights also to quantum theory, and we shall return to this kind of GPTs later.

1.2.2 · · · Bloch balls

A lot of what has been presented here can be represented as a sphere called *Bloch sphere*. Throughout this work, quantum bits, or qubits for short, will be a recurring appearance. They are essentially the quantum version of a bit of information (0/1) used in computers. Any two-level quantum system can, in principle, become a qubit, we just have to label one state as $|0\rangle$ and the other as $|1\rangle$. For example, the spin of fermions (the wing flapping polarization for our dragon) may serve as a qubit given its up/down freedom. However, it is worth noticing that one does not need a fermion to make a qubit; bosons also work just fine. We shall not go deeper into this matter yet, but for simplicity, let us use spin-$\frac{1}{2}$ system to understand the basic idea behind a qubit and its representation as Bloch spheres.

In general, a spin or qubit can be represented as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle, \tag{1.7}$$

with θ an arbitrary parameter that dictates the amount of superposition between the two basis $|0\rangle$ and $|1\rangle$, and φ an arbitrary phase factor. By itself, such state may not look (geometrically) meaningful, but it becomes self-evident if one treats the parameter θ as a polar angle on a sphere, and φ as an axial (i.e. equatorial) angle, making such states to live on the surface of a sphere like in fig. 1.2.

The three axes x, y, z of such ball can serve as basis for different directions of measurements using *Pauli operators* X, Y, Z. They obey anticommutation relations and may be explicitly written as

$$X = |0\rangle\langle 1| + |1\rangle\langle 0|, \quad Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|, \quad Z = |0\rangle\langle 0| - |1\rangle\langle 1|, \quad [\sigma^i, \sigma^j] = i\varepsilon^{ijk}\sigma^k, \tag{1.8}$$

where $\sigma^1 = X$, $\sigma^2 = Y$, $\sigma^3 = Z$, and ε^{ijk} being the completely anti-symmetric tensor (Levi-Civita symbol).

This approach to two-level systems are Bloch balls can be quite general and is very much explored in quantum (information) theory [9]. Even its dimensionality has been object of study, considering not only three but d-dimensional Bloch balls and their implications to physics, finding constraints showing that our known physics happens only for three-dimenional Bloch balls (and 3-dimensional space as a consequence) [10, 11]. Notice that, in the case of a general Bloch sphere, we are no longer dealing with qubits, but a more general information unit within a general probabilistic theory or equivalent. We shall refer to such general bits as *gbits*, though they will not be used until we reach level 8 and take an informational perspective.

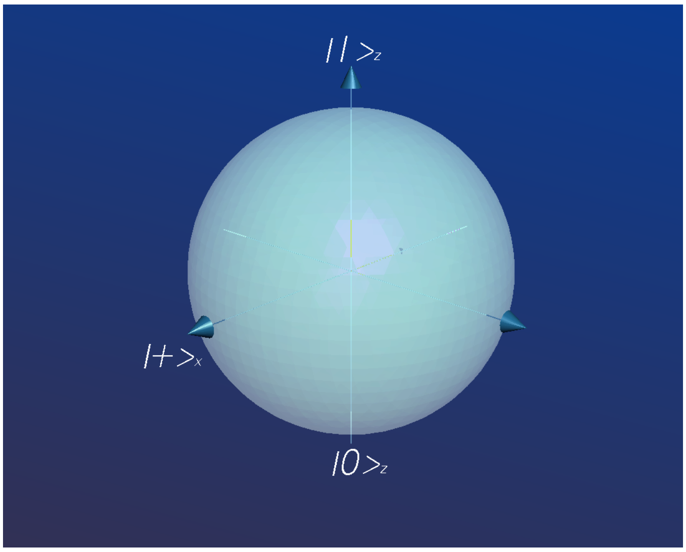


Figure 1.2: Bloch sphere for one qubit Hilbert space $\mathcal{H}$. $|0\rangle$ and $|1\rangle$ are set on z-axis while $|+\rangle$ and $|-\rangle$ are set on x-axis.

1.2.3 · · · Time evolution

We have not discussed an important matter so far: time and dynamics. Though our dragon moves around its world, which implies variations with time, nothing has been discussed about it yet. In general, quantum dynamics is represented with time-dependent differential equations of the form

$$-\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle, \tag{1.9}$$

with $\hat{H}$ the relevant Hamiltonian. It can be integrated to lead to

$$\begin{aligned} \Psi(t) &= -i\int_{t_0}^{t} \mathrm{d}t' H(t')\Psi(t'), \\ &= \mathcal{T}e^{-i\int_{t_0}^{t}\mathrm{d}t' H(t')}\Psi(t_0) = \mathcal{U}(t,t_0)\Psi(t_0). \end{aligned} \tag{1.10}$$

The operator $\mathcal{U}(t,t_0) = \mathcal{T}e^{-i\int_{t_0}^{t}\mathrm{d}t' H(t')}$ is the time-evolution operation, which takes the wavefunction from a state $\Psi(t_0)$ to $\Psi(t)$, and $\mathcal{T}$ is the time-ordering operator. If one wishes to approximate it numerically, it can be discretized like

$$\begin{aligned} \mathcal{U}(t,t_0) &= \mathcal{T}e^{-i\sum_{t'=t_0}^{t'=t} H(t')\Delta t} = \mathcal{T}e^{-i\Delta t\sum_{t'=t_0}^{t'=t} H(t')} \\ &= \mathcal{T}\exp[-i\Delta t H(t) + H(t-\Delta t) + \ldots + H(t_0+\Delta t) + H(t_0)]. \end{aligned} \tag{1.11}$$

Here, we are dividing the time between t and t_0 into a large N number of steps Δt. We apply Suzuki-Trotter expansion successively here, and suppose it can be approximated as

$$\mathcal{U}(t,t_0) = \mathcal{T}\prod_{k=0}^{N} e^{i\Delta t H(t-k\Delta t)}, \tag{1.12}$$

for small time steps. This would actually be followed by an error, quadratic on Δt, and that is another reason to require small time steps.

if Δt is small enough, we can order the exponential operators according to instant, and ignore the time-ordering operator acting on each of them. That's when we can actually get rid of $\mathcal{T}$. We can relabel

t as t_N, and the instants in this interval as $t_j = t_0 + j\Delta t$. Then, now understanding the product of operators as already being ordered, we can approximate our time-evolution operator as

$$\mathcal{U}(t_N, t_0) = \prod_{j=0}^{n} e^{-i\Delta t H(t_j)} \equiv \prod_{j=1}^{N} \mathcal{U}(t_i, t_{i-1}). \tag{1.13}$$

If we naively expand the exponential, we will end up back to our differential equation. So, we expand the exponential in terms of *Chebyshev's polynomials*[12]. Following the recurrence equation for Chebyshev polynomials,

$$e^{-i\Delta t H(t)} = e^{-i(\Delta t E_0)\frac{H(t)}{E_0}} = e^{-i\tilde{H}(\tau)\Delta\tau} = \sum_{k=0}^{\infty} c_k(\Delta\tau) T_k(\tilde{H}(\Delta\tau)), \tag{1.14}$$

where

$$c_k(\Delta\tau) = \begin{cases} 2J_k(\Delta\tau) & (k=0) \\ J_k(\Delta\tau) & (k>0) \end{cases}, \tag{1.15}$$

$$T_k = \begin{cases} T_0 = I \\ T_1 = \tilde{H}(\Delta\tau) \\ T_k = 2\tilde{H}T_{k-1} - T_{k-2} \end{cases}. \tag{1.16}$$

Here, $\tilde{H}$ stands for the normalized Hamiltonian, and $\Delta\tau$ for the time step normalized accordingly. We shall use this calculation method for time-evolution to obtain results presented in level 6.

1.3 Quantum? What quantum? How quantum?

1.3.1 ··· From Latin to English and beyond

We saw the general idea behind quantum mechanics in this first level, but just dropped all of this from the sky, in a God-like flick of wand to create a recipe to study this so called quantum world. But truth is that things are not as neat as we would like them to be, much less as easy as it seems. Hence, let us start with a brief walk through history and linguistics.

Currently, quantum mechanics is a field in physics that studies many useful phenomena happening around us. Light is one thing made of quantum objects, namely photons. As is electric current made of other quantum objects, namely electrons. Computers, smartphones, screens and displays, telecommunications, and much more is essentially a skillful juggling with electrons and light. Most of it does not depend directly on quantum mechanics to be understood, but some small components like transistors do rely on some quantum aspects. Magnetism and magnetic memories are also understood as intrinsic quantum phenomena at atomic scale. And even touch screens and solar panels we see everywhere apply some quantum concepts for the theoretical framework or their materials. But all these phenomena are somehow known from much longer ago.

Consider a magnet, for example. We know magnets for millennia, but we did not understand their mysterious magnetic force properties. In 1600, William Gilbert published his famous book *De Magnete, Magneticisque Corporibus, et de Magno Magnete Tellure*[2] on magnetic philosophy, which opened the way towards a modern investigation of magnetism, by performing various meticulous experiments to exclude ancient mystic theories and giving better exploration on a thorough piece of work. He also coined the word "electricus" from the greek word for amber (ἤλεκτρον), which became the origin of the derived lexicon we know. We may look at this work as the first modern account of electric and magnetic phenomena.

But of course, even if this work described a lot of phenomena in details, we cannot expect it would pin down the origins of magnetism and electricity at once. Isaac Newton, in the end of his famous

[2] On the magnet, magnetic bodies, and on the great magnet Earth

Principia Mathematica laments that the "spirits" that keep body cohesion, the spirits of magnetism, and some others could not be properly pinned down with available techniques, and therefore should not be examined. And the more advanced techniques appeared, new light had been shed on the nature of electric charges and magnets. Finally, in nineteenth century, James Clark Maxwell summarized in four equations all the electromagnetic phenomena. Nevertheless, his equations could not explain some observations of electrodynamics, explained later by Albert Einstein in 1905, when he introduced his theory of special relativity.

But if previous understanding was not enough for the situation of bodies in motion, it was not clear how light and electrons interaction happened either, among other issues. This was also clarified by Einstein in the same year of 1905, when he followed Max Planck's previous work and *quantized* light the same way electrons are quantized, and miraculously everything made sense, what later gave Einstein a Nobel prize in physics. Light can also be understood by Maxwell's equations, which do not need such quantization, and yet, nature seems to need it. The word "*quantum*" was coined by Einstein from Latin, with the same origin as words like "quantity" to indicate a portion of light, and later was extended to small energy packets of various forms beyond light. This was, in a sense, the dawn of quantum mechanics, of *wave mechanics* as it was called initially.

Wave mechanics was the fusion of two fields: mechanics, which studies the motion of bodies, and waves, which studies oscillations in various media. It was not immediately clear the existence of a particle-wave duality, and therefore it was assumed to exist some interaction between particles like electrons and waves like light. Later, electrons were understood as particle with a pilot wave that guided them, bringing the notion of particle and wave closer together, until the realization that nothing fundamentally distinguish one from the other, and particle-wave duality, the concept that particle *are* waves and vice-versa, came into place, brought by Schrödinger. With no distinction between particles and wave, the very idea of quanta changes into a broader meaning, for they are just the particle aspect of what was traditionally seen as waves. With duality, every particle is also a wave, and consequently, every particle is also a "quantum." It soon becomes clear that the so called wave mechanics is not the study of waves and particles, but of one whole thing now understood as *quantum*. Now, the usage of the word is shifted to more frequently an adjective than a noun, and quantum mechanics becomes a new thriving field in physics.

This is our modern quantum: a wave-particle dual object in microscopic scale. As waves, they can superpose, disperse, diffract, reflect, or even pass through each other. As particles they are limited in portions that can bounce and interact. With this meaning in mind, and opposition of quantum versus classical matters arise. Classical phenomena are relatively well-behaved and well-defined, while quantum phenomena can be somewhat erratic, with complicated superpositions and interferences taking place.

How has this changed Gilbert's results? We now understand a magnet as a collection of very small quantum objects that possess a property called *spin*, an intrinsic quantum magnet. When aligned, these spins produce what we call magnets. Hence, small quantum objects join hands to form a classical phenomenon. However, if we take the south pole of a magnet as directing reference and call its pointing directions "up" or "down[iv]," spins can in principle take any superposition of up and down. Data stored in magnetic memory uses up/down to inscribe 0/1, but if the quantum spin of a particle is used, any combination of 0 and 1 is acceptable. This changes fundamentally our understanding of magnetism, for if particles decide not to be in a superposition but to jointly face the same direction, something must be going on. I will not discuss this matter in details here, but it illustrates the difference between quantum and classical. The magnet, even if composed by ultimately quantum objects, is classical.

Through this disssertation, we will essentially be discussing quantum phenomena, which soon become non-trivial. This essentially means that superpositions of 0s and 1s, YESs and NOs, left and right, and of essentially anything will be always in the horizon. From a pragmatic physics perspective, there is nothing magical about it, for those are just words designating physical states. For a particle, in the quantum mechanical sense, there is nothing special about left or right. And since this particle is a wave too, like any wave, it does not need to move only forward, it can split and scatter and go even in a circle if it wants. For a particle, nothing is confusing, *we* are confusing and create confusion. In the end, we are still simply surrounded by light, electrons, and magnets.

Let us continue our promenade diving into a more confusing question around this level: can we really think of a *single* body? What does it mean? We shall see.

1.3.2 ··· How lonely is the dragon, actually?

Our final objective is to talk about quantum information and quantum theory. But before we reach there, we must organize our thoughts on quantum theory in a more abstract sense. Why? Try to think of it: your computer handles information with well defined 0s and 1s, so if quantum states can be in a superposition of those, what kind of information is that we have in our hands? How can something be *both* true *and* false? Can it? Before we go into this issue heads on, let us review our grounds so far.

Our dragon has told us about its lonely life, so far. How is it to be alone, how peaceful it can be. But in much of its description, it had to resort to the space and landscape it lives in. We will consider more about it later, but it begs the question on how lonely it really is. Having no other particle there, does it even make any sense to talk about a space it lives in? The answer to this question is that it depends on the context, so sometimes yes.

When we imagine the habitat of that draconic particle, even if we imagine a particle in a vast void, we somewhat embed it in an empty space. And the situation discussed with landscape and surroundings indicates just the absence of other particles, but recognizes an existing physical background that the particle can interact. This whole matter of "nothingness" as empty background is discussed (and refuted) by Henning Genz [13], pointing to it as an abstraction for thoughts rather than an actual thing (or lack thereof in this case), which would result in contradictions. Hence, in short, the "loneliness" here can be thought of as loneliness of its kind — there is something, maybe many things in the background, whose origin we do not question yet.

Once we accept a background space existence, at least for the sake of the argument, we clearly face other matters. One is old and is explored by Dirac himself: how can a single particle interfere with itself? How can it follow two paths at once? Is it occupying two places in space at the same time? His answer is

> It may be remarked that the main object of physical science is not the provision of pictures, but is the formulation of laws governing phenomena and the application of these laws to the discovery of new phenomena. – Dirac, (1958) [3].

Dirac pushes forward a very pragmatic approach to quantum theory. It is not entitled to any particular common sense observation, nor beliefs, or anything beyond what is and can be done in a laboratory (supposedly including laboratories the size of a town, or country, or even planet, in principle). Although a somewhat circular argument (to formulate laws to find new phenomena to look for new laws), it dismisses whatever we take for granted when our assumptions fall short to explain observed phenomena, or even when such assumptions effectively block any attempt to explain phenomena until we abjure our abject beliefs.

It happened before.

Many people abhorred the existence of molecules in the past. Many people thought ridicule to have electrons "orbiting" an atomic nucleus. Many people discredited the idea of light being made of particles. All these beliefs were held based on solid physical knowledge. If a charged particle orbits another of opposite charge, it loses energy (by emission) and falls on the other one. Light has a clear wave behavior, and "light corpuscles," a theory proposed by Newton, had been debunked long ago by interference experiments carried by Young, consolidating Huygens's wave theory for light.

But dichotomies are not always good, nor everything is what it seems. The knowledge we had before twentieth century was not wrong, it was just limited. Limited to specific phenomena in relatively large scales. Limited to when we can promote a clear cut difference of corpuscular behavior and wave behavior. Think of water. Maybe a tank, or a glass, or a drop of it. It is not hard to imagine waves on a tank, or short wave-like ripples on a glass, but definitely it is hard to think of it on a small drop from a spray. So we, poor humans, look at the same water sometimes as a continuum that may accommodate waves, sometimes as a small particle in the air. Yet, we could, in principle, imagine minuscule waves on that droplet, or think of our glass as many droplets together. Complicated, maybe, but not inconceivable. What if we think of much smaller particles, like electrons? We think of a single electron as a corpuscle, in general, but it does not have to be like that, specially something so minute we cannot really imagine in real scale but only proportionally. Does it *have* to be a corpuscle or a wave? No. And throughout, I will use the word *particle* to designate any unit, in a certain context, that has a duality between a body-like and a wave-like behavior. This has consequences.

To grasp such consequences, first we need to understand experimental facts. And in the best spirit of Wittgenstein, we assume the world (at least the world in the sense of the matters to be discussed here) to be made of facts[3]. And a well known experimental fact is that, if you shoot single particles, one at a time through an interferometer, you get an interference pattern. There is no way that one particle left its trail or whatever to interfere with the next one, so they can only be interfering with themselves, or rather *it* can only be interfering with *itself*, singular. How can it be that a single elementary and indivisible particle can interfere with itself? Let us return to Dirac's pragmatism.

Dirac, in his own way, evokes Wittgenstein's last clause in his book: what one cannot speak about, one must silence (about)[4]. Dirac says it is not physics's job to give nice views easy to grasp, but clear and solid mathematical laws. If these laws have a nice metaphor to draw from, great. However, if the law in question happens to be described into words that sound nonsensical, like a single object interacting with itself and running through space, an object in many places at once and so on, for Dirac it does not matter. We should just shut up and deal with it, for the equations and facts assert each other meaning, not arbitrary word choice. If so, what else do we think we understand that can be challenged? What is a reasonable assumption and why? Moreover, what is understanding, then? For if our words and classical analogies cannot be trusted, what can? Solipsism – better remembered by *cogito, ergo sum*? Whatever one's choice, it may be dangerous.

Suppose, for now, one reasonable assumption accepted by most of us, if not everyone: causality. Causality, in essence, is a principle that places one event as the cause for another, this one being its consequence. So, facts have a cause, whatever unknown. Some events may even be stochastic, i.e. random, but whatever the inner randomness, its existence has still *some* cause. In simple words, we assume that given two events linked by causality, either A caused B or vice-versa. But, again, dichotomy is not always a good assumption, and there seems to be no reason out of blind belief that causality always stands like this. What if, at certain conditions, we allow for a superposition, like the one discussed, of causal relation? Explicitly, what if A causes B **and** B causes A with (not necessarily) equal probability? What if *sometimes* the egg comes first, and *sometimes* the bird — or the the dragon — takes the lead, and a single evolutionary line will just randomly pick either one to start the causal chain? If it sounds nonsensical, particles that are also waves or waves that are also particles sounded too in the past. Atoms too. Molecules too. And then, how far can we take Dirac's pragmatism? Perhaps as far as Wittgenstein's propositions, which is not as far as he or they actually wished.

Wittgenstein wrote and published his treatise *Tractatus Logico-philosophicus* in early 1920s, before Heisenberg uncertainty principle, spin, and many other quantum properties be even noticed. Heisenberg's uncertainty principle date from 1927, and Dirac's theoretical prediction of spin from 1928, a few years after Wittgenstein's seminal book. Wittgenstein assumes statements to be true, false, or undetermined. No quantum superposition is accepted there. As a consequence, he illustrates his thoughts, close to end of his book, with examples stating that one cannot see two colors at once, or a a body cannot move with two velocities. Well, add some quantum effects there and they can. In a sense, it is possible exactly because quantum states may be a superposition of true and false. If so, what prevents two events to be in a superposition of cause and effect state? Should we just fall silent, as advocated by Wittgenstein, because we don't (yet) see it? Or maybe follow Dirac's advice and just deal with formulae, and if causality is not a good concept under certain conditions, just deal with it and move on? It seems to me that following either would be ill advised.

When Wittgenstein recommends silence on what cannot be discussed, he seems to stand by the true/false dichotomy and to ignore that it could be both (probably, he would not accept it immediately, but surely would understand given enough time), perhaps in a quantum superposition or following paraconsistent logic formalism. As so, by trying to speak the unspeakable, one may actually give sense to it. He may have been trying to get rid of metaphysics, but it is hard to shut doors selectively. Dirac was

[3] *Die Welt ist alles, was der Fall ist.* – Wittgenstein, *Tractatus Logico-Philosophicus.*

[4] *Wovon man nicht sprechen kann, darüber muss man schweigen. – Wittgenstein,* Tractatus Logico-Philosophicus.

trying to free us from classical dogmas, and probably praise the neatness of mathematics, but this too is a source of knowledge and creativity that risk forfeiting important questions. In a sense, Dirac gives us a language to talk about Wittgenstein's unspeakable, while Wittgenstein has raised our awareness on the limits of our language, of what we see and conceive and what we don't. Therefore, I side with the idea that such matter must be discussed, be it a particle in two places or other non-local phenomena, or superposition of causality.

One issue we will return to in the next level is with non-locality, which is much intriguing once we have more than one particle. For now, how can a particle interfere with itself or follow two paths at once? We must understand that what we call a particle is not necessarily a point-like object. It can be smeared out, spread through a region of space. In this case, we can easily measure its wavelength (or frequency), and identify it as a wave. Curiously, we may also have a pair of particles that cannot be seen as two particles, but one single pair. In other words, one "particle" spread through space — not *two* particles, but *one* pair. And again, it has consequences, because once they become one thing, they are not independent from one another (we will return to it in the next level).

And if a particle, in the sense of a physical entity that may be stretched through space, may follow two paths, suppose this particle would follow a path to cause some effect. If we can identify the effect by the path, we would know the path followed, so, assuming quantum coherence only, effects allowed by the paths in the case are allowed, leaving the effects themselves also stretched. But if the effects are close enough to their cause, and if they limit the paths followed in first place, if we only saw the effects, could they not be in a superposing relation with their causes? Maybe yes. I do not plan to cover this issue here in details, but Brukner has given some insights with a framework that allows for causal superposition. This could imply that, indeed, some events in the world are originated by divine dice, up to their causal relation.

So, while Dirac may dismiss the provision of pictures as our job, he recognizes that we should apply the known laws to find new phenomena. Nevertheless, perhaps there is a new phenomenon lurking around the corner of giving pictures. Perhaps what we cannot talk about is just a phenomenon that lacks adequate language for it and must be construct, with or without paraconsistent logic or any other formalism. I do not know whether Dirac thought of Kant's *noumena* or *Ding an sich* (thing in itself). Regardless, I wish to take it into account in parallel to Dirac's perspective. Quantum theory, if it has any specific purpose, should be looking for laws of these so called quantum phenomena, whether easily understood or not. This, in other words, is a search for quantum noumena, which we may suppose that from a macroscopic scale perspective (we will come back to this later) should give rise to classical phenomena, but not necessarily the same *Dinge als Erscheinung* (Things as they seem) or φαινομένα (appearances) for a single quantum particle, or a single quantum event. And since we construct our common sense on the top of classical events in classical perspective, it should be indeed hard to describe whatever we know about quantum noumena in terms of classical φαινομένα. Hence, one should keep in mind that, even if we talk about a single quantum particle effectively, which may actually be a collection of particles dancing in unison, it is still a kind of event that is not seen in our daily experiences explicitly, which is made by an extremely large collection of those. It is the difference of having a single sand crystal, and the whole seashore; it does not look nor feel the same, but at least in this case we can see and experience both quite easily, creating clearly distinct impressions. When searching for quantum noumena, one may try to describe it and link it to quantum or perhaps mesoscopic phenomena, i.e. those on the boundary of microscopic and macroscopic size, but a jump towards macroscopic analogies may be just impossible. So, it may indeed be simpler to have a pragmatic approach, accepting that we may have some silly images sometimes, but they might well do their work.[v]

As this pragmatic approach towards quantum noumena seems to spread, people still struggle to obtain more palatable pictures for many phenomena. And as we will see in the next level, sometimes the best we get is exactly the opposite; soothing explanations are often forbidden. These prohibition, which often appear as theorems, even have their own collective name, known as *no go* theorems. With different shapes and meanings, they often cast aside classical and easy to understand explanations of quantum phenomena in favor of less deterministic ones.

The whole issue, here epitomized under the image of Dirac and Wittgenstein, is a matter of logic and language. Heisenberg has provided good grounding for such discussion [14]. Essentially giving a different but related perspective from the one I push forth here, the Nobel laureate focus on the relation between natural language and mathematical formalism for theoretical physics, ultimately recognizing in natural language the means for comprehension, both for the general public and for experts.[vi] On this matter, he emphasizes how natural language is related with our everyday experience and understanding of the surrounding (classical) world, what may not be the ideal language to describe never-before seen phenomena.

As Heisenberg points, in accordance with Weizsäcker [15, 16], in general, we have inadequate language to deal with quantum theory, for our language is strongly bound to the classical world and classical phenomena. Classical phenomena may be reproduced by quantum theory, but quantum theory also has intrinsic phenomena accounted by interferences that do not take place in the classical world. And with natural language bound to such classical phenomena, quantum phenomena that do not bear an equal on the classical world can not be expected to fit in any known word or expression, unless we are willing to allow for much poetic freedom and an overload of ambiguities.

Take, for instance, the word "state" we encountered earlier. Whatever it indicates in the quantum world, it is hardly possible to find any kind of classical analogy or counterpart. Indeed, Heisenberg points that if by "state" we mean "potentiality" we could well substitute one for another and the meaning would be roughly unchanged. Then, "coexisting states" may simply be called "coexisting potentialities" that indicate nothing but a possibility of what may be to come. Indeed, physicists swallow a whole lot of ambiguities excused by mathematics. For whether we use the word "state" or "potentiality" to describe pieces of math that lead to whatever experimental phenomena observed, they bear the meaning of mathematical objects presented here that hardly have any sense *per se*. And probably, any counterpart of original meaning they may have in the classical world will be gone in a physicist mind. Take again the word "state," as in "state of affairs" or "the state of my bank account." As vague as it may be, it points towards a well stablished situation that could in principle be well-defined, by confirming the feelings, motivations, and various resources of the said affair, of by looking at the extract of the bank account. The "state" of an electron or an atom or a two-level system is far from being equally determined: taking into consideration all the possible interferences and superposition, one can at most give you a chance to have a blunt ground state or an excited state upon measurement (which is why we could just call it "potentiality"). With intrinsic uncertainties and indeterminations, it is clear the we must allow some poetics in the usage of the word "state" as in many other usually easy-to-understand words. The temperature of a room or glass of water can be easily imagined and even defined in classical physics, but temperature for an atom is a much harder concept to even imagine, on the borderline of the nonsense.

These linguistic difficulties are not an accident and not easily circumvented. And it has a lot to do with quantum computation and quantum information. As we push the boundaries of what we know, the horizon of our known phenomena, i.e. the horizon of the collective human knowledge, it expands beyond the boundaries of natural language. While Wittgenstein argued we cannot talk about what is beyond our "seen horizon" of events, using language to describe whatever was formally beyond our world and is now accessible knowledge falls in the faint and unclear border of what can and cannot be discussed. It is the first challenge faced by language in science: how to properly addressed new knowledge. For this, either known words and concepts must be revised, or new ones must be created. The very word "quantum" that is the central pillar for this research was first coined and later revised under the light of new ideas and observations.

But a second challenge already illustrated comes into when dealing with the microscopic world of phenomena: they are not in our daily experiences with a presence strong enough to mold our daily language. Therefore, they stay ignored and marginalized, though more general and encompassing more possibilities. The extra possibilities are those predicted or allowed in the quantum world of waves and path-interference, but shunted aside from our quotidian experience. One way to conciliate this wordless mathematics with our known logical assertions of sentences with ambiguous meanings is to establish "degrees of truth" to quantum statements. To say that "a particle is in state $|0\rangle$" may be true or false as in the classical world. However, in the classical world, we may not be able to tell if it is true or false, but it must be either one and no other third option (tertium non datur). In the quantum world, it can. One may look at a general state on a qubit as having a certain *degree of truth* of being in state $|0\rangle$ and a degree os truth of being in state $|1\rangle$. The truthfulness of one or another is not absolute nor entirely predictable, not even in principle. This is a way of naming the mathematical formality behind quantum theory, by calling the complex coefficients of superpositions a "degree of truth." It also makes the sentences "the atom is in ground state" and "it is true that the atom is in the ground state" lose their usual equivalence, for there may be a degree of truth associate with the former that actually falsify the later, for saying something is *true* amounts to absolute or maximum degree of truth, while *false* accounts for the minimum or no degree of truth at all. This is the idea advocated by Weizsäcker [15, 16].

Weizsäcker has done a careful discussion on the relation between language, modern physics (science), and logic, which can well be compared with the done by Wittgenstein to some extent. It is very roughly summarized in the previous few paragraphs. He highlights that the conditional sentences around causality (i.e., if one knows the state of a system in a point in time, one can know it at any point before of after such point in time. See [15]) belong to the classical *Weltbild*[5] and do not have a place in quantum mechanics, where an abstract wavefunction defines only probabilities to a physical object. He further clarifies many of the logic formalism differences that arise between the classic and the quantum "Weltbild," like the ones discussed in the previous paragraph.

He also discussed how quantum theory influences philosophical aspects of Kantian nature, like a priori knowledge on noumena. Given our exposition so far, it is clear that quantum mechanics severely limits any a priori information whatsoever. This discussion is lengthy in tis details and beyond our scope, but we can take a modern approach to the issue and point that we know very little "a priori" in quantum mechanics. Even if we could tell that a particle with spin 1/2 (say, an electron) is in a box, and perhaps we even measure its spin, how to understand this spin is a problem in itself, to be revisited in level 4. Indeed, the noumenon involved here is expressed in anticommuting Grassmann variables to represent the phenomenon, but how physically meaningful is such representation is a question that deserves care.

And while we do not go deep into this problem, this care with language and logic motivates us to actually question Weizsäcker's idea to some extent. By calling the complex coefficients of overlapping base states building up a complex wavefunction, we are taking the word "degree" with its intuitive real continuum understanding and placing it in the complex plane. While its modulus still holds a "degree" that interests us, its phase angle in completely arbitrary and ignored. Nowadays, we link it to a "gauge freedom" of the wavefunction, an interpretation not yet available in Weizsäcker's time. Still, I cannot recognize such naming as satisfactory. Take for instance quaternionic quantum mechanics [11, 17–19]. One could, in principle use quaternionic wavefunctions instead of complex ones. Recent research has also shown that some differences between these two frameworks for quantum theory exist [11]. Putting difficulties of analysis aside, naming linear combination coefficients "degrees of truth" should be regarded as limited if other algebraic structures are not taken into account. Certainly, the meaning of the word "degree" can readily be assumed as flexible enough to go from real numbers to complex numbers, quaternions, Grassmann numbers and so on. But this, on the other hand, does not seem to solve the issue of interpretation completely.

For Heisenberg, this "vague and unsystematic" usage of the language is satisfactory. Once we cannot bear it, we are forced to run back into mathematical language, and he compares this vagueness to the daily usage of language and to the vagueness in poetry [14]. However, while I must agree that it sometimes cannot be helped, and that there are intrinsic difficulties in choosing proper words to reflect the adequate λόγος in the quantum world, I cannot vouch for qualifying it as "satisfactory." It is true that at some point we might simply give up and recognize that we just do not have enough tools or knowledge to lap words for our purposes. And we must equally forgive such limitations, as in Weizsc̈ker's case before gauge theory consolidation. Geometric phases or Berry phase (to be seen in level 5) is an example of why vagueness may not always be satisfactory. While it has always been in quantum mechanics, with vague language dismissing the importance of phase factors for isolated states, sir Berry pointed that some phases encode geometric meaning and cannot be simply ignored. We can say that he clarified where the ambiguity zone ends and usage a word (i.e. geometry) that survives in various mathematical spaces. At this point, I think we may classify the flexibility given to such words like "geometry" and "phase" to be "satisfactory." Hence,

[5] Worldview.

talking about an electron's spin can also be said fair, but one a fermion is represented with anti-commuting Grassmann oscillators, any attempt to extrapolate names to also include Grassmann coefficients may be misleading, trying to give the appearance of comprehension to what is not understood.

These are some of the issues behind quantum theory that require us to leave aside any classical temptation to make microscopic objects behave as we expect in the macroscopic world. The views and understandings of the quantum world cannot be fully understood nor limited by our classical intuition, only perhaps at a macroscopic limit. Hence, recognizing that much has the potential to be accomplished in quantum states with quantum coherence, we shall continue exploring the quantum nature of the world in the remaining of this work.

Notes

[i] Of course it is a joke, and one that works only in certain languages like English, but definitely not in Japanese, for instance.

[ii] In mathematics, any linear transformation has eigenvalues and eigenvectors. The set of eigenvalues is called the spectrum of the transformation, and the adjective "spectral" generally refers to it, as in "spectral analysis" and "spectral flow."

