Operator splitting for dispersion-generalized Benjamin-Ono equations

(分散項を一般化したBenjamin-Ono方程式系に対する作用素分割)

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Contents

0. Introduction to the splitting method

In this paper, we analize the error of the operator splitting for a class of differential equations. Section 0 is an introduction to the operator splitting for beginners. The main part of this paper starts from Section 1. In Section 1, we introduce the dispersion-generalized Benjamin-Ono equations, and we mention some related equations and known results. We also mention the main results in this paper at the end of Section 1. In Section 2, we mention some preliminary lemmas for Section 3 and 4. In Section 3 and 4, we prove the main results for the Godunov and the Strang splittings, respectively.

The basic idea of the splitting method is to approximate the exact flow of the differential equation by combining simpler flows. In this paper, we use two schemes of the operator splitting, the Godunov splitting and the Strang splitting. As the author know, the Strang splitting was first introduced by G. Strang in [13], and it is expected to give a second-order approximation in Δt, where Δt is a time interval for one cycle of the calculation.

The rest of this section is organized as follows: In Section 0.1, we prepare time domains for the splitted equations. In Section 0.2, we introduce the Godunov splitting and the Strang splitting.

0.1. **Preliminaries for explaining the splitting method.** Before introducing the splitting method, we prepare some notations. Let $n \in \mathbb{N}$, $0 < \Delta t \ll 1$, and $T > 0$. We define $t_n = n\Delta t$ and $t_{n+1/2} = (n+1/2)\Delta t$. We also define $N = N(\Delta t, T)$ as the largest integer such that $N\Delta t \leq T$ for given $T > 0$ and $\Delta t > 0$.

For the Godunov splitting, we define $\Pi_G^{(n)} = \Sigma_{G,1}^{(n)} \cup \Sigma_{G,2}^{(n)}$ and

$$\Sigma_{G,1}^{(n)} = \bigcup_{l=1}^{n} \Omega_1^{(l)}\,,\ \Sigma_{G,2}^{(n)} = \bigcup_{l=1}^{n} \Omega_2^{(l)},$$

and

$$\Omega_1^{(n)} = (t_{n-1}, t_n] \times \{t_{n-1}\}\,,\ \Omega_2^{(n)} = [t_{n-1}, t_n] \times (t_{n-1}, t_n].$$

For the Strang splitting, we define $\Pi_S^{(n)} = \Sigma_{S,1}^{(n)} \cup \Sigma_{S,2}^{(n)}$ and

$$\Sigma_{S,1}^{(n)} = \bigcup_{l=1}^{n} \left(\Omega_{1,1}^{(l)} \cup \Omega_{1,2}^{(l)}\right)\,,\ \Sigma_{S,2}^{(n)} = \bigcup_{l=1}^{n} \left(\Omega_{2,1}^{(l)} \cup \Omega_{2,2}^{(l)}\right).$$

and

$$\Omega_{1,1}^{(n)} = (t_{n-1}, t_{n-1/2}] \times \{t_{n-1}\}\,,\ \Omega_{1,2}^{(n)} = (t_{n-1/2}, t_n] \times [t_{n-1/2}, t_n]\,,$$
$$\Omega_{2,1}^{(n)} = [t_{n-1}, t_{n-1/2}] \times (t_{n-1}, t_{n-1/2}]\,,\ \Omega_{2,2}^{(n)} = \{t_{n-1/2}\} \times (t_{n-1/2}, t_n].$$

0.2. **Godunov and Strang splitting.** Now we introduce the splitting method. Let $\Phi_C[t]u_0 \in X$, where X is some normed space, denotes the solution of the differential equation

$$\begin{cases} \partial_t u = C(u), & t \in [0, n\Delta t], \\ u(\cdot, 0) = u_0. \end{cases} \tag{0.1}$$

Typical C includes a differential operator in the spatial variables. But we also consider the cases C does not have any differential operators in the spatial variables, which are the cases of ordinary differential equations. The exact solution of (0.1) is

$$u(t_n) = \Big(\Phi_C(\Delta t)\Big)^n u_0.$$

Here, we assume that $C = A + B$. [1] In the Godunov operator splitting, we consider

$$\begin{cases} \partial_t v(t,\tau) = B(v(t,\tau)), & (t,\tau) \in \Sigma_{G,1}^{(n)}, \\ \partial_\tau v(t,\tau) = A(v(t,\tau)), & (t,\tau) \in \Sigma_{G,2}^{(n)}, \\ v(0,0) = u_0. \end{cases} \tag{0.2}$$

The solution of this equation is

$$v(t_n, t_n) = \Big(\Phi_A(\Delta t) \circ \Phi_B(\Delta t)\Big)^n u_0.$$

In the Strang operator splitting, we consider

$$\begin{cases} \partial_t v(t,\tau) = B(v(t,\tau)), & (t,\tau) \in \Sigma_{S,1}^{(n)}, \\ \partial_\tau v(t,\tau) = A(v(t,\tau)), & (t,\tau) \in \Sigma_{S,2}^{(n)}, \\ v(0,0) = u_0. \end{cases} \tag{0.3}$$

The solution of this equation is

$$v(t_n, t_n) = \Big(\Phi_B(\frac{\Delta t}{2}) \circ \Phi_A(\Delta t) \circ \Phi_B(\frac{\Delta t}{2})\Big)^n u_0.$$

We call the estimate for $v(t,t) - u(t)$ in some Banach space X such that $u(t)$, $v(t,t) \in X$ the error estimate. In general, the error estimate for the Godunov splitting is expected to the first order of Δt, and the error estimate for the Strang splitting is expected to the second order of Δt. Here we give an example.

[1]For the Korteweg-de Vries (KdV) equation, the linear part is the linear Airy equation $\partial_t u + \partial_x^3 u = 0$. On the other hand, the nonlinear term uu_x is concerned with the inviscid Burgers equation $\partial_t u = uu_x$. Thus, we put A an operator to be a linear part, and B an operator to be a nonlinear part.

Example 0.1. *As [5], let us consider an ordinary differential equation*

$$\begin{cases} u'(t) = u(t)(1-u(t)), & t \in (0, n\Delta t], \\ u(0) = u_0 > 0, \end{cases} \tag{0.4}$$

which is called the logistic equation and is a simple model of demography. The exact solution of (0.4) *is*

$$u(t) = \frac{u_0}{u_0 - e^{-t}(u_0 - 1)}. \tag{0.5}$$

(A) A Godunov splitting for the logistic equation is

$$\begin{cases} \partial_t v(t,\tau) = -v(t,\tau)^2, & (t,\tau) \in \Sigma_{G,1}^{(n)}, & (0.6) \\ \partial_\tau v(t,\tau) = v(t,\tau), & (t,\tau) \in \Sigma_{G,2}^{(n)}, & (0.7) \\ v(0,0) = u_0 > 0. & & (0.8) \end{cases}$$

Let $j \in \mathbb{N}$ such that $(t,\tau) \in [t_j, t_{j+1}]^2$. Since $v(t,t_j) = v(t_j,t_j)/(1 + v(t_j,t_j)(t-t_j))$ and $v(t,\tau) = e^{\tau - t_j} v(t,t_j)$ satisfies (0.6)–(0.8), *we have*

$$v(\Delta t, \Delta t) = \frac{u_0}{e^{-\Delta t} + u_0 \Delta t e^{-\Delta t}}. \tag{0.9}$$

Therefore, by (0.5) *and* (0.9), *we have*

$$v(\Delta t, \Delta t) - u(\Delta t) = \frac{u_0^2\{1 - e^{-\Delta t} - \Delta t e^{-\Delta t}\}}{(e^{-\Delta t} + u_0 \Delta t e^{-\Delta t})(u_0 - e^{-\Delta t}(u_0 - 1))}.$$

By Taylor expansion at $t = 0$, we have

$$1 - e^{-\Delta t} - \Delta t e^{-\Delta t} = -\frac{1}{2}(\Delta t)^2 + o((\Delta t)^2). \tag{0.10}$$

Thus we have $|v(\Delta t, \Delta t) - u(\Delta t)| \lesssim (\Delta t)^2$. Next, we estimate the error at $t = n\Delta t$. By induction, we have

$$v(n\Delta t, n\Delta t) = \frac{u_0}{e^{-n\Delta t} + u_0 \Delta t e^{-\Delta t} \sum_{j=0}^{n-1} e^{-j\Delta t}}. \tag{0.11}$$

Therefore, by (0.5) *and* (0.11), *we have*

$$\begin{aligned} &v(n\Delta t, n\Delta t) - u(n\Delta t) \\ &= \frac{u_0^2(1 - e^{-n\Delta t})}{(e^{-n\Delta t} + u_0 \Delta t \sum_{j=0}^{n-1} e^{-j\Delta t})(u_0 - e^{-n\Delta t}(u_0 - 1))} \times \frac{1 - e^{-\Delta t} - \Delta t e^{-\Delta t}}{1 - e^{-\Delta t}}. \end{aligned}$$

By Taylor expansion at $t = 0$, we have

$$1 - e^{-\Delta t} = \Delta t + o(\Delta t). \tag{0.12}$$

Therefore, by (0.10)*, we have* $|v(n\Delta t, n\Delta t) - u(n\Delta t)| \lesssim \Delta t$*.*
(B) A Strang splitting for the logistic equation is

$$\begin{cases} \partial_t v(t,\tau) = -v(t,\tau)^2, & (t,\tau) \in \Sigma_{S,1}^{(n)}, & (0.13) \\ \partial_\tau v(t,\tau) = v(t,\tau), & (t,\tau) \in \Sigma_{S,2}^{(n)}, & (0.14) \\ v(0,0) = u_0 > 0. & & (0.15) \end{cases}$$

In the same manner as (A), we have

$$v(\Delta t, \Delta t) = \frac{u_0}{e^{-\Delta t} + ((u_0 \Delta t)/2)(1 + e^{-\Delta t})}. \tag{0.16}$$

Therefore, by (0.5) *and* (0.16)*, we have*

$$v(\Delta t, \Delta t) - u(\Delta t) = \frac{u_0^2\{1 - (\Delta t)/2 - e^{-\Delta t} - (\Delta t/2)e^{-\Delta t}\}}{(e^{-\Delta t} + ((u_0\Delta t)/2)(1 + e^{-\Delta t}))(u_0 - e^{-\Delta t}(u_0 - 1))}.$$

By Taylor expansion at $t = 0$*, we have*

$$1 - \frac{\Delta t}{2} - e^{-\Delta t} - \frac{\Delta t}{2} e^{-\Delta t} = -\frac{(\Delta t)^3}{12} + o((\Delta t)^3). \tag{0.17}$$

