A Study on the Relations for Special Values of Some Multiple Zeta Functions
(様々な多重ゼータ関数の特殊値の関係式についての研究)

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Abstract

In this thesis, we give some results on multiple zeta values, multiple zeta star values and zeta functions of root systems. In the first section, we give an overview of the study of multiple zeta values, multiple zeta star values and zeta functions of root systems. In the second section which is based on [18], we give relations among multiple zeta values with the proof. In the third section, we give relations among multiple zeta star values. These relations can be regarded as analogues of relations for multiple zeta values introduced in the second section. In the fourth section which is based on [19], we give an evaluation of double series of Tornheim type. From this evaluation, we can obtain the parity result for zeta values of the root system of type G_2 which was suggested by Komori–Matsumoto–Tsumura in [27]. In the final section, we show a property for general multiple series. This property can be regarded as a generalized parity result. Moreover, we can obtain the parity result for Mordell–Tornheim type of multiple zeta values as a special case of Theorem 5.1

Contents

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1 Introduction

1.1 Multiple zeta values

1.1.1 Euler's work

The special values of the Riemann zeta function

$$\zeta(s) = \sum_{n>0} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

have interesting properties and have been recognized as an important subject of study in number theory for a long time. In particular, the following famous formula for the Riemann zeta values at positive even integers is well known. For any positive integer k, we have

$$\zeta(2k) = -\frac{1}{2}\frac{B_{2k}}{(2k)!}(2\pi\sqrt{-1})^{2k}, \tag{1.1}$$

where B_n is the n-th Seki–Bernoulli number defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} B_n \frac{t^n}{n!}.$$

This formula was given by Euler. Though the formula for the Riemann zeta values at even integers is given explicitly, any formula for those at odd integers is not known yet.

In 1776, Euler [10] considered the following series which can be regarded as a generalization of the Riemann zeta values:

$$s_{m,n} = 1 + \frac{1}{2^m}\left(1 + \frac{1}{2^n}\right) + \frac{1}{3^m}\left(1 + \frac{1}{2^n} + \frac{1}{3^n}\right) + \frac{1}{4^m}\left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n}\right) + \dots. \tag{1.2}$$

In his paper, he showed the relations

$$s_{2,1} = 2\zeta(3)$$

and

$$s_{m,n} + s_{n,m} = \zeta(m)\zeta(n) + \zeta(m+n).$$

1.1.2 Multiple zeta values

It is said that Euler's studies are the origin of the study of multiple zeta values. After his work, multiple zeta values have not been studied for a long time. However, it has begun to be known that multiple zeta values are closely related to knot theory, arithmetic geometry, mathematical physics and so on from around 1990, thence, multiple zeta values have been studied and are still studied very actively even today. Multiple zeta values were first introduced by Hoffman [13] and Zagier [53] independently.

Definition 1.1 (Multiple zeta values)**.** For positive integers $k_1, \dots, k_{r-1}$ and k_r with $k_r \geq 2$, the multiple zeta value $\zeta(k_1, \dots, k_r)$ is defined by the following r-ple series:

$$\begin{aligned}\zeta(k_1, \dots, k_r) :&= \sum_{1 \leq n_1 < n_2 < \cdots < n_r} \frac{1}{n_1^{k_1} n_2^{k_2} \dots n_r^{k_r}} \\ &= \sum_{m_1, \dots, m_r > 0} \frac{1}{m_1^{k_1}(m_1 + m_2)^{k_2} \dots (m_1 + \cdots + m_r)^{k_r}}.\end{aligned}$$

The series with replacing k_i by a complex variable s_i is called the Euler–Zagier type of multiple zeta function. For an index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$, we define two quantities. One of these is called the weight which is defined by the sum of entries $k = k_1 + \cdots + k_r$ and denoted by $\mathrm{wt}(\mathbf{k})$. The other is called the depth which is defined by the number r and denoted by $\mathrm{dep}(\mathbf{k})$. For multiple zeta values $\zeta(k_1, \dots, k_r)$, we also call $k_1 + \cdots + k_r$ weight and r depth. As we can see, multiple zeta values of depth one coincide with the Riemann zeta values. From easy calculations, we can see that the number of indices of weight k and depth r is $\binom{k-2}{r-1}$, and the number of indices of weight k is 2^{k-2}. Here, we give some examples of multiple zeta values.

r \ k	2	3	4	5
1	$\zeta(2)$	$\zeta(3)$	$\zeta(4)$	$\zeta(5)$
2	-	$\zeta(1,2)$	$\zeta(1,3), \zeta(2,2)$	$\zeta(1,4), \zeta(2,3), \zeta(3,2)$
3	-	-	$\zeta(1,1,2)$	$\zeta(1,1,3), \zeta(1,2,2), \zeta(2,1,2)$
4	-	-	-	$\zeta(1,1,1,2)$

Similar to the Riemann zeta values, there are some multiple zeta values whose values are known concretely. For example, for positive integers k and n, we have

$$\zeta(\underbrace{2k, 2k, \dots, 2k}_{n}) = C_n^{(k)} \frac{(2\pi\sqrt{-1})^{2nk}}{(2nk)!}, \tag{1.3}$$

where $C_n^{(k)}$ is the rational number defined recursively by

$$C_0^{(k)} = 1, C_n^{(k)} = \frac{1}{2n} \sum_{m=1}^{n} (-1)^m \binom{2nk}{2mk} B_{2mk} C_{n-m}^{(k)} \quad (n \geq 1).$$

Besides the above formula, for positive integer n, the formula

$$\zeta(\underbrace{1, 3, \dots, 1, 3}_{2n}) = \frac{2\pi^{4n}}{(4n+2)!} \tag{1.4}$$

is also known which was discovered by Borwein–Bradley–Broadhurst–Lisoněk [3] and Kontsevich–Zagier [29]. Furthermore, Bowman–Bradley [2] and Muneta [34] also gave an explicit formula for certain multiple zeta values.

1.1.3 $\mathbb{Q}$-vector space spanned by multiple zeta values

Here, let us introduce a $\mathbb{Q}$-vector space $\mathcal{Z}_k$.

Definition 1.2. For a non negative integer k, we define the $\mathbb{Q}$-vector space $\mathcal{Z}_k$ by

$$\mathcal{Z}_0 := \mathbb{Q}, \quad \mathcal{Z}_1 := \{0\},$$
$$\mathcal{Z}_k := \sum_{\substack{r \geq 1 \\ k_1, \ldots, k_{r-1} \geq 1, k_r \geq 2 \\ k_1 + \cdots + k_r = k}} \mathbb{Q} \cdot \zeta(k_1, \ldots, k_r) \quad (k \geq 2).$$

Namely, for $k \geq 2$, $\mathcal{Z}_k$ is a $\mathbb{Q}$-vector space spanned by multiple zeta values of weight k. Let us see the examples of $\mathcal{Z}_k$ in the case of low weight. $\mathcal{Z}_2$ is spanned by $\zeta(2)$. Hence, we have $\mathcal{Z}_2 = \mathbb{Q}\zeta(2)$. $\mathcal{Z}_3$ is spanned by $\zeta(3)$ and $\zeta(1,2)$. However, since $\zeta(1,2) = \zeta(3)$ (see equation (1.5) with $k = 3$), we have $\mathcal{Z}_3 = \mathbb{Q}\zeta(3)$. $\mathcal{Z}_4$ is spanned by $\zeta(4), \zeta(1,3), \zeta(2,2)$ and $\zeta(1,1,2)$. From the formula (1.3) with $k = 1$ and $n = 2$, we have $\zeta(2,2) \in \mathbb{Q}\pi^4$, and from the formula (1.4) with $n = 1$, we also have $\zeta(1,3) \in \mathbb{Q}\pi^4$. Moreover, since $\zeta(1,1,2) = \zeta(4)$ (see equation (1.7)), we have $\mathcal{Z}_4 = \mathbb{Q}\zeta(4)$ from these observations and the equation (1.1) with $k = 2$.

Zagier [53] suggested the following conjecture on the dimension of $\mathcal{Z}_k$ from numerical computations.

Conjecture 1.3 (Zagier [53]). *For any non negative integer k,*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k \stackrel{?}{=} d_k,$$

where $\{d_k\}_{k \geq 0}$ is a sequence defined by

$$d_0 := 1, d_1 := 0, d_2 := 1,$$
$$d_k := d_{k-2} + d_{k-3} \quad (k \geq 3).$$

Let us compare d_k and the number of indices of weight k.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2^{k-2}	-	-	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192
d_k	1	0	1	1	1	2	2	3	4	5	7	9	12	16	21	28

Table 1.1: 2^{k-2} and d_k

From the above observations, the conjecture is true for $k = 0, 1, 2, 3$ and 4. So far, the conjecture has not been solved for $k \geq 5$. It is to be stressed that even a single number $k \geq 5$ which satisfies $\dim_{\mathbb{Q}} \mathcal{Z}_k \geq 2$ has never been discovered yet. However, as an affirmative evidence for the conjecture, the upper bound of the dimension of $\mathcal{Z}_k$ is already proved.

Theorem 1.4 (Goncharov [11], Terasoma [44], Deligne–Goncharov [7]). *For any non negative integer k, we have*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k.$$

Here, let us introduce one more interesting $\mathbb{Q}$-vector space $\mathcal{Z}$.

Definition 1.5. We define $\mathcal{Z}$ as follows:

$$\mathcal{Z} := \sum_{k \geq 0} \mathcal{Z}_k.$$

As we will check later, we can see that $\mathcal{Z}_k \cdot \mathcal{Z}_\ell \subset \mathcal{Z}_{k+\ell}$ for any non negative integers k and ℓ. Hence, the $\mathbb{Q}$-vector space $\mathcal{Z}$ has a structure as a $\mathbb{Q}$-algebra. Moreover, the following conjecture was suggested by Zagier.

Conjecture 1.6 (Zagier [53]).

$$\mathcal{Z} \overset{?}{=} \bigoplus_{k \geq 0} \mathcal{Z}_k.$$

This conjecture means that there does NOT exist any rational linear relations among multiple zeta values of different weight. Besides the above result on the dimension of $\mathcal{Z}_k$, the very important and interesting result on the generators of $\mathcal{Z}$ conjectured by Hoffman [14] is known.

Theorem 1.7 (Brown [6]). *Let H_k be the set of all multiple zeta values of weight k and entries consist of only 2 and/or 3, and H be the union of H_k. Namely,*

$$H_k := \{\zeta(k_1, \ldots, k_r) \mid r \geq 1, k_1 + \cdots + k_r = k, k_i \in \{2, 3\}\},$$
$$H := \bigcup_{k \geq 2} H_k.$$

Then any multiple zeta values can be written as a rational linear combination of the elements of H.

Let h_k be the number of elements of H_k. To see an interesting point of this result, let us count h_k. By definition, $h_2 = 1, h_3 = 1$ and $h_4 = 1$. The all elements of H_5 are obtained by adding 3 as a last entry to the all elements of H_2 and adding 2 as a last entry to the all elements of H_3. Hence, $h_5 = h_3 + h_2$. In the same way, we obtain the recurrence formula $h_k = h_{k-2} + h_{k-3}$ for positive integer $k \geq 5$. From this recurrence formula and the initial values, we have $h_k = d_k$. Therefore, the result of Brown means that we could get the candidate of the basis of $\mathcal{Z}_k$ if we believe Conjecture 1.6. Actually, Hoffman conjectured the above statement from $h_k = d_k$ and the numerical experiments. Moreover, the following conjecture which is called {2, 3}-basis conjecture was suggested by Hoffman based on the numerical experiments.

Conjecture 1.8 (Hoffman [14]). *The set $H \cup \{1\}$ is a basis of $\mathcal{Z}$ over $\mathbb{Q}$.*

1.1.4 Relations among multiple zeta values

Here, let us consider the meaning of the Table 1.1 . As we can see, the number of generators of $\mathcal{Z}_k$ (namely 2^{k-2}) is very large, while the expected dimension of $\mathcal{Z}_k$ (namely d_k) is very small. What does this observation mean? That is there exists a lot of independent relations among multiple zeta values. So far, since some applications to the problem on modular forms and arithmetic geometry are known, to find and study the relations among multiple zeta values is recognized as an important problem, and indeed, many relations among them have been discovered. For instance, let us consider the product of the Riemann zeta values

$$\zeta(k)\zeta(\ell) = \sum_{m>0} \frac{1}{m^k} \sum_{n>0} \frac{1}{n^\ell}.$$

Decomposing the sum into $m < n, n < m$ and $m = n$, we have

$$\begin{aligned}\zeta(k)\zeta(\ell) &= \left(\sum_{0<m<n} + \sum_{0<n<m} + \sum_{0<m=n} \right) \frac{1}{m^k n^\ell} \\ &= \zeta(k,\ell) + \zeta(\ell,k) + \zeta(k+\ell).\end{aligned}$$

This formula is called a harmonic (stuffle) product formula. In general, calculating the product of two multiple zeta values of weight k and ℓ in the same rule, we obtain $\mathcal{Z}_k \cdot \mathcal{Z}_\ell \subset \mathcal{Z}_{k+\ell}$. Then, we see that $\mathcal{Z}$ is a $\mathbb{Q}$-algebra with this product. As other relations, Euler discovered the relation which connects double zeta values and a Riemann zeta value. For any positive integer k, we have

$$\sum_{\substack{k_1\geq 1, k_2 \geq 2 \\ k_1+k_2=k}} \zeta(k_1,k_2) = \zeta(k). \tag{1.5}$$

This formula is called Euler's sum formula and means that the sum of all double zeta values of weight k is equal to $\zeta(k)$. The simplest case of this formula is when $k = 3$, and that is $\zeta(1,2) = \zeta(3)$. This contributes to the fact $\mathcal{Z}_3 = \mathbb{Q}\zeta(3)$. How about the case of general depth? In 1997, Granville [12] gave a generalization of Euler's sum formula to general depth r, and that is called just the sum formula.

Theorem 1.9 (Sum formula, Granville [12]). *For positive integers r and k with $k > r$, we have*

$$\sum_{\substack{k_1,\ldots,k_{r-1}\geq 1, k_r \geq 2 \\ k_1+\cdots+k_r=k}} \zeta(k_1,\ldots,k_r) = \zeta(k).$$

The author feels that the sum formula is a really beautiful and simple relation, and has deep affection to this formula. This is because the starting point of the author's study of multiple zeta values is to have learned this formula in a talk of Professor Yasuo Ohno on 23rd July, 2013.

