Measures and K-Theory on the Boundary of Trees

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Contents

Abstract

The aim of this thesis is to study measures and K-theory on the boundary of semi-homogeneous trees. For a semi-homogeneous tree $\mathcal{T}$, we introduce the notion of the space of rational ends $\partial_R\mathcal{T}$ and represent measures on the boundary $\partial\mathcal{T}$ of $\mathcal{T}$ as Alexander-Spanier cocycles on $\partial_R\mathcal{T}$. Moreover, if an action of a group G on $\partial\mathcal{T}$ descends to an action on $\partial_R\mathcal{T}$, then we prove that the above representation is G-equivariant. We also define modular symbols for Hecke triangle groups and investigate their properties based on the theory of Alexander-Spanier 1-cocycles on the space of rational ends of the trees of Hecke triangle groups. It will also be shown that for a free subgroup $\Gamma \subset G$, the natural isomorphism between the group of Γ-invariant $\mathbb{Z}$-valued measures $\mathrm{Meas}(\partial\mathcal{T}, \mathbb{Z})^\Gamma$ on $\partial\mathcal{T}$ and the K-homology group $K^0(C(\partial\mathcal{T}) \rtimes_r \Gamma)$ of the reduced crossed product C^*-algebra is compatible with the action of Hecke operators.

This work is motivated by the works of Manin and Marcolli [15, 16] and Mesland and Şengün [18] which study the "noncommutative boundary" of quotient spaces by arithmetic groups. Indeed, some of the results of this thesis are generalizations of those of [16]. While they consider only the case of $\mathrm{PSL}_2(\mathbb{Z})$ and its tree, we treat the case of arbitrary groups acting on semi-homogeneous trees nicely.

Introduction

For a finite index subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$, Manin and Marcolli showed that there is a one-to-one correspondence between Γ-equivariant modular symbols for $\mathrm{PSL}_2(\mathbb{Z})$ and Γ-equivariant measures on the boundary of the tree of $\mathrm{PSL}_2(\mathbb{Z})$. In this thesis, we generalize this correspondence to the case of semi-homogeneous trees.

Our starting point is the simple observation that modular symbols for $\mathrm{PSL}_2(\mathbb{Z})$ are nothing but Alexander-Spanier 1-cocycles on $\mathbb{P}^1(\mathbb{Q})$. This observation allows us to interpret that the result of Manin and Marcolli states that we can represent measures on the boundary of the tree of $\mathrm{PSL}_2(\mathbb{Z})$ as Alexander-Spanier 1-cocycles on $\mathbb{P}^1(\mathbb{Q})$. Then it is natural to try to represent measures on the boundary $\partial\mathcal{T}$ of a semi-homogeneous tree $\mathcal{T}$ as Alexander-Spanier 1-cocycles on some space associated with the tree. To do this, we introduce the notion of the space of rational ends $\partial_R\mathcal{T}$, which plays the same role as $\mathbb{P}^1(\mathbb{Q})$ does for the tree of $\mathrm{PSL}_2(\mathbb{Z})$. The following theorem is one of our main results:

Theorem A. (Theorem 2.2.1, 2.5.1) *Let $\mathcal{T}$ be a semi-homogeneous tree and M an additive group. We denote by $\mathrm{Meas}(\partial\mathcal{T}, M)$ the group of M-valued measures on $\partial\mathcal{T}$, and $Z^1_{AS}(\partial_R\mathcal{T}; M)$ denotes the group of Alexander-Spaner 1-cocycles on $\partial_R\mathcal{T}$ with coefficients in M. Then there is an isomorphism*

$$\mathrm{Meas}(\partial\mathcal{T}, M) \simeq Z^1_{AS}(\partial_R\mathcal{T}; M).$$

Moreover, if an action of a group G on $\partial\mathcal{T}$ descends to an action on $\partial_R\mathcal{T}$, then we have an isomorphism

$$\mathrm{Meas}_G(\partial\mathcal{T}, M) \simeq Z^1_{AS,G}(\partial_R\mathcal{T}; M),$$

where $\mathrm{Meas}_G(\partial\mathcal{T}, M)$ denotes the group of G-equivariant M-valued measures on the boundary $\partial\mathcal{T}$, and $Z^1_{AS,G}(\partial_R\mathcal{T}; M)$ denotes the group of G-equivariant Alexander-Spaner 1-cocycles on the space of rational ends $\partial_R\mathcal{T}$ with coefficients in M.

Alexander-Spanier 1-cocycles on the space of rational ends enjoy the same properties as modular symbols for $\mathrm{PSL}_2(\mathbb{Z})$. For example, for the natural action of $G := \mathbb{Z}_{q+1} * \mathbb{Z}_{\ell+1}$ on a semi-homogeneous tree $\mathcal{T}$ of degree (q, ℓ) (see Section 2.6), we have the following explicit description of $Z^1_{AS,G}(\partial_R\mathcal{T}; M)$, which is an extension of [16, Theorem 2.3]:

Theorem B. (Theorem 2.6.1) *Let $q \geq 2$ and $\ell \geq 1$ be two integers. Assume $G := \mathbb{Z}_{q+1} * \mathbb{Z}_{\ell+1}$ acts on a semi-homogeneous tree $\mathcal{T}$ of degree (q, ℓ) naturally. For a G-module M, define a map $(\tilde{\tau}, \tilde{\sigma}) : M \to M \times M$ by $\tilde{\tau}(m) := m + \tau m + \cdots + \tau^q m$ and $\tilde{\sigma}(m) := m + \sigma m + \cdots + \sigma^\ell m$, where τ and σ are generators of $\mathbb{Z}_{q+1}$ and $\mathbb{Z}_{\ell+1}$, respectively. Then there is an isomorphism*

$$Z^1_{AS,G}(\partial_R\mathcal{T}; M) \simeq \ker(\tilde{\tau}, \tilde{\sigma}).$$

We use this special example to investigate modular symbols for Hecke triangle groups, which will be defined in Chapter 3. In fact, for a Hecke triangle group G_q, we prove that modular symbols for G_q coincide with Alexander-Spanier 1-cocycles on the space of rational ends of the tree of G_q. Then the above description shows that rational period functions for G_q in the sense of [20] are modular symbols for G_q.

For a semi-homogeneous tree $\mathcal{T}$, the space of rational ends $\partial_R\mathcal{T}$ is constructed from the special points of the boundary $\partial\mathcal{T}$. However, as already mentioned in [16, Theorem 2.3] for the tree of $\mathrm{PSL}_2(\mathbb{Z})$, we can reconstruct the whole boundary from the space of rational ends when we embed $\partial_R\mathcal{T}$ densely into the unit circle S^1. To do this, we use the following construction due to Spielberg [26]: For a dense subset F of S^1, let B_F denote the norm closure of the $*$-algebra generated by $\{p(\alpha,\beta) \mid \alpha,\beta \in F,\ \alpha \neq \beta\}$, where $p(\alpha,\beta)$ denotes the characteristic function of the half-open interval $[\alpha,\beta)$. Then B_F is a unital commutative C^*-algebra, and hence there is a compact Hausdorff space D_F such that $B_F \simeq C(D_F)$. The compact Hausdorff space D_F is called the disconnection of S^1 on F.

Theorem C. (Theorem 2.3.1) *Let $\Phi : \partial_R \mathcal{T} \to S^1$ be an orientation preserving dense embedding. Then there is a homeomorphism*

$$D_{\Phi(\partial_R \mathcal{T})} \simeq \partial \mathcal{T}.$$

The notion of disconnection was originally introduced by Spielberg to investigate the crossed product C^*-algebras arising from the action of Fuchsian groups on the boundary of the unit disk, that is, the circle S^1. Such C^*-algebras are called boundary crossed products and have received attention by the researchers.

When a group G acts on some bulk space Y, the action of the group on the boundary ∂Y may become very chaotic. In such a case, the quotient space $\partial Y/G$ usually becomes a poor object, but the (reduced) crossed product C^*-algebra $C(\partial Y) \rtimes_r G$ has a rich structure, and it is interesting to consider its invariants (e.g. K-theory). This phenomenon was mentioned by Connes in his book [5].

In [15], for the case of a finite index subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ acting on the upper-half plane $\mathbb{H}$, Manin and Marcolli called the crossed product $C(\mathbb{P}^1(\mathbb{R})) \rtimes_r \Gamma$ the noncommutative boundary of the modular curve $X_\Gamma = \overline{\Gamma \setminus \mathbb{H}}$ and investigated its relation to the bulk space, that is, the modular curve X_Γ, where $\mathbb{P}^1(\mathbb{R}) = \partial\mathbb{H}$. They proved that there are natural isomorphisms $K_0(C(\mathbb{P}^1(\mathbb{R})) \rtimes_r \Gamma) \simeq H_1(X_\Gamma, \mathrm{cusps}; \mathbb{Z}) \oplus \mathbb{Z} \oplus T$ and $K_1(C(\mathbb{P}^1(\mathbb{R})) \rtimes_r \Gamma) \simeq H_1(X_\Gamma, \mathrm{cusps}; \mathbb{Z}) \oplus \mathbb{Z}$, where T denotes the torsion part. This thesis as well as [16] is motivated by this investigation.

Inspired by this work, Mesland and Şengün [18] studied the noncommutative boundary of a Bianchi manifold $C(\partial\mathbb{H}_3) \rtimes_r \Gamma$, where Γ is a torsion-free finite index subgroup of a Bianchi group $\mathrm{PSL}_2(\mathcal{O}_K)$, and $\mathcal{O}_K$ is the ring of integers of an imaginary quadratic field K. They investigated the K-homology of the crossed product and its relation to the group cohomology of Γ. In particular, they constructed the Hecke operators in K-theory and proved that there is a Hecke equivariant isomorphism $K^1(C(\partial\mathbb{H}_3) \rtimes_r \Gamma) \simeq H^1(\Gamma,\mathbb{Z}) \oplus H^1(\Gamma,\mathbb{Z})$. Using the same methods to prove the Hecke equivariant isomorphism, they also showed that for a torsion-free finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ the isomorphisms described above are compatible with the action of Hecke operators.

We now return to the case of the boundary of trees. Let $\mathcal{T}$ be a semi-homogeneous tree of degree (q,ℓ) with q or $\ell \geq 2$. Then the boundary of the tree is a totally disconnected compact Hausdorff space, and hence we can consider $\mathrm{Meas}(\partial\mathcal{T},\mathbb{Z})$ as a subgroup of the K-homology group $K^0(C(\partial\mathcal{T}))$. Let G be a group acting on $\mathcal{T}$ and $\Gamma \subset G$ a free subgroup. Using the Pimsner-Voiculescu exact sequence for free groups and the universal coefficient theorem, we can compute the K-homology group of the crossed product $C(\partial\mathcal{T}) \rtimes_r \Gamma$. That is to say, we have an isomorphism

$$K^0(C(\partial\mathcal{T}) \rtimes_r \Gamma) \simeq \mathrm{Meas}_\Gamma(\partial\mathcal{T},\mathbb{Z}) = \mathrm{Meas}(\partial\mathcal{T},\mathbb{Z})^\Gamma = H^0(\Gamma; \mathrm{Meas}(\partial\mathcal{T},\mathbb{Z})),$$

where we consider $\mathbb{Z}$ as a trivial Γ-module. If the commensurator $C_G(\Gamma)$ is nonempty, since the construction of the Hecke operators in K-theory is also valid for our case, we can consider the Hecke operators

$$T_g : \mathrm{Meas}(\partial\mathcal{T},\mathbb{Z})^\Gamma = H^0(\Gamma; \mathrm{Meas}(\partial\mathcal{T},\mathbb{Z})) \to H^0(\Gamma; \mathrm{Meas}(\partial\mathcal{T},\mathbb{Z})) = \mathrm{Meas}(\partial\mathcal{T},\mathbb{Z})^\Gamma$$

and

$$T_g : K^0(C(\partial\mathcal{T}) \rtimes_r \Gamma) \to K^0(C(\partial\mathcal{T}) \rtimes_r \Gamma),$$

where $g \in G$. We prove that these Hecke operators are compatible with the above isomorphism:

Theorem D. (Theorem 4.3.1) *For any $g \in C_G(\Gamma)$, the diagram*

$$\begin{array}{ccc} K^0(C(\partial\mathcal{T}) \rtimes_r \Gamma) & \xrightarrow{T_g} & K^0(C(\partial\mathcal{T}) \rtimes_r \Gamma) \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{Meas}(\partial\mathcal{T},\mathbb{Z})^\Gamma & \xrightarrow{T_g} & \mathrm{Meas}(\partial\mathcal{T},\mathbb{Z})^\Gamma \end{array}$$

commutes.

Note that we prove the above theorem for totally disconnected compact Hausdorff spaces since the proof of the theorem remains valid for that case. In the case where Γ is a torsion-free finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, the group of Γ-equivariant $\mathbb{Z}$-valued modular symbols for $\mathrm{PSL}_2(\mathbb{Z})$ coincides with the group $\mathrm{Meas}(\partial\mathcal{T}, \mathbb{Z})^\Gamma$, and its Hecke module structure described in [16] is also compatible with Theorem D.

About this thesis

We now turn to the description of this thesis. In Chapter 1, we introduce the basic notions and terminologies which will be needed for the main part of this thesis.

In Chapter 2, we investigate measures on the boundary of trees. Section 2.1 contains a brief summary of measures on the boundary of trees. In Section 2.2, we introduce the notion of the space of rational ends and represent measures on the boundary of a tree as Alexander-Spanier 1-cocycles on the space of rational ends. The remaining sections are devoted to the study of the space of rational ends and Alexander-Spanier cocycles on the space.

Modular symbols for Hecke triangle groups are the subject of Chapter 3. After reviewing basic properties of Hecke triangle groups and their relation to Rosen continued fractions, we define modular symbols for Hecke triangle groups and discuss their properties based on the results in Chapter 2.

In Chapter 4, we study the relationship between measures and K-theory. In fact, we compute the K-theory and the K-homology of the crossed product $C(X) \rtimes_r \Gamma$. We present the definition of the Hecke operators in K-theory and prove the Hecke equivariant isomorphism in Theorem D.

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Chapter 1

Preliminaries

1.1 Trees

In this section, we review some of the standard facts on trees. For a through treatment we refer the reader to [7].

1.1.1 Basic Definitions of Trees

An *(unoriented) graph* $\mathcal{G}$ is a pair $(\mathcal{V}, \mathcal{E})$ consisting of a set $\mathcal{V}$ and a family $\mathcal{E}$ of two-element subsets of $\mathcal{V}$. An element $v \in \mathcal{V}$ is called a *vertex* of $\mathcal{G}$ and an element $e \in \mathcal{E}$ is called an *(unoriented) edge.* Two vertices v, w are said to be *adjacent* if they belong to the same edge. An orientation of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a family of maps $\{f_e : \{0,1\} \to e\}_{e \in \mathcal{E}}$. Set $s(e) := f_e(0)$ and $r(e) := f_e(1)$. We call $s(e)$ (resp. $r(e)$) the *source* of e (resp. the *range* of e). An ordered pair $(s(e), r(e)) \in \mathcal{V} \times \mathcal{V}$ is called an oriented edge and we will denote by $\mathcal{E}^+$ the set of oriented edges with respect to a given orientation.

Let $e = (v, w)$ be an oriented edge. Then $\bar{e}$ denotes the ordered pair $(w, v) \in \mathcal{V} \times \mathcal{V}$ and is called the oriented edge wtih *reverse orientation.* We denote the set of orientation reversed edges by $\mathcal{E}^-$ and set $\mathcal{E}^\pm := \mathcal{E}^+ \cup \mathcal{E}^- = \{(v, w) \in \mathcal{V} \times \mathcal{V} \mid \{v, w\} \in \mathcal{E}\}$.

A finite sequence of vertices $(v_0, \cdots, v_n)$ is called a *path* in the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if $\{v_i, v_{i+1}\} \in \mathcal{E}$ for $i = 0, \cdots n-1$. A *chain* is a path $(v_0, \cdots, v_n)$ such that $v_i \neq v_{i+2}$ for $i = 0, \cdots, n-2$, that is, a chain is a path without backtracking. We denote a chain $(v_0, \cdots, v_n)$ by $v_0 \cdots v_n$. A chain $v_0 \cdots v_n$ with $v_n = v_0$ is called a *circuit.* A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is said to be *connected* if, for any two vertices $v, w \in \mathcal{V}$, there exists a path $(v_0, \cdots, v_n)$ such that $v_0 = v$ and $v_n = w$.

Definition 1.1.1. A *tree* is a connected graph without circuits.

We will use the letter $\mathcal{T}$ to denote a tree instead of $\mathcal{G}$. Since a tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ has no circuit, for any given two vertices v and w, there exists a unique chain $v_0 \cdots v_n$ such that $v_0 = v$ and $v_n = w$. This chain is called the *geodesic* joining v to w and denoted by $[v, w]$. If a geodesic $[v, w]$ is of the form $v_0 \cdots v_n$, then the number n is called the *distance* between v and w or the *length* of the geodesic. The distance between v and w is denoted by $d(v, w)$, and the length of the geodesic $[v, w]$ is denoted by $\ell([v, w])$. We extend the notion of distance to the case of $v = w$ by setting $d(v, v) := 0$. Then the map $d : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$; $(v, w) \mapsto d(v, w)$ defines a natural metic on $\mathcal{V}$.

For any vertex v, the cardinality of the set of edges which contain v is called the *degree* of v and denoted by $\deg(v)$. A tree is said to be *locally finite* if the degree of any vertex is finite. If every vertex has the same degree, a tree is said to be *homogeneous*, and the common degree of all vertices of the homogeneous tree is called the *degree of the tree.* For a locally finite homogeneous tree, the degree of the tree is generally denoted by $q + 1$.

Let q and ℓ be positive integers with $\ell \neq q$. A tree is said to be *semi-homogeneous* of degree (q, ℓ) if every vertex has degree $q + 1$ or $\ell + 1$ and two adjacent vertices have different degrees.

Example 1.1.1. (The tree of $\mathrm{PSL}_2(\mathbb{Z})$) Let $G := \mathrm{PSL}_2(\mathbb{Z})$ be the modular group (see Example 3.1.1). Let $\mathcal{V}$ be the union of two orbits $G(i)$ and $G(\rho)$, where $i = \sqrt{-1}$ and $\rho = e^{\pi i/3}$. Define

$\mathcal{E} := \{g(i), g(\rho)\}_{g \in G}$ and $\mathcal{T} := (\mathcal{V}, \mathcal{E})$. We call $\mathcal{T}$ *the tree of* $\mathrm{PSL}_2(\mathbb{Z})$. This is a semi-homogeneous tree of degree $(2, 1)$.

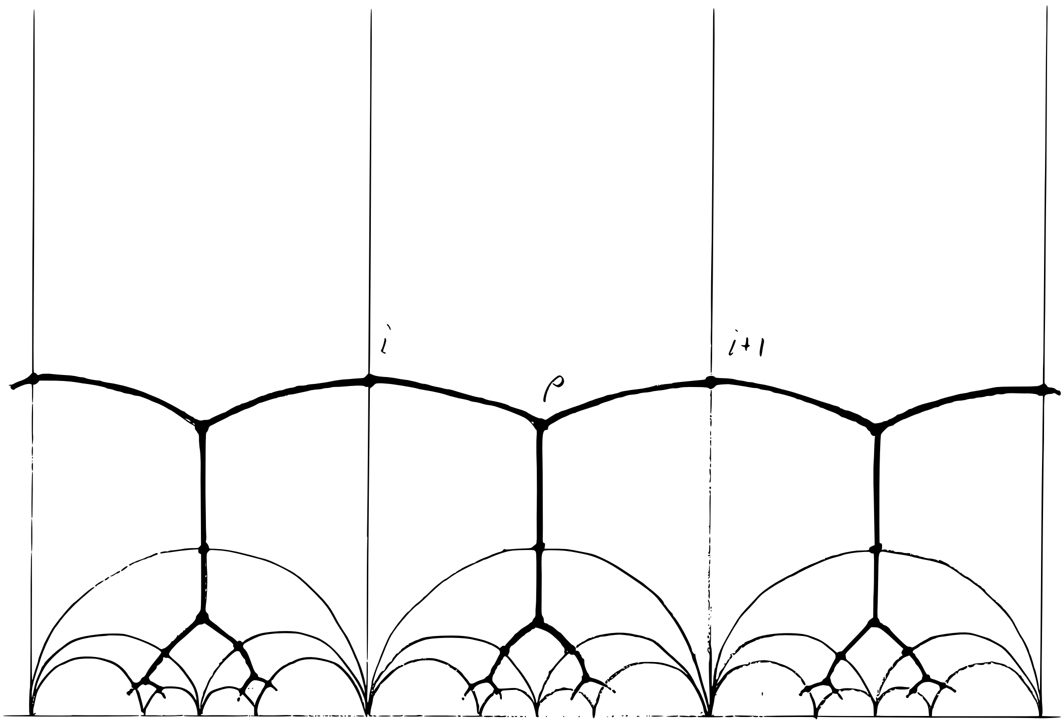

Figure 1.1: The tree of $\mathrm{PSL}_2(\mathbb{Z})$

We use the following convention and regard locally finite homogeneous trees as semi-homogeneous trees.

Definition 1.1.2. By a *semi-homogeneous tree of degree* (q, q) we mean a locally finite homogeneous tree of degree $q + 1$.

In this paper we will treat mainly semi-homogeneous trees. Note that the cardinality of the set of vertices of semi-homogeneous tree is always infinite. Such a tree is called an *infinite tree*, and we can define its "boundary", which is the main subject of this paper.

To treat the boundary of trees, whose definition is given in the next subsection, it is useful to fix a special vertex.

Definition 1.1.3. A *rooted tree* is a pair $(\mathcal{T}, 0)$, where $\mathcal{T}$ is a tree and 0 is a vertex of $\mathcal{T}$. The specified vertex 0 is called the *root*.