Thus we have $|v(\Delta t, \Delta t) - u(\Delta t)| \lesssim (\Delta t)^3$*. Next, we estimate the error at* $t = n\Delta t$*. By induction, we have*

$$v(n\Delta t, n\Delta t) = \frac{u_0}{e^{-n\Delta t} + ((u_0\Delta t)/2)(1 + e^{-\Delta t}) \sum_{j=0}^{n-1} e^{-j\Delta t}}. \tag{0.18}$$

Therefore, by (0.5) *and* (0.18)*, we have*

$$\begin{aligned} &v(n\Delta t, n\Delta t) - u(n\Delta t) \\ &= \frac{u_0^2(1 - e^{-n\Delta t})}{(e^{-n\Delta t} + ((u_0\Delta t)/2)(1 + e^{-\Delta t}) \sum_{j=0}^{n-1} e^{-j\Delta t})(u_0 - e^{-n\Delta t}(u_0 - 1))} \\ &\quad \times \frac{1 - (\Delta t)/2 - e^{-\Delta t} - (\Delta t/2)e^{-\Delta t}}{1 - e^{-\Delta t}}. \end{aligned}$$

Therefore, by (0.12) *and* (0.17)*, we have* $|v(n\Delta t, n\Delta t) - u(n\Delta t)| \lesssim (\Delta t)^2$*.*

Now we explain it for the general case formally. It is difficult to write down the case $n \geq 2$, therefore we only explain the case $n = 1$ here. Note that Exercises 9 in Section 6.5 in [1] is to estimate the error of the Strang splitting at $t = \Delta t$ for the case A and B are $d \times d$ matrices. Let A and B be some operators. Let us consider the partial differential equation

$$\begin{cases} \partial_t u = A(u) + B(u), & t \in [0, \Delta t], \\ u(\cdot, 0) = u_0. \end{cases} \tag{0.19}$$

For example, for (1.1) and (1.6)–(1.8), $A(u) = Ku$ and $B(u) = -uu_x$. The solution of (0.19) in $[0, \Delta t]$ is written as $e^{(A+B)\Delta t}u_0$ formally. Then by Taylor expansion at $t = 0$, we have

$$\begin{aligned}
&e^{(A+B)\Delta t}u_0 \\
&\quad = u_0 + (\Delta t)(A+B)u_0 + \frac{(\Delta t)^2}{2}(A^2 + AB + BA + B^2)u_0 \\
&\qquad + \frac{(\Delta t)^3}{6}(A^3 + B^3 + AB^2 + B^2A + ABA + BAB + A^2B + BA^2)u_0 \\
&\qquad + o((\Delta t)^3),
\end{aligned} \tag{0.20}$$

Now we consider a Godunov splitting of (0.19) as follows.

$$\begin{cases}
\partial_t v(t,0) = B(v(t,0)), & t \in (0, \Delta t], \\
\partial_\tau v(t,\tau) = A(v(t,\tau)), & (t,\tau) \in \Omega_2^{(1)}, \\
v(0,0) = u_0.
\end{cases} \tag{0.21}$$

The solution of (0.21) at $(t,\tau) = (\Delta t, \Delta t)$ is written as $e^{A\Delta t}e^{B\Delta t}u_0$ formally. Then by Taylor expansion at $t = 0$, we have

$$\begin{aligned}
e^{A\Delta t}e^{B\Delta t}u_0 &= u_0 + (\Delta t)(A+B)u_0 \\
&\quad + \frac{(\Delta t)^2}{2}(A^2 + 2AB + B^2)u_0 + o((\Delta t)^2).
\end{aligned} \tag{0.22}$$

Then, by (0.20) and (0.22), the error estimate for the Godunov splitting at $t = \Delta t$ is expected to the second order in Δt. Next, we consider an Strang splitting of (0.19) as follows.

$$\begin{cases}
\partial_t v(t,\tau) = B(v(t,\tau)), & (t,\tau) \in \Omega_{1,1}^{(1)} \cup \Omega_{1,2}^{(1)}, \\
\partial_\tau v(t,\tau) = A(v(t,\tau)), & (t,\tau) \in \Omega_{2,1}^{(1)} \cup \Omega_{2,2}^{(1)}, \\
v(0,0) = u_0.
\end{cases} \tag{0.23}$$

The solution of (0.23) at $(t,\tau) = (\Delta t, \Delta t)$ is written as $e^{(B\Delta t)/2}e^{A\Delta t}e^{(B\Delta t)/2}u_0$ formally. Then by Taylor expansion at $t = 0$, we have

$$\begin{aligned}
&e^{(B\Delta t)/2}e^{A\Delta t}e^{(B\Delta t)/2}u_0 \\
&\quad = u_0 + (\Delta t)(A+B)u_0 + \frac{(\Delta t)^2}{2}(A^2 + AB + BA + B^2)u_0 \\
&\qquad + \frac{(\Delta t)^3}{24}(4A^3 + 4B^3 + 3AB^2 + 3B^2A + 6BAB + 6A^2B + 6BA^2)u_0 \\
&\qquad + o((\Delta t)^3).
\end{aligned} \tag{0.24}$$

Then, by (0.20) and (0.24), the error estimate for the Strang splitting at $t = \Delta t$ is expected to the third order in Δt.

1. Introduction to the diepersion-generalized Benjamin-Ono equations

We consider the operator splitting of a class of nonlinear equation

$$\begin{cases} \partial_t u + uu_x - Ku = 0, & (x,t) \in \mathbb{R} \times [0,T], \\ u(\cdot,0) = u_0 \in H^s(\mathbb{R}) \end{cases} \tag{1.1}$$

where $K = \mathcal{F}^{-1}[k(\xi)\mathcal{F}]$ is a Fourier multiplier, and u_0 and u are $\mathbb{R}$-valued. Throughout this paper, we assume that $p \in [0,\infty)$ and k satisfies

$$\Re(k(\xi)) \le 0\,, \quad k(-\xi) = \overline{k(\xi)}\,, \quad |k(\xi)| \lesssim \langle \xi \rangle^p\,, \tag{1.2}$$

$$|(\xi+\eta)k(\xi+\eta) - \eta k(\eta) - \xi k(\xi)| \lesssim |\xi|\langle \eta \rangle^p + |\eta|\langle \xi \rangle^p. \tag{1.3}$$

for all ξ , $\eta \in \mathbb{R}$, where $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$. The first inequality means that $\langle f, Kf \rangle_{H^s} + \langle Kf, f \rangle_{H^s} \le 0$. The second equality means that $Kf \in \mathbb{R}$ if $f \in \mathbb{R}$. Thus, we have $\langle f, Kf \rangle_{H^s} \le 0$. Note that (1.3) is satisfied if $k \in C^1(\mathbb{R})$, which is proved by the mean-value theorem.

(1.1) is the generalization of the Korteweg-de Vries (KdV) equation, the Benjamin-Ono (BO) equation, and dispersion-generalized Benjamin-Ono (dBO) equation

$$\begin{cases} \partial_t u + uu_x - \partial_x |D_x|^{p-1} u = 0, & (x,t) \in \mathbb{R} \times [0,T], \\ u(\cdot,0) = u_0 \in H^s(\mathbb{R}), \end{cases} \tag{1.4}$$

where $|D_x| = \mathcal{F}^{-1}[|\xi|\mathcal{F}]$ and $2 < p < 3$. The KdV, the BO, and the dBO equation are all the examples of the dispersive equations, but we also consider the non-dispersive equations. (1.1) also includes the Burgers equation,

$$\begin{cases} \partial_t u + uu_x - \partial_x^2 u = 0, & (x,t) \in \mathbb{R} \times [0,T], \\ u(\cdot,0) = u_0 \in H^s(\mathbb{R}). \end{cases} \tag{1.5}$$

Now we introduce some examples such that p is a fractional number. The extended Whitham equation, which is a model for surface water waves (see [12] and [11]), is

$$u_t + uu_x - \int_{\mathbb{R}} e^{ix\xi}(1+\beta|\xi|^2)^{\frac{1}{2}} \Big(\frac{\tanh \xi}{\xi}\Big)^{\frac{1}{2}} i\xi\widehat{u}(\xi)d\xi = 0,$$

where β is a measure of surface tension. (see [11]) When $\beta > 0$, this satisfies (1.2) and (1.3) with $p = 3/2$. The case $\beta = 0$ is the case of no surface tension and called the Whitham equation (see [12]), and also the model for purely gravitational waves (see [11]), and this satisfies (1.2) and (1.3) with $p = 1/2$.

The difficulty for (1.1) is that nonlinear term uu_x has the derivative. For example, let us conseder the nonlinear term with no derivative u^p for $p \in \mathbb{N}$. By the Sobolev theorem, we have $\|u^p\|_{H^\sigma}(\mathbb{R}) \le C\|u\|_{H^\sigma}^p$ for $\sigma > 1/2$. But the nonlinear term with derivative does not have such a good property. For example, there exists $u \in H^\sigma(\mathbb{R})$ such that $uu_x \notin H^\sigma(\mathbb{R})$.

The Godunov splitting for (1.1) is defined in $(x, t, \tau) \in \mathbb{R} \times \Pi_G^{(n)}$ as

$$\begin{cases} \partial_t v + vv_x = 0, \quad x \in \mathbb{R}\,,\ (t, \tau) \in \Sigma_{G,1}^{(n)}, & (1.6) \\ \partial_\tau v - Kv = 0, \quad x \in \mathbb{R}\,,\ (t, \tau) \in \Sigma_{G,2}^{(n)}, & (1.7) \\ v(0, 0) = u_0 \in H^s(\mathbb{R}). & (1.8) \end{cases}$$

The Strang splitting for (1.1) is defined in $(x, t, \tau) \in \mathbb{R} \times \Pi_S^{(n)}$ as

$$\begin{cases} \partial_t v + vv_x = 0, \quad x \in \mathbb{R}\,,\ (t, \tau) \in \Sigma_{S,1}^{(n)}, & (1.9) \\ \partial_\tau v - Kv = 0, \quad x \in \mathbb{R}\,,\ (t, \tau) \in \Sigma_{S,2}^{(n)}, & (1.10) \\ v(0, 0) = u_0 \in H^s(\mathbb{R}). & (1.11) \end{cases}$$

Since the splitting method requires the existence of the solution for the full equation (1.1), we give some known results. For the KdV equation, Colliander-Keel-Steffilani-Takaoka-Tao proved global-wellposedness (GWP) in H^s for $s > -3/4$ (see [2]) and Kishimoto proved GWP in $H^{-3/4}$ (see [10]). For the Benjamin-Ono equation, Tao proved GWP in H^1 (see [14]) and Ionescu-Kenig proved GWP in L^2 (see [7]). For $k(\xi) = -i\xi|\xi|^{p-1}$, Guo proved the local-wellposedness (LWP) in H^{3-p} for $p \in [2, 3]$ and the GWP in $H^{(p-1)/2}$ for $p \in (7/3, 3]$ are also proved in [4].