Let us return to the main subject. In 2008, Ohno–Zudilin [39] gave an interesting analogy of Euler's sum formula. That is the sum of double zeta values with some weights is equal to a rational multiple of the Riemann zeta value. The formulas like this are called weighted sum formulas.

Theorem 1.10 (Ohno–Zudilin [39]). *For any positive integer k with $k \geq 3$, we have*

$$\sum_{\substack{k_1,k_2\geq 1\\ k_1+k_2=k-1}} 2^{k_2+1}\zeta(k_1,k_2+1)=(k+1)\zeta(k).$$

So far, some generalizations of the above relation are known. One of them was given by Eie–Liaw–Ong [9], and their relation is as follows.

Theorem 1.11 (Eie–Liaw–Ong [9]). *For positive integers k and r with $k > 2r$, we have*

$$\sum_{\substack{k_i\geq 1\\ k_1+\cdots+k_{2r}=k-1}} \sum_{j=1}^{r} 2^{k_{2j}+1}\zeta(k_1,\ldots,k_{2r-1},k_{2r}+1)=(k+2r-1)\zeta(k).$$

In Section 2, we give a relation which is a generalization of the relation of Eie–Liaw–Ong.

To state another important relation which is called the duality relation, we introduce the integral representation of multiple zeta values. One can check that multiple zeta values have the following integral representation:

$$\zeta(k_1,k_2,\ldots,k_r)=\int\cdots\int_{0<t_1<t_2<\cdots<t_k<1}\omega_1(t_1)\omega_2(t_2)\ldots\omega_k(t_k),$$

where $k=k_1+\cdots+k_r$,

$$\omega_i(t)=\begin{cases}\dfrac{dt}{1-t}, & i\in\{1,k_1+1,k_1+k_2+1,\ldots,k_1+\cdots+k_{r-1}+1\},\\[2mm] \dfrac{dt}{t}, & \text{otherwise.}\end{cases}$$

This integral representation is the typical example of iterated path integrals which was introduced by Chen. We give some examples of the integral representation of multiple zeta values.

Example 1.12.

$$\begin{aligned}\zeta(4)&=\int\cdots\int_{0<t_1<t_2<t_3<t_4<1}\frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}\frac{dt_4}{t_4},\\ \zeta(1,1,2)&=\int\cdots\int_{0<t_1<t_2<t_3<t_4<1}\frac{dt_1}{1-t_1}\frac{dt_2}{1-t_2}\frac{dt_3}{1-t_3}\frac{dt_4}{t_4},\\ \zeta(1,3,2)&=\int\cdots\int_{0<t_1<t_2<t_3<t_4<t_5<t_6<1}\frac{dt_1}{1-t_1}\frac{dt_2}{1-t_2}\frac{dt_3}{t_3}\frac{dt_4}{t_4}\frac{dt_5}{1-t_5}\frac{dt_6}{t_6}.\end{aligned} \tag{1.6}$$

This integral representation plays an important role in the study of multiple zeta values though it is not difficult to derive that. To see this point, let us change variables $t_i = 1 - s_i$ for $i = 1, 2, 3$ and 4 in equation (1.6). Now, we can see that the modulus of Jacobian is 1 and the integral region becomes $0 < s_4 < s_3 < s_2 < s_1 < 1$. Hence, we have

$$\begin{aligned} \zeta(1,1,2) &= \int\cdots\int_{0<t_1<t_2<t_3<t_4<1} \frac{dt_1}{1-t_1}\frac{dt_2}{1-t_2}\frac{dt_3}{1-t_3}\frac{dt_4}{t_4} \\ &= \int\cdots\int_{0<s_4<s_3<s_2<s_1<1} \frac{ds_1}{s_1}\frac{ds_2}{s_2}\frac{ds_3}{s_3}\frac{ds_4}{1-s_4} \\ &= \zeta(4). \end{aligned}$$

We could get the relation

$$\zeta(1,1,2) = \zeta(4). \tag{1.7}$$

The family of relations which is obtained by changing variables $t_i = 1 - s_i$ in the integral representation is the duality relation. To state the duality theorem precisely, we define the dual index.

Definition 1.13 (Dual index)**.** Any index $\mathbf{k} = (k_1, k_2, \ldots, k_r) \in \mathbb{N}^r$ with $k_r \geq 2$ can be written as follows uniquely by using positive integers a_i and b_i:

$$\mathbf{k} = (\underbrace{1,\ldots,1}_{a_1-1}, b_1+1, \underbrace{1,\ldots,1}_{a_2-1}, b_2+1, \ldots, \underbrace{1,\ldots,1}_{a_s-1}, b_s+1).$$

Namely, we express an index so that the entries which are greater than or equal to 2 do not appear side by side. Then, we define the dual index $\mathbf{k}'$ of $\mathbf{k}$ by

$$\mathbf{k}' = (\underbrace{1,\ldots,1}_{b_s-1}, a_s+1, \underbrace{1,\ldots,1}_{b_{s-1}-1}, a_{s-1}+1, \ldots, \underbrace{1,\ldots,1}_{b_1-1}, a_1+1).$$

By definition, we can check the following properties.

$$\begin{aligned} (\mathbf{k}')' &= \mathbf{k}, \\ \mathrm{wt}(\mathbf{k}) &= \mathrm{wt}(\mathbf{k}'), \\ \mathrm{dep}(\mathbf{k}) + \mathrm{dep}(\mathbf{k}') &= \mathrm{wt}(\mathbf{k}) = \mathrm{wt}(\mathbf{k}'). \end{aligned}$$

Theorem 1.14 (Duality theorem)**.** *For an index $\mathbf{k} = (k_1, \ldots, k_r)$ with $k_r \geq 2$, we write*

$$\zeta(\mathbf{k}) = \zeta(k_1, \ldots, k_r)$$

under the above notation. Then, for any index $\mathbf{k} = (k_1, \ldots, k_r)$ with $k_r \geq 2$ and the dual index $\mathbf{k}'$ of $\mathbf{k}$, we have

$$\zeta(\mathbf{k}) = \zeta(\mathbf{k}').$$

Here, let us give two easy examples of the dual indices and duality relations.

Example 1.15. (i) For positive integers $r, \ell \geq 1$, the dual index of $(\underbrace{1,\dots,1}_{r-1}, \ell+1)$ is $(\underbrace{1,\dots,1}_{\ell-1}, r+1)$. Hence, we have

$$\zeta(\underbrace{1,\dots,1}_{r-1}, \ell+1) = \zeta(\underbrace{1,\dots,1}_{\ell-1}, r+1). \tag{1.8}$$

In particular, putting $\ell = 1$, we have

$$\zeta(\underbrace{1,\dots,1}_{r-1}, 2) = \zeta(r+1). \tag{1.9}$$

This example is a generalization of the relation (1.7).

(ii) The index $\mathbf{k} = (1,2,2)$ can be written as

$$\mathbf{k} = (\underbrace{1,\dots,1}_{2-1}, 1+1, \underbrace{1,\dots,1}_{1-1}, 1+1),$$

namely, $s = 2, a_1 = 2, b_1 = 1, a_2 = 1$ and $b_2 = 1$. Hence, the dual index of $(1,2,2)$ is

$$(\underbrace{1,\dots,1}_{1-1}, 1+1, \underbrace{1,\dots,1}_{1-1}, 2+1) = (2,3).$$

Therefore, from the duality theorem, we have $\zeta(1,2,2) = \zeta(2,3)$.

At the end of this subsection, we introduce the formula which is called Ohno's relation and given by Ohno [38]. This formula is a generalization of the sum formula, duality theorem and Hoffman's relation [13].

Theorem 1.16 (Ohno's relation, Ohno [38])**.** *For an index $\mathbf{k} = (k_1, \dots, k_r)$ with $k_r \geq 2$ and non negative integer ℓ, we have*

$$\sum_{\substack{\varepsilon_i \geq 0 \\ \varepsilon_1 + \cdots + \varepsilon_r = \ell}} \zeta(k_1 + \varepsilon_1, \dots, k_r + \varepsilon_r) = \sum_{\substack{\varepsilon'_i \geq 0 \\ \varepsilon'_1 + \cdots + \varepsilon'_{r'} = \ell}} \zeta(k'_1 + \varepsilon'_1, \dots, k'_{r'} + \varepsilon'_{r'}),$$

where $(k'_1, \dots, k'_{r'})$ is the dual index of $\mathbf{k}$.

For positive integers k and r with $k > r$, putting $\mathbf{k} = (\underbrace{1,\dots,1}_{r-1}, 2)$ and $\ell = k - r - 1$, we obtain the sum formula. Putting $\ell = 0$, we can obtain the duality relation.

1.1.5 Parity result

We conclude this section with the explanation of the parity result. What is the parity result? It is said that the origin of the parity result is the following question which Euler considered.

Question 1.17. *What sort of multiple zeta values or the sum of multiple zeta values can be written in terms of the Riemann zeta values?*

Euler gave the following expression as one of the answers to this question.

$$\zeta(1,k-1)=\frac{k-1}{2}\zeta(k)-\frac{1}{2}\sum_{j=2}^{k-2}\zeta(j)\zeta(k-j)\quad (k\geq 3).$$

Furthermore, we have already given some answers to this question, for instance, equations (1.3), (1.4), the sum formula, the weighted sum formulas, the special case of the duality relation (1.9) and so on. We can classify these examples into two categories. The first is the sum formula and the weighted sum formulas. The second is the Euler's expression, equations (1.3), (1.4) and (1.9). In the former category, those give the expression of certain sum of multiple zeta values. On the other hand, in the latter category, those give the expression of each multiple zeta value itself as a polynomial of the Riemann zeta values. The following famous result which can be regarded as a generalization of the latter category is known.

Theorem 1.18 (Parity result, Tsumura [47], Ihara–Kaneko–Zagier [17])**.** *When k and r are of different parity, multiple zeta values of weight k and depth r can be written as a rational linear combination of the products of multiple zeta values of depth lower than r.*

They proved this property in totally different way from each other. Tsumura used a very analytic method and Ihara–Kaneko–Zagier used a very algebraic method. Let us give some examples.

Example 1.19.

$$\begin{aligned}
\zeta(1,6)&=3\zeta(7)-\zeta(2)\zeta(5)-\zeta(3)\zeta(4),\\
\zeta(1,2,5)&=\frac{7}{4}\zeta(2,6)+\frac{7}{2}\zeta(5)\zeta(3)-\zeta(2)\zeta(3)^2-\frac{289}{144}\zeta(8),\\
\zeta(1,2,7)&=\frac{9}{4}\zeta(2,8)-\zeta(2)\zeta(2,6)+\frac{9}{2}\zeta(7)\zeta(3)+\frac{9}{4}\zeta(5)^2-\zeta(4)\zeta(3)^2-\frac{377}{60}\zeta(10).
\end{aligned}$$

Historically, the above theorem in the case $r=2$ was proved by Tornheim [45] in 1950. About 50 years later, Borwein–Borwein–Girgensohn [4] in 1995 and Huard–Williams–Zhang [15] in 1996 gave an explicit formula of double zeta values independently. Moreover, in 1996, Borwein–Girgensohn proved the case $r=3$. The reason why the above theorem is called the parity result is that the theorem is the result for the property of multiple zeta values when weight and depth have different parity. Mysteriously, it is not yet revealed what is the essential connection between the phenomenon like this and the difference of the parity of weight and depth.

1.2 Multiple zeta star values

1.2.1 Multiple zeta star values

As one of analogues of multiple zeta values, the notion of multiple zeta star values are known. These are also real numbers and also defined by multiple series which is obtained by replacing the strict inequality by the inequality with the equality.

Definition 1.20 (Multiple zeta star values)**.** For positive integers $k_1, \ldots, k_{r-1}$ and k_r with $k_r \geq 2$, the multiple zeta star value $\zeta^\star(k_1, \ldots, k_r)$ is defined by the following r-ple series:

$$\zeta^\star(k_1, \ldots, k_r) := \sum_{1 \leq n_1 \leq n_2 \leq \cdots \leq n_r} \frac{1}{n_1^{k_1} n_2^{k_2} \ldots n_r^{k_r}}.$$

The series $s_{m,n}$ (see equation (1.2)) which Euler considered is nothing but the double zeta star values $\zeta^\star(n, m)$. From definitions, we can check that multiple zeta values and multiple zeta star values can be expressed as a linear combination of the others with rational coefficients by decomposing the inequality into the strict inequality and the equality. For example,

$$\begin{aligned}\zeta^\star(k_1, k_2) &= \sum_{1 \leq n_1 \leq n_2} \frac{1}{n_1^{k_1} n_2^{k_2}} = \left(\sum_{1 \leq n_1 < n_2} + \sum_{1 \leq n_1 = n_2} \right) \frac{1}{n_1^{k_1} n_2^{k_2}} \\ &= \zeta(k_1, k_2) + \zeta(k_1 + k_2),\end{aligned} \tag{1.10}$$

$$\begin{aligned}\zeta^\star(k_1, k_2, k_3) &= \sum_{1 \leq n_1 \leq n_2 \leq n_3} \frac{1}{n_1^{k_1} n_2^{k_2} n_3^{k_3}} \\ &= \left(\sum_{1 \leq n_1 < n_2 < n_3} + \sum_{1 \leq n_1 < n_2 = n_3} + \sum_{1 \leq n_1 = n_2 < n_3} + \sum_{1 \leq n_1 = n_2 = n_3} \right) \frac{1}{n_1^{k_1} n_2^{k_2} n_3^{k_3}} \\ &= \zeta(k_1, k_2, k_3) + \zeta(k_1, k_2 + k_3) + \zeta(k_1 + k_2, k_3) + \zeta(k_1 + k_2 + k_3).\end{aligned} \tag{1.11}$$

As mentioned above, multiple zeta star values are analogues of multiple zeta values. Hence, it is natural that we want to know whether there exists the properties and relations of multiple zeta star values similar to those of multiple zeta values or not. Indeed, the following two analogies are already known. For any positive integers k and n, we have

$$\zeta^\star(\underbrace{2k, \ldots, 2k}_{n}) \in \mathbb{Q}\pi^{2nk}$$

and

$$\zeta^\star(\underbrace{1, 3, \ldots, 1, 3}_{2n}) \in \mathbb{Q}\pi^{4n}.$$

The first one was proved by Hoffman [13], Zlobin [58] and Muneta [33], and the second one was proved by Muneta [33]. Moreover, Kondo–Saito–Tanaka [28] gave multiple zeta star analogue of the result of Bowman–Bradley [2].