By abuse of notation, we sometimes write $\mathcal{T}$ instead of $(\mathcal{T}, 0)$. Let v and w be two vertices of a rooted tree $\mathcal{T}$. If (as sets of vertices) the geodesic $[0, w]$ contains $[0, v]$, then w is called a *descendant* of v, and v is called an *ancestor* of w (written as $v \succeq w$). If, in addition, v, w are adjacent, then w is called a *child* of v, and v is called the *father* of w. The binary relation $\succeq$ defines a partial order on the vertices of $\mathcal{T}$ and this partial order defines an orientation of $\mathcal{T}$: For an edge $e = \{v, w\} \in \mathcal{E}$ with $v \succeq w$, we define $s(e) := v$ and $r(e) := w$. We call this orientation the *natural orientation of a rooted tree.*

1.1.2 The Boundary of Trees

By an *infinite chain* we mean an infinite sequence $(v_0, v_1, \cdots)$ of vertices such that $\{v_i, v_{i+1}\}$ is an edge and $v_i \neq v_{i+2}$ for every $i \geq 0$. As in the case of finite chains, we denote an infinite chain

$(v_0, v_1, \cdots)$ by $v_0v_1\cdots$. We define an equivalence relation on the set of all infinite chains as follows. Let $v_0v_1\cdots$ and $w_0w_1\cdots$ be two infinite chains. Then they are said to be equivalent if (as sets of vertices) they have an infinite intersection. Equivalently, $v_0v_1\cdots$ and $w_0w_1\cdots$ are equivalent if there exists an integer $n \in \mathbb{Z}$ such that $v_k = w_{k+n}$ for sufficiently large k. We say an infinite chain $v_0v_1\cdots$ passes through a vertex v if the infinite chain contains the vertex v as sets of vertices (i.e. $v \in \{v_0, v_1, \cdots\}$).

Definition 1.1.4. The *boundary* of a tree $\mathcal{T}$ is the set of equivalence classes of infinite chains. We denote the boundary of $\mathcal{T}$ by $\partial\mathcal{T}$.

By definition, if the set of vertices of a tree is finite, then $\partial\mathcal{T}$ is empty. Therefore, in what follows, we will only consider infinite trees.

Note that if we fix a vertex $0 \in \mathcal{V}$ of a tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$, then the set of all infinite chains starting at the root 0 can be identified with $\partial\mathcal{T}$. This interpretation of the boundary of trees is fairly useful to investigate the property of boundaries. It is said that a point $x \in \partial\mathcal{T}$ *passes through* a vertex v if the infinite chain starting at the root which gives the point x passes through v.

We now define a topology of the boundary of trees. Fix a vertex $0 \in \mathcal{V}$ of a tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ and regard the tree as the rooted tree $(\mathcal{T}, 0)$. For every vertex v of the tree, let $V(v) \subset \partial\mathcal{T}$ denote the set of all infinite chains starting at the root 0 and passing through v. Then the family $\{V(v)\}_{v\in\mathcal{V}}$ becomes a basis of open sets for a topology of $\partial\mathcal{T}$.

Proposition 1.1.1. *Let $\mathcal{T} = (\mathcal{T}, 0)$ be a rooted tree and $\mathcal{T}' = (\mathcal{T}, 0')$ a rooted tree with a different root $0'$. Then there is a natural homeomorphism between $\partial\mathcal{T}$ and $\partial\mathcal{T}'$.*

Let $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ be a tree and choose a vertex $0 \in V$. As we already mentioned, we can identify the boundary of the non-rooted tree $\mathcal{T}$ and the boundary of the rooted tree $(\mathcal{T}, 0)$. Using this identification, we define a topology of the boundary $\partial\mathcal{T}$ of the non-rooted tree $\mathcal{T}$. This topology is independent of the choice of root by the previous proposition. For an oriented edge $e = (v, w) \in \mathcal{E}^{\pm}$, $V(e)$ denotes the set of all infinite chains starting at v and containing w. Note that the set $V(e)$ is clopen.

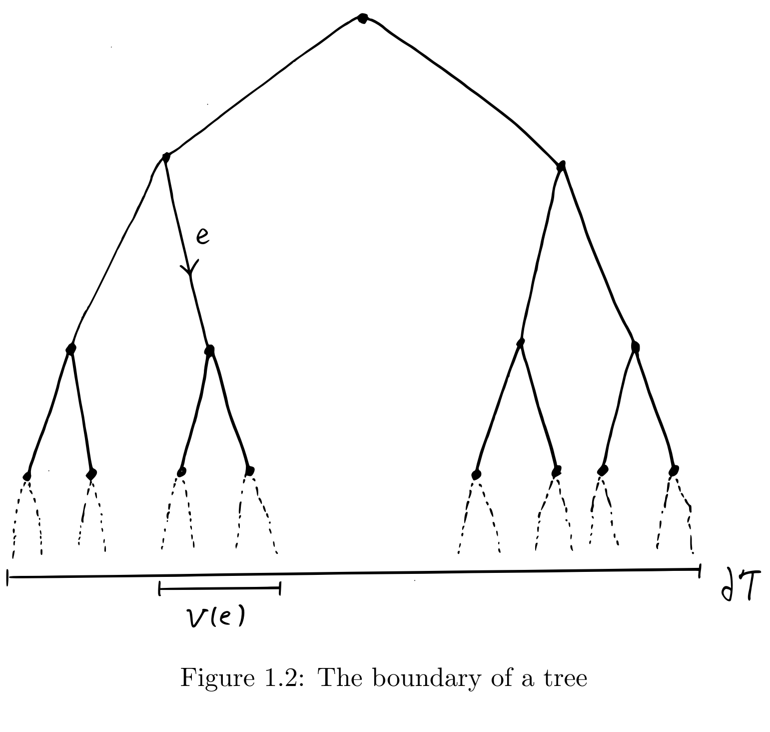

Figure 1.2: The boundary of a tree

Proposition 1.1.2. *Let $\mathcal{T}$ be a rooted tree and $\partial\mathcal{T}$ an its boundary endowed with the topology described above.*

(i) *$\partial\mathcal{T}$ is a totally disconnected Hausdorff space.*

(ii) *If $\mathcal{T}$ is a semi-homogeneous rooted tree of degree (q, ℓ), where q and ℓ are positive integers. Then the boundary $\partial\mathcal{T}$ of the tree is compact. Moreover, if q or $\ell \geq 2$, then $\partial\mathcal{T}$ has no isolated points, and consequently it is homeomorphic to a Cantor set.*

Proposition 1.1.3. *Let $\mathcal{T}$ be a semi-homogeneous rooted tree.*

(i) *If $V(e)$ and $V(e')$ are two clopen subsets associated with $e, e' \in \mathcal{E}^+$, then either $V(e) \cap V(e') = \emptyset$ or one of them is contained in another.*

(ii) *$V(e) \subset V(e')$ if and only if $r(e) \preceq r(e')$.*

(iii) *Every clopen subset V of $\partial\mathcal{T}$ can be written in the form $\bigsqcup_{i=1}^{n} V(e_i)$ (disjoint union).*

1.1.3 Group Actions on Trees

By an *automorphism of a tree* we mean a bijective map of the set of vertices onto itself which preserves the edges. An automorphism is also an isometry of $\mathcal{T}$ with respect to its natural metric structure. Conversely, every isometry of a tree is also an automorphism. Let $\mathrm{Aut}(\mathcal{T})$ denote the group of all isomorphisms of a tree $\mathcal{T}$.

Proposition 1.1.4. *Every automorphism of a tree $\mathcal{T}$ defines a homeomorphism of the boundary $\partial\mathcal{T}$.*

By this proposition, we have a natural inclusion $\mathrm{Aut}(\mathcal{T}) \subset \mathrm{Homeo}(\partial\mathcal{T})$, where $\mathrm{Homeo}(\partial\mathcal{T})$ denotes the group of all homeomorphisms of $\partial\mathcal{T}$.

Let G be a discrete group. Since the above inclusion preserves the group structure of $\mathrm{Aut}(\mathcal{T})$, if G acts on a tree $\mathcal{T}$, then G also acts on the boundary $\partial\mathcal{T}$. Note that $\mathrm{Homeo}(\partial\mathcal{T})$ is bigger than $\mathrm{Aut}(\mathcal{T})$ in general, and hence there exists a group which does not act on $\mathcal{T}$ but acts on $\partial\mathcal{T}$.

1.2 Alexander-Spanier Cocycles and Group Cohomology

1.2.1 Alexander-Spanier Cocycles

We introduce the notion of Alexander-Spanier cocycles in this subsection. For a thorough treatment we refer the reader to [25].

Definition 1.2.1. Let X be a set and M an additive group. For any integer $q \geq 0$, we define the group of *Alexander-Spanier q-cochains* of X with coefficients in M as the set of functions from X^{q+1} to M:

$$C^q(X; M) := \{\varphi : X^{q+1} \to M\}.$$

We sometimes leave out the coefficient group M from the notation and simply write $C^q(X)$.

The *coboundary homomorphism* $\delta^q : C^q(X) \to C^{q+1}(X)$ is defined by

$$\delta^q\varphi(x_0, \cdots x_{q+1}) := \sum_{0 \leq i \leq q+1} (-1)^i \varphi(x_0, \cdots, \hat{x_i}, \cdots x_{q+1}).$$

Here $\hat{x_i}$ means omitting x_i.

Definition 1.2.2. For an integer $q \geq 0$, we set $Z^q(X; M) := Z^q_{AS}(X; M) := \ker \delta^q$, the group of *Alexander-Spanier q-cocycles* of X with coefficient in M.

The group structure of $Z^q_{AS}(X; M)$ is given by the point-wise addition:

$$(\varphi + \varphi')(x, y) := \varphi(x, y) + \varphi'(x, y).$$

For a ring R and R-module M, $Z^q_{AS}(X; R)$ (resp. $Z^q_{AS}(X; M)$) becomes a ring (resp. an R-module) as well.

Remark 1.2.1. *For any $q \geq 0$, one has $\delta^{q+1} \circ \delta^q = 0$. This shows that $C^\bullet(X;M) := (C^q(X;M), \delta^q)$ is a cochain complex over M, and we have a cohomology group $H^q(C^\bullet(X;M))$. Note that this cohomology group is not the Alexander-Spanier cohomology group and does not contain any interesting information about X. In fact, if X is a nonempty space, then $H^0(X;M) \cong M$ and $H^q(X;M) = 0$ for $q \geq 1$.*

By definition, we have

$$Z^1_{AS}(X;M) = \{\varphi : X \times X \to M \mid \varphi(x,y) + \varphi(y,z) - \varphi(x,z) = 0 \ for \ all \ x,y,z \in X\}$$

In particular, the 1-cocycle condition implies

$$\varphi(x,x) = 0, \ \varphi(y,x) = -\varphi(x,y),$$

for all $x, y \in X$.

Let G be a group acting on a set X from the left and M a G-module. We define a right action of G on $Z^q_{AS}(X;M)$ by

$$(\varphi g)(x_0, \cdots, x_q) := \varphi(gx_0, \cdots, gx_q).$$

An Alexander-Spanier q-cocycle $\varphi \in Z^q_{AS}(X;M)$ is said to be *G-equivariant* if it satisfies

$$(\varphi g)(x_0, \cdots, x_q) = g \cdot \varphi(x_0, \cdots, x_q)$$

for all $g \in G$ and $(x_0, \cdots, x_q) \in X^{q+1}$. Let us denote by $Z^q_{AS,G}(X;M)$ the group of *G-equivariant Alexander-Spanier cocycles.*

1.2.2 Group Cohomology

For more detailed treatment about group cohomology, we refer the reader to [1] and [12].

Definition 1.2.3. Let G be a group and M a left G-module. For any integer $q \geq 0$, define the *q-cochains* of G with coefficients in M as the set of functions from G^q to M:

$$C^q(G;M) := \{\psi : G^q \to M\}.$$

The qth *differential* $d^q : C^q(G;M) \to C^{q+1}(G;M)$ is defined by

$$\begin{aligned} d^q(\psi)(g_0, \cdots, g_q) \quad &:= \quad g_0 \cdot \psi(g_1, \cdots, g_q) \\ &+ \quad \sum_{i=1}^{q} (-1)^i \psi(g_0, \cdots, g_{i-2}, g_{i-1}g_i, g_{i+1}, \cdots, g_q) + (-1)^{q+1}\psi(g_0, \cdots, g_{q-1}). \end{aligned}$$

Note that $C^0(G;M)$ is taken to be M, as G^0 is a singleton set. It can be shown that $d^{q+1} \circ d^q = 0$ for $q \geq 0$. Thus $C^\bullet(G;M)$ is a cochain complex.

Definition 1.2.4. Let $q \geq 0$.

(i) Set $Z^q(G;M) := \ker d^q$, the group of *q-cocycles* of G with coefficients in M.

(ii) Set $B^0(G;M) := 0$ and $B^q(G;M) := \mathrm{im} d^{q-1}$ for $q \geq 1$, the group of *q-coboundaries* of G with coefficients in M.

(iii) Define $H^q(G;M) := Z^q(G;M)/B^q(G;M)$, the *qth cohomology group* of G with coefficients in M.

The cohomology group of a group with coefficients in a right module is defined similarly. We give some examples of cohomology groups of low degree.

Lemma 1.2.1. *Let M be a G-module.*

(i) $H^0(G;M) = M^G := \{m \in M \mid gm = m\}$*, the group of G-invariants of M.*

(ii) $Z^1(G;M) = \{\psi : G \to M \mid \psi(g_1g_2) = g_1\psi(g_2) + \psi(g_1) \ for \ all \ g_1, g_2 \in G\}$.

(iii) $B^1(G;M) = \{\psi : G \to M \mid \textit{ there exists an } m \in M \textit{ such that } \psi(g) = gm - m \textit{ for all } g \in G\}$.

Let G be a group and M a left G-module. We can also define the *homology group* of G with coefficients in M. Here we give only the definition of 0th homology groups.

Definition 1.2.5. Let G be a group and M a G-module. Define the 0*th homology group* as the quotient

$$H_0(G;M) := M_G := M/\langle m - gm \mid m \in M, g \in G\rangle,$$

where $\langle m - gm \mid m \in M, g \in G\rangle$ denotes the submodule of M generated by the elements of the form $m - gm$ for $m \in M$ and $g \in G$. The quotient module M_G is usually called the group of G-*coinvariants* of M.

Example 1.2.1. Let G be a group acting on a set X from the left. For given $x \in X$ and $\varphi \in Z^1_{AS,G}(X;M)$, define

$$\psi^\varphi_x(g) := \varphi(gx, x).$$

Then the cocycle condition and the G-equivariance of Alexander-Spanier cocycles imply

$$\psi^\varphi_x(gh) = \varphi(ghx, x) = \varphi(ghx, gx) + \varphi(gx, x) = g\varphi(hx, x) + \varphi(gx, x) = g\psi^\varphi_x(h) + \psi^\varphi_x(g).$$

Thus the map

$$\psi^\varphi_x : G \to M \ ; \ g \mapsto \psi^\varphi_x(g)$$

defines a 1-cocycle of G with coefficients in M. For another $y \in X$, we have

$$\begin{aligned} \psi^\varphi_x(g) - \psi^\varphi_y(g) &= \varphi(gx, x) - \varphi(gy, y) \\ &= \varphi(gx, x) + \varphi(x, gy) - \varphi(x, gy) - \varphi(gy, y) \ = \ g\varphi(x, y) - \varphi(x, y), \end{aligned}$$

that is, $\psi^\varphi_x - \psi^\varphi_y$ is a coboundary. Hence we obtain a well-defined map

$$Z^1_{AS,G}(X;M) \to H^1(G;M) \ ; \ \varphi \mapsto \psi^\varphi := [\psi^\varphi_x].$$

Since the group structure of $Z^1_{AS,G}(X;M)$ and that of $H^1(G;M)$ are given by point-wise addition, the map is a group homomorphism.

1.2.3 Hecke Operators on Group Cohomology

The notion of Hecke operators for group cohomology is defined by Y.H.Rhie and G.Whaples [21].

Let G be a group. Two subgroups Γ, Γ' of G are said to be *commensurable* if

$$[\Gamma : \Gamma \cap \Gamma'] < \infty, \quad [\Gamma' : \Gamma \cap \Gamma'] < \infty.$$

We write $\Gamma \approx \Gamma'$ when Γ and Γ' are commensurable. Note that $\approx$ defines an equivalence relation. For a subgroup Γ of G, we set $C_G(\Gamma) := \{g \in G \mid g^{-1}\Gamma g \approx \Gamma\}$, the *commensurator* of Γ in G.

Let G be a group and $\Gamma \subset G$ a subgroup. Suppose $C_G(\Gamma) \neq \emptyset$. Set $\Gamma_{g^{-1}} := \Gamma \cap g^{-1}\Gamma g$ for $g \in C_G(\Gamma)$. Let $\{\delta_1, \cdots, \delta_d\}$ be a complete set of representatives of $\Gamma/\Gamma_{g^{-1}}$. For a left (resp. right) G-module M, we define the *Hecke operator* on the 0th homology group (resp. cohomology group) by

$$T_g : H_0(\Gamma;M) = M_\Gamma \to M_\Gamma = H_0(\Gamma;M) \ ; \ [m]_\Gamma \mapsto \sum_{i=1}^d [g\delta_i^{-1}m]_\Gamma$$

and

$$T_g : H^0(\Gamma;M) = M^\Gamma \to M^\Gamma = H^0(\Gamma;M) \ ; \ m \mapsto \sum_{i=1}^d m\delta_i g^{-1},$$

where $[m]_\Gamma$ denotes the equivalence class of m in M_Γ.

1.3 C^*-Algebras and K-Homology

In this section, we present basic notions and theorems about C^*-algebras and their K-theory. For more details we refer the reader to [3], [9], and [22].

1.3.1 Gelfand-Naimark Duality

Definition 1.3.1. A $*$-*algebra* is a $\mathbb{C}$-algebra equipped with an involution $a \mapsto a^*$ satisfying $(ab)^* = b^*a^*$, $(\lambda a)^* = \bar{\lambda}a^*$ for $a, b \in A$ and $\lambda \in \mathbb{C}$. A norm $\|\cdot\|$ on a $*$-algebra A is said to be *multiplicative* if it satisfies $\|ab\| \leq \|a\|\|b\|$ for $a, b \in A$. A C^*-*algebra* is a $*$-algebra equipped with a complete multiplicative norm $\|\cdot\|$ satisfying $\|a^*a\| = \|a\|^2$ for $a \in A$. We call this condition the C^*-identity. If a C^*-algebra A has a unit 1_A, we call A a *unital* C^*-algebra.

Example 1.3.1. Let H be a Hilbert space. Define an involution on the set of all bounded linear operators $\mathcal{B}(H)$ on H by taking the adjoint operator $T \mapsto T^*$ for $T \in \mathcal{B}(H)$, and endow $\mathcal{B}(H)$ with the norm

$$\|T\| := \sup_{\|v\| \leq 1} \|Tv\|,$$

where $T \in \mathcal{B}(H)$ and $v \in H$. Then $\mathcal{B}(H)$ is a C^*-algebra.

Example 1.3.2. For a compact Hausdorff space X, $C(X)$ denotes the algebra of all continuous $\mathbb{C}$-valued functions on X. We define an involution by $f^*(x) := \overline{f(x)}$ for $x \in X$, and set a norm by

$$\|f\| := \sup_{x \in X} |f(x)|.$$

Then $C(X)$ is a unital commutative C^*-algebra.

Next, we define morphisms in the category of C^*-algebras.

Definition 1.3.2. Let A and B be C^*-algebras. A map $\phi : A \to B$ is called a $*$-homomorphism if it is a $\mathbb{C}$-algebra homomorphism and preserves the involutions. In addition, if A and B are unital and $\phi(1_A) = 1_B$, we call ϕ *unital*. An isometry is a map $\phi : A \to B$ such that $\|\phi(a)\| = \|a\|$ for all $a \in A$.

Note that an injective $*$-homomorphism between C^*-algebras is always an isometry. Thus, in the category of C^*-algebras, a morphism means a $*$-homomorphism.

For any C^*-algebra A, let us denote by A^* the dual space of A as a Banach space. We can endow A^* with the weak $*$-topology; the weakest topology such that $f_a : A^* \to \mathbb{C}$; $\phi \mapsto \phi(a)$ becomes continuous for all $a \in A$.

Definition 1.3.3. For a commutative C^*-algebra A, we call

$$\hat{A} := \{\phi : A \to \mathbb{C} : \text{continuous multiplicative linear map} \mid \phi \neq 0\}$$

the *dual* of A.

$\hat{A}$ is a closed set contained in the closed unit ball $(A^*)_1$ of A^*. On the other hand, since the definition of weak $*$-topology and the Alaoglu's theorem imply that $(A^*)_1$ is a relatively compact Hausdorff space, we have the following lemma:

Lemma 1.3.1. *Let A be a unital commutative C^*-algebra. Then the dual $\hat{A}$ of A is a compact Hausdorff space.*

For each $a \in A$, define a function $\hat{a} : \hat{A} \to \mathbb{C}$ by $\hat{a}(\psi) := \psi(a)$. Note that $\hat{a}$ is continuous by the definition of the weak $*$-topology.

Definition 1.3.4. Let A be a unital commutative C^*-algebra. Then the map

$$A \to C(\hat{A}) \ ; \ a \mapsto \hat{a}$$

is called the *Gelfand transform*.

The Gelfand transform is in fact a $*$-homomorphism. Moreover, using the C^*-identity, one can show that the Gelfand transform is injective, and the Stone-Weierstrass theorem implies the surjectivity. Thus we have the following theorem:

Theorem 1.3.1. (Gelfand-Naimark)
The Gelfand transform $A \to C(\hat{A})$ is an isometric $*$-isomorphism.

The above theorem gives a contravariant categorical equivalence between the category of unital commutative C^*-algebras and the category of compact Hausdorff spaces. We refer this duality as *Gelfand-Naimark duality.*

1.3.2 K-Theory and K-Homology for C^*-Algebras

Definition 1.3.5. Let A be a unital C^*-algebra. An element $p \in A$ is called a *projection* if it satisfies $p = p^2 = p^*$. A *unitary* is an element $u \in A$ such that $u^*u = uu^* = 1$. Let us denote by $\mathcal{P}(A)$ (resp. $\mathcal{U}(A)$) the set of all projections (resp. unitaries) in A.

Let a and b be two elements in a C^*-algebra A. If there exists a continuous map $[0,1] \to A$ such that $f(0) = a,\ f(1) = b$, then a and b are said to be *homotopic* (written $a \sim_h b$). In particular, for a subset $J \subset A$, if there exists a continuous map $[0,1] \to J$ such that $f(0) = a,\ f(1) = b$, then a and b are said to be homotopic in J. Note that $\sim_h$ defines an equivalence relation.