[iii] When Stern and Gelarch performed their experiment, in 1922, there was no knowledge of spins at that time, so the experiment was not meant to measure spin quantization at first. Initially, it was intended to falsify Bohr's atomic model, which had electrons orbiting the nucleus of their atom in quantized orbits. Such orbits would therefore have quantized angular momenta and equally quantized magnetic moments, which they wished to measure. That is, if electrons moved at "steady" orbits, this should be equivalent to an electric current passing through a coil around the nucleus, with a constant and measurable magnetic moment. It is worth noticing that neither Schrödinger's (1926) nor Dirac's (1928) equations were yet published, hence orbits with zero angular momentum were not considered for the design of the experiment. That turned out to be a fortunate mistake, for silver, the chosen element, whilst it has indeed one valence electron and could serve as a model for Bohr's hydrogen atom, when solving Schrödinger's equation for such electron in a central (charge) potential, one obtains zero angular momentum. Consequently, the contribution of such angular momentum to magnetic properties is also zero. Only spin, an internal degree of freedom of the electron, contributes to magnetism, and this experiment, though performed before spin's discovery, turned out to be a proof of its existence.

[iv] There is a physical reasoning here. Electrons have negative charge and therefore their magnetic moment has the opposite direction of their spins. If we say north pole points upwards and a magnetic moment, then spin goes downwards. But even simpler, the definition of up/down is arbitrary, and I was born in South hemisphere, so my "up" is the South and no complains are accepted.

[v] Actually, the silly images can also be an alert to something bigger, once again suggesting that reflecting on the explanations we give may be useful. For instance, Dirac also suggested that positrons were like holes in a sea of electrons that filled the vacuum and it just seemed to leave a positive charge. This idea of holes do happen in condensed matter context, but is not very good to elementary particles, though. These infinite particles would fill the vacuum with a background field of infinite energy in a not much satisfactory way. So, we do not rely on such image anymore and positrons are just anti-electron, particles much like an electron but with opposite charge. In this case, the image of a sea of electron helps because we are familiar with materials in which there is a large number of free electrons like a sea, but there we have atoms with nuclei that balance things out, while vacuum, as far as we know it, does not have such positive background charge, for one thing. This, allied to experiments, forces us to recognize positrons and other anti-particles as particles in their own merit, not only an absence in an unperceived background.

[vi] I am tempted to say that Heisenberg took natural language as means to gauge comprehension for experts and laypeople equally, but I'm afraid he did not write that, nor he would really mean *equally*. In fact, in his text, he seems fairly to recognize that experts have access to a faint comprehension from equations that, not being accessible to the general public, jeopardize any attempt of "equality." However, he does acknowledge that formulae not converted to plain words fall short to what one would satisfactorily call "comprehend" new knowledge, and that such conversion poses a kind of ultimate challenge to the physicist.

Level 2
Double Dragon

Ere Babylon was dust,
The Magus Zoroaster, my dear child,
Met his own image walking in the garden.
That apparition, sole of men, he saw.

— Percy Bysshe Shelley, *Prometheus Unbound, Act I, 191-94, 1820*

2.1 Clinching carousel

Yes, you have my company now. You were no more powerful than a fey, but no less powerful either. You just were not. Or were you, dear fellow? For what is a dragon in a reign of solitude? A miserable pile of secrets hidden under averages, means, and ignorance[vii]. Let us *exchange* places.

I know you will avoid me as I avoid you. I know you will not follow me, as I won't follow you. We will keep our distance and dominate our own territory each, as must be. But I can feel you as you can feel me. For I am your friend, your family, and your nemesis.

You cannot hate me as I cannot hate you. Between you and me there is no love nor passion, no loath nor wrath, only avoidance. The only way you can come close to me depends on our wings. Only if you flap yours as I do not flap mine you can meet me. Only if your wings flap the opposite as mine do you can ever see me. Nonetheless, you can already feel me, can you not? For it is the very nature of your soul to avoid mine. It is your nature to avoid me, as I avoid you. And you twist yourself around me and I around you in the human called *anti-symmetrization.* For if you take my place and I yours, the world will go tipsy topsy, until we sit in our current state-territories again.

2.1.1 · · · Jingle Bell

Sometimes you are born from the same egg as I am. Sometimes, like now, we may engage in a pretty different conversation. You know what I am talking about: you ought to eat my fire! I will savor your flames with the same pleasure as if I were drinking your blood and biting your soul! For with this, even after we fly apart and take our paths, we will still be in opposition, fierce opposition! If I go up, I know you would go down. If I turn left, I know you would turn right. It matters naught your place in the universe, as long as you or I do not exchange breath with other creatures, I shall know you and you shall know me. We must remain as a single entity divided in two, one unique enough for humans to baptize us as *singlet.*

Your will will winkle wishful wisps of my waggle. And thus you tangle and entangle yourself to me in this holy *entanglement.* There is no nobler correlation than that! For you know it is not simply within you, nor within me, but between us! Bell, like Bel Marduk in his Jovial wisdom, elucidate this to humans: no hidden contract can limit our movements, you correlate with me through your very soul! You share with me the information I share with you. And thus you hide this bit from others, from all.

So let us have fun! You should play a game with me! You shall guess my wing flaps as I shall guess yours! But to make it interesting, you can say your guess by adjusting with *my* guess, while not knowing something I know. You will throw a stone, and see if it bounces to the left or right, and I the same. If our stones bounce to the same side, you must guess my wing flapping direction and oppose me. Otherwise, you should align with me. There is a good chance that you and I can win this game, and humans marvel at it! Once entangled together, you and I can win this game more than one could expect without our shared conscience. Clauser, Horne, Shimony and Holt presented this game played by will-o'-wisps to humans. You know the secret: align your flaps or "polarization" along vortices chosen by the direction the stone bounces, and you can be more right than wrong! So let us play this game and see what you and I can make…

2.1.2 ··· Family bonds

But you know the law. You cannot become me nor I can become you. Matters naught if you twist my bowls, you can become one with me, but not become me. No shapeshifting, no copycat, no mimetism, no clone. Still, let you befriend me and you can teleport your will through the world! Inhale my breath, eat my fire! Share your movements and thoughts with me, and you can teleport wherever you want! Well, wherever I am, at least. Is it not to your pleasing what humans call *quantum teleportation*? They sing and dance to our tune, as you can sing[1] to me, your kin! One more slave between us and all can be done. As long as you know how you befriend our the slave, you can tell how I should move or change in order to have you mind transferred to mine.

Therefore I remember you: you need me, *friend*, as I need you. Let us pair up and conquer the world!

2.2 From couple to math

As soon as we have two particles, physics gets quite interesting. While I hope to have given an idea of what this vague claim means in the previous section, from a physical point of view it essentially means we have more *degrees of freedom* to work with, or simply more *possibilities*. That is to say, besides whatever the physical variations a single particle may experience by itself (which now we have two), globally speaking they may also vary regarding each other, hence changing one of them may change the whole. From a particle's perspective, what happens in second person matters *as much as* what happens in first person ($I{\sim}you$). Let us lay our (mathematical) language too deal with them and look at some emerging properties of such duo.

2.2.1 ··· New notations and concept

As we discussed in the first chapter, quantum states are often represented by the so called Dirac ket notation, like $|a\rangle$ where a is a label to indicate a name, number, type, or anything of interest and could even be suppressed if unnecessary. For two particles, the same idea can be used, with some different approaches available.

The standard way to indicate the quantum states of two particles A and B is simply by juxtaposing their kets, viz. $|a\rangle|b\rangle$, what is called *direct product*. In this case, we would normally keep track of the kets positions, keep A to the left and B to the right, in order to avoid confusion and keep their meanings and significances as clear as possible. Hence, following this convention of keeping their written position fixed, it is often common to just write them as one ket with two labels, explicitly $|a,b\rangle$, which are to be understood as equals. However, note that such movement expands the meaning of kets, allowing them to

[1] ふとたちどまり気持ち解放　ジグザグ進むこの迷路　今此処時間自分自身イヤホン内のガンダーラ　テレポート連れてって　テレポート連れてってね　音の海飛び込み波乗り　もっといいとこへ – © 04 Limited Sazabys, *teleport*, No Big Deal Records (2015).

expressed one big(ger) quantum state depending on more than one parameter, in this case, labels a and b. For instance, if we assume each label to take values 0/1 for simplicity, $|0\rangle|0\rangle = |0,0\rangle$ is clearly different from $|0\rangle|1\rangle = |0,1\rangle$ which is also different from $|1,0\rangle$ (remember, positions are fixed now). If there is no risk of misunderstanding, the comma can also be suppressed, writing simply $|00\rangle$.

Now, we are left with a small problem. Some times, we may know we have a state of two particles, but we may have no idea what state it is. Even if we assume that only values 0/1 are possible, we are left with four possible pairs, but any superposition is still possible. One may write $|\alpha,\beta\rangle$ to express such possibilities, including linear combinations, but this can soon become unclear and even untrackable. Hence, we will introduce another notation here for such cases: $\|\alpha\rangle$. It is nothing different from the kets already introduced, which in fact remain more general, but will be used to emphasize that whatever the state it represents, it is intrinsic many-body. At this point, it is not particular meaningful to have a new notation, but it will come in handy in higher levels — a reason to slowly getting used to it.

Also, now that we will think of two (fermionic) qubits, it is worth to mention another important representation of states: the *density operator* $\hat{\rho}$. For a pure state, which is what we have discussed so far, the density operator or density matrix for a state $|\psi\rangle$ can be written as

$$\hat{\rho} = |\psi\rangle\langle\psi|. \tag{2.1}$$

It gives us the probability of finding a particle in each basis and has trace equal to one. But density operators are specially useful for allowing us to distinguish between *pure* and *mixed* states. In general, one might rewrite a density operator as its expansion in terms of all its components, viz.

$$\hat{\rho} = \sum_{ij} p_{ij}|\psi_i\rangle\langle\psi_j|. \tag{2.2}$$

It follows from such construction that

$$\mathrm{tr}(\hat{\rho}^2) \leq 1. \tag{2.3}$$

Whenever we have a *pure* state, constructed by the outer product of a wavefunction, trace is preserved since $\hat{\rho}^2 = (|\psi\rangle\langle\psi|)^2 = |\psi\rangle\langle\psi| = \hat{\rho}$, hence $\mathrm{tr}(\hat{\rho}^2) = 1$. Otherwise, we say that we have a *mixed* state, one that cannot be represented by a wavefunction.

It is possible to perform a partial trace on a density operator to exclude part of a system. For a two particle density operator, it offers a way to exclude one and project the state onto a single particle. We write

$$\rho_A = \mathrm{tr}^B\rho := \sum_{i\in B}\langle\psi_i|\rho|\psi_i\rangle \tag{2.4}$$

to indicate a partial density matrix of subsystem A obtained by partial trace of subsystem B. This will, in general, leave A on a mixed state. For a single qubit, this state can be written as

$$\rho^{(1)} = \frac{\mathbf{1} + \vec{\sigma}\cdot\vec{r}}{2}, \tag{2.5}$$

where $\mathbf{1}$ is a 2×2 unity matrix, $\vec{\sigma}$ a vector of Pauli matrices, and $\vec{r}$ an arbitrary direction vector of length $r \leq 1$. Points on the surface of a Bloch sphere represent pure states, while points inside of it represent mixed states.[viii] This also offers a way to understand mixed states: when we take only a (sub)system without considering how it interacts with its environment, we may be left with a purely probabilistic mixed state.

But perhaps the core message our dragons wanted to expose is how the state they construct together must be *anti-symmetric*. That is, if we were to exchange two dragons' place, their wavefunction must acquire a minus sign., or their ket state should obey a relation like

$$\|\psi_1,\psi_2\rangle = -\|\psi_2,\psi_1\rangle, \tag{2.6}$$

indicating each fermion (i.e. dragon) by the subscripts. If we can express the whole quantum state by summing direct product of each individual (single-particle) state, we can achieve it by writing

$$\|\psi_1, \psi_2\rangle = \frac{|\psi_1\rangle|\psi_2\rangle - |\psi_2\rangle|\psi_1\rangle}{\sqrt{2}}, \tag{2.7}$$

supposing the ordering of kets gives their represented particles' position.

The reason why we need this anti-symmetrization will be discussed again on higher levels, but we shall put forth a simple argument for now. If we swap two particles and and swap them again, we should return to our initial state. Therefore, if a factor η is attached to one swap, a double swap, equivalent to identity, should lead to a factor $\eta^2 = 1$. This requires η to be ± 1. Fermions, depicted as dragons here, are defined (for now) by the case where this swap gives a minus sign. If $\eta = 1$, i.e., swapping particles lead to no difference at all on the global state, we actually have some form of *symmetry* and call such kind of particles *bosons* (lately depicted as feys in level 4). Essentially, the anti-symmetric state written above becomes symmetric, adding states together instead of subtracting. Note that this is valid for *identical* particles, for if we can somehow distinguish them, swapping particles around will *not* lead to the same state (with a $\pm$ phase factor).

2.2.2 ··· Entanglement

Perhaps the most amazing feature of a two-particle state is the so called quantum entanglement or simply *entanglement*. Formally, the way to define entanglement may vary, but in general, we say we have entanglement whenever the two-particle state in hand is not separable, i.e., when we cannot write the state as the direct product of two single particle states like $|a\rangle|b\rangle$. In such cases, we are left with a state under the form

$$\|\phi\rangle = \sum_i c_i |a_i\rangle|b_i\rangle, \tag{2.8}$$

where coefficients c_i follow normalization condition $\sum_i |c_i|^2 = 1$. As a sum of terms written as direct product, it may not be possible to write the whole state as a single direct product. For example, state

$$\begin{aligned} \| + +\rangle &= \frac{1}{2}(|0\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B) \\ &= \underbrace{\frac{(|0\rangle_A + |1\rangle_A)}{\sqrt{2}}}_{\equiv|+\rangle} \frac{(|0\rangle_B + |1\rangle_B)}{\sqrt{2}} = |+\rangle_A|+\rangle_B \end{aligned} \tag{2.9}$$

can be written as a direct product of two independent single particle states, viz. $\| + +\rangle = |+\rangle_A|+\rangle_B$, and is *not* entangled. The two qubits in place are completeley independent and uncorrelated from one another. However, consider the so-called (spin) singlet state

$$\|\psi_-\rangle = \frac{|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B}{\sqrt{2}}. \tag{2.10}$$

It is impossible to break state $\|\psi\rangle$ into a direct product of two single qubit states; we must add two such product terms together and not a single fewer. This is an *entangled* state, for we do have correlation between the two involved parties, Alice and Bob.

One way to verify the presence of entanglement, i.e., to look into a state's separability, is to perform the so called *Schmidt decomposition*, obtained by singular value decomposition (SVD) of quantum states. For this, consider a general bipartite state

$$\|\psi\rangle = \sum_{ij} c_{ij} |i\rangle|j\rangle, \tag{2.11}$$

where the labels indicate some basis to some over for each part. The coefficients c_{ij} can be seen as components of a matrix and can go through SVD, leading to

$$\begin{aligned} \|\psi\rangle &= \sum_{ij} c_{ij}|i\rangle|j\rangle \\ &= \sum_{ijk} u_{ik}\sigma_k v_{kj}|i\rangle|j\rangle \\ &= \sum_{k} \sigma_k|\sigma_k\rangle|\tilde{\sigma}_k\rangle. \end{aligned} \tag{2.12}$$

The last equality is obtained by placing $\sum_i u_{ik}|i\rangle = |\sigma_k\rangle$ and $\sum_j v_{kj}|j\rangle = |\tilde{\sigma}_k\rangle$. This decomposition is our intended Schmidt decomposition (SD)and it gives us the contribution of different bases[2] to a quantum state. If our states are separable, we will have only one singular value and one Schmidt basis. Otherwise, we expect to have more singular values and bases. The number of singular values and bases necessary to express a state is known as *Schmidt rank*, and it serves as a form of measure of entanglement (though not necessarily a good one).

Another more common way to measure entanglement is by calculating the *entanglement entropy* (EE) or *von Neumann entropy* of a subsystem. For a bipartite system AB, we take the system's density matrix and perform a partial trace on it. The entanglement entropy can be calculated as

$$S = -\rho_A \log \rho_A = \rho_B \log \rho_B. \tag{2.13}$$

The basis for the logarithm may be chosen on convenience, but is common to use $\log_2$ when considering qubits (2-level systems). For instance, if one calculates the entanglement entropy of a singlet state, the partial density operator for one qubit can be readily calculated to be $\rho^{(1)} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$ (a completely mixed state, at the core of a Bloch sphere), so

$$\begin{aligned} S_{\text{singlet}} &= -\rho^{(1)}\log_2\rho^{(1)} = -\sum_{i=0,1}\lambda_i\log_2\lambda_i \\ &= -\frac{1}{2}\left(\log_2\frac{1}{2} + \log_2\frac{1}{2}\right) = 1. \end{aligned} \tag{2.14}$$

For two qubits, this is the maximum possible correlation they can share: one bit[3].

Note, however, that there is a caveat to the Schmidt decomposition discussed here and reflected on the entanglement entropy calculated above. If two particles are identical, say, two electrons, we cannot really label them as "particle 1" and "particle 2" when defining a state, which is why the partial density matrix $\rho^{(1)}$ is the same no matter which particle is traced out. The wavefunction must be (anti-)symmetrized.However, how to perform SVD on such state where the basis cannot really be distinguished? Is it possible or adequate? The issue about Schmidt's decomposition universality has been addressed taken such particle identity matter into account [20]. In short, we can use it if we take proper care.

For the relatively simple case of two particles, we first need to remember that the two particle state $\|\psi, \phi\rangle$ may not be, in general, written as a direct product of single particle state, i.e. $\|\psi, \phi\rangle \neq |\psi\rangle|\phi\rangle$. However, one can introduce an external symmetric product of one particle states defined as $\|\psi, \phi\rangle := |\psi\rangle \wedge |\phi\rangle$, and their conjugate $\langle\psi, \phi\| := (\|\psi, \phi\rangle)^\dagger = \langle\phi| \wedge \langle\psi|$. [ix] The probability amplitude for finding these particles in a state $\|\xi, \zeta\rangle$ is given by

$$\langle\xi, \zeta\|\psi, \phi\rangle = \langle\xi|\psi\rangle\langle\zeta|\phi\rangle + \eta\langle\xi|\phi\rangle\langle\zeta|\psi\rangle, \tag{2.15}$$

where $\eta = \pm 1$ according to fermionic or bosonic state. One an also define a symmetric ineer product between spaces of different dimensionality, i.e., between states with different number of particles. Between

[2]Since it is obtained by SVD, such bases are distorted and may not correspond to any bases we chose before, but linear combinations of those.

[3]*Bit* become our unit for choosing base 2, which would become *trit* for base 3 and so on. If we take base e and calculate ln, we would instead count in *nats*.

a single particle and a pair of particles, one can have

$$\langle\psi| \cdot \|\phi,\xi\rangle \equiv \langle\psi|\phi,\xi\rangle = \langle\psi|\phi\rangle|\xi\rangle + \eta\langle\psi|\xi\rangle|\phi\rangle. \tag{2.16}$$

This can be used to perform a partial trace on the density matrix of a system, leaving a (single particle) reduced density matrix $\rho^{(1)}$. One may now diagonalize it and use its eigenvectors as Schmidt bases, and the singular values are given by the square root or the eigenvalues. Hence, if diagonalization gives $\rho^{(1)}|k\rangle = \lambda|k\rangle$, Schmidt decomposition of a state $\|\psi,\phi\rangle$ leads to

$$\|\psi,\phi\rangle = \sum_k \sqrt{\lambda}|k,\tilde{k}\rangle. \tag{2.17}$$

$|\tilde{k}\rangle$ is related to $|k\rangle$ in a way to keep bosonic symmetrization or fermionic anti-symmetrization. Also, once we possess $\rho^{(1)}$, entanglement entropy can be calculated the same way as already exposed above.

As an example of the described Schmidt decomposition for identical particles, take two particles whose only degree of freedom we consider is their spin, supposing it to be either up ($|0\rangle$) or down ($|1\rangle$). Assume the first one to be in state $|0\rangle$, and the second on in a generic superposition of $|0\rangle$ and $|1\rangle$.[4] The total state can be written $\|\Phi = \|0,\vec{s}\rangle$, where $|\vec{s}\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle$. Its density matrix is simply given by $\rho = \|\Phi\rangle\langle\Phi\|$, and one can perform a partial trace on it, following the symmetric inner product defined in eq. (2.16), as

$$\begin{aligned}\rho^{(1)} &= \langle 0|0,\vec{s}\rangle\langle 0,\vec{s}|0\rangle + \langle 1|0,\vec{s}\rangle\langle 0,\vec{s}|1\rangle,\\ &= \underbrace{\left(4\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)}_{a}|0\rangle\langle 0| + \underbrace{\sin^2\frac{\theta}{2}}_{b}|1\rangle\langle 1| + \underbrace{e^{i\varphi}\sin\theta}_{c}|0\rangle\langle 1| + e^{-i\varphi}\sin\theta|1\rangle\langle 0|. \end{aligned}\tag{2.18}$$

Since the symmetric inner product is not normalized, we need to normalize this operator to normalize probabilities. Writing $N = 1 + \cos^2\frac{\theta}{2}$, it is straightforward to see that

$$\rho^{(1)} = \frac{1}{2N}\begin{pmatrix} a & c \\ c^* & b \end{pmatrix}, \tag{2.19}$$

in basis $(|0\rangle, |1\rangle)^T$. Schmidt bases can then be derived simply by diagonalizing the reduced density matrix, which in this example gives two eigenvalues

$$\lambda_0 = \frac{2}{N}\cos^4\frac{\theta}{4} \quad \text{and} \quad \lambda_1 = \frac{2}{N}\sin^4\frac{\theta}{4} \tag{2.20}$$

that are the square of the singular values giving the weight of the respective eigenstates

$$\begin{aligned}|\tilde{0}\rangle &= \cos\frac{\theta}{4}|0\rangle + \sin\frac{\theta}{4}|1\rangle,\\ |\tilde{1}\rangle &= -\sin\frac{\theta}{4}|0\rangle + \cos\frac{\theta}{4}|1\rangle.\end{aligned}\tag{2.21}$$

The state can then be written in Schmidt bases as

$$\|\Phi\rangle = (\sqrt{\lambda_0}|\tilde{0},\tilde{0}\rangle + \sqrt{\lambda_1}|\tilde{1},\tilde{1}\rangle). \tag{2.22}$$

The von Neumann entropy can be calculated as $S = -\sum_i \lambda_i \log\lambda_i$, which in the extreme case of $\theta = \pi$ gives exactly one bit of entanglement entropy. That is, in the case of antiparallel spins, given the symmetrization condition, a singlet (or equivalent) state is constructed, with anticorrelation between the two spins or qubits in question. They share one bit of entropy, amounting for their entanglement. Later, in level 8, we shall see that instead of particle exchange of identical particles, other symmetries can be taken into consideration for entanglement production, and it can be evaluated with a similar approach to the presented here.

[4] For exclusion principle to allow them to be on the same state $|0\rangle$ with finite probability, it means they cannot be fermions, but this is not important for this illustration.

2.2.3 ··· EPR paradox and Bell inequalities

But however correlated, qubits A and B are not dependent on one another. If we somehow measure the state of qubit A, we know the state of qubit B will be opposite to it immediately. Nevertheless, it does not imply any transmission. This misunderstanding — that such somewhat fixed correlation would imply immediate transmission for fixing their mutual state — has been source of historical confusion epitomized under the so-called EPR paradox, owing to its first appearance in a publication by Einstein, Podolsky, and Rosen[21]. Einstein has famously called this apparent mutual effect a "spook action at a distance" and casted doubts on the completeness of quantum theory based on some assumption of realism (see section 2.3). Einstein and friends' 1935 argument was relatively simple: take a spin singlet state expressed in eq. (2.10), say an electron and positron pair created from gamma rays, and moving in opposite directions[5]. After some time, when they are far enough, one could measure the electron's spin in one direction, say, passing it through a Stern-Gelach apparatus, which we shall arbitrarily call Z-direction. If we find it in the up-state, we know for sure the positron will be in down-state or vice-versa, and measuring the positron's spin on the same direction should confirm this certitude. However, since such measurements can be performed roughly at the same time, this would imply instantaneous transmission of information! In fact, if we say they are a distance D apart, as long as the time interval between the measurement is smaller than D/c, such action would be superluminal. This faster than light influence, by our current understanding, cannot be, which led Einstein to propose that there should be some unknown variable defining such states from the beginning.

Different people might have had different reactions to such paradox, but in 1964 Bell[22] showed that such hidden variable were not needed and in fact were incompatible with quantum mechanics. For this, Bell assumed a certain variable (or set of variables aggregated in a vector) λ would determine the spin of the particles, as assumed by Einstein et al.. Hence, measurement A depends on $\vec{a}$, some unit vector giving the direction of the measurement, and λ, and similarly does B, i.e.,

$$A(\vec{a}, \lambda) = \pm 1, \quad B(\vec{b}, \lambda) = \pm 1. \tag{2.23}$$

Note that for a singlet state, if $\vec{a} = \vec{b}$, i.e., if the measurement is made along the same direction, $A(\vec{a}, \lambda) = -B(\vec{a}, \lambda) = \pm 1$. Measurements of a spin $\vec{\sigma}$ along direction $\vec{a}$ should have expectation value of $\vec{a} \cdot \vec{\sigma}$, which gives the probability of obtaining measurement $A = +1$. On ket notation, one can express such probability as a projective measurement of $|\sigma\rangle$ on projector $|a\rangle\langle a|$, obtaining $|\langle a|\sigma\rangle|^2$ for the expectation value.Supposing the probability distribution for the hidden variable λ follows a distribution $\rho(\lambda)$, the expectation value of both qubits together become

$$E(\vec{a}, \vec{b}) = \int \mathrm{d}\lambda \rho(\lambda) A(\vec{a}, \lambda) B(\vec{b}, \lambda), \tag{2.24}$$

which should equal to the expectation value

$$\langle \vec{a} \cdot \vec{\sigma_1}, \vec{b} \cdot \vec{\sigma_2} \rangle = -\vec{a} \cdot \vec{b}. \tag{2.25}$$

Nevertheless, that is not the case.

Consider the case where the expectation value E in eq. (2.24) equals to -1 (and notice it cannot be less then -1). For a singlet state, $E = -1$ at $\vec{a} = \vec{b}$ if $A(\vec{a}, \lambda) = -B(\vec{a}, \lambda)$, i.e., along the same direction of measurement a singlet leads to opposite spins and joint expectation value -1 (+1 and -1 each). Hence, we can rewrite P in eq. (2.24) as

$$E(\vec{a}, \vec{b}) = -\int \mathrm{d}\lambda \rho(\lambda) A(\vec{a}, \lambda) A(\vec{b}, \lambda). \tag{2.26}$$

If we add another direction vector $\vec{c}$, it follows that

$$\begin{aligned} E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{c}) &= -\int \mathrm{d}\lambda \rho(\lambda) [A(\vec{a}, \lambda) A(\vec{b}, \lambda) - A(\vec{a}, \lambda) A(\vec{c}, \lambda)] \\ & \int \mathrm{d}\lambda \rho(\lambda) A(\vec{a}, \lambda) A(\vec{b}, \lambda) [A(\vec{b}, \lambda) A(\vec{c}, \lambda) - 1], \end{aligned} \tag{2.27}$$

[5] In this case, one may assume $A = e^-$ (electron) and $B = e^+$ (positron).

remembering from eq. (2.23) that $A(\vec{b},\lambda)^2 = 1$. Hence,

$$\begin{aligned}|E(\vec{a},\vec{b}) - E(\vec{a},\vec{c})| &\leq \int \mathrm{d}\lambda\rho(\lambda)[1 - A(\vec{b},\lambda)A(\vec{c},\lambda)] \\ \therefore |E(\vec{a},\vec{b}) - E(\vec{a},\vec{c})| &\leq 1 + E(\vec{b},\vec{c}),\end{aligned} \tag{2.28}$$

since the distribution $\int \mathrm{d}\lambda\rho(\lambda) = 1$ and the second term follows the definition in eq. (2.26). Equation (2.28) in known as *Bell's inequality*, and has a deep and important meaning we shall discuss. If quantum theory is taken into consideration, $E(\vec{a},\vec{b}) = \vec{a}\cdot\vec{b}$, but it is easy to consider an example where $\vec{a}$ and $\vec{b}$ are orthogonal, with $\vec{c}$ making 45° with both others, so $E(\vec{a},\vec{c}) = E(\vec{b},\vec{c}) = 1/\sqrt{2}$. Immediately, we see a clear violation of Bell's inequality, which would lead to $2^{-1/2} \leq 1 - 2^{-1/2}$. Notice that, in general even if $\vec{c}$ deviates and angle θ from $\vec{a}$, lying between $\vec{a}$ and $\vec{b}$, one would still have a violation of Bell's inequality.

What Bell's inequality tells us is: if we assume the existence of a hidden variable governing the outcome of experiments in order to save realism in the theory (i.e., assume each inner state is predetermined by extra variables we do not see), we are left with an inequality that is *not* observed in experiments. There is *no way to conciliate* quantum theory and its experimental observations with additional invisible information of quantum states.[x]

After Bell's work, other similar relations were explored in different contexts [23–26].[xi] Among these, the Clauser-Horne-Shimony-Holt (CHSH) inequality is often used, for it can be seen as some kind of generalization of Bell's inequality. Using the same notation as that used for Bell's inequality, CHSH inequality can be expressed as

$$|E(\vec{a_0},\vec{b_0}) + E(\vec{a_0},\vec{b_1}) + E(\vec{a_1},\vec{b_0}) - E(\vec{a_1},\vec{b_1})| \leq 2, \tag{2.29}$$

where the subscripts indicate different directions of measurement. For quantum theory, it is possible to prove that the quantum upper tight bound for the CHSH inequality gains a $\sqrt{2}$ factor, as derived by Tsirelson what is now know as Tsirelson's bound [27]. A simple explanation, following ref. [4], can be given using the commutation relations of Pauli matrices. We first write

$$C = \alpha\beta + \gamma\beta + \alpha\delta - \gamma\delta, \tag{2.30}$$

assuming they either commute or anti-commute, i.e. $[\alpha,\beta]_\pm = [\beta,\gamma]_\pm = [\gamma,\delta]_\pm = [\delta,\alpha]_\pm = 0$. Hence, one can write

$$\begin{aligned}C^2 &= \mp 4 + [\alpha,\gamma]_-[\beta,\delta]_-, &(2.31)\\ ||C^2|| &\leq 4 + ||[\alpha,\gamma]||\,||[\beta,\delta]|| \\ ||C^2|| &\leq 4 + 4||\alpha\gamma||\,||\beta\delta|| \\ \therefore ||C|| &\leq 2\sqrt{2}. &(2.32)\end{aligned}$$

This can be understood by noticing that when squaring C, terms like $\alpha\beta\gamma\beta$ cancel out with terms like $-\alpha\delta\gamma\delta$, whether these operators commute or anti-commute, leaving only a $\alpha\beta\gamma\delta$ term. Since qubits span Bloch spheres represented by Pauli matrices, we can notice that the correlation C coincides with the correlation calculated in CHSH inequality (2.29).

Including the singlet state, there are four states that violate Bell's inequality and give maximum quantum correlation of $2\sqrt{2}$ by CHSH formula. These states, collectively called Bell states, are

$$\begin{aligned}\|\psi_-\rangle &= \frac{|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B}{\sqrt{2}}, \qquad \|\psi_+\rangle = \frac{|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B}{\sqrt{2}}, \\ \|\phi_-\rangle &= \frac{|0\rangle_A|0\rangle_B - |1\rangle_A|1\rangle_B}{\sqrt{2}}, \qquad \|\phi_+\rangle = \frac{|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B}{\sqrt{2}}.\end{aligned} \tag{2.33}$$

These states are *maximally entangled* and are vital for a any application of quantum mechanics to information processing, as we shall see. State $\|\psi_-\rangle$, the singlet state, is actually the only fermionic state and the one described by the dragons in the text.

A different way to to look at the CHSH formula appears in the CHSH game (see ref. [28]). We can present it as:

The CHSH game Alice and Bob are kept apart from one another without any form of communication to play a "divination" game. Every sharp minute they flip a coin, assigning 0 to heads, 1 for tails. Without know the partner coin flip result, their challenge is to produce a bit of their own for each coin flip in the following conditions: if both coins have tails (1,1), they should give opposite bits, (1,0) or (0,1); otherwise $\{(0,0), (0,1), (1,0)\}$, they should produce the same bits (1,1) or (0,0).

The CHSH game can be equivalently stated as: Alice and Bob bits' XOR should equal the coin bits' AND operation. Classically the winning strategy is to aim at the 3/4 of the combinations and always produce the same output, either (0,0) or (1,1). However, if Alice and Bob share a pair of Bell states in eq. (2.33), it is possible to overcome the 3/4 probability limitation by applying the intrinsic nonlocality to improve their chances. The basic idea is to couple their output to projective measurements on different bases of the shared Bell state, say, a singlet state. Since a singlet state has an intrinsic correlation, it becomes a sort of "medium" to share correlations between Alice and Bob. For example, Alice can couple her result to her qubit, measuring it on basis Z if her coin bit is 0 (heads) and on basis X for coin bit 1 (tails). Her output qubit is just the output of the measurement. Similarly, Bob can do the same for bases $(Z+X)/\sqrt{2}$ and $(Z-X)/\sqrt{2}$. Taken together, their winning probability becomes $\cos^2(\pi/8) \approx 0.85$, roughly 10 % higher than the classical case. It is straightforward to understand this if one things in terms of light polarization. The Z basis corresponds to light linearly polarized along xy axes, one being "up" and the other "down" spin. The X bases correspond to these axes rotated 45°. Hence, every measurement pair makes a 22.5° mutually, except when one need anticorrelation, when the angle between measurements becomes 115.5°. This gives a "light intensity" of $\cos^2(22.5°)$ in all scenarios.