1.2.2 $\mathbb{Q}$-vector space spanned by multiple zeta star values

Since multiple zeta values and multiple zeta star values can be written as a rational linear combination of the others, the $\mathbb{Q}$-vector space spanned by multiple zeta star values is the same as the $\mathbb{Q}$-vector space spanned by multiple zeta values. Although it is not clear that multiple zeta values whose components consist of only 2 and/or 3 can be expressed as a rational linear combination of multiple zeta star values whose components also consist of only 2 and/or 3, the following interesting conjecture is known.

Conjecture 1.21 (Ihara–Kajikawa–Ohno–Okuda [16])**.** *Let $H_k^\star$ be the set of all multiple zeta star values of weight k and all entries are 2 and/or 3, and $H^\star$ be the union of $H_k^\star$. Namely,*

$$H_k^\star := \{\zeta^\star(k_1, \ldots, k_r) \mid r \geq 1, k_1 + \cdots + k_r = k, k_i \in \{2,3\}\},$$
$$H^\star := \bigcup_{k \geq 2} H_k^\star.$$

Then, the set $H^\star \cup \{1\}$ is a basis of $\mathcal{Z}$ over $\mathbb{Q}$.

They have already the reliable evidence of this conjecture up to weight 16 by the numerical experimentation.

1.2.3 Relations among multiple zeta star values

As in the case of multiple zeta values, we want to find some relations among multiple zeta star values. In particular, we want to know whether there exist some relations among multiple zeta star values which are analogues of relations among multiple zeta values or not. As one of the answers to this question, the following sum formula is known.

Theorem 1.22 (Sum formula)**.** *For positive integers r and k with $k > r$, we have*

$$\sum_{\substack{k_1, \ldots, k_{r-1} \geq 1, k_r \geq 2 \\ k_1 + \cdots + k_r = k}} \zeta^\star(k_1, \ldots, k_r) = \binom{k-1}{r-1} \zeta(k).$$

The interesting point is that this formula is quite similar to the sum formula for multiple zeta values but we cannot obtain this formula from that for multiple zeta values by only rewriting like equations (1.10) and (1.11). So far, not so many relations among multiple zeta star values which are analogues of relations among multiple zeta values are known. In that situation, the author discovered a relation among multiple zeta star values which can be regarded as an analogue of the relation among multiple zeta values which were also given by the author. We discuss them in Section 3.

At the end of this subsection, we introduce the integral representation of multiple zeta star values which plays an important role even in the proof of the author's result. From the series which defines multiple zeta star values, it is easy to see that multiple zeta star values also have the integral representation similar to that of multiple zeta values.

$$\zeta^\star(k_1,k_2,\dots,k_r) = \int\cdots\int_{0<t_1<t_2<\cdots<t_k<1} \omega_1(t_1)\omega_2(t_2)\dots\omega_k(t_k),$$

where $k = k_1+\cdots+k_r$,

$$\omega_i(t) = \begin{cases} \dfrac{dt}{1-t} & , \quad i=1, \\ \dfrac{dt}{t(1-t)} & , \quad i\in\{k_1+1, k_1+k_2+1, \dots, k_1+\cdots+k_{r-1}+1\}, \\ \dfrac{dt}{t} & , \quad \text{otherwise.} \end{cases}$$

Example 1.23.

$$\zeta^\star(1,1,2) = \int\cdots\int_{0<t_1<t_2<t_3<t_4<1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2(1-t_2)}\frac{dt_3}{t_3(1-t_3)}\frac{dt_4}{t_4},$$
$$\zeta^\star(1,3,2) = \int\cdots\int_{0<t_1<t_2<t_3<t_4<t_5<t_6<1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2(1-t_2)}\frac{dt_3}{t_3}\frac{dt_4}{t_4}\frac{dt_5}{t_5(1-t_5)}\frac{dt_6}{t_6}.$$

As we see from the above examples, changing variables $t_i = 1-s_i$ does not give the duality relations for multiple zeta star values. However, the result which is regarded as "duality" for multiple zeta star values is known.

Theorem 1.24 (Kaneko [20], Kaneko–Ohno [21])**.** *For positive integers $r,\ell \geq 2$, we have*

$$\zeta^\star(\underbrace{1,\dots,1}_{r-1},\ell) - (-1)^{r+\ell}\zeta^\star(\underbrace{1,\dots,1}_{\ell-1},r) \in \mathbb{Q}[\zeta(2),\zeta(3),\zeta(5),\zeta(7),\dots].$$

This theorem implies that the sum of suitable multiple zeta star values with sign is a polynomial of the Riemann zeta values and can be regarded as an analogue of equation (1.8).

1.3 Zeta functions of root systems

1.3.1 The Witten zeta function

In the paper which Zagier introduced multiple zeta values, he also defined the Witten zeta function $\zeta_W(s,\mathfrak{g})$ for a complex semisimple Lie algebra $\mathfrak{g}$ and a complex variable s as follows:

$$\zeta_W(s,\mathfrak{g}) := \sum_{\varphi} \frac{1}{(\dim\varphi)^s},$$

where φ runs over all finite dimensional irreducible representations of $\mathfrak{g}$. Why did Zagier name it the Witten zeta function? The special values of the Witten zeta function at positive even integers express the volume of a certain moduli space which appears in quantum gauge theory considered by Witten. Moreover, Witten [52] gave a volume formula. The number theoretical meaning of Witten's volume formula can be written as the following statement:

Theorem 1.25 (Witten [52])**.** *Let n be the number of positive roots of $\mathfrak{g}$. For any positive integer k, we have*

$$\zeta_W(2k, \mathfrak{g}) \in \mathbb{Q}\pi^{2nk}.$$

Let us see some examples of the Witten zeta function which Zagier gave in his paper.

Example 1.26.

$$\begin{aligned}
\zeta_W(s, \mathfrak{sl}(2)) &= \sum_{m>0} \frac{1}{m^s}, \\
\zeta_W(s, \mathfrak{sl}(3)) &= \sum_{m,n>0} \frac{2^s}{m^s n^s (m+n)^s}, \\
\zeta_W(s, \mathfrak{so}(5)) &= \sum_{m,n>0} \frac{6^s}{m^s n^s (m+n)^s (m+2n)^s}.
\end{aligned}$$

Gazing at the above examples, we find some factors with the same variable in the denominator. From this observation, we may think out multi variable version of these examples. That is exactly zeta functions of root systems which were defined by Komori–Matsumoto–Tsumura. Moreover, their argument is based on the fact that simple Lie algebras have several nice properties. Let us see the process by which zeta functions of root systems are defined.

1.3.2 Zeta functions of root systems

First, when $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\zeta_W(s, \mathfrak{g})$ is equal to the product of $\zeta_W(s, \mathfrak{g}_1)$ and $\zeta_W(s, \mathfrak{g}_2)$, namely, we have

$$\zeta_W(s, \mathfrak{g}_1 \oplus \mathfrak{g}_2) = \zeta_W(s, \mathfrak{g}_1)\zeta_W(s, \mathfrak{g}_2).$$

Second, any semisimple Lie algebra can be decomposed into a finite direct sum of simple Lie algebras. From these two facts, we can see that to study the Witten zeta functions for semisimple algebras, it is sufficient to study those for simple Lie algebras. From now on, let $\mathfrak{g}$ be a simple Lie algebra. By the Killing–Cartan theory, it is known that any simple Lie algebra can be categorized into types A_r, B_r, C_r, D_r which are called the classical types or types E_6, E_7, E_8, F_4, G_2 which are called the exceptional types. Here, let us introduce some notations. Let r be the rank of $\mathfrak{g}$, and denote by Δ the set of all roots of $\mathfrak{g}$ and

Δ_+ the set of all positive roots of $\mathfrak{g}$. For any $\alpha \in \Delta$, we denote by $\alpha^\vee$ the coroot of α. Let $\lambda_1, \dots, \lambda_r$ be the fundamental weights, and we denote λ the dominant weight. It is known that the dominant weight can be written as a linear combination of $\lambda_1, \dots, \lambda_r$ with non negative integer coefficients uniquely, and there exists one-to-one correspondence between the dominant weight λ and the irreducible representation φ. By these properties and substituting Weyl's dimension formula

$$\dim \varphi = \prod_{\alpha \in \Delta_+} \frac{\langle \alpha^\vee, \lambda + \lambda_1 + \cdots + \lambda_r \rangle}{\langle \alpha^\vee, \lambda_1 + \cdots + \lambda_r \rangle}$$

into the definition of the Witten zeta function, we have

$$\begin{aligned} \zeta_W(s, \mathfrak{g}) &= \sum_{\varphi} (\dim \varphi)^{-s} \\ &= \sum_{\lambda} \left(\prod_{\alpha \in \Delta_+} \frac{\langle \alpha^\vee, \lambda + \lambda_1 + \cdots + \lambda_r \rangle}{\langle \alpha^\vee, \lambda_1 + \cdots + \lambda_r \rangle} \right)^{-s} \\ &= K(\mathfrak{g})^s \sum_{m_1, \dots, m_r > 0} \prod_{\alpha \in \Delta_+} \langle \alpha^\vee, m_1\lambda_1 + \cdots + m_r\lambda_r \rangle^{-s}, \end{aligned}$$

where

$$K(\mathfrak{g}) = \prod_{\alpha \in \Delta_+} \langle \alpha^\vee, \lambda_1 + \cdots + \lambda_r \rangle.$$

Komori–Matsumoto–Tsumura defined zeta functions of root systems by removing the factor $K(\mathfrak{g})^s$ and introducing the independent several variables.

Definition 1.27 (Zeta functions of root systems)**.** For a simple Lie algebra $\mathfrak{g}$ with rank r and complex variables $\mathbf{s} = (s_\alpha)_{\alpha \in \Delta_+}$, the zeta function of the root system $\zeta_r(\mathbf{s}, \mathfrak{g})$ are defined by the following series:

$$\zeta_r(\mathbf{s}, \mathfrak{g}) := \sum_{m_1, m_2, \dots, m_r > 0} \prod_{\alpha \in \Delta_+} \langle \alpha^\vee, m_1\lambda_1 + m_2\lambda_2 + \cdots + m_r\lambda_r \rangle^{-s_\alpha}.$$

When $\mathfrak{g}$ is of type X_r, we write $\zeta_r(\mathbf{s}, \mathfrak{g})$ just $\zeta_r(\mathbf{s}, X_r)$. Then the Witten zeta function $\zeta_W(s, \mathfrak{sl}(3))$ and $\zeta_W(s, \mathfrak{so}(5))$ correspond to

$$\zeta_2(s_1, s_2, s_3, A_2) = \sum_{m,n>0} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3}}$$

and

$$\zeta_2(s_1, s_2, s_3, s_4, B_2) = \sum_{m,n>0} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3} (m+2n)^{s_4}},$$

respectively. Historically, first, in 1950, Tornheim [45] treated the type A_2. Of course, he did not recognize this double series as a special case of zeta functions of root systems. Second, in 2003, Matsumoto [30] introduced the type B_2 and general type A_r was introduced by Matsumoto–Tsumura [31] in 2006. Finally, Komori–Matsumoto–Tsumura [22] gave a general definition of zeta functions of root systems in 2007. As one of the important analytical properties, it is known that zeta function of the root system $\zeta_r(\mathbf{s}, X_r)$ absolutely converges in the region defined by $\Re(s_i) > 1$ for any i and can be continued meromorphically to the whole space. The latter fact can be obtained as a corollary of Matsumoto's result [30, Theorem 3]. We give some more examples of zeta functions of root systems.

Example 1.28.

$$\begin{aligned}
&\zeta_2(s_1, \ldots, s_6, G_2) \\
&= \sum_{m,n>0} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3} (m+2n)^{s_4} (m+3n)^{s_5} (2m+3n)^{s_6}}, \qquad (1.12)\\
&\zeta_3(s_1, \ldots, s_6, A_3) \\
&= \sum_{m_1,m_2,m_3>0} \frac{1}{m_1^{s_1} m_2^{s_2} m_3^{s_3} (m_1+m_2)^{s_4} (m_2+m_3)^{s_5} (m_1+m_2+m_3)^{s_6}}, \qquad (1.13)\\
&\zeta_3(s_1, \ldots, s_9, B_3) \\
&= \sum_{m_1,m_2,m_3>0} \frac{1}{m_1^{s_1} m_2^{s_2} m_3^{s_3} (m_1+m_2)^{s_4} (m_2+m_3)^{s_5} (2m_2+m_3)^{s_6}} \qquad (1.14)\\
&\quad \times \frac{1}{(m_1+m_2+m_3)^{s_7} (m_1+2m_2+m_3)^{s_8} (2m_1+2m_2+m_3)^{s_9}}.
\end{aligned}$$

Komori–Matsumoto–Tsumura [23] gave the explicit form of zeta functions of root systems of types A_r, B_r, C_r and D_r. Here, we enumerate those formulas.

Proposition 1.29 (type A_r). *For* $\mathbf{s} = (s_{ij}) \in \mathbb{C}^{r(r+1)/2}$,

$$\zeta_r(\mathbf{s}, A_r) = \sum_{m_1,m_2,\ldots,m_r>0} \prod_{1\le i<j\le r+1} (m_i + \cdots + m_{j-1})^{-s_{ij}}.$$

Proposition 1.30 (type B_r). *For* $\mathbf{s} = ((s_i), (s_{ij}^-), (s_{ij}^+)) \in \mathbb{C}^{r^2}$,

$$\begin{aligned}
\zeta_r(\mathbf{s}, B_r) = \sum_{m_1,m_2,\ldots,m_r>0} &\prod_{1\le i\le r} (2(m_i + \cdots + m_{r-1}) + m_r)^{-s_i} \\
&\times \prod_{1\le i<j\le r} (m_i + \cdots + m_{j-1})^{-s_{ij}^-} \\
&\times \prod_{1\le i<j\le r} (m_i + \cdots + m_{j-1} + 2(m_j + \cdots + m_{r-1}) + m_r)^{-s_{ij}^+}.
\end{aligned}$$

Proposition 1.31 (type C_r)**.** *For* $\mathbf{s} = ((s_i), (s_{ij}^-), (s_{ij}^+)) \in \mathbb{C}^{r^2}$,

$$\zeta_r(\mathbf{s}, C_r) = \sum_{m_1, m_2, \dots, m_r > 0} \prod_{1 \le i \le r} (m_i + \cdots + m_r)^{-s_i} \times \prod_{1 \le i < j \le r} (m_i + \cdots + m_{j-1})^{-s_{ij}^-} \times \prod_{1 \le i < j \le r} (m_i + \cdots + m_{j-1} + 2(m_j + \cdots + m_r))^{-s_{ij}^+}.$$

Proposition 1.32 (type D_r)**.** *For* $\mathbf{s} = ((s_{ij}^-), (s_{ij}^+)) \in \mathbb{C}^{r(r-1)}$,

$$\zeta_r(\mathbf{s}, D_r) = \sum_{m_1, m_2, \dots, m_r > 0} \prod_{1 \le i \le r} ((m_i + \cdots + m_{r-2}) + m_r)^{-s_{ir}^+} \times \prod_{1 \le i < j \le r} (m_i + \cdots + m_{j-1})^{-s_{ij}^-} \times \prod_{1 \le i < j \le r} (m_i + \cdots + m_{j-1} + 2(m_j + \cdots + m_{r-2}) + m_{r-1} + m_r)^{-s_{ij}^+}.$$

So far, Komori–Matsumoto–Tsumura have given several results on zeta functions of root systems by using the following classical Mellin–Barnes formula.