Definition 1.3.6. Let A be a unital C^*-algebra. Define $K_0(A)$ as the abelian group generated by the symbols $[p]$ each of which corresponds to a projection p in each matrix algebra $M_n(A)$ $(n = 1, 2, \cdots)$, and satisfying the following relations:

(i) If p, q are projections in $M_n(A)$ for some n, and $p \sim_h q$ in $\mathcal{P}(M_n(A))$, then $[p] = [q]$,

(ii) $[0] = 0$ for any size of zero matrix,

(iii) $[p] + [q] = [p \oplus q]$ for any size of projection matrices p and q.

Here, for two projections $p \in M_n(A)$ and $q \in M_m(A)$, $p \oplus q$ denotes the projection $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ in $M_{n+m}(A)$.

Example 1.3.3. Let us consider $\mathbb{C}$ as a C^*-algebra. Then the map

$$K_0(\mathbb{C}) \to \mathbb{Z}\ ;\ [p] - [q] \mapsto \mathrm{rank}(p) - \mathrm{rank}(q)$$

is an isomorphism.

Each unital $*$-homomorphism $\phi : A \to B$ between unital C^*-algebras induces a homomorphism between $K_0(A)$ and $K_0(B)$:

$$\phi_* := K_0(\phi) : K_0(A) \to K_0(B)\ ;\ [p] \mapsto [\phi(p)].$$

Thus K_0 is a functor from the category of unital C^*-algebras to the category of abelian groups.

There is another description of the elements in $K_0(A)$. Let E be a finitely generated projective right A-module. Then E is isomorphic to $A^n p$ for some n and projection $p \in M_n(A)$. Using such a projection, we can associate each isomorphism class of finitely generated projective A-module E to an element of $K_0(A)$. Let us denote by $[E]$ the isomorphism class of E, and we regard this class as an element of $K_0(A)$:

$$[E] = [A^n p] = [p] \in K_0(A).$$

Then taking the direct sum of such modules corresponds to the addition in $K_0(A)$.

Next, we define the group $K_1(A)$.

Definition 1.3.7. Let A be a unital C^*-algebra. Define

$$K_1(A) := \pi_0(\bigcup_{n=1}^{\infty} \mathcal{U}_n(A)).$$

Here π_0 denotes the connected components. Since $\bigcup_{n=1}^{\infty} \mathcal{U}_n(A)$ is a group, so is $K_1(A)$.

Example 1.3.4. Since a unitary matrix in $M_n(\mathbb{C})$ is homotopic to the identity matrix in $\mathcal{U}_n(\mathbb{C})$, we have

$$K_1(\mathbb{C}) = 0.$$

Definition 1.3.8. Let A be a separable C^*-algebra. A *Fredholm module* over A is given by the following data:

(i) a separable Hilbert space H,

(ii) a $*$-homomorphism $\rho : A \to \mathcal{B}(H)$,

(iii) an operator $F : H \to H$

satisfying

$$(F^2 - 1)\rho(a),\ (F - F^*)\rho(a),\ F\rho(a) - \rho(a)F \in \mathcal{K}(H)$$

for all $a \in A$. Here $\mathcal{K}(H)$ denotes the set of all compact operators on H. A Fredholm module (ρ, H, F) is said to be *even* if there is an operator $\gamma : H \to H$ such that $\gamma^2 = 1, \gamma = \gamma^*, \gamma F + F\gamma = 0$ and $\gamma\rho(a) = \rho(a)\gamma$ for all $a \in A$. An *odd* Fredholm module is a Fredholm module which is not even.

If a Fredholm module is even, then we can write

$$H = H_+ \oplus H_-,\ \rho = \begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \end{pmatrix},\ F = \begin{pmatrix} 0 & F_- \\ F_+ & 0 \end{pmatrix}.$$

Let (ρ, H, F) and (ρ', H', F') be two Fredholm modules. Then they are said to be *unitarily equivalent* if there exist a unitary $U : H' \to H$ such that $U^*\rho U = \rho', U^*FU = F'$, and $U^*\gamma U = \gamma'$ if γ and γ' are grading operators of (ρ, H, F) and (ρ', H', F'), respectively. Two Fredholm modules (ρ, H, F) and (ρ, H, F') are said to be *operator homotopic* if there exists a norm continuous map $[0, 1] \to \mathcal{B}(H); t \mapsto F_t$ with $F_0 = F,\ F_1 = F'$ such that (ρ, H, F_t) is a Fredholm module for each $t \in [0, 1]$.

Definition 1.3.9. The *K-homology group* $K^0(A)$ (resp. $K^1(A)$) is defined as the abelian group generated by the symbols $[x]$ each of which corresponds to a unitary equivalence class of even (resp. odd) Fredholm modules, and satisfying the following relations:

(i) If x_0 and x_1 are operator homotopic even (resp. odd) Fredholm modules, then $[x_0] = [x_1]$ in $K^0(A)$ (resp. $K^1(A)$),

(ii) If x_0 and x_1 are two even (resp. odd) Fredholm modules, then $[x_0 \oplus x_1] = [x_0] + [x_1]$ in $K^0(A)$ (resp. $K^1(A)$).

Here, for two Fredholm modules $x_0 = (\rho_0, H_0, F_0)$ and $x_1 = (\rho_1, H_1, F_1)$, $x_0 \oplus x_1$ denotes the Fredholm module $(\rho_0 \oplus \rho_1, H_0 \oplus H_1, F_0 \oplus F_1)$.

We can define a bi-linear pairing between K-theory and K-homology. Note that if (ρ, H, F) is a Fredholm module over A, then $(\rho \otimes I_k, H^k, F \otimes I_k)$ is a Fredholm module over $M_k(A)$, where I_k denotes the identity matrix in $M_k(\mathbb{C})$. Then the pairing between $[p] \in K_0(A)$ and $[(\rho, H, F)] \in K^0(A)$ is given by the Fredholm index

$$\langle [p], [(\rho, H, F)] \rangle := \mathrm{Index}(p(F^+ \otimes I_k)p : pH^k \to pH^k),$$

where $p \in \mathcal{P}_k(A)$.

For $[u] \in K_1(A)$ and $[(\rho, H, F)] \in K^1(A)$, we define the pairing by

$$\langle [u], [(\rho, H, F)] \rangle := \mathrm{Index}(P_k u P_k : H^k \to H^k),$$

where $u \in \mathcal{U}_k(A)$ and $P_k := \frac{1}{2}(1 + F) \otimes I_k$.

Definition 1.3.10. For a separable C^*-algebra A, the pairing between $K_i(A)$ and $K^i(A)$ $(i = 0, 1)$ defined above is called the *index pairing.*

For $i = 0, 1$, the index pairing induces a group homomorphism

$$\mathrm{Index} : K^i(A) \to \mathrm{Hom}(K_i(A), \mathbb{Z})\ ;\ y \mapsto \langle \cdot, y \rangle,$$

called the *index map.*

To compute the K-homology, the following theorem is useful.

Theorem 1.3.2. (Universal Coefficient Theorem, Rosenberg-Schochet [24])

Let A be a separable C^*-algebra which belongs to the bootstrap category. Then there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}(K_i(A), \mathbb{Z}) \longrightarrow K^{i+1}(A) \xrightarrow{\mathrm{Index}} \mathrm{Hom}(K_{i+1}(A), \mathbb{Z}) \longrightarrow 0$$

for $i = 0, 1$, where $i + 1$ is considered as an element of $\mathbb{Z}/2\mathbb{Z}$.

1.3.3 Discrete Crossed Products

Let A be a C^*-algebra and G a discrete group. Then we can construct a new C^*-algebras $A \rtimes_r G$, called reduced crossed product C^*-algebra. In this subsection, we give the construction of this C^*-algebra and introduce an exact sequence of K-theory which allows one to compute the K-theory of crossed products by free groups (see [2] and [3] for more details).

Definition 1.3.11. A *(discrete) C^*-dynamical system* (A, G, α) consists of a C^*-algebra A together with a group homomorphism $\alpha : G \to \mathrm{Aut}(A)$.

For a C^*-dynamical system (A, G, α), we will denote an automorphism $\alpha(g)$ by α_g for $g \in G$. Define $C_c(G, A)$ as the set of all finitely supported functions on G with values in A. We write an element $S \in C_c(G, A)$ as a finite sum $S = \sum_{g\in G} a_g g$. We equip $C_c(G, A)$ with a product and an involution which make $C_c(G, A)$ a $*$-algebra. Let $S = \sum_{s\in G} a_s s$ and $T = \sum_{t\in G} b_t t$ be two elements in $C_c(G, A)$. Define

$$ST := \sum_{s,t\in G} a_s \alpha_s(b_t) st \quad \text{and} \quad S^* := \sum_{s\in G} \alpha_{s^{-1}}(a_s^*) s^{-1}.$$

We can construct a $*$-homomorphism $C_c(G, A) \to \mathcal{B}(H\otimes\ell^2(G))$ from an injective $*$-homomorphism $A \hookrightarrow \mathcal{B}(H)$ as follows. First, we fix an injecticve $*$-homomorphism $A \hookrightarrow \mathcal{B}(H)$ and regard A as a subalgebra of $\mathcal{B}(H)$. Define a new $*$-homomorphism $\pi : A \to \mathcal{B}(H \otimes \ell^2(G))$ by

$$\pi(a)(v \otimes \delta_g) := (\alpha_{g^{-1}}(a)(v)) \otimes \delta_g,$$

where $\{\delta_g\}$ denotes the canonical orthonormal basis of $\ell^2(G)$. Let $\lambda : G \to \ell^2(G)$ be the left regular representation. Then for $S = \sum_{s\in G} a_s s \in C_c(G, A)$, set

$$\tilde{\pi}(S) := \sum_{s\in G} \pi(a_s)(id_H \otimes \lambda_s).$$

This defines a $*$-homomorphism $\tilde{\pi} : C_c(G, A) \to \mathcal{B}(H \otimes \ell^2(G))$, called a *regular representation.* It can be seen that a regular representation is injective, and so we can consider $C_c(G, A)$ as a subalgebra of $\mathcal{B}(H \otimes \ell^2(G))$.

Definition 1.3.12. The norm closure of $C_c(G, A)$ in $\mathcal{B}(H \otimes \ell^2(G))$ is called the *reduced crossed product* of (A, G, α) and denoted by $A \rtimes_{\alpha,r} G$. If it is clear which $*$-homomorphism α has been chosen, we leave out the α from the notation and simply call it the reduced crossed product of A by G.

Note that the reduced crossed product $A \rtimes_{\alpha,r} G$ does not depend on the choice of the injective $*$-homomorphism $A \hookrightarrow \mathcal{B}(H)$.

Generally, computing the K-theory of a crossed product $A \rtimes_r G$ is difficult. However, the following theorem shows that there are some cases one can gain some information about it from the K-theory of A.

Theorem 1.3.3. (Pimsner-Voiculescu [19])

Let A be a C^*-algebra and Γ a free group with n generators $\gamma_1, \cdots, \gamma_n$. Then the following sequence is exact:

$$\begin{array}{ccccc} \bigoplus_{i=1}^n K_0(A) & \xrightarrow{\varrho} & K_0(A) & \longrightarrow & K_0(A \rtimes_r \Gamma) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_r \Gamma) & \longleftarrow & K_1(A) & \underset{\varrho}{\longleftarrow} & \bigoplus_{i=1}^n K_1(A), \end{array}$$

where $\varrho := \sum_{i=1}^n (1 - \gamma_{i*})$, and the morphism $K_0(A) \to K_0(A \rtimes_r \Gamma)$ is induced by the natural inclusion $A \hookrightarrow A \rtimes_r \Gamma$.

Chapter 2

Alexander-Spanier Cocycles on the Space of Rational Ends

In this chapter, we study measures on the boundary of trees. After reviewing some of the standard facts on measures, we introduce the notion of *the space of rational ends* and relate Alexander-Spanier cocycles on the space with measures. Our construction is motivated by the work of Manin and Marcolli [16], where they studied the tree of $\mathrm{PSL}_2(\mathbb{Z})$ and related measures on the boundary of the tree with modular symbols for $\mathrm{PSL}_2(\mathbb{Z})$. We generalize their results to semi-homogeneous trees by replacing modular symbols for $\mathrm{PSL}_2(\mathbb{Z})$ with Alexander-Spanier cocycles on the space of rational ends.

2.1 Measures on the Boundary

In this section, we introduce the notion of measures on totally disconnected spaces and mention its relation to the notion of currents on a tree (see for instance [29]).

Definition 2.1.1. Let X be a totally disconnected compact Hausdorff space and M an additive group. Define an M-valued *measure* on X as a function μ from the set of all clopen subsets of X to M satisfying

(M1) $\mu(V \cup V') = \mu(V) + \mu(V')$ for clopen subsets $V, V' \subset X$ with $V \cap V' = \emptyset$,

(M2) $\mu(X) = 0$.

We denote by $\mathrm{Meas}(X, M)$ the set of all M-valued measures on X.

We define a group structure on $\mathrm{Meas}(X, M)$ by

$$(\mu + \mu')(V) := \mu(V) + \mu(V').$$

Note that an M-valued measure is not a measure in the usual sense, since it can take negative values. But it is usually called a measure in the literature, so we use the term "measure". An M-valued measure is also called an M-valued *distribution.*

Let $C(X, M) := C(X, \mathbb{Z}) \otimes_{\mathbb{Z}} M$ denote the set of all locally constant M-valued functions on X. Then an M-valued measure on X can be interpreted as a functional on $C(X, M)$.

Definition 2.1.2. Let M be an additive group. Let us denote by $\mathrm{Hom}^0(C(X, M), M)$ the set of all M-module homomorphism ϕ from $C(X, M)$ to M such that $\phi(1) = 0$.

Let $\phi \in \mathrm{Hom}^0(C(X, M), M)$. Then for any clopen subset $V \subset X$, we set

$$\mu_\phi(V) := \phi(\chi_V),$$

where χ_V denotes the characteristic function on V. It is easy to check that μ_ϕ is an M-valued measure. The following proposition can be considered as the Riesz representation theorem for totally disconnected compact Hausdorff spaces.

Proposition 2.1.1. *Let X be a totally disconnected compact Hausdorff space and M an additive group. Then the map*

$$\mathrm{Hom}^0(C(X,M),M) \xrightarrow{\sim} \mathrm{Meas}(X,M) \ ; \ \phi \mapsto \mu_\phi$$

is an isomorphism.

Proof. Note that every continuous function $f \in C(X,M)$ is of the form $f = \sum_{i=1}^n a_i \chi_{V_i}$. We define

$$\phi_\mu(f) := \int f d\mu := a_i \mu(\chi_{V_i}).$$

It is easily seen that ϕ_μ is well-defined. Since $\mu(X) = 0$, $\phi_\mu \in \mathrm{Hom}^0(C(X,M),M)$. Obviously, $\mu_{\phi_\mu} = \mu$ and $\phi_{\mu_\phi} = \phi$. □

Let G be a group acting on a totally disconnected compact Hausdorff space X from the left, and let M be a left G-module. Then we define a right action of G on $\mathrm{Meas}(X,M)$ and $\mathrm{Hom}^0(C(X,M),M)$ by

$$(\mu g)(V) := \mu(gV), \quad (\phi g)(f) := \phi(gf),$$

where $V \subset X$ is a clopen subset and $f \in C(X,M)$. Note that the action of G on $C(X,M)$ is given by $(gf)(x) := f(g^{-1}x)$. We have the following *change of variable formula*:

Lemma 2.1.1. *Let $\mu \in \mathrm{Meas}(X,M)$. Then for any $g \in G$, we have*

$$\phi_{\mu g}(f) = \int f d(\mu g) = \int g f d\mu = \phi_\mu(gf) = (\phi_\mu g)(f).$$

Definition 2.1.3. Let G be a group acting on a totally disconnected compact Hausdorff space X and M a G-module.

(i) We set $\mathrm{Meas}_G(X,M) := \{\mu \in \mathrm{Meas}(X,M) \mid \mu(gV) = g(\mu(V))\}$, the set of M-valued G-*equivariant measures.*

(ii) We set $\mathrm{Hom}^0_G(C(X,M),M) := \{\phi \in \mathrm{Hom}^0(C(X,M),M) \mid \phi(gf) = g(\phi(f))\}$, the set of G-equivariant M-module homomorphisms from $C(X,M)$ to M with $\phi(1) = 0$.

The following proposition follows immediately from Proposition 2.1.1.

Proposition 2.1.2. *Let G be a group acting on a totally disconnected compact Hausdorff space X and M a G-module. Then the map in Proposition 2.1.1 induces an isomorphism*

$$\mathrm{Hom}^0_G(C(X,M),M) \xrightarrow{\sim} \mathrm{Meas}_G(X,M) \ ; \ \phi \mapsto \mu_\phi.$$

Proof. By Lemma 2.1.1, we have

$$\mu_\phi(gV) = \phi(\chi_{gV}) = \phi(g\chi_V)$$

for any clopen set $V \subset X$ and $g \in G$. Hence μ_ϕ is a G-equivariant measure if and only if ϕ is a G-module homomorphism, and the proposition follows. □

In the case of $X = \partial\mathcal{T}$ for some tree $\mathcal{T}$, we can interpret M-valued measures on $\partial\mathcal{T}$ as M-valued currents on $\mathcal{T}$, which are defined as follows.

Definition 2.1.4. Let $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ be a tree and M an additive group. Fix an orientation of $\mathcal{T}$. Then a *current* on $\mathcal{T}$ is a function $c : \mathcal{E}^\pm \to M$ satisfying

(C1) $c(e) = -c(\bar{e})$ for $e \in \mathcal{E}^\pm$,

(C2) $\sum_{s(e)=v} c(e) = 0$ for $v \in \mathcal{V}$.

Let $\mathcal{C}(\mathcal{T}, M)$ denote the set of all M-valued currents on $\mathcal{T}$.

Currents are also called *harmonic cocycles*, and the condition (C2) is referred as the *harmonicity condition*.

Let $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ be a tree and M an additive group. For an M-valued measure μ on $\partial\mathcal{T}$, we define a current on $\mathcal{T}$ by

$$c_\mu(e) := \mu(V(e)),$$

where $e \in \mathcal{E}^\pm$.

Proposition 2.1.3. *Let $\mathcal{T}$ be a tree and M an additive group. Then the map*

$$\mathrm{Meas}(\partial\mathcal{T}, M) \xrightarrow{\sim} \mathcal{C}(\mathcal{T}, M) \ ; \ \mu \mapsto c_\mu$$

is an isomorphism.

Proof. Let c be a current on $\mathcal{T}$. For a clopen subset $V(e)$ ($e \in \mathcal{E}^\pm$), define

$$\mu_c(V(e)) := c(e).$$

Since the family $\{V(e)\}_{e\in\mathcal{E}^\pm}$ forms a basis of clopen sets, μ_c defines a measure on $\partial\mathcal{T}$. It is easy to check that the mappings

$$\mu \mapsto c_\mu \text{ and } c \mapsto \mu_c$$

are inverses of each other, which completes the proof. □

Let G be a group acting on a tree $\mathcal{T}$ and M a G-module. A current c on $\mathcal{T}$ is said to be *G-equivariant* if $c(ge) = g \cdot c(e)$ for $e \in \mathcal{E}^\pm$ and $g \in G$. Let us denote by $\mathcal{C}_G(\mathcal{T}, M)$ the set of all G-equivariant currents on $\mathcal{T}$.

Proposition 2.1.4. *Let G be a group acting on a tree $\mathcal{T}$ and M a G-module. Then the map in Proposition 2.1.3 induces an isomorphism*

$$\mathrm{Meas}_G(\partial\mathcal{T}, M) \xrightarrow{\sim} \mathcal{C}_G(\mathcal{T}, M) \ ; \ \mu \mapsto c_\mu.$$

Proof. Let μ be a measure on $\partial\mathcal{T}$. Since $\mu(gV(e)) = \mu(V(ge)) = c_\mu(ge)$ for any $e \in \mathcal{E}^\pm$ and $g \in G$, μ is G-equivariant if and only if c_μ is G-equivariant, and the proposition follows. □

2.2 The Space of Rational Ends

Let $q \geq \ell \geq 1$ be two integers. In the remainder of this chapter, unless otherwise stated, we assume that a tree $\mathcal{T} = (\mathcal{T}, v_0) = ((\mathcal{V}, \mathcal{E}), v_0)$ is a naturally oriented semi-homogeneous rooted tree of degree (q, ℓ) with $\deg(v_0) = q + 1$. For each vertex $v \in \mathcal{V}$, let $\mathcal{E}(v) := \{e \in \mathcal{E}^\pm \mid s(e) = v\}$.

Definition 2.2.1. A *label* of edges of $\mathcal{T}$ means a family of bijective maps

$$\lambda_v : \mathcal{E}(v) \to \{0, \cdots, deg(v) - 1\}$$

for $v \in \mathcal{V}$. Set $\lambda_v(w) = \lambda_v((v, w))$ for an oriented edge $(v, w) \in \mathcal{E}^+$. We say that a label $\{\lambda_v\}_{v\in\mathcal{V}}$ is *compatible with the natural orientation* if it satisfies the following condition:

For $v \in \mathcal{V} \setminus \{v_0\}$, $\lambda_v(w) = 0$ if w is the father of v.

The image $\lambda_v(e)$ is called the *label of* $e \in \mathcal{E}(v)$.

When we fix a label of edges of a tree, we can interpret points of the boundary as sequences of numbers as follows. Let $q \geq \ell \geq 1$ be integers. We define the *set of infinite sequences of type* (q, ℓ) as the set

$$\bar{\Lambda}(q, \ell) := \{(\lambda_0, \lambda_1, \cdots) \in \prod^\infty \mathbb{Z} \mid 0 \leq \lambda_0 \leq q,\ 1 \leq \lambda_{2k-1} \leq \ell,\ 1 \leq \lambda_{2k} \leq q,\ k \in \mathbb{Z}_{\geq 1}\}.$$

Let $\{\lambda_v\}_{v\in\mathcal{V}}$ be a label on a tree $\mathcal{T} = (\mathcal{T}, v_0)$. For an infinite chain $v_0v_1v_2 \cdots v_k \cdots \in \partial\mathcal{T}$, we have

$$0 \leq \lambda_{v_0}(v_1) \leq q,\ 1 \leq \lambda_{v_{2k-1}}(v_{2k}) \leq \ell,\ 1 \leq \lambda_{v_{2k}}(v_{2k+1}) \leq q$$

for $k = 1, 2, \cdots$. Hence we have a well-defined map

$$\Lambda : \partial\mathcal{T} \to \bar{\Lambda}(q, \ell) \ ; \ v_0 v_1 v_2 \cdots v_k \cdots \mapsto (\lambda_{v_0}(v_1), \lambda_{v_1}(v_2), \cdots, \lambda_{v_k}(v_{k+1}), \cdots).$$

Obviously, Λ is bijective. Moreover, if we restrict this map to finite sequences, then we have a one-to-one correspondence between naturally oriented edges and finite sequences. We will denote by $(\lambda_0(e), \cdots, \lambda_k(e))$ the image of $e \in \mathcal{E}^+$ under this correspondence, where $k = \ell([v_0, r(e)])$.