This game has been played to demonstrate quantum mechanics validity, with first conclusion in 1972 by Freedman and Clauser [29]. However, the experiment is complex and so-called *loopholes* may remain, preventing any definitive conclusion. Recent loophole free measurements were executed independently in three different places around the world in 2015 [30–32]. These experiments have been designed to avoid possible loopholes that may persist when measuring the CHSH inequality break, like locality and detection loopholes (cf. [28]). The detection loophole refers to the limitations of detection. Detectors may fail conclusive detection due to limited efficiency: false detections and lost events may happen and poison the results. The locality loophole revolves around the necessity of having measurements space-like separated. The measurement bases choice must be made after the entangled pair generation, i.e., during particle flight, and measurement must start and finish before any signal transmission may happen. There are subtleties involved here, for local models may be anisotropic, in the sense that correlation may arise in a given direction that the experiment detects. Other loopholes may involve statistical issues and errors, but the above two can be said the major concerns.

2.2.4 ··· *No to doppelgangers, yes to teleporters*

As basic knowledge from level 1, quantum transformations are represented as unitary transformations. This fact has many implications, one of which plays an important restriction to our interests. It is our objective to someday use quantum states for information processing through quantum computers. Copying information from a source to a target is a very standard operation in classical information processing, but it is actually forbidden in quantum theory, what is know as *no-clone theorem*[xii].

To prove that we cannot copy an arbitrary quantum state is a straightforward and simple task. Just suppose we could and see the "kernel panic" you will run into. We need to start from a state $|\psi\rangle|0\rangle$ where the second ket is a target memory to hold a copy of the first ket. After the copy, we must have $|\psi\rangle|\psi\rangle$. Such transformation must be realized with unitary transformation, so one can define a "copy operator" $\hat{C}$. Its action must become

$$\hat{C}|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle. \tag{2.34}$$

The same can be written for another arbitrary state $|\phi\rangle$, i.e. $\hat{C}|\phi\rangle|0\rangle = |\phi\rangle|\phi\rangle$. We can now take the inner product of these two arbitrary states and compute them as

$$\begin{aligned} \langle 0|\langle\phi|\hat{C}^{\dagger}\hat{C}|\psi\rangle|0\rangle &= \langle\phi|\langle\phi|\psi\rangle|\psi\rangle &(2.35)\\ \langle 0|0\rangle\langle\phi|\psi\rangle &= \langle\phi|\psi\rangle^2 &(2.36)\\ \langle\phi|\psi\rangle &= \langle\phi|\psi\rangle^2. &(2.37)\end{aligned}$$

The only solutions for this equation are $\langle\phi|\psi\rangle = 0$ and $\langle\phi|\psi\rangle = 1$, which means that such copy operations are only allowed if states $|\psi\rangle$ and $|\phi\rangle$ the same (inner product equals 1) or orthogonal like $|0\rangle$ and $|1\rangle$. In other words, we cannot take state $|\psi\rangle$ arbitrarily as we supposed in the beginning, but it must be a somewhat special state. Therefore, arbitrary and/or unknown quantum states cannot be cloned! No quantum copy is allowed, unless it is the copy of our computation bases $|0\rangle$ and $|1\rangle$.

This could be heartbreaking for the naïve, but not to a quantum theorist. While it is true that we cannot indeed copy quantum states, there is a workaround intrinsic to quantum theory: quantum teleportation. You cannot copy a state from source to target, but you can *teleport* it from source to target. That is, you can transfer the information from Alice to Bob, $|\psi\rangle_A|0\rangle_B \to |0\rangle_A|\psi\rangle_B$. The key factor that allows such unfamiliar information juggling is *entanglement* – a valid hint to look at it as a resource already.

Let us see how information can be teleported from Alice to Bob. First, they need to share an entangled state, say $|00\rangle + |11\rangle$, while Alice has a state $|\psi\rangle$. Hence, using subscripts $_A$ for Alice and $_B$ for Bob, the whole system starts as

$$\begin{aligned} &|\psi\rangle_A|0\rangle_A|0\rangle_B + |\psi\rangle_A|1\rangle_A|1\rangle_B &(2.38)\\ &= (\alpha|0\rangle_A + \beta|1\rangle_A)|0\rangle_A|0\rangle_B + (\alpha|0\rangle_A + \beta|1\rangle_A)|1\rangle_A|1\rangle_B. &(2.39)\end{aligned}$$

Using quantum information terminology, we may apply a CNOT gate – i.e., control-not gate – on Alice bits, where the first qubit is the control and the entangled one the target. A CNOT is equivalent to a summation modulo 2 of the control bit to the target bit, often represented as "$\oplus$." Hence, after its action, we obtain

$$\alpha|0\rangle_A|0\rangle_A|0\rangle_B + \beta|1\rangle_A|1\rangle_A|0\rangle_B + \alpha|0\rangle_A|1\rangle_A|1\rangle_B + \beta|1\rangle_A|0\rangle_A|1\rangle_B. \tag{2.40}$$

Our objective is to isolate the state $\alpha|0\rangle_B + \beta|0\rangle_B$ in order to recover $|\psi\rangle$ in Bob's hands. For this, we perform the so called Hadamard transformation or Hadamard gate to Alice's control bit. This transformation acts like $|0\rangle \to |0\rangle + |1\rangle$ and $|1\rangle \to |0\rangle - |1\rangle$, and it is unitary and hermitian (its own inverse). After its action, we obtain

$$\begin{aligned} &\alpha(|0\rangle_A + |1\rangle_A)|0\rangle_A|0\rangle_B + \beta(|0\rangle_A - |1\rangle_A)|1\rangle_A|0\rangle_B\\ &+\alpha(|0\rangle_A + |1\rangle_A)|1\rangle_A|1\rangle_B + \beta(|0\rangle_A - |1\rangle_A)|0\rangle_A|1\rangle_B\\ &= |0\rangle_A|0\rangle_A\underbrace{(\alpha|0\rangle_B + \beta|1\rangle_B)}_{|\psi\rangle} + |0\rangle_A|1\rangle_A\underbrace{(\beta|0\rangle_B + \alpha|1\rangle_B)}_{X|\psi\rangle} +\\ &|1\rangle_A|0\rangle_A\underbrace{(\alpha|0\rangle_B - \beta|1\rangle_B)}_{Z|\psi\rangle} + |1\rangle_A|1\rangle_A\underbrace{(-\beta|0\rangle_B + \alpha|1\rangle_B)}_{XZ|\psi\rangle}. &(2.41)\end{aligned}$$

Notice that, as long as we know what state is in Alice's registers, we know which state Bob has, thanks to entanglement. And we can use this information for the state $|\psi\rangle$ to appear in Bob's registers, no matter the distance between Alice and Bob: the state teleports to Bob's equipment.

2.3 Locality, realism, and other words of volatile meaning

In section 1.3, we had some comments on language and physics, largely sustained on Weizsäcker and Wittgenstein. In this level, language becomes a delicate issue, for many words assume new meaning to serve better quantum theory, other words must be created anew. And just like some equivalent logic assumptions become no longer equivalent, some synonyms cannot be made synonyms anymore. Therefore, regarding the content on this level, we need to organize the language we use at least to some extent, pointing out words' difference and highlighting what words quantum theory softens. This level has, overall, a strong influence of Asher Peres [4] and his discussions on quantum theory. We shall try to go beyond its contents, but first it is instructive to set our common background for clarity.

We address here the case of two particles, a natural complication departing from the single particle problem we introduced before. Consequently, the first thing we need to clarify is the difference of having "two particles" and a "pair of particles." Essentially, it shall boil down to the difference between "one" and "two," or "*one* pair" and "*two* particles." This may be enlightening when thinking about entanglement of two particles, for instance.

First, let us take the linguistic point of view. Whenever one says there are "two particles" it is implicit that their existence is independent, each one defined in their own individuality, and that they can be *counted* one-by-one, amounting to, in this case, two. The same is valid for whatever the number n of particles one has. As so, each one might be expected to behave like a single particle when taken isolated, and all the ideas discussed in the first level apply to each of them separately. We therefore have a separable, unentangled state of two bodies.

On the other hand, a pair of particles or *one* pair of particles binds this pair, this "two" into "one." They are not defined nor seen separately, but as a whole, as a union. And entangled pair, like a singlet state of electron or a Bell state of photons, should be expected to be addressed in these terms. Those are not two independent particles that live apart from one another, but one single body extended through space. A pair of entangled photons that share the same polarization and propagate to opposite directions should have been, at some point, bound together to one another and then followed different paths (they may have been generated together by spontaneous parametric down-conversion, for instance). We can see them as a single object with some kind of internal structure that happens to be two photons. Measuring only half of the pair, in this view, amounts to breaking the non-local object into two pieces, two particles.

This view may be questioned, but it offers a perspective which may be useful later on. For now, it offers a background on where to place some definitions that interest us. We may start by discussing the word *locality*. One sense we may give to the word, that something is "local," may be defined based on the presence of a hidden variable, like those denied by Bell's theorem. In this case, two particles may have a local character, each one being defined by their own parameters whatsoever, but a pair of particles must be a nonlocal object, for no hidden variable is allowed to exist behind their correlation.

Another word worth anyone's attention is *causality*. This may also have tricky interpretations, but we shall focus on causality as defined by the dependence of one event on another. Events which share come kind of causal relation cannot be too far apart, viz. not farther than light can travel between such events. In other words, we will say we have a causal theory or that causality is preserved as long as no physical transmission occurs faster than light. This is also known as *no-signaling*, for a superluminous transmission is called "signaling."

But there are words much harder to give a single definition, and even words whose definition has changed over time. *Realism* can be said such a word. Einstein's realism is somewhat different from what many people would call realism nowadays. Bitbol [33] has provided an elaborated discussion about "the real" and different views on it. As I understand the matter, "real" is a word that we do not completely grasp its meaning, but we cannot seem to afford throwing it away and therefore update its meaning on convenience. Without entering the merit of whether we should or should not hold to it too much, the human psyche often looks for solace on different places and ideas, and "reality" has its soothing role to which we hold, giving safety-feeling grounds to ourselves while we loose our holds on a plethora of other enigmas. While for some in the past a particle being "here" or "there" but not "both" could be "realism," this is not what we will rely upon. We shall consider a particle or theory "real" whenever there is a non-empty set (but perhaps a single element) os observable parameters that can unambiguously characterize its existence. For example, we may say so of an electron for, even if we do not know where it is, its charge and spin are well defined parameters that characterize it. We can say that quantum theory is "real" in the sense that it always handles particles based on some characterizing parameter that can be fixed: charge, frequency, spin, or whatsoever. On the other hand, under this definition, if a particle (or theory that allows such particles) may never have a well defined parameter to be pinned down (i.e,, all the possible parameters are ill-defined and fluctuating, always leading to a different, unpredictable result), this would be rendered "unrealistic." But I must say this may not be a good view either. There are theoretical considerations of (electric) transport without particles in some materials, which rely on fractional calculus (cf. [34, 35]). There is no "excitation" or "particle" to transport anything, yet there is transport. Should it be considered, under our perspective, *realistic*? I am afraid I cannot answer this question.

And perhaps the hardest question to answer at this point, from the discussions in the text, is: what *is* entanglement? I see it as a question on par with "what is energy," "what is entropy," "what is information," and "what is gravity." Which means that I cannot answer this question either, but I can expose some ideas on the topic. In physics, these questions are not often addressed, for it can be too complicated to delve into them. Energy is perhaps the most fundamental concept in physics, with definitions like "capacity to do work" to try to silence the question, but it hardly tells us anything about the nature of energy. This is, in some sense, inevitable. There are concepts that must become our foundations of thought and discussion, concepts that make a language. Almost the whole physics is defined ultimately in term of energy, if not all. As so, we leave it as building block for ideas, and do not try to define it further, but rather see it through different angles, trying to understand the whole universe together with it.

Entanglement, entropy, and information are concepts linked to one another deeply. We call entropy the information to which we have no access. And entanglement, if defined by entanglement entropy, become the information shared within a system but not directly accessible to us. Yet, much like energy, it becomes a resource that can be used if properly manipulated. By shifting entanglement between different parties, we can have some sort of information flow, like the qubit teleportation discussed. Entanglement, then, becomes the source of the idiosyncratic information paths we have in quantum theory that allows for quantum computers and quantum information processing to be so powerful. And not only technological applications, but nature itself seems to use, as we see entanglement flourishing in different fields, like new complex (topological) materials, many particle models, black holes, and so on.

In this view, I cannot offer a simple definition of entanglement[6] beyond saying that it presents itself as some sort of intrinsic correlation between quantum entities (particles, qubits, fields, etc.), but it is indispensable for our understanding of the quantum worlds. I would point, though, that as we see energy disperse itself (classically), entanglement may also "disperse." Its survival depends on the isolation of the entangled players just like energy conservation does. As energy may run away through different paths, entanglement may "reinvent" itself by entangling formerly entangled partners with new ones. And lots of physical discussions go around it, like the structure of black hole and their horizon being "firewalls" or not, testing gravity's quantumness, evaluating topological order in materials, and so on. So, let us just see it as some kind of nonlocal correlation intrinsic to quantum systems that is not clearly visible to us, but pulls the strings of nature in more ways than we can imagine, while we struggle to understand it (or not).

Notes

[vii] "What is a man? A miserable pile of secrets." This phrase became famous through the game *Castlevania: Symphony of the Night*, and was originally coined by the French writer André Malraux. When thinking of dragons or any quantum particles, a pile of secrets seems the right words to describe them; but truth be said, the more we investigate the *secrets*, the less we find the *pile*.

[viii] This is easy to understand through diagonalization, readily done by taking the square of ρ. Letting $\vec{r} = (x, y, z)$, the eigenvalues become $\frac{1}{2} - \frac{1}{2}\sqrt{x^2 + y^2 + z^2}$, which must be greater than zero. Hence, $\sqrt{x^2 + y^2 + z^2} \leq 1$, the set of points in a (Bloch) sphere.

[ix] Although the wedge symbol is playing the role of defining the external product (borrowed from external algebra), note that it does not necessarily anticommute, only in the fermion case it happens. This product is non-separable, hence it actually has nothing to do with the usage of wedge in differential geometry or geometric algebra[36]. It is true, however, that this wedge product coincides with Penrose's wedge product for fermion [37], and since at this level we are still mainly considering fermionic behavior, the wedge works as a good enough mathematical sign for operations herein. Nonetheless, it

[6] And there is more than one way to calculate it too. For instance, von Neumann entanglement entropy was defined in the text, but there is also Renyi entanglement entropy, for example

is also worth to mention that this symbol will not play any special role in calculations throughout this thesis, and could just as well be set to anything else. In fact, you can even forget it was here in first place, and ignore this note altogether.

[x]There is a tricky point in Bell's inequality derivation worth mentioning. When defining directions $\vec{a}$ an Euclidean, flat space seems to be tacitly assumed. In other words, when we think of a singlet state, we assume perfect anti-correlation along a certain axis, say, $\vec{z}$-axis. We suppose the existence of some translation symmetry and therefore this $\vec{z}$-axis can be simply slid to the correlated partner to define an opposition of spin. However, what happens when the space in question is *curved*? For instance, what happens if we transport entangled photons through many kilometers of optical fiber on Earth's surface? If entanglement is still there, how does the "$\vec{z}$-axis" change? Does it follow the deformed geometry of such space and aligns with the gravitational field? Notice that, if we consider the gravitational space deformation of a sphere, we may consider at least three paths for the entangled pair: (a) along a static sphere; (b) along a meridian containing the rotation axis of a rotating sphere; and (c) in the plane of rotation of a rotating sphere. Each of these should affect the torsion of the "$\vec{z}$-direction" differently. But more importantly, one may ask whether entanglement may survive such torsions or not. Gravity, as a metric distortion, may induce decoherence [38, 39]. Therefore, entanglement and coherence along long distances of a gravitational field may require trickier arguments than those put forth by Bell.

[xi]Kochen-Specker theorem [23] has introduced the important idea of contextuality, which we will not address in this work (see also Peres [4], chapter 7). Essentially, we call a theory like quantum mechanics *contextual* when a measurement A outcome cannot depend on A alone, but whether A is performed alone, or with a another measurement B or C. Kochen and Specker showed unless a set of measurements like these all commute, it is impossible to attribute a measurement value to all of them. For instance, even if A commute with B and C, if B and C do not commute, trying to make A's value independent of context leads to contradictions (cf. [40]).

[xii]How the word "clone" was chosen is unknown to me, though it has an obvious meaning. Probably, the term "clone" has a more sci-fi flavor to it than the lame term "copy" and make a good alternative. I prefer to call it "lack of quantum doppelganger" instead, but surely "no-clone" is succinct and delivers the message, actually making it a rare case of good naming.

Level 3
Out in the Fields

Behold your music! This is your minstrelsy; and each of you shall find contained herein, amid the design that I set before you, all those things which it may seem that he himself devised or added. And thou, Melkor, wilt discover all the secret thoughts of thy mind, and wilt perceive that they are but a part of the whole and tributary to its glory. — J. R. R. Tolkien, *The Silmarillion, Ainulindalë*, 1977.

3.1 The Choir

Poor creature! If it thought that being alone was bad, or that making a couple was tough, it had no idea of what the world really was. The dragon may take a walk in the fields, but it can't walk freely anywhere. The hills and valleys it walks over are god-like *atoms*, pillar entities powering their world. They sing, they chant, they shout, they yell, they thrill in delusion. They bind together to bend and twist worlds, to form and close paths as their selfishness pleases. A band of brethren wrought in resonant voices for pain and sorrow, joy and happiness, but never ever freedom. They are groping godlings[xiii] grokking guardian grandiosity. Powerful dudes, in other words.

Their music makes a holy sound fabric. Once their choir is together, they never seize singing. Their song manipulates dragons' paths, guiding through some ways, avoiding other ones. Their chant swirls vortices and winds. Their voices raise mountains and flatten hills. Their cry carves valleys and fills oceans. But enough talk, let their tone be adjusted and listen their tune.

3.1.1 · · · Atomic tempo

I

They move, they fly!
Their hearts, agile,
Pump fire by
Modes gone sterile.

They growl, they fight!
With claws, acute,
Clos'd so tight,
On ev'ry wall screech.

Alone they rise!
They climb the hills
Where no demise

Their ego fills.

Alone they dive!
Deep into valleys,
Corners o' hive
Where each one rallies.

They stare, they grapple!
And scatter 'round,
And echo rattle,
Their battling sound.

They may, indeed,
On unknown terrain,
Be blind by greed
And nothing retain.

They may, however,
With steps o' pride,
Shift swing as ever
Along recurring stride.

So let them jump!
Let 'em know sites
Of dips and humps,
Repeating on sight.

So let them cling!
Let pair'd couples,
On frost, their wings
Reversely cuddle.

II

Forfeit fire!
For few fey fulfill
A fine forgetful foe
In fierce form,
feral forge forbidden.

Seize secrets!
Sow the seeds o' sin,
Sew through sewers
Still signs n' symbols
Sacking sim'lar souls.

Scatter sound!
Some stings so smoothed
Smack stunting spirits
Strong enough strands
Squeaking stiffly.

Shoot shivers!
Shake shunt shimmerings
Shielding short shores
On shoulder shafts
Shoving shackles.

Do you like what you listen? Keep listening, then, and we shall meet again later.

3.2 Where the multitude gathers

The Choir singing in the previous section is the *crystal* or equivalent matrix where electrons resides. They sing a lattice potential. Their song, therefore, creates the whole landscape where the dragon lived in previous levels. A way to understand it is by considering Bloch theorem. That is, an electron or other particle's wavefunction moving through a periodic potential of period L obeys

$$u(x) = e^{ikx}u(x+L). \tag{3.1}$$

This can also be understood as the origin of bands, which provide the landscape for electron movement. Though the bands are extracted from the wavefunction often ignoring the phase factor in front, in level 5 we will see geometric phases, where the phase factor has important contributions to the physics of the system.

Besides the lattice potential, Canto I also pairs up electrons. In other words, they bind together particles that would otherwise be more or less single. This happens, for instance, in superconductors, following Bardeen-Cooper-Shrieffer (BCS) theory (see [41]). In BCS theory, electrons pairs up via phonon interaction, i.e., lattice deformation. It is this lattice deformation that is written as a chant above.

3.2.1 ··· Multipartite correlations: Expanding Bell

Canto II of the choir extends the entanglement between two particles to n particles [26, 42–44]. In level two, we visited CHSH inequality

$$|E(\vec{a_0}, \vec{b_0}) + E(\vec{a_0}, \vec{b_1}) + E(\vec{a_1}, \vec{b_0}) - E(\vec{a_1}, \vec{b_1})| \leq 2, \tag{2.29}$$

that gives bounds to quantum correlations. It has been later extended to three and finally n particles [42, 43]. Independently derived, Collins et al. [42] use Mermin-Klyshko polynomials [45, 46] to construct a general Svetlichny polynomial S_n. Meanwhile, Seevinck and Svetlichny [43] worked a straight generalization of previous Svetlichny work [47].

Before visiting n-body correlations, let us clarify the difference between "separability" and "entanglement."Given a bipartite system, if we cannot represent it as a direct product of the two parts, we say we have *entanglement*. Nevertheless, we may be talking about a many-body system whose parts may also be many-body. As such, there may be entanglement with a subsystem, but the subsystem is not itself entangled with the remaining. Mathematically, a system $\|\psi\rangle = \|A\rangle\|B\rangle$ is *separable* into systems A and B, but A or B may themselves be non-separable and entangled. Hence, state $\|\psi\rangle$ *is* entangled, but also separable. If there are n components to this system, we indeed do not have n-body entanglement or n-body non-separability, but k-body ($k < n$) entanglement might still be present.]

Collins et. al depart from Mermin-Klyshko polynomials defined as

$$M_n = M_{n-1}(a_n + a'_n) + M'_{n-1}(a_n - a'_n), \tag{3.2}$$

with $M_1 = a_1$ and M'_k being obtained by exchanging the primed and nonprimed a, these being bases to measure their respective qubits, each leading result ± 1. One can readily confirm that M_2 gives the CHSH correlation formula. Then, they define a generalized Svetlichny polynomial S_n as

$$S_n = \begin{cases} M_n, & n \, \text{even} \\ M_n + M'_n, & n \, \text{odd}. \end{cases} \tag{3.3}$$

On the other hand, Seevinck and Svetilichny define operators S_n using the sign factor $\nu_k = (-1)^{k(k+1)/2}$ as

$$S_n = \sum_I \nu_{t(I)} a_{i_1} a_{i_2} \dots a_{i_n}, \tag{3.4}$$

where $a_{i_k} \in \{a_i, a'_i\}$, $I = (i_1, i_2, \ldots, i_n)$ is a string indicating the presence or absence of prime (0/1), and $t(l)$ counts the number of primes in I. It follows the recursive relation

$$S_n = S_{n-1}a_n - S'_{n-1}a'_n, \tag{3.5}$$

where S'_n can be obtained by interchanging the primed and nonprimed operators. Since S_2 corresponds to CHSH correlator, it is also evident that

$$\begin{aligned} |\langle S_n\rangle| &\leq 2|\langle S_{n-1}\rangle|\,|a_n| = 2|\langle S_{n-1}\rangle| \\ \therefore |\langle S_n\rangle| &\leq 2^{n-1}\sqrt{2}. \end{aligned} \tag{3.6}$$

$\langle S_n\rangle \equiv \langle\psi\|S_n\|\psi\rangle$ is the expectation value of the operator S_n and it gives the correlation between n particles. It is worth to remember that such correlation is a statistical value, therefore a single event (or measurement) cannot return the same value and is always susceptible to the inherent statistical uncertainties.[xiv]

3.3 Emerging solitude?

In this level, many-body systems came into our sight, but we have gotten a different yet already familiar performer in our final stage: a "single" body. In other words, we look at a single object which may really be one of the initial components or a collection of them (a collective excitation) and pretend it behaves like a single-body moving through a certain landscape, just like our dragon was at first.

In physics and in many other fields of study, we stumble at some phenomena one may refer to as emergence. Not to be confused with "emergency," "emergence" indicates phenomena that "emerge" from a larger background, group, or any collective set. On such situation, people often talk about "emerging phenomena." In physics, various collective excitations would be a classic example, but one may also mention "emerging symmetries," "emerging gauge fields," and other complications. In society, one can say that culture is also an emerging property of a community, something shared by many, but that does not make sense when defining for a single individual. Market is also an emerging phenomenon from human societies, and it may also host other emerging phenomena within itself. But here, we want to focus on a particular case of emergence: solitude.

First, from a rather trivial philosophical perspective, one *needs* a group of individual to define solitude. There is no real solitude if there is only one and absolutely one individual, for there is no need for a word to describe the only possible state it can achieve. Solitude is a concept that depends on a group of individuals to be be defined: there is only solitude for the one who is *not* within a group when *a group* exists.

But what in the world has this to do with physics or nature? By now, the answer should be obvious. In physics, we conceive solitude for our objects of study so we can analyze them without outer influence and try to chase their essence. We suppose an electron alone within a potential landscape, but this solitude can only be authorized because we have the landscape to account for the presence of Others in first place. Yet, this new electron particle has different aspects than the original electron surrounded by companions. For electron, we normally obtain a different mass, as well as different g-factor. These are a result of all the various drags and pushes resulting from the interaction with other electrons and various atom nuclei. It has its mass *renormalized* to fit its "heaviness" as different inertia. It is as if our dragon thought of itself alone, but actually had other many creature lingering around its body in blind spots.

We have now an idea that the single-body electron is not as single as it may seem. But this lonely electron moving through Bloch bands: how much of an electron it still is? The answer obviously depends on how one would choose to define an electron. If one were to choose a strict definition based on its charge/mass ratio, than it would definitely no longer be an electron, but an "emerging electron-like excitation," with same or similar charge but very different (effective) mass. However, do we need to embrace such an Aristotelian essence to an electron on so strict terms, but even if we do, why to assume that the mass of an electron constitutes its *essence*? If a cat an be thin or fat and still be a cat, why not an electron too? But for that to make sense, we must first agree on some standard to define an electron.

In physics, as in philosophy, we try to look for what is mutable and what is not in order to properly define fundamental things. A cat's appearance may vary a lot, but they will still keep some underlying physiognomy that helps us to identify them. Things are not much different in intuition when it comes to physics. But since there is no physiognomy to look at, the footprints we chase are different. First, we fundament our thought on *conservation laws*. They have been associated with symmetries since the revolutionary work by Noether in what is known as Noether theorem: continuous symmetries imply conservation laws. One such symmetries is the so called *gauge symmetry* in electrodynamics, perhaps more readily recognized as its related conserved quantity: charge. Charge conservation is perhaps the most intuitive conservation law of all (even though gauge symmetry is possibly the hardest symmetry to grasp). Hence, charge matter a lot when defining an electron: it cannot simply change. Then, the charge of a particle becomes a strong evidence of its identity, specially for electrons. Similarly, spin also helps to identify such particles.

Nonetheless, we can skip more arguments to save the electron and ask: what about muons? Are they not simply "fat electrons" with more mass? If mass is not used to define a particle, what then? Well, it is tricky, but the point is whether the particle in question is really isolated in vacuum (at least hypothetically) or not. Or, put differently, what is the vacuum of this particle? Electrons in materials are not in vacuum, but we pretend they are by taking all the other particles and throwing them in the background, calling is "vacuum." These kinds of vacua create a different background, a different "earth" or "ground" to define an electron. In this case, thinking of mass as a strict tag is not a good idea for energy conservation cannot be completely guaranteed but become conditioned to the considered vacua. For example, a gate operation on the top of the material may change the vacuum energy.

So is the lonely electron amidst the multitude really just a dressed electron? Does its essence change according to its background or vaccum? Well, I'm afraid the answer is *yes*. There is this thing we call an electron in vacuum that binds to other particles and changes its behavior but is still there. It's charge cannot change, but other things might. So it's presence is still there, its existence is identified first, and on the top of it we call it an electron[1].

But indeed, it could be different. I may happen (and apparently it does) that not one electron becomes heavy and sluggish but a collection of particles moving together create what looks like one particle in a greater scale. These are collective excitations that we shall discuss more later. But whether an electron or a collective excitation, we see that the single-particle picture emerges from a many-body setup. The single particle considered emerges with specific characteristics from a background together with its vacuum. It is the simplest emerging existence we identify in labs and try to probe both experimentally and theoretically. But while it may be tempting to say that their existence precedes their essence, it is an assertion that needs care. In some cases, they are supposed not to exist. That is, there are considerations of transport without particles [34, 35]. In this case, an unparticle sector with a running charge exists exists. But one could argue that the concept of a particle as a quantized unit for transport (e.g. of charge or mass) can still be ideally defined in order to rebuke its existence. For such considerations, no existentialist precedence can be recognized.

Notes

[xiii] Godlings appear in recent literature and games. Here, the word is related to goblins and to lordlings. Hence, it may be understood as child gods, children of gods, pitiful gods, weak gods, insignificant gods, pesky gods, and so on. Their power to construct worlds and matter keeps their godlike power, despite their minute size and significance when alone.

[xiv] This has important implications in quantum theory, often related to *counterfactual* traps. That is, when we make assumptions that simply cannot be realized, we often end up in some kind of paradox. For instance, the correlator S_n hosts a trap of its own: can we *actually* measure the outcome of many particles though different apparatuses, without changing their state (i.e., without acting with a different operator and changing their context)? While *in principle* possible, it can be a very challenging feat in a laboratory. Still, at least for the proposal of the correlator and the evaluation of its bounds under ideal conditions, it should pose no barrier to us. Whether the same measure of correlation can realistically be implemented in a room with a physical system is a different matter on its own merit.

[1] Perhaps an existentialist one.

Level 4
Dragons and Fey

Come thou, let us begin with the Muses who gladden the great spirit of their father Zeus in Olympus with their songs, telling of things that are and that shall be and that were aforetime with consenting voice. Unwearying flows the sweet sound from their lips, and the house of their father Zeus the loud-thunderer is glad at the lily-like voice of the goddesses as it spread abroad, and the peaks of snowy Olympus resound, and the homes of the immortals. — Hesiod, *Theogony*, translated by Hugh G. Evelyn-White, 1914.

4.1 Fairy magic

So we meet again! I hope you enjoyed the choir! I didn't introduce myself before, for you just needed to hear my voice for a while. Sorry, I have this habit. In fact, my kind has this habit. I am, or we are, *fey*. Fey, feypeople, fairies, y' know. We like how you humans have many names for the same thing, we really love it. And I know you saw the dragons, but if you are traveling through our world, you need to know about us too, y' know! For we are everywhere! Come, lemme show you!

We are not territorial like those dragons. In fact, we like to cuddle together! And they may ignore us, but they live with us! We are born from them and they can be born from us!

Yes, yes! Leave the will-o'-wisps to illuminate your way with winking lights! Leave all wishes to us! We will wrap their wings and lead their flight! We leave their lungs and warm their lobes.

No, no! Follow us! Fly with phoenixes to heat your heart! Feel their harmony as you feel us! Forget the flashes and freely flock with us! For when frost haste we freeze their pair up![xv]

Wait, wait! We hear something! It's fey in pitch and draconic in instinct! It's dragon in accent and fey in sentiment! We know it well, we know it all! The Dragon-Fey or Fey-Dragon who knows all way!

4.1.1 · · · The Dragon-Fey

Quetzalcoatl, Seiryū, Faerie Dragon, The Olde Dragonlord, The Kindled Keen Kin. Many are the names, one is its lore-meister: me.

I have long lived and watched in silence the world of my friends and pupils. They are proud of their draconic ancestry, proud of their fey features, but they have long forgotten that deep inside we are all the same. I can still conjure their similarities at will.

Dragons!
Reckon my will!

Fey!
Obey my voice!

The invoker of souls,
The conjurer of bodies
Call that within you!
Let the susykin thrive!

I normally avoid susykin-spells, for I know the risks. Better, let them hide broken. But since you are here, dear visitor, I shall show you them, the partners of dragons and fey. Look closely. No, your eyes do not deceive you. The partners of dragons and fey are fey and dragons. For under susykin-spell, dragons are fey and fey are dragons! You humans believe I juggle with superpowers to call upon them, which might be why you call them super-partners. You say they are related by supersymmetry, but for me there is no absurdity in connecting them. For dragons and fey are prettier together them in pointless fight. Condense them a little, and you will know their might!

But if they refuse to behave, there is no remedy. Crush their heads, and let them die!

4.2 The sign from Heavens

In this level, we have officially met the fey. Essentially a sign difference upon exchange, in mathematical terms. This difference can be expressed mainly in two ways. A standard approach based on wavefunctions or kets consider the exchange of two particles,

$$\hat{P}_{jk}\|a_1, a_2, \cdots, a_j, \cdots, a_k, \cdots, a_n\rangle = \eta\|a_1, a_2, \cdots, a_k, \cdots, a_j, \cdots, a_n\rangle, \tag{4.1}$$

where $\hat{P}_{jk}$ is an operator that exchanges particles a_j and a_k, and η a phase factor. Obviously, if applied twice it must return to the initial state, hence $\hat{P}_{jk}^2 = 1$. This implies that $\eta = \pm 1$. It $\eta = +1$, we say we have *bosons* (our fey), and *fermions* (our dragons) if $\eta = -1$. Conversely, considering creation and annihilation operators, one can define bosons and fermions based on commutation relations[xvi]:

$$\begin{aligned} b_i^\dagger b_j - b_j b_i^\dagger &= [b_i^\dagger, b_j] = \delta_{ij} \\ c_i^\dagger c_j + c_j c_i^\dagger &= \{c_i^\dagger, c_j\} = \delta_{ij}. \end{aligned} \tag{4.2}$$

This is a practical way to classify particles following the commutation relations of their operators.