Proposition 1.33. *For complex numbers s and λ with $\Re(s) > 0, |\arg \lambda| < \pi$ and $\lambda \neq 0$, we have*

$$(1+\lambda)^{-s} = \frac{1}{2\pi\sqrt{-1}} \int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)} \lambda^z \, dz,$$

where c is a real number with $-\Re(s) < c < 0$.

Using this formula, they give a certain integral representation of zeta functions of root systems and a recursive structure of them. Moreover, they showed that $\zeta_r(\mathbf{s}, X_r)$ can be continued meromorphically to the whole space again and determined the possible singularities by using that recursive structure. In their papers, the recursive structure can be described in terms of Dynkin diagrams. (For the type A_r in [31], for the types B_r, C_r, D_r in [23]) Besides the above properties, they also discovered functional relations by using analytical method and the Weyl group symmetry. For the types A_2, A_3 and B_3, Nakamura [36] also gave some functional relations.

Here, let us observe three examples in Example 1.28. In Eq. (1.12), putting $s_2 = s_4 = s_5 = s_6 = 0$, we have

$$\zeta_2(s_1, 0, s_3, 0, 0, 0, G_2) = \zeta(s_1, s_3).$$

In (1.13), for example, putting $s_2 = s_3 = s_5 = 0$, we have

$$\zeta_3(s_1, 0, 0, s_4, 0, s_6, A_3) = \zeta(s_1, s_4, s_6).$$

In (1.14), for example, putting $s_1 = s_3 = s_5 = s_6 = s_8 = s_9 = 0$, we have

$$\zeta_3(0, s_2, 0, s_4, 0, 0, s_7, 0, 0, B_3) = \zeta(s_2, s_4, s_7).$$

As these observations, we can obtain the Euler–Zagier type of multiple zeta function from zeta functions of root systems by putting some variables 0. Namely, $\zeta_r(\mathbf{s}, X_r)$ can be regarded as a generalization of the Euler–Zagier type of multiple zeta function. It is natural to consider whether there exists a generalization or analogue of the results for the Euler–Zagier type of multiple zeta functions and their special values. Actually, there are some known results on special values of zeta functions of root systems, but in this thesis, we especially focus on the parity result. Namely, we discuss about $\zeta_r(\mathbf{k}, X_r)$ when $\mathrm{wt}(\mathbf{k})$ and r have different parity.

1.3.3 Parity result for zeta values of root systems

Tornheim [45] proved that $\zeta_2(k_1, k_2, k_3, A_2)$ can be written as a polynomial of the Riemann zeta values when $k_1 + k_2 + k_3$ is odd, and Borwein–Borwein–Girgensohn [4] and Huard–Williams–Zhang [15] gave the explicit formula for that. In 2004, Tsumura [48] gave the explicit evaluation of $\zeta_2(k_1, k_2, k_3, k_4, B_2)$ in terms of the Riemann zeta values when $k_1 + k_2 + k_3 + k_4$ is odd. Incidentally, Zhao [55] obtained the explicit evaluation of $\zeta_2(k_1, k_2, k_3, k_4, B_2)$ in terms of double alternating Euler sums and the Riemann zeta values when $\zeta_2(k_1, k_2, k_3, k_4, B_2)$ converges for non negative integers k_1, k_2, k_3 and k_4. The remaining zeta values of root systems with rank 2 is of type G_2 since $B_2 \simeq C_2$. The first study in this direction is also given by Zhao [54]. He got the explicit evaluation of $\zeta_2(k_1, \ldots, k_6, G_2)$ in terms of the special values of double polylogarithms at twelfth roots of unity and the Riemann zeta values when $\zeta_2(k_1, \ldots, k_6, G_2)$ converges for non negative integers $k_1, \ldots, k_5$ and k_6. After his study, Okamoto [40] proved the following theorem in 2012.

Theorem 1.34 (Okamoto [40]). *For positive integer n and real number x, we define the Clausen type functions $S_n(x)$ and $C_n(x)$ as follows:*

$$S_n(x) := \sum_{m>0} \frac{\sin(2\pi m x)}{m^n}, \quad C_n(x) := \sum_{m>0} \frac{\cos(2\pi m x)}{m^n}.$$

For positive integers $k_1, \ldots, k_5$ and k_6, if $k_1 + \cdots + k_6$ is odd, $\zeta_2(k_1, \ldots, k_6, G_2)$ can be written as a polynomial of

$$\zeta(n), L(n, \chi_3), S_n(\tfrac{j}{\ell}) \text{ and } C_n(\tfrac{j}{\ell})$$

with $\ell = 4, 12, 0 < j < \ell$ and $\gcd(j, \ell) = 1$, where χ_3 is the Dirichlet character modulo 3.

This result can be regarded as an analogue of the parity theorem since the above theorem means that double series can be expressed in terms of some single series when weight and depth have different parity. At that time, we are not sure that the Clausen type functions are really required or not. In 2015, Komori–Matsumoto–Tsumura [27] proposed the following conjecture based on the observation of several examples.

Conjecture 1.35. *For* $\mathbf{k} = (k_1, \ldots, k_6) \in \mathbb{N}^6$*, if* $\mathrm{wt}(\mathbf{k})$ *is odd, then*

$$\zeta_2(k_1, \ldots, k_6, G_2) \overset{?}{\in} \mathbb{Q}[\{\zeta(j+1), L(j, \chi_3) \mid j \in \mathbb{N}\}].$$

The author, Okamoto and Tasaka [19] completely solved this conjecture. Moreover, we showed that $\zeta_2(k_1, \ldots, k_6, G_2)$ can be expressed as a rational linear combination of the product of two Riemann zeta values and the product of two Dirichlet L-values when $\mathrm{wt}(\mathbf{k})$ is odd. We discuss them in Section 4. As a higher rank case, Zhao–Zhou [56] showed that $\zeta_3(k_1, \ldots, k_6, A_3)$ can be written as a rational linear combination of multiple zeta values of same weight and depth three or less when $\zeta_3(k_1, \ldots, k_6, A_3)$ converges for non negative integers $k_1, \ldots, k_5$ and k_6. Moreover, recently, Okamoto proved that $\zeta_3(\mathbf{k}, A_3)$ when $\mathrm{wt}(\mathbf{k})$ is even can be expressed in terms of zeta values of root system of type A_2 and the Riemann zeta values. In section 5, we discuss the parity theorem for more general multiple series not limited to zeta values of root systems.

2 Certain weighted sum formulas for multiple zeta values with some parameters

2.1 Certain weighted sum formulas for multiple zeta values with some parameters

As mentioned in Section 1, for an r-tuple of natural numbers $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ ($\alpha_r \geq 2$), the multiple zeta value (MZV) $\zeta(\boldsymbol{\alpha}) = \zeta(\alpha_1, \alpha_2, \ldots, \alpha_r)$ is defined as the following series.

$$\zeta(\boldsymbol{\alpha}) = \zeta(\alpha_1, \alpha_2, \ldots, \alpha_r) := \sum_{0<n_1<n_2<\cdots<n_r} \frac{1}{{n_1}^{\alpha_1}{n_2}^{\alpha_2}\ldots{n_r}^{\alpha_r}}.$$

We call r the depth of $\zeta(\boldsymbol{\alpha})$, $w = \alpha_1 + \cdots + \alpha_r$ the weight of $\zeta(\boldsymbol{\alpha})$. Easily one find that MZV has the following integral representation:

$$\zeta(\alpha_1, \alpha_2, \ldots, \alpha_r) = \int\cdots\int_{0<t_1<t_2<\cdots<t_w<1} \omega_1(t_1)\omega_2(t_2)\ldots\omega_w(t_w),$$

where

$$\omega_i(t) = \begin{cases} \dfrac{dt}{1-t} & , \ i \in \{1, \alpha_1+1, \alpha_1+\alpha_2+1, \ldots, \alpha_1+\cdots+\alpha_{r-1}+1\}, \\ \dfrac{dt}{t} & , \ \text{otherwise}. \end{cases}$$

This integral representation plays an important role in this thesis.

First, MZVs were studied by Euler [10] in the special case $r = 2$, and Hoffman [13] gave the above general definition. It is known that MZVs are closely related to knot theory, arithmetic geometry and mathematical physics. Zagier [53] proposed the conjecture for the dimension of the $\mathbb{Q}$-vector spaces $\mathcal{Z}_k$ spanned by MZVs with weight k by numerical calculations. According to that conjecture we can expect that there are many relations among MZVs. So in the study of MZVs, one of the main topic is to obtain various relations among them. The present section is devoted to this topic, and the aims of the present section are to give two relations which have some parameters(see Theorems 2.1 and 2.2) and to discuss the position of those relations in the families of relations among MZVs.

Before stating theorems, we introduce some notations. Let $\mathfrak{S}_n$ be the symmetric group of n-th order. For indices $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{N}^r$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_s) \in \mathbb{N}^s$ (α_r and β_s can be 1), we define $\boldsymbol{\alpha}_+$, $\boldsymbol{\alpha} \circ \boldsymbol{\beta}$ and $|\boldsymbol{\alpha}|$ as follows:

$$\boldsymbol{\alpha}_+ = (\alpha_1, \alpha_2, \ldots, \alpha_r + 1),$$
$$\boldsymbol{\alpha} \circ \boldsymbol{\beta} = (\alpha_1, \alpha_2, \ldots, \alpha_r + \beta_1, \beta_2, \ldots, \beta_s),$$
$$|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_r.$$

Theorem 2.1. *For non-negative integers k and ℓ, and four parameters (indeterminates) μ_1, μ_2, ξ_1 and ξ_2, we have*

$$\begin{aligned}
&\sum_{\substack{a_1+a_2=k\\ b_1+b_2=\ell}} \mu_1^{a_1}\mu_2^{a_2}\xi_1^{b_1}\xi_2^{b_2}\zeta(a_1+b_1+2)\zeta(a_2+b_2+2)\\
&=\sum_{\sigma\in\mathfrak{S}_2}\Bigg[\sum_{\substack{a_1+a_2=k\\ b_1+b_2=\ell}} \mu_{\sigma(1)}^{a_1}\mu_{\sigma(2)}^{a_2}\xi_{\sigma(1)}^{b_1}\xi_{\sigma(2)}^{b_2}\sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1\\ |\boldsymbol{\beta}|=a_2+b_2+1}}\zeta(\boldsymbol{\alpha}_+,\boldsymbol{\beta}_+)\\
&\quad+\sum_{\substack{a_1+a_2+a_3=k\\ b_1+b_2+b_3=\ell}} \mu_{\sigma(1)}^{a_1}\xi_{\sigma(1)}^{b_1}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}(\mu_{\sigma(1)}^{a_3}\xi_{\sigma(1)}^{b_3}+\mu_{\sigma(2)}^{a_3}\xi_{\sigma(2)}^{b_3})\\
&\quad\times\sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1\\ |\boldsymbol{\beta}|=a_2+b_2+1\\ |\boldsymbol{\gamma}|=a_3+b_3+1}}\zeta(\boldsymbol{\alpha},\boldsymbol{\beta}\circ\boldsymbol{\gamma}_+)\Bigg],
\end{aligned}$$

where a_i and b_i run over non-negative integers with the conditions that the sum of those are k and ℓ, and $\boldsymbol{\alpha}=(\alpha_0,\alpha_1,\ldots,\alpha_{a_1}), \boldsymbol{\beta}=(\beta_0,\beta_1,\ldots,\beta_{a_2})$ and $\boldsymbol{\gamma}=(\gamma_0,\gamma_1,\ldots,\gamma_{a_3})$. Therefore, for example $\displaystyle\sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1\\ |\boldsymbol{\beta}|=a_2+b_2+1}}$ means that $\alpha_0,\ldots,\alpha_{a_1}$ and $\beta_0,\ldots,\beta_{a_2}$ run over all positive integers with $\alpha_0+\cdots+\alpha_{a_1}=a_1+b_1+1$, $\beta_0+\cdots+\beta_{a_2}=a_2+b_2+1$.

In Theorem 2.1, when k is an even number, putting $(\mu_1,\mu_2,\xi_1,\xi_2)=(1,-1,1,1)$ we can derive the results of Eie, Liaw and Ong [9, Main Theorem] which is a generalization of the weighted sum formula of Ohno–Zudilin [39, Theorem 3]. We will derive [9, Main Theorem] from Theorem 2.1 in Section 2.3. The next one is the main theorem in this section.