On the other hand, we have no results for the global existence of the solution for the full equation (1.1) for general k. Throughout this paper, we assume that there exists a solution $u \in C([0, T] : H^s)$ of (1.1) and a constant $C_0 > 0$ satisfies

$$\sup_{t \in [0,T]} \|u(t)\|_{H^s(\mathbb{R})} \le C_0. \tag{1.12}$$

The splitting method also requires the solvability for (1.6), (1.7), (1.9), and (1.10). The global existence for the linear equations are obvious. In [8] Kato proved the existence and the uniqueness of a local solution for a class of nonlinear equations, which includes the nonlinear equations, (1.6) and (1.9). (Theorem 2.1)

In this paper, we prove two error estimates in the Sobolev space. These proofs are based on the method used in [5]. The first is the first-order approximation in Δt for the Godunov splitting, that is the following.

Theorem 1.1. *Let $T > 0$ and $s > 3/2 + \max\{1, p\}$. Assume that $u \in C([0, T] : H^s(\mathbb{R}))$ satisfies (1.1) on $[0, T]$ and $C_0 > 0$ satisfies (1.12). Then there exists $\overline{\Delta t} = \overline{\Delta t}(s, C_0, T) > 0$ such that for all $\Delta t \in [0, \overline{\Delta t}]$, there exists a unique solution $v \in$*

$C(\Pi_G^{(N)} : H^s)$ *of* (1.6)–(1.8). *In addition, there exists* $C = C(C_0, s, T) > 0$ *such that*

$$\sup_{t\in[0,N\Delta t]} \|v(t,t) - u(t)\|_{H^{s-\max\{1,p\}}} \le C\Delta t.$$

Remark 1.2. Theorem 1.1 is also true for the Cauchy problem of

$$\begin{cases} \partial_t v - Kv = 0, & x \in \mathbb{R}\,,\ (t,\tau) \in \Sigma_{G,1}^{(N)}, \\ \partial_\tau v + vv_x = 0, & x \in \mathbb{R}\,,\ (t,\tau) \in \Sigma_{G,2}^{(N)}, \\ v(0,0) = u_0 \in H^s(\mathbb{R}). \end{cases} \tag{1.13}$$

We will prove this remark at the end of Section 3 briefly.

The second is the second-order approximation in Δt for the Strang splitting, that is the following.

Theorem 1.3. *Let* $T > 0$ *and* $s > 3/2 + 3\max\{1, p\}$. *Assume that* $u \in C([0,T] : H^s(\mathbb{R}))$ *satisfies* (1.1) *on* $[0,T]$ *and* $C_0 > 0$ *satisfies* (1.12). *Then there exists* $\overline{\Delta t} = \overline{\Delta t}(s, C_0, T) > 0$ *such that for all* $\Delta t \in [0, \overline{\Delta t}]$, *there exists a unique solution* $v \in C(\Pi_S^{(N)} : H^s)$ *of the Cauchy problem* (1.9)–(1.11). *In addition, there exists* $C = C(C_0, s, T) > 0$ *such that*

$$\sup_{t\in[0,N\Delta t]} \|v(t,t) - u(t)\|_{H^{s-3\max\{1,p\}}} \le C(\Delta t)^2.$$

Remark 1.4. Theorem 1.3 is also true for the Cauchy problem of

$$\begin{cases} \partial_t v - Kv = 0, & x \in \mathbb{R}\,,\ (t,\tau) \in \Sigma_{S,1}^{(N)}, \\ \partial_\tau v + vv_x = 0, & x \in \mathbb{R}\,,\ (t,\tau) \in \Sigma_{S,2}^{(N)}, \\ v(0,0) = u_0 \in H^s(\mathbb{R}). \end{cases} \tag{1.14}$$

The proof is the same as that of Theorem 1.3.

The proof of Theorems 1.1 and 1.3 works for the case $x \in \mathbb{R}^d$ for $d \ge 2$. For instance, we have Theorems 1.1 (resp. Theorem 1.3) for the Zakharov-Kuznetsov equation below

$$\begin{cases} \partial_t u + (\partial_x + \partial_y)u^2 + (\partial_x^3 + \partial_y^3)u = 0, & (x,y,t) \in \mathbb{R}^2 \times [0,T], \\ u(\cdot,0) = u_0 \in H^s(\mathbb{R}^2), \end{cases} \tag{1.15}$$

if we replace $s > 3/2 + \max\{1,p\}$ (resp. $s > 3/2 + 3\max\{1,p\}$) with $s > 2 + \max\{1,p\}$ (resp. $s > 2 + 3\max\{1,p\}$), and we use (A) and (B) in Remark 2.6 instead of (A) and (B) in Proposition 2.4.

Next, we mention the privious results for the splitting method. Holden-Karlsen-Risebro-Tao proved the first-order approximation in Δt for the KdV equation in

H^{s-3} when $u_0 \in H^s$ and $s \geq 5$ is an odd integer in [5]. In the same paper, they also proved the second-order approximation for the KdV equation in H^{s-9} when $u_0 \in H^s$ and $s \geq 17$ is an odd integer. Our result is a generalization of these results because we can apply Theorem 1.1 for the KdV equation for $s > 9/2$ and Theorem 1.3 for the KdV equation for $s > 21/2$, respectively. Holden-Karlsen-Risebro proved the first-order approximation in Δt in H^{s-p} and the second-order approximation in H^{s-2p+1} for the case k is a polynomial, where s is sufficiently large, $u_0 \in H^s$, and $p \geq 2$ is the degree of k (see [6]). Dutta-Holden-Koley-Risebro proved the first-order approximation in Δt in L^2 for the Benjamin-Ono equation for $u_0 \in H^{5/2}$ and the second-order approximation in L^2 for the Benjamin-Ono equation for $u_0 \in H^{9/2}$ in [3]. As far as the author know, no previous results of the splitting method exist for the extended Whitham equation mentioned above. On the other hand, our results include that for the extended Whitham equation, because our theorems work for $p \geq 0$.

2. Preliminaries for the error estimate

First, we mention the solvability of the inviscid Burgers equation

$$\begin{cases} \partial_t v + v v_x = 0, \quad x \in \mathbb{R}\,,\, t \in [0, T'], & (2.1) \\ v(0) = v_0 \in H^s(\mathbb{R}). & (2.2) \end{cases}$$

Theorem 2.1. *Let $s > 3/2$. Then, there exists $T' = T'(s, \|v_0\|_{H^s(\mathbb{R})}) > 0$ such that for all $T_0 \le T'$ there exists a unique solution $v \in C([0, T_0] : H^s(\mathbb{R})) \cap C^1([0, T_0] : H^{s-1}(\mathbb{R}))$ of (2.1) and (2.2) defined on $t \in [0, T_0]$.*

We have Theorem 2.1 from Theorem II in [8] by Kato.

Corollary 2.2. *Let $\sigma > 3/2$, $\tau_0 \ge 0$, $v(\tau_0, \tau_0) \in H^\sigma(\mathbb{R})$, and $M > 0$ satisfy $\|v(\tau_0, \tau_0)\|_{H^\sigma} \le M$. Then, there exists $\overline{\Delta t}_B = \overline{\Delta t}_B(\sigma, M) > 0$ such that for all $\Delta t \le \overline{\Delta t}_B$ there exists a unique solution $v \in C([\tau_0, \tau_0 + \Delta t]^2 : H^\sigma(\mathbb{R})) \cap C^1([\tau_0, \tau_0 + \Delta t]^2 : H^{\sigma-1}(\mathbb{R}))$ of*

$$\begin{cases} \partial_t v + v v_x = 0, \quad x \in \mathbb{R}\,,\, (t, \tau) \in (\tau_0, \tau_0 + \Delta t] \times \{\tau_0\}, & (2.3) \\ \partial_\tau v - K v = 0, \quad x \in \mathbb{R}\,,\, (t, \tau) \in [\tau_0, \tau_0 + \Delta t] \times (\tau_0, \tau_0 + \Delta t]. & (2.4) \end{cases}$$

We can solve (2.4) as $v(t, \tau, x) = \mathcal{F}_\xi^{-1}[e^{k(\xi)(\tau - \tau_0)} \mathcal{F}_x[v(t, \tau_0, x)]]$. Therefore we have Corollary 2.2 from Theorem 2.1.

The following lemma is a variant of the Kato-Ponce commutator estimate (see Lemma X1 in [9]).

Lemma 2.3. *Let $s \ge \sigma > 3/2$. Then there exists $C = C(s) > 0$ such that for all f, $g \in H^s(\mathbb{R})$*

$$\|\partial_x \langle \partial_x \rangle^s (fg) - (\partial_x \langle \partial_x \rangle^s f) g - f(\partial_x \langle \partial_x \rangle^s g)\|_{L^2} \le C(\|f\|_{H^s} \|g\|_{H^\sigma} + \|f\|_{H^\sigma} \|g\|_{H^s}),$$

where $\langle \partial_x \rangle^s = \mathcal{F}_\xi^{-1} \langle \xi \rangle^s \mathcal{F}_x$.