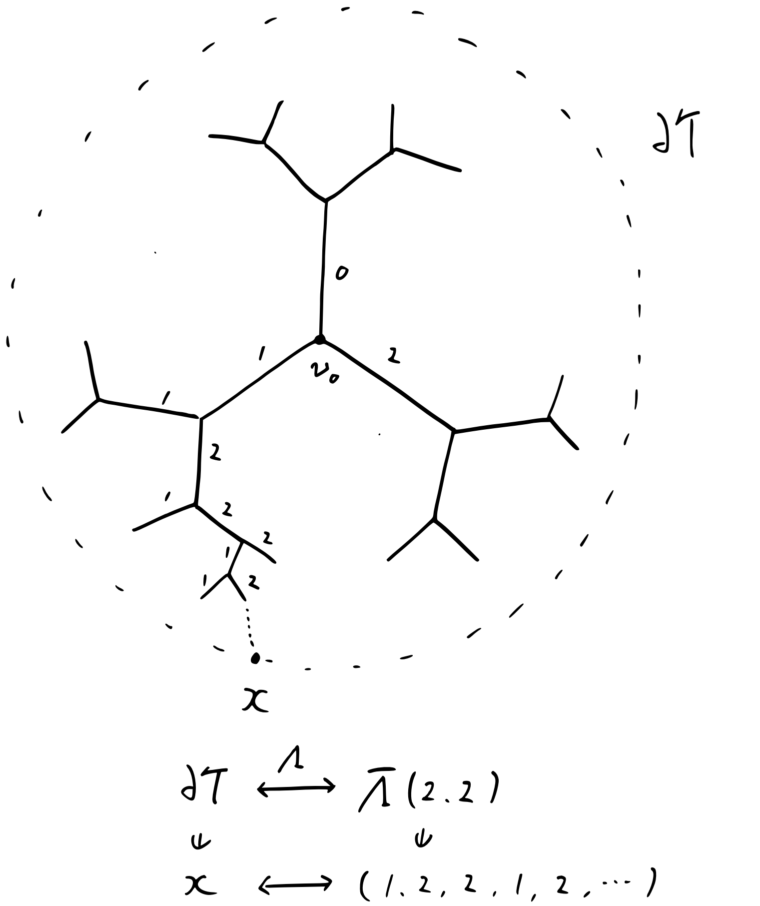

Figure 2.1: A label of edges

Definition 2.2.2. Let $\Lambda(q, \ell)$ be the subset of $\bar{\Lambda}(q, \ell)$ consisting of the sequences which satisfies the following condition (L) or (R);

(L) there exists an integer $k \geq 0$ such that $\lambda_{k+j} = 1$ for $j \geq 1$,

(R) there exists an integer $k \geq 0$ such that

$$\begin{cases} \lambda_{k+2j-1} = \ell, \ \lambda_{k+2j} = q \text{ for } j \geq 1, \text{ if } k \text{ is even}, \\ \lambda_{k+2j-1} = q, \ \lambda_{k+2j} = \ell \text{ for } j \geq 1, \text{ if } k \text{ is odd}. \end{cases}$$

We call a sequence in $\Lambda(q, \ell)$ is *type (L)* (resp. *type (R)*) if it satisfies the condition (L) (resp. (R)).

By definition, a sequence of type (L) or (R) is of the form

$$\text{type (L)} \qquad (\lambda_0, \cdots, \lambda_k, 1, 1, 1, 1, \cdots),$$

$$\text{type (R)} \qquad \begin{cases} (\lambda_0, \cdots, \lambda_k, \ell, q, \ell, q, \cdots) & \text{if } k \text{ is even}, \\ (\lambda_0, \cdots, \lambda_k, q, \ell, q, \ell, \cdots) & \text{if } k \text{ is odd}. \end{cases}$$

For any finite sequence of integers $(\lambda_0, \cdots, \lambda_k)$ with $0 \le \lambda_0 \le q,\ 1 \le \lambda_{2j-1} \le \ell,\ 1 \le \lambda_{2j} \le q$, let us denote by $L_{(\lambda_0,\cdots,\lambda_k)}$ and $R_{(\lambda_0,\cdots,\lambda_k)}$ the infinite sequence of the above form:

$$\begin{aligned} L_{(\lambda_0,\cdots,\lambda_k)} &:= (\lambda_0, \cdots, \lambda_k, 1, 1, 1, 1, \cdots). \\ R_{(\lambda_0,\cdots,\lambda_k)} &:= \begin{cases} (\lambda_0, \cdots, \lambda_k, \ell, q, \ell, q, \cdots) & \text{if } k \text{ is even}, \\ (\lambda_0, \cdots, \lambda_k, q, \ell, q, \ell, \cdots) & \text{if } k \text{ is odd}. \end{cases} \end{aligned}$$

We define an equivalence relation on $\Lambda(q, \ell)$. Let $(\lambda_0, \lambda_1, \cdots)$ and $(\lambda'_0, \lambda'_1, \cdots)$ be two sequences in $\Lambda(q, \ell)$. Suppose $(\lambda_0, \lambda_1, \cdots)$ is type (R) and $(\lambda'_0, \lambda'_1, \cdots)$ is type (L). Then they are said to be equivalent if there exists an integer $k \ge 0$ such that $\lambda_i = \lambda'_i$ for $i < k$ and $\lambda'_k = \lambda_k + 1$, or $(\lambda_0, \lambda_1, \cdots) = (q, \ell, q, \ell, q, \ell, \cdots)$ and $(\lambda'_0, \lambda'_1, \cdots) = (0, 1, 1, 1, 1, 1, \cdots)$. We write this equivalence relation by $\sim$.

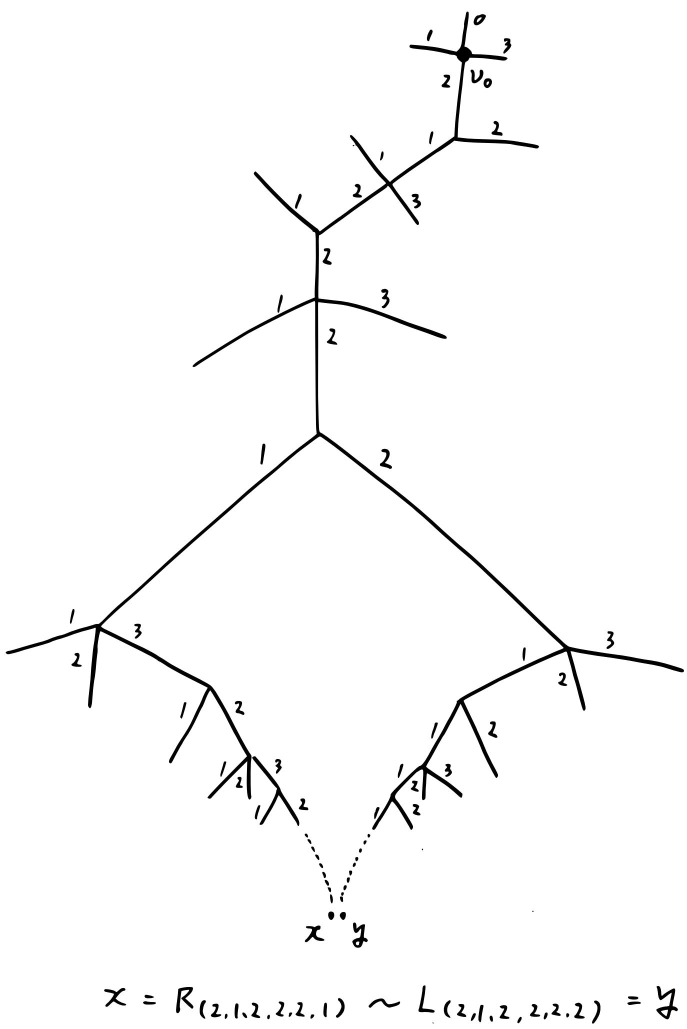

Figure 2.2: Sequences of type (L) and type (R)

Definition 2.2.3. Define the *space of rational ends of type* (q, ℓ) (denoted by $\mathcal{RE}(q, \ell)$) as the quotient space of $\Lambda(q, \ell)$ by the equivalence relation $\sim$:

$$\mathcal{RE} := \mathcal{RE}(q, \ell) := \Lambda(q, \ell)/\sim.$$

Note that for any pair $(\xi, \eta) \in \mathcal{RE} \times \mathcal{RE}$, we can take a representative of ξ (resp. η) with type (L) (resp. type (R)).

The next proposition justifies the name of "rational ends".

Proposition 2.2.1. *Let $q \geq 2$ and $\ell \geq 1$ be two integers. Then there is a one-to-one correspondence between the space of rational ends $\mathcal{RE}(q,\ell)$ and the set of rational numbers r with $0 \leq r < 1$, i.e., $[0,1) \cap \mathbb{Q}$.*

Proof. We only consider the case $q = \ell$, that is, the tree is homogeneous. The semi-homogeneous case follows from the same argument with minor modifications.

Let $(\lambda_0, \cdots, \lambda_k)$ be a finite sequence, and let $L_{(\lambda_0,\cdots,\lambda_k)}$ and $R_{(\lambda_0,\cdots,\lambda_k)}$ be corresponding infinite sequences of type (L) and type (R), respectively. Define

$$\begin{aligned}
\Phi(L_{(\lambda_0,\cdots,\lambda_k)}) &:= \frac{\lambda_0}{q+1} + \frac{\lambda_1 - 1}{(q+1)q} + \cdots + \frac{\lambda_k - 1}{(q+1)q^k} \\
&= \frac{\lambda_0}{q+1} + \sum_{i=1}^{k} \frac{\lambda_i - 1}{(q+1)q^i}, \\
\Phi(R_{(\lambda_0,\cdots,\lambda_k)}) &:= \frac{\lambda_0}{q+1} + \frac{\lambda_1 - 1}{(q+1)q} + \cdots + \frac{\lambda_k - 1}{(q+1)q^k} + \frac{q-1}{(q+1)q^{k+1}} + \cdots \\
&= \frac{\lambda_0}{q+1} + \sum_{j=1}^{k} \frac{\lambda_j - 1}{(q+1)q^j} + \sum_{j=k+1}^{\infty} \frac{q-1}{(q+1)q^j},
\end{aligned}$$

and $\Phi(R_{(q)}) := 0$. Then for any $R_{(\lambda_0,\cdots,\lambda_{k-1},\lambda_k)}$ with $\lambda_k \neq q$, we have

$$\begin{aligned}
\Phi(R_{(\lambda_0,\cdots,\lambda_{k-1},\lambda_k)}) &= \frac{\lambda_0}{q+1} + \sum_{j=1}^{k} \frac{\lambda_j - 1}{(q+1)q^j} + \sum_{j=k+1}^{\infty} \frac{q-1}{(q+1)q^j} \\
&= \frac{\lambda_0}{q+1} + \sum_{j=1}^{k} \frac{\lambda_j - 1}{(q+1)q^j} + \frac{1}{(q+1)q^k} \\
&= \frac{\lambda_0}{q+1} + \sum_{j=1}^{k-1} \frac{\lambda_j - 1}{(q+1)q^j} + \frac{\lambda_k}{(q+1)q^k} \\
&= \Phi(L_{(\lambda_0,\cdots,\lambda_{k-1},\lambda_k+1)}).
\end{aligned}$$

Hence the map Φ factors through the quotient map $\Lambda(q,q) \to \mathcal{RE}(q,q)$, and the image of Φ lies in $[0,1) \cap \mathbb{Q}$.

Conversely, using base-q system, every rational number in $[0,1)$ can be written uniquely in that form. Thus Φ defines a one-to-one correspondence between $\mathcal{RE}(q,q)$ and $[0,1) \cap \mathbb{Q}$. □

Fix $q, \ell \in \mathbb{Z}_{\geq 1}$. We will consider Alexander-Spanier 1-cocycles on $\mathcal{RE} = \mathcal{RE}(q,\ell)$.

Lemma 2.2.1. *Let M be an additive group. For any Alexander-Spanier 1-cocycle $\varphi \in Z^1_{AS}(\mathcal{RE}; M)$ and $k \geq 1$, we have*

$$\begin{aligned}
\varphi([L_{(\lambda_0,\cdots,\lambda_{k-1},\lambda_k)}], [R_{(\lambda_0,\cdots\lambda_{k-1},\lambda_k)}]) + \varphi([L_{(\lambda_0,\cdots\lambda_{k-1},\lambda_k+1)}], [R_{(\lambda_0,\cdots\lambda_{k-1},\lambda_k+1)}]) \\
= \varphi([L_{(\lambda_0,\cdots,\lambda_{k-1},\lambda_k)}], [R_{(\lambda_0,\cdots,\lambda_{k-1},\lambda_k+1)}]),
\end{aligned}$$

where $[\cdot]$ stands for the equivalence class. Moreover, the following hold:

(i) $\sum_{i=1}^{n_k-1} \varphi([L_{(\lambda_0,\cdots,\lambda_k,i)}], [R_{(\lambda_0,\cdots\lambda_k,i+1)}]) = \varphi([L_{(\lambda_0,\cdots,\lambda_k)}], [R_{(\lambda_0,\cdots,\lambda_k)}])$ *for any $k \geq 0$, where $n_k = \ell$ if k is even, $n_k = q$ if k is odd.*

(ii) $\sum_{i=0}^{q} \varphi([L_{(i)}], [R_{(i)}]) = 0$.

Proof. By definition, $L_{(\lambda_0,\cdots,\lambda_{k-1},\lambda_k)} \sim R_{(\lambda_0,\cdots\lambda_{k-1},\lambda_k)}$. Therefore

$$\begin{aligned}
&\varphi([L_{(\lambda_0,\cdots,\lambda_{k-1},\lambda_k)}], [R_{(\lambda_0,\cdots\lambda_{k-1},\lambda_k)}]) + \varphi([L_{(\lambda_0,\cdots\lambda_{k-1},\lambda_k+1)}], [R_{(\lambda_0,\cdots\lambda_{k-1},\lambda_k+1)}]) \\
=\ &\varphi([L_{(\lambda_0,\cdots,\lambda_{k-1},\lambda_k)}], [R_{(\lambda_0,\cdots\lambda_{k-1},\lambda_k)}]) + \varphi([R_{(\lambda_0,\cdots\lambda_{k-1},\lambda_k+1)}], [R_{(\lambda_0,\cdots\lambda_{k-1},\lambda_k+1)}]) \\
=\ &\varphi([L_{(\lambda_0,\cdots,\lambda_{k-1},\lambda_k)}], [R_{(\lambda_0,\cdots,\lambda_{k-1},\lambda_k+1)}]).
\end{aligned}$$

The last equation follows from the cocycle condition of Alexander-Spanier cocycles. □

Definition 2.2.4. A pair of rational ends $(\xi, \eta) \in \mathcal{RE} \times \mathcal{RE}$ is said to be *primitive* if there exists a finite sequence $(\lambda_0, \cdots, \lambda_k)$ such that $\xi = [L_{(\lambda_0,\cdots,\lambda_k)}]$ and $\eta = [R_{(\lambda_0,\cdots,\lambda_k)}]$. It is said that a primitive pair $([L_{(\lambda_0,\cdots,\lambda_k)}], [R_{(\lambda_0,\cdots,\lambda_k)}])$ is type q (resp. type ℓ) if k is even (resp. odd). For a primitive pair $(\xi, \eta) = ([L_{(\lambda_0,\cdots,\lambda_k)}], [R_{(\lambda_0,\cdots,\lambda_k)}])$, define the *length* of (ξ, η) (written $\ell(\xi, \eta)$) as the length of the finite sequence which defines the primitive pair;

$$\ell([L_{(\lambda_0,\cdots,\lambda_k)}], [R_{(\lambda_0,\cdots,\lambda_k)}]) := k + 1.$$

Lemma 2.2.2. *Let φ be an Alexander-Spanier 1-cocycle on $\mathcal{RE}$ with coefficients in an additive group M. Then for any pair $(\xi, \eta) \in \mathcal{RE} \times \mathcal{RE}$, there exist primitive pairs $(\xi_0, \eta_0), \cdots, (\xi_m, \eta_m)$ such that $\varphi(\xi, \eta) = \varphi(\xi_0, \eta_0) + \cdots + \varphi(\xi_m, \eta_m)$.*

Proof. Let $\varphi \in Z^1_{AS}(\mathcal{RE}; M)$ be an Alexander-Spanier 1-cocycle. We may assume $\xi = [(0, 1, 1, \cdots)] = [L_{(0)}]$, since $\varphi(\xi, \eta) = \varphi([L_{(0)}], \eta) - \varphi([L_{(0)}], \xi)$ by the cocycle condition for φ. Let $\eta = [R_{(\lambda_0,\cdots,\lambda_k)}]$ for some finite sequence $(\lambda_0, \cdots, \lambda_k)$. We prove the lemma by induction on k.

If $k = 0$, then

$$\varphi([L_{(0)}], [R_{(\lambda_0)}]) = \sum_{i}^{\lambda_0} \varphi([L_{(i)}], [R_{(i)}])$$

by the definition of the equivalence relation. Suppose that the statement holds for $k' \leq k$. Since

$$\begin{aligned}\varphi([L_{(0)}], [R_{(\lambda_0,\cdots,,\lambda_k,\lambda_{k+1})}]) + \sum_{i=\lambda_{k+1}}^{n_k} ([L_{(\lambda_0,\cdots,\lambda_k,i)}], [R_{(\lambda_0,\cdots,\lambda_k,i)}]) \\ = \varphi([L_{(0)}], [R_{(\lambda_0,\cdots,,\lambda_k,n_k)}]) = \varphi([L_{(0)}], [R_{(\lambda_0,\cdots,,\lambda_k)}]),\end{aligned}$$

we have

$$\varphi([L_{(0)}], [R_{(\lambda_0,\cdots,,\lambda_k,\lambda_{k+1})}]) = \varphi([L_{(0)}], [R_{(\lambda_0,\cdots,,\lambda_k)}]) - \sum_{i=\lambda_{k+1}}^{n_k} ([L_{(\lambda_0,\cdots,\lambda_k,i)}], [R_{(\lambda_0,\cdots,\lambda_k,i)}]),$$

where $n_k = \ell$ if k is even, $n_k = q$ if k is odd. By the induction hypothesis, $\varphi([L_{(0)}], [R_{(\lambda_0,\cdots,,\lambda_k)}])$ can be written as the sum of the values on the primitive pairs, and the lemma follows. □

This lemma asserts that every Alexander-Spanier 1-cocycle can be determined by its values on primitive pairs. The converse statement also holds:

Lemma 2.2.3. *Let $\tilde{\varphi}$ be a function defined on the set of primitive pairs which satisfies the following properties:*

(i) $\tilde{\varphi}(\xi, \eta) = -\tilde{\varphi}(\eta, \xi)$ *for any primitive pair* (ξ, η).

(ii) $\sum_{i=1}^{n_k} \tilde{\varphi}([L_{(\lambda_0,\cdots,\lambda_k,i)}], [R_{(\lambda_0,\cdots\lambda_k,i)}]) = \tilde{\varphi}([L_{(\lambda_0,\cdots,\lambda_k)}], [R_{(\lambda_0,\cdots,\lambda_k)}])$ *for any* $k \geq 0$*, where* $n_k = \ell$ *if* k *is even,* $n_k = q$ *if* k *is odd.*

(iii) $\sum_{\lambda_0=0}^{q} \tilde{\varphi}([L_{(\lambda_0)}], [R_{(\lambda_0)}]) = 0$.

Then $\tilde{\varphi}$ can be extended to an Alexander-Spanier 1-cocycle on $\mathcal{RE}(q, \ell)$.

Proof. Let $\tilde{\varphi}$ be a function satisfying the properties. As the same argument of the previous lemma, for every pair $(\xi, \eta) \in \mathcal{RE} \times \mathcal{RE}$, there is a sequence of primitive pairs $(\xi_0, \eta_0), \cdots, (\xi_m, \eta_m)$ such that $\xi_0 = \xi, \eta_m = \eta$ and $\eta_i = \xi_{i+1}$ for $i = 0, \cdots, m-1$. We refer to such a sequence of primitive pairs as *a primitive chain of length m connecting ξ and η.*

We set

$$\varphi(\xi, \eta) := \sum_{i=0}^{m} \tilde{\varphi}(\xi_i, \eta_i).$$

This function satisfies the Alexander-Spanier cocycle condition. In fact, if $(\xi_0, \eta_0), \cdots, (\xi_m, \eta_m)$ (resp. $(\xi'_0, \eta'_0), \cdots, (\xi'_{m'}, \eta'_{m'})$) is a primitive chain connecting ξ and η (resp. η and ζ), then $(\xi_0, \eta_0), \cdots, (\xi_m, \eta_m), (\xi'_0, \eta'_0), \cdots, (\xi'_{m'}, \eta'_{m'})$ is a primitive chain connecting ξ and ζ. Hence

$$\varphi(\xi, \eta) + \varphi(\eta, \zeta) = \sum_{i=0}^{m} \tilde{\varphi}(\xi_i, \eta_i) + \sum_{i=0}^{m'} \tilde{\varphi}(\xi'_i, \eta'_i) = \varphi(\xi, \zeta).$$

Thus we only have to check that φ is well-defined. To prove this, it suffices to show that

(∗) for any primitive chain $(\xi_0, \eta_0), \cdots, (\xi_m, \eta_m)$ such that $\xi_0 = \eta_m = \xi$ for some ξ, referred to as *a primitive loop starting at* ξ, we have $\sum_{i=0}^{m} \tilde{\varphi}(\xi_i, \eta_i) = 0$.