Theorem 2.2. *For non-negative integers k,ℓ and six parameters (indeterminates) μ_1,μ_2, μ_3,ξ_1,ξ_2 and ξ_3, we have the following relation:*

$$\begin{aligned}
&\sum_{\substack{a_1+a_2+a_3=k\\ b_1+b_2+b_3=\ell}} \mu_1^{a_1}\mu_2^{a_2}\mu_3^{a_3}\xi_1^{b_1}\xi_2^{b_2}\xi_3^{b_3}\zeta(a_1+b_1+2)\zeta(a_2+b_2+2)\zeta(a_3+b_3+2)\\
&=\sum_{\sigma\in\mathfrak{S}_3}\Bigg[\sum_{\substack{a_1+a_2+a_3=k\\ b_1+b_2+b_3=\ell}} P_{1,\sigma}\times\sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1\\ |\boldsymbol{\beta}|=a_2+b_2+1\\ |\boldsymbol{\gamma}|=a_3+b_3+1}}\zeta(\boldsymbol{\alpha}_+,\boldsymbol{\beta}_+,\boldsymbol{\gamma}_+)\\
&\quad+\sum_{\substack{a_1+\cdots+a_4=k\\ b_1+\cdots+b_4=\ell}}(P_{2,\sigma}+P_{3,\sigma})\sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1\\ \vdots\\ |\boldsymbol{\delta}|=a_4+b_4+1}}\zeta(\boldsymbol{\alpha},\boldsymbol{\beta}\circ\boldsymbol{\gamma}_+,\boldsymbol{\delta}_+)\\
&\quad+\sum_{\substack{a_1+\cdots+a_5=k\\ b_1+\cdots+b_5=\ell}}(P_{4,\sigma}+P_{5,\sigma}+P_{7,\sigma}+P_{12,\sigma})\sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1\\ \vdots\\ |\boldsymbol{\varepsilon}|=a_5+b_5+1}}\zeta(\boldsymbol{\alpha},\boldsymbol{\beta}\circ\boldsymbol{\gamma},\boldsymbol{\delta}\circ\boldsymbol{\varepsilon}_+)
\end{aligned}$$

$$
+ \sum_{\substack{a_1+\cdots+a_4=k\\ b_1+\cdots+b_4=\ell}} (P_{6,\sigma}+P_{11,\sigma}) \sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1\\ \vdots \\ |\boldsymbol{\delta}|=a_4+b_4+1}} \zeta(\boldsymbol{\alpha}_+,\boldsymbol{\beta},\boldsymbol{\gamma}\circ\boldsymbol{\delta}_+)
$$

$$
+ \sum_{\substack{a_1+\cdots+a_5=k\\ b_1+\cdots+b_5=\ell}} (P_{8,\sigma}+P_{9,\sigma}+P_{10,\sigma}+P_{13,\sigma}+P_{14,\sigma}+P_{15,\sigma}) \sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1\\ \vdots \\ |\boldsymbol{\varepsilon}|=a_5+b_5+1}} \zeta(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\circ\boldsymbol{\delta}\circ\boldsymbol{\varepsilon}_+)\Bigg],
$$

where

$$
\begin{aligned}
\boldsymbol{\alpha} &= (\alpha_0,\alpha_1,\ldots,\alpha_{a_1}),\\
\boldsymbol{\beta} &= (\beta_0,\beta_1,\ldots,\beta_{a_2}),\\
\boldsymbol{\gamma} &= (\gamma_0,\gamma_1,\ldots,\gamma_{a_3}),\\
\boldsymbol{\delta} &= (\delta_0,\delta_1,\ldots,\delta_{a_4}),\\
\boldsymbol{\varepsilon} &= (\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_{a_5}),
\end{aligned}
$$

$$
\begin{aligned}
P_{1,\sigma} &= \mu_{\sigma(1)}^{a_1}\mu_{\sigma(2)}^{a_2}\mu_{\sigma(3)}^{a_3}\xi_{\sigma(1)}^{b_1}\xi_{\sigma(2)}^{b_2}\xi_{\sigma(3)}^{b_3},\\
P_{2,\sigma} &= \mu_{\sigma(1)}^{a_1}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}\mu_{\sigma(2)}^{a_3}\mu_{\sigma(3)}^{a_4}\xi_{\sigma(1)}^{b_1}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}\xi_{\sigma(2)}^{b_3}\xi_{\sigma(3)}^{b_4},\\
P_{3,\sigma} &= \mu_{\sigma(1)}^{a_1+a_3}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}\mu_{\sigma(3)}^{a_4}\xi_{\sigma(1)}^{b_1+b_3}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}\xi_{\sigma(3)}^{b_4},\\
P_{4,\sigma} &= \mu_{\sigma(1)}^{a_1+a_3}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}(\mu_{\sigma(1)}+\mu_{\sigma(3)})^{a_4}\mu_{\sigma(3)}^{a_5}\\
&\quad\times \xi_{\sigma(1)}^{b_1+b_3}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}(\xi_{\sigma(1)}+\xi_{\sigma(3)})^{b_4}\xi_{\sigma(3)}^{b_5},\\
P_{5,\sigma} &= \mu_{\sigma(1)}^{a_1+a_3+a_5}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}(\mu_{\sigma(1)}+\mu_{\sigma(3)})^{a_4}\xi_{\sigma(1)}^{b_1+b_3+b_5}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}(\xi_{\sigma(1)}+\xi_{\sigma(3)})^{b_4},\\
P_{6,\sigma} &= \mu_{\sigma(1)}^{a_1}\mu_{\sigma(2)}^{a_2}(\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_3}\mu_{\sigma(3)}^{a_4}\xi_{\sigma(1)}^{b_1}\xi_{\sigma(2)}^{b_2}(\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_3}\xi_{\sigma(3)}^{b_4},\\
P_{7,\sigma} &= \mu_{\sigma(1)}^{a_1}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}\mu_{\sigma(2)}^{a_3}(\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_4}\mu_{\sigma(3)}^{a_5}\\
&\quad\times \xi_{\sigma(1)}^{b_1}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}\xi_{\sigma(2)}^{b_3}(\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_4}\xi_{\sigma(3)}^{b_5},\\
P_{8,\sigma} &= \mu_{\sigma(1)}^{a_1}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}(\mu_{\sigma(1)}+\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_3}(\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_4}\mu_{\sigma(3)}^{a_5}\\
&\quad\times \xi_{\sigma(1)}^{b_1}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}(\xi_{\sigma(1)}+\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_3}(\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_4}\xi_{\sigma(3)}^{b_5},\\
P_{9,\sigma} &= \mu_{\sigma(1)}^{a_1}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}(\mu_{\sigma(1)}+\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_3}(\mu_{\sigma(1)}+\mu_{\sigma(3)})^{a_4}\mu_{\sigma(3)}^{a_5}\\
&\quad\times \xi_{\sigma(1)}^{b_1}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}(\xi_{\sigma(1)}+\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_3}(\xi_{\sigma(1)}+\xi_{\sigma(3)})^{b_4}\xi_{\sigma(3)}^{b_5},\\
P_{10,\sigma} &= \mu_{\sigma(1)}^{a_1+a_5}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}(\mu_{\sigma(1)}+\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_3}(\mu_{\sigma(1)}+\mu_{\sigma(3)})^{a_4}\\
&\quad\times \xi_{\sigma(1)}^{b_1+b_5}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}(\xi_{\sigma(1)}+\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_3}(\xi_{\sigma(1)}+\xi_{\sigma(3)})^{b_4},\\
P_{11,\sigma} &= \mu_{\sigma(1)}^{a_1}\mu_{\sigma(2)}^{a_2+a_4}(\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_3}\xi_{\sigma(1)}^{b_1}\xi_{\sigma(2)}^{b_2+b_4}(\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_3},\\
P_{12,\sigma} &= \mu_{\sigma(1)}^{a_1}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}\mu_{\sigma(2)}^{a_3+a_5}(\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_4}\\
&\quad\times \xi_{\sigma(1)}^{b_1}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}\xi_{\sigma(2)}^{b_3+b_5}(\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_4},\\
P_{13,\sigma} &= \mu_{\sigma(1)}^{a_1}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}(\mu_{\sigma(1)}+\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_3}(\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_4}\mu_{\sigma(2)}^{a_5}
\end{aligned}
$$

$$\times\, \xi_{\sigma(1)}^{b_1}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}(\xi_{\sigma(1)}+\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_3}(\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_4}\xi_{\sigma(2)}^{b_5},$$

$$P_{14,\sigma}=\mu_{\sigma(1)}^{a_1}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2+a_4}(\mu_{\sigma(1)}+\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_3}\mu_{\sigma(2)}^{a_5}$$
$$\times\, \xi_{\sigma(1)}^{b_1}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2+b_4}(\xi_{\sigma(1)}+\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_3}\xi_{\sigma(2)}^{b_5},$$

$$P_{15,\sigma}=\mu_{\sigma(1)}^{a_1+a_5}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2+a_4}(\mu_{\sigma(1)}+\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_3}$$
$$\times\, \xi_{\sigma(1)}^{b_1+b_5}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2+b_4}(\xi_{\sigma(1)}+\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_3}.$$

Remark 2.3. The formulas of Theorems 2.1 and 2.2 have several parameters, so one may get relations among MZVs by comparing the parameters of the both sides or differentiating partially with respect to parameters. (See Remark 2.7.)

2.2 Proof of Theorems 2.1 and 2.2

In this subsection, we prove Theorem 2.2 and then, briefly indicate how to prove Theorem 2.1. The basic structure of the proof is the same as in [9]. For $k,\ell\in\mathbb{Z}_{\geq 0}$, and 6 parameters $\mu_1,\mu_2,\mu_3,\xi_1,\xi_2$ and ξ_3, we define $I_{k,\ell}(\mu_1,\mu_2,\mu_3,\xi_1,\xi_2,\xi_3)$ as the following integral:

$$I_{k,\ell}(\mu_1,\mu_2,\mu_3,\xi_1,\xi_2,\xi_3)$$
$$:=\frac{1}{k!\ell!}\int\limits_{\substack{0<s_1<s_2<1\\0<t_1<t_2<1\\0<u_1<u_2<1}}\left(\mu_1\log\frac{1-s_1}{1-s_2}+\mu_2\log\frac{1-t_1}{1-t_2}+\mu_3\log\frac{1-u_1}{1-u_2}\right)^k$$
$$\times\left(\xi_1\log\frac{s_2}{s_1}+\xi_2\log\frac{t_2}{t_1}+\xi_3\log\frac{u_2}{u_1}\right)^\ell\frac{ds_1ds_2dt_1dt_2du_1du_2}{(1-s_1)s_2(1-t_1)t_2(1-u_1)u_2}.$$

To obtain Theorem 2.2, we calculate this integral in two ways.

• The first calculations

First, we expand the factors of the integrand simply as follows:

$$\left(\mu_1\log\frac{1-s_1}{1-s_2}+\mu_2\log\frac{1-t_1}{1-t_2}+\mu_3\log\frac{1-u_1}{1-u_2}\right)^k$$
$$=\sum_{a_1+a_2+a_3=k}\frac{k!}{a_1!a_2!a_3!}\mu_1^{a_1}\mu_2^{a_2}\mu_3^{a_3}\left(\log\frac{1-s_1}{1-s_2}\right)^{a_1}\left(\log\frac{1-t_1}{1-t_2}\right)^{a_2}\left(\log\frac{1-u_1}{1-u_2}\right)^{a_3},$$

$$\left(\xi_1\log\frac{s_2}{s_1}+\xi_2\log\frac{t_2}{t_1}+\xi_3\log\frac{u_2}{u_1}\right)^\ell$$
$$=\sum_{b_1+b_2+b_3=\ell}\frac{\ell!}{b_1!b_2!b_3!}\xi_1^{b_1}\xi_2^{b_2}\xi_3^{b_3}\left(\log\frac{s_2}{s_1}\right)^{b_1}\left(\log\frac{t_2}{t_1}\right)^{b_2}\left(\log\frac{u_2}{u_1}\right)^{b_3}.$$

Substituting these expansions, we get

$$
\begin{aligned}
I_{k,\ell}(\mu_1,\mu_2,\mu_3,\xi_1,\xi_2,\xi_3) = &\sum_{\substack{a_1+a_2+a_3=k\\ b_1+b_2+b_3=\ell}} \mu_1^{a_1}\mu_2^{a_2}\mu_3^{a_3}\xi_1^{b_1}\xi_2^{b_2}\xi_3^{b_3}\\
&\times\left\{\frac{1}{a_1!b_1!}\int\limits_{0<s_1<s_2<1}\left(\log\frac{1-s_1}{1-s_2}\right)^{a_1}\left(\log\frac{s_2}{s_1}\right)^{b_1}\frac{ds_1ds_2}{(1-s_1)s_2}\right\}\\
&\times\left\{\frac{1}{a_2!b_2!}\int\limits_{0<t_1<t_2<1}\left(\log\frac{1-t_1}{1-t_2}\right)^{a_2}\left(\log\frac{t_2}{t_1}\right)^{b_2}\frac{dt_1dt_2}{(1-t_1)t_2}\right\}\\
&\times\left\{\frac{1}{a_3!b_3!}\int\limits_{0<u_1<u_2<1}\left(\log\frac{1-u_1}{1-u_2}\right)^{a_3}\left(\log\frac{u_2}{u_1}\right)^{b_3}\frac{du_1du_2}{(1-u_1)u_2}\right\}.
\end{aligned}
$$

Here let us consider $\left(\log\dfrac{1-s_1}{1-s_2}\right)^a$. We can rewrite it as

$$
\begin{aligned}
\left(\log\frac{1-s_1}{1-s_2}\right)^a &= \left(\int_{s_1}^{s_2}\frac{dp}{1-p}\right)^a\\
&= \int\limits_{\substack{s_1<p_1<s_2\\ \vdots\\ s_1<p_a<s_2}}\prod_{i=1}^{a}\frac{dp_i}{1-p_i}\\
&= \sum_{\sigma\in\mathfrak{S}_a}\int\limits_{s_1<p_{\sigma(1)}<\cdots<p_{\sigma(a)}<s_2}\prod_{i=1}^{a}\frac{dp_i}{1-p_i}\\
&= a!\int\limits_{s_1<p_1<\cdots<p_a<s_2}\prod_{i=1}^{a}\frac{dp_i}{1-p_i}.
\end{aligned}
$$

We can rewrite $\left(\log\dfrac{s_2}{s_1}\right)^b$ similarly, so we get the following lemma.

Lemma 2.4. *For $a,b\in\mathbb{Z}_{\geq 0}$, and $0<s_1<s_2<1$ we have*

$$
\left(\log\frac{1-s_1}{1-s_2}\right)^a = a!\int\limits_{s_1<p_1<\cdots<p_a<s_2}\prod_{i=1}^{a}\frac{dp_i}{1-p_i}
$$

and

$$
\left(\log\frac{s_2}{s_1}\right)^b = b!\int\limits_{s_1<q_1<\cdots<q_b<s_2}\prod_{j=1}^{b}\frac{dq_j}{q_j},
$$

where the empty product is to be understood as 1.