Proof. We put $h(\xi) = \xi \langle \xi \rangle^s$. First, we prove

$$|h(\xi) - h(\xi - \xi_1) - h(\xi_1)| \le C(|\xi_1| \langle \xi - \xi_1 \rangle^s + |\xi - \xi_1| \langle \xi_1 \rangle^s). \tag{2.5}$$

By symmetry, we only need to prove the case $|\xi_1| \le |\xi - \xi_1|$. We have $|h'(\xi)| = |1 + (s+1)\xi^2| \langle \xi \rangle^{s-2} \le C \langle \xi \rangle^s$. By the mean-value theorem, we have

$$|h(\xi) - h(\xi - \xi_1)| = |\xi_1| |h'(\xi - \theta \xi_1)| \le C |\xi_1| \langle \xi - \theta \xi_1 \rangle^s \le C |\xi_1| \langle \xi - \xi_1 \rangle^s,$$

where $\theta \in (0, 1)$. Since $|\xi_1| \le |\xi - \xi_1|$, we have $|h(\xi_1)| \le |\xi_1| \langle \xi - \xi_1 \rangle^s$. Thus we have (2.5). Next, we prove Lemma 2.3. By the Sobolev inequality, the Plancherel equality, and (2.5), we have

$$\begin{aligned}L.H.S. &\le C\Big\| \int_{\mathbb{R}} \{h(\xi) - h(\xi-\xi_1) - h(\xi_1)\}\widehat{f}(\xi-\xi_1)\widehat{g}(\xi_1)d\xi_1\Big\|_{L^2} \\ &\le C\Big\| \int_{\mathbb{R}} \{|\xi_1|\langle\xi-\xi_1\rangle^s + |\xi-\xi_1|\langle\xi_1\rangle^s\}|\widehat{f}(\xi-\xi_1)||\widehat{g}(\xi_1)|d\xi_1\Big\|_{L^2} \\ &\le C(\|\langle\xi\rangle^s|\widehat{f}|\|_{L^2}\||\xi||\widehat{g}|\|_{L^1} + \|\langle\xi\rangle^s|\widehat{g}|\|_{L^2}\||\xi||\widehat{f}|\|_{L^1}) \\ &\le C(\|f\|_{H^s}\|g\|_{H^\sigma} + \|g\|_{H^s}\|f\|_{H^\sigma}).\end{aligned}$$

□

Proposition 2.4. *Let $s \ge \sigma > 3/2$. Assume that f and g are $\mathbb{R}$-valued functions.*
(A) Then there exists $C = C(s) > 0$ such that for all $f,\ g \in H^s(\mathbb{R})$

$$|\langle f, (fg)_x\rangle_{H^s}| \le C\|f\|_{H^s}^2\|g\|_{H^{s+1}}. \tag{2.6}$$

(B) Then there exists $C = C(s,\sigma) > 0$ such that for all $f \in H^s(\mathbb{R})$

$$|\langle f, ff_x\rangle_{H^s}| \le C\|f\|_{H^s}^2\|f\|_{H^\sigma}. \tag{2.7}$$

Proof. First, we show (A). By the definition of inner product of $H^s(\mathbb{R})$, we have

$$\begin{aligned}\langle f, (fg)_x\rangle_{H^s} &= \langle\langle\partial_x\rangle^s f, (\partial_x\langle\partial_x\rangle^s f)g\rangle_{L^2} + \langle\langle\partial_x\rangle^s f, f(\partial_x\langle\partial_x\rangle^s g)\rangle_{L^2} \\ &\quad + \langle\langle\partial_x\rangle^s f, \partial_x\langle\partial_x\rangle^s(fg) - (\partial_x\langle\partial_x\rangle^s f)g - f(\partial_x\langle\partial_x\rangle^s g)\rangle_{L^2}.\end{aligned} \tag{2.8}$$

For the first term, by integration by parts and the Sobolev inequality, we have

$$\begin{aligned}|\langle\langle\partial_x\rangle^s f, (\partial_x\langle\partial_x\rangle^s f)g\rangle_{L^2}| &= |-\frac{1}{2}\langle\langle\partial_x\rangle^s f, (\langle\partial_x\rangle^s f)(\partial_x g)\rangle_{L^2}| \\ &\le C\|f\|_{H^s}^2\|g\|_{H^\sigma}.\end{aligned} \tag{2.9}$$

For the second term, by the Sobolev inequality, we have $|\langle\langle\partial_x\rangle^s f, f(\partial_x\langle\partial_x\rangle^s g)\rangle_{L^2}| \le C\|f\|_{H^s}\|f\|_{H^{\sigma-1}}\|g\|_{H^{s+1}}$. By Lemma 2.3, the third term in (2.8) is bounded by

$$\begin{aligned}&\|f\|_{H^s}\|\partial_x\langle\partial_x\rangle^s(fg) - (\partial_x\langle\partial_x\rangle^s f)g - f(\partial_x\langle\partial_x\rangle^s g)\|_{L^2} \\ &\quad \le C\|f\|_{H^s}(\|f\|_{H^s}\|g\|_{H^\sigma} + \|f\|_{H^\sigma}\|g\|_{H^s}).\end{aligned} \tag{2.10}$$

Since $s \ge \sigma$, we have the desired result. Next, we prove (B). We put $g = f$ in (2.8). Then the second term in (2.8) is equal to the first term. Therefore, we have the desired result by (2.9) and (2.10). □

Remark 2.5. Lemma 2.3 is extended to the $H^s(\mathbb{R}^d)$ functions as follows: let $s \ge \sigma > 1 + d/2$ and $j = 1, 2, \dots, d$. Assume that f and g are $\mathbb{R}$-valued functions. Then there exists $C = C(s) > 0$ such that for all $f,\ g \in H^s(\mathbb{R}^d)$,

$$\|\partial_{x_j}\langle\partial_x\rangle^s(fg) - (\partial_{x_j}\langle\partial_x\rangle^s f)g - f(\partial_{x_j}\langle\partial_x\rangle^s g)\|_{L^2} \le C(\|f\|_{H^s}\|g\|_{H^\sigma} + \|f\|_{H^\sigma}\|g\|_{H^s}).$$

The proof is the same as Lemma 2.3.

Remark 2.6. Proposition 2.4 is extended to the $H^s(\mathbb{R}^d)$ functions as follows: let $s \geq \sigma > 1 + d/2$ and $j = 1\,,\,2\,,\ldots\,,\,d$. Assume that f and g are $\mathbb{R}$-valued functions.
(A) Then there exists $C = C(s) > 0$ such that for all $f,\, g \in H^s(\mathbb{R}^d)$

$$|\langle f, (fg)_{x_j}\rangle_{H^s}| \leq C\|f\|_{H^s}^2\|g\|_{H^{s+1}}.$$

(B) Then there exists $C = C(s, \sigma) > 0$ such that for all $f \in H^s(\mathbb{R}^d)$

$$|\langle f, ff_{x_j}\rangle_{H^s}| \leq C\|f\|_{H^s}^2\|f\|_{H^\sigma}.$$

The proof is the same as Proposition 2.4 (A) and (B) by Remark 2.5.

The following lemma is so called bootstrap lemma, which follows from the continuity of v and the connectivity of $\Pi^{(n)}$.

Lemma 2.7. *(bootstrap lemma) Let $\sigma > 3/2$, $\Delta t > 0$, $n \in \mathbb{N}$, and $\Pi^{(n)} = \Pi_G^{(n)}$ or $\Pi_S^{(n)}$. Assume that $v \in C(\Pi^{(n)} : H^\sigma)$ and $C_1 > 0$ satisfy the following two conditions.*
(A) $\|v(0,0)\|_{H^\sigma} \leq C_1$.
(B) $\sup_{(t,\tau)\in\Pi^{(n)}} \|v(t,\tau)\|_{H^\sigma} \leq C_1/2$ holds if $\sup_{(t,\tau)\in\Pi^{(n)}} \|v(t,\tau)\|_{H^\sigma} \leq C_1$.
Then we have

$$\sup_{(t,\tau)\in\Pi^{(n)}} \|v(t,\tau)\|_{H^\sigma} \leq C_1/2.$$

Lemma 2.8. *Let $\Delta t > 0$, $T > 0$, $n \in \mathbb{N}$ such that $n\Delta t \leq T$, $\Pi^{(n)} = \Pi_G^{(n)}$ (resp. $\Pi_S^{(n)}$) and $s \geq \sigma > 3/2$. Assume that $v \in C(\Pi^{(n)} : H^\sigma)$ satisfy (1.6)–(1.8) (resp. (1.9)–(1.11)) on $\Pi^{(n)}$. Assume that $C_1 > 0$ and v satisfy $\sup_{(t,\tau)\in\Pi^{(n)}} \|v(t,\tau)\|_{H^\sigma} \leq C_1$. Then there exists $C_1' = C_1'(\|u_0\|_{H^s}, C_1, s, \sigma, T) > 0$ such that*

$$\sup_{(t,\tau)\in\Pi^{(n)}} \|v(t,\tau)\|_{H^s} \leq C_1'. \tag{2.11}$$

Proof. We prove the case $\Pi^{(n)} = \Pi_G^{(n)}$. From (1.2) and (1.7), $\|v(t,\tau)\|_{H^s}$ is monotonically decresing with respect to τ in $\Omega_2^{(n)}$. So we only need to prove the boundness of $\|v(t,t_{n-1})\|_{H^s}$ in $\Omega_1^{(n)}$. By (1.6), (2.7), and $\sup_{(t,\tau)\in\Pi_G^{(n)}} \|v(t,\tau)\|_{H^\sigma} \leq C_1$, we have

$$\begin{aligned} \frac{d}{dt}\|v(t,t_{n-1})\|_{H^s}^2 &= -2\langle v(t,t_{n-1}), v(t,t_{n-1})v_x(t,t_{n-1})\rangle_{H^s} \\ &\leq C\|v(t,t_{n-1})\|_{H^s}^2\|v(t,t_{n-1})\|_{H^\sigma} \\ &\leq CC_1\|v(t,t_{n-1})\|_{H^s}^2. \end{aligned} \tag{2.12}$$

We have $\|v(t,t_{n-1})\|_{H^s} \le \|v(t_{n-1},t_{n-1})\|_{H^s} e^{CC_1(t-t_{n-1})}$ by applying the Gronwall inequality to (2.12). Therefore it follows that

$$\begin{aligned}\|v(t,\tau)\|_{H^s} &\le \|v(t_{n-1},t_{n-1})\|_{H^s} e^{CC_1(t-t_{n-1})} \\ &\le \|v(t_{n-2},t_{n-2})\|_{H^s} e^{CC_1(t_{n-1}-t_{n-2})} e^{CC_1(t-t_{n-1})} \\ &\le \cdots \le \|v(0,0)\|_{H^s} e^{CC_1 t} \le \|u_0\|_{H^s} e^{CC_1 T}. \qquad (2.13)\end{aligned}$$

Similar arguments apply to the case $\Pi^{(n)} = \Pi_S^{(n)}$. $\square$

3. Error estimate for the Godunov splitting

The main estimate in this section is Proposition 3.1 below.

Proposition 3.1. *Let $\Delta t > 0$, $T > 0$, $n \in \mathbb{N}$ such that $n\Delta t \le T$, and $s_1 = s - \max\{1, p\} > 3/2$. Assume that there exists a unique solution $v \in C(\Pi_G^{(n)} : H^s)$ of (1.6)–(1.8) and a constant $C_1' > 0$ satisfies (2.11). Then there exists $C_2 = C_2(C_0, C_1', s, s_1, T) > 0$ such that*

$$\sup_{t\in[0,n\Delta t]} \|v(t,t) - u(t)\|_{H^{s_1}} \le C_2\Delta t.$$

Proposition 3.1 follows from Lemmas 3.2 and 3.3 below.