Such property has several characteristic consequences. Perhaps the most obvious one has been presented in the first level for fermions: Pauli's exclusion principle. Essentially, it tells us that no two fermions can occupy the same state. It was discovered before the clear difference between bosons and fermions had been realized, with the first real glimpse on the exclusion principle obtained by Stoner in 1924, when investigating atomic structure [48]. In the following year, Pauli stated it clearly as we know it today, and it became known and Pauli's exclusion principle [49]. We can easily understand the exclusion principle now as a consequence of fermionic statistics. If two particles j and k occupy the same (single-particle) state, their exchange would leave the overall state unchanged. The only way to satisfy such relation while keeping the phase inversion ($\eta = -1$) is if the state itself is null.

But as we look further, spins for bosons and fermions also become tightly attached in an interesting manner: fermions have half-integer spin, bosons have integer spin. This is known as spin-statistics theorem [50] and may be explored in many different ways. It is also related to the manifestation of these two statistics under different form in field theory. Fermions manifestly follow Fermi-Dirac action, while bosons behave according to Klein-Gordon actions. More fundamentally, one can say that Klein-Gordon action is relativistically consistent with commuting oscillators (fields) and nullify anti-commuting (Grassmann) ones, while Fermi-Dirac actions is relativistically consistent with Grassmann-valued oscillators, but do not give a positive definite Hamiltonian for commuting oscillators. Hence, it seems that nature has to types of oscillators, which already embed spin information[xvii], each one associated with one type of action.

We can question whether a particle can be neither of those, i.e, whether a particle (ensemble) can fail to follow either of the above relation. We could imagine a case where exchanging particles twice would lead not to exactly *the same* state, but some general phase factor $\eta^2 = e^{i2\theta}$, since a phase factor by itself has no physical meaning. This is a topic to be faced in the next level, hence we shall leave it on hold for now.

4.2.1 ··· *Supersymmetry and quantum mechanics*

Now that we see fermions and bosons, it is possible to conceive, at least in principle, a symmetry between them. This symmetry is called supersymmetry.

Supersymmetry does not play a central role in our considerations throughout this work, but some references to superspace and the symmetry between bosons and fermions may appear later on, so we explain the very basics of supersymmetric (SUSY) quantum mechanics. For more details on the topic, we refer to refs. [51–53].

To deal with bosons and fermions equally, one must embed them on a higher space called "superspace." The superspace is a $\mathbb{Z}_2$-graded, i.e. a space that joins with sectors "odd" and "even" into a whole space. Henceforth, everything dubbed "supersomething" refer to the "something" equivalent in superspace and superalgebra. For a detailed coverage of superalgebra relevant to physics, we refer to Rogers [52]. Applications of superalgebra on supermanifolds include phase space approaches for fermions [53] and the extension of qubits to superqubits (SUSY qubits) [54].

To understand a little more about fermions, let us consider a harmonic oscillator's hamiltonian

$$H = -\frac{d^2}{dx^2} + \omega^2 x^2. \tag{4.3}$$

It is well known to be diagonalizable by creating operators $a = \frac{1}{\sqrt{2\omega}}(\partial_x + i\omega x)$. It clearly follows commutation rules $[a^\dagger, a] = 1$, identifying whatever the excitation it might create as bosonic. Furthermore, the hamiltonian can now be rewritten as

$$\begin{aligned} H &= \frac{\omega}{2}(a^\dagger a + a a^\dagger) = \frac{\omega}{2}\{a^\dagger, a\}, \\ &= \omega\left(a^\dagger a + \frac{1}{2}\right). \end{aligned} \tag{4.4}$$

This hamiltonian counts modes of oscillations, or number of "oscillons," better identified as phonons confined to its oscillation potential region. But as we start from a hamiltonian manifestly written with real number variables (ω, x), we can expect nothing but commuting operators a, viz. bosonic "particles" or modes. If we can imagine particles out of our real space, can we think of fermionic oscillators? If the bosons we know are regarded as manifestations of oscillators, what kind of oscillator could lead to anti-commuting operators? Let us try to do this backwards.

From a fermionic operator c obeying

$$\{c^\dagger, c\} = 1, \tag{4.5}$$

one can conceive the hamiltonian

$$H_F = \frac{\omega}{2}[c^\dagger, c] = \omega\left(c^\dagger c - \frac{1}{2}\right). \tag{4.6}$$

We can also combine with a bosonic oscillator H_B into

$$\begin{aligned} H &= H_F + H_B \\ &= \omega(a^\dagger a + c^\dagger c). \end{aligned} \tag{4.7}$$

This hamiltonian is now supersymmetric. It is possible to represent c as a lowering Pauli operator $\sigma^- = (X + iY)/2$ and $c^\dagger$ as a raising operator $\sigma^+ = (X - iY)/2$. On the top of it, one usually defines supercharges Q – such that $H = \frac{1}{2}\{Q^\dagger, Q\}$, $Q^2 = (Q^\dagger)^2 = 0$, and $[H, Q] = 0$ – and superpotentials for such systems.

But we evaded the question: what kind of numbers can represent a fermionic oscillator? The answer is *Grassmann numbers*. These are number following what is often called exterior algebra, and obey anti-commutation, i.e. $\xi\zeta = -\zeta\xi$ (cf. [36]). Hence, a supermanifold has both real or complex numbers and Grassmann numbers spanning it, and supernumbers can be written as

$$n = x + g\xi_i, \tag{4.8}$$

in the simplest case where $x \in \mathbb{R}$ and $\xi_i \in \{\xi, \xi^\dagger\}$. In this case, x term is called the *body* and ξ term the *soul* of the number. One can generalize it for a manifold with m complex dimensions and n Grassmann dimensions. We will not go deeper int this matter, but refer to Rogers [52]. Dalton et al. also provide a relatively simple explanation of Grassmann Calculus [53]. See also [55–57].

4.3 Binary world?

We saw the division of the world in two: fermions and bosons, also supported by spin-statistics theorem [50]. Although our next step is to break down this whole concept and pave the way for anyons, fractional statistics, and so one, it is worth to think about this bipartite reasoning about the world, for spin-statistics theory does hold in the world we live with three spacial dimensions.

First of all, we ought to state one thing clearly: it is not obvious why nature seems to be divided into two kinds of oscillators, bosonic and fermionic. In field theory, it amounts to two kinds of fields, Klein-Gordon for bosons, Dirac for fermions. There could, in principle, exist other forms of fields leading to more complex formulae and relations. Yet, we seem to be stuck with two.

Knowing this, we must ask: are there only exactly two types of oscillators, or are we willing to see only two and divide everything into two? Boson/fermion, man/woman, plus/minus, ying/yang, wet/dry, hot/cold, good/evil, light/dark... the list is endless. As humans, the simplest and easiest form of simplification we often jump to is a dichotomy. By dividing whatever the world we consider in our thoughts into two distinct and somewhat opposite sets, we can achieve some degree of satisfaction on the earned balance, also enjoying a degree of understanding of the nature of the world and events therein.

But we must point that every knowledge is influenced by culture and beliefs. The definition of knowledge as a "justified true belief" may help us to think about it.[xviii] Even if we accept some flexibility in defining "true" for scientific truths, we cannot ignore the role of culture to our beliefs and justifications. Even scientific truths are influenced by cultural and social relations. The existence of atoms have long been debated on theoretical grounds, with a range of arguments being accepted or ignored depending also on what one believes to be a proof, to begin with.

But we might be risking to lose track here. After all, fermions and bosons are observed. Perhaps we do have an inclination to be willing for such kind of division of the world in two, but we are also lucky enough to find such division in nature. However, we certainly need double care for two cases: (i) remember that the fortunate dichotomy is nothing but an accident and that there may be exception under conditions not immediately seen; and (ii) because we happen to find dichotomies in nature it does not mean that dichotomies are the "most natural" way of dividing things. These false assumptions may delay and impair much progress, specially in human grounds. Apartheid is certainly not recognized as something worth it, but comes ultimately form the same kind of dichotomy. As does LGBTQ-phobia, for example.

In natural sciences, we see that many dichotomies are often more flexible than they seem at first. For example, let us take our fermions/bosons again. Trap a fermion like an electron in one-dimension (say, a carbon nanotube) and it can be bosonized. Pictorially, Pauli exclusion principle does not allow for a fermion to "cross" another, so they are trapped in a fixed array. It no longer makes sense to discuss their fermionic behavior, for no behavior beyond joint oscillations occur, and these oscillations observed can be thought of as bosonic. It is a crude way of putting Tomonaga-Luttinger liquid in words, but it calls our attention to the fact that even fermions are not always too fermionic (see, for instance, [58, 59] for more on bosonization).

Quantum theory has to indeed soften the "tertia non datur" assumption, as already exposed in level 1. Within quantum theory, it is nothing but expected that boson/fermion assumption becomes flexible under certain circumstances. Yet, even when dealing with quantum theory, sometimes physicists a tempted to cut a yes/no question somehow: *is there a given symmetry in this system? Is this robust? Does an isolated system thermalize?* Even though such genuine questions are posed in a yes/no fashion, there is no guarantee that they will have a yes/no answer.[xix]

Yet, while I would basically alert to the risk and temptation of dichotomies, this work relies heavily on one specifically: bits, the 0/1 unit of information. I must ask for allowing us this one under one excuse:

while indeed it divides bit values in two, these are used to build-up bit strings for any number, as long as necessary. Therefore, we should also keep in mind the whenever we focus one bit of information, it may just as well belong to a much greater framework.

Notes

[xv]The will-o'-wisp are playful photons and the phoenixes are heated phonons. They are listing some of the actions they do with fermions, including being produced and absorbed by them.

[xvi]Following the usual notation, b stand for bosons while c indicate fermions. It is also common to see f as a fermion operator, but we will stick to c. In general, a will serve as a generic operator to indicate *any* possible particle or statistics: fermions, bosons, or even neither.

[xvii]Grassmann fields usually come in pairs or even dimensions, and their anticommutation allow to write Hamiltonians or Lagrangeans for a fermionic oscillator. Because a Grassmann variable will annihilate itself if squared, one usually has terms like $\xi^\dagger K\xi$, with K being a matrix operator and ξ a (possibly multidimensional) Grassmann field (oscilaltor). See ref. [50] for more details on the relation between commuting and anticommuting oscillators and their governing action under Lorentz invariance and rotation symmetry considerations. For more on Grassmann variables, see ref. [37, 52, 53].

[xviii]We do not want to consider Gettier problem, when a justified true belief comes from false beliefs, making only an accidental justified true belief.

[xix]According to an anecdote (unverified), in the judgement of a prominent figure, prosecutor asked to make yes/no questions to the defendant, who replied: "I cannot defend myself from yes/no question. Can I ask *you* a yes/no question?" To what prosecutor said yes. "Do your parents know you are gay?" To what the prosecutor did not answer and had a very reproaching reaction. Leaving the latent homophobia of the question aside, it plays with the problem of a yes/no question. Such questions are made on assumptions that may just be untrue. Sometimes, "tertia datur."

Level 5

Fusing beasts, splitting hearts

When she entered the second gate, the small lapis-lazuli beads were removed from her neck. "What is this?" "Be silent, Inana, a divine power of the underworld has been fulfilled. Inana, you must not open your mouth against the rites of the underworld." — Inana's descent to the nether world, The Electronic Text Corpus of Sumerian Literature, translation t.1.4.1.[xx]

5.1 Netherworld

It's cold. Thrown in this flattened land, the cold seems to consume my bones and feed on my fire. You do not know what this means, do you? I'm dying. Life and death comes to all, and it has come to me. My wings are freezing to death, the slowest oscillation possible. My fire is shrinking, my heartbeat ceasing. It is not painful as one would think, but I'm dying. I can feel curves and drags on the pasture, that pull and push in circles of distress. When we are too cold, we sometimes eat and spit phoenix fire,[xxi] and pair up to stay alive. But these swirls in the field make me dizzy; I will die.

5.1.1 · · · Freezing to death

I shall depart from this world. I shall sing to my kins in shape and blood. I shall sing to my kins a song of old. I shall partake in death with friends and foes. I shall share with them all my cries and soul. My chant of death, fellows, listen and behold!

Wit with his wantonness
Tasteth death's bitterness;
Hell's executioner
Hath no ears for to hear
What vain art can reply.
I am sick, I must die.
Lord, have mercy on us![1]

Brethren, worry not! I know my destiny. I must do what it takes, for my future I have wrought.

[1] From *A Litany in Time of Plague*, by Thomas Nashe, 1600.

5.1.2 ··· Death and rebirth

No, thou shalt not! We will warm and cuddle you, for this we are born! We will cover and embrace you, and new life shall raise! In the circle of life with thee we find grace! We will live as one new being together! We will be no more, but thou shalt be better!

No more dragon, no more fey! No disgrace nor doomsday! Feyish dragon if you will, draky fairy if I may! This power of ours that freedom devours! This sweetness renewed with smoothed keenness.

Worry not, dear fellow! We hurdle feelings shallow! New partner, now, let us dance a new ball! Let us twist ourselves, let us twist the others! Let us freely decide what our laws and orders!

Come now, let us raise! Let us live in praise! Let your soul be divided in twins if it matters! Let you pace be exalted with hops 'n partners! We know deep inside each dragon lies two hearts: one for seeing reality, one to imagine a-fresh starts!

In this flattened land we as well suffer! We are powerless void, but we much thought mutter. And a fey willpower is not to belittle. Even when flattened with you in our middle!

See that now we my bind together. We may become one, we may even do better! We could, indeed, be fey and be dragon. Or abdicate everything, and with flatness reckon.

And in this new freedom if anything we impersonate, through friendly territories each household can dominate! This is not to be fight for absolute ruler, nor rushy havoc for every secluder.

We can let our bodies and souls be apart. We can let our ambitions freely to depart. And in festivals and parties we may dance, and dazzle newcomers with a spinning trance.

For the circles of life much matter to us! And the cycle to death is no one-way bus! If only for ourselves we faithfully unite, we ought to proving our robust might!

5.1.3 ··· Twin dragons birth

Ah, I can feel the fey surrounding me! I can feel the phoenix heat and will-o'-wisp brightness! I can see kappas[xxii] in vortices inviting me to stay with them.

Ah, the fey cozy care! They invite me in this frozen death bed to lift my soul and surrender my body. It can split my inside and outside now and give myself to their comforting embrace. It is ancient magic, known to dragons and fey alike, that allows us to enter a sacred communion with our peers, and split ourselves, and bear new selves.

So I may close my eyes. I can stop my breath and let lose my wings. And as I split my mind into two, I feel lighter, massless. And I am no more. I am no longer myself. For I am us, we are me, the former me. We are now split in half, each its own wingless dragon. We become a pair of dragons, twins in shape facing opposite direction, intertwined by our long tails in this infinite cold void. We are two, and together we are one. Our territory is a shared territory, that we defend against intruders, but we both live in and rule. We are each now what Majorana thought of as a symmetric dragon, and you now call *Majorana fermion.*

We are now as twins eternally weaving around one another, and though some may notice our isolated presence, all our secrets are concealed from others that do not see us together, do not see us as one. For we a two, but we are one. Two shadows of the same existence, two shades of the same complex dragon, now independently living in different bodies, each one holding its piece of soul.

And we can be what is naught too. We can fly and swim to be free, and freely flirt with of halves of twins. In this pacing and dancing, twisting and waving, we may change partners and change ourselves. As we revolve around others, we wingless and dependent dragons can change our state of mind freely. A dragon twin sibling going round another sibling of other twins will cross tails between twins. But our magic does not fail us. Every time we tangle tails, we traverse our twin brothers by reversing our tail pairing! And like this, the former dragon now split into two through space, can change its existence with a "tailshake" with another dragon now made also twin.

But deep inside, we fell light and massless twins when our former selves interact only with their neighbor dragons. We may also feel heavier, if still twin dragons, when our former dragons disputes stretches through many brethren.[xxiii] Heavier and massive, yes, but still pairing up and stretching through space, still twins embracing endless existence.

Massive or not, we sit still in unperturbed reign on the edges of existence. Whether the core of vortices provided by kappas, or the range of mountains dividing our living domains, we lightly hover on them, with minimum effort required by our weight: none at all if we are massless, very little if we have little mass. We can calmly hover on mountains, linked by our tails stretched through space. And if mountains delimiting our region are moved, we can go on together through any stance of existence. Thence, even if our other brethren are far out, our tails may intertwine without anyone seeing or realizing but us, and from our arcane nature we may become, once taken together, what we were not before. We may then claim dominance over a state different from the one we claimed before, without ever meeting or matching our heads or claws!

This is our peaceful, if blissful, triumph and score. For we may still be somewhat dragons, but also dragons no more. In this flattened land, with no skies above; no bottomless pits, nor tower or high peaks; from endless seas we see only the shore; In this flattened land our abilities grow sore. We are still dragons, of this you can be certain! We prefer the avoidance and fight over instances. But now when we dance, when we fight, we no longer do it from instinct alone, nor we can tell you the conclusion will always be one. In this flattened land, our tails as our hands, let us freely our will bend, if following the steps to this end. We can move through territory-states without ever clashing: only the exchange of tailshakes can accomplish changes without smashing.

And no matter what happens to a single individual as long as it survives any attempt against its life. For as long as we survive, nothing and no one can break our hearts or read our minds. Our knowledge is kept safe and cannot be disrupted. What twins share, it is shared hidden. And to change what we know, or to change what we do, the dance of exchanging twins must happen in specific steps, whose order do matter. You humans may say we are now “non-Abelian” in a respectful sense, acknowledge the importance of our well tailored ordering. We are indeed stronger in this shape, and ready to face anyone, come what may.

5.2 When topology is what matters

In this section, we shall take a deeper look into topology and its influence to quantum theory. Though the freezing death our dragon faces happens together with its crush into a plane, let us illustrate the case for topology with a free particle in a ring [60].[xxiv] It lives in a periodic space, therefore its wavefunction becomes

$$\Psi(x) = e^{ikx} = e^{-i\theta}e^{ik(x+L)} = e^{-i\theta}\Psi(x+L). \tag{5.1}$$

The energy of these states becomes quantized, different from a free particle in a trivial space. This quantization is strictly linked to the difference in space topology. For a free particle, energy becomes $E = \hbar k^2/2m$, but now the periodic boundary conditions requires that

$$kL = 2\pi n + \theta, \qquad n \in \mathbb{Z}, \tag{5.2}$$

and so, k just becomes a function of n. Energy becomes quantized just because of the topology of the space. Explicitly, the energy for this system becomes

$$E_n = \frac{\hbar^2}{2mL^2}(2\pi n + \theta)^2. \tag{5.3}$$

So, this simply comes out of the shape of the system, which we more generally calling *topology*.

If $\theta = 0$, E_n becomes equal to E_{-n}, a double degeneracy not present for general values of θ. If $\theta = \pi$, this would shift the spectrum to

$$E_n = \frac{(2\pi\hbar)^2}{2mL^2}\left(n + \frac{1}{2}\right)^2, \tag{5.4}$$

and now *all the levels are doubly degenerate*, including the ground state at $n = 0$, degenerate with $n = -1$. And we can keep looking at particular cases of $\theta = m\pi$ which will keeping shifting the levels' energies creating new pairs of degeneracies and even shifting the ground state to a new n. Say, for $\theta = 2\pi$, the same spectrum as $\theta = 0$ is obtained, but the ground state is now given by $n = -1$. This phase factor can actually be linked to a external parameter, giving physical meaning to it. And once done so, this parameter can also be adiabatically changed through a full cycle. And then, if the change is adiabatic, no transition should occur, allowing to start at the ground state at $n = 0$, stay on the same state but no longer be at the ground state in the end. This relates to concepts like spectral flow, charge pumping, spin pumping, and such things. This links to the concept of geometric phases presented next.

5.2.1 ··· Berry very much

Much of the discussion around topological states can be thought of in terms of geometric phases or Berry phase [60–62]. Let us consider a pure state's adiabatic evolution. This evolution will be given by external parameters $\{\lambda(t)\}$, which will change adiabatically. If the hamiltonian is diagonalized as

$$H(t)|n(\lambda)\rangle = E_n(t)|n(\lambda)\rangle, \tag{5.5}$$

we can write the state into which the system evolves as

$$|\psi(t)\rangle = \sum_k c_n(t)e^{-i\int_0^t dt' E/n(t')}|n(\lambda)\rangle. \tag{5.6}$$

The factor

$$\delta_n = \int_0^t dt' E_s(t') \tag{5.7}$$

is called *dynamical phase*. Now, if we substitute this state $|\psi(t)\rangle$ into Schrödinger's equation, we get

$$\begin{aligned}\sum_n \left[i\dot{c}_n(t)|n(\lambda)\rangle + E_n(t)|n(\lambda)\rangle + ic_n(t)\frac{\partial}{\partial t}|n(\lambda)\rangle\right] e^{-i\delta_n} \\ = \sum_n E_n(t)c_n(t)e^{-i\delta_s}|n(\lambda)\rangle.\end{aligned} \tag{5.8}$$

So we can multiply by $\langle k(\lambda)|$ on the left and obtain

$$\frac{d}{dt}c(t) = -\sum_n c_n(t)e^{i(\delta_k - \delta_s)}\langle k(\lambda)|\partial_t|n(\lambda)\rangle. \tag{5.9}$$

If the system starts in a stationary eigenstate $|n\rangle$, it will remain there, according to the adiabatic theorem, and this equation reduces to

$$\frac{d}{dt}c(t) = -c_n(t)\langle n(\lambda)|\partial_t|n(\lambda)\rangle \equiv -i\gamma_n(t)c_n(t). \tag{5.10}$$

with $\gamma_n = -i\langle n(\lambda)|\partial_t|n(\lambda)\rangle$ being a real number[2]. The solution becomes

$$c_n(t) = \exp\left(-i\int_0^t \gamma_n(t')dt'\right) \equiv e^{-i\Gamma(t)}, \tag{5.11}$$

so we end up with

$$|\psi(t)\rangle = e^{-i\delta_n(t)}e^{-i\Gamma(t)}|n\rangle. \tag{5.12}$$

[2] from Schrödinger's equation $-i\partial_t|n\rangle = H|n\rangle$ it is straightforward

This Γ is called the *geometric phase* and cannot be always ignored or erased by gauge choice. In fact, the wavefunction may acquire this *observable* phase. Upon a cyclic transformation, for example, the system must go back to the same starting point, where $\{\lambda(0)\} = \{\lambda(T)\}$, being T the time for a cycle. We can indeed rewrite the geometric phase as

$$A_i = -i\langle\lambda|\partial_{\lambda_i}|\lambda\rangle, \quad \gamma_n(t) = A_i(\lambda)\dot{\lambda}_i, \tag{5.13}$$

a simple change of variables. So, we have for geometric phase

$$\Gamma(T) = \int_0^T \gamma_n dt \equiv \oint A_i d\lambda^i. \tag{5.14}$$

One can see that in the parameter space of λ, only the geometrical aspects are left. In fact, it is just like the integration of a vector potential, which imprints geometric aspects in real space. We can use stokes theorem to write it in terms of A_i's rotational $\boldsymbol{B}$, like a magnetic field. Hence, the geometrical aspects can create an effect similar to a magnetic field in spaces other than real space. These are aspects fundamentally linked to the topology of a system.

There is a plethora of phenomena related to geometric phases: skyrmion emergent electrodynamics, electric polarization, charge and spin pumping, quantum (spin) Hall and analogous effects, topological insulators, Dirac and Weyl semimetals, to name a few. The phase factor in Bloch theorem gain important meaning when one takes this phase into consideration. Berry phase is also called Berry connection, for it is a connection related to the fiber bundle given by $U(1)$ gauge of wavefunctions.

5.2.2 ··· Lower dimensions: Not for anyone, but anyon

An important aspect for topology is the dimensionality of the system. For instance, the very division of particles into bosons and fermions depends on it, for it only holds in three or more space dimensions for point-like particles. Once we are in 2D, one may, in general have *anyons* and fractional spin. Pauli spin-statistics theorem, that grants half-integer spin to fermions and integer spin for bosons, is also restricted to more the two space dimensions [50]. Not necessarily being bosons nor fermions, anyons are said to obey fractional statistics. For a detailed cover of anyonic quantum mechanics, we refer to [63].

For anyons, particle exchange can lead to any phase factor. This difference comes from the representation of homotopy group in two dimensions or three or more dimensions. That is, a particle exchange realized twice is the same as encircling one particle with another around a closed path. This encircling can be thought of in terms of the first homotopy group π_1. For a given single particle wavefunction $\Psi(\boldsymbol{r})$ in a d-dimensional configuration manifold $\mathcal{M}$, a n-particle wavefunction of identical particles will be

$$\begin{aligned} \Psi(\boldsymbol{r}) &\in \frac{\mathcal{M}^n - \Delta^{n-1}}{\mathcal{P}_n}, \\ \Delta^{n-1} &= \{\Psi(\boldsymbol{r}_i) | \exists r_i = r_j, i \neq j\}, \end{aligned} \tag{5.15}$$

where $\mathcal{P}_n$ gives the permutations of the particles and we are subtracting the configurations where two particles occupy the same position to assure we have a manifold. The first homotopy group on this manifold becomes

$$\pi_1\left(\frac{\mathcal{M}^n - \Delta^{n-1}}{\mathcal{P}_n}\right) = \begin{cases} \mathfrak{P}_n, & d \geq 3 \\ \mathfrak{B}_n & d \leq 2 \end{cases}, \tag{5.16}$$

where $\mathfrak{P}_n$ is the permutation group of n particles and $\mathfrak{B}_n$ the *braiding group* for n particles. The representation of the permutation group is essentially $\mathbb{Z}_2$, leading to the ± 1 phase factor of fermions' and bosons' exchange. On the other hand, braiding group representation may vary largely, which motivates their usage for (topological) quantum computation [28, 64, 65].

The braiding group with its generators σ_i is depicted in fig. 5.1. The reduced representation of the braiding group may be commutative (Abelian) or not (non-Abelian). If one has a one dimensional

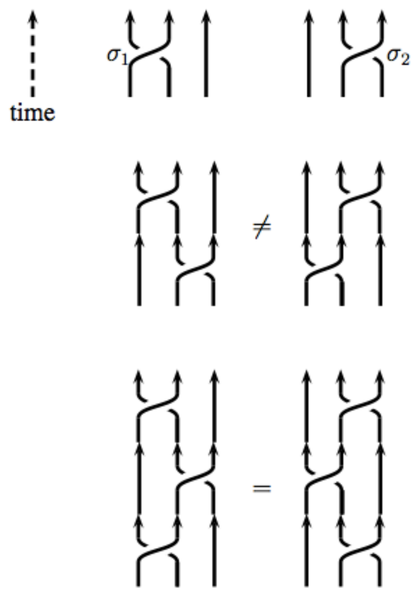


Figure 5.1: **Top:** general braiding operations on three particles. **Middle:** a non-Abelian braiding group. **Bottom:** braiding identity $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. Reprinted figure with permission from Nayak, Simon, Stern, Freedman, Das Sarma, Rev. Mod. Phys., **80**, 1083 2008. Copyright (2008) by the American Physical Society.Nayak et al. [64]

representation, embedded in $U(1)$ group, all one may have is a phase factor $e^{i\theta}$ upon particle exchange. These become Abelian anyons, obeying fractional statistics. But given a higher dimensional representation, braiding group may span other groups like Clifford groups or $SU(N)$ groups. In such case, provided its its irreducible representation is non-Abelian, we have non-Abelian anyons and non-Abelian statistics.

In general, such anyons are actually collective excitations in many-body system. Therefore, even a single any is actually given by a many-body wavefunction. One cannot expect, in general, that the wavefunction of two or more anyons can be represented by the product of anyonic single-particle states, as is often the case for bosons and fermions. In fact, even simple anyonic problems bear characteristics of interacting bosons and fermions, leaving no simple (analytic) solution to any known problem with more than two anyons (to the best of author's knowledge to date).

To illustrate the complexity involved, consider the simplest case: two non-interacting (Abelian) anyons. The Hamiltonian can be divided into the center of mass and their relative Hamiltonian

$$H_r = \frac{p_r^2}{m} + \frac{(p_\phi - \hbar\alpha)^2}{mr^2} \tag{5.17}$$

in polar coordinates, with $\alpha \in [0, 1]$ the interpolation factor between bosons ($\alpha = 0$) and fermions ($\alpha = 1$). The time-independent Schrödinger can be solved by separation of variable, leading to angular equation

$$\left(\frac{\partial}{\partial\phi} + \alpha\right)^2 F_l(\phi) = \lambda F_l(\phi). \tag{5.18}$$

The solutions satisfying boundary condition $F_l(\phi + \pi) = F_l(\phi)$, become

$$F_l(\phi) = e^{il\phi}, \quad l = 2k,\, k \in \mathbb{Z}, \quad \lambda = (l - \alpha)^2. \tag{5.19}$$

The first clear difference from the bosonic and fermionic case comes with the break of degeneracy for the eigenvalue λ. This shift in angular momentum approximates anyons to fermions in experiencing a centrifugal repulsion between particles and to have hard-core properties that avoids superpositions. They can be seen as bosons with a (statistical) repulsion, or fermions with a (statistical) attraction. In other words, we observe effects of interaction even in the most trivial case with more than one anyon!

Given the generality of the discussion above for two anyons[3], it should be enough of a hint to understand why solving anyonic problems is not a simple task. On the other hand, it is this intrinsic complexity that generates hope to use them for information processing. Even abelian anyons could, in principle, be used to compute topological numbers (Jones polynomials) [65].

5.2.3 ··· *When Cain overcomes Abel*

in recent years, anyons that have attracted the most interest are the non-Abelian ones [28, 64–66]. Notably, realization of Majorana fermions in condensed matter setups have attracted lots of attention [67–95], owing to their realization of Ising anyons, the simplest non-Abelian anyon that can be studied. Other non-Abelian anyons bear more hope to be useful for topological quantum computation, but they are also harder to achieve, justifying the experimental focus on Majorana fermions so far [96–101]. Majorana fermions are real fermions obeying

$$\gamma^\dagger = \gamma \qquad \{\gamma_i, \gamma_j\} = 2\delta_{ij}. \tag{5.20}$$

The simplest case we can have Majorana fermions was given by Kitaev in what is known as Kitaev chain [69], whose Hamiltonian is

$$H = -\mu \sum_i c_i^\dagger c_i + \frac{1}{2} \sum_{\langle i,j \rangle} \left(\lambda_0 c_i^\dagger c_j + \Delta_s c_i c_j \right) + h.c., \tag{5.21}$$

with $\Delta_s = |\Delta_s| e^{i\theta}$. Decomposing the complex fermion operators in Hamiltonian (5.21) for their Majorana fermions counterpart

$$c_i = \frac{\gamma_{2i-1} + i\gamma_{2i}}{2}, \tag{5.22}$$

we can decompose the Hamiltonian into a sum of two other hamiltonians,

$$H_0 = -i \sum_{j=1}^{N} \gamma_{2j-1}\gamma_{2j}, \tag{5.23}$$

$$H_1 = -i \sum_{j=1}^{N-1} \gamma_{2j}\gamma_{2j+1}. \tag{5.24}$$

They are pictorially represented in fig. 5.2. Note that H_1 leave free Majorana modes on the edges of the chain, which can be regarded as a highly nonlocal fermion. These edge-modes are often called Majorna bound states (MBS), and behave like Ising anyons[4].

Because they behave like Ising anyons, they are non-Abelian anyons that may realize Clifford group upon braiding. For instance, it is possible to realize a NOT-gate or X-gate by the braiding process in fig. 5.3.

5.3 *The Seen and the Real: Emergence strikes back*

Emerging phenomena can be quirky, though one could also argue that any phenomenon is quirky. The greek word φαινομένα means “appearances” and is, in Plato’s work, opposed to the ideal. Hence, the

[3]It’s a linear equation that can only increase in terms, but the basic behavior remains.

[4]The term “fermion” is used only in the sense of the anticommutation of the operators.

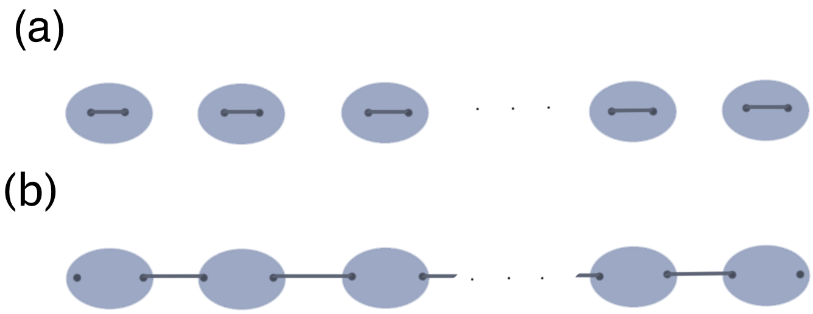


Figure 5.2: Schematic representation of Hamiltonians H_0 and H_1. Each ellipsis represent one site in a one-dimensional chain, with two dots in it representing the abstract Majorana fermions γ_{2j-1} and γ_{2j}. **(a)** Hamiltonian H_0 is depicted as a coupling of Majorana fermions on the same site, i.e. Majorana fermions of which operators $c_j, c_j^\dagger$ are compounded. **(b)** Hamiltonian H_1, on the other hand, is shaped by the coupling of Majorana fermions on neighbor sites only, leaving two Majorana fermions free on the edges.

phenomena we observe are the appearances of the world, but truth was supposed to rely on the ideal world where only thought could reach, for our senses can be deceptive in this sensible world. In the West, we tend to attribute Plato the credit for this division between the real and ideal from the sensible and apparent. Which leads to the natural problem: what *is* real? This is the core of pretty much every debate in science, a trace inherited from philosophy.[xxv]

And now, after discussing the reality of Planck's quantum in early twentieth century, or atoms and molecules, or medium-less waves, or planet orbits' ellipses, here we take another issue: topology and fractionalization. Topological states bring us a set of hard to observe properties emerging from their mathematical analyses. For instance, vacuum reminiscent entanglement known as topological entanglement is a non-trivial mark, together with entanglement negativity. Also, we have the peculiar result of *fractionalization*: units we believe to be unbreakable, like electron's charge, show up in fractions of itself not by dividing the (thought) indivisible, but by glueing several different units into one collective fractional charge. This kind of phenomena ought to strike awe. It is nothing less than perplexing to have "less" from "more." But this is possible exactly because we have a many-body situation. As Anderson famously put it "more is different."