Using the above lemma, we get

$$
\begin{aligned}
&\frac{1}{a_1!b_1!}\int\limits_{0<s_1<s_2<1}\left(\log\frac{1-s_1}{1-s_2}\right)^{a_1}\left(\log\frac{s_2}{s_1}\right)^{b_1}\frac{ds_1ds_2}{(1-s_1)s_2}\\
&=\int\limits_{\substack{0<s_1<s_2<1\\ s_1<p_1<\cdots<p_{a_1}<s_2\\ s_1<q_1<\cdots<q_{b_1}<s_2}}\frac{ds_1}{1-s_1}\prod_{i=1}^{a_1}\frac{dp_i}{1-p_i}\prod_{j=1}^{b_1}\frac{dq_j}{q_j}\frac{ds_2}{s_2}\\
&=\sum_{(r_1,\ldots,r_{a_1+b_1})}\int\limits_{\substack{0<s_1<s_2<1\\ s_1<r_1<\cdots<r_{a_1+b_1}<s_2}}\frac{ds_1}{1-s_1}\prod_{i=1}^{a_1}\frac{dp_i}{1-p_i}\prod_{j=1}^{b_1}\frac{dq_j}{q_j}\frac{ds_2}{s_2},
\end{aligned}
\tag{2.1}
$$

where the summation runs over all tuples $(r_1,\ldots,r_{a_1+b_1})$ such that

$$\{r_1,\ldots,r_{a_1+b_1}\}=\{p_1,\ldots,p_{a_1}\}\cup\{q_1,\ldots,q_{b_1}\}$$

and

$$p_1<\cdots<p_{a_1},\quad q_1<\cdots<q_{b_1}.$$

Then each integral gives a multiple zeta value, which implies that the above is

$$=\sum_{|\boldsymbol{\alpha}|=a_1+b_1+1}\zeta(\alpha_0,\alpha_1,\ldots,\alpha_{a_1}+1).$$

By using the sum formula [12, Proposition], we have

$$
\begin{aligned}
&I_{k,\ell}(\mu_1,\mu_2,\mu_3,\xi_1,\xi_2,\xi_3)\\
&=\sum_{\substack{a_1+a_2+a_3=k\\ b_1+b_2+b_3=\ell}}\mu_1^{a_1}\mu_2^{a_2}\mu_3^{a_3}\xi_1^{b_1}\xi_2^{b_2}\xi_3^{b_3}\zeta(a_1+b_1+2)\zeta(a_2+b_2+2)\zeta(a_3+b_3+2).
\end{aligned}
$$

This is the end of the first calculations.

Remark 2.5. We can get the above conclusion without using the sum formula by changing variables

$$x=\log\frac{1-s_1}{1-s_2},\quad y=\log\frac{s_2}{s_1}$$

in the integral on the left hand side of (2.1).

- The second calculations

We divide the region $0<s_1<s_2<1, 0<t_1<t_2<1, 0<u_1<u_2<1$ to 90 regions, according to the order of magnitude of variables, and calculate the integral on each regions. But, as we will see just below, because of the symmetry of variables s, t and u, it is sufficient to calculate on the following 15 regions:

$$
\begin{aligned}
&D_1 : 0 < s_1 < s_2 < t_1 < t_2 < u_1 < u_2 < 1, \quad D_2 : 0 < s_1 < t_1 < s_2 < t_2 < u_1 < u_2 < 1,\\
&D_3 : 0 < s_1 < t_1 < t_2 < s_2 < u_1 < u_2 < 1, \quad D_4 : 0 < s_1 < t_1 < t_2 < u_1 < s_2 < u_2 < 1,\\
&D_5 : 0 < s_1 < t_1 < t_2 < u_1 < u_2 < s_2 < 1,\\
&D_6 : 0 < s_1 < s_2 < t_1 < u_1 < t_2 < u_2 < 1, \quad D_7 : 0 < s_1 < t_1 < s_2 < u_1 < t_2 < u_2 < 1,\\
&D_8 : 0 < s_1 < t_1 < u_1 < s_2 < t_2 < u_2 < 1, \quad D_9 : 0 < s_1 < t_1 < u_1 < t_2 < s_2 < u_2 < 1,\\
&D_{10} : 0 < s_1 < t_1 < u_1 < t_2 < u_2 < s_2 < 1,\\
&D_{11} : 0 < s_1 < s_2 < t_1 < u_1 < u_2 < t_2 < 1, \quad D_{12} : 0 < s_1 < t_1 < s_2 < u_1 < u_2 < t_2 < 1,\\
&D_{13} : 0 < s_1 < t_1 < u_1 < s_2 < u_2 < t_2 < 1, \quad D_{14} : 0 < s_1 < t_1 < u_1 < u_2 < s_2 < t_2 < 1,\\
&D_{15} : 0 < s_1 < t_1 < u_1 < u_2 < t_2 < s_2 < 1.
\end{aligned}
$$

Now, we define $I_{k,\ell,m}(\mu_1, \mu_2, \mu_3, \xi_1, \xi_2, \xi_3)$ as the integral on the region D_m ($m = 1, \ldots, 15$):

$$
\begin{aligned}
&I_{k,\ell,m}(\mu_1, \mu_2, \mu_3, \xi_1, \xi_2, \xi_3)\\
&:= \frac{1}{k!\ell!} \int\limits_{D_m} \left(\mu_1 \log \frac{1-s_1}{1-s_2} + \mu_2 \log \frac{1-t_1}{1-t_2} + \mu_3 \log \frac{1-u_1}{1-u_2} \right)^k\\
&\quad \times \left(\xi_1 \log \frac{s_2}{s_1} + \xi_2 \log \frac{t_2}{t_1} + \xi_3 \log \frac{u_2}{u_1} \right)^\ell \frac{ds_1 ds_2 dt_1 dt_2 du_1 du_2}{(1-s_1)s_2(1-t_1)t_2(1-u_1)u_2}.
\end{aligned}
$$

How to treat the remaining 75 regions? For example, we can see that the integration on the region $0 < u_1 < u_2 < s_1 < s_2 < t_1 < t_2 < 1$ which is one of the remaining 75 regions is written by using the integration on D_1 as follows:

$$
\begin{aligned}
&\frac{1}{k!\ell!} \int\limits_{0<u_1<u_2<s_1<s_2<t_1<t_2<1} \left(\mu_1 \log \frac{1-s_1}{1-s_2} + \mu_2 \log \frac{1-t_1}{1-t_2} + \mu_3 \log \frac{1-u_1}{1-u_2} \right)^k\\
&\quad \times \left(\xi_1 \log \frac{s_2}{s_1} + \xi_2 \log \frac{t_2}{t_1} + \xi_3 \log \frac{u_2}{u_1} \right)^\ell \frac{ds_1 ds_2 dt_1 dt_2 du_1 du_2}{(1-s_1)s_2(1-t_1)t_2(1-u_1)u_2}\\
&= I_{k,\ell,1}(\mu_3, \mu_1, \mu_2, \xi_3, \xi_1, \xi_2).
\end{aligned}
$$

Therefore the integrals on the remaining 75 regions are obtained by changing parameters in the integrals on $D_1, D_2, \ldots, D_{14}$ or D_{15}. After all, we obtain

$$
I_{k,\ell}(\mu_1, \mu_2, \mu_3, \xi_1, \xi_2, \xi_3) = \sum_{\sigma \in \mathfrak{S}_3} \sum_{m=1}^{15} I_{k,\ell,m}(\mu_{\sigma(1)}, \mu_{\sigma(2)}, \mu_{\sigma(3)}, \xi_{\sigma(1)}, \xi_{\sigma(2)}, \xi_{\sigma(3)}).
$$

To save pages, we present only the calculations on the region $D_4 : 0 < s_1 < t_1 < t_2 < u_1 < s_2 < u_2 < 1$ in this section. We want to write the integral as an explicit sum of MZVs, hence we need to modify some terms in the integrand before we expand the integrand. Those modifications are the most important point in this proof. We expand the integrand as follows:

$$\begin{aligned}
&\left(\mu_{\sigma(1)}\log\frac{1-s_1}{1-s_2}+\mu_{\sigma(2)}\log\frac{1-t_1}{1-t_2}+\mu_{\sigma(3)}\log\frac{1-u_1}{1-u_2}\right)^k\\
&=\left\{\mu_{\sigma(1)}\log\frac{1-s_1}{1-t_1}+(\mu_{\sigma(1)}+\mu_{\sigma(2)})\log\frac{1-t_1}{1-t_2}+\mu_{\sigma(1)}\log\frac{1-t_2}{1-u_1}\right.\\
&\quad\left.+(\mu_{\sigma(1)}+\mu_{\sigma(3)})\log\frac{1-u_1}{1-s_2}+\mu_{\sigma(3)}\log\frac{1-s_2}{1-u_2}\right\}^k\\
&=\sum_{a_1+\cdots+a_5=k}\frac{k!}{a_1!a_2!a_3!a_4!a_5!}\mu_{\sigma(1)}^{a_1+a_3}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}(\mu_{\sigma(1)}+\mu_{\sigma(3)})^{a_4}\mu_{\sigma(3)}^{a_5}\\
&\quad\times\left(\log\frac{1-s_1}{1-t_1}\right)^{a_1}\left(\log\frac{1-t_1}{1-t_2}\right)^{a_2}\left(\log\frac{1-t_2}{1-u_1}\right)^{a_3}\\
&\quad\times\left(\log\frac{1-u_1}{1-s_2}\right)^{a_4}\left(\log\frac{1-s_2}{1-u_2}\right)^{a_5},
\end{aligned}\tag{2.2}$$

and

$$\begin{aligned}
&\left(\xi_{\sigma(1)}\log\frac{s_2}{s_1}+\xi_{\sigma(2)}\log\frac{t_2}{t_1}+\xi_{\sigma(3)}\log\frac{u_2}{u_1}\right)^\ell\\
&=\left\{\xi_{\sigma(1)}\log\frac{t_1}{s_1}+(\xi_{\sigma(1)}+\xi_{\sigma(2)})\log\frac{t_2}{t_1}+\xi_{\sigma(1)}\log\frac{u_1}{t_2}\right.\\
&\quad\left.+(\xi_{\sigma(1)}+\xi_{\sigma(3)})\log\frac{s_2}{u_1}+\xi_{\sigma(3)}\log\frac{u_2}{s_2}\right\}^\ell\\
&=\sum_{b_1+\cdots+b_5=\ell}\frac{\ell!}{b_1!b_2!b_3!b_4!b_5!}\xi_{\sigma(1)}^{b_1+b_3}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}(\xi_{\sigma(1)}+\xi_{\sigma(3)})^{b_4}\xi_{\sigma(3)}^{b_5}\\
&\quad\times\left(\log\frac{t_1}{s_1}\right)^{b_1}\left(\log\frac{t_2}{t_1}\right)^{b_2}\left(\log\frac{u_1}{t_2}\right)^{b_3}\left(\log\frac{s_2}{u_1}\right)^{b_4}\left(\log\frac{u_2}{s_2}\right)^{b_5}.
\end{aligned}\tag{2.3}$$

The above modifications are based on the following observation. First consider $\log\frac{1-s_1}{1-s_2}$. In this case, there are t_1, t_2 and u_1 between s_1 and s_2, so we modify $\log\frac{1-s_1}{1-s_2}$ to the following form.

$$\log\frac{1-s_1}{1-s_2}=\log\frac{1-s_1}{1-t_1}+\log\frac{1-t_1}{1-t_2}+\log\frac{1-t_2}{1-u_1}+\log\frac{1-u_1}{1-s_2}.$$

There is s_2 between u_1 and u_2, so we modify $\log\frac{1-u_1}{1-u_2}$ similarly:

$$\log\frac{1-u_1}{1-u_2}=\log\frac{1-u_1}{1-s_2}+\log\frac{1-s_2}{1-u_2}.$$

These modifications give (2.2), and similarly we can show (2.3). Substituting the above modified expansions and arguing in the same way as (2.1), we get

$$
\begin{aligned}
&I_{k,\ell,4}(\mu_{\sigma(1)},\mu_{\sigma(2)},\mu_{\sigma(3)},\xi_{\sigma(1)},\xi_{\sigma(2)},\xi_{\sigma(3)})\\
&=\sum_{\substack{a_1+\cdots+a_5=k\\ b_1+\cdots+b_5=\ell}} P_{4,\sigma}\int_{D'_4}\frac{ds_1}{1-s_1}\prod_{i=1}^{a_1}\frac{dp_i}{1-p_i}\prod_{j=1}^{b_1}\frac{dq_j}{q_j}\frac{dt_1}{1-t_1}\prod_{i=a_1+1}^{a_1+a_2}\frac{dp_i}{1-p_i}\prod_{j=b_1+1}^{b_1+b_2}\frac{dq_j}{q_j}\frac{dt_2}{t_2}\\
&\quad\times\prod_{i=a_1+a_2+1}^{a_1+a_2+a_3}\frac{dp_i}{1-p_i}\prod_{j=b_1+b_2+1}^{b_1+b_2+b_3}\frac{dq_j}{q_j}\frac{du_1}{1-u_1}\prod_{i=a_1+a_2+a_3+1}^{a_1+\cdots+a_4}\frac{dp_i}{1-p_i}\prod_{j=b_1+b_2+b_3+1}^{b_1+\cdots+b_4}\frac{dq_j}{q_j}\frac{ds_2}{s_2}\\
&\quad\times\prod_{i=a_1+\cdots+a_4+1}^{a_1+\cdots+a_5}\frac{dp_i}{1-p_i}\prod_{j=b_1+\cdots+b_4+1}^{b_1+\cdots+b_5}\frac{dq_j}{q_j}\frac{du_2}{u_2}\\
&=\sum_{\substack{a_1+\cdots+a_5=k\\ b_1+\cdots+b_5=\ell}} P_{4,\sigma}\sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1\\ |\boldsymbol{\beta}|=a_2+b_2+1\\ |\boldsymbol{\gamma}|=a_3+b_3+1\\ |\boldsymbol{\delta}|=a_4+b_4+1\\ |\boldsymbol{\varepsilon}|=a_5+b_5+1}}\zeta(\alpha_0,\dots,\alpha_{a_1},\beta_0,\beta_1,\dots,\beta_{a_2}+\gamma_0,\gamma_1,\dots,\gamma_{a_3},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\delta_0,\delta_1,\dots,\delta_{a_4}+\varepsilon_0,\varepsilon_1,\dots,\varepsilon_{a_5}+1)\\
&=\sum_{\substack{a_1+\cdots+a_5=k\\ b_1+\cdots+b_5=\ell}} P_{4,\sigma}\sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1\\ |\boldsymbol{\beta}|=a_2+b_2+1\\ |\boldsymbol{\gamma}|=a_3+b_3+1\\ |\boldsymbol{\delta}|=a_4+b_4+1\\ |\boldsymbol{\varepsilon}|=a_5+b_5+1}}\zeta(\boldsymbol{\alpha},\boldsymbol{\beta}\circ\boldsymbol{\gamma},\boldsymbol{\delta}\circ\boldsymbol{\varepsilon}_+),
\end{aligned}
$$

where D'_4 is the set of all the points $(s_1,s_2,t_1,t_2,u_1,u_2,p_1,\dots,p_k,q_1,\dots,q_\ell)\in(0,1)^{k+\ell+4}$ satisfying

$$
\begin{gathered}
0<s_1<t_1<t_2<u_1<s_2<u_2<1,\\
s_1<p_1<\cdots<p_{a_1}<t_1,\\
s_1<q_1<\cdots<q_{b_1}<t_1,\\
t_1<p_{a_1+1}<\cdots<p_{a_1+a_2}<t_2,\\
t_1<q_{b_1+1}<\cdots<q_{b_1+b_2}<t_2,\\
t_2<p_{a_1+a_2+1}<\cdots<p_{a_1+a_2+a_3}<u_1,\\
t_2<q_{b_1+b_2+1}<\cdots<q_{b_1+b_2+b_3}<u_1,\\
u_1<p_{a_1+a_2+a_3+1}<\cdots<p_{a_1+\cdots+a_4}<s_2,\\
u_1<q_{b_1+b_2+b_3+1}<\cdots<q_{b_1+\cdots+b_4}<s_2,\\
s_2<p_{a_1+\cdots+a_4+1}<\cdots<p_{a_1+\cdots+a_5}<u_2,\\
s_2<q_{b_1+\cdots+b_4+1}<\cdots<q_{b_1+\cdots+b_5}<u_2,
\end{gathered}
$$

and

$$
P_{4,\sigma}=\mu_{\sigma(1)}^{a_1+a_3}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}(\mu_{\sigma(1)}+\mu_{\sigma(3)})^{a_4}\mu_{\sigma(3)}^{a_5}\xi_{\sigma(1)}^{b_1+b_3}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}(\xi_{\sigma(1)}+\xi_{\sigma(3)})^{b_4}\xi_{\sigma(3)}^{b_5}.
$$