Lemma 3.2. *Let $F(t,\tau) = v_t + vv_x$ and $F(t) = F(t,t)$. Under the same assumptions of Proposition 3.1, there exists $C = C(C_0, C_1', s, s_1, T) > 0$ such that*

$$\|v(t,t) - u(t)\|_{H^{s_1}} \le C \int_0^t \|F(t')\|_{H^{s_1}} dt' \tag{3.1}$$

for all $t \in [0, n\Delta t]$.

Proof. Let $w(t) = v(t,t) - u(t)$. By (1.1), (1.6), and the definition of F, we have

$$\begin{aligned}\frac{\partial}{\partial t} w(t,x) &= v_t(t,\tau,x)|_{\tau=t} + v_\tau(t,\tau,x)|_{\tau=t} - u_t(t,x)\\ &= -ww_x - (uw)_x + Kw + v_t + vv_x\\ &= -ww_x - (uw)_x + Kw + F\end{aligned} \tag{3.2}$$

In view of (3.2), we call F the forcing term. Then we have

$$\frac{d}{dt}\|w(t)\|_{H^{s_1}}^2 = 2\langle w, -ww_x - (uw)_x + Kw + F\rangle_{H^{s_1}}. \tag{3.3}$$

Note that $\langle w, Kw\rangle_{H^s} \le 0$ from (1.2). By the Schwarz inequality, Proposition 2.4, Lemma 2.8, and (1.12), we get

$$\begin{aligned}\frac{d}{dt}\|w(t)\|_{H^{s_1}} &\le C\{\|w\|_{H^{s_1}}(\|w\|_{H^{s_1}} + \|u\|_{H^{s_1+1}}) + \|F\|_{H^{s_1}}\}\\ &\le C(\|w\|_{H^{s_1}} + \|F\|_{H^{s_1}}).\end{aligned} \tag{3.4}$$

Here we used $\|w\|_{H^{s_1}} \le \|u\|_{H^s} + \|v\|_{H^s} \le C_0 + C_1'$ and $\|u\|_{H^{s_1+1}} \le C_0$. Applying the Gronwall inequality and $w(0) = 0$ to (3.4), and we have

$$\|w(t)\|_{H^{s_1}} \le C \int_0^t \|F(\sigma)\|_{H^{s_1}} d\sigma.$$

□

Lemma 3.3. *Let $F(t,\tau)=v_t+vv_x$ and $F(t)=F(t,t)$. Under the same assumptions of Proposition 3.1, there exists $C=C(\|u_0\|_{H^s},C_1',s,s_1,T)>0$ such that*

$$\sup_{t\in[0,n\Delta t]}\|F(t)\|_{H^{s_1}}\leq C\Delta t. \tag{3.5}$$

Proof. By (1.7) and the definition of F, the forcing term F satisfies

$$\partial_\tau F(t,\tau)-KF(t,\tau)=X(t,\tau) \tag{3.6}$$

in $(t,\tau)\in\Pi_G^{(n)}$, where $X(t,\tau)=X_1(t,\tau)+X_2(t,\tau)$ and $X_1(t,\tau)$ and $X_1(t,\tau)$ are defined as below.

$$X_1(t,\tau)=-\left\{\frac{1}{2}K(v^2)_x-vKv_x\right\},\ X_2(t,\tau)=-v_xKv$$

By (3.6), we get

$$\partial_\tau\|F(t,\tau)\|_{H^s}^2=2\langle F,KF\rangle_{H^s}+2\langle F,X\rangle_{H^s}. \tag{3.7}$$

The first term in (3.7) is equal to or less than 0 because of (1.2). For the second term, by (1.3), the Sovolev inequality, and Lemma 2.8, we have

$$\begin{aligned}\|X_1\|_{H^{s_1}}&=\frac{1}{2}\|\langle\xi\rangle^{s_1}\int_{\mathbb{R}}\{\xi k(\xi)-\xi_1k(\xi_1)-(\xi-\xi_1)k(\xi-\xi_1)\}\widehat{v}(t,\tau,\xi-\xi_1)\widehat{v}(t,\tau,\xi_1)d\xi_1\|_{L^2}\\&\leq C(\|\langle\xi\rangle^{s_1}|\xi||\widehat{v}|\|_{L^2}\|\langle\xi\rangle^p|\widehat{v}|\|_{L^1}+\|\langle\xi\rangle^{s_1}\langle\xi\rangle^p|\widehat{v}|\|_{L^2}\||\xi||\widehat{v}|\|_{L^1})\\&\leq C(\|v\|_{H^{s_1+1}}\|v\|_{H^{s_1+p}}+\|v\|_{H^{s_1+p}}\|v\|_{H^{s_1+1}})\leq C,\end{aligned} \tag{3.8}$$

where we used $s=s_1+\max\{1,p\}$ and $s_1>3/2$ to have the last inequality in (3.8). Under the same assumption of s and s_1 written above, by the Sobolev inequality, we have

$$\|X_2\|_{H^{s_1}}\leq C\|v\|_{H^{s_1+1}}\|v\|_{H^{s_1+p}}\leq C. \tag{3.9}$$

Therefore, we have

$$\partial_\tau\|F(t,\tau)\|_{H^s}\leq C. \tag{3.10}$$

Take $k\in\mathbb{N}$ such that $(t,\tau)\in[t_k,t_{k+1}]^2$. Since $F(t,t_k)=0$,

$$\begin{aligned}\|F(t,\tau)\|_{H^s}&=\|F(t,\tau)\|_{H^s}-\|F(t,t_k)\|_{H^s}\\&=\int_{t_k}^{\tau}\partial_\sigma\|F(t,\sigma)\|_{H^s}d\sigma\leq C|\tau-t_k|\leq C\Delta t.\end{aligned}$$

□

Remark 3.4. The condition (1.3) is used only for (3.8), which is the estimate for X_1 in H^{s_1}.

As a corollary of Proposition 3.1, we have the following.

Corollary 3.5. *Let $T > 0$ and $s_1 = s - \max\{1, p\} > 3/2$. Then there exist $\overline{\Delta t}_* = \overline{\Delta t}_*(C_0, s, s_1, T) > 0$ and $C_* = C_*(C_0, s, s_1, T) > 0$ such that, for all $\Delta t \leq \overline{\Delta t}_*$, $n \in \mathbb{N}$ satisfying $n\Delta t \leq T$, and $v \in C(\Pi_G^{(n)} : H^{s_1})$ satisfying (1.6)–(1.8) on $\Pi_G^{(n)}$, it follows that $v \in C(\Pi_G^{(n)} : H^s)$ and*

$$\sup_{(t,\tau)\in\Pi_G^{(n)}} \|v(t,\tau)\|_{H^{s_1}} \leq 2C_0, \tag{3.11}$$

$$\sup_{t\in[0,n\Delta t]} \|v(t,t) - u(t)\|_{H^{s_1}} \leq C_*\Delta t. \tag{3.12}$$

Proof. First, we prove (3.11). For that purpose, we only need to prove (A) and (B) of Lemma 2.7 with $C_1 = 4C_0$. Obviously, (A) holds by (1.12). Next, we prove (B). Assume that $\sup_{(t,\tau)\in\Pi_G^{(n)}} \|v(t,\tau)\|_{H^{s_1}} \leq 4C_0$. By Lemma 2.8 with $C_1 = 4C_0$, there exists $C = C(C_0, s, s_1, T) > 0$ such that $\sup_{(t,\tau)\in\Pi_G^{(n)}} \|v(t,\tau)\|_{H^s} \leq C$. By $s_1 + p \leq s$, (1.2), and (1.7), we have

$$\begin{aligned}\|v(t,\tau) - v(t,t)\|_{H^{s_1}} &\leq \int_t^\tau \|\partial_\sigma v(t,\sigma)\|_{H^{s_1}} d\sigma \\ &\leq |t-\tau| \sup_{(t,\tau)\in\Pi_G^{(n)}} \|v(t,\tau)\|_{H^{s_1+p}} \leq C\Delta t. \end{aligned} \tag{3.13}$$

Then, by Proposition 3.1, it follows that for all $\Delta t \leq \overline{\Delta t}_*$,

$$\begin{aligned}\|v(t,\tau)\|_{H^{3/2+\epsilon}} &\leq \|v(t,\tau) - v(t,t)\|_{H^{s_1}} + \|v(t,t) - u(t)\|_{H^{s_1}} + \|u(t)\|_{H^s} \\ &\leq C\Delta t + C_0 \leq 2C_0.\end{aligned}$$

Here we take $\overline{\Delta t}_*$ such that $C\overline{\Delta t}_* = C_0$. Thus, we obtain (B). Therefore, we have (3.11) by Lemma 2.7.

By applying Lemma 2.8 to (3.11), it is also proved that $v \in C(\Pi_G^{(n)} : H^s)$ and there exists $C = C(C_0, s, s_1, T) > 0$ such that $\sup_{(t,\tau)\in\Pi_G^{(n)}} \|v(t,\tau)\|_{H^s} \leq C$. Therefore, by Proposition 3.1, we have (3.12). □

Finally we prove Theorem 1.1 and Remark 1.2.

Proof of Theorem 1.1. Let $\overline{\Delta t} = \min\{\overline{\Delta t}_*, \overline{\Delta t}_B(s_1, 2C_0)\}$, where $\overline{\Delta t}_B = \overline{\Delta t}_B(\sigma, M)$ is defined in Corollary 2.2. Note that $\overline{\Delta t} = \overline{\Delta t}(C_0, s, s_1, T)$.

We put conditions $(A)_n$ and $(B)_n$ for $n \in \mathbb{N}$ satisfying $1 \leq n \leq N$ as below.

$(A)_n$: For any $\Delta t \leq \overline{\Delta t}$, there exists a unique solution $v \in C(\Pi_G^{(n)} : H^{s_1}(\mathbb{R}))$ which satisfies (1.6)–(1.8).

$(B)_n$: For any $\Delta t \leq \overline{\Delta t}$, the solution $v \in C(\Pi_G^{(n)} : H^{s_1}(\mathbb{R}))$ of (1.6)–(1.8) satisfies $v \in C(\Pi_G^{(n)} : H^s(\mathbb{R}))$, (3.11), and (3.12).

The proof is by induction on n. Obviously, $(A)_1$ and $(B)_1$ are true. Let $l \in \mathbb{N}$ such that $1 \leq l \leq N-1$. We assume $(A)_l$ and $(B)_l$, and prove $(A)_{l+1}$ and $(B)_{l+1}$. First, we prove that $(A)_{l+1}$ holds. Since $\Delta t \leq \overline{\Delta t} \leq \overline{\Delta t}_{B*}$, we have $(A)_{l+1}$ by (3.11) with $n = l$ and Corollary 2.2. Next, we prove $(B)_{l+1}$ from $(A)_{l+1}$, but this has already been proved as Corollary 3.5.