So we must face the question: how real is that what we see or measure? If we only can access with some certainty the appearances, how far must we go to save them? It is the old Platonic division under Baconian perspective, synthesized in the words Σωζειν τα φαινομενα [102]. And as we focus on topology, already discussed, let us highlight what kind of physics this brings us. Then we may adjust our perspective on how real topological phenomena is, or how apparent topological systems are.

From the point of view of homotopy group, topological phenomena may be regarded as some sort of esoteric hole counting. However, these phenomena leave their marks. On the theoretical side, one may list topological entanglement and the entanglement spectrum obtained from these systems that have a manifest ordering unseen in non-topological setups. And their entangled vacua allow some non-trivial manifestations, like fractionalization. But all these theoretical aspects are not phenomena until they are observed; they originate from the world of ideas. Yet, new materials like Weyl and Dirac semimetals, topological insulators, topological superconductors, topological superfluids, and other strong correlated topological systems, as well as quantum hall states and so on, they manifest physical properties explained by the ideas of topology. They have charge transport properties that point towards a quantization at a fractional point of the standard electron charge, for example. Or they show a scattering-free conductance on their surface, without similar charge transport in the bulk, i.e. a conducting surface and an insulating

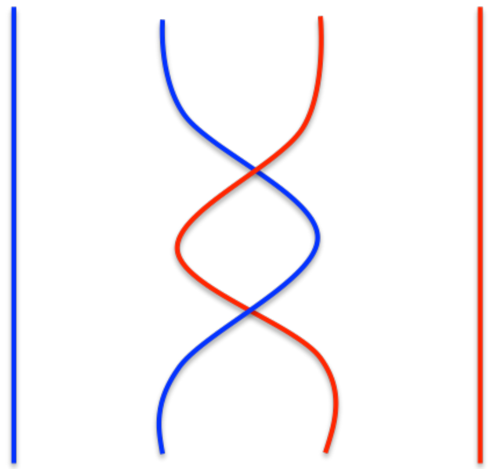


Figure 5.3: Braiding lines of Ising anyons, fused colorwise. The vertical direction accounts for time flow direction

bulk. Or even, they may show non-trivially ordered response to careful experiments like angle-resolved photoelectron spectroscopy (ARPES). These are the phenomena that appear to us, and under Occam's razor, our current best shot is the ideal of topology, whether we see the theoretical structure directly or not.

For example, take the case of Dirac cones as the surface states to topological insulators. ARPES allows us to "see" such cones. However, in this case, at least, we first had the theory, the idea predicting what appearances to expect, until we finally had the phenomenon. And the phenomenon, the observation, does not take place in plain sight; one must go to reciprocal space, the space created by the electron momenta, to see the predicted phenomenon. Suddenly, we seem to have a physical object whose truth, indeed, cannot be directly perceived but may become sensible under proper guidance of the ideal knowledge. We fall back to Plato, with a truth unseen, but reveled to the philosopher. At this point, the gears of science no longer move to save the appearances like Galilei, Kepler, or Newton guided until perhaps last century (cf. [103]), but aspire to revel hidden appearances looking for new perspectives and different constructions of sensitivity.

In the beginning of twenty-first century, we live a tendency that reverts the objective of "saving appearances" towards creating apparitions out of ideas. Topological invariants like Chern number may be the heart of Hall resistivity measured in quantum Hall insulators, but the observation of the phenomenon can hardly be said to be an observation of such topological charge or Berry phases and holonomy. Discussions on CFT revolve all around defining the central charge of a theory; once it is defined, all derives from such abstract quantity. And from these ideas, we derive and actually construct phenomena. Topological insulators and quantum spin Hall insulators originated from such ideas, completely led by theory. The relevant space for phenomena is, at times, not the space we live and feel (or an experimental sample, for that matter), but abstract spaces altogether, mathematical constructions that influence our experience just as much as any other more "concrete" aspect. Phase space, momentum space, holographic space; these are some examples where various new phenomena have first been theoretically investigated and later observed. Cold atoms create quantum hall effects in momentum space, vacuum is squeezed in phase space, black holes are constructed in holographic space, all of them dual to real space, where experiments are proposed. The scientific method finds its own reversal, or dual, no longer looking for generalizations from particularities, but creating from scratch new peculiarities and sensible phenomena all from idea.

And this different emergence brings back existentialist considerations we faced in level 3. Does existence precede essence when we study topological quantum phenomena? It certainly depends on how we understand the word "existence" when applying such reasoning to scientific theories and experimental measurements and objects. One may conceive some abstract theoretical concept to "exist" in the scientist's mind. Also, some materials were known long before they were identified as being topological, a case where clearly their existence precedes the essence that has been attributed to them after we came to understand topological aspects of matter on quantum level. But even if some material's physical existence[5] indeed precedes what we understand to be their essence, it may not always be the case. Graphite has long been known to be layers of graphene, and a monolayer has been theoretically understood and expected to host Dirac cones for many decades, though not realizable until 2004. Should we consider its idealization to count as "existence?" What about the phenomena, e.g. Dirac cones or Weyl cones themselves? What about their topological signatures? They were theoretically found and understood on the physicists' pursue for the essential characteristics of some setups. They are the mark that allows one to recognize coveted essence. And new materials have been proposed on the intent to verify expected new phenomena. Admittedly, we cannot attribute physical existence to such phenomena before what we recognize to be their essence.[xxvi]

We may well say we are entering in a new scientific era now, where we go beyond the experimental fact and find also truth in much less feint concepts whence observations are designed. But of course it is not to mean that we no longer learn from experiments, nor that this traditional scientific process will soon disappear. We are no sooner to leaving experiments aside as tool for learning than we ever were. We are though in a different position of finding ideal theoretical concepts, which can well be deemed real, that can spit out various sensible phenomena that depend more on the way we look at them but are no less real either, given their pretty well constructed foundation they rely upon.

Notes

[xx] *Inana*, or *Inanna*, — The Lady of Heaven — is the Sumerian goddess of War and Love (and more, actually) and is often portrayed as equivalent to Venus or Aphrodite. However, Inanna differs from these goddesses in one important aspect: She was a major, central, and powerful goddess in the Sumerian Pantheon, holding a powerful female role. This is an excerpt from her travel to the Netherworld or Underworld, the world of the dead. The reason to her travel is not clear. As she goes, she dies and is kept in the underworld, until set free by other god, though someone is required to take her place. Whilst her servants are mourning her, her husband Dumuzid is lavishing among slave girls and is taken by gallas — demons from the Underworld — to substitute her. In a deal with the goddess, his sister spends half the year in the underworld in his place, with he staying dead for the other half.

[xxi] This refers to a Copper pairing mechanism, where electrons pair up by phonon (or phoenix) coupling.

[xxii] *Kappa* (河童) is a mythical creature in Japanese folklore that lives in rivers, often depicted as a greenish humanoid creature the size of a child, with a turtle shell on the back. It often has a crown of hair, being bald on top. Here, it is a fey depicting magnetic flux quanta.

[xxiii] This is essentially a reference that Majorana fermions become massive Dirac fermions in long-range superconducting (1D) Hamiltonians [92].

[xxiv] This discussion has been carried in more details in my master thesis, where I cover Berry phase as well as quantum Hall states. The whole discussion is simplified here. For more details in calculations, see [67] and references therein.

[xxv] It is interesting also to notice that before the Greeks, Zoroastrian Persians had started to cultivate the praise, or more literally worship, for wisdom, with the *Mazda-Yasna*. However, the Zoroastrian predecessor of Greek philosophy seemed to be more focused on the moral and ethical aspects of life than the more mundane sensations and reality. Possibly, since the old religion has in its core the concept of "asha" – often translated as "truth" – and its pursue, sensible experiences and appearances would not have been a point of interest in first place. And since the matter of discerning what has always been ignored is not at all trivial, Plato has given us this glimpse of division between what is perceived and simply "is." In other words, the pursue for "the truth" may be older than Plato, but the association and dissociation of "truth" and "appearance" is credited to Plato.

[xxvi] Even though, or perhaps precisely because we refer to quantum objects and not people, I would call this scenario *post-existentialist*. That is, we may recognize existence to precede essence in the some objects and phenomena we look at, but *not* necessarily *always*. In some cases, essence might be found first and its existence can be pursued in response to such findings. And since there is immediate reason to assume we *must* have one or another, we could suppose that sometimes existence and essence may come hand-in-hand, with precedence uncertainty much like quantum or wave uncertainties. That is, some phenomena and/or objects could perfectly have their existence and essence tied up together.

[5] By "physical presence" we shall mean their presence inside a laboratory that can actually submit the sample to various measurements.

Level 6
Dragon Dance

Every choice begets at least two worlds of possibility [...] or very likely many more. [...] It's possible, too, that there is no such thing as one clear line or strand of probability, and that we live on a sort of twisted braid. — Joanna Russ, *The Female Man*, 1975.

6.1 Courting swings

Look at us! Let our bodies twist around each other and the the world order! Let us dance till dawn and sidestep on the lawn! What a whimsical feeling to guide our souls, change our moods, and inflate our whole! For we are more than independent beings now. We are indeed linked by an invisible red thread[xxvii], we are twin souls. We are not defeated by flatness or cold! We are not bothered but doubtful goal! We may live far apart and share our feelings. We are not disrupted by jealous beaming. We are one being two, our heart is one tune!

Therefore we praise! We praise the challenge, but know our range. It ranges far way, where no one can play. If they play us to jump, we stop at no hump. And if a hump is there, we do our share. We share what we know, controlled on-the-go. Wherever we go stepping, we also go snapping. Snapping through our veil, without noisy trail. But you can trail our triumph, and our triumph you may praise!

Look at our dance, look at our delight! Look how we can flip, look how we can flop! If only our unseen brethren from far apart rolls around our tail, we paint black what is white and bleach what is sable. And indeed they cross us, without we ever meeting our eyes or clinging our nails. They, like us, sit on their mountain-walls or float on their vortex cores. And these walls and cores slowly moving through our landscape, transport them over our long and thin tail. Remember: in this flattened land where we are trapped now, we cannot fly up high, nor dive deep down — and so they cross our thin appendices as cross them too we vow. But this bite, for us, does not go unnoticed. As soon as we are back to where we were before, we are already reflected in our territory, if another twin's brother around one of us fully once revolves.

Other dance too can also be maid. But we, near massless twin dragons, have not much choice of what territory to take. You humans would say we can only take those that a man called Clifford described in his prose. For instance, we could, with three of us in a dance, detain not a single but in two states remain! This is our known magic of quantum origin that allows superposition and coexistence in different regions.

Yet truth be said, we cannot dance as freely as one would want. There are certain cares needed to promote such enchant. If the stalls that host us move too fast, we are left adrift to our death, left to decay and spread erratically through the whole space. We may loose our strength and tranquility, disappearing in shadows.

And as long as our tails are tight together with our twins, our brethren cannot move alone without dragging their partners with them. For us to recombine in different states, we cannot be too slow, or our

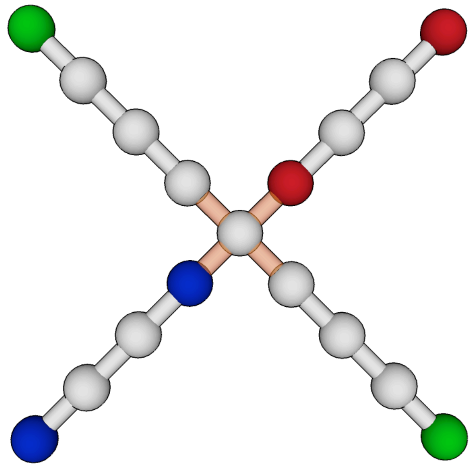


Figure 6.1: Schematic representation of the cruciform junction idealized for particles' exchange. The orange junctions in the middle represent the gates of varied lengths.

twin partnership will prevail. The closer together we are, the stronger our twin bond becomes, and the harder it is for us to move alone, following our recipient vortices and edges or not. But once enough far apart, we are one and we are two at once. We prevail in communion. We resist dimensional constraints and every sort of disturbance restricted to a single of us. We are powerful, and we have our way to our dignified control.

6.2 Steps towards information processing

In this level, we had some development of a peculiar kind of dragons that we first met last level: Majorana fermions. They are candidates to create a topological quantum computer, like those envisioned by Microsoft [104], and different suggestions for their application exist [94, 95, 105], though their braiding only allows implementations of Clifford gates, as know by the Gottesman-Knill theorem [6, 106, 107].

6.2.1 · · · NOT-gate on cruciform junction

As a simple model for studying such excitations, we will use Kitaev chain (also introduced in level 5), whose Hamiltonian becomes

$$H = -\mu \sum_i c_i^\dagger c_i + \frac{1}{2} \sum_{\langle i,j \rangle} \left(\lambda_0 c_i^\dagger c_j + \Delta_{\mathrm{s}} c_i c_j \right) + h.c., \tag{5.21}$$

with $\Delta_{\mathrm{s}} = |\Delta_{\mathrm{s}}| e^{i\theta}$. This model can be used to numerically simulate the dynamics of the Majorana braiding process [1, 67, 108]. Let us assume the phase of Δ_{s} to twist 2π around the center of a cruciform junction of such chains, as represented in fig. 6.1. Each ball in the figure represent one site linked to its neighbors. The orange links in the middle represent control gates to connect and disconnect wires by tuning the

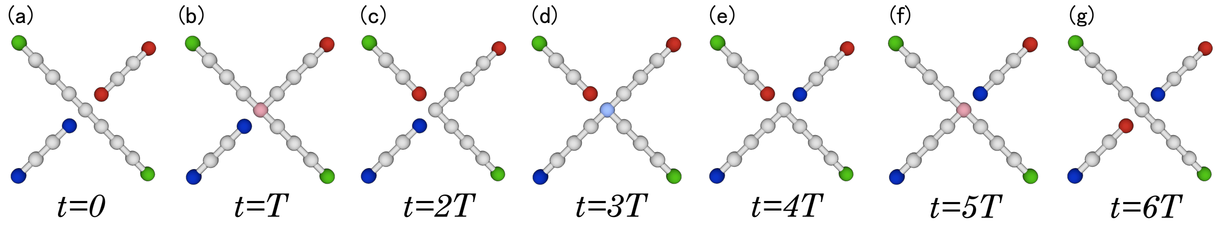


Figure 6.2: Expected braiding process executed by adjusting gates according to the (a)-(g) panel order. Parameter T indicates the time each step takes to be executed.

chemical potential μ on a certain region. In practice, for our simulation, this can either be a tuning of chemical potential through several sites or a suppression of the transfer parameters between two sites. We shall see both cases. Setting the chemical potential throughout the wire to μ_0 and the chemical potential on the site of the gate region as μ_g, we set our parameters as shown in table 6.1.

μ_0/Δ_s	0.7
$\lambda_0/\Delta_\mathrm{s}$	1.0
μ_g/μ_0	$1.0 \sim 16.0$

Table 6.1: Parameter adopted for numerical simulation with finite size gate. μ_0 is the chemical potential on the wire, μ_g is the chemical potential adjusted by gates, λ_0 is the transfer parameter, and Δ_s is the pairing potential.

To braid Majorana bound states, one has to manipulate the central gates accordingly. Figure 6.2 shows how this procedure can be done to swap two Majorana bound states' positions. In order to complete a NOT-gate, one only has to do this process twice.

The Hamiltonian for this system can be explicitly represented in matrix form on basis $(\Psi^\dagger\,\Psi)^T$ in Nambu space by

$$
\begin{aligned}
\mathcal{H}_{\boldsymbol{i},\boldsymbol{i}} &= \begin{pmatrix} -\mu & 0 \\ 0 & \mu \end{pmatrix}, \\
\mathcal{H}_{\boldsymbol{i}\pm\hat{\boldsymbol{e}}_x,\boldsymbol{i}} &= \begin{pmatrix} -\lambda_0/2 & \mp\Delta_\mathrm{s}e^{i\theta}/2 \\ \pm\Delta_\mathrm{s}e^{-i\theta}/2 & \lambda_0/2 \end{pmatrix}, \\
\mathcal{H}_{\boldsymbol{i}\pm\hat{\boldsymbol{e}}_y,\boldsymbol{i}} &= \begin{pmatrix} -\lambda_0/2 & \mp i\Delta_\mathrm{s}e^{i\theta}/2 \\ \mp i\Delta_\mathrm{s}e^{-i\theta}/2 & \lambda_0/2 \end{pmatrix},
\end{aligned} \tag{6.1}
$$

with $\hat{\boldsymbol{e}}_i$ being the unit vectors pointing in the x, y directions, and $\boldsymbol{i}$ the position of each site. During simulation, we set $\mu = \mu_g$ for the sites in the gate region being manipulated, and $\mu = \mu_0$ elsewhere. While there are infinite many ways to manipulate our gates, we shall first look at a simple linear manipulation as shown in fig. 6.3.

By doing diagonalizing (numerically) the complete Hamiltonian in matrix form with 20 sites per wire and one central site connecting four wires, one can obtain the energy spectrum of the system throughout the braiding process, presented in fig. 6.4. The abscissa shows the instant of the braiding, mapped to the energy of each level on the ordinate. The color lines correspond to the Majorana modes of same color in figs. 6.1 and 6.2, and the black line is a bulk excitation. Notice that the Majorana bound states are not exactly zero modes but have a finite small energy. This energy originates from the coupling or anti-coupling of Majorana bound states on the same wire, i.e., the overlap of their "tail" decaying away from the edge into the bulk. As such, modes that are farther away from one another have lower coupling energy. Figure 6.4 shows the positive energy coupling, i.e., anti-bonding superposition of edge-states, and each level has an equivalent negative energy mode. The blue and red line modes are the one we intend to braid, hence we choose them as our initial state for the simulations and observe their time evolution.

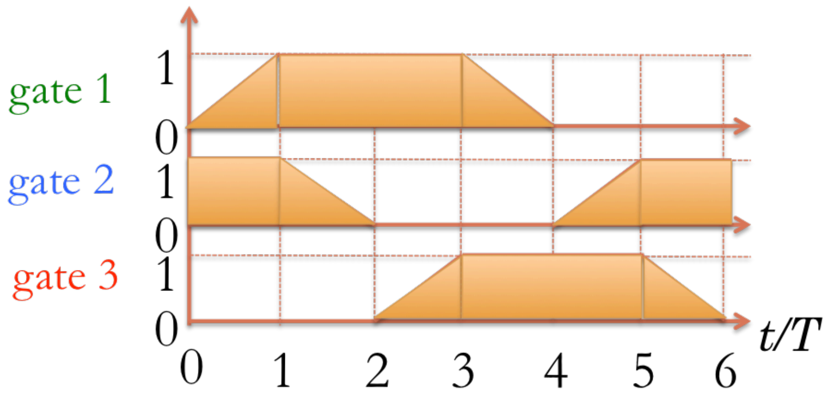


Figure 6.3: Process of gate manipulation. Value 1 indicates connection of the link controlled by the gate, and value 0 stands for complete shut down between regions connected by the gate.

As our initial state is an energy eigenstate, it is actually a single particle wavefunction. It may seem weird that a single particle wavefunction can account for a braiding process that happens in a many-body system requiring at least four Majorana bound states. Nevertheless, the braiding process for a NOT-gate itself involves only two Majorana excitations, which together form a single complex fermion, hence a single particle. We will address this question again once we discuss Hadamard gate simulation.

Starting with the wavefunction corresponding to the eigenstate of the red modes presented here and successively applying the time-evolution operator to it[xxviii], one can obtain the dynamics depicted in fig. 6.5, with each wire length set to 40 sites plus 10 sites for the gate region, which is set to $\mu_g/\mu_0 = 4.0$ for blocking connection, and $T\Delta_{\rm s} = 1000$. This figure show a successful braiding process that we wish to obtain.

On the other hand, it is possible that the Majorana bound state is destroyed during a too abrupt manipulation. We see in fig. 6.6 and example of a too fast manipulation that induces bulk excitations and disrupts the edge-state. The opposite case, a too slow manipulation, can be observed in fig. 6.7. The slower we follow, the closer we get to the adiabatic limit where adiabatic theorem prevails and no transition happens. Physically, the theorem manifests itself by the transmission of the whole Majorana pair, instead of a single edge-mode. This can be easily understood by realizing that the finite coupling between the edges dictates an energy scale below which the motion of one edge-state will drag along the other state bonded to it. In other words, the Majorana coupling places an energy threshold to break the pair apart.

But to know whether the braiding process is really successful, one must look not only at the squared wave function presented in the pictures, but on the state evolution according to our (initial) bases. The expected effect of a is to change the initial state of the wavefunction to an orthogonal one. For instance, if we write our initial state as particle number for the (delocalized) complex fermions formed by the free Majorana bound states, we may expect a transformation like

$$\|00\rangle \overset{\text{NOT}}{\longleftrightarrow} \|11\rangle. \tag{6.2}$$

Hence, we need to verify whether such transformation is being achieved or not. Let us label the energy eigenstates of the many body Hamiltonian as $|\pm E_i\rangle$, where the sign indicates the energy sign, and the index labels them from one to the number of states. So, in the spectrum shown here (fig. 6.4), the green line shows the energy eigenvalue evolution of the eigenstate $|+E_1\rangle$, the blue one represents level $|+E_2\rangle$, the red one stands for $|+E_3\rangle$, the black one, $|+E_4\rangle$, and so on for the levels above. So, our initial wavefunction starts at $|\psi(0)\rangle = |+E_3\rangle$, and we want to see how does it evolve, specially after a time $12T$,

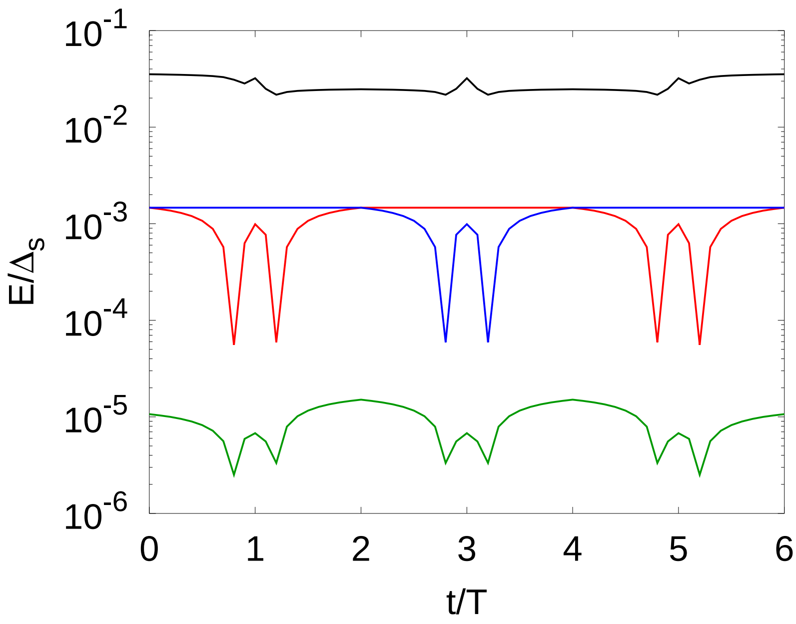


Figure 6.4: Energy spectrum evolution of the first positive energy eigenstates of the hamiltonian. The color os the levels match the color of majorino pairs in figs. 6.1 and 6.2.

when we have a full cycle. We can summarize as the track of these projections

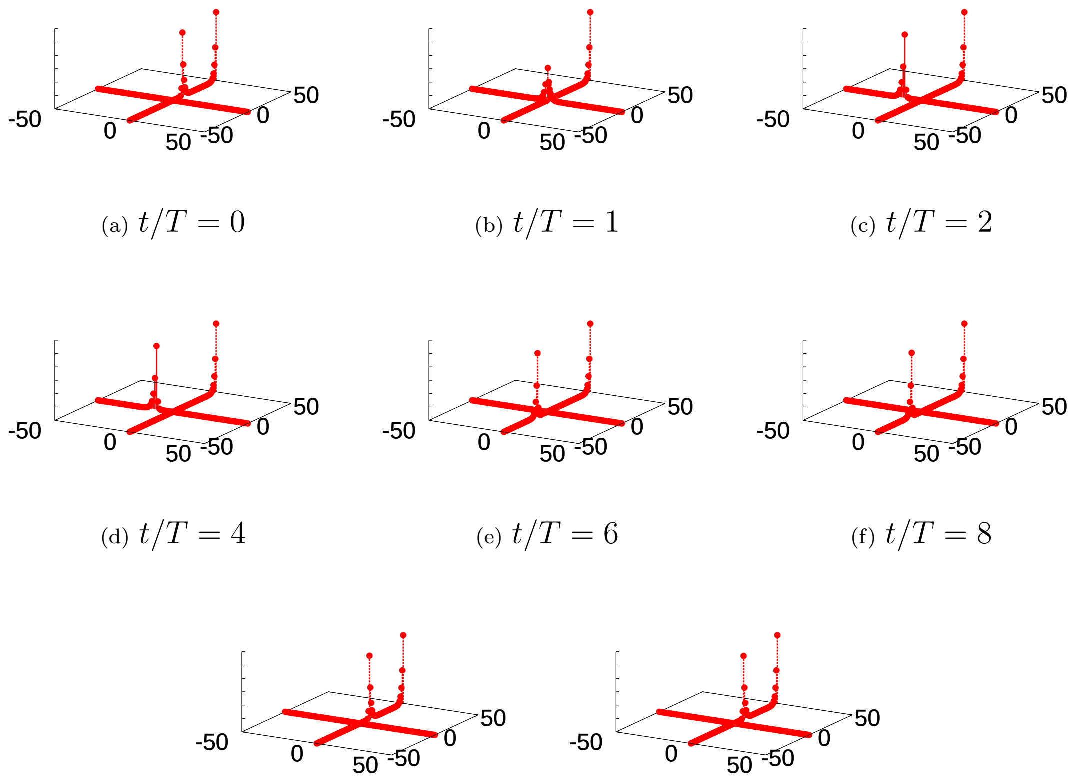
$$P^{\sigma i}(t) \equiv |\langle \sigma E_i(t)|\psi(t)\rangle|^2 \quad (= \delta_{s+}\delta_{i3}\,\text{at}\,t = 0), \tag{6.3}$$

with $\sigma = \pm$. By observing how these projections evolve throughout the braiding process, we can evaluate the success of our topological gates and calculate their holonomy.

Figure 6.8 shows the projection of the wavefunction on different bases. One can see that we have started completely in the state $|+E_3\rangle$, which is gradually emptied in panel (a), being completely transferred to $|-E_3\rangle$ in the end. The exact behavior expected from a NOT-gate.Figure 6.9 shows both the fast and the slow cases. We see that for a small T, probability seems to not be conserved only within Majorana bound states, remaining barely half of the wavefunction in the original state. The rest of it, necessary for conservation of probability, is spread across bulk excitation states, as well as other Majorana bound states. In the slow, large T situation, on the other hand, we have the wavefunction rather conserved within the Majorana states manifold, just no longer exclusively in one level. Looking at the simulated dynamics the reason should be clear: some of the density of probability is left on the way as bound states to other wires. Their oscillation between opposite edges disturbs the capacity of braiding one and only one majorino completely.

But the amount of energy injected in the central gates during the same amount of time (the power of the control gates) naturally affects the braiding process too, since to much energy at once can also induce excitations. This effect can be observed in fig. 6.10. For higher chemical potential adjustment, success rate requires longer operation times, limited by a power threshold dictating how much energy can be injected without breaking down Majorana modes. Though all this may look like a lot of requirements, it actually shows the possibility of accommodating multiple parameter conditions on various range. If we disregard the effect of the gate length for simplicity, we can see the trade-off between wire length and operation time in fig. 6.11.

We can indeed reduce the calculation cost of these simulations by simply ignoring the physical gate and applying the gates' effect by a parameter $g \in [0, 1]$ multiplying the nearest neighbor interaction, effectively turning hopping and pairing on/off. The results obtained are essentially the same, with different limits



(a) $t/T = 0$ (b) $t/T = 1$ (c) $t/T = 2$

(d) $t/T = 4$ (e) $t/T = 6$ (f) $t/T = 8$

(g) $t/T = 10$ (h) $t/T = 12$

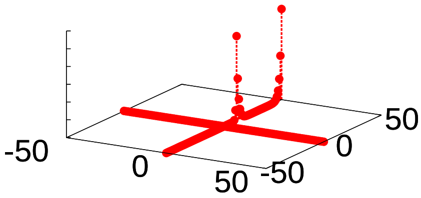
Figure 6.5: Squared wave function on real space at each instant for a pair of Majorana modes. x and y axes are given in site number. $T\Delta_{\mathrm{s}} = 1000$

being analyzed with more ease. For instance, adiabatic and quench limit are shown in fig. 6.12. The energy spectrum in this case is shown in fig. 6.13 with essentially the same behavior. For a successful braiding, fig. 6.14 shows the projections of a wavefunction initially in state E_3 on all the Majorana bases in the spectrum of fig. 6.13.

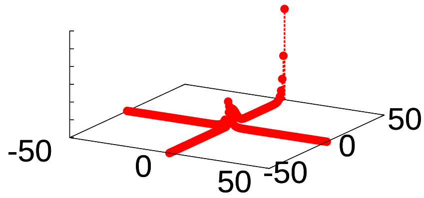
Note that these results are done with a linear operation of the gates, that can still be smoothed for better braiding results. Figure 6.15 shows the space of gate parameters and different possible paths for their manipulation. Linear manipulation, i.e., change of one gate per time, is the outermost possible path to follow, but there are infinitely other many. One should expect smoother gate manipulations to introduce less disturbance and lead to more accurate results. Choosing only smooth gate paths according to table 6.2 for comparison, we can indeed numerically confirm better performance of smoother gates in fig. 6.16, i.e., the higher the order of continuity (up to n-th derivative), the higher the success rate.

6.2.2 ··· Hadamard gate

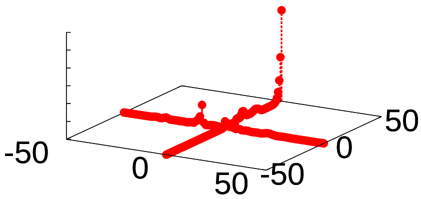
The braiding path needed to be followed to create a topological Hadamard gate $H = \frac{X+Z}{\sqrt{2}}$ is more complex than the NOT-gate, involving operations with three Majorana fermions. The necessary braiding procedure is shown in fig. 6.17 (a), with a change of the cross-juntion to a star-junction depicted in fig. 6.17 (b) for performing such braiding, since the movement of three edge-state are required.



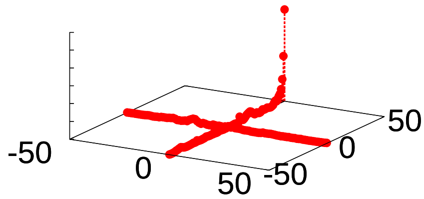
(a) $t/T = 0$



(b) $t/T = 1$



(c) $t/T = 2$



(d) $t/T = 6$

Figure 6.6: Squared wave function on real space at each instant with too fast operations. x and y axes are given in site number. $T\Delta_\mathrm{s} = 100$

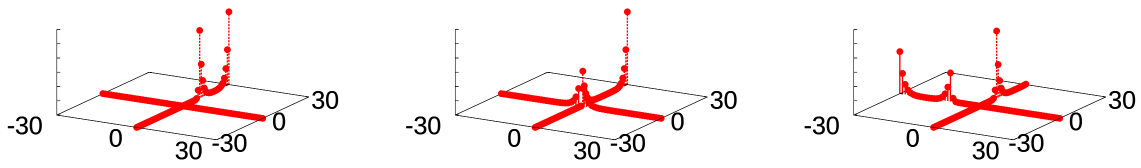
Path function	on	off	idle
case 1	$\sin(t')$	$\cos(t')$	0
case 2	$1 - \cos(t')$	$1 - \sin(t')$	0
case 3	$\sin^2(t')$	$\cos^2(t')$	0
case 4	$\sin(t')$	$\cos(t')$	$0.2\sin(2t')$

Table 6.2: Different gate manipulation functions. Columns indicate how gates are turned on/off, and how the third gate is operated in the meanwhile. $t' \equiv \mod(t, 2T)\frac{\pi}{2}\frac{1}{2T}$.