In fact, if $(\xi_0, \eta_0), \cdots, (\xi_m, \eta_m)$ and $(\xi'_0, \eta'_0), \cdots, (\xi'_m, \eta'_m)$ are two primitive chains connecting ξ and η, then $(\xi_0, \eta_0), \cdots, (\xi_m, \eta_m), (\eta'_m, \xi'_m), \cdots, (\eta'_0, \xi'_0)$ is a primitive loop starting at ξ. Hence the condition (∗) implies

$$\sum_{i=0}^{m} \tilde{\varphi}(\xi_i, \eta_i) = \sum_{i=0}^{m} \tilde{\varphi}(\xi'_i, \eta'_i)$$

by the property (i) of $\tilde{\varphi}$.

Let us first observe that we can assume that there is no subchain of the form $(\xi, \eta), (\eta, \xi)$ in a given loop, since we can delete this chain from the given loop without changing the value $\sum_{i=0}^{m} \tilde{\varphi}(\xi_i, \eta_i)$ by the property (i). Moreover, we can also assume that there is no subloop of the following form:

(a) $([L_{(\lambda_0,\cdots,\lambda_k,1)}], [R_{(\lambda_0,\cdots,\lambda_k,1)}]), \cdots, ([L_{(\lambda_0,\cdots,\lambda_k,n_k)}], [R_{(\lambda_0,\cdots,\lambda_k,n_k)}]), ([R_{(\lambda_0,\cdots,\lambda_k)}], [L_{(\lambda_0,\cdots,\lambda_k)}])$ or its cyclic permutation.

(b) $([R_{(\lambda_0,\cdots,\lambda_k,n_k)}], [L_{(\lambda_0,\cdots,\lambda_k,n_k)}]), \cdots, ([R_{(\lambda_0,\cdots,\lambda_k,1)}], [L_{(\lambda_0,\cdots,\lambda_k,1)}]), ([L_{(\lambda_0,\cdots,\lambda_k)}], [R_{(\lambda_0,\cdots,\lambda_k)}])$ or its cyclic permutation.

Here n_k denotes q or ℓ depending on k is odd or even, respectively. In fact, if such a subloop exists, we can remove such a loop from the original one without changing the value $\sum_{i=0}^{m} \tilde{\varphi}(\xi_i, \eta_i)$ by the property (ii).

We prove that the assertion (∗) holds by iterating the following reduction procedure. Let $(\xi_0, \eta_0), \cdots (\xi_m, \eta_m)$ be a primitive loop starting at some ξ. We refer this loop as c. Note that each primitive pair (ξ_i, η_i) is of the form $([L_{(\lambda_0,\cdots,\lambda_{k_i})}], [R_{(\lambda_0,\cdots,\lambda_{k_i})}])$. Set $\mathbb{L}_c(k)$ the set of primitive chains with length k which are contained in the loop c:

$$\mathbb{L}_c(k) := \{(\xi_i, \eta_i) \mid \ell(\xi, \eta) = k\}.$$

Since the loop consists of finitely many primitive pairs, there are numbers $k_c^{\max} := \max\{k \mid \mathbb{L}_c(k) \neq \emptyset\}$ and $k_c^{\min} := \{k \mid \mathbb{L}_c(k) \neq \emptyset\}$. If we choose a primitive pair (ξ, η) with $\ell(\xi, \eta) = k_c^{max}$, then there must exist a subchain constructed from primitive pairs with length $k_c^{\max}$ containing (ξ, η) which is of the form

$$([L_{(\lambda_0,\cdots,\lambda_{k_c^{\max}},1)}], [R_{(\lambda_0,\cdots,\lambda_{k_c^{\max}},1)}]), \cdots, ([L_{(\lambda_0,\cdots,\lambda_{k_c^{\max}},n)]}, [R_{(\lambda_0,\cdots,\lambda_{k_c^{\max}},n}])$$

or

$$([R_{(\lambda_0,\cdots,\lambda_{k_c^{\max}},n)]}, [L_{(\lambda_0,\cdots,\lambda_{k_c^{\max}},n}]), \cdots, ([R_{(\lambda_0,\cdots,\lambda_{k_c^{\max}},1)}], [L_{(\lambda_0,\cdots,\lambda_{k_c^{\max}},1)}]),$$

where n is q if $k_c^{\max}$ is odd or ℓ if $k_c^{\max}$ is even. In that case, we can replace these with

$$([L_{(\lambda_0,\cdots,\lambda_{k_c^{\max}})}], [R_{(\lambda_0,\cdots,\lambda_{k_c^{\max}})}])$$

or

$$([R_{(\lambda_0,\cdots,\lambda_{k_c^{\max}})}], [L_{(\lambda_0,\cdots,\lambda_{k_c^{\max}})}]),$$

respectively, without changing $\sum_{i=0}^{m} \tilde{\varphi}(\xi_i, \eta_i)$, by the property (ii).

When we apply this procedure to all primitive subchains consisting of primitive pairs with length $k_c^{\max}$ of the above form. Then we get a new loop $(\xi'_0, \eta'_0), \cdots, (\xi'_{m'}, \eta'_{m'})$ (denoted by c'), such that $\sum_{i=0}^{m} \tilde{\varphi}(\xi_i, \eta_i) = \sum_{i=0}^{m'} \tilde{\varphi}(\xi'_i, \eta'_i)$ and $k_{c'}^{\max} = k_c^{\max} - 1$.

Applying this reduction procedure recursively, we finally obtain a loop which is one of the following form:

(a) $([L_{(\lambda_0,\cdots,\lambda_k,1)}], [R_{(\lambda_0,\cdots,\lambda_k,1)}]), \cdots, ([L_{(\lambda_0,\cdots,\lambda_k,n_k)}], [R_{(\lambda_0,\cdots,\lambda_k,n_k)}]), ([R_{(\lambda_0,\cdots,\lambda_k)}], [L_{(\lambda_0,\cdots,\lambda_k)}])$ or its cyclic permutation.

(b) $([R_{(\lambda_0,\cdots,\lambda_k,n_k)}], [L_{(\lambda_0,\cdots,\lambda_k,n_k)}]), \cdots, ([R_{(\lambda_0,\cdots,\lambda_k,1)}], [L_{(\lambda_0,\cdots,\lambda_k,1)}]), ([L_{(\lambda_0,\cdots,\lambda_k)}], [R_{(\lambda_0,\cdots,\lambda_k)}])$ or its cyclic permutation.

(c) $([L_0],[R_0]),\cdots,([L_q],[R_q])$ or its cyclic permutation.

(d) $([R_q],[L_q]),\cdots,([R_0],[L_0])$ or its cyclic permutation.

Here $k=k^{\max}=k^{\min}$, and n_k denotes q or ℓ depending on k is odd or even, respectively. By the property (ii) and (iii), the resulting loop has the value $\sum_{i=0}^{m}\tilde{\varphi}(\xi_i,\eta_i)=0$. This completes the proof. □

If we fix a label of edges of a semi-homogeneous rooted tree $\mathcal{T}$ of degree (q,ℓ), then we can consider a rational end in $\mathcal{RE}(q,\ell)$ as an equivalence class of boundary points in $\partial\mathcal{T}$. Let us denote by $\partial_R\mathcal{T}$ the set of these equivalence classes. We call $\partial_R\mathcal{T}$ *the space of rational ends of* $\mathcal{T}$.

For a naturally oriented edge $e\in\mathcal{E}$, let L_e and R_e be infinite sequences defined by

$$\begin{aligned} L_e &:= L_{(\lambda_0(e),\cdots,\lambda_k(e))}=(\lambda_0(e),\cdots,\lambda_k(e),1,1,1,1,\cdots),\\ R_e &:= \begin{cases} R_{(\lambda_0(e),\cdots,\lambda_k(e))}=(\lambda_0(e),\cdots,\lambda_k(e),\ell,q,\ell,q,\cdots) & \text{if } k \text{ is even},\\ R_{(\lambda_0(e),\cdots,\lambda_k(e))}=(\lambda_0(e),\cdots,\lambda_k(e),q,\ell,q,\ell,\cdots) & \text{if } k \text{ is odd},\end{cases}\end{aligned}$$

where $(\lambda_0(e),\cdots,\lambda_k(e))$ is a finite sequence which corresponds to the edge e under the bijection $\Lambda:\partial\mathcal{T}\to\bar{\Lambda}(q,\ell)$. It is obvious that L_e and R_e belong to $\Lambda(q,\ell)$.

The next lemma is a direct consequence of the definition of L_e and R_e.

Lemma 2.2.4. *Let e,e',e'' be oriented edges with $r(e)=s(e')=s(e'')=:v$ such that $\lambda_v(e')=1$ and $\lambda_v(e'')=deg(v)-1$. Then $L_e=L_{e'}$ and $R_e=R_{e''}$.*

By definition, a pair $(\xi,\eta)\in\partial_R\mathcal{T}\times\partial_R\mathcal{T}$ is primitive if and only if there exists an oriented edge $e\in\mathcal{E}^+$ such that $\xi=[L_e]=[L_{(\lambda_0(e),\cdots,\lambda_k(e))}]$ and $\eta=[R_e]=[R_{(\lambda_0(e),\cdots,\lambda_k(e))}]$, where k denotes the length of the geodesic $[v_0,r(e)]$. Then the Alexander-Spanier cocycles on $\partial_R\mathcal{T}$ can be considered as cocycles on $\mathcal{RE}$, and the following lemmas are restatements of Lemma 2.2.1, 2.2.2, 2.2.3.

Lemma 2.2.5. *Let $n:=deg(v)-1$, and let $e,e_1,\cdots,e_n$ be oriented edges with $r(e)=s(e_1)=\cdots=s(e_n)=:v$ such that $\lambda_v(e_i)=i$ for $i=1,\cdots,n$. Then for any Alexander-Spanier cocycle $\varphi\in Z^1_{AS}(\partial_R\mathcal{T};M)$, we have*

$$\varphi([L_{e_i}],[R_{e_i}])+\varphi([L_{e_{i+1}}],[R_{e_{i+1}}])=\varphi([L_{e_i}],[R_{e_{i+1}}]).$$

Moreover, the following hold:

(i) $\sum_{i=1}^{n}\varphi([L_{e_i}],[R_{e_i}])=\varphi([L_e],[R_e])$.

(ii) *If $s(e_0)=\cdots=s(e_q)=v_0$, then $\sum_{i=0}^{q}\varphi([L_{e_i}],[R_{e_i}])=0$.*

Lemma 2.2.6. *Each Alexander-Spanier 1-cocycle $\varphi\in Z^1_{AS}(\partial_R\mathcal{T};M)$ can be determined by its values on the primitive pairs. Conversely, for any function $\tilde{\varphi}$ defined on the set of primitive pairs which satisfies*

(i) $\tilde{\varphi}([L_e],[R_e])=-\tilde{\varphi}([R_e],[L_e])$ *for any $e\in\mathcal{E}^+$,*

(ii) $\sum_{i=1}^{n}\tilde{\varphi}([L_{e_i}],[R_{e_i}])=\tilde{\varphi}([L_e],[R_e])$ *for any $e,e_1,\cdots e_n\in\mathcal{E}^+$ with $r(e)=s(e_1)=\cdots=s(e_n)$, where $n=\deg(r(e))-1$, and*

(iii) $\sum_{i=0}^{q}\tilde{\varphi}([L_{e_i}],[R_{e_i}])=0$, *for $e_0,e_1,\cdots e_q\in\mathcal{E}^+$ with $s(e_0)=\cdots=s(e_q)=v_0$*

can be extended to an Alexander-Spanier 1-cocycle on $\partial_R\mathcal{T}$.

Theorem 2.2.1. *Let $\mathcal{T}$ be a naturally oriented semi-homogeneous rooted tree of degree (q,ℓ) and M an additive group. Fix a label of edges of $\mathcal{T}$ and identify $\partial_R\mathcal{T}$ with $\mathcal{RE}(q,\ell)$. Then there is an isomorphism*

$$\mathrm{Meas}(\partial\mathcal{T},M)\simeq Z^1_{AS}(\partial_R\mathcal{T};M).$$

Proof. Let $\varphi \in Z^1_{AS}(\partial_R \mathcal{T}; M)$ be an Alexander-Spanier 1-cocycle on $\partial_R \mathcal{T}$. By Proposition 1.1.3, every clopen subset V of $\partial \mathcal{T}$ is of the form $V = \bigsqcup_{i=0}^{k} V(e_i)$. Then we define a measure μ_φ on $\partial \mathcal{T}$ by

$$\mu_\varphi(V) := \sum_{i=1}^{k} \varphi([L_{e_i}], [R_{e_i}]).$$

We have to check that μ_φ is well-defined. To prove this, it suffices to consider the case where

$$V(e) = \bigsqcup_{i=1}^{n} V(e_i) \text{ with } r(e) = s(e_i) \text{ for } i = 1, \cdots n, \text{ where } n = \deg(r(e)).$$

Then we have $\mu_\varphi(V(e)) = \varphi([L_e], [R_e])$ and $\mu_\varphi(\bigsqcup_{i=1}^{n} V(e_i)) = \sum_{i=1}^{n} \varphi([L_{e_i}], [R_{e_i}])$ by definition. By changing the indices if necessary, we may assume that the label of e_i is i. Then, by Lemma 2.2.5, we have

$$\sum_{i=1}^{n} \varphi([L_{e_i}], [R_{e_i}]) = \varphi([L_e], [R_e]).$$

Thus $\mu_\varphi(V(e)) = \mu_\varphi(\bigsqcup_{i=1}^{n} V(e_i))$, and hence μ_φ is well-defined.

We next show that μ_φ is a measure. The condition (M1) holds immediately from the definition of μ_φ. Let $e_0, \cdots, e_q$ be oriented edges with $s(e_i) = v_0$ for $i = 0, \cdots, q$. Then we can write $\partial \mathcal{T} = \bigsqcup_{i=0}^{q} V(e_i)$. Lemma 2.2.6 now implies

$$\mu_\varphi(\partial \mathcal{T}) = \sum_{i=0}^{q} \mu_\varphi(V(e_i)) = \sum_{i=0}^{q} \varphi([L_{e_i}], [R_{e_i}]) = 0.$$

This proves μ_φ satisfies (M2), and consequently μ_φ is a measure on $\partial \mathcal{T}$.

Conversely, let $\mu \in \mathrm{Meas}(\partial_R \mathcal{T}, M)$ be a measure on $\partial \mathcal{T}$. Define

$$\tilde{\varphi}_\mu([L_e], [R_e]) := \mu(V(e))$$

for any edge $e \in \mathcal{E}^+$. Then $\tilde{\varphi}_\mu$ is a function on the set of primitive pairs which satisfies the conditions of Lemma 2.2.6. In fact, for any $e, e_1, \cdots e_n \in \mathcal{E}^+$ with $r(e) = s(e_1) = \cdots = s(e_n)$, where $n = \deg(v) - 1$,

$$\sum_{i=1}^{n} \tilde{\varphi}_\mu([L_{e_i}], [R_{e_i}]) = \sum_{i=1}^{n} \mu(V_{e_i}) = \mu(V(e)),$$

and for $e_0, e_1, \cdots e_n \in \mathcal{E}^+$ with $s(e_0) = \cdots = s(e_q) = v_0$,

$$\sum_{i=0}^{n} \tilde{\varphi}_\mu([L_{e_i}], [R_{e_i}]) = \sum_{i=0}^{n} \mu(\partial \mathcal{T}) = 0$$

since μ is a measure. Hence $\tilde{\varphi}_\mu$ extends to an Alexander-Spanier 1-cocycle φ_μ on $\partial_R \mathcal{T}$ by Lemma 2.2.6. By construction, it is obvious that $\mu_{\varphi_\mu} = \mu$ and $\varphi_{\mu_\varphi} = \varphi$, which completes the proof. □

Remark 2.2.1. *Note that we can identify the space of rational ends of the tree of* $\mathrm{PSL}_2(\mathbb{Z})$ *with* $\mathbb{P}^1(\mathbb{Q})$. *Then Alexander-Spanier* 1*-cocycles on* $\mathbb{P}^1(\mathbb{Q})$ *coincide with modular symbols for* $\mathrm{PSL}_2(\mathbb{Z})$ *(see Proposition 3.2.1). Thus the above results are extensions of the work of Manin and Marcolli [16].*

2.3 Dense Embeddings and Disconnection

In this section, we consider the relation between the boundary of trees and the spaces of rational ends. The argument of this section is a fairly straightforward generalization of [16].

Definition 2.3.1. Let X be a set. An injective map $X \to S^1$ with dense image is called a *dense embedding* of X into S^1.

We only consider a dense embedding which preserves orientation. To clarify the notion of "orientation preserving", we introduce the signature function defined as follows. Under the natural identification $S^1 = \mathbb{R}/\mathbb{Z}$, we consider an element in S^1 as a real number lying in $[0, 1)$. Then for $(\alpha, \beta) \in S^1 \times S^1$, we define

$$\mathrm{sign}((\alpha,\beta)) \quad := \quad \begin{cases} 1 & \text{if } \alpha < \beta \\ 0 & \text{if } \alpha = \beta \\ -1 & \text{if } \beta < \alpha. \end{cases}$$

On the other hand, when we endow $\Lambda(q, \ell)$ with the lexicographic order, this order descends to equivalence classes and defines an order on $\mathcal{RE}$ (written as $\prec$). For $(\xi, \eta) \in \mathcal{RE} \times \mathcal{RE}$, set

$$\mathrm{sign}((\xi,\eta)) \quad := \quad \begin{cases} 1 & \text{if } \xi \prec \eta, \\ 0 & \text{if } \xi = \eta, \\ -1 & \text{if } \eta \prec \xi. \end{cases}$$

Definition 2.3.2. A dense embedding $\Phi : \mathcal{RE} \to S^1$ is said to be *orientation preserving* if

$$\mathrm{sign}(\xi, \eta) = \mathrm{sign}(\Phi(\xi), \Phi(\eta))$$

for any $\xi, \eta \in \mathcal{RE}$.

Remark 2.3.1. *Note that Lemma 2.2.1 says that there is an orientation preserving dense embedding into S^1.*

We introduce the notion of *disconnections*. The following construction is due to Spielberg [26]. Let F be a subset of S^1. Define a C^*-algebra B_F as the closure in the supremum norm of the $*$-algebra generated by $C(S^1) \cup \{p(\alpha, \beta) \mid \alpha, \beta \in F,\ \alpha \neq \beta\}$, where $p(\alpha, \beta)$ denotes the characteristic function of the half-open interval $[\alpha, \beta)$. Since B_F is commutative and unital, by the Gelfand-Naimark duality, there is a compact Hausdorff space D_F such that $B_F \simeq C(D_F)$.

Definition 2.3.3. We call D_F the *disconnection of S^1 on F.*

As mentioned in [26], if a subset F is dense in S^1, we have the following characterization of D_F.

Lemma 2.3.1. *D_F is totally disconnected if and only if F is dense in S^1. Moreover, in this case, $C(D_F) = \overline{\mathrm{span}}\{p(\alpha, \beta) \mid \alpha, \beta \in F,\ \alpha \neq \beta\}$, where we identify $C(D_F)$ with B_F.*

The following theorem states that we can reconstruct $\partial\mathcal{T}$ from $\partial_R\mathcal{T}$ using a dense embedding.

Theorem 2.3.1. *Let $\Phi : \mathcal{RE}(q, \ell) \to S^1$ be an orientation preserving dense embedding. If we fix a label of edges of a semi-homogeneous tree of degree (q, ℓ), then we have*

$$D_{\Phi(\partial_R\mathcal{T})} \simeq \partial\mathcal{T},$$

where we identify $\mathcal{RE}(q, \ell)$ with $\partial_R\mathcal{T}$.

Proof. Note that the family $\{V(e)\}_{e\in\mathcal{E}^+}$ forms a basis of open sets for the topology of $\partial\mathcal{T}$. Since $\partial\mathcal{T}$ is totally disconnected, $C(\partial\mathcal{T})$ is generated by the characteristic functions $\chi_{V(e)}$ for $e \in \mathcal{E}^+$. Then we can define an injective morphism $C(\partial\mathcal{T}) \to C(D_{\Phi(\partial_R\mathcal{T})})$ by $\chi_{V(e)} \mapsto p([L_e], [R_e])$ since Φ preserves the orientation.

On the other hand, any projection $p(\xi, \eta) \in C(D_{\Phi(\partial_R\mathcal{T})})$ with $\xi, \eta \in \partial_R\mathcal{T}$ can be written by the sum of projections of the form $p([L_e], [R_e])$ by the same argument as in the proof of Lemma 2.2.2. Hence the morphism $C(\partial\mathcal{T}) \to C(D_{\Phi(\partial_R\mathcal{T})})$ is an isomorphism. Then the Gelfand-Naimark duality yields $D_{\Phi(\partial_R\mathcal{T})} \simeq \partial\mathcal{T}$. □

2.4 Change of Labels

Let $\mathcal{T} = (\mathcal{T}, v_0)$ be a semi-homogeneous rooted tree of degree (q, ℓ) and $\{\lambda_v\}_{v \in \mathcal{V}}$ a label of edges of $\mathcal{T}$. We consider changing roots. Let w_0 be a vertex adjacent to the root v_0 and $\mathcal{T}'$ denotes the rooted tree with root w_0 (i.e. $\mathcal{T}' = (\mathcal{T}, w_0)$). Then we can define a label $\{\lambda'_v\}_{v \in \mathcal{V}}$ of edges of $\mathcal{T}'$ by applying the following procedure:

1. Define a label of edges of $\mathcal{T}$ by $\lambda''_{v_0}((v_0, w)) := \lambda_{v_0}((v_0, w)) - \lambda_{v_0}((v_0, w_0)) \bmod (q+1)$ and $\lambda''_v((v, w)) := \lambda_v((v, w))$ for $v \neq v_0$.
2. Set $\lambda'_{w_0}((w_0, v_0)) := 0$ and $\lambda'_w((w, v)) = \lambda''_v((v, w))$ for $v \neq v_0$.