By induction on n, we have Theorem 1.1 for $C = C_2(C_0, C_1'|_{C_1=2C_0}, s, s_1, T)$, where C_2 is the constant in Proposition 3.1. □

Proof of Remark 1.2. Let $G = v_t - Kv$. We obtain the first order error estimate in Δt for (1.13) in the same manner as in this section if we use

$$\sup_{t\in[0,n\Delta t]} \|G(t)\|_{H^{s_1}} \leq C\Delta t. \tag{3.14}$$

instead of Lemma 3.3. Now we prove (3.14). By $v_\tau + vv_x = 0$, we have

$$\partial_\tau G - (vG)_x = X, \tag{3.15}$$

where X is defined in the proof of Lemma 3.3. By (3.15), we have

$$\partial_\tau \|G\|_{H^{s_1}}^2 = 2\langle G, (vG)_x\rangle_{H^{s_1}} + 2\langle G, X\rangle_{H^{s_1}}. \tag{3.16}$$

By Proposition 2.4 (A), we have $|\langle G, (vG)_x\rangle_{H^{s_1}}| \leq C\|G\|_{H^{s_1}}^2 \|v\|_{H^{s_1+1}}$. We have $|\langle G, X\rangle_{H^{s_1}}| \leq C\|G\|_{H^{s_1}}$ by (3.8) and (3.9). Therefore, by the Gronwall inequality and $G(t, t_{l-1}) = 0$ for $t \in [t_{l-1}, t_l)$, we have (3.14). □

4. Error estimate for the Strang splitting

In this section, we prove Theorem 1.3. We put $w(t) = v(t,t) - u(t)$, $\Lambda_1^{(n)} = \cup_{l=1}^{n}(\Omega_{1,1}^{(l)} \cup \Omega_{2,1}^{(l)})$, $\Lambda_2^{(n)} = \cup_{l=1}^{n}(\Omega_{2,2}^{(l)} \cup \Omega_{1,2}^{(l)})$, and tilde is a time-shift operator, that is $\widetilde{f}(t,x) = f(t+\Delta t/2, x)$.

The main proposition in this section is Proposition 4.1 below. We obtain Theorem 1.3 in the same manner as in Section 3 if we use Proposition 4.1 instead of Proposition 3.1. Therefore, we only need to prove Proposition 4.1.

Proposition 4.1. *Let $\Delta t > 0$, $T > 0$, $n \in \mathbb{N}$ such that $n\Delta t \leq T$, and $s_2 = s - 3\max\{1,p\} > 3/2$. Assume that there exists a unique solution $v \in C(\Pi_S^{(n)} : H^s)$ of* (1.9)–(1.11) *and a constant $C_1' > 0$ satisfies* (2.11)*. Then, there exists $C_2 = C_2(C_0, C_1', s, s_2, T) > 0$ such that*

$$\sup_{t\in[0,n\Delta t]} \|w(t)\|_{H^{s_2}} \leq C_2(\Delta t)^2. \tag{4.1}$$

For the proof of Proposition 4.1, we only need to prove

$$\sup_{t\in[0,t_{n-1/2}]} \|w(t) + \widetilde{w}(t)\|_{H^{s_2}} \leq C(\Delta t)^2 \tag{4.2}$$

$$\sup_{t\in[0,t_{n-1/2}]} \|w(t) - \widetilde{w}(t)\|_{H^{s_2}} \leq C(\Delta t)^2 \tag{4.3}$$

instead of (4.1), since

$$\begin{aligned}\sup_{t\in[0,t_n]} \|w(t)\|_{H^{s_2}} &\leq \sup_{t\in[0,t_{n-1/2}]} \|w(t)\|_{H^{s_2}} + \sup_{t\in[0,t_{n-1/2}]} \|\widetilde{w}(t)\|_{H^{s_2}} \\ &\leq \sup_{t\in[0,t_{n-1/2}]} \|w(t) + \widetilde{w}(t)\|_{H^{s_2}} + \sup_{t\in[0,t_{n-1/2}]} \|w(t) - \widetilde{w}(t)\|_{H^{s_2}}.\end{aligned}$$

First, we prepare some notations to prove (4.2). We put

$$F(t,\tau) = \begin{cases} v_t(t,\tau) + v(t,\tau)\, v_x(t,\tau), & (t,\tau) \in \Lambda_1^{(n)}, \\ 0, & (t,\tau) \in \Lambda_2^{(n)}, \end{cases} \tag{4.4}$$

$$G(t,\tau) = \begin{cases} 0, & (t,\tau) \in \Lambda_1^{(n)}, \\ v_\tau(t,\tau) - Kv(t,\tau), & (t,\tau) \in \Lambda_2^{(n)}. \end{cases} \tag{4.5}$$

We also define the total forcing term H in $\Pi_S^{(n)}$ as $H(t,\tau) = F(t,\tau) + G(t,\tau)$ and $H(t) = H(t,t)$. By (1.1), (1.10), and (4.4) for the case $(t,t) \in \Lambda_1^{(n)}$ and (1.1), (1.9), and (4.5) for the case $(t,t) \in \Lambda_2^{(n)}$, w' is written in $t \in [0, t_n]$ as

$$w' = -ww_x - (uw)_x + Kw + H(t). \tag{4.6}$$

For simplicity, we put $z(t) = w(t) + \widetilde{w}(t)$ in $t \in [0, t_{n-1/2}]$.

Next, we prepare some lemmas to estimate $\|z(t)\|_{H^{s_2}}$.

Lemma 4.2. *Let $H(t,\tau) = F(t,\tau) + G(t,\tau)$ and $H(t) = H(t,t)$, where F satisfies* (4.4) *and G satisfies* (4.5). *Under the same assumption of Proposition 4.1, there exists $C = C(C_0, C_1', s, s_2, T) > 0$ such that for all $t \in [0, t_{n-1/2}]$,*

$$\begin{aligned}
&\|z(t)\|_{H^{s_2}} \\
&\quad \le \|z(0)\|_{H^{s_2}} e^{Ct} + Ct\{\sup_{t\in[0,t_{n-1/2}]} \|H(t) + \widetilde{H}(t)\|_{H^{s_2}} \\
&\qquad + (\sup_{t\in[0,t_n]} \|w\|_{H^{s_2+1}})^2 + \sup_{t\in[0,t_n]} \|w\|_{H^{s_2+1}} \sup_{t\in[0,t_{n-1/2}]} \|\widetilde{u} - u\|_{H^{s_2+1}}\}. \qquad (4.7)
\end{aligned}$$

Proof. By (4.6), it follows that for $t \in [0, t_{n-1/2}]$,

$$\widetilde{w}' = \widetilde{H}(t) + K\widetilde{w} - (\widetilde{u}\widetilde{w})_x - \widetilde{w}\widetilde{w}_x. \qquad (4.8)$$

By (4.6) and (4.8), we have the equation for z' in $t \in [0, t_{n-1/2}]$ as below.

$$z' = H(t) + \widetilde{H}(t) - \left(\frac{1}{2}z^2 + uz\right)_x + Kz - \left\{\widetilde{w}(\widetilde{u} - u) + w\widetilde{w}\right\}_x. \qquad (4.9)$$

Then it follows that

$$\frac{d}{dt}\|z(t)\|_{H^{s_2}}^2 = \left\langle z, H(t) + \widetilde{H}(t) - \left(\frac{1}{2}z^2 + uz\right)_x + Kz - \left\{\widetilde{w}(\widetilde{u} - u) + w\widetilde{w}\right\}_x \right\rangle_{H^{s_2}}.$$

Note that $\langle z, Kz\rangle_{H^{s_2}} \le 0$ from (1.2). By the Sobolev inequality, Proposition 2.4, Lemma 2.8, and (1.12), it follows that for $t \in [0, t_{n-1/2}]$,

$$\begin{aligned}
\frac{d}{dt}\|z(t)\|_{H^{s_2}} \le C\{&\|z\|_{H^{s_2}} + \|H(t) + \widetilde{H}(t)\|_{H^{s_2}} \\
&+ \|w\|_{H^{s_2+1}}\|\widetilde{w}\|_{H^{s_2+1}} + \|\widetilde{w}\|_{H^{s_2+1}}\|\widetilde{u} - u\|_{H^{s_2+1}}\}. \qquad (4.10)
\end{aligned}$$

Here we used the following inequality.

$$\begin{aligned}
\|z\|_{H^{s_2+1}} &\le \|w\|_{H^{s_2+1}} + \|\widetilde{w}\|_{H^{s_2+1}} \\
&\le \|u\|_{H^{s_2+1}} + \|v\|_{H^{s_2+1}} + \|\widetilde{u}\|_{H^{s_2+1}} + \|\widetilde{v}\|_{H^{s_2+1}} \le C.
\end{aligned}$$

Applying the Gronwall inequality to (4.10), we have (4.7). □

Lemma 4.3. *Let $X = -\{K(v^2)_x/2 - v_x Kv - vKv_x\}$. Under the same assumption of Proposition 4.1 , there exists $C = C(\|u_0\|_{H^s}, C_1', s, s_2, T) > 0$ such that*

$$\|X(t_1) - X(t_2)\|_{H^{s_2}} \le C|t_1 - t_2| \qquad (4.11)$$

for all t_1, $t_2 \in [0, t_n]$.

Remark 4.4. Since $F = v_t + vv_x$ in $\Lambda_1^{(n)}$, $G = v_\tau - Kv$ in $\Lambda_2^{(n)}$, and v satisfies (1.9) and (1.10), we have

$$F_\tau - KF = X\,,\ (t,\tau) \in \Lambda_1^{(n)},$$
$$G_t + (vG)_x = -X\,,\ (t,\tau) \in \Lambda_2^{(n)}.$$

Before proving Lemma 4.3, we estimate v_t and v_τ.