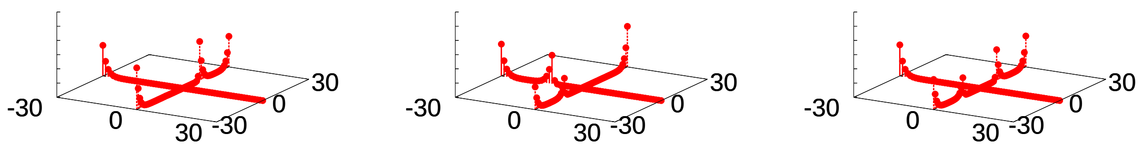
Knowing the necessary braiding process to be analyzed, it is straightforward to change our implementation so far in order to simulate a Hadamard gate. Nevertheless, as our simulation use what is actually a single-particle wavefunction, a caveat remains. Supposing our initial state is given by $\|\tilde{0}\rangle = \|00\rangle$ (with tilde representing logical bits), the action of a logical Hadamard gate becomes

$$\begin{aligned} H\|\tilde{0}\rangle &= \frac{\|\tilde{0}\rangle + \|\tilde{1}\rangle}{\sqrt{2}} \\ &= \frac{\|00\rangle + \|11\rangle}{\sqrt{2}}. \end{aligned} \tag{6.4}$$

Different from the case of a NOT-gate, this is a non-separable state which we cannot evaluate completely with a single-particle wavefunction. In other words, the holonomy for this kind of topological gate mixes the Majorana pairs. In order to obtain such holonomy's components, we need to look at the density



(a) $t/T = 0$ (b) $t/T = 1$ (c) $t/T = 4$



(d) $t/T = 6$ (e) $t/T = 8$ (f) $t/T = 12$

Figure 6.7: Squared wave function on real space at each instant for the slow case of majorino exchange. x and y axes are given in site number. $T\Delta_{\mathrm{s}} = 10000$

operator for this system, i.e.

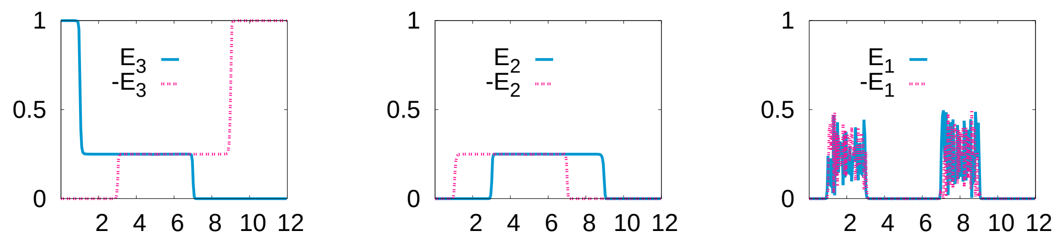
$$\rho_+ = \|+\rangle\langle+\| = \frac{1}{2}(\|00\rangle\langle00\| + \|00\rangle\langle11\| + \|11\rangle\langle00\| + \|11\rangle\langle11\|). \tag{6.5}$$

It is straightforward to calculate the partial density matrix for a single particle (a single Majorana pair), calling each Majorana pair A and B, as

$$\rho_A = \mathrm{tr}^A \rho_+ = \mathrm{tr}^B \rho_+ = \rho_B = \frac{1}{2}(|0\rangle\langle0| + |1\rangle\langle1|). \tag{6.6}$$

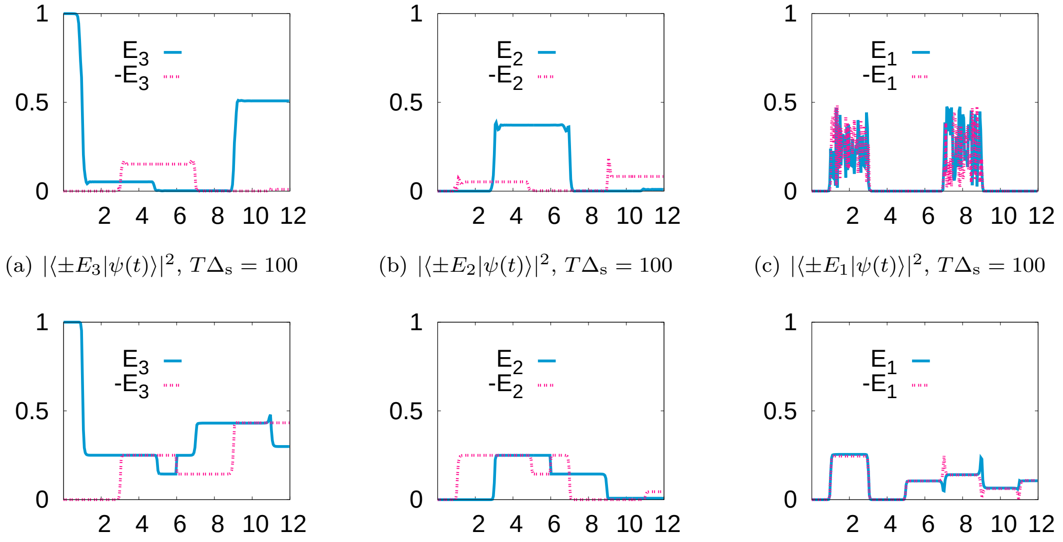
In other words, $\langle i^\alpha|\rho_\alpha|j^\alpha\rangle = \delta_{ij}/2$. As such, we expect that, starting with state $|0^a\rangle$, once its braiding is done and we are left with state $\Gamma_H|0^A\rangle$ (Γ_H being Hadamard gate's holonomy), its projection to other bases must lead to

$$|\langle i^\alpha|\Gamma_H\ |0^A\rangle| = \frac{1}{2}, \tag{6.7}$$



(a) $|\langle\pm E_3|\psi(t)\rangle|^2$ (b) $|\langle\pm E_2|\psi(t)\rangle|^2$ (c) $|\langle\pm E_1|\psi(t)\rangle|^2$

Figure 6.8: Projection of the wave function on the instantaneous eigenstates $|\langle\sigma E_i(t)|\psi(t)\rangle|^2$



(a) $|\langle \pm E_3|\psi(t)\rangle|^2$, $T\Delta_\mathrm{s} = 100$
(b) $|\langle \pm E_2|\psi(t)\rangle|^2$, $T\Delta_\mathrm{s} = 100$
(c) $|\langle \pm E_1|\psi(t)\rangle|^2$, $T\Delta_\mathrm{s} = 100$
(d) $|\langle \pm E_3|\psi(t)\rangle|^2$, $T\Delta_\mathrm{s} = 10000$
(e) $|\langle \pm E_2|\psi(t)\rangle|^2$, $T\Delta_\mathrm{s} = 10000$
(f) $|\langle \pm E_1|\psi(t)\rangle|^2$, $T\Delta_\mathrm{s} = 10000$

Figure 6.9: Projection of the wave function on the instantaneous eigenstates $|\langle \sigma E_i(t)|\psi(t)\rangle|^2$ for (a)-(c) fast and (d)-(f) fast gate manipulation.

or conversely $|\langle i^\alpha|\Gamma_H\|0^A\rangle|^2 = \frac{1}{4}$. Also, we may confirm Hadamard gate's hermiticity in order to assure we are not simply stopping half-way of a NOT-gate, for example. We can verify it by following the projection of the wavefunction on the Majorana bases through the whole braiding process executed twice. These projections can be seen in fig. 6.18. As expected, applying Hadamard gate once distributes the wavefunction on four different bases (here, $\pm E_i$, with $i = 3, 4$), and executing this braiding process twice returns the wavefunciton to its initial state.

As Hadamard gate is also a Clifford gate like the NOT-gate, the simulations lead to essentially the same results as already discussed for the NOT-gate in terms of control parameters like time and length scale, hence we shall not repeat to the same discussion as the previous exposition still holds true.

6.3 Delocalization: the ethereal being

All the discussion carried here was essentially a presentation of simulation results, an exposition of the braiding properties of zero dimensional Majorana bound states on one dimensional chains (restricted to nearest neighbor sites interaction, whence the adequate usage of the word *chain*). Nevertheless, it would be blatantly dishonest, in my opinion, not to address the fuzziness of the situation involved: divide a particle (itself somewhat elementary) into two and let each half hold a piece of information unity (half unity?), keeping it completely "hidden" from the environment (though in principle observable). While hidden, move the pieces around to manipulate information without actually having any access to it, even if the difference of the initial and final state is not completely clear. This is probably the perspective one would have from the present scheme, admittedly nothing intuitive and not trivial in the least. Hence, we shall discuss this phenomenon called *delocalization*.

In section 2.3 we discussed the conceptual difference of having a pair of particles and having two (somewhat existentially[xxix] independent) particles. With such difference in mind, we must first point that we have *pairs* of Majorana particles. As such, a pair is not really two, but one single particle somehow extended through space. And once this is a two-dimensional space like the surface with two wires we explored here, this makes a great deal of difference, making the word "braiding" considerably literal in

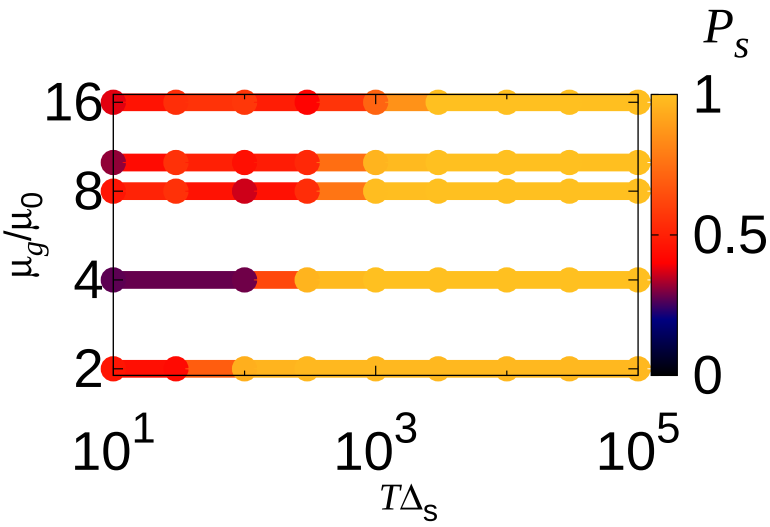


Figure 6.10: Fraction of the wavefunction successfully braided as a NOT-gate with gate length $L_g = 10$ sites and wire length $L_w = 40$ sites.

character. If we just think of such Majorana bound states as ends of threads extending through each wire, the act of twisting different wires' edges around does indeed produce a braid. Nevertheless, though this picture makes it easy to understand the act of braiding (two threads being braided), the image of the "thread" itself may not be completely clear.

What *is* this thread we talk about? Mathematically, it is the breaking down of one complex subspace into two pieces. It is the transformation of one complex algebra into a real algebra (as $\mathbb{C}$ and $\mathbb{R}$ algebras). But this separation of complex fermions do not have to carry physical meaning, as any other mathematical transformation. It is worth remembering that, though wavefunctions and phase factors have no physical meaning per se, they do imply some physical aspects that we actually observe. To expect that the real and imaginary part of a complex variable whose phase is irrelevant would bear an absolute meaning is naïve. However, it cannot be discarded without consideration either. Just like geometric phases cannot be just thrown away and account for real observable physics, so may such real and imaginary part of operators. And this happens with Majorana fermions in condensed matter. In a sense, the geometric phase or holonomy in question gives birth to very real edge-modes corresponding to such real and imaginary parts.

But what does it has to do with the thread we can imagine links different edge-states? Everything. These edge-modes are the product of a many-body system moving "differently" according to direction. They are a pair linked in the bulk of the whole system. Though may be seen independently one by one, they are a happily married couple from their conception. Mathematically, if they are taken together, they will look like a single complex fermion, a single body with each "foot" in the opposite extremity of space. In this sense, we have *delocalization*: the split of what can be regarded as one body extended through space. But on a different view, where we think of each "foot" as independent from each other, we could indeed forget the whole body and think only of the (apparently) independent "feet." But in reality, they are bound to one another.

The idea of a particle extended through space is not specially peculiar. A photon with a well-defined frequency ν, for instance, would also have a well-defined wavelength, which would mean the photon is infinitely extended through space. In this case, too, we have a particle (this time a fermion) extended through the whole space, except that the space itself (i.e. the wire) is confined to certain limits. Now, we do not speak of frequency[1], but we have one particle with pronounced complimentary characteristics

[1] In principle, the zero-modes are at the Γ-point with zero momentum, zero wave number, zero wavelength, just sitting

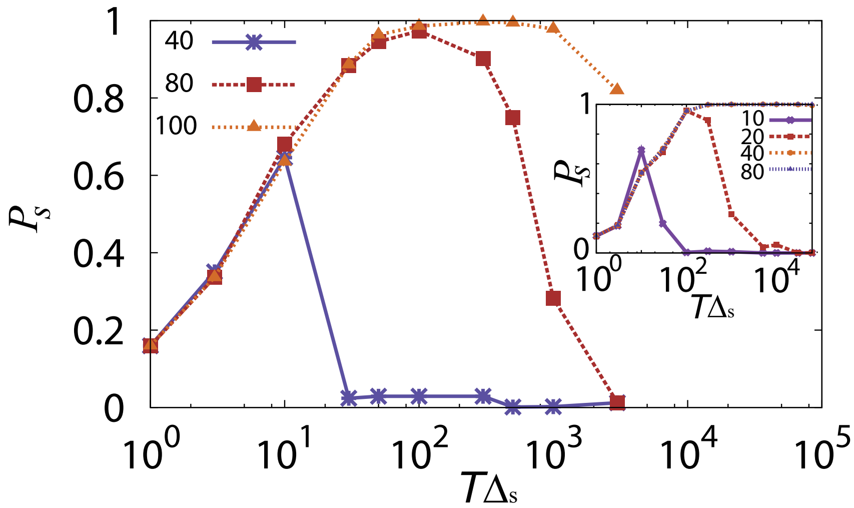


Figure 6.11: Fraction of success of the braiding process for different lengths given in number of sites, when gate length is disregarded. Outset is calculated with $\lambda_0/\Delta_s = 10$, inset with $\lambda_0/\Delta_s = 1$

(peaks) on opposite extremities. There is where we have our invisible thread.

But what happens with braiding? How can, without any touching or direct interaction, one have state transitions? Again, we must remember it is not an idle pair that we have, but one of many-body interaction in the bulk, which is not really static. By moving one edge around with the system's Hamiltonian, it is forced to pick a (non-Abelian) geometric phase in accordance with the bulk interactions it crosses. As a result, the correlations two edges might share may flip or recombine in different ways, according to the holonomy of the path followed.

Now, if we simply assume that such Majorana edges can mutually correlate or anti-correlate (or a linear combination of both), we obtain two possible states that may be thought of as a qubit, for instance. As so, the information will be encoded not on each "half" but on the whole, since it is their mutual correlation that defines the whole state. This gives us a delocalized information unit, one that relied on the relation of two pieces that are not disconnected. If they were independent, each one could change freely, but once they are somehow fixed through the bulk, one has to do a more intrusive operation (like braiding) to change their relative state. In other words, delocalization happens exactly because we have a many-body system creating a "synchronous" behavior extended through space that we pretend to be a single body.

Notes

[xxvii] In Japan, 赤い糸 (*akai ito*, a red thread or destiny knot) is said to link two people destined to one another, two souls meant to complete each other. They should, therefore, be a perfect match, a perfect and inseparable couple.

[xxviii] Here, we use Chebyshev polynomials to expand the time-evolution operator $e^{iH\delta t}$ for a small time interval $\delta t\Delta_s = 0.001$. In fact, given the small variation of the wavefunction, it is possible to speed-up simulations by using mixed precision to calculate only the variation of the wavefunction in single precision, and adding to the whole wavefunction in double precision.

[xxix] And here I throw the word "existentially" without giving proper account of it. Does it has anything to do with *existentialism*? Well, as a quantum phenomenology, yes and no. Yes, in the sense that the particles we talk about, entangled or not, will indeed have their existence preceding essence, in whatever "existence" we might want to define, but I would stick with particle number conservation for sculpting a well-defined "existence." However, we are not talking about humans here, hence this may have quite different nuance and implications. Still, if we are to refer to Sartre, two particles' correlation and coherence can be readily disrupt by other various sources in the environment, fitting pretty in a possible definition of "hell" being "the Others" (*L'enfer, c'est les autres* – Jean Paul Sartre, *Huis Clos*, 1944).

still.

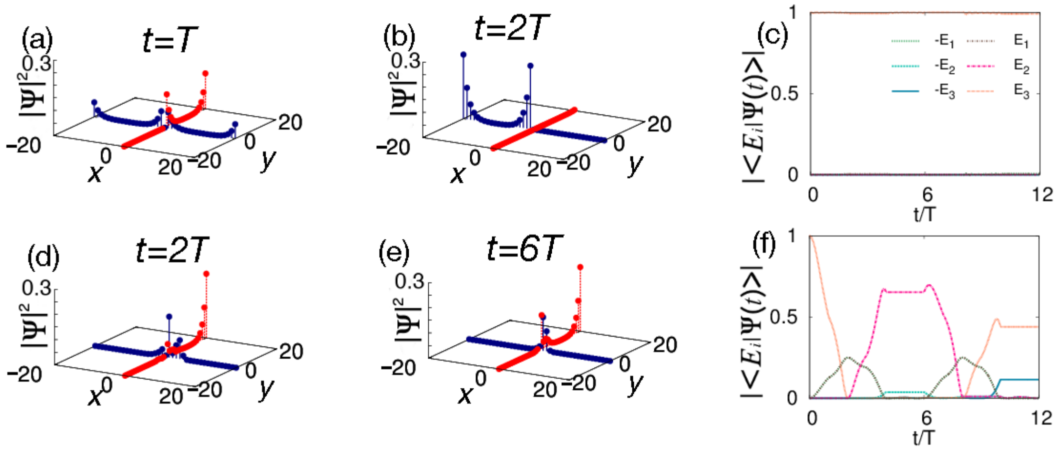


Figure 6.12: **Top:** Adiabatic limit with $T\Delta_\mathrm{s} = 100000$. (a), (b) Evolution of the wavefunction. (c) The projection of the wavefunction on eigenstates remain almost static in the initial state. **Bottom:** Sudden limit with $T\Delta_\mathrm{s} = 1$. (d), (e) Evolution of the wavefunction. (f) Projection of the wavefunction on Majorana eigenstates is seen to have an erratic spreading. The color of the wire in x-direction has been changed for clarity, but represent the same wavefunction.

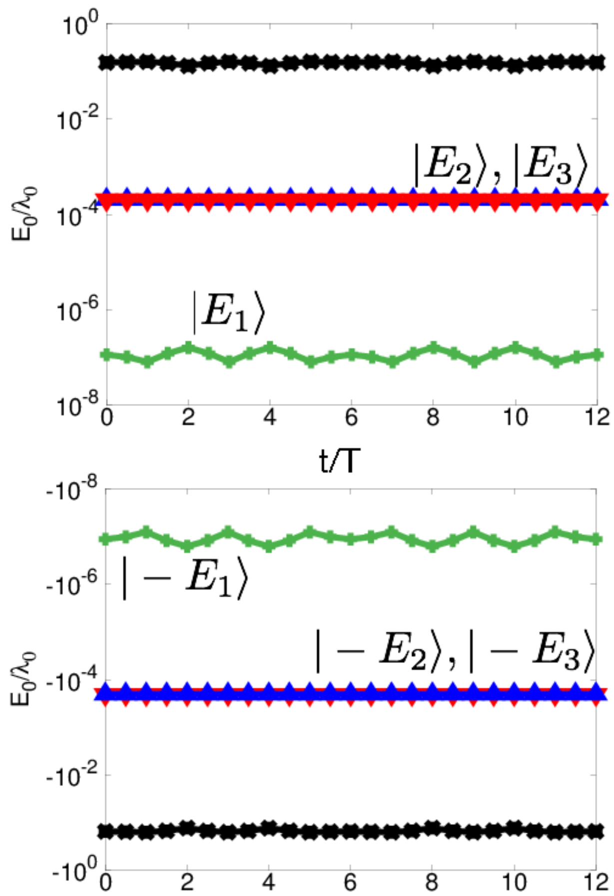


Figure 6.13: Energy spectrum for finite wires. Horizontal axis shows the time-evolution of each level, and vertical axis shows the eigenenergies.

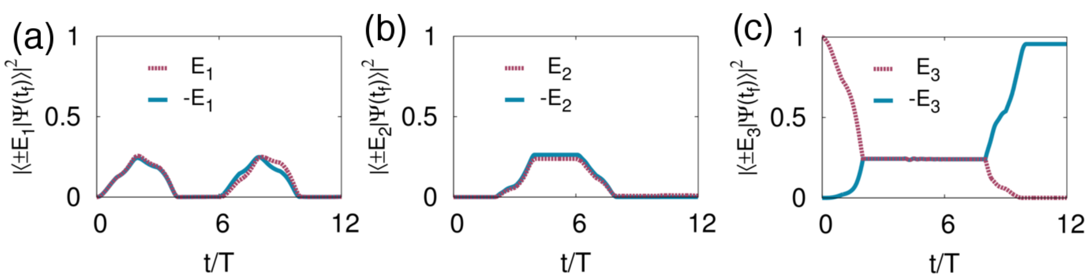


Figure 6.14: Projections $|\langle \sigma E_i|\psi(t)\rangle|^2$ for zero-size gates. $T\Delta_\mathrm{s} = 100$.

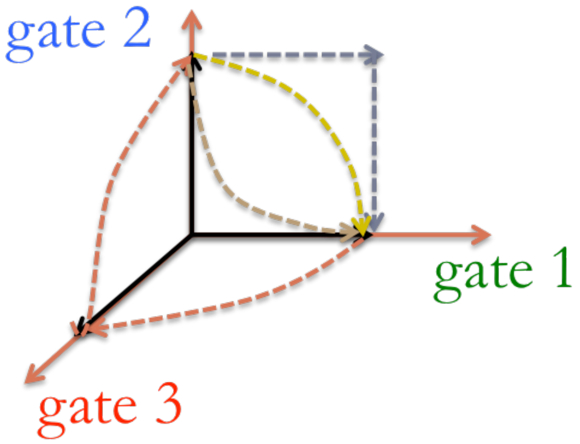
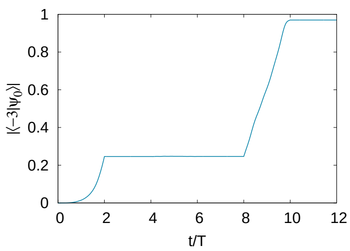
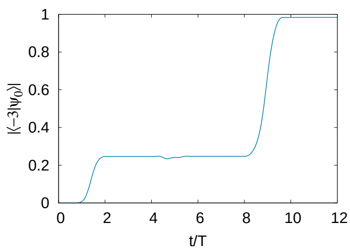


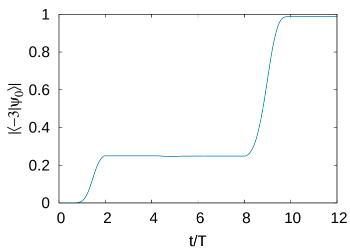
Figure 6.15: Parameter space formed by gates g_i. The gray arrows are followed for a linear manipulation of gates, while colored arrows indicate other possible paths.



(a) case 1

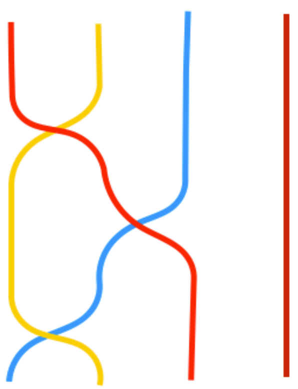


(b) case 2

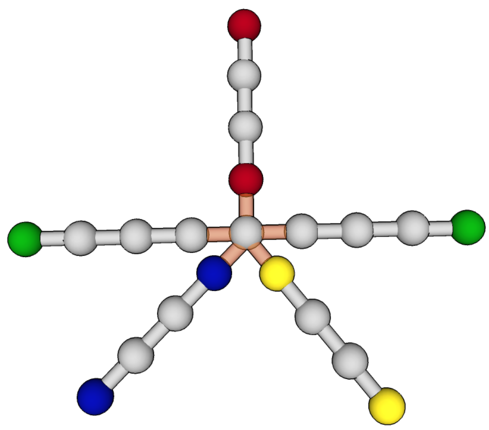


(c) case 3

Figure 6.16: Projections $|\langle E_3|\psi(t)\rangle|^2$ for smooth gates. $T\Delta_\mathrm{s} = 100$, 20 sites/wire.

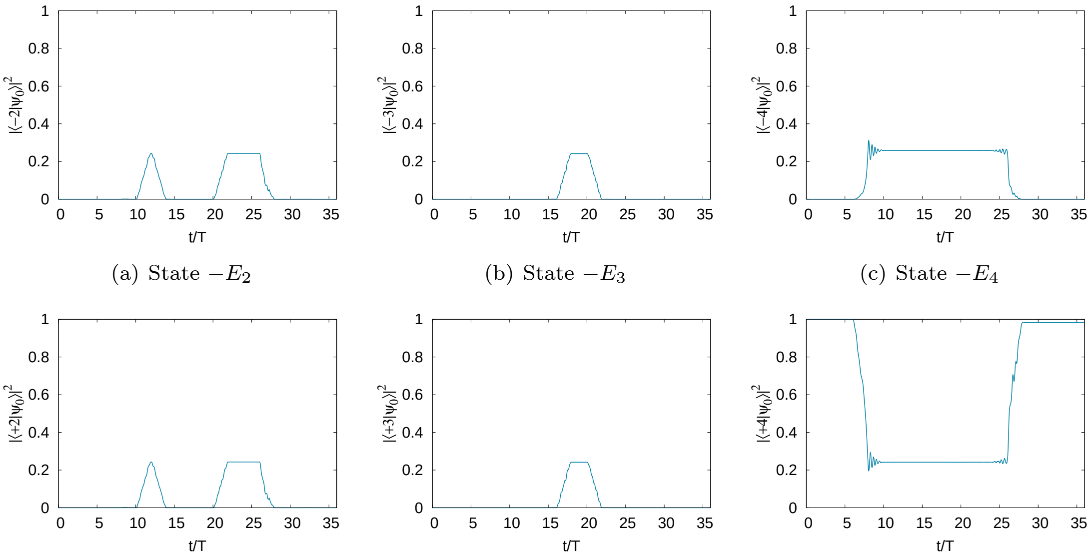


(a) Topological Hadamard gate



(b) Star junction

Figure 6.17: (a) Hadamard gate braiding process. (b) Star junction for Hadamard simulation



(a) State $-E_2$ (b) State $-E_3$ (c) State $-E_4$

(d) State E_2 (e) State E_3 (f) State E_4

Figure 6.18: Projection of the wavefunction during Hadamard braiding process done twice, starting from state E_4. Upper panels show negative binding energy bases and lower panels positive energy ones. At $t/T = 18$, one Hadamard gate braiding is complete.

Level 7
Knowing souls

"Tell me one last thing," said Harry. "Is this real? Or has this been happening inside my head?"
"Of course it is happening inside your head, Harry, but why on earth should that mean that it is not real?"

— J. K. Rowling, *Harry Potter and the Deathly Hallows*, 2007.

7.1 Anewment enlight

WHAT A METEMPSYCHOSIS! — One should thing when dealing with those haughty lizards. But we know them well. We know them as we know ourselves, for we live with them and in them even if they do not realize. Dragons and fey alike bear with us, in their innermost mood; changes in their state of mind that reflect on every bit of their actions, like wing beating for one thing. And as far as we know, you humans started to treat us in their inner mood under something called *information theory*. This inner mood connects all of us together, or isolates us, just like we ourselves do. Since you already know about them, it is time for us to show you what is it you call *information* from our perspective.

On the light of a new day, here we are back to where we began; dragons and fey alike. But how can you tell? How can you know one is one and other is other? How can you learn? How can you forget? This is where we manipulate their desires and their strength! Even when they gather as one to live in peace, they cannot forget us. We are the ghostly presence guiding them, pointing the way to and from death, pushing their will to and fro. We let them stay alive even near death, and let them live back following our will. There is no fey or dragon when it comes to reverie! There are only the directions to be taken, the paths to be chosen. And behind that: *the chooser*.

We may be less vicious or less gracious, but not because we are *less*, yet because we simply *are*. The valleys and cascades may guide a dragon, but we are all that is to be. The joy and freedom may exalt fey, but dwell everywhere. You may have a hard time to see us for we are in all and with all! How can one see it?

Let the dragons return, let the fey disperse, and I will tell the wings where to flap. Let them eat their fire, let their spell fail or prevail, and we shall tell where to find them! For we know it all. We are all that is to know.

7.1.1 · · · Puppetiers

Here we are to let them resurrect, but only as long as it fits our intensions. We are the steerers of nature's wheels, and these beasts are nothing better than you or they might think they are. They all live in accordance to our determination. We tell them where to go and how to go. They may have their own freedom you probably call or associate with energy, but if it were not for us, they would know nothing about what to do with it.

So, here we enter in their lives. We guide and control them. When they have energy to move, we tell them to move left or right, for instance. All decision of life and death is in our hands; we are gods in our way! Humans needed some time to realize our presence. And after they finally did, they still needed to understand that we pervade what they call *the Universe*. And we define universe's structure, and we guide al the events by guiding all the players in our games.

Even the dragon fey and his susykin spells is not above us: we too have our "superfellows" to reach every instance of their existence. We may go through body and soul to keep hold of any strings we wish to pull. And in this case, we can keep an eye on each and every particle under our control with stronger ties than you could possibly imagine!

And don't be fooled by their bragging! When they say they entangle and know each other, they know nothing! At least, nothing that we do not allow them to know. They must first realize our presence to know anything, or they know nothing at all. Whatever they pretend to know, we limit it as we please. They cannot know more that whatever they know about as; they can only know from other as much as they know from us. Humans started calling it *information causality*, for obvious reasons. They only know what they know from us beforehand, if anything at all. So much so that humans started to recognize in us the true meaning of things, the true meaning of nature, the true meaning of the universe. Humans simply assume we exist before all: units of information glaring at any defiance.

So waste not your time with those creatures' caprice! We are the ones you should look at, the real rulers of high purity! These dragons and fey think they can overcome us. They think they are better than us. They think they are more important than us. And they believe they can overcome us, in whatever the new shape they take, as if they could shape-shift as they please. But they can't.

We will not leave them to their will. How could such pitiful creatures possibly expect to juggle us and suppose we could passively be sent back and forth as they please? They are the ones who do whatever *we* please. If they fall to ghastly dragons split in space, it's because we want. If the ones of us inside them move around and shift, it is our will. And we will track them and trap them to our will whatsoever!

We are what is *real*.

7.2 Information physicality

In the previous levels, we have been looking at physics, and brought some aspects of information science when looking at it, notably when using words like "qubits." But the center of the stage has been occupied almost entirely by physics-motivated considerations. Now, it is time to lean towards information science, bringing information and information units closer to physics and physical contents.

This topic is covered with considerable information theoretical details in refs. [28, 109–111].

7.2.1 · · · From cbits to qubits

1936 was an important year for our modern era. It was the year when Church and Turing independently developed the Church-Turing thesis [112, 113]. Their work targets computable functions or computable numbers, and may be seen as the advent of modern computing that we are familiar with. Church used λ-calculus to study computable functions, while turing idealized a machine, now called Turing machine, to crunch computable numbers, and showed their equivalence. Following Turing's definition, a computable number is any real number that can be calculated up to its last decimal in finite steps by a Turing machine. This Turing machine is essentially what we now know as a *computer* in a idealized form: it has a tape on which it can print symbols, and a table of rules to manipulate the tape and the symbols. Both Church and Turing showed the relation between computable functions, i.e., functions on from natural to natural numbers,[1] and the "decision problem" or *Entscheidungsproblem*. This problem essentially asks whether

[1] These functions' domain and image have cardinality $\aleph_0$, where $\aleph_0$ is the cardinality of natural numbers (in other words, the "number" of countable numbers).

an algorithm can take a logic statement as input and answer whether this logic statement completely satisfies a finite number of given axioms. Church and Turing showed that there is no general solution to this problem, and it became linked with what is now more widely known as the *halt problem*: given an input to a computer, can we tell whether it will halt (reach a conclusion) *before* running the program? In general, there is no way to know it except by running the program, the algorithm,

Turing's work organized the idea of a computing machine and paved the way for all the boom of computers and computer science that followed. In the 1980's, computers had already improved considerably in reliance and performance, and new questions were hovering in the air. Among them, what is the limit of computation? For Feynman, the answer was: when every particle in a computer is computing, that is the limit. He also pointed that to understand the quantum world properly, one would need a quantum computer, as one may already understand from the logic and mathematics exposed so far. And in 1985, Deutsch generalized the Turing machine to a quantum version in the Church-Turing-Deutsch thesis [114]. In fact, before Deutsch, it was known that an analog Turing machine or a stochastic Turing machine could leverage more computational power and efficiently solve more hard problems than a standard classical Turing machine, but Deutsch formalized how a quantum Turing machine could realize those. And as the turing machine became the grounds for modern computer machines, Deutsch's machine asserted the possibility of extracting information power from quantum system.

The interest in quantum computers grew bigger and bigger, even more after concrete algorithms for concrete applications were proposed, for instance by Shor [115] and Grover [116]. Shor's algorithm provided a way to efficiently factorize integers into its prime factors, a task that can normally only be realized by a laborious and inefficient scanning process of candidate prime numbers. Shor's algorithm perform the same task much faster by applying what is known as *quantum Fourier transform* (see refs. [6, 109]), being able to factorize numbers in polynomial time. Grover's algorithm allows efficient database search, another tedious task that classically requires to check entry by entry looking for a match. On the other hand, Grover's algorithm can solve this problem efficiently, with order $\sqrt{N}$ steps for N entries.