This is the end of the calculations for $I_{k,\ell,4}(\mu_{\sigma(1)},\mu_{\sigma(2)},\mu_{\sigma(3)},\xi_{\sigma(1)},\xi_{\sigma(2)},\xi_{\sigma(3)})$. When one calculate the other $I_{k,\ell,m}(\mu_{\sigma(1)},\mu_{\sigma(2)},\mu_{\sigma(3)},\xi_{\sigma(1)},\xi_{\sigma(2)},\xi_{\sigma(3)})$ by the same way of modifications, then the $P_{m,\sigma}$ which is in Theorem 2.2 appears as the coefficient of modified integrand for the integral of D_m. Namely, one obtain

$$I_{k,\ell,m}(\mu_{\sigma(1)},\mu_{\sigma(2)},\mu_{\sigma(3)},\xi_{\sigma(1)},\xi_{\sigma(2)},\xi_{\sigma(3)}) = \sum_{a,b} P_{m,\sigma} \times (\text{the sum of MZVs})$$

as above. Moreover, several types of the sum of MZVs will appear in the calculations, but one can notice that these types of the sum of MZVs depend only on the subscript of divided regions. Namely, the sum of MZVs which appears after the calculations for D_2 and D_3 are the same, for D_4, D_5, D_7 and D_{12} are the same, for D_6 and D_{11} are the same, and for $D_8, D_9, D_{10}, D_{13}, D_{14}$ and D_{15} are the same. Noting this point, we obtain

$$\begin{aligned}
&I_{k,\ell}(\mu_1,\mu_2,\mu_3,\xi_1,\xi_2,\xi_3) \\
&= \sum_{\sigma\in\mathfrak{S}_3}\Bigg[\sum_{\substack{a_1+a_2+a_3=k\\ b_1+b_2+b_3=\ell}} P_{1,\sigma} \times \sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1\\ |\boldsymbol{\beta}|=a_2+b_2+1\\ |\boldsymbol{\gamma}|=a_3+b_3+1}} \zeta(\boldsymbol{\alpha}_+,\boldsymbol{\beta}_+,\boldsymbol{\gamma}_+) \\
&\quad + \sum_{\substack{a_1+\cdots+a_4=k\\ b_1+\cdots+b_4=\ell}} (P_{2,\sigma}+P_{3,\sigma}) \sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1\\ \vdots\\ |\boldsymbol{\delta}|=a_4+b_4+1}} \zeta(\boldsymbol{\alpha},\boldsymbol{\beta}\circ\boldsymbol{\gamma}_+,\boldsymbol{\delta}_+) \\
&\quad + \sum_{\substack{a_1+\cdots+a_5=k\\ b_1+\cdots+b_5=\ell}} (P_{4,\sigma}+P_{5,\sigma}+P_{7,\sigma}+P_{12,\sigma}) \sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1\\ \vdots\\ |\boldsymbol{\varepsilon}|=a_5+b_5+1}} \zeta(\boldsymbol{\alpha},\boldsymbol{\beta}\circ\boldsymbol{\gamma},\boldsymbol{\delta}\circ\boldsymbol{\varepsilon}_+) \\
&\quad + \sum_{\substack{a_1+\cdots+a_4=k\\ b_1+\cdots+b_4=\ell}} (P_{6,\sigma}+P_{11,\sigma}) \sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1\\ \vdots\\ |\boldsymbol{\delta}|=a_4+b_4+1}} \zeta(\boldsymbol{\alpha}_+,\boldsymbol{\beta},\boldsymbol{\gamma}\circ\boldsymbol{\delta}_+) \\
&\quad + \sum_{\substack{a_1+\cdots+a_5=k\\ b_1+\cdots+b_5=\ell}} (P_{8,\sigma}+P_{9,\sigma}+P_{10,\sigma}+P_{13,\sigma}+P_{14,\sigma}+P_{15,\sigma}) \sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1\\ \vdots\\ |\boldsymbol{\varepsilon}|=a_5+b_5+1}} \zeta(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\circ\boldsymbol{\delta}\circ\boldsymbol{\varepsilon}_+)\Bigg].
\end{aligned}$$

Combining the conclusions of the first calculations and the second calculations, we obtain the asserted relations of Theorem 2.2.

To prove Theorem 2.1, it is sufficient to calculate similarly the following integrals:

$$\begin{aligned}
&I_{k,\ell}(\mu_1,\mu_2,\xi_1,\xi_2) \\
&:= \frac{1}{k!\ell!} \int\limits_{\substack{0<s_1<s_2<1\\ 0<t_1<t_2<1}} \left(\mu_1\log\frac{1-s_1}{1-s_2} + \mu_2\log\frac{1-t_1}{1-t_2}\right)^k
\end{aligned}$$

$$\times \left(\xi_1 \log \frac{s_2}{s_1} + \xi_2 \log \frac{t_2}{t_1}\right)^\ell \frac{ds_1 ds_2 dt_1 dt_2}{(1-s_1)s_2(1-t_1)t_2}.$$

We omit the details of the proof.

2.3 Deduction of the result of Eie, Liaw and Ong, and some remarks

Let us consider a special case of Theorem 2.1. Putting $\xi_1 = \xi_2 = \xi \neq 0$ in the formula of Theorem 2.1, and using the harmonic product $\zeta(a)\zeta(b) = \zeta(a,b) + \zeta(b,a) + \zeta(a+b)$ for the left hand side, we find that ξ^ℓ parts of the both sides are canceled with each other, and

$$\begin{aligned}
&(\ell+1)\zeta(k+\ell+4) \sum_{a_1+a_2=k} \mu_1^{a_1}\mu_2^{a_2} \\
&\quad + \sum_{\substack{a_1+a_2=k \\ b_1+b_2=\ell}} (\mu_1^{a_1}\mu_2^{a_2} + \mu_2^{a_1}\mu_1^{a_2})\zeta(a_2+b_2+2, a_1+b_1+2) \\
&= \sum_{\substack{a_1+a_2=k \\ b_1+b_2=\ell}} (\mu_1^{a_1}\mu_2^{a_2} + \mu_2^{a_1}\mu_1^{a_2}) \sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1 \\ |\boldsymbol{\beta}|=a_2+b_2+1}} \zeta(\alpha_0, \alpha_1, \ldots, \alpha_{a_1}+1, \beta_0, \beta_1, \ldots, \beta_{a_2}+1) \\
&\quad + \sum_{\substack{a_1+a_2+a_3=k \\ b_1+b_2+b_3=\ell}} (\mu_1^{a_1} + \mu_2^{a_1})(\mu_1+\mu_2)^{a_2} 2^{b_2} (\mu_1^{a_3} + \mu_2^{a_3}) \\
&\times \sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1 \\ |\boldsymbol{\beta}|=a_2+b_2+1 \\ |\boldsymbol{\gamma}|=a_3+b_3+1}} \zeta(\alpha_0, \ldots, \alpha_{a_1}, \beta_0, \beta_1, \ldots, \beta_{a_2}+\gamma_0, \gamma_1, \ldots, \gamma_{a_3}+1).
\end{aligned} \tag{2.4}$$

Applying Ohno's relation ([38]) for the index $(\underbrace{1,\ldots,1}_{a_1}, 2, \underbrace{1,\ldots,1}_{a_2}, 2)$, we can see that

$$\begin{aligned}
&\sum_{b_1+b_2=\ell} \sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1 \\ |\boldsymbol{\beta}|=a_2+b_2+1}} \zeta(\alpha_0, \alpha_1, \ldots, \alpha_{a_1}+1, \beta_0, \beta_1, \ldots, \beta_{a_2}+1) \\
&= \sum_{b_1+b_2=\ell} \zeta(a_2+b_2+2, a_1+b_1+2).
\end{aligned}$$

Then the first term on the right hand side of (2.4) is same as the the second term on the left hand side of (2.4). The second term on the right hand side of (2.4) can be written as follows:

$$
\begin{aligned}
&\sum_{b_1+b_2+b_3=\ell}\ \sum_{\substack{|\boldsymbol{\alpha}|=a_1+b_1+1\\|\boldsymbol{\beta}|=a_2+b_2+1\\|\boldsymbol{\gamma}|=a_3+b_3+1}} 2^{b_2}\zeta(\alpha_0,\ldots,\alpha_{a_1},\beta_0,\beta_1,\ldots,\beta_{a_2}+\gamma_0,\gamma_1,\ldots,\gamma_{a_3}+1)\\
&=\sum_{\substack{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+|\boldsymbol{\gamma}|\\-(a_1+a_2+a_3)-3=\ell}} 2^{|\boldsymbol{\beta}|-1-a_2}\zeta(\alpha_0,\ldots,\alpha_{a_1},\beta_0,\beta_1,\ldots,\beta_{a_2}+\gamma_0,\gamma_1,\ldots,\gamma_{a_3}+1)\\
&=\sum_{|\boldsymbol{\alpha}'|=k+\ell+3} 2^{\alpha'_{a_1+1}+\cdots+\alpha'_{a_1+a_2}-1-a_2}\left(\sum_{\beta_{a_2}+\gamma_0=\alpha'_{a_1+a_2+1}} 2^{\beta_{a_2}}\right)\zeta(\alpha'_0,\alpha'_1,\ldots,\alpha'_{k+1}+1)\\
&=\sum_{|\boldsymbol{\alpha}'|=k+\ell+3} 2^{\alpha'_{a_1+1}+\cdots+\alpha'_{a_1+a_2+1}-1-a_2}(1-2^{1-\alpha'_{a_1+a_2+1}})\zeta(\alpha'_0,\alpha'_1,\ldots,\alpha'_{k+1}+1).
\end{aligned}
$$

Here we used $a_1+a_2+a_3=k$. Then we have

$$
\begin{aligned}
&\sum_{|\boldsymbol{\alpha}|=k+\ell+3}\ \sum_{a_1+a_2+a_3=k}(\mu_1^{a_1}+\mu_2^{a_1})(\mu_1+\mu_2)^{a_2}(\mu_1^{a_3}+\mu_2^{a_3})\\
&\quad\times 2^{\alpha_{a_1+1}+\cdots+\alpha_{a_1+a_2+1}-1-a_2}(1-2^{1-\alpha_{a_1+a_2+1}})\zeta(\alpha_0,\alpha_1,\ldots,\alpha_{k+1}+1)\\
&=(\ell+1)\zeta(k+\ell+4)\sum_{a_1+a_2=k}\mu_1^{a_1}\mu_2^{a_2}.
\end{aligned}
$$

In particular, when $\mu_1=-\mu_2=\mu\neq 0$ and k is even, the right hand side is

$$
\mu^k(\ell+1)\zeta(k+\ell+4),
$$

and the left hand side is

$$
\mu^k\left\{\sum_{|\boldsymbol{\alpha}|=k+\ell+3}\sum_{j=0}^{\frac{k}{2}}2^{\alpha_{2j+1}+1}\zeta(\alpha_0,\alpha_1,\ldots,\alpha_{k+1}+1)-2(k+2)\zeta(k+\ell+4)\right\}.
$$

by the sum formula. Therefore we have

$$
\sum_{|\boldsymbol{\alpha}|=k+\ell+3}\sum_{j=1}^{\frac{k}{2}+1}2^{\alpha_{2j+1}+1}\zeta(\alpha_1,\alpha_2,\ldots,\alpha_{k+2}+1)=(2k+\ell+5)\zeta(k+\ell+4).
$$

For positive integers m,n with $m>2n$, setting $k=2n-2$, $\ell=m-2n-1$, we get [9, Main Theorem]. When $m=2n$, [9, Main Theorem] is derived from the duality theorem. Therefore we now conclude that [9, Main Theorem] can be deduced from our Theorem 2.1.

Remark 2.6. More generally, by calculating integrals

$$
\begin{aligned}
&I_{k,\ell}(\mu_1,\dots,\mu_n,\xi_1,\dots,\xi_n)\\
&=\frac{1}{k!\ell!}\int\limits_{\substack{0<x_1<y_1<1\\ \vdots \\ 0<x_n<y_n<1}}\left(\sum_{i=1}^{n}\mu_i\log\frac{1-x_i}{1-y_i}\right)^k\left(\sum_{j=1}^{n}\xi_j\log\frac{y_j}{x_j}\right)^\ell\prod_{h=1}^{n}\frac{dx_h dy_h}{(1-x_h)y_h},
\end{aligned}
$$

we could get the same type of relations, but it seems difficult to write down the general form explicitly.