Lemma 4.5. *Let $j = 1, 2, 3$ and $3/2 < \sigma \le s - j\max\{1,p\}$ for each j. Under the same assumption of Proposition 4.1, there exists $C = C(\|u_0\|_{H^s}, C_1', s, s_2, T) > 0$ such that*

$$\|(\partial_t)^j v(t,\tau)\|_{H^\sigma} \le \prod_{m=0}^{j} \sup_{(t,\tau)\in\Pi_S^{(n)}} \|v(t,\tau)\|_{H^{\sigma+m}}, \tag{4.12}$$
$$\|(\partial_\tau)^j v(t,\tau)\|_{H^\sigma} \le C \sup_{(t,\tau)\in\Pi_S^{(n)}} \|v(t,\tau)\|_{H^{\sigma+jp}}. \tag{4.13}$$

Proof. First, we prove (4.12) for the case $(t,\tau) \in \Lambda_1^{(n)}$ with $j = 1$. Since (1.2) and (1.10), it follows that for $(t,\tau) \in \Lambda_1^{(n)} \cap \Sigma_{S,2}^{(n)}$ and $3/2 < \sigma \le s$,

$$\partial_\tau \|v_t(t,\tau)\|_{H^\sigma}^2 = 2\langle v_t, Kv_t\rangle_{H^\sigma} \le 0. \tag{4.14}$$

By (1.9), it follows that for $(t,\tau) \in \Lambda_1^{(n)}$, $l \in \mathbb{N}$ such that $t \in [t_{l-1}, t_{l-1/2}]$, and $3/2 < \sigma \le s-1$,

$$\|v_t(t,\tau)\|_{H^\sigma} \le \|v_t(t,t_{l-1})\|_{H^\sigma} \le \|v(t,t_{l-1})\|_{H^\sigma}\|v(t,t_{l-1})\|_{H^{\sigma+1}} \le \prod_{j=0}^{1} \sup_{(t,\tau)\in\Pi_S^{(n)}} \|v(t,\tau)\|_{H^{\sigma+j}}.$$

Next, we prove (4.12) for the case $(t,\tau) \in \Lambda_1^{(n)}$ with $j = 2, 3$. By induction argument, we have that $\partial_t(vv_x)$ becomes the $(p+1)$-th polynomial with p derivatives. By (1.2), we have $\partial_\tau\|(\partial_t)^j v(t,\tau)\|_{H^\sigma}^2 \le 0$. Therefore, the same argument as the case $j = 1$ works and we have (4.12).

Next, we prove (4.12) for the case $(t,\tau) \in \Lambda_2^{(n)}$ with $j = 1$. By (1.10), it follows that for $(t,\tau) \in \Lambda_2^{(n)}$ and $3/2 < \sigma \le s-1$,

$$\|v_t(t,\tau)\|_{H^\sigma} \le \|v(t,\tau)\|_{H^\sigma}\|v(t,\tau)\|_{H^{\sigma+1}} \le \prod_{j=0}^{1} \sup_{(t,\tau)\in\Pi_S^{(n)}} \|v(t,\tau)\|_{H^{\sigma+j}}.$$

Then, we have (4.12). (4.12) for the case $(t,\tau) \in \Lambda_2^{(n)}$ with $j = 2, 3$ is easily proved since $\partial_t(vv_x)$ becomes the $(p+1)$-th polynomial with p derivatives.

Next, we prove (4.13) for the case $(t,\tau) \in \Lambda_2^{(n)}$ with $j = 1$. By (1.9) and Proposition 2.4 (A), it follows that for $(t,\tau) \in \Lambda_2^{(n)} \cap \Sigma_{S,1}^{(n)}$ and $3/2 < \sigma \le s-1$,

$$\partial_t \|v_\tau(t,\tau)\|_{H^\sigma}^2 = 2\langle v_\tau, (vv_x)_\tau\rangle_{H^\sigma} \le C\|v_\tau\|_{H^\sigma}^2 \|v\|_{H^{\sigma+1}}.$$

By the Gronwall inequality, (1.2), and (1.10), it follows that for $(t,\tau) \in \Lambda_2^{(n)}$, $l \in \mathbb{N}$ such that $\tau \in [t_{l-1/2}, t_l]$, and $3/2 < \sigma \le s - \max\{1,p\}$,

$$\begin{aligned}\|v_\tau(t,\tau)\|_{H^\sigma} &\le e^{CC_1'(t-t_{l-1/2})}\|v_\tau(t_{l-1/2},\tau)\|_{H^\sigma}\\ &\le e^{CC_1'\Delta t}\|v(t_{l-1/2},\tau)\|_{H^{\sigma+p}} \le e^{CC_1'\Delta t} \sup_{(t,\tau)\in\Pi_S^{(n)}} \|v(t,\tau)\|_{H^{\sigma+p}}.\end{aligned}$$

Next, we prove (4.13) for the case $(t,\tau) \in \Lambda_2^{(n)}$ with $j = 2, 3$. In the same manner as the case $j = 1$, by the Gronwall inequality, we have $\|(\partial_\tau)^j v(t,\tau)\|_{H^\sigma} \le e^{CC_1'(t-t_{l-1/2})}\|(\partial_\tau)^j v(t_{l-1/2},\tau)\|_{H^\sigma}$. Thus, by (1.10), we have (4.13). Finally, we prove (4.13) for the case $(t,\tau) \in \Lambda_1^{(n)}$ with $j = 1, 2, 3$. By (1.2) and (1.10), it follows that for $(t,\tau) \in \Lambda_1^{(n)}$ and $3/2 < \sigma \le s - j$,

$$\|(\partial_\tau)^j v(t,\tau)\|_{H^\sigma} \le \|v(t,\tau)\|_{H^{\sigma+jp}} \le \sup_{(t,\tau)\in\Pi_S^{(n)}} \|v(t,\tau)\|_{H^{\sigma+jp}}.$$

Therefore, we have (4.13). □

Next, we prove Lemma 4.3.

Proof. For the proof of Lemma 4.3, we only need to prove the boundness of X_t and X_τ in H^{s_2}. We have the boundness for $(t,\tau) \in \Lambda_2^{(n)}$ in the same manner as for $(t,\tau) \in \Lambda_1^{(n)}$, so we only give the proof for the case $(t,\tau) \in \Lambda_1^{(n)}$. First, we prove the boundness of X_t in H^{s_2}. By the definition of X, we have

$$X_t = -K(v_t v_x) - K(vv_{xt}) + v_{xt}(Kv) + v_x(Kv_t) + v_t(Kv_x) + v(Kv_{xt}).$$

Since $s_2 + 2 + p \le s$, we have

$$\|X_t\|_{H^{s_2}} \le C\|v\|_{H^{s_2+1+p}}\|v\|_{H^{s_2+2+p}} \le C. \tag{4.15}$$

Next, we prove for the boundness of X_τ in H^{s_2}. Since $s_2 + 1 + 2p \le s$, (1.10), and Remark 4.4, in the same manner as (4.15), we have

$$\begin{aligned}\|X_\tau\|_{H^{s_2}} \le{}& \|K((Kv)v_x)\|_{H^{s_2}} + \|K(v(Kv_x))\|_{H^{s_2}}\\ &+ 2\|(Kv)(Kv_x)\|_{H^{s_2}} + \|v_x(K^2 v)\|_{H^{s_2}} + \|v(K^2 v_x)\|_{H^{s_2}}\\ \le{}& C.\end{aligned}$$

□

Next, we estimate $H + \widetilde{H} = F + G + \widetilde{F} + \widetilde{G}$. In view of $\widetilde{F} = G = 0$ in $\Lambda_1^{(n)}$, $F = \widetilde{G} = 0$ in $\Lambda_2^{(n-1)}$, and Remark 4.4, it is natural to estimate $\widetilde{F} + G$ and $F + \widetilde{G}$.

Lemma 4.6. *Let $l \in \mathbb{N}$, F satisfy* (4.4)*, and G satisfy* (4.5)*. Under the same assumptions of Proposition 4.1, there exists $C = C(\|u_0\|_{H^s}, C_1', s, s_2, T) > 0$ such that*

$$\left\| F(t) + G(t + \frac{\Delta t}{2}) \right\|_{H^{s_2}} \leq C(\Delta t)^2$$

for all $t \in [t_{l-1}, t_{l-1/2}] \subset [0, t_{n-1/2}]$, and

$$\left\| F(t) + G(t - \frac{\Delta t}{2}) \right\|_{H^{s_2}} \leq C(\Delta t)^2$$

for all $t \in [t_{l-1}, t_{l-1/2}] \subset [t_1, t_{n-1/2}]$.

Proof. Let $\Phi = F_{tt} + 2F_{t\tau} + F_{\tau\tau}$ and $\Psi = G_{tt} + 2G_{t\tau} + G_{\tau\tau}$. By Taylor expansion at $t = t_{l-1}$, we have

$$\begin{aligned} &F(t) + G(t \pm \frac{\Delta t}{2}) \\ &\quad = \{F(t_{l-1}) + G(t_{l-1\pm\frac{1}{2}})\} + (t - t_{l-1})\{(F_t + F_\tau)(t_{l-1}) + (G_t + G_\tau)(t_{l-1\pm\frac{1}{2}})\} \\ &\qquad + \frac{(t - t_{l-1})^2}{2} \int_0^1 \{\Phi(\theta(t - t_{l-1}) + t_{l-1}) + \Psi(\theta(t \pm \frac{\Delta t}{2} - t_{l-1\pm\frac{1}{2}}) + t_{l-1\pm\frac{1}{2}})\} d\theta. \end{aligned}$$

Since $F = 0$ in $\Lambda_2^{(n)} \cup \Sigma_{S,1}^{(n)}$ because of (1.9) and (4.4), and $G = 0$ in $\Lambda_1^{(n)} \cup \Sigma_{S,2}^{(n)}$ because of (1.10) and (4.5), we have $F(t_{l-1}) = G(t_{l-1\pm 1/2}) = 0$ and $F_t(t_{l-1}) = G_\tau(t_{l-1\pm 1/2}) = 0$. By Remark 4.4 and $F(t_{l-1}) = G(t_{l-1\pm 1/2}) = 0$, we have $F_\tau(t_{l-1}) + G_t(t_{l-1\pm\frac{1}{2}}) = X(t_{l-1}) - X(t_{l-1\pm\frac{1}{2}})$. Then, we have the second order estimate in Δt of the second term in H^{s_2} space by Lemma 4.3. $\|\Phi\|_{H^{s_2}} + \|\Psi\|_{H^{s_2}} \leq C$ is proved by (1.9)–(1.11), Proposition 2.4, (2.11), (4.12) , and (4.13). Therefore it follows that for $t \in [t_{l-1}, t_{l-1/2}]$,

$$\begin{aligned} \|F(t) + G(t \pm \frac{\Delta t}{2})\|_{H^{s_2}} &\leq |t - t_{l-1}| \|X(t_{l-1\pm\frac{1}{2}}) - X(t_{l-1})\|_{H^{s_2}} + \frac{C}{2}|t - t_{l-1}|^2 \\ &\leq C(\Delta t)^2. \end{aligned} \tag{4.16}$$

□

Lemma 4.7. *Let $s_2 = s - 3\max\{1, p\} > 3/2$. Assume that $u \in C([0, t_{1/2}] : H^s)$ satisfies* (1.1) *on $[0, t_{1/2}]$ and $v \in C(\Lambda_1^{(1)} : H^s)$ satisfies* (1.9)*–*(1.11) *on $\Lambda_1^{(1)}$. Assume that a constant $C_1' > 0$ satisfies $\sup_{(t,\tau)\in\Lambda_1^{(1)}} \|v(t, \tau)\|_{H^s} \leq C_1'$. Then, there exists $C = C(\sup_{[0,t_{1/2}]} \|u(t)\|_{H^s}, C_1', s, s_2) > 0$ such that*

$$\|w(t)\|_{H^{s_2}} \leq C(\Delta t)^2$$

for all $t \in [0, t_{1/2}]$.