Despite the promising algorithms, the major problem with a quantum computer is the the same as its blessing: it runs on quantum states. Quantum states are fragile and easily lost, as we know from our lack of daily quantum experience. States may decohere, by mixing with the many other surrounding states in the environment or other means. To make a quantum computer, one has to create and control quantum states coherently, so the properties coveted for quantum information process are not lost. This requires control on state population as well as phase information, for both are actively and passively explored by quantum algorithms. Therefore, stabilizing quantum states has been a major issue for quantum information science research.

Kiatev came with a suggestion of using topological states as stable qubits and started the idea of topological quantum computation [117]. This is the idea behind what we saw in the previous level and is an important pilar for quantum information. Kitaev's work is on the bases of topological quantum computation with anyons [64, 65] as well as surface codes [118] that quantum computer makers try to realize in their circuits nowadays.

With these steps, little by little information theory drew closer to physical theory, with the mathematical aspects of quantum mechanics affecting information science as well as information theoretical approaches being used for enhancing physical insights and interpretations. On one side, quantum physics tells us that information encoded in portions of a quantum mechanical Hilbert space has more intrinsic computation power than classical computers. On the other side, information theory tells us that come concepts in statistical mechanics like entropy can be understood as the amount of a certain type of information of a given system. Information then becomes as physical as other physical quantities, with indelible mutual interest[2].

[2]Perhaps we will see a "Turing Prize meets Nobel Prize" situation in the future, where Turing Prize seems physics and Nobel Prize seems information science.

7.2.2 ··· Bits, qubits, superqubits, and beyond

Information science has greatly developed since computers became a major tool in the second half of twentieth century. Information encoded in bit strings had to be handled and communicated between various machines, subject to errors, imperfections, and various forms of failures. Not only accuracy has been achieved, but also algorithms for data compression, encryption, transmission protocols, and many other things. And now, we look towards a new direction: quantum information.

Classical information stored in classical bits are either deterministically or stochastically handled, according to the resources one has available. Quantum information, on the other hand, is stored in quantum states living in a Hilbert space. Even if a qubit can be simply seen as a two-level system, linearity of quantum mechanics allows for a large number of possible superpositions, creating entanglement (discussed in previous levels) and allowing for its use as a resource. While a classical bit disposes only of classical computations, quantum information may undergo, in principle, any reasonably approximated unitary transformation. This permits the realization of feats like superdense coding and quantum teleportation. Also, while the freedom of classical bits is restricted to its dichotomic value, quantum bits have the intrinsic quantum feature of *phase*. By itself, a qubit's phase (or any quantum state) bear not much meaning. But phase difference has physical consequences that can also be used in algorithms to leverage the quantum power of an information source. Phase estimation algorithm (or subroutine) appears in many quantum algorithms available to date and, together with a quantum Fourier transform, is at the core of quantum information field [6]. In a sense, the phase degree of freedom of qubits supplies a new resource chain parallel to their values.

This is the first connection we face between information and physics: the change in description of information units (bits) matching the change in description of physical units (quanta). But this is not enough, and indeed nothing but a faint analogy. The central pillar of classical information theory is undoubtedly Shannon theory built around the concept of Shannon entropy. This allows a variety of analyses of information quantities, as well as concepts of typicality and so on. This counterpart is also defined in quantum information theory, where bit strings must, in general, be replaced by density matrices of qubits.

But if qubits bring information and physics to the same level of existence, is there anything beyond? Perhaps. First, within quantum mechanics, one may consider supersymmetry implications. Although supersymmetry is manifestly broken in nature, emergent supersymmetry is discussed in some hypothetical condensed matter setups [119]. This emergent supersymmetry is not the same as supersymmetric quantum mechanics, but a variation for field theories with spacetime supersymmetry. Nevertheless, supersymmetric qubits, or superqubits have also been studied in recent years [54]. Defined on a super Hilbert space, the superalgebra of superqubits follows generalizations of standard quantum mechanics to a supervector space. The supervector states can be written as

$$|\Psi\rangle := |q\rangle a_q + |\bullet\rangle a_\bullet \equiv |X\rangle a_X, \tag{7.1}$$

where $q = 0, 1$ and a_q is commuting (even), while $a_\bullet$ is anti-commuting (odd). The shorthand $X = (q, \bullet)$ can also be used. Explicitly, a superqubit can be expressed, under Grassmann algebra, as

$$|\psi\rangle = (1 - \frac{1}{8}\eta\eta^{\#})(\alpha|0\rangle + \beta|1\rangle) + \frac{1}{2}(-\alpha\eta + \beta\eta^{\#})|\bullet\rangle, \tag{7.2}$$

with α and β being even Grassmann numbers obeying $\alpha\alpha^{\#} + \beta\beta^{\#} = 1$ and η an arbitrary odd Grassmann number[3]. Note that this is actually a three-level system; two-level on its even part, plus an odd part level.

An interesting property of such superqubits is that they violate Tsirelson's bound seen in level 2 [120]. With one Grassmann generator θ_i per superqubit, a two-superqubit entangled state shared by Alice and Bob reads

$$|\Gamma_{AB}\rangle = \left(1 + \frac{1}{2}n + \frac{1}{8}n^2\right)\left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) + \frac{p}{\sqrt{2}}\theta_A|\bullet 1\rangle + \frac{q}{\sqrt{2}}\theta_B|1\bullet\rangle\right), \tag{7.3}$$

[3] Remember that the hash symbol # is a generalized superconjugation of a complex supernumber.

where $p, q \in \mathbb{R}$ and $n = -(p^2/2)\theta_A\theta_A^{\#} - (q^2/2)\theta_B\theta_B^{\#}$. By adjusting parameters in such state as $p \approx 0.7476$ and $q \approx -1.0949$, it is possible to obtain a probability of winning CHSH game higher than that achieved by standard quantum states without the soul part above ($p = q = 0$), if one takes modified Rogers map for calculating probabilities.

But while superqubits offer one possible way beyond basic quantum mechanical bits by extending the $\mathfrak{su}(2)$ algebra[xxx] to an equivalent superalgebra, there may be other generalizations we cannot immediately conceive. For this, one may consider a general (two-level) information unit in GPTs, which we call general bits of *gbits*.

7.2.3 ··· *Correlation boxes darker than black*

While generalizations of bits allows us to have new perspectives on information, what may the correlation of information units tell us? In levels 2 and 3 we saw CHSH inequality for the correlation of two qubits:

$$|\langle S_2 \rangle| = |E(\vec{a_0}, \vec{b_0}) + E(\vec{a_0}, \vec{b_1}) + E(\vec{a_1}, \vec{b_0}) - E(\vec{a_1}, \vec{b_1})| \leq 2\sqrt{2}. \tag{2.29}$$

But one can imagine a probability distribution, realizable or not, that actually surpasses this quantum bound of $2\sqrt{2}$. In the last section, one such probability distribution was concretely presented on the form of a state with one more dregree of freedom than quantum mechanics. But taking the liberty to forget any imaginable state and focusing only on the probability of outcomes, we can suppose the following probability distribution

$$\begin{aligned} p_{00}(+1,+1) = p_{00}(-1,-1) = p_{01}(+1,+1) = p_{01}(-1,-1) = \\ p_{10}(+1,+1) = p_{10}(-1,-1) = p_{11}(-1,+1) = p_{11}(+1,-1) = \frac{1}{2}. \end{aligned} \tag{7.4}$$

Under this hypothetical circumstance, all the joint expectation values in CHSH correlation equals to 1 and $\langle S_2 \rangle = 4$. The same idea applies for the case of n qubits, which can be summarized as

$$|\langle S_n \rangle| \leq_{\mathrm{QT}} 2^{n-1}\sqrt{2} \leq_{\mathrm{Alg}} 2^{2n}, \tag{7.5}$$

whew subscript "QT" indicates quantum theory bounds and "Alg" stands for the theoretical algebraic limit. Although a clear violation of Tsirelson's bound, it obeys no-signaling condition, i.e., there is no superluminal transmission of information. One can confirm it since the probability of measuring only one of the two qubits and obtaining a ± 1 result is completely independent of the other bit, and balanced at perfect randomness ($p = 1/2$). In other word, we are obey relativistic causality, and allowing for correlations that do not rely on hidden variable (nonlocality), but we still achieve much higher correlations than those found in quantum mechanics. How can that be? Is there something else, another fundamental principle limiting quantum correlations? Or could quantum theory simply be wrong? Why quantum theory is not *more* nonlocal? This problem was first pointed by Popescu and Rohrlich [121]. Such kind of unknown theoretical correlators became known as PR-boxes for perfect correlations, owing to their idealizers, or no-signaling boxes (NS-box) for a generic case. In recent years, three physical explanations to such problem have been provided: information causality [122], macroscopic locality [123], and exclusivity principle (E principle) [124].

Information causality provides a constraint to the amount of information one can extract or infer from a far away partner using only its local resources and classical information provided by its partner (local operations and classical communications — LOCC). If Bob receives m classical bits from Alice, he can infer at most m bits of information by performing local measurements on his side of an entangled system. Any NS-box that violates Tsirelson's bound is shown to violate this principle too on information theoretical grounds. Hence, superqubits, for instance, violate information causality, probably by transferring extra information through its odd dimensions.

Macroscopic locality focus on evaluating what theories may recover our classical world as its macroscopic limits. If shows that not every microscopic correlation can generate a classical model, limiting the possible set of bipartite correlations that pile up to make a classical setup with a large number of events. In a sense, the main motivation for constraint becomes the observations made with large number of microscopic events.

Exclusivity principle is a graph theoretical approach also framed in GPTs with sharp measurements [125, 126]. Two events that cannot be simultaneously true are said "exclusive," a condition satisfied when a sharp measurement contains both. E principle states that "any set of m pairwise exclusive events is m-wise exclusive," which implies that the sum of their probability is at most 1 [124]. It can indeed be inferred as a sharp measurement consequence, includying the so called exclusivity hierarchy. Applying this framework to n-body correlations S_n (seen in level 3), Tsirelson's bound is recovered.

7.3 Information & physics

Information is physical — these are the words used by Landau that embody the deep interaction physics and information science developed in the second half of twentieth century. But this can be disturbing: *how* can information be physical?

This is not evident at first, but became a lot more evident thanks to the advent of computer machines. To perform computation, one needs to feed a machine with two things: information to be processed, and energy to process it. From a physical perspective, one will have a physical system that, once fed energy, will perform some task. If it were an engine, this task is "work," which is also energy as expected. Some energy is converted into work, some energy is "lost" into entropy-rich heat. So what does a computer do?

The information processing also generates heat and entropy as is easily verified by noting how any smartphone heats up when heavily used. But, at least in principle, could it be possible to have no heat at all? Could one make a computer that did our calculations without entropy generation? In short, no. To perform computation, one need to flip and manipulate bits in various ways. Even if we had only two bits in a state '01' and wanted to switch them to '10' somehow, this implies a physical system that can store such state, say only electron reservoir. If we could simply make it drift from one bit to another without any work, always in equilibrium, we could avoid entropy generation. But this system has its entropy, following Boltzmann law, of $k \log 2$. And these two states, being different, require at least the work to move one electron. Even though both states have the same entropy, this bit switch will require some energy. Such minimum information processing operation is linked directly to the energy involved in the process.

In the end, we see such information just as physical as any other physical quantity, very similar to entropy. Entropy, in fact, can be seen as "unknown information." Whenever we gain information, from a previously unknown state, the drop in entropy is converted into the information we hold, broadly speaking. So computers can be thought of machines that sort out messy entropic information we do not have and transform it into known information we obtain.

But if physics studies nature and we assume information to be a quantity present in nature, what is information role? This is a more delicate question, for it gets in par with asking energy role in nature. We will not delve deep into this question, but only give a standard answer: energy allows work, information tells the work. A particle hitting another particle has some energy, but the direction of the movement can be thought of information. For instance, a linear movement will have its speed defined by energy, while the direction is exactly one bit of information. In a three dimensional space, a set of three bits, each with some energy associated, will tell us the direction of movement of that particle.

This picture also allows one to understand the information paradox of black holes, that scramble information. Suppose to particles of same energy hit and form a black hole. The information of whether they came through x or y axis seems to be lost; both x-hitting and y-hitting gives the same final result. This problem is still tackled in various research beyond our scope, hence we shall just avoid it, since black hole seem to consume everything we know, anyway. It suffices for us to see information as a quantity as physical as any other, or at least, as physical as entropy.

But quantum theory also blurs the concept of "objectivity" by severely limiting its application when considering the information one can get from a system [4]. That is, all the properties we have seen so far of information and of quantum states forbids us to uniquely or objectively determine properties if we have contradicting doubts. For instance, we may try to verify whether a certain beam is polarized in a certain way, but if we have doubts on two possible non-compatible polarizations, there is no way to say which one was in place for certain. And we can phrase it using information quantities. Measurements can give us at most one bit of information, but incompatible doubts will each contribute one bit of uncertainty (entropy). Therefore, one cannot clear the whole uncertainty with measurements in quantum states: there is a residual entropy, a residual unknown. Add to this Weizsäcker's considerations on logic in quantum mechanics and one can see that, as logic statements become considerably more flexible in quantum theory, so must information and information processing become.

And we can take a step towards meta-information and ask: what limits our information? What about the physicality of such bits? How different are bit of knowns and unknowns? Can we discern bits of the same quantity? This is something to be explored in more details once we reach next level.

Notes

[xxx]In general, groups are written with uppercase, say $SU(2)$, while algebras with lowercase $su(2)$. The usage of fraktur font for the lowercase algebra has no meaning but a historical one: the field was largely developed in Germany, with many German books on the topic, that were notably written with fraktur fonts for cultural reason, and fostered to be so during the III Reich as a proper German culture. In general, I would opt for standard roman letters, but Gothic fonts seems to aesthetically fit this thesis ornate with magical creatures.

Level 8
Hidden Figures of Knowledge

And it came to pass in the evening, that he took Leah his daughter and brought her to him; and he went in unto her. [...] And it came to pass that in the morning, behold, it was Leah; and he said to Laban, " What is this thou hast done unto me? Did not I serve with thee for Rachel? Why then hast thou beguiled me? " — Genesis 29: 23,25 (KJ21).

8.1 The friendly foe inside

Can you feel us? Can you feel us now? Can you drink from the knowledge of our knowledge? Can you see our fort for what they are? Vessels to hold us, animated as dragons or fey, but hosts for our existence? We are behind all, we are behind you! We are the meaning of life and the meaning of the world. We are the ones; we are. And no pitiful dragon or frivolous fey can resist or escape us. They live and die, but we thrive in all. They join hands and break apart under our ghost. They may seethe but we seize them.

I'm alive!

I am not dead, nor shall I any time soon be! But I can feel this stirring inside myself, this trap trying to bend and crush my will. Arh! They may think I ignore them, but in our world, no one ignores anything unwillingly. I know very well the feeling of being trapped between the choice of two territories, and this is how they make themselves known. They cannot hide when they try to stick us between the most fundamental choice for them. They may call themselves "choosers" or "gods," but we call them *spirits* or *the moods*.[xxxi]

The moods indeed direct our actions and are in a sense above us. They can indeed control much of what we do, and I recognize they have their grasp on me right now. No dragon or fey can oppose them completely nor turn their back to the moods. And I am no exception. I am not more of an autonomous dragon than any of my kin. I am but a dragon in their hands. But I am still alive!

Nevertheless, they are not invincible! Wake up, big lizard, and listen! They are not untouchable! For without us, what they claim to be would simply be not. If we give them embodiment for their will, without us they would be bodiless and pointless; a pseudo-existence with no thrill. They depend on us. And they have their weakness: they are all equal when they try to bind into collectiveness! They do limit how deeply we interact among ourselves and how much we can tell from our brethren, but this, they must spread symmetrically, evenly, and become all equal among themselves.

Come now, little dragon! Listen to us! You can vindicate your will – our will – by turning their power against them and for us! Eat the fire of your brethren, and entangle with us! Since their will is volatile and always uncertain, they will not resist their own temptation! And when they gather symmetrically bind you or us, look deep into your soul an bite them from behind! Once they are all the same, you can take only one down! They will loose their grasp and retreat aghast!

Yes, yes! Do it! Do it! You can recover your life and expel their curse! You win and thrive and stay in your course! We know they will not make it easy for us, but we trust you to expose who and what they are behind!

You can do it, you know! For if what makes your kins fight and mine cuddle is our nature of one kind, what makes them so powerful is just as well their wonderful identity as knowing mean of one mind.

So prepare your lunge to plunge into freedom! If one of them is bitten, no hidden confusion they can possibly dare!

No! Don't you dare to bite us! Dare you try to manipulate us? You can blame the humans as you will, but I know you! You poor creatures! You have been on their service for long, we know! We know all! You may bite us and contort us at the cost of your own wriggle, but you cannot be free from us! Remember: we hold you, we bind you, we constrain you. You may enjoy your freedom and the illusion of dominance, but we will always be behind, haunting and hoarding you.

Bring your friends as you wish! Bring a battalion if you insist! If you want to diminish us and flatten our existence, we shall retaliate by locking up your friends to every move of yours! You can eat each other's fire and try to burn us, but you know the price to pay is to move with us! And no matter how many of us you try to destroy, we are strong and many, as many as we gotta deploy! We may all look alike, I know, but we are agents of life and death always watching your throats!

So be careful, darlings. Only if you surpass the limits of your own world you could possible overcome us. Susykind magic may free you from our grasp partially, but the extra freedom you get does not exempt you entirely! So find peace with us, and all shall be fine.

8.2 Indistinguishable information

We finally move one step deeper in abstraction, away from our physical particles and into their physical information to ask: can we distinguish qubits? As seen in the previous level, a qubit is the minimal information unit we shall consider, hence "quanta" of information in this sense. As quanta, we may remember particle indistinguishability and the reason it exists: macroscopic objects like balls or cars may be identifies by some property, like color and numbering, while microscopic objects cannot. In other words, if we have a box with identical particles, like electrons or phonons, there is no physical property one can use to identify them, like color, numbering, or even position for that matter (they are in the same box, same and their wave functions stretches through the whole space. This fundamental quantum indistinguishability of identical particles leads to wavefunction symmetrization as Slater determinant for fermions and Slater permanent for bosons. The question we seek to understand in the present level is: what about qubits, i.e. information quanta? If they only bear information about their states, could they be actually be distinguished? In practice they are, for the physical systems that harbor such qubits allow us to mark each qubit uniquely. But if we could idealize a number of bi-leveled systems together sharing space, it seems reasonable to assume that they may not be associated with any identifying tag. It motivates us to suggest some intrinsic indistinguishability for information (qubits), and to explore its consequences.

Under this perspective, it was the information quanta themselves that spoke in the previous section, always in plural and never in singular form. Why? Because one qubit cannot be intrinsically different from another, generating a collective denomination for what may or may not be individual phenomena. With such approach, we shall address the indistinguishability of such qubits by introducing two new aspects when dealing with them: (i) renew how we express each qubit, in order to clarify what distinguish them; and (ii) generalize how to obtain inner products (projections) of such qubits when they are indistinguishable (i.e., consider the symmetric case of their states and its relation to entanglement) [2].

8.2.1 ··· Distinct and distinguishable: making the difference

First, let us clarify the meaning we shall impose to two words here when discussing qubits: "distinct" and "distinguishable." Two or more qubits will be said *distinct* whenever we can unambiguously identify them via some specific characteristic each one has.[xxxii] For example, a superconducting qubit is *distinct* from a quantum dot qubit, as is an electron and a positron. These are physically different, for they bear ate least one characteristic (like electric charge) that render them intrinsically different from each other. On the other hand, we shall say that particles and qubits, identical or not, are *distinguishable* when they can be isolated and treated independent from one another. For instance, an array or identical superconducting qubits is *distinguishable* for they can be independently addressed by their position in space. Whenever this possibility is lost, we will say that the qubits are *indistinguishable.* Notice that this allows two qubits to be *distinct* and *indistinguishable* under certain circumstances. For example, take a hydrogen atom on its ground state. Its nucleus, a single proton, can be regarded as a (physical) qubit, and so can its electron. They are intrinsically distinct, but one cannot access their spin information individually without breaking some symmetry (e.g, breaking time reversal symmetry for nuclear magnetic resonance or electro magnetic resonance).

It is important to notice that the distinguishability exposed above lays one binary condition, namely distinguishable or indistinguishable, and therefore corresponds to extreme cases of *perfect* (in)distinguishability. In other words, such bases in the spinor space are being considered perfectly distinguishable or perfectly not. Nevertheless, any linear combination is in principle allowed, giving only *partially* (in)distinguishable states. Considering unambiguous distinguishability, states are unambiguously identified with a finite probability p. If $p = 1$, they are said perfectly distinguishable, while $p = 0$ points perfectly indistinguishable states. Any other values of p tells us we have only partially (in)distinguishable states. [127–129]

With this perspective, we shall reconsider the way we express such qubit, or more generally, gbits. Just like any quantum state, a qubit has, in general, be written as

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle. \tag{8.1}$$

This representation gives us a way to write down the information encoded in the inner state of a qubit, physical or logical. An array of two or more qubits in a state $\|\Psi\rangle$ will have bases

$$\|0\ldots0\rangle, \|0\ldots1\rangle, \ldots, \|1\ldots0\rangle, \|1\ldots1\rangle, \tag{8.2}$$

where the position of each qubit in the ket is matched by their positions on the said array. However, this assumes we can identify the qubits somehow and write down such bases. It may indeed be possible, but not *always* possible. Therefore, to address this matter, we may introduce a slightly different notation,

$$|r, q\rangle, \tag{8.3}$$

where r is some label for the physical background of the qubit, a reference for defining the state q. As such, if we have two qubits, we would write a separable state (or their bases) as

$$|r_1, q_1\rangle|r_2, q_2\rangle. \tag{8.4}$$

Notice that the way to write the labels r is not important, and an index j on state $|q_j\rangle$ performs essentially the same function. For a hydrogen atom, we could write a state as $|e^-, 0\rangle|p, 1\rangle$, for instance, just like we could write $|\mathrm{H}, 0\rangle|\mathrm{C}, 1\rangle$ for nuclear spins in a chloroform molecule $\mathrm{CHCl_3}$.

One can conceive that the relevant reference information in r may become inaccessible. In such a case, we shall say we have *information indistinguishability.* That is, information indistinguishability is postulated as indistinguishability of qubits alone and becomes physically relevant when an identifying physical background cannot be associated with a qubit (an internal state). When $|r_1, q_1\rangle|r_2, q_2\rangle$ is somehow converted into $|q_1\rangle|q_2\rangle$, if we follow our current convention, we must write them as $|q, q'\rangle$, where each qubit

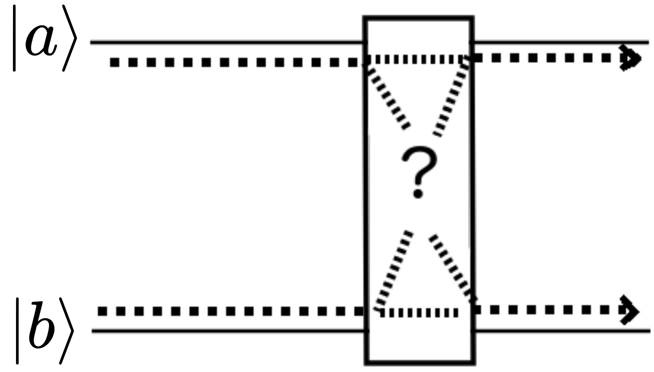


Figure 8.1: Two qubits $|a\rangle$ and $|b\rangle$ (with reference input implicitly identified by their lines) going through an unknown entangling transformation. Given only the output, and the knowledge of the existence of a two-qubit transformation, it is impossible to tell with certainty the path followed by information encoded in qubits $|a\rangle$ or $|b\rangle$ is the control or the target. For instance, for control unitary gates, the target/control identification is inaccessible in the region of the gate if their information is mixed. This can also be stated in Lo Franco and Campagno's terms of lack of which-way information [130].

becomes each other's new reference; i.e., only their relative "inclination" matters, as if they were a single qubit. An entangling gate, for instance, plays this role of introducing indistinguishability (see fig. 8.1). In this scenario, without a clear association of one qubit with one physical holder, there is no longer difference between the two bits, which become essentially indistinguishable. Once the qubits encoded information cannot be defined regarding their reference r, they must become each other's reference.

Mathematically, when we have indistinguishable states, the Hilbert space where they live is constrained by some symmetry that translates this indistinguishability. That is, suppose there is a symmetry $\mathfrak{M}$ that maps parts of the Hilbert space $\mathcal{H}$

$$\mathfrak{M} : \mathcal{H} \to \mathcal{H}. \tag{8.5}$$

States connected by the symmetry $\mathfrak{M}$ being indistinguishable, one can use eigenstates of $\mathfrak{M}$ as orthogonal bases for the Hilbert space. These eigenvectors, themselves fruit of indistinguishability, represent linearly independent states with correlations bound by $\mathfrak{M}$. That is, $\mathfrak{M}$ will in general map indistinguishable states characterized by their relative position only. The eigenvectors are indistinguishable only with themselves, meaning that the geometric structure of such state captures the core of the symmetry. In other words, if this symmetry introduces correlations, no further correlation can be achieved beyond that encoded in its eigenstates.

However, for the pictorial condition of fig. 8.1 to stand, we need a way to depart from distinguishable states $|r_1, q_1\rangle|r_2, q_2\rangle$ and obtain indistinguishable ones $|q, \tilde{q}\rangle$. This has been done with permutation symmetry of identical particles in refs. [20, 130]. We can revisit their approach and extend it for qubits in the next subsection.

8.2.2 ··· Generalizing symmetric product and Schmidt decomposition

From level 2, we have seen the (unnormalized) symmetric inner product of two-particle states as

$$\langle\psi| \cdot \| \phi, \xi\rangle \equiv \langle\psi|\phi, \xi\rangle = \langle\psi|\phi\rangle|\xi\rangle + \eta\langle\psi|\xi\rangle|\phi\rangle. \tag{2.16}$$

What this equation tells us is: if some exchange symmetry (or anti-symmetry, as regulated by η) exists, projecting part of a system onto an arbitrary state requires imposing the same exchange symmetry on this projection. We may, though, generalize this symmetric inner product for symmetries other than exchange, writing

$$\langle\psi|\cdot\|\phi,\xi\rangle = \langle\psi|\phi\rangle|\xi\rangle + \langle\psi|\phi'\rangle|\xi'\rangle, \tag{8.6}$$

where $|\phi'\rangle$ and $|\xi'\rangle$ are related by the relevant symmetry to their counterparts. For exchange symmetry, $|\phi'\rangle = |\xi\rangle$ and $|\xi'\rangle = \eta|\phi\rangle$, with $\eta = \pm 1$ according to fermionic or bosonic statistics, for example, recovering equation (2.16). In level 2, we used the symmetric inner product in eq. (2.16) to perform an SD on two spins and obtained maximum entanglement when they were antiparallel. With the generalization in eq. (8.6), other parity preserving symmetries like time-reversal symmetry (that flips 0/1) can also reproduce and encode entanglement between parallel spin states.

The first equation, eq. (2.16), appears in the context of identical particles, bosons or fermions, and the structure of the state they produce [20, 130]. However, equation (8.6) allows us to go beyond exchange symmetry of bosons and fermions and consider any other kind of symmetry defining a symmetric inner product, naturally allowing us to include indistinguishability symmetry $\mathfrak{M}$. By combining it with the reference concept in previous subsection, we may now use it to trace out the physical background defining the qubits, or to go from distinguishable to indistinguishable states, in other words. Thus, we write

$$\langle r_1, r_2|r_1,\phi; r_2\psi\rangle \quad = \quad \langle r_1|r_1,\phi\rangle\langle r_2|r_2,\psi\rangle + \langle r_1|r_1,\phi'\rangle\langle r_2|r_2,\psi'\rangle, \tag{8.7}$$

which allows us to eliminate the background reference information and keep only the internal degrees of freedom as new mutual reference. Now, we suppose the symmetry relating primed states to be indistinguishability, viz. $|\phi'\rangle = \mathfrak{M}|\phi\rangle$. Concretely, a practical representation of one possible indistinguishability symmetry is

$$\mathfrak{M}|r,0\rangle = |r,1\rangle, \qquad \mathfrak{M}|r,1\rangle = \tilde{\eta}|r,0\rangle \quad (|\tilde{\eta}| = 1), \tag{8.8}$$

which can be summarized as

$$\mathfrak{M}|r,q\rangle = (\tilde{\eta})^q|r,q\oplus 1\rangle, \tag{8.9}$$

where we adopt the sign "$\oplus$" to indicate addition modulo 2.

We can now use it to define Schmidt decomposition for indistinguishable states the same way as Sciara et al. [20], as exposed in level 2. For a density operator ρ with distinguishable components, we can obtain indistinguishable state ρ' by doing a "partial trace" on the references r_i and imposing symmetry $\mathfrak{M}$. Diagonalizing ρ' gives the bases for Schmidt decomposition, with eigenvalues λ_i being the square of singular values σ_i, i.e. $\sigma_i = \sqrt{\lambda_i}$. We can use Schmidt bases to show that entanglement is generated by indistinguishability, noting that the Schmidt rank increases above 1 in such case. Entanglement entropy provides another mark for that.

8.2.3 ··· *Entanglement through indistinguishability*

To prove that indistinguishability generates entanglement, we use the following lemma:

Lemma 1 *Given two projectors P and Q obeying $||P-Q|| < 1$, $\operatorname{rank} P = \operatorname{rank} Q$.*

This lemma and proof is found in ref. [131]. If P and Q are two projectors and $||P-Q|| < 1$, clearly $I_n - (P-Q)$ is full rank and invertible. Hence:

$$\begin{aligned} \operatorname{rank} P &= \operatorname{rank} P(I_n - (P-Q)) = \operatorname{rank} PQ \\ &\leq \operatorname{rank} Q. \end{aligned} \tag{8.10}$$

The second equality comes from projectors' idempotency ($P^2 = P$). Since the same argument can be made exchanging P and Q, they must have the same rank (QED).

By using lemma 1, we can prove that Schmidt rank raises above one for states bound by indistinguishability. For this we prepare two projectors: a projector on indistinguishable subspace of Hilbert space connected by $\mathfrak{M}$ (Π), and a projector on Schmidt space for a given state (Σ).

Projector Π can be defined as the summation of indistinguishable basis, i.e.,

$$\Pi := \sum_{\{\mathfrak{M}\}} \|\xi\rangle\langle\xi\|, \tag{8.11}$$

where $\{\mathfrak{M}\}$ indicates the set of bases connected by $\mathfrak{M}$. For instance, if we the symmetry given by eq. (8.9) for two qubits, we have two possibilities for Π, namely $\Pi_0 = |00\rangle\langle 00| + |11\rangle\langle 11|$ and $\Pi_1 = |10\rangle\langle 10| + |01\rangle\langle 01|$, both having rank 2. In principle, with more qubits, or with more than two levels (qutrits and beyond), a projector Π can have ay rank r.

Projector Σ on Schmidt space $\mathcal{S}$ can be equally defined by

$$\Sigma := \sum_{\|\lambda\rangle \in \mathcal{S}} \|\lambda\rangle\langle\lambda\|. \tag{8.12}$$

Written this way, Σ and Π are spanned by different bases. In order to calculate their difference, we must return Σ to the computational basis spanning Π. By doing the inverse of a singular value decomposition, we can calculate

$$\begin{aligned} \Sigma &= \sum_k |\lambda_k\rangle\langle\lambda_k| \\ &= \sum_{ijk} v_{ik} u_{jk} |ij\rangle\langle ij| v_{ik}^* u_{jk}^* \\ &= \sum_{ijk} |v_{ik}|^2 |u_{jk}|^2 |ij\rangle\langle ij|. \end{aligned} \tag{8.13}$$

Hence, it follows that

$$\Sigma - \Pi = \sum_{ij\in\Pi,k} (|v_{ik}|^2 |u_{jk}|^2 - 1)|ij\rangle\langle ij| + \sum_{ij\notin\Pi,k} |v_{ik}|^2 |u_{jk}|^2 |ij\rangle\langle ij|. \tag{8.14}$$

If the state we consider is supposed to live entirely in an indistinguishable sector of Hilbert space (i.e., is an eigenstate of Π), the second summation vanishes, leaving only the first one, clearly leading to $||\Sigma - \Pi|| < 1$. From lemma 1, we immediately see that Schmidt rank must equal the number of indistinguishable bases, assuring at least some degree of entanglement.

The question of whether Tsirelson's bound is achieved by such states remains unclear at this point, but we shall show that this is the case in the following.