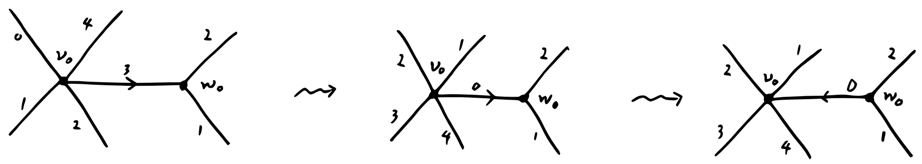

Figure 2.3: Change of labels

Then the label $\{\lambda'_v\}_{v \in \mathcal{V}}$ induces a bijection $\Lambda' : \partial\mathcal{T}' \to \bar{\Lambda}(q, \ell)$ which coincides with the bijection $\Lambda : \partial\mathcal{T} \to \bar{\Lambda}(q, \ell)$ under the identification $\partial\mathcal{T} \leftrightarrow \partial\mathcal{T}'$, that is, the following diagram commutes:

$$\begin{array}{ccc} \partial\mathcal{T} & \longrightarrow & \partial\mathcal{T}' \\ {\scriptstyle\Lambda}\downarrow & & \downarrow{\scriptstyle\Lambda'} \\ \bar{\Lambda}(q, \ell) & = & \bar{\Lambda}(q, \ell). \end{array}$$

Definition 2.4.1. The label $\{\lambda'_v\}_{v \in \mathcal{V}}$ is called *the label associated with the change of roots* $v_0 \mapsto w_0$.

Since a tree is connected, by iterating the above procedure, we can define a label of edges of a rooted tree $(\mathcal{T}, v)$ for any vertex $v \in \mathcal{V}$. Since the label associated with $v_0 \mapsto v_0$ is the original label, the resulting label does not depend on the process of changing roots. Thus we call the resulting label *the label associated with the change of roots* $v_0 \mapsto v$.

Let $\mathcal{T} = (\mathcal{T}, v_0)$ be a naturally oriented semi-homogeneous rooted tree, and fix a label of edges of $\mathcal{T}$. Then for any $w_0 \in \mathcal{V}$, we assume that the rooted tree $(\mathcal{T}, w_0)$ is endowed with the label associated with the change of roots $v_0 \mapsto w_0$. Under this assumption, we can identify $\partial_R\mathcal{T}$ with $\partial_R\mathcal{T}'$ by the commutativity of the above diagram.

Concerning the group actions on trees, it is convenient to introduce the following convention.

Definition 2.4.2. For any $v \in \mathcal{V}$ and $e \in \mathcal{E}^{\pm}$ with $v = s(e)$, L^v_e (resp. R^v_e) denotes the infinite sequence of type(L) (resp. type(R)) associated with e when v is considered as the root. By abuse of notation, if it is clear which root v has been chosen, we sometimes omit v from the notation.

Using this convention, we can restate Lemma 2.2.6:

Lemma 2.4.1. *Let $\mathcal{T} = (\mathcal{T}, v_0)$ be a naturally oriented semi-homogeneous rooted tree of degree (q, ℓ) and M an additive group. Fix a label of edges of $\mathcal{T}$. Let $\tilde{\varphi}$ be a function defined on the set of primitive pairs which satisfies*

$(*)$ $\sum_{i=0}^{\deg(v)-1} \tilde{\varphi}([L^v_{e_i}], [R^v_{e_i}]) = 0$, *for any $v \in \mathcal{V}$ and $e_0, \cdots e_{\deg(v)-1} \in \mathcal{E}^{\pm}$ with $s(e_0) = \cdots = s(e_{\deg(v)-1}) = v$*

can be extended to an Alexander-Spanier 1-cocycle on $\partial_R\mathcal{T}$.

2.5 Equivariant Alexander-Spanier Cocycles on $\partial_R\mathcal{T}$

Fix a label of edges of a naturally oriented semi-homogeneous rooted tree $\mathcal{T}$. An automorphism $h : \mathcal{T} \to \mathcal{T}$ is said to be *preserving rationals* if it preserves the equivalence relation $\sim$ on $\partial_R\mathcal{T}$, that is, $L_{h(e)} \sim R_{h(e')}$ when $L_e \sim R_{e'}$ for any $e, e' \in \mathcal{E}$.

Example 2.5.1. Let $(\mathcal{T}, v_0)$ be a naturally oriented semi-homogeneous rooted tree of degree (q, ℓ) with fixed label $\{\lambda_v\}_{v\in\mathcal{V}}$. Define an automorphism $h : \mathcal{T} \to \mathcal{T}$ by

$$\begin{aligned} h(v_0) &:= v_0, \\ h(v_{(\lambda_0,\lambda_1,\cdots,\lambda_k)}) &:= v_{(\lambda_0+1 (\mathrm{mod}\ q+1),\lambda_1,\cdots,\lambda_k)}. \end{aligned}$$

This automorphism preserves rationals, and we call it *the label preserving rotation around the root* v_0.

Let G be a group acting on a tree $\mathcal{T}$, and let us denote by h_g the automorphism defined by $g \in G$. Then the action of G on $\mathcal{T}$ is said to be *preserving rationals* if h_g preserve rationals for all $g \in G$.

Example 2.5.2. The natural action of $\mathrm{PSL}_2(\mathbb{Z})$ on the tree of $PSL_2(\mathbb{Z})$ preserves rationals.

An automorphism of a tree $\mathcal{T}$ which preserves rationals induces a bijection from the set of rational ends $\partial_R\mathcal{T}$ onto itself. Thus if an action of G on $\mathcal{T}$ preserves rationals, then G also acts on $\partial_R\mathcal{T}$ by $g[L_e] := [L_{ge}]$ for any $g \in G$ and $e \in \mathcal{E}^+$. Then we have the equivariant version of Theorem 2.2.1.

Theorem 2.5.1. *Let G be a group acting on a naturally oriented semi-homogeneous tree $\mathcal{T}$ and M a G-module. If the action of G preserves rationals, then there is an isomorphism*

$$\mathrm{Meas}_G(\partial\mathcal{T}, M) \simeq Z^1_{AS;G}(\partial_R\mathcal{T}, M).$$

Proof. The correspondence between $\mu \in \mathrm{Meas}(\partial\mathcal{T}, M)$ and $\varphi \in Z^1_{AS}(\partial_R\mathcal{T}; M)$ is given by

$$\mu(V(e)) = \varphi([L_e], [R_e])$$

for $e \in \mathcal{E}^+$. Then, by definition of the action of G on $\partial_R\mathcal{T}$, we have

$$\mu(gV(e)) = \mu(V(ge)) = \varphi([L_{ge}], [R_{ge}]) = \varphi(g[L_e], g[R_e]).$$

Hence μ is G-equivariant if and only if φ is G-equivariant, and the theorem follows. □

Note that this theorem is an extension of [16], where G is a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ and $\mathcal{T}$ is the tree of $\mathrm{PSL}_2(\mathbb{Z})$. The following propositions are also mentioned in [16] in the case of the tree of $\mathrm{PSL}_2(\mathbb{Z})$.

Proposition 2.5.1. *There exists a homomorphism*

$$\mathrm{Meas}_G(\partial\mathcal{T}, M) \xrightarrow{\sim} Z^1_{AS,G}(\partial_R\mathcal{T}; M) \to H^1(G; M)\ ;\ \mu \mapsto \varphi_\mu \mapsto \psi^{\varphi_\mu}.$$

Proof. By Example 1.2.1, there exists a homomorphism from $Z^1_{AS,G}(X; M)$ to $H^1(G; M)$. Combining this fact with the above theorem, we have the desired homomorphism. □

If a group G acts on $\partial_R\mathcal{T}$, any subgroup Γ of G also acts on $\partial_R\mathcal{T}$. Then the following proposition, which looks like Shapiro's lemma, holds.

Proposition 2.5.2. *Let Γ be a subgroup of G and M a Γ-module. Define*

$$\widehat{M} := \{\vartheta : G \to M \mid \vartheta(\gamma g) = \gamma\vartheta(g)\ \textit{for all}\ \gamma \in \Gamma\ \textit{and}\ g \in G\}$$

and give $\widehat{M}$ a G-module structure by pointwise addition and $(g\vartheta)(\gamma) := \vartheta(\gamma g)$. Then we have

$$Z^1_{AS,\Gamma}(\partial_R\mathcal{T}; M) \simeq Z^1_{AS,G}(\partial_R\mathcal{T}; \widehat{M}).$$

Proof. For $\varphi \in Z^1_{AS,\Gamma}(\partial_R\mathcal{T}; M)$, define

$$\widehat{\varphi}(\xi,\eta)(g) := \varphi(g\xi, g\eta),$$

where $(\xi,\eta) \in \partial_R\mathcal{T} \times \partial_R\mathcal{T}$ and $g \in G$. Then $\widehat{\varphi} \in Z^1_{AS,G}(\partial_R\mathcal{T}; \widehat{M})$.

Conversely, let $\widehat{\varphi} \in Z^1_{AS,G}(\partial_R\mathcal{T}; \widehat{M})$. Set

$$\varphi(\xi,\eta) := (\widehat{\varphi}(\xi,\eta))(1_G).$$

This defines an element in $Z^1_{AS,\Gamma}(\partial_R\mathcal{T}; M)$. The isomorphism is given by

$$Z^1_{AS,\Gamma}(\partial_R\mathcal{T}; M) \ni \varphi \longleftrightarrow \widehat{\varphi} \in Z^1_{AS,G}(\partial_R\mathcal{T}; \widehat{M}).$$

□

2.6 Example: the Case $\mathbb{Z}_{q+1} * \mathbb{Z}_{\ell+1}$

We will consider special examples with $G = \mathbb{Z}_{q+1} * \mathbb{Z}_{\ell+1}$, which include the tree of $\mathrm{PSL}_2(\mathbb{Z})$ as a special case. We fix generators τ, σ of $\mathbb{Z}_{q+1}$ and $\mathbb{Z}_{\ell+1}$, respectively. Thus these elements satisfy $\tau^{q+1} = \sigma^{\ell+1} = 1$. Let $\mathcal{T} = (\mathcal{T}, v_0)$ be a naturally oriented semi-homogeneous rooted tree of degree (q,ℓ) with $q \geq 2$, and fix a label of edges of $\mathcal{T}$. We define the action of G on $\mathcal{T}$ as follows. The action of $\tau \in \mathbb{Z}_{q+1}$ is defined by the label preserving rotation around the root v_0. Let w_0 be a vertex of $\mathcal{T}$ adjacent to v_0. Then we change the root v_0 with w_0 and define the action of $\sigma \in \mathbb{Z}_{\ell+1}$ as the label preserving rotation around the root w_0. Then the automorphisms $\tau : \mathcal{T} \to \mathcal{T}$ and $\sigma : \mathcal{T} \to \mathcal{T}$ preserve rationals. Since the composition of two rational preserving automorphisms also preserves rationals, the action of $\mathbb{Z}_{q+1} * \mathbb{Z}_{\ell+1}$ defined above preserves rationals, and hence descends to an action on $\partial_R\mathcal{T}$. We call this action *the natural action of* $\mathbb{Z}_{q+1} * \mathbb{Z}_{\ell+1}$. Note that when we consider the case of the tree of $\mathrm{PSL}_2(\mathbb{Z})$, the natural action of $\mathrm{PSL}_2(\mathbb{Z})$ coincides with the natural action defined above. Thus the results of this section are also generalizations of [16].

Lemma 2.6.1. *Let $\mathcal{T}$ and G be as above, and let $e_0 = \{v_0, w_0\} \in \mathcal{E}$. Then for every edge $e \in \mathcal{E}$, there exists a unique element $g \in G$ such that $e = ge_0 = \{gv_0, gw_0\}$.*

This lemma implies that for any primitive pair $(\xi,\eta) \in \partial_R\mathcal{T} \times \partial_R\mathcal{T}$, there exists a unique $g \in G$ such that $(\xi,\eta) = (g[L_{e_0}], g[R_{e_0}])$. Combining this fact with Lemma 2.2.6, we have the following proposition.

Proposition 2.6.1. *Each G-equivariant Alexander-Spanier cocycle $\varphi \in Z^1_{AS,G}(\partial_R\mathcal{T}; M)$ can be determined by its single value $\varphi([L_{e_0}], [R_{e_0}])$.*

Proof. Let $(\xi,\eta) \in \partial_R\mathcal{T} \times \partial_R\mathcal{T}$. By Lemma 2.2.6, we can assume that (ξ,η) is a primitive pair. Set $(\xi,\eta) = ([L_e], [R_e])$. Then the above lemma asserts that there exists $g \in G$ such that $e = ge_0$. Thus we have

$$\varphi(\xi,\eta) = \varphi([L_{ge_0}], [R_{ge_0}]) = \varphi(g[L_{e0}], g[R_{e_0}]) = g\varphi([L_{e_0}], [R_{e_0}]).$$

This proves the proposition. □

Similarly, for any finite index subgroup $\Gamma \subset G$, let $\{g_1, \cdots, g_d\}$ denote the coset representatives of $\Gamma \setminus G$. Then $\varphi(g_1[L_{e_0}], g_1[R_{e_0}]), \cdots, \varphi(g_d[L_{e_0}], g_d[R_{e_0}])$ determine the values of any $\varphi \in Z^1_{AS,G}(\partial_R\mathcal{T}; M)$.

By Proposition 2.6.1, for any $\varphi \in Z^1_{AS,G}(\partial_R\mathcal{T}; M)$, $\varphi([L_{e_0}], [R_{e_0}]) = 0$ implies $\varphi = 0$. Thus we have an injective group homomorphism

$$\iota : Z^1_{AS,G}(\partial_R\mathcal{T}; M) \to M \ ; \ \varphi \mapsto \varphi([L_{e_0}], [R_{e_0}]).$$

Let

$$\tilde{\tau} : M \to M \ ; \ m \mapsto m + \tau m + \cdots + \tau^q m,$$
$$\tilde{\sigma} : M \to M \ ; \ m \mapsto m + \sigma m + \cdots + \sigma^\ell m$$

be group homomorphisms. The following description of $Z^1_{AS,G}(\partial_R\mathcal{T}; M)$ is an extension of [16, Theorem 2.3].

Theorem 2.6.1. *There exists an exact sequence*

$$0 \longrightarrow Z^1_{AS,G}(\partial_R \mathcal{T}; M) \xrightarrow{\ \iota\ } M \xrightarrow{(\tilde{\tau},\tilde{\sigma})} M \times M$$

and, therefore, $Z^1_{AS,G}(\partial_R \mathcal{T}; M) \simeq \ker(\tilde{\tau}, \tilde{\sigma})$.

Proof. Since $L_{\tau e_0} \sim R_{e_0}$ and $L_{e_0} \sim R_{\tau^q e_0}$, we have

$$\tilde{\tau}\varphi([L_{e_0}], [R_{e_0}]) = \sum_{i=0}^{q} \varphi([L_{\tau^i e_0}], [R_{\tau^i e_0}]) = \varphi([L_{\tau e_0}], [R_{\tau^q e_0}]) = 0.$$

Similarly, the relations $L_{\sigma e_0} \sim R_{e_0}$ and $L_{e_0} \sim R_{\sigma^q e_0}$ imply

$$\tilde{\sigma}\varphi([L_{e_0}], [R_{e_0}]) = 0.$$

Hence $\mathrm{im}\iota \subset \ker(\tilde{\tau}, \tilde{\sigma})$.

Conversely, let $m \in \ker(\tilde{\tau}, \tilde{\sigma})$. By Lemma 2.6.1, every primitive pair is of the form $([L_{ge_0}], [R_{ge_0}])$ with a unique $g \in G$. Define a function $\tilde{\varphi}$ on the set of primitive pairs by

$$\tilde{\varphi}([L_{ge_0}], [R_{ge_0}]) := gm.$$

We prove that the function $\tilde{\varphi}$ satisfies the conditions in Lemma 2.4.1. Let v be a vertex of degree $q+1$. Then there exists a unique $g \in G$ such that $v = gv_0$. Then the element $g\tau g^{-1} \in G$ acts as the label preserving rotation around v, and the edges $e \in \mathcal{E}(v)$ are of the form $g\tau^i e_0$, where $i = 0, \cdots, q$. Thus

$$\sum_{i=0}^{q} \tilde{\varphi}([L^v_{e_i}], [R^v_{e_i}]) = \sum_{i=0}^{q} \tilde{\varphi}([L_{g\tau^i e_0}], [R_{g\tau^i e_0}]) = \sum_{i=0}^{q} \tau^i m = \tilde{\tau}(m) = 0.$$

The case of $\deg(v) = \ell + 1$ can also be treated in the same manner. Hence the function $\tilde{\varphi}$ extends to an Alexander-Spanier cocycle $\varphi \in Z^1_{AS}(\partial_R \mathcal{T}; M)$. Moreover, since

$$\varphi(g'[L_{ge_0}], g'[R_{ge_0}]) = \varphi([L_{g'ge_0}], g[R_{g'ge_0}]) = g'gm = g'\varphi([L_{ge_0}], [R_{ge_0}])$$

for any $g' \in G$, φ is G-equivariant. Therefore $\ker(\tilde{\tau}, \tilde{\sigma}) \subset \mathrm{im}\iota$, which completes the proof. □

Chapter 3

Modular Symbols for Hecke Triangle Groups

In this chapter, we introduce the notion of modular symbols for Hecke triangle groups which is a natural generalization of modular symbols for $\mathrm{PSL}_2(\mathbb{Z})$. We also introduce the trees of Hecke triangle groups and discuss the relation between Alexander-Spanier 1-cocycles on the space of rational ends of the trees and modular symbols for Hecke triangle groups.

3.1 Hecke Triangle Groups and Rosen Continued Fractions

In this section, we review some of the standard facts on Hecke triangle groups and Rosen continued fractions. The classical work here is due to Rosen [23].

Let $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper half-plane. A *Fuchsian group* is a discrete subgroup of

$$\mathrm{PSL}_2(\mathbb{R}) := \{\mathbb{H} \to \mathbb{H}\ ;\ z \mapsto \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{R},\ ad-bc=1\}.$$

If a Fuchsian group G satisfies $\mathrm{vol}(G \backslash \mathbb{H}) < \infty$, then G is said to be of the first kind. Note that we can identify $\mathrm{PSL}_2(\mathbb{R})$ with $\mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$. In abuse of notation, when we write a matrix, it means its class modulo scalar matrices.

Example 3.1.1. We define the *modular group* as

$$\mathrm{PSL}_2(\mathbb{Z}) := \{\mathbb{H} \to \mathbb{H}\ ;\ z \mapsto \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{Z},\ ad-bc=1\} = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}.$$

This is a Fuchsian group of the first kind. It is well-known that the modular group is generated by

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

that is, $\mathrm{PSL}_2(\mathbb{Z}) = \langle S, T \rangle$. Moreover, if we let

$$\sigma := S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \tau := ST^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

then $\mathrm{PSL}_2(\mathbb{Z})$ can be identified with the free product $\langle \sigma \rangle * \langle \tau \rangle \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

For any $\lambda > 0$, the group

$$G_\lambda := \langle S, \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \rangle$$

is a natural generalization of $\mathrm{PSL}_2(\mathbb{Z})$. The following classical result of Hecke tells us whether the group G_λ is a Fuchsian group of the first kind or not:

Theorem 3.1.1. (Hecke [8])
G_λ is a Fuchsian group of the first kind if and only if $\lambda = 2\cos(\pi/q)$ with $q \in \mathbb{Z}_{\geq 3}$ or $\lambda = 2$.

Note that when $q = 3$, the group $G_{2\cos(\pi/3)} = G_1$ is nothing less than the modular group. The group G_2 is a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ and we ignore this case.

Definition 3.1.1. For an integer $q \geq 3$, we define the *q-th Hecke triangle group G_q* as the subgroup of $\mathrm{PSL}_2(\mathbb{R})$ generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T_q := \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix},$$

where $\lambda_q := 2\cos(\pi/q)$.

By Theorem 3.1.1, the group G_q is a Fuchsian group of the first kind. As in the case of the modular group, we can write G_q as the free product of finite cyclic groups. Let

$$\sigma := S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \tau_q := ST_q^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & -\lambda_q \end{pmatrix}.$$

Then we have $\sigma^2 = \tau_q^q = id$ and $G_q = \langle\sigma\rangle * \langle\tau_q\rangle \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$.

It is well-known that the modular group is closely related to the theory of continued fractions. In his study on Hecke triangle groups ([23]), Rosen introduced another class of continued fractions, called Rosen continued fractions (or λ-continued fractions).

Definition 3.1.2. Let $\lambda > 0$. A formal expression

$$\lambda a_0 + \cfrac{-1}{\lambda a_1 + \cfrac{-1}{\lambda a_2 + \cfrac{-1}{\cfrac{\cdots}{+\cfrac{-1}{\lambda a_n}}}}} \tag{3.1}$$

is called a *finite λ-continued fraction* of order n. We will denote by $[a_0; a_1, a_2, \cdots, a_n]$ the λ-continued fraction of the form (3.1).

For a given λ-continued fraction $[a_0; a_1, \cdots, a_n]$ and $0 \leq k \leq n$, we call the λ-continued fraction

$$s_k := [a_0; a_1, \cdots, a_k]$$

a kth *segment* of the continued fraction. A *reminder* of the continued fraction is the λ-continued fraction

$$r_k := [a_k; a_{k+1}, \cdots, a_n].$$

Note that we can write

$$[a_k; a_1, \cdots, a_n] = [a_0; a_1, \cdots, a_{k-1}, r_k].$$

If $a_0, a_1, \cdots, a_n$ are actual numbers, one can write a λ-continued fraction as an ordinary fraction $\frac{p}{q}$. But such a representation is not unique. Thus we have to define how to represent λ-continued fractions as usual fractions. The representation we will use is called the *canonical representation* and defined inductively as follows. For a 0th order λ-continued fraction $[a_0] = \lambda a_0$, we take the fraction $\frac{\lambda a_0}{1}$ as our canonical representation. Suppose the canonical representations are defined for λ-continued fractions of order less than n. An nth order λ-continued fraction can be written as the form

$$[a_0; a_1, \cdots, a_n] = [a_0; r_1] = \lambda a_0 + \frac{-1}{r_1},$$

where $r_1 = [a_1; a_2, \cdots, a_n]$ is a λ-continued fraction of order $n-1$. By the induction hypothesis, we can represent r_1 as its canonical representation;

$$r_1 = \frac{p'}{q'}.$$

Then we have

$$[a_0; a_1, \cdots, a_n] = \lambda a_0 - \frac{q'}{p'} = \frac{\lambda a_0 p' - q'}{p'}.$$

We take this last fraction as our canonical representation of the λ-continued fraction $[a_0; a_1, \cdots, a_n]$. Thus if we set

$$[a_0; a_1, \cdots, a_n] = \frac{p}{q}, \quad r_1 = [a_1; a_0, \cdots, a_n] = \frac{p'}{q'},$$

we obtain the following expressions:

$$p = \lambda a_0 p' - q', \quad q = q'.$$

Hence, by induction, we have uniquely defined canonical representations of λ-continued fractions of all orders.