Proof. We use the Taylor expantion of $w(t)$ at $t=0$. That is

$$\begin{aligned} w(t) &= w(0) + t\big(\partial_t v(t,\tau)|_{t=\tau=0} + \partial_\tau v(t,\tau)|_{t=\tau=0} - u_t(0)\big) \\ &\quad + \frac{t^2}{2}\int_0^1 \{\partial_{tt} v(\sigma t,\sigma t) + 2\partial_{t\tau} v(\sigma t,\sigma t) + \partial_{\tau\tau} v(\sigma t,\sigma t) - u_{tt}(\sigma t)\} d\sigma. \end{aligned}$$

The term of the 0th order of t is 0 because of $w(0) = v(0,0) - u(0)$ and $u(0) = v(0,0)$. Applying (1.1), (1.9), (1.10), and $u(0) = v(0,0)$ to the term of the 1st order of t, we have

$$\begin{aligned} &\partial_t v(t,\tau)|_{t=\tau=0} + \partial_\tau v(t,\tau)|_{t=\tau=0} - u'(0) \\ &\qquad = -v(0,0)v_x(0,0) + Kv(0,0) + u(0)u_x(0) - Ku(0) = 0. \end{aligned}$$

By (4.12) and (1.10), it follows that

$$\begin{aligned} &\|\partial_{tt} v(\sigma t,\sigma t) + 2\partial_{t\tau} v(\sigma t,\sigma t) + \partial_{\tau\tau} v(\sigma t,\sigma t)\|_{H^{s_2}} \\ &\quad \le \|\partial_{tt} v(\sigma t,\sigma t)\|_{H^{s_2}} + 2\|K\partial_t v(\sigma t,\sigma t)\|_{H^{s_2}} + \|K^2 v(\sigma t,\sigma t)\|_{H^{s_2}} \\ &\quad \le \prod_{j=0}^{2} \max_{(t,\tau)\in\Lambda_1^{(1)}} \|v(t,\tau)\|_{H^{s_2+j}} + \prod_{j=0}^{1} \max_{(t,\tau)\in\Lambda_1^{(1)}} \|v(t,\tau)\|_{H^{s_2+p+j}} + \max_{(t,\tau)\in\Lambda_1^{(1)}} \|v(t,\tau)\|_{H^{s_2+2p}} \\ &\quad \le C \end{aligned} \tag{4.17}$$

Therefore, $\|w(t)\|_{H^{s_2}} \le Ct^2$. Since $t \le (\Delta t)/2$, we have the desired result. □

Next, we prove (4.2) by Lemmas 4.2, 4.7, Theorem 1.1, and Remark 1.2.

Proof. First, we prove three inequalities

$$\|z(0)\|_{H^{s_2}} \le C(\Delta t)^2,\ \|\widetilde{u} - u\|_{H^{s_2+1}} \le C\Delta t,\ \|w\|_{H^{s_2+1}} \le C\Delta t. \tag{4.18}$$

We have the first inequality in (4.18) by applying Lemma 4.7 for $t = (\Delta t)/2$. Note that $z(0) = w(0) + \widetilde{w}(0) = w(\frac{\Delta t}{2})$. To prove the second one, we use (1.1), (1.12), and $\tilde{u}(t) - u(t) = \int_0^{\frac{\Delta t}{2}} u'(t+\sigma)d\sigma$. The third one is already proved as Theorem 1.1 and Remark 1.2.

Finally, we prove (4.2). For $(t,t) \in \Lambda_1^{(n)}$, since $\widetilde{F} = G = 0$, we have $\|H + \widetilde{H}\|_{H^{s_2}} = \|F + \widetilde{G}\|_{H^{s_2}} \le C(\Delta t)^2$ by Lemma 4.6. For $(t,t) \in \Lambda_2^{(n-1)}$ (not $\Lambda_2^{(n)}$), since $F = \widetilde{G} = 0$, we have $\|H + \widetilde{H}\|_{H^{s_2}} = \|\widetilde{F} + G\|_{H^{s_2}} \le C(\Delta t)^2$ by Lemma 4.6. Thus, it follows that for $(t,t) \in \Lambda_1^{(n)} \cup \Lambda_2^{(n-1)}$,

$$\|H(t) + \widetilde{H}(t)\|_{H^{s_2}} \le C(\Delta t)^2. \tag{4.19}$$

Since $(t,t) \in \Lambda_1^{(n)} \cup \Lambda_2^{(n-1)}$ is equivalent to $t \in [0, t_{n-1/2}]$, (4.2) follows by applying (4.18) and (4.19) to Lemma 4.2. □

Next, we prove (4.3). We use the following lemma to prove (4.3) later.

Lemma 4.8. *Let $l \in \mathbb{N}$ and assumptions of Proposition 4.1 hold.*
(A) Then there exists $C = C(C_0, C_1', s, s_2, T) > 0$ such that for all $t \in [t_{l-1}, t_{l-1/2}] \subset [0, t_n]$,

$$\|w(t) - w(t_{l-1})\|_{H^{s_2}} \leq C(\Delta t)^2.$$

(B) Then there exists $C = C(C_0, C_1', s, s_2, T) > 0$ such that for all $t \in [t_{l-1/2}, t_l] \subset [0, t_n]$,

$$\|w(t) - w(t_{l-1/2})\|_{H^{s_2}} \leq C(\Delta t)^2.$$

Proof. We have (B) in the same manner as (A), so we only prove (A). Let $V(t,x)$ satisfy

$$\begin{cases} \partial_t V - VV_x + KV = 0, & (x,t) \in \mathbb{R} \times [t_{l-1}, T], \\ V(\cdot, t_{l-1}) = v(\cdot, t_{l-1}, t_{l-1}) \in H^s(\mathbb{R}). \end{cases} \tag{4.20}$$

Note that the first equations in (1.1) and (4.20) are the same, and we may also apply (1.12) to (4.20). First, we decompose $w(t) - w(t_{l-1})$ as

$$\begin{aligned} &w(t) - w(t_{l-1}) \\ &\quad = \{v(t) - V(t)\} + \{V(t) - v(t_{l-1}, t_{l-1})\} - \{u(t) - u(t_{l-1})\}. \end{aligned} \tag{4.21}$$

We have the second-order approximation in Δt from the first term in (4.21) by Lemma 4.7. The second and the third terms are

$$\begin{aligned} &\{V(t) - v(t_{l-1}, t_{l-1})\} - \{u(t) - u(t_{l-1})\} \\ &= \int_{t_{l-1}}^{t} \{V_t(t') - u_t(t')\} dt' \\ &= \int_{t_{l-1}}^{t} \{V(t')V_x(t') - KV(t') - u(t')u_x(t') + Ku(t')\} dt' \\ &= \int_{t_{l-1}}^{t} \{V(t')(V(t') - u(t'))_x + (V(t') - u(t'))u_x(t') - K(V-u)(t')\} dt'. \end{aligned}$$

Let $s_2' = s_2 + \max\{1, p\} (= s - 2\max\{1, p\})$. Since $\|V\|_{H^{s_2}} \leq C$ and $\|u_x\|_{H^{s_2}} \leq C_0$, we have

$$\begin{aligned} &\Big\| \{V(t) - v(t_{l-1}, t_{l-1})\} - \{u(t) - u(t_{l-1})\} \Big\|_{H^{s_2}} \\ &\quad \leq C \int_{t_{l-1}}^{t} \|V(t') - u(t')\|_{H^{s_2'}} dt' \leq C\Delta t \sup_{t \in [t_{l-1}, t_{l-1/2}]} \|V(t) - u(t)\|_{H^{s_2'}}. \end{aligned}$$

$\|V(t)-u(t)\|_{H^{s'_2}}$ is estimated from (1.1), (1.12), (4.20), Theorem 1.1, and Remark 1.2 as

$$\begin{aligned} &\|V(t)-u(t)\|_{H^{s'_2}} \\ &\quad \le \|V(t)-v(t_{l-1},t_{l-1})\|_{H^{s'_2}} + \|w(t_{l-1})\|_{H^{s'_2}} + \|u(t_{l-1})-u(t)\|_{H^{s'_2}} \\ &\quad \le \int_{t_{l-1}}^{t} \|V'(t')\|_{H^{s'_2}} dt' + C\Delta t + \int_{t_{l-1}}^{t} \|u'(t')\|_{H^{s'_2}} dt' \le C\Delta t. \end{aligned} \tag{4.22}$$

Thus, we have the second-order approximation in Δt from the second and the third terms in (4.21). Therefore we have Lemma 4.8 (A). □

Finally, we prove (4.3).

Proof. Let $l \in \mathbb{N}$. By Lemma 4.8, it follows that for $t \in [t_{l-1}, t_{l-1/2}] \subset [0, t_{n-1/2}]$,

$$\begin{aligned} &\|\widetilde{w}(t)-w(t)\|_{H^{s_2}} \\ &\le \|w(t+\Delta t/2)-w(t_{l-1/2})\|_{H^{s_2}} + \|w(t_{l-1/2})-w(t_{l-1})\|_{H^{s_2}} + \|w(t_{l-1})-w(t)\|_{H^{s_2}} \\ &\le C(\Delta t)^2, \end{aligned}$$

and for $t \in [t_{l-1/2}, t_l] \subset [0, t_{n-1/2}]$,

$$\begin{aligned} &\|\widetilde{w}(t)-w(t)\|_{H^{s_2}} \\ &\le \|w(t+\Delta t/2)-w(t_l)\|_{H^{s_2}} + \|w(t_l)-w(t_{l-1/2})\|_{H^{s_2}} + \|w(t_{l-1/2})-w(t)\|_{H^{s_2}} \\ &\le C(\Delta t)^2. \end{aligned}$$

□

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