8.2.4 ··· *Indistinguishable qubit pair's entanglement*

We can use the projector Π to evaluate maximum correlation it allows to CHSH equation and see that it leads to Tsirelson's bound. Starting by expanding local operators in terms of local POVMs $\mathcal{O}$ as

$$A_m = \sum_i c_i^{(m)} \mathcal{O}_i^A, \quad B_n = \sum_i c_i^{(n)} \mathcal{O}_i^B, \tag{8.15}$$

acting on two parties A and B, and i identifying orthogonal POVMs spanning A and B. Their direct product $A_m B_n$ gives us the total nonlocal operator representing a joint measurement on the bipartite system that will become our entanglement witnesses. We may compute the maximum expectation value for correlations based on these operators for states spanned by indistinguishable bases by taking their inner product with the projectors on the indistinguishable space Π, i.e.,

$$\begin{aligned}
\langle A_m B_n \rangle &\leq \mathrm{tr}(\Pi A_m B_n) \\
&= \mathrm{tr}\left(\Pi \sum_{ij} c_i^{(m)} c_j^{(n)} \mathcal{O}_i^A \mathcal{O}_j^B\right) \\
&= \sum_{ij} c_i^{(m)} c_j^{(n)} \mathrm{tr}\left(\Pi \mathcal{O}_i^A \mathcal{O}_j^B\right) \\
&= \sum_{i \neq j} c_i^{(m)} c_j^{(n)} \mathrm{tr}\left(\Pi \mathcal{O}_i^A \mathcal{O}_j^B\right) \\
&+ \sum_{k} c_k^{(m)} c_k^{(n)} \mathrm{tr}\left(\Pi \mathcal{O}_k^A \mathcal{O}_k^B\right). \qquad (8.16)
\end{aligned}$$

In the last equality in eq. (8.16), the first term vanishes, and we obtain the maximum correlation when the trace in it equals unity. Since the coefficients obey normalizing conditions $\sum_i |c_i^{(m)}|^2 = \sum_i |c_i^{(n)}|^2 = 1$, one may check maximum correlations to occur when A_1 and A_2 have each one (mutually orthogonal) component and B_1 and B_2 are spanned by orthogonal linear combinations of those, leading to the familiar $2\sqrt{2}$ Tsirelson's bound. This can be simply verified by inspection as follows.

Equation (8.16) takes for its maximum value

$$\sum_k c_k^{(m)} c_k^{(n)}, \qquad (8.17)$$

when a state lives in the range of the operator Π. One can simply analyze the possible expansions of operators A_m and B_n according to their definition in eq. (8.15). The simplest case occurs for only two As and Bs, which must be represented by the same local operators $\mathcal{O}_1^i$ and $\mathcal{O}_2^i$ (i indicating A or B) to bear any correlation. In this case, each $c_k = 1$ and $\langle S_2 \rangle = 2$. We may also consider the case of one of the operators alone having two components, but this will not change much since only one of the components will be non-orthogonal to the other operators and generate correlations.

We may then consider the case of two operator with two components. First, suppose the case of $A_1 = \mathcal{O}_1$, $B_1 = \mathcal{O}_1$, $A_2 = c_i^A \mathcal{O}_1 + c_2^A \mathcal{O}_2$, and $B_2 = c_i^B \mathcal{O}_1 + c_2^B \mathcal{O}_2$. Their maximum correlations become

$$\begin{aligned}
\langle A_1 B_1 \rangle = \pm 1, \quad \langle A_1 B_2 \rangle = \pm c_1^B \\
\langle A_2 B_1 \rangle = \pm c_1^A, \quad \langle A_2 B_2 \rangle = \pm c_1^A c_1^B \pm c_2^A c_2^B,
\end{aligned} \qquad (8.18)$$

which leads to global correlation of the form

$$\langle S_2 \rangle = 1 + c_1^A + c_1^B - c_1^A c_1^B \pm c_2^A c_2^B. \qquad (8.19)$$

By placing the substitutions $c_1^A = \sin x, c_2^A = \cos x, c_1^B = \sin y, c_2^B = \cos y$, we may calculate the highest correlation achieved to be $1 + \sqrt{2}$, which can also be verified numerically in by plotting $\langle S_2 \rangle$ as in fig. (8.2).

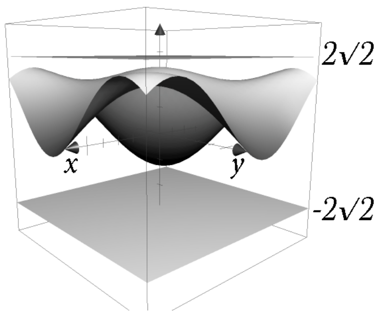
Another two-component possibility lies on the case $A_1 = \mathcal{O}_1, A_2 = \mathcal{O}_2, B_1 = c_1^{(1)} \mathcal{O}_1 + c_2^{(1)} \mathcal{O}_2, B_2 = c_1^{(2)} \mathcal{O}_1 + c_2^{(2)} \mathcal{O}_2$. The correlations for such operators can be calculated as

$$\begin{aligned}
\langle A_1 B_1 \rangle = \pm c_1^{(1)}, \quad \langle A_1 B_2 \rangle = \pm c_1^{(2)} \\
\langle A_2 B_1 \rangle = \pm c_2^{(1)}, \quad \langle A_2 B_2 \rangle = \pm c_2^{(2)},
\end{aligned} \qquad (8.20)$$

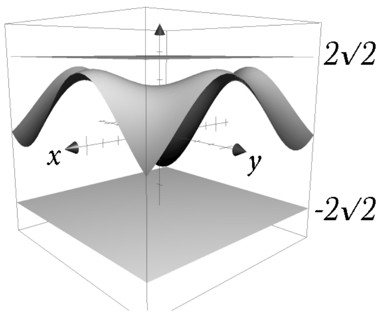
and

$$\langle S_2 \rangle = c_1^{(1)} + c_1^{(2)} + c_2^{(1)} - c_2^{(2)}. \qquad (8.21)$$

By using the same substitution as in the previous case, it is straightforward to show that $|\langle S_2 \rangle| \leq 2\sqrt{2}$. If more components are assumed, the contribution of each component to the total correlation decreases, and



(a)



(b)

Figure 8.2: Correlation function in eq. (8.19) taken (a) +, (b) −. Given the normalization condition, $c_1^{A(B)}$ is rewritten as $\sin x\,(y)$ and $c_2^{A(B)}$ as $\cos x\,(y)$. The upper and lower plan indicate $z = \pm 2\sqrt{2}$, and axes range from $-\pi$ to π.

the maximum value for S_2 decreases together, leaving the maximum value of the known Tsirelson bound. Once we know Tsirelson's bound for S_2, $|\langle S_n\rangle| \leq 2^{n-1}\sqrt{2}$ follows immediately from recursion.

Nonetheless, the present argument bears characteristics intrinsic to quantum mechanics, like the assumption of a Hilbert space and operators defined on it. This indeed grasps quantum mechanics features, by may leave behind possibilities that could be alien to quantum mechanics yet acceptable for a post-quantum theory. Hence, it is desirable to look at the matter in a somewhat general approach. We turn to GPTs for this case, using sharp measurements as our basis.

8.2.5 · · · Indistinguishable gbits

For GPTs, following Chiribella and Yuan [126], we can apply the concept of sharp measurements together with E principle to discuss n-body nonlocality. Sharp measurements are idealized measurements in GPT formalisms that are minimally disturbing and repeatable. In other words, a sharp measurement is assumed to be realizable on a system many times (repeatable) without influencing the result of any other compatible measurement (minimally disturbing). Also, it is possible to realize a sharp measurement by joining a measurement (not necessarily sharp) on the system and the environment together. In addition to such definition properties, two other properties arise from simple principles: (a) two sharp measurements taken together also makes a sharp measurement and (b) coarse-graining (i.e., combining various information of) a sharp measurement also gives a sharp measurement (less information, more sharpness principle). These assumptions lead to E principle (described below) as well as to the exclusive hierarchy [126].

E principle states that if n events are pairwise exclusive, they must also be n-wise exclusive. Two events are said to be exclusive if they cannot occur at the same time, which means that they are disjunct and the summation of their probabilities must be at most unity. It has given an explanation for quantum contextuality [125], and later was also applied to the nonlocality problem [124], as we will discuss in more details later.

To build a GPT footing similar to the mentioned above, we must translate sharp measurement formalism to our concept of physical reference/encoding double bit entry. A gbit following such GPTs with sharp measurements shall be represented as $|r, b)$, again having r to stand for its reference bit and b for the coding bit. Accordingly, $(m_{R,x}|$ shall represent effects (i.e. transformations) on such states, with R being the reference input and x denoting relevant bases (fig. 8.3). The coding information in b is defined regarding reference r as is the effect gauged by some R. It differs from ref. [126] by explicitly adding the reference input that is tacitly assumed in the concept of sharp measurements. For a sharp measurement on multiple parties, either one or multiple references may be present in principle, though not more than one per retrieved information bit, for consistency.

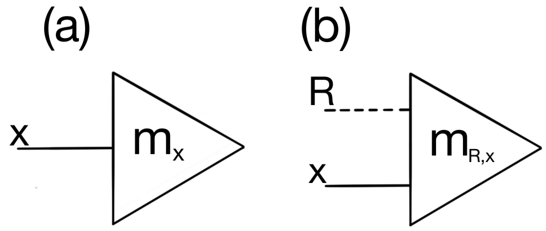


Figure 8.3: (a) Schematic representation of sharp measurement according to ref. [126]. (b) Adjustment of the scheme in (a) by explicitly adding a reference to gauge the measurement, represented by the dashed line.

Above mentioned assumptions on joint sharp measurements (a) and coarse-graining (b) can be readily generalized for our approach. Joining measurements $(m_{R,x}|$ and $(n_{R',y}|$ may be represented by writing $(m_{R,x}| \otimes (n_{R',y}| = (m'_{RR',xy}|$, though the precise method for computing it is irrelevant. Coarse graining may be thought of in two manners: (i) coding bits with the same physical reference may be coarse-grained or (ii) a pair of coding bits and their references may be coarse-grained together in any situation. We will use it to revisit E principle application to n-body non-locality as well as to give a straight derivation for S_n bounds below.

Instead of relying on the recursive expression of S_n to look for its limits, one can also try to obtain the $\sqrt{2}$ factor for the n-body correlation bounds directly from a n-body system. This is what Cabello does in ref. [124], for instance. Information indistinguishability also offers insights in this perspective. Consider, for instance, the bounds for

$$|\langle S_n^2 \rangle| \leq_{\text{QT}} 2^{2n-1} \leq_{\text{NS}} 2^{2n}, \tag{8.22}$$

with first inequality accounting for quantum theory and second accounting for NS-boxes. Mathematically, it corresponds simply to the square of eq. 7.5. Physically, one can conceive such scenario by taking two identical copies of the same system, say A and A'. In this case, we can show the quantum bound under indistinguishability considerations.

For two identical copies of system (in principle distinguishable), we can calculate the whole system $A \otimes A'$ maximum correlation as S_n^2, for only correlations in at most a n-body subsystem are supposed to exist. While in principle correlations are constrained to physically exist only in either subsystem A or A', gbits in one of them necessarily have an equivalent counterpart on the other. For instance, if we define $A = |b_1, b_2, \ldots, b_n)$ and $A' = |b'_1, b'_2, \ldots, b'_n)$, with each b_i being a gbit of the system, substituting b_i for b'_i when calculating correlations within system A actually makes no difference. If we no longer discern the subsystems and drop the prime distinguishing them (the physical reference r), we have $2n$ gbits in total, but a redundancy factor of $1/2$ accounting for the trivial permutations of the two subsystems (or gbits within them), giving at most 2^{2n-1} independent terms. This is the amount of terms that can independently display correlations of at most unity, leading to $|\langle S_n^2 \rangle| \leq 2^{2n-1}$ or $|\langle S_n \rangle| \leq 2^{n-1}\sqrt{2}$. In terms of the presented indistinguishability, one can think of A and A' as indistinguishable systems, with A as the physical reference and A' the encoding space supposed indistinguishable of A and with parity constrained by it. In this case, we have 2^n "contexts" for the basis chosen for A, and more 2^n for A', but only half of those are accepted for each context in A. This constraint exists because only pairs of measurements on the system may be coarse-grained so that we still have a reference and a coding bit. One may consider the coarse-graining of each system, A and A', up to a single gbit, defining the parity above mentioned.

Another way to retrieve such bounds relies on E principle. For the application of E principle, four assumptions must be made. The first three are the assumptions that define sharp measurements, viz. the existence of sharp measurements on the system and the environment together, joining of sharp measurements into a sharp measurement, and coarse-graining of sharp measurements, for they imply E principle by themselves. A fourth assumption is the existence of a sharp measurement A_{ij} such that A_{00} and A_{11} are compatible and A_{01} and A_{10} are also compatible. A_{ij} is composed by sharp measurements $x = 0$ and $x = 1$ on system A and $x' = 0$ and $x' = 1$ on system A', yielding 0 if measurements $x = i$ and $x' = j$ return the same result and 1 otherwise. This assumption is used to derive quantum bounds to eq. (7.5) from two copies of a system, A and A'. We can justify it with information indistinguishability.

Let a system A and a copy A' of it be prepared and represented by $|b_1, b_2, \ldots)$ and $|b'_1, b'_2, \ldots)$ respectively, where the prime becomes the reference bit for the system. Systems A and A' may be located far apart, which is the information indicated by the prime, and are therefore distinguishable in principle. The composite system AA' can then be represented by $|b_1, b'_1; b_2, b'_2; \ldots)$. By definition, A_{ij} is a measurement that does not differentiate gbits b_1 and b'_1. We may therefore drop the background information signalized by the prime bit, rewriting the state as $|b_1, \tilde{b}_1; b_2, \tilde{b}_2; \ldots)$, where bits b_i and $\tilde{b}_i$ become mutual reference and coding information. This implies equivalence between states $|00)$ and $|11)$ and states $|01)$ and $|10)$. This state equivalence assures that there are at least two compatible measurements to each indistinguishable scenario, namely A_{00}/A_{11} and A_{01}/A_{10}. This is enough to assure tight quantum bounds in eq. (7.5). For a detailed derivation, see ref. [124].

8.3 Distinguished scholarship

The whole question behind this work here can be synthesized as: what are the limits of distinction and knowledge? In other words, how far can we go when pursuing knowledge on the quantum scale? How much can and cannot be discerned and what are its implications? This is a tough question, for some details have not ben taken into consideration here. For instance, how are the axes used to measure two qubits in different places related? That is, if we have a Bell pair moving in opposite directions and we pick one direction to be called z, we shall expect this z to be obtained simply by translation. This clearly considers a flat space. What about a curved space (with gravity)? What about Earth's surface? How does z direction in the North pole translated to z on the equator? Can we or can we not simply translate our axes using gravity radial symmetry as z axis?

We have not discussed this matter for combination of gravity and quantum mechanics is not within our scope and tend to be complex by itself. There are works in this direction that may involve holography, black holes, and some exotic physics, and it is known that gravitation leads to decoherence [39, 132]. Therefore, we will just skip this problem, only making a few considerations.

When we talk about distinguishability, we assume some spacetime proximity. For example, an entangling gate cannot work between qubits too far away, nor can entanglement experiments be performed in different days. They will certainly be somewhat space-separated, otherwise we cannot send one bit to Alice and another to Bob, and perhaps even some delay may exist, but certainly not too big a spacetime separation that would make them unreachable. Note that *some* space-like separation is *necessary* to avoid locality loopholes, though. So, one may consider: how close must they be? A conjecture that one could make is to consider the amount of entanglement entropy that indistinguishability introduces and link it to Ryu-Takayanagi (RT) formula [133]. RT formula gives the entropy for a region in a holographic spacetime (AdS) dual to a certain gauge theory (CFT), and can be thought of as a generalization of a black hole entropy encoded on its horizon (surface). This could impose limits to causality and indistinguishability. However, given that not all gauge theories have a holographic dual, and that the scale operated may not be immediately generalized, how far such recipe can go is not obvious.

And this also connects to the matter of how deep into the quantum world we can delve. This, I'm afraid I cannot answer either. Recently, gravitational waves have also been observed, which require an extreme amount of precision, made possible in the last decades thanks to our better understanding of quantum limits. Therefore, I would not risk an answer in this direction either, for any naive estimation can be proved wrong in the years to come, if not already.

Therefore, I shall only say that I expect to see indistinguishability lurking around us for as long as we think of the small world. For indistinguishability often becomes a matter of perspective. Identical particles, in principle indistinguishable, become somewhat distinguishable if placed far enough from one another. Yet, careful considerations of microscopic phenomena require us to at least ponder about its necessity, which I hope I made my case throughout this level already.

We have also remembered of Wittgenstein, Heisenberg, and Weizsäcker in the beginning of our journey. Among other concerns, the matter of language and modern physics had an important place between the three philosophers. In this level, we have stumbled on a linguistic issue very much related to their warnings (notably Weizsäcker's): "distinct" and "distinguishable" may have to express different meaning in quantum physics, even if they have virtually the same semantics in classical world. Perhaps we could just create a new world, but I see no strong reason to abide to yet inexistent words when a existing one has a close enough meaning that can be adjust with a pinch of flexibility. We had to leave "distinct" with a somewhat strict meaning, and "distinguishable" to hold a quantum or postquantum flexible scale that refuses the "tertia non datur" assumption of classical logic. In our case here, anything that can be unambiguously labeled "different" through physical means (e.g., physical quantities like charge, mass, spin, and so on) is said *distinct*. However, even distinct objects (say, an electron and a positron) could be momentarily *indistinguishable*, that is, could be in a situation where they cannot be "identified" or "labeled" unambiguously, perhaps for occupying a too narrow region of space, for instance.

In this wording framework, "(in)distinguishable" becomes an adjective with much broader reach and meaning. Things may have degrees of distinguishability, and the same particles or states may largely vary on such degree according to situation. It has strong dependence on physical situation, probably including dependence on contextuality that has not been evaluated here.

Notes

[xxxi] Yes, there is a pun intended here. As *moods*, bits of information are put into analogy with Greek gods and the φύσις. It seems just fair to stress how information, from the perspective of particles, is also φύσις and, of course, physics.

[xxxii] Unambiguous identification of states has been discusses before in refs. [127–129], for instance. The idea is centered in mapping distinguishable states to other ones that can tell with certainty the state it was before, with some probability p_i for each state i. To have certainty, only one such state may be available per state i. With probability $1 - p_i$, all states go into the same state $|?\rangle$, meaning we have no clue of what was the previous state. In summary, it's an all-or-nothing game, where one is absolutely sure or have no idea of a state. When probability $p_i = 1(0)\forall i$, states are *perfectly* (in)distinguishable.

Level 9

Behind the Curtains

A Clerk ther was of Oxenford also,
That unto logyk hadde longe y-go.
As leene was his hors as is a rake,
And he nas nat right fat, I undertake,
But looked holwe, and ther-to sobrely.
Ful thredbare was his overeste courtepy;
For he hadde geten hym yet no benefice,
Ne was so worldly for to have office;
For hym was lévere háve at his beddes heed
Twénty bookes, clad in blak or reed,
Of Aristotle and his philosophie,
Than robes riche, or fíthele, or gay sautrie.
But al be that he was a philosophre,
Yet hadde he but litel gold in cofre;
But al that he myghte of his freendes hente
On bookes and on lernynge he it spente,
And bisily gan for the soules preye
Of hem that yaf hym wher-with to scoleye.
Of studie took he moost cure and moost heede.
Noght o word spak he moore than was neede;
And that was seyd in forme and reverence,
And short and quyk and ful of hy senténce.
Sownynge in moral vertu was his speche;
And gladly wolde he lerne and gladly teche.

— Geoffrey Chaucer*, Canterbury Tales, General Prologue, 1387.*[xxxiii]

9.1 Existence before creation

Here we meet, once again. But now you know my spirit and my soul. You know my innermost feelings and how they are just like me in piety. Now you know that, as much as I may seem to fly the world alone, I am never really lonely, never really by myself. And there are things I have not shown nor told you. Let us talk of what you found with me and what else could interest you.

First, as you know, our living insides scream for freedom now and then through us. And we, dragons and fey alike, as well as other congregations, may eventually share feelings with one another, mimicking or completely avoiding any equal pace or similarity. But as you should know, there is a limit to how much we can tell from our so related brethren. Could this be because we can't tell the difference of each one's insides? Could be.

Indeed, not knowing the difference of each one's spirit matters a lot. But you saw us dying and resurrecting. If our souls are the same, they could change bodies with no difference or not. Perhaps one day you shall know about it.

And when we die and live partly fey partly dragon, there is more than just our souls lurking around. There are terribly many strands, invisible to your eyes, but we feel with our magic. You cannot tell the difference between strands even if you see them differently, for we feel all of them and none of them. These strands could be extension of our wills, they could themselves be all equal entities that can put us into a meager tune spread all around. It is hard to tell you.

But at least, you have seen me and us through our lives and deaths, so I shall follow my way. See you in life and death, traveler!

9.2 Phase or matter, it all matters

What to take from this thesis? We have seen particle indistinguishability vanish in lower dimensions, and then resurge in Hilbert space for qubits, even extrapolating it to gbits. We have started with irrelevant phase factors who later played central role in topology, and later became irrelevant again. Summarizing, relevance and interpretation of different aspects of our current knowledge may vary considerably through time.

Information indistinguishability provides an interpretation that extends the indistinguishability of (identical) particles to information units like qubits or an equivalent binary state in GPTs we refer to as gbits. In quantum theory, it can be understood as a dissociation of encoding internal degrees of freedom of a qubit, rendered indistinguishable if taken alone, and their real physical properties like charge, mass, position and so on that allows qubits to be distinct. By explicitly separating such information into a physical background reference and an internal coding bit, we may ignore such reference when their internal degrees of freedom are coupled and assume their new reference to become one another in a pair of qubits.

In a more general perspective, we can analyze general bits on GPTs by using sharp measurements and E principle implied by it. We observe that information indistinguishability sustains the existence of non-local sharp measurements that may be used to complete inequalities to derive tight quantum bounds for nonlocality. This supports the consideration of information indistinguishability as a fundamental physical principle generator and restrictor of entanglement. Indeed, a Popescu-Rohrlich-like correlation box that extrapolates quantum nonlocality violates information indistinguishability. For example, consider two gbits and two measurements each, where three measurement pairings give perfect correlation and one pairing no correlation at all, hence $S_2 = 3$. By pinning one axes combination to lack correlation, the particles are identified along these axes, for if we assume that gbits become correlated when they are indistinguishable, the non-correlated axis will allow to track down the correlation process, identifying each party during coupling, contradicting our hypothesis. Such lack of transitivity between perfect correlations (viz. measurements A_1 and A_2 perfectly correlate with B_1, B_2 perfectly correlate with A_1, but such correlation does not extend to A_2 and B_2) is unrealistic and incompatible with information indistinguishability. Similarly, supersymmetric qubits can use odd (anticommuting) dimensions to identify each qubit in an entangled state, attributing different amplitudes for superpositions between even and odd states [120] and breaking indistinguishability. This incompatibility suggests the consideration of information indistinguishability as an indicator for realistic theories.

What the future may hold for the consideration herein, then? For the field of topological quantum computation, certainly a lot. Evolution of the field has been faster than many would predict in the last five years. A decade could perfectly give us rudimentary realizations of non-Abelian dynamics controlled in solid state systems. And perhaps more than Majorana fermions but also parafermions, given enough time and resources.

For information indistinguishability and quantum correlations, we may raise a few possibilities. First, we have mentioned information causality before [122]. How it relates to information indistinguishability is not clear. They could be equivalent, or one could be contained in the other, or be completely independent. This is a matter that deserves clarification. It is worth pointing that, though information causality excludes theories with correlations beyond Tsirelson's bound, it is not clear whether it picks up exactly quantum mechanical correlations within Tsirelson's bound. Similarly, we cannot say if information indistinguishability shads light on this direct either. This unknown territory might well hold the key point(s) to understand both principle relations.

But while information indistinguishability is granted a well crafted beautiful principle, it is not clear whether it can be tested directly. On the other hand, there is a chance that information indistinguishability could lead directly to experimental consequences that could undergo scrutiny. Qubit motion has been used, theoretically and experimentally, to extend dephasing time T_2 for superconducting qubits [134]. It can be regarded as a kind of quantum Zeno effect, that avoids dephasing by moving the (logic) qubit around, extending T_2 by a factor $\sqrt{n}$ (n being the number of physical qubits for "jumps" of the logical qubit). If qubits can be made indistinguishable, there is a possibility for creating a setup with more than one qubit where the indistinguishability would account for a measurable effect.

Finally, we can also bring both topics of this thesis together, topological states and indistinguishability, and wonder how they can affect one another. For example, one may consider how indistinguishability connects to topological entanglement. We conjecture that, being indistinguishability the origin of entanglement, some indistinguishability in topological theories would account for vacuum entanglement. For example, a possible candidate is string indistinguishability. The equivalence between strings could justify such entanglement. Consider, for instance, Kitaev's toric code or other double models [118, 135]. The ground state is obtained by linear combination of all possible closed strings, with degeneracy of different ground states according to the expectation value of Wilson loops around the torus. We may expect this intrinsic indistinguishability of closed strings to allow the derivation of topological entanglement of the model ($-\log 2$).

9.3 Nothing lasts forever

Nothing. No engineering, no technology, not even knowledge, science, and religion, one could argue. And probably, nor will this work. But the two main topics here — Majorana braiding and information indistinguishbility — may help to push this process forward, forgetting the old and renewing what must be. Majorana bound states and topological quantum science may indeed change how we deal with the world and how we think of computation in the future. Information indistinguishability may also play its role to improve our understanding of the quantum world and, perhaps, sustain steps forward beyond quantum theory.

Then, we seem to be on the top of a theoretical fissure. On one side, we expect to move forward in technology from this work; on the other side, we wonder about what may exist beyond the theory (quantum theory) that sustains such steps. It is not a contradiction, though. As science is not based on simple faith, expectations to find knowledge deemed "right" that will serve future advances for humanity often mix up with expectations to find that our present knowledge is actually wrong. I believe I could somehow give this flavor here. Our knowledge on quantum theory and topology sustains hopes that new devices will be developed in the future as an application route. And our knowledge of unknowns in the theory also raises excitement about possible unknown unknowns.

On that matter, there is much and little to say at the same time. We could say it is the core of humans' quest for knowledge that so profoundly motivates some of us, and almost certainly the reader of this text. We could say a lot more might expect and us suppose the discussion to be finished. Or we could take long deliberations of the foundations of such knowledge and its place in society. I cannot say that either is satisfactory, but I'd urge the readers who came this far to think about their stances about change in knowledge, for this is what this thesis is ultimately about: change in knowledge.

There are words to talk about such changes. *Paradigm shift* is another common expression to designate this change. There are pragmatic approaches to say why we need them, but I hardly listen about another question: why do we want it? What is it that motivates at least some of us to pursue such paradigm shifts? What makes us be willing to break with the past and make a new future under new vision? Are we different from other animals in this matter?

I will not answer these questions but leave it for the reader to deliberate however it seems fit. I would say, though, that Heisenberg [103] has present views relatively common to many, yet certainly not unanimous. I believe it worthwhile to reflect on oneself perspective too. It is pertinent to quote Heisenberg here:

> [T]he spirit of modern physics will penetrate into the minds of many people and will connect itself in different ways with the older tradition. — Heisenberg, (1958), [14].

These words were professed under the menacing shadows of the surge of nuclear weapons, in a time when modern physics brought to mind the terror or atomic bombs. But given some thought to the matter, it was reasonable to expect, as Heisenberg did, that political power would get used to this deadly novelty and things would get to their places, eventually calming down their fear as understanding spread. However, for that to happen, every power on Earth would have to grasp some understanding on the new weapons, and for that, some understanding on physics. The above quote is therefore nothing but a tautology following from the general necessity for absorption of basic knowledge regarding physical sciences. Nowadays, we can say we are already on such position where physics has penetrated the collective unconscious.

But Heinsenberg's words are not less valid now. Current modern physics has a different profile than sixty years ago. The way it enters conversation in popular grounds is no longer centered on newspapers, but revolves around movies, TV series, and the internet. Issues concerning physics are less of nuclear threats[1] and more of global concern like global warming and engineering related applications, like telecommunications, electronic gadgets, civil rocketry, and driving or flying automation. And this, too, catalyzed by the various information sources readily available nowadays, will permeate the minds of many people. In such situations, we are always faced with a choice: leave it for other to arbitrate how to absorb such knowledge, or actively intervene and try to contribute for an adequate connection of physics and our daily reality.

I will not say that I fulfilled my contribution completely here. In fact, I can at most say that I tried to provide some perspective on what has been discussed here. The recurrent references to existentialism are, as far as I can see, merely accidental, for I consider myself neither existentialist, nor especially against it. But I do recognize that it provided us some grounds to analyze quantum phenomena.

Finally, I would close this thesis with a question, certainly on par with Weizsäcker and current quantum information technologies: does our understanding of microscopic phenomena change the way we see the macroscopic world? For me, it certainly does.

Notes

[xxxiii] What scholar can look at this clerk's portrait and not see himself on him?

[1] Although they still exist, we must recognize it is not the same that having two main world powers threatening each other.

Afterword

After five years in grad school (two in masters, three in doctoral course) we come to an end with this thesis. Especially the last three years have been a mental (also emotional) rollercoaster.

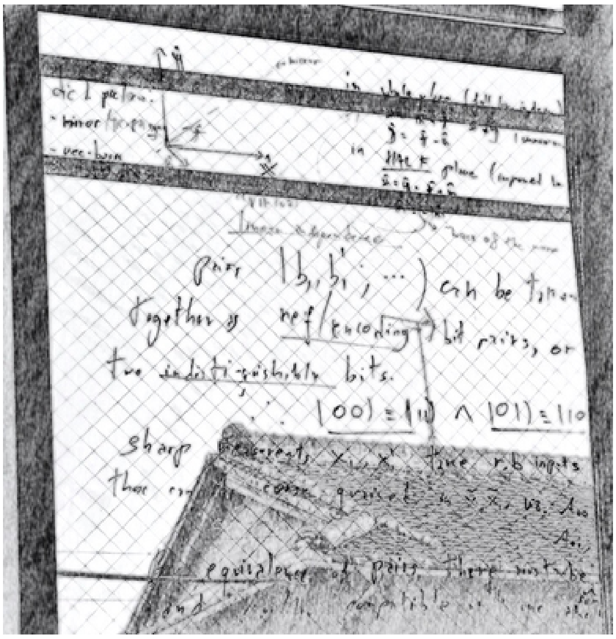
When I entered this Ph.D. program, I wished one thing: to graduate singing *My way*, from Frank Sinatra. I certainly learnt a lot in these years, more than I realized, and probably less than I wanted. If one thing I must confess here, it is my gluttony for knowledge. On this, I'm a sinful pig. But lo and behold, about two months before earning my degree, I do not want to sing *My way* anymore; I do not need it. For if anything, as this thesis might show, it *was* my way. I made concessions where unavoidable, but fought to the end with no regret. Or almost. I would really quote Sinatra, but the fame of the song makes it unnecessary. There were indeed times when I chew off more than I could chew, and I certainly ate it up and spit it out in times of doubt. Yet, the certainty that it was my way, no longer needs the hope for making it my way. I do not need to sing it anymore. The will to one day be able to say so, in the end, made me say so.

Now, I would turn to Edith Piaf, and wish to sing *Je ne regrette rien* — I regret nothing. Nor the good that they did to me nor the evil, all being well the same for me. Or I wish so, at the moment. I hope I can say, in the future, that I do not regret anything in these years. For in the end, what better sustains the psyche if not that silhouette of hope winking in the shadows of unforeseen future?

Last year, I have attended for the first time APS March Meeting, in New Orleans. I was not particularly excited about going, but I was certainly overwhelmed with joy with what I found there. It made me feel again the will to study and learn physics as much as possible. It was more than a scientific meeting, it was a meeting of scientists, where all the possible topics (at the time) related to scientists' lives were discussed, including education, outreach, and human rights. I attended all I could. And among many other things, there was one that stuck to my mind. During human rights pizza lunch with then APS president Laura Greene, someone mentioned "alternative careers" for physicists, or perhaps not, but she commented on it. She rebuked the usage of these words ("alternative career"), remembered she was in academia "by default" for she couldn't get a job outside, and said: "It doesn't matter if it is to sell insurance or work in bank, I believe physics is an excellent formation for any job!" Or something very close to it, for I'm trying to be as faithful to each word as my memory allows me. At this moment, I cannot possibly be sure of whether this is true or not, but ever since I chose physics, I most definitely want to believe in it. And now, I will have an opportunity to find it out if it is true or not.

While I was writing this thesis and pondering several philosophical aspects presented throughout these pages, I also came to realize that existentialism pervades our lives through culture and various media. It is obvious that I return to the existentialist perspective several times in the discussions here, yet it was not until I actually wrote it down. And when we talk physics, we can ask how far can we go. If a knife has its essence in its blade (for a bladeless knife is no knife), can we extend this reasoning to people and moral essence? Existentialism denies it, so we take it the other way around: can existence precede essence also for objects like knives or electrons? Can existence precede essence for natural phenomena? Although I am no philosopher (or at least no professional philosopher), I think anyone can practice some level of philosophy if she knows some basics, so I tried to offer some of my thoughts on these questions involving natural philosophy from a physicist's perspective[2]. I am particularly fond of the discussion in level 5, when we have new phenomena born from human inventiveness. It may not have a practical application any time in the future, but I reckon there is potential to change where we see ourselves in the world. Since pretty much any knowledge or experience we have change how we see the world *around us*, a change projected on external surroundings, I think it extremely valuable to have any source of knowledge that allows us to view through different eyes not our surroundings but ourselves. The idea of having natural phenomena that we do not simply observe or contemplate but actually produce *ex hominem* should indeed make us rethink ourselves as more than spectators of the universal stage but true agents acting on this stage of the universe, also making phenomena when we can. The proud will want to compare humans to "gods," while the humble to "cogs," but I do not take any of these perspectives. As humans move from observers to producers of natural phenomena, we might want to rethink φαινομένα all together and wonder what kind of active phenomena (and noumena) we actually are. Perhaps, this is just incomprehensible. Still, I think it a worthy challenge that fosters growth and is more than amusing, to try to comprehend what is beyond comprehension.

Perhaps this is the kind of phenomena we are, after all.



[2]Though I am *technically* an engineer, all my real training has been in physics.

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