For a segment s_k of given λ-continued fraction $[a_0; a_1, \cdots, a_n]$, we denote by $\frac{p_k}{q_k}$ its canonical representation and call it the kth order *convergent* of the λ-continued fraction. Note that convergents of a λ-continued fraction $[a_0; a_1, \cdots, a_n]$ satisfy

$$\begin{cases} p_k = \lambda a_k p_{k-1} - p_{k-2}, \\ q_k = \lambda a_k q_{k-1} - q_{k-2} \end{cases} \tag{3.2}$$

and

$$p_k q_{k-1} - p_{k-1} q_k = -1, \tag{3.3}$$

where $1 \leq k \leq n$ and put $p_{-1} := 1, \ q_{-1} := 0$.

Definition 3.1.3. A real number $x \in \mathbb{R}$ is said to be a *finite λ-fraction* if there are integers $a_0, a_1, \cdots, a_n$ such that x is the canonical representation of the finite λ-continued fraction $[a_0; a_1, \cdots, a_n]$.

Note that we regard the point at infinity in $\mathbb{P}^1(\mathbb{R})$ as also a finite λ-fraction. In what follows, we discuss the relationship between Hecke triangle groups and Rosen continued fractions.

Proposition 3.1.1. *Let G_q be a Hecke triangle group and $\infty \in \mathbb{P}^1(\mathbb{R})$ the point at infinity. Then for any $g \in G_q$, $g(\infty)$ is a finite λ_q-fraction.*

Proof. Let $g \in G_q$. Since $G_q = \langle S, T_q \rangle$, g must be

$$g = T_q^{a_0} S T_q^{a_1} S \cdots T_q^{a_n}$$

or

$$g = T_q^{a_0} S T_q^{a_1} S \cdots T_q^{a_n} S,$$

where $a_0 \in \mathbb{Z}$ and $a_1, \cdots, a_n \in \mathbb{Z} \setminus \{0\}$. When $g = T_q^{a_0} S T_q^{a_1} S \cdots T_q^{a_n}$, we can write

$$g(\infty) = \lambda_q a_0 + \cfrac{-1}{\lambda_q a_1 + \cfrac{-1}{\cfrac{\cdots}{+\cfrac{-1}{\lambda_q a_n}}}}.$$

Thus $g(\infty)$ is the canonical representation of $[a_0; a_1, \cdots, a_n]$.

On the other hand, if $g = T_q^{a_0} S T_q^{a_1} S \cdots T_q^{a_n} S$, then

$$g(\infty) = \lambda_q a_0 + \cfrac{-1}{\lambda_q a_1 + \cfrac{-1}{\cfrac{\cdots}{+\cfrac{-1}{\lambda_q a_{n-1}}}}},$$

and, therefore, $g(\infty)$ is the canonical representation of $[a_0; a_1, \cdots, a_{n-1}]$. Hence $g(\infty)$ is a finite λ_q-fraction. $\square$

Lemma 3.1.1. *Let $\frac{p_k}{q_k}$ be the kth order convergent of $[a_0; a_1, \cdots, a_n]$, for $0 \leq k \leq n$. Then for any $z \in \mathbb{H}$, we have*

$$T_q^{a_0} S T_q^{a_1} S \cdots T_q^{a_n} S(z) = \frac{p_n z - p_{n-1}}{q_n z - q_{n-1}}.$$

Proof. We can write

$$T_q^{a_0} S T_q^{a_1} S \cdots T_q^{a_n} S(z) = \lambda_q a_0 + \cfrac{-1}{\lambda_q a_1 + \cfrac{-1}{\cfrac{\cdots}{+\cfrac{-1}{\lambda_q a_n + \cfrac{-1}{z}}}}}.$$

This can be considered as the canonical representation of $[a_0; a_1, \cdots, a_n, \frac{z}{\lambda_q}]$. On the other hand, by (3.2), the canonical representation of this finite λ_q-continued fraction is

$$\frac{\lambda_q (z/\lambda_q) p_n - p_{n-1}}{\lambda_q (z/\lambda_q) q_n - q_{n-1}} = \frac{p_n z - p_{n-1}}{q_n z - q_{n-1}},$$

and the lemma follows. □

Proposition 3.1.2. *Let $a_0 \in \mathbb{Z}$ and $a_1, \cdots, a_n \in \mathbb{Z} \backslash \{0\}$. Assume that the canonical representation of $[a_0; a_1, \cdots, a_n]$ is $\frac{a}{c}$ and that of $[a_0; a_1, \cdots, a_{n-1}]$ is $\frac{b}{d}$, respectively. Then*

$$\begin{pmatrix} a & -b \\ c & -d \end{pmatrix} \in G_q.$$

Proof. Let $g = \begin{pmatrix} a & -b \\ c & -d \end{pmatrix}$. Then, by the equation (3.3),

$$\det(g) = -(ad - bc) = -1(-1) = 1.$$

Hence $g \in \mathrm{SL}_2(\mathbb{R})$. Through the identification $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$, g gives an element of $\mathrm{PSL}_2(\mathbb{R})$.

Let us consider $T_q^{a_n} S T_q^{a_n} \cdots T_q^{a_n} S \in G_q$. Since the nth (resp. $(n-1)$th) convergent of $[a_0; a_1, \cdots, a_n]$ is $\frac{a}{c}$ (resp. $\frac{b}{d}$), Lemma 3.1.1 implies

$$T_q^{a_n} S T_q^{a_n} \cdots T_q^{a_n} S(z) = \frac{az - b}{cz - d} = \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} (z)$$

for any $z \in \mathbb{H}$. This shows that $g = T_q^{a_n} S T_q^{a_n} \cdots T_q^{a_n}$ as an element of $\mathrm{PSL}_2(\mathbb{H})$, and consequently $g \in G_q$. □

3.2 Modular Symbols for Hecke Triangle Groups

Modular symbols for $\mathrm{PSL}_2(\mathbb{Z})$ is first introduced by Y.I.Manin [14] to study the special values of L-functions of modular forms. In this section, we introduce the notion of modular symbols for Hecke triangle groups which is a natural generalization of modular symbols for $\mathrm{PSL}_2(\mathbb{Z})$. Modular symbols for Hecke triangle groups are also studied by G.Wiese [30]. His approach relies on the theory of group cohomology, whereas we derive almost all theorems about modular symbols from the properties of the boundary of trees obtained in the previous chapter.

Definition 3.2.1. Let G_q be a Hecke triangle group and M an additive group. Then an *M-valued modular symbol* for G_q is a function

$$\mu : G_q(\infty) \times G_q(\infty) \to M$$

satisfying

(i) $\mu(\alpha,\alpha) = 0$, $\mu(\alpha,\beta) + \mu(\beta,\alpha) = 0$,

(ii) $\mu(\alpha,\beta) = \mu(\beta,\gamma) = \mu(\gamma,\alpha) = 0$,

for all $\alpha,\beta,\gamma \in G_q(\infty)$.

Let us denote by $\mathbb{M}(M)$ the set of all M-valued modular symbols for G_q. Then $\mathbb{M}(M)$ becomes an additive group by point-wise addition. The Hecke triangle group G_q acts on $\mathbb{M}(M)$ by the rule

$$(\mu g)(\alpha,\beta) := \mu(g\alpha, g\beta).$$

For a finite index subgroup $\Gamma \subset G_q$ and a Γ-module M, a *Γ-equivariant M-valued modular symbol* is an M-valued modular symbol μ satisfying $(\mu\gamma)(\alpha,\beta) = \gamma \cdot \mu(\alpha,\beta)$ for any $\gamma \in \Gamma$. We denote by $\mathbb{M}_\Gamma(M)$ the set of all Γ-equivariant modular symbols.

Example 3.2.1. Let Γ be a finite index subgroup of G_q. For a cusp form f of weight 2 with respect to Γ, define

$$\mu_f(\alpha,\beta) := \int_\alpha^\beta f(z)dz.$$

Then μ_f is a Γ-equivariant $\mathbb{C}$-valued modular symbol, where we regard $\mathbb{C}$ as a trivial Γ-module.

To relate modular symbols to the results of the previous chapter, we introduce trees associated with Hecke triangle groups. Let $\mathcal{V}_q$ be the union of two orbits $G_q(i)$ and $G_q(\rho_q)$, where $i = \sqrt{-1}$ and $\rho_q = e^{\pi i/q}$ are fixed points of σ and τ_q, respectively. Define $\mathcal{E}_q := \{g(i), g(\rho_q)\}_{g\in G_q}$ and $\mathcal{T}_q := (\mathcal{V}_q, \mathcal{E}_q)$. We call $\mathcal{T}_q$ *the tree of the Hecke triangle group G_q*. This is a semi-homogeneous tree of degree $(q-1,1)$.

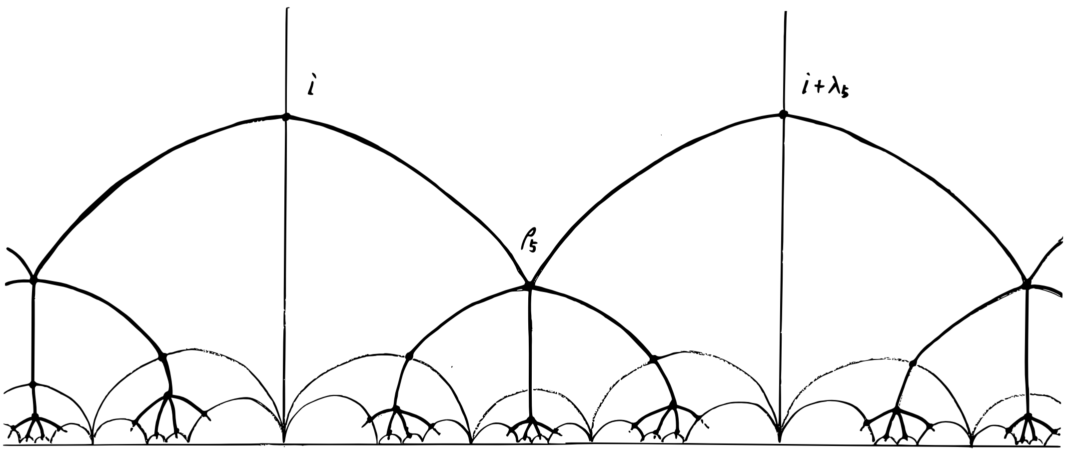

Figure 3.1: The tree of the Hecke triangle group G_5

The proof of the following proposition is a fairly straightforward generalization of that of [16, Lemma 5.2.1].

Proposition 3.2.1. *We can identify the space of rational ends of $\mathcal{T}_q$ with $G_q(\infty)$. Moreover, for any $q \geq 3$ and an additive group M, there is an isomorphism*

$$Z^1_{AS}(\partial_R\mathcal{T}_q; M) \simeq \mathbb{M}(M).$$

Proof. Consider the Farey tessellation of the hyperbolic plane $\mathbb{H}$ by G_q which translates the ideal q-gon with vertices $\{0, \tau_q(0), \tau_q^2(0), \cdots, \tau_q^{q-1}(0) = \infty\}$. Then the tree $\mathcal{T}_q$ has a vertex of degree q in each q-gon, a vertex of degree two bisecting each edge of the q-gons and q-edges in each q-gon joining the degree q vertex to each of the degree two vertices of the sides of the q-gon. If we fix a vertex v_0 (e.g. $v_0 := \rho_q$), then the space of rational ends coincides with the set of all vertices of the q-gons which is nothing less than $G_q(\infty)$.

Hence we can identify the set of rational ends and the orbit G_q. Then the cocycle condition for functions on $\partial_R\mathcal{T}_q \times \partial_R\mathcal{T}_q$ corresponds to the condition for modular symbols which are functions on $G_q(\infty) \times G_q(\infty)$. □

Since the above identification $\partial_R \mathcal{T} = G_q(\infty)$ is compatible with the action of G_q, we have the equivariant version of the above proposition:

Proposition 3.2.2. *For any finite index subgroup $\Gamma \subset G_q$ and Γ-module M, there is an isomorphism*

$$Z^1_{AS,\Gamma}(\partial_R \mathcal{T}_q; M) \simeq \mathbb{M}_\Gamma(M).$$

Note that the tree $\mathcal{T}_q$ is a special case of the examples we discussed in Section 2.6. Thus we can apply the theorems obtained in the previous chapter to modular symbols for G_q.

Proposition 3.2.3. *Let μ be an M-valued modular symbol for G_q. Then for any pair $(\alpha, \beta) \in G_q(\infty) \times G_q(\infty)$, there exist $g_1, \cdots, g_n \in G_q$ such that*

$$\mu(\alpha, \beta) = \mu(g_1(\infty), g_1(0)) + \cdots + \mu(g_n(\infty), g_n(0)).$$

Proof. This follows from Lemma 2.2.2, since a primitive pair corresponds to $(g(\infty), g(0))$ for some $g \in G_q$. □

Remark 3.2.1. *This proposition is known as Manin's trick when $q = 3$ which uses continued fraction expansions of rational numbers. For $q \geq 3$, we derive the above result from the consideration of the property of the semi-homogeneous tree. However, we can also prove the proposition by using Rosen continued fractions as follows, which is a natural extension of the method of Manin.*

Let $(\alpha, \beta) \in G_q(\infty) \times G_q(\infty)$. We can assume $\beta = \infty$. Since $\alpha \in G(\infty)$, there exists a finite λ_q-continued fraction $[a_0; a_1, \cdots, a_n]$ such that its canonical representation is α. Then let $\frac{p_k}{q_k}$ be the kth convergent of this λ_q-fraction and define

$$g_k := \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}$$

for $0 \leq k \leq n$, where $p_{-1} := 1, q_{-1} := 0$. Since $g_n(\infty) = \frac{p_n}{q_n} = \alpha$ and $g_0(0) = \frac{1}{0} = \infty$, we have

$$\mu(\alpha, \infty) = \mu(g_n(\infty), g_n(0)) + \cdots + \mu(g_1(\infty), g_1(0)),$$

where $g_k \in G_q$, by Proposition 3.1.2.

The following theorem, which is an extension of Theorem 2.3 of [16], is a direct consequence of Theorem 2.6.1 and Proposition 3.2.1.

Theorem 3.2.1. *For a G_q-module M, define a map $(\tilde{\tau}_q, \tilde{\sigma}) : M \to M \times M$ by*

$$\tilde{\tau}_q(m) := m + \tau_q m + \cdots + \tau_q^{q-1} m, \ \tilde{\sigma}(m) := m + \sigma m.$$

Then there exists an exact sequence

$$0 \longrightarrow \mathbb{M}_{G_q}(M) \xrightarrow{\ \iota\ } M \xrightarrow{(\tilde{\tau}, \tilde{\sigma})} M \times M$$

and, therefore, $\mathbb{M}_{G_q}(M) \simeq \ker(\tilde{\tau}, \tilde{\sigma})$.

Example 3.2.2. We introduce the notion of rational period functions for G_q, following Culp-Ressler [20]. Let $\mathbb{C}(z)$ be the set of rational functions. For $f \in \mathbb{C}(z)$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_q$, define the weight $2k$ slash operator $f|g$ by

$$(f|g)(z) := (cz + d)^{-2k} f(gz).$$

Then $\mathbb{C}(z)$ becomes a G_q-module. We call $f \in \mathbb{C}(z)$ a *rational period function of weight $2k$ for G_q* if it satisfies

$$f + f|\sigma = 0$$

and

$$f + f|\tau_q + \cdots f|\tau_q^{q-1} = 0.$$

By Theorem 3.2.1, this can be considered as a G_q-equivariant $\mathbb{C}(z)$-valued modular symbol for G_q.

Chapter 4

K-Theory for Boundary Crossed Products

4.1 K-Theory for Totally Disconnected Spaces

Let X be a totally disconnected compact Hausdorff space. In this section we compute the K-theory and the K-homology of the C^*-algebra $C(X)$. The following proposition is well-known (see [22]).

Proposition 4.1.1. *Let X be a totally disconnected compact Hausdorff space. Then we have*

(i) $K_0(C(X)) \xrightarrow{\sim} C(X, \mathbb{Z})$; $[p] \mapsto \mathrm{rank}(p)$,

(ii) $K_1(C(X)) = 0$.

In the sequel, we identify $K_0(C(X))$ with $C(X, \mathbb{Z})$ through the above isomorphism, and write $K_0(C(X)) = C(X, \mathbb{Z})$.

Let A be a separable C^*-algebra. Note that we have the index map

$$\mathrm{Index} : K^i(A) \to \mathrm{Hom}(K_0(A)), \mathbb{Z}); \ y \mapsto \mathrm{Index}(y) := (x \mapsto \langle x, y \rangle)$$

defined by the index pairing $K_i(A) \times K^i(A) \to \mathbb{Z}$.

Proposition 4.1.2. *For any totally disconnected compact Hausdorff space X, the index map is an isomorphism for $i = 0, 1$, that is,*

(i) $\mathrm{Index} : K^0(C(X)) \xrightarrow{\sim} \mathrm{Hom}(C(X, \mathbb{Z}), \mathbb{Z})$; $x \mapsto \mathrm{Index}(x)$,

(ii) $K^1(C(X)) = 0$.

Proof. By the universal coefficient theorem (Theorem 1.3.2), we have an exact sequence

$$0 \longrightarrow \mathrm{Ext}(K_i(C(X)), \mathbb{Z}) \longrightarrow K^{i+1}(C(X)) \xrightarrow{\mathrm{Index}} \mathrm{Hom}(K_{i+1}(C(X)), \mathbb{Z}) \longrightarrow 0$$

for $i = 0, 1$. Since $K_0(C(X)) = C(X, \mathbb{Z})$ and $K_1(C(X)) = 0$ are both free, $\mathrm{Ext}(K_i(C(X)), \mathbb{Z}) = 0$ for $i = 0, 1$. Hence the index maps are isomorphisms. □

For the boundary of a semi-homogeneous rooted tree, the following identifications hold by the results of Chapter 2.

Proposition 4.1.3. *Let $\mathcal{T}$ be a semi-homogeneous rooted tree and $\partial_R\mathcal{T}$ the space of rational ends for a fixed label of edges. Then we have*

$$\begin{aligned} Z^1_{AS}(\partial_R\mathcal{T}; \mathbb{Z}) &\simeq \mathrm{Meas}(\partial\mathcal{T}, \mathbb{Z}) \simeq \mathcal{C}(\mathcal{T}; \mathbb{Z}) \\ &\simeq \mathrm{Hom}^0(C(\partial\mathcal{T}, \mathbb{Z}), \mathbb{Z}) \simeq \{x \in K^0(C(\partial\mathcal{T})) \mid \mathrm{Index}(x)(1) = 0\}. \end{aligned}$$

Remark 4.1.1. *It is proved in [10] that the subgroup $\{x \in K^0(C(\partial\mathcal{T})) \mid \mathrm{Index}(x)(1) = 0\}$ coincides with the set of K-homology classes given by Pearson-Bellissard spectral triples. In particular, we can construct a Fredholm module over $C(\partial\mathcal{T})$ whose K-homology class corresponds to a given Alexander-Spanier cocycle on $\partial_R\mathcal{T}$.*

4.2 K-Theory for Crossed Products

In this section, we compute the K-theory and the K-homology of a reduced crossed product $C(X) \rtimes_r \Gamma$, where X is a totally disconnected compact Hausdorff space and Γ is a free group. The following computations are adapted from [4].

Proposition 4.2.1. *Let Γ be a free group acting on a totally disconnected compact Hausdorff space X. Then there is an isomorophism*

$$K_0(C(X) \rtimes_r \Gamma) \xrightarrow{\sim} C(X, \mathbb{Z})_\Gamma \ ; \ [p] \mapsto [\mathrm{rank}(p)]_\Gamma,$$

where $[\mathrm{rank}(p)]_\Gamma$ denotes the equivalence class of $\mathrm{rank}(p)$ in $C(X, \mathbb{Z})_\Gamma$.

Proof. Let $\{\gamma_1, \cdots, \gamma_n\}$ be a set of generators of Γ. Then the Pimsner-Voiculescu exact sequence for free groups (Theorem 1.3.3) gives

$$\begin{array}{ccccc} \bigoplus_{i=1}^n K_0(C(X)) & \xrightarrow{\varrho} & K_0(C(X)) & \longrightarrow & K_0(C(X) \rtimes_r \Gamma) \\ \uparrow & & & & \downarrow \\ K_1(C(X) \rtimes_r \Gamma) & \longleftarrow & K_1(C(X)) & \longleftarrow & \bigoplus_{i=1}^n K_1(C(X)), \end{array}$$

where $\varrho = \sum_{i=1}^n (1 - \gamma_{i*})$, and the morphism $K_0(C(X)) \to K_0(C(X) \rtimes_r \Gamma)$ is induced by the natural inclusion $C(X) \hookrightarrow C(X) \rtimes_r \Gamma$. Since $K_1(C(X)) = 0$, the above exact sequence becomes

$$0 \to K_1(C(X) \rtimes_r \Gamma) \to \bigoplus_{i=1}^n K_0(C(X)) \xrightarrow{\varrho} K_0(C(X)) \to K_0(C(X) \rtimes_r \Gamma) \to 0.$$

Hence

$$K_0(C(X))/\mathrm{im}\varrho \xrightarrow{\sim} K_0(C(X) \rtimes_r \Gamma) \ ; \ \langle [p] \rangle \mapsto [p],$$

where $\langle [p] \rangle$ denotes the equivalence class of $[p]$ in $K_0(C(X))/\mathrm{im}\varrho$.

Since $\mathrm{im}\varrho = \langle 1 - \gamma \mid \gamma \in \Gamma \rangle$ and the identification $K_0(C(X)) = C(X, \mathbb{Z})$ is given by $[p] \mapsto [\mathrm{rank}(p)]$, we have an isomorphism

$$K_0(C(X))/\mathrm{im}\varrho \xrightarrow{\sim} C(X, \mathbb{Z})_\Gamma \ ; \ \langle [p] \rangle \mapsto [p]_\Gamma.$$

It follows that

$$K_0(C(X) \rtimes_r \Gamma) \xrightarrow{\sim} C(X, \mathbb{Z})_\Gamma \ ; \ [p] \mapsto [\mathrm{rank}(p)]_\Gamma.$$

□

Remark 4.2.1. *The above proof gives more, namely for any equivalence class of a projection in $K_0(C(X) \rtimes_r \Gamma)$, we can take a projection in $C(X)$ as its representative.*

Let $\phi : C(X, \mathbb{Z})_\Gamma \to \mathbb{Z}$ be a group homomorphism. Define $\mu_\phi(V) := \phi([\chi_V]_\Gamma)$ for any clopen set $V \subset X$, where $[\chi_V]_\Gamma$ denotes the equivalence class of χ_V in $C(X, \mathbb{Z})_\Gamma$. If $V, V' \subset X$ are two clopen sets with $V \cap V' = \emptyset$, then

$$\mu_\phi(V \cup V') = \phi([\chi_{V \cup V'}]_\Gamma) = \phi([\chi_V + \chi_{V'}]_\Gamma) = \phi([\chi_V]_\Gamma) + \phi([\chi_V']_\Gamma) = \mu_\phi(V) + \mu_\phi(V').$$

Since $\gamma\chi_X = \chi_X$ for any $\gamma \in \Gamma$, we have $[\chi_X]_\Gamma = 0$. Hence

$$\mu_\phi(X) = \phi([\chi_X]_\Gamma) = 0.$$

This shows that μ_ϕ is a measure on X. Moreover, since $[\gamma\chi_V] = [\chi_V]$ for any $\gamma \in \Gamma$, it follows that

$$\mu_\phi(\gamma V) = \phi([\chi_{\gamma V}]) = \phi([\gamma\chi_V]) = \phi([\chi_V]) = \mu_\phi(V).$$

Therefore $\mu_\phi \in \mathrm{Meas}(X, \mathbb{Z})^\Gamma$.

Proposition 4.2.2. *Let X be a totally disconnected compact Hausdorff space and Γ a free group acting on X. Then the maps*

$$K^0(C(X) \rtimes_r \Gamma) \xrightarrow[x \mapsto Index(x)_*]{\sim} \mathrm{Hom}(C(X,\mathbb{Z})_\Gamma, \mathbb{Z}) \xrightarrow[\phi \mapsto \mu_\phi]{\sim} \mathrm{Meas}(X,\mathbb{Z})^\Gamma$$

are isomorphisms.

Proof. By the universal coefficient theorem (Theorem 1.3.2), there exists a short exact sequence

$$0 \longrightarrow \mathrm{Ext}(K_1(C(X) \rtimes_r \Gamma), \mathbb{Z}) \longrightarrow K^0(C(X) \rtimes_r \Gamma) \xrightarrow{\mathrm{Index}} \mathrm{Hom}(K_0(C(X) \rtimes_r \Gamma), \mathbb{Z}) \longrightarrow 0.$$

Since the exact sequence in the proof of Proposition4.2.1 shows that $K_1(C(X) \rtimes_r \Gamma)$ is free, and $K_0(C(X) \rtimes_r \Gamma) \simeq C(X,\mathbb{Z})_\Gamma$, we have

$$K^0(C(X) \rtimes_r \Gamma) \xrightarrow{\sim} \mathrm{Hom}(C(X,\mathbb{Z})_\Gamma, \mathbb{Z}) \ ; \ x \mapsto \mathrm{Index}(x)_*.$$

We will show that the map $\mathrm{Hom}(C(X,\mathbb{Z})_\Gamma, \mathbb{Z}) \to \mathrm{Meas}(X;\mathbb{Z})^\Gamma \ ; \ \phi \mapsto \mu_\phi$ is an isomorphism.

Let $\mu \in \mathrm{Meas}(X,\mathbb{Z})^\Gamma$. Then we have a homomorphism $\phi_\mu : C(X,\mathbb{Z}) \to \mathbb{Z} \ ; \ f \mapsto \int f d\mu$ defined in Proposition 2.1.1. Let us first prove that the homomorphism ϕ_μ factors through $C(X,\mathbb{Z})_\Gamma$. By the change of variable formula (Proposition 2.1.1) and the Γ-invariance of μ, we have

$$\phi_\mu(\gamma f) = \int \gamma f d\mu = \int f d(\mu\gamma) = \int f d\mu = \phi_\mu(f).$$

This gives $\phi_\mu(\gamma f - f) = 0$ for any $f \in C(X,\mathbb{Z})$ and $\gamma \in \Gamma$. Hence ϕ_μ factors through $C(X,\mathbb{Z})_\Gamma$. By abuse of notation, we use the same letter ϕ_μ for the induced map $C(X,\mathbb{Z})_\Gamma \to \mathbb{Z}$.

For any Γ-invariant measure μ and clopen set $V \subset X$, we have

$$\mu_{\phi_\mu}(V) = \phi_\mu([\chi_V]_\Gamma) = \phi_\mu(\chi_V) = \int \chi_V d\mu = \mu(V).$$

On the other hand, for any $\phi \in \mathrm{Hom}(C(X,\mathbb{Z})_\Gamma, \mathbb{Z})$ and $\chi_V \in C(X,\mathbb{Z})$, we have

$$\phi_{\mu_\phi}([\chi_V]_\Gamma) = \int \chi_V d\mu_\phi = \mu_\phi(V) = \phi([\chi_V]_\Gamma).$$

This shows that the mappings

$$\mu \mapsto \phi_\mu \text{ and } \phi \mapsto \mu_\phi$$

are inverses of each other, which completes the proof. □

4.3 Hecke Equivariant Isomorphisms

The notion of Hecke operators in K-theory was introduced by Mesland and Şengün [18]. In this section, we show that the isomorphisms proved in Proposition 4.2.1 and 4.2.2 are compatible with the action of the Hecke operators.

Let G be a group acting on a compact Hausdorff space X, and let Γ be a subgroup of G. We introduce the notion of Hecke operators on $K_0(C(X) \rtimes_r \Gamma)$ and $K^0(C(X) \rtimes_r \Gamma)$, following Mesland and Şengün [18]. Suppose $C_G(\Gamma) := \{g \in G \mid g^{-1}\Gamma g \approx \Gamma\} \neq \emptyset$. Set $A := C(X)$. Then, for any $g \in C_G(\Gamma)$, we can define a conditional expectation $\rho_g : A \rtimes_r \Gamma \to A \rtimes_r \Gamma_{g^{-1}}$ by $\sum_{\gamma \in \Gamma} a_\gamma \gamma \mapsto \sum_{\gamma \in \Gamma_{g^{-1}}} a_\gamma \gamma$, where $\Gamma_g := \Gamma \cap g\Gamma g^{-1}$. We thus get a Hilbert $(A \rtimes_r \Gamma, A \rtimes_r \Gamma_{g^{-1}})$-bimodule $(A \rtimes_r \Gamma)_{\rho_{g^{-1}}}$. Then we define the *Hecke operators* T_g on $K_0(A \rtimes_r \Gamma)$ and $K^0(A \rtimes_r \Gamma)$ by

$$T_g : K_0(A \rtimes_r \Gamma) \xrightarrow{\otimes (A \rtimes_r \Gamma)_{\rho_{g^{-1}}}} K_0(A \rtimes_r \Gamma_{g^{-1}}) \xrightarrow{\mathrm{Ad}_{g*}} K_0(A \rtimes_r \Gamma_g) \to K_0(A \rtimes_r \Gamma)$$

and

$$T_g : K^0(A \rtimes_r \Gamma) \to K^0(A \rtimes_r \Gamma_g) \xrightarrow{\mathrm{Ad}_g^*} K^0(A \rtimes_r \Gamma_{g^{-1}}) \xrightarrow{(A \rtimes_r \Gamma)_{\rho_{g^{-1}}} \otimes} K^0(A \rtimes_r \Gamma).$$

Remark 4.3.1. *By the composition of $*$-homomorphisms $A \rtimes_r \Gamma_g \xrightarrow{\mathrm{Ad}_{g^{-1}}} A \rtimes_r \Gamma_{g^{-1}} \hookrightarrow A \rtimes_r \Gamma$, we can define the interior tensor product $T_g^{\Gamma} := (A \rtimes_r \Gamma)_{\rho_{g^{-1}}} \otimes_{\mathrm{Ad}_{g^{-1}}} A \rtimes_r \Gamma$, which is an $A \rtimes_r \Gamma$-bimodule. Then the Hecke operator defined above is just the Kasparov product with the class $[T_g^{\Gamma}] \in KK_0(A \rtimes_r \Gamma, A \rtimes_r \Gamma)$. The associativity of the Kasparov product implies $\langle T_g x, y\rangle = \langle x, T_g y\rangle$ for any $x \in K_0(A \rtimes_r \Gamma)$ and $y \in K_0(A \rtimes_r \Gamma)$, where $\langle \cdot, \cdot \rangle$ denotes the index pairing.*

Lemma 4.3.1. *Let G be a group acting on a totally disconnected compact Hausdorff space X and $\Gamma \subset G$ a subgroup. Suppose Γ is a free group and $C_G(\Gamma) \neq \emptyset$. Then for any $g \in C_G(\Gamma)$, the map*

$$K_0(C(X) \rtimes_r \Gamma) \xrightarrow{\otimes (C(X)\rtimes_r \Gamma)_{\rho_{g^{-1}}}} K_0(C(X) \rtimes_r \Gamma_{g^{-1}})$$

is given by

$$[p] \mapsto \sum_{i=1}^{d} [\delta_i^{-1}(p)],$$

where $\{\delta_i\}_{i=1}^d$ denotes a complete set of representatives of $\Gamma/\Gamma_{g^{-1}}$.

Proof. Let $\{\delta_1, \cdots, \delta_d\}$ be a complete set of representatives of the coset $\Gamma/\Gamma_{g^{-1}}$. Set $A := C(X)$. Then the map

$$C_c(\Gamma, A) \to \bigoplus_{i=1}^{d} C_c(\Gamma_{g^{-1}}, A) \ ; \ \sum_{\gamma\in\Gamma} a_\gamma \gamma \mapsto \left(\sum_{t \in \Gamma_{g^{-1}}} \delta_i^{-1}(a_{\delta_i t})\delta_i t \right)_{i=1,\cdots,d}$$

extends to an isomorphism $A \rtimes_r \Gamma \to \bigoplus_{i=1}^d A \rtimes_r \Gamma_{g^{-1}}$ between Hilbert $(A \rtimes_r \Gamma_g)$-modules. Since X is totally disconnected compact Hausdorff, each element in $K_0(A \rtimes_r \Gamma)$ is given by a projection in A (see Remark 4.2.1). On the other hand, the above isomorphism sends a projection $p \in A$ to $(\delta_i^{-1}(p))_{i=1}^d$. Thus we have an isomorphism between Hilbert $(A \rtimes_r \Gamma_g)$-modules:

$$p(A \rtimes_r \Gamma)_{\rho_{g^{-1}}} \simeq \delta_1^{-1}(p)(A \rtimes_r \Gamma_{g^{-1}}) \oplus \cdots \oplus \delta_d^{-1}(p)(A \rtimes_r \Gamma_{g^{-1}}).$$

This shows that the map $\otimes(A \rtimes_r \Gamma)_{\rho_{g^{-1}}} : K_0(A \rtimes_r \Gamma) \to K_0(A \rtimes_r \Gamma_{g^{-1}})$ is of the form $[p] \mapsto \sum_{i=1}^d [\delta_i^{-1}(p)]$, which completes the proof. □

From this lemma, we can describe the action of Hecke operators on $K_0(C(X) \rtimes_r \Gamma)$ explicitly.

Corollary 4.3.1. *Let G be a group acting on a totally disconnected compact Hausdorff space X and $\Gamma \subset G$ a subgroup. Suppose Γ is a free group and $C_G(\Gamma) \neq \emptyset$. Let $\{\delta_i\}_{i=1}^d$ be a complete set of representatives of $\Gamma/\Gamma_{g^{-1}}$. Then the composition*

$$T_g : K_0(C(X) \rtimes_r \Gamma) \xrightarrow{\otimes (C(X)\rtimes_r \Gamma)_{\rho_{g^{-1}}}} K_0(C(X) \rtimes_r \Gamma_{g^{-1}}) \xrightarrow{\mathrm{Ad}_{g*}} K_0(C(X) \rtimes_r \Gamma_g) \to K_0(C(X) \rtimes_r \Gamma)$$

is given by

$$[p] \longmapsto \sum_{i=1}^{d} [\delta_i^{-1}(p)] \longmapsto \sum_{i=1}^{d} [g\delta_i^{-1}(p)] \longmapsto \sum_{i=1}^{d} [g\delta_i^{-1}(p)].$$

In Proposition 4.2.1 and 4.2.2, we obtain the isomorphisms $K(C(X) \rtimes_r \Gamma) \simeq C(X, \mathbb{Z})_\Gamma$ and $K(C(X) \rtimes_r \Gamma) \simeq \mathrm{Meas}(X, \mathbb{Z})^\Gamma$ for a free group Γ acting on a totally disconnected compact Hausdorff space X. Note that we can write the invariants and the coinvariants in terms of group cohomology:

$$C(X, \mathbb{Z})_\Gamma = H_0(\Gamma, C(X, \mathbb{Z})), \ \mathrm{Meas}(X, \mathbb{Z})^\Gamma = H^0(\Gamma, \mathrm{Meas}(X, \mathbb{Z})).$$

Thus there are Hecke operators (see Subsection 1.2.3)

$$T_g : C(X, \mathbb{Z})_\Gamma = H_0(\Gamma, C(X, \mathbb{Z})) \to H_0(\Gamma, C(X, \mathbb{Z})) = C(X, \mathbb{Z})_\Gamma$$

and

$$T_g : \mathrm{Meas}(X, \mathbb{Z})^\Gamma = H^0(\Gamma, \mathrm{Meas}(X, \mathbb{Z})) \to H^0(\Gamma, \mathrm{Meas}(X, \mathbb{Z})) = \mathrm{Meas}(X, \mathbb{Z})^\Gamma.$$

Theorem 4.3.1. *Let G be a group acting on a totally disconnected compact Hausdorff space X and $\Gamma \subset G$ a subgroup. Suppose Γ is a free group and $C_G(\Gamma) \neq \emptyset$. Then, for any $g \in C_G(\Gamma)$, the isomorphisms in Proposition 4.2.1 and 4.2.2 are Hecke equivariant, that is, the following diagrams commute*

$$\begin{array}{ccccccc} K_0(C(X) \rtimes_r \Gamma) & \xrightarrow{T_g} & K_0(C(X) \rtimes_r \Gamma) & \quad & K^0(C(X) \rtimes_r \Gamma) & \xrightarrow{T_g} & K^0(C(X) \rtimes_r \Gamma) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ C(X,\mathbb{Z})_\Gamma & \xrightarrow{T_g} & C(X,\mathbb{Z})_\Gamma, & & \mathrm{Meas}(X,\mathbb{Z})^\Gamma & \xrightarrow{T_g} & \mathrm{Meas}(X,\mathbb{Z})^\Gamma. \end{array}$$

Proof. Let $g \in C_G(\Gamma)$. By Corollary 4.3.1, the action of the Hecke operator T_g on $K_0(C(X) \rtimes_r \Gamma)$ is of the form

$$T_g([p]) = \sum_{i=1}^{d} [g\delta_i^{-1}(p)],$$

where $p \in C(X)$ is a projection. On the other hand, the Hecke operator T_g on $H_0(\Gamma, C(X,\mathbb{Z})) = C(X,\mathbb{Z})_\Gamma$ is of the form

$$T_g([f]_\Gamma) = [\sum_{i=1}^{d} g\delta_i^{-1}(f)]_\Gamma = \sum_{i=1}^{d} [g\delta_i^{-1}(f)]_\Gamma,$$

where $f \in C(X,\mathbb{Z})$. Since the isomorphism $K_0(C(X) \rtimes_r \Gamma) \xrightarrow{\sim} C(X,\mathbb{Z})_\Gamma$ is given by $[p] \mapsto [\mathrm{rank}(p)]_\Gamma$, we have

$$T_g([p]) = \sum_{i=1}^{d} [g\delta_i^{-1}(p)] \mapsto \sum_{i=1}^{d} [g\delta_i^{-1}(\mathrm{rank}(p))]_\Gamma = T_g([\mathrm{rank}(p)]_\Gamma).$$

This shows that the diagram of the left commute.

To prove the commutativity of the diagram of the right, we consider the maps

$$T_g^* : \mathrm{Hom}(K_0(A \rtimes_r \Gamma), \mathbb{Z}) \to \mathrm{Hom}(K_0(A \rtimes_r \Gamma), \mathbb{Z}) \ ; \ \phi \mapsto \phi \circ T_g,$$
$$T_g^* : \mathrm{Hom}(C(X,\mathbb{Z})_\Gamma, \mathbb{Z}) \to \mathrm{Hom}(C(X,\mathbb{Z})_\Gamma, \mathbb{Z}) \ ; \ \phi \mapsto \phi \circ T_g.$$

Since Hecke operators are compatible with the index pairing, we have a commutative diagram

$$\begin{array}{ccc} K^0(C(X) \rtimes_r \Gamma) & \xrightarrow{T_g} & K^0(C(X) \rtimes_r \Gamma) \\ {\scriptstyle \mathrm{Index}} \downarrow \wr & & {\scriptstyle \mathrm{Index}} \downarrow \wr \\ \mathrm{Hom}(K_0(A \rtimes_r \Gamma), \mathbb{Z}) & \xrightarrow{T_g^*} & \mathrm{Hom}(K_0(A \rtimes_r \Gamma), \mathbb{Z}) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{Hom}(C(X,\mathbb{Z})_\Gamma, \mathbb{Z}) & \xrightarrow{T_g^*} & \mathrm{Hom}(C(X,\mathbb{Z})_\Gamma, \mathbb{Z}). \end{array}$$

Here $\mathrm{Hom}(C(X,\mathbb{Z})_\Gamma, \mathbb{Z})$ is isomorphic to $\mathrm{Meas}(X,\mathbb{Z})^\Gamma$ as proved in Proposition 4.2.2. This isomorphism is given by $\mu \mapsto \phi_\mu$, where $\phi_\mu([f]_\Gamma) := \int f d\mu$ for $[f]_\Gamma \in C(X,\mathbb{Z})_\Gamma$. Then

$$\phi_\mu(T_g([f]_\Gamma)) = \phi_\mu(\sum_{i=1}^{d} [g\delta_i^{-1}(f)]_\Gamma) = \sum_{i=1}^{d} \int g\delta_i^{-1} f d\mu = \sum_{i=1}^{d} \int f d(\mu g \delta_i^{-1}) = \sum_{i=1}^{d} \phi_{\mu g \delta_i^{-1}}([f]_\Gamma)$$

by the change of variable formula (Lemma 2.1.1). On the other hand, the Hecke operator

$$T_g : \mathrm{Meas}(X,\mathbb{Z})^\Gamma = H^0(X, \mathrm{Meas}(X,\mathbb{Z})) \to H^0(X, \mathrm{Meas}(X,\mathbb{Z})) = \mathrm{Meas}(X,\mathbb{Z})^\Gamma$$

is of the form

$$T_g(\mu) = \sum_{i=1}^{d} \mu g \delta_i^{-1}.$$

Since the map $\mu \mapsto \phi_\mu$ is a group homomorphism, we have

$$\phi_{T_g(\mu)} = \phi_{\sum_{i=1}^{d} \mu g \delta_i^{-1}} = \sum_{i=1}^{d} \phi_{\mu g \delta_i^{-1}}.$$

This shows that the following diagram commute:

$$\begin{array}{ccc} \mathrm{Hom}(C(X,\mathbb{Z})_\Gamma, \mathbb{Z}) & \xrightarrow{T_g^*} & \mathrm{Hom}(C(X,\mathbb{Z})_\Gamma, \mathbb{Z}) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{Meas}(X,\mathbb{Z})^\Gamma & \xrightarrow{T_g} & \mathrm{Meas}(X,\mathbb{Z})^\Gamma. \end{array}$$

Hence the diagram of the right commute, which completes the proof. □

Remark 4.3.2. *In the case where* Γ *is a torsion-free finite index subgroup of* $\mathrm{PSL}_2(\mathbb{Z})$*, Proposition 3.2.2 and Theorem 2.5.1 show that there is an isomorphism* $\mathbb{M}_\Gamma(\mathbb{Z}) \simeq \mathrm{Meas}(\partial\mathcal{T}, \mathbb{Z})^\Gamma$*, where* $\mathcal{T}$ *is the tree of* $\mathrm{PSL}_2(\mathbb{Z})$*, and we consider* $\mathbb{Z}$ *as a trivial* Γ*-module. Since the Hecke operators on* $\mathbb{M}_\Gamma(\mathbb{Z})$ *described in [16] coincide with the Hecke operators on* $\mathrm{Meas}(\partial\mathcal{T}, \mathbb{Z})^\Gamma = H^0(\Gamma; \mathrm{Meas}(\partial\mathcal{T}, \mathbb{Z}))$ *under the isomorphism* $\mathbb{M}_\Gamma(\mathbb{Z}) \simeq \mathrm{Meas}(\partial\mathcal{T}, \mathbb{Z})^\Gamma$*, the following diagram commute:*

$$\begin{array}{ccc} K^0(C(\partial\mathcal{T}) \rtimes_r \Gamma) & \xrightarrow{T_g} & K^0(C(\partial\mathcal{T}) \rtimes_r \Gamma) \\ \downarrow \wr & & \downarrow \wr \\ \mathbb{M}_\Gamma(\mathbb{Z}) & \xrightarrow{T_g} & \mathbb{M}_\Gamma(\mathbb{Z}). \end{array}$$

On the other hand, Manin and Marcolli investigated the relations between Γ*-equivariant modular symbols and the* K*-theory of the boundary crossed product* $C(\partial\mathcal{T}) \rtimes_r \Gamma$*. Note that these relations remain valid for* $Z^1_{AS,G}(\partial_R\mathcal{T}; M)(\simeq \mathrm{Meas}_G(\partial\mathcal{T}, M))$ *and the* K*-theory of* $C(\partial\mathcal{T}) \rtimes_r G$*, where* G *is a group acting on the boundary of a semi-homogeneous tree* $\mathcal{T}$*. It would be interesting to investigate these relations combining with the action of Hecke operators.*

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