Remark 2.7. The author has shown that the sum of MZVs which is obtained as the coefficient of the parameter $\mu_1^{m_1}\mu_2^{m_2}\xi_1^{n_1}\xi_2^{n_2}$ (m_1,m_2,n_1,n_2 are non negative integers and satisfy $m_1+m_2=k, n_1+n_2=\ell$) on the right hand side of Theorem 2.1 is equal to the shuffle product of

$$\sum_{|\boldsymbol{\alpha}|=m_1+n_1+1}\zeta(\alpha_0,\alpha_1,\dots,\alpha_{m_1}+1)$$

and

$$\sum_{|\boldsymbol{\alpha}|=m_2+n_2+1}\zeta(\alpha_0,\alpha_1,\dots,\alpha_{m_2}+1).$$

Hence our theorems may be regarded as generating functions of those relations. In particular, we find that the relation [9, Main Theorem] is obtained by using the shuffle product, the harmonic product and Ohno's relation. Moreover, it seems that the same type of phenomena would appear in the general case mentioned in Remark 2.6.

3 Weighted sum formulas for multiple zeta star values

3.1 Weighted sum formulas for multiple zeta star values

As mentioned in Section 1, for an r-tuple of natural numbers $(k_1, k_2, \ldots, k_r)$ with $k_r \geq 2$, the multiple zeta value (for short MZV) $\zeta(k_1, k_2, \ldots, k_r)$ and the multiple zeta-star value (for short MZSV) $\zeta^\star(k_1, k_2, \ldots, k_r)$ are defined as the following r-ple series, respectively:

$$\zeta(k_1, k_2, \ldots, k_r) := \sum_{1 \leq \ell_1 < \ell_2 < \cdots < \ell_r} \frac{1}{\ell_1^{k_1} \ell_2^{k_2} \ldots \ell_r^{k_r}},$$
$$\zeta^\star(k_1, k_2, \ldots, k_r) := \sum_{1 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_r} \frac{1}{\ell_1^{k_1} \ell_2^{k_2} \ldots \ell_r^{k_r}}.$$

The number r is called the depth, and the sum $k = k_1 + \cdots + k_r$ is called the weight. By definition, we can find that MZVs and MZSVs can be expressed as the linear combination of the others with rational coefficients. We can find that MZSV has the following iterated integral representation:

$$\zeta^\star(k_1, k_2, \ldots, k_r) = \int \cdots \int_{0 < t_1 < t_2 < \cdots < t_k < 1} \omega_1(t_1) \omega_2(t_2) \ldots \omega_k(t_k),$$

where

$$\omega_i(t) = \begin{cases} \dfrac{dt}{1-t} & , \quad i = 1, \\ \dfrac{dt}{t(1-t)} & , \quad i \in \{k_1 + 1, k_1 + k_2 + 1, \ldots, k_1 + \cdots + k_{r-1} + 1\}, \\ \dfrac{dt}{t} & , \quad \text{otherwise.} \end{cases} \tag{3.1}$$

Euler [10] studied the properties of MZVs and MZSVs in the case $r = 2$ and gave some relations among them, for example $\zeta(1, 2) = \zeta(3)$ and Euler's sum formula that is the sum of MZVs with weight k and depth 2 is equal to $\zeta(k)$. In 1990's, Hoffman [13] and Zagier [53] defined MZVs with general depth independently. Then, many people started to study MZVs and MZSVs (cf.[16]), and a lot of relations have been known, for example the sum formula which is an extension of Euler's sum formula to general depth and proved by Granville [12], the weighted sum formula which is the weighted analogue of Euler's sum formula and proved by Ohno and Zudilin [39, Theorem 3]. Eie, Liaw and Ong [9, Main Theorem] gave one generalization of the relation of Ohno and Zudilin. Moreover, the author [18] obtained a relation for MZVs which is a certain generalization of [9, Main Theorem]. In this section, we will obtain MZSV's version of [18] by using a similar method.

Before stating theorems, we introduce some notations. Let $\mathfrak{S}_n$ be the symmetric group of n-th order. For indices $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{N}^r$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_s) \in \mathbb{N}^s$ (α_r and β_s can be 1), we define $|\boldsymbol{\alpha}|$, $d(\boldsymbol{\alpha})$, $\boldsymbol{\alpha}_+$ and $\boldsymbol{\alpha} \circ \boldsymbol{\beta}$ as follows:

$$\begin{aligned} |\boldsymbol{\alpha}| &= \alpha_1 + \alpha_2 + \cdots + \alpha_r, \\ d(\boldsymbol{\alpha}) &= r, \\ \boldsymbol{\alpha}_+ &= (\alpha_1, \alpha_2, \ldots, \alpha_r + 1), \\ \boldsymbol{\alpha} \circ \boldsymbol{\beta} &= (\alpha_1, \alpha_2, \ldots, \alpha_r + \beta_1, \beta_2, \ldots, \beta_s). \end{aligned}$$

Theorem 3.1. *For non-negative integers m and n, and four parameters (indeterminates) μ_1, μ_2, ξ_1 and ξ_2, we have*

$$\begin{aligned} &\sum_{\substack{a_1+a_2=m \\ b_1+b_2=n}} \prod_{i=1}^{2} \mu_i^{a_i} \xi_i^{b_i} \binom{a_i + b_i + 1}{a_i} \zeta(a_i + b_i + 2) \\ &= \sum_{\sigma \in \mathfrak{S}_2} \Bigg[\sum_{\substack{a_1+a_2=m \\ b_1+b_2=n}} \mu_{\sigma(1)}^{a_1} \mu_{\sigma(2)}^{a_2} \xi_{\sigma(1)}^{b_1} \xi_{\sigma(2)}^{b_2} \sum_{\substack{\square = \text{ , or } \circ \\ |\boldsymbol{\alpha}| = a_1+b_1+1, d(\boldsymbol{\alpha}) = a_1+1 \\ |\boldsymbol{\beta}| = a_2+b_2+1, d(\boldsymbol{\beta}) = a_2+1}} (-1)^{N(\circ)} \zeta^\star(\boldsymbol{\alpha}_+ \,\square\, \boldsymbol{\beta}_+) \\ &\quad + \sum_{\substack{a_1+a_2+a_3=m \\ b_1+b_2+b_3=n}} \mu_{\sigma(1)}^{a_1} \xi_{\sigma(1)}^{b_1} (\mu_{\sigma(1)} + \mu_{\sigma(2)})^{a_2} (\xi_{\sigma(1)} + \xi_{\sigma(2)})^{b_2} (\mu_{\sigma(1)}^{a_3} \xi_{\sigma(1)}^{b_3} + \mu_{\sigma(2)}^{a_3} \xi_{\sigma(2)}^{b_3}) \\ &\quad \sum_{\substack{\square = \text{ , or } \circ \\ |\boldsymbol{\alpha}| = a_1+b_1+1, d(\boldsymbol{\alpha}) = a_1+1 \\ |\boldsymbol{\beta}| = a_2+b_2+1, d(\boldsymbol{\beta}) = a_2+1 \\ |\boldsymbol{\gamma}| = a_3+b_3+1, d(\boldsymbol{\gamma}) = a_3+1}} (-1)^{N(\circ)} \zeta^\star(\boldsymbol{\alpha} \,\square\, \boldsymbol{\beta} \circ \boldsymbol{\gamma}_+) \Bigg], \end{aligned}$$

where $N(\circ)$ is the number of $\circ$ in $\square$ [1]*, a_i and b_i run over non-negative integers with the conditions that the sum of those are m and n, and the summation $\sum_{\substack{|\boldsymbol{\alpha}| = a_1+b_1+1, d(\boldsymbol{\alpha}) = a_1+1 \\ |\boldsymbol{\beta}| = a_2+b_2+1, d(\boldsymbol{\beta}) = a_2+1}}$ runs over all indices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ which satisfy $|\boldsymbol{\alpha}| = a_1 + b_1 + 1$, $d(\boldsymbol{\alpha}) = a_1 + 1$, $|\boldsymbol{\beta}| = a_2 + b_2 + 1$, and $d(\boldsymbol{\beta}) = a_2 + 1$.*

Here, we give some relations which are obtained by comparing the coefficients of $\mu_1^0\mu_2^0\xi_1^0\xi_2^0$, $\mu_1^1\mu_2^1\xi_1^0\xi_2^0$, $\mu_1^0\mu_2^0\xi_1^1\xi_2^1$, $\mu_1^1\mu_2^0\xi_1^1\xi_2^0$ and $\mu_1^1\mu_2^0\xi_1^0\xi_2^1$ of the both sides of equation in Theorem 3.1.

- (coefficients of $\mu_1^0\mu_2^0\xi_1^0\xi_2^0$)

$$4\zeta^\star(1,3) + 2\zeta^\star(2,2) - 6\zeta(4) = \zeta(2)\zeta(2),$$

[1]The author does not use the Kronecker symbol although $N(\circ)$ is 0 or 1 in this theorem. The reason is that $N(\circ)$ is 0, 1, or 2 in the next theorem.

- (coefficients of $\mu_1^0\mu_2^0\xi_1^1\xi_2^1$)

$$12\zeta^\star(1,5)+6\zeta^\star(2,4)+2\zeta^\star(3,3)-20\zeta(6)=\zeta(3)\zeta(3),$$

- (coefficients of $\mu_1^1\mu_2^0\xi_1^1\xi_2^0$)

$$\begin{aligned}&9\zeta^\star(1,1,4)+8\zeta^\star(1,2,3)+5\zeta^\star(1,3,2)+5\zeta^\star(2,1,3)+3\zeta^\star(2,2,2)\\&\quad-6\zeta^\star(1,5)-9\zeta^\star(2,4)-9\zeta^\star(3,3)-6\zeta^\star(4,2)=3\zeta(4)\zeta(2),\end{aligned}$$

- (coefficients of $\mu_1^1\mu_2^0\xi_1^0\xi_2^1$)

$$\begin{aligned}&9\zeta^\star(1,1,4)+5\zeta^\star(1,2,3)+2\zeta^\star(1,3,2)+2\zeta^\star(2,1,3)+\zeta^\star(2,2,2)+\zeta^\star(3,1,2)\\&\quad-4\zeta^\star(1,5)-6\zeta^\star(2,4)-6\zeta^\star(3,3)-4\zeta^\star(4,2)=2\zeta(3)\zeta(3),\end{aligned}$$

- (coefficients of $\mu_1^1\mu_2^1\xi_1^0\xi_2^0$)

$$\begin{aligned}&6\zeta^\star(1,1,1,3)+3\zeta^\star(1,1,2,2)+\zeta^\star(1,2,1,2)\\&\quad-2\zeta^\star(1,2,3)-2\zeta^\star(1,3,2)-4\zeta^\star(2,1,3)-2\zeta^\star(2,2,2)=2\zeta(3)\zeta(3),\end{aligned}$$

The next theorem is the main theorem of this section.

Theorem 3.2. *For non-negative integers m and n, and six parameters (indeterminates) $\mu_1,\mu_2,\mu_3,\xi_1,\xi_2$ and ξ_3, we have the following relation:*

$$\begin{aligned}&\sum_{\substack{a_1+a_2+a_3=m\\b_1+b_2+b_3=n}}\prod_{i=1}^{3}\mu_i^{a_i}\xi_i^{b_i}\binom{a_i+b_i+1}{a_i}\zeta(a_i+b_i+2)\\
&=\sum_{\sigma\in\mathfrak{S}_3}\Bigg[\sum_{\substack{a_1+a_2+a_3=m\\b_1+b_2+b_3=n}}P_{1,\sigma}\sum_{\substack{\square=\ ,\ \text{or}\ \circ\\|\boldsymbol{\alpha}|=a_1+b_1+1,d(\boldsymbol{\alpha})=a_1+1\\|\boldsymbol{\beta}|=a_2+b_2+1,d(\boldsymbol{\beta})=a_2+1\\|\boldsymbol{\gamma}|=a_3+b_3+1,d(\boldsymbol{\gamma})=a_3+1}}(-1)^{N(\circ)}\zeta^\star(\boldsymbol{\alpha}_+\ \square\ \boldsymbol{\beta}_+\ \square\ \boldsymbol{\gamma}_+)\\
&\quad+\sum_{\substack{a_1+\cdots+a_4=m\\b_1+\cdots+b_4=n}}(P_{2,\sigma}+P_{3,\sigma})\sum_{\substack{\square=\ ,\ \text{or}\ \circ\\|\boldsymbol{\alpha}|=a_1+b_1+1,d(\boldsymbol{\alpha})=a_1+1\\\vdots\\|\boldsymbol{\delta}|=a_4+b_4+1,d(\boldsymbol{\delta})=a_4+1}}(-1)^{N(\circ)}\zeta^\star(\boldsymbol{\alpha}\ \square\ \boldsymbol{\beta}\circ\boldsymbol{\gamma}_+\ \square\ \boldsymbol{\delta}_+)\\
&\quad+\sum_{\substack{a_1+\cdots+a_5=m\\b_1+\cdots+b_5=n}}(P_{4,\sigma}+P_{5,\sigma}+P_{7,\sigma}+P_{12,\sigma})\\
&\qquad\times\sum_{\substack{\square=\ ,\ \text{or}\ \circ\\|\boldsymbol{\alpha}|=a_1+b_1+1,d(\boldsymbol{\alpha})=a_1+1\\\vdots\\|\boldsymbol{\varepsilon}|=a_5+b_5+1,d(\boldsymbol{\varepsilon})=a_5+1}}(-1)^{N(\circ)}\zeta^\star(\boldsymbol{\alpha}\ \square\ \boldsymbol{\beta}\circ\boldsymbol{\gamma}\ \square\ \boldsymbol{\delta}\circ\boldsymbol{\varepsilon}_+)\\
&\quad+\sum_{\substack{a_1+\cdots+a_4=m\\b_1+\cdots+b_4=n}}(P_{6,\sigma}+P_{11,\sigma})\sum_{\substack{\square=\ ,\ \text{or}\ \circ\\|\boldsymbol{\alpha}|=a_1+b_1+1,d(\boldsymbol{\alpha})=a_1+1\\\vdots\\|\boldsymbol{\delta}|=a_4+b_4+1,d(\boldsymbol{\delta})=a_4+1}}(-1)^{N(\circ)}\zeta^\star(\boldsymbol{\alpha}_+\ \square\ \boldsymbol{\beta}\ \square\ \boldsymbol{\gamma}\circ\boldsymbol{\delta}_+)\end{aligned}$$

$$+ \sum_{\substack{a_1+\cdots+a_5=m \\ b_1+\cdots+b_5=n}} (P_{8,\sigma} + P_{9,\sigma} + P_{10,\sigma} + P_{13,\sigma} + P_{14,\sigma} + P_{15,\sigma})$$

$$\times \sum_{\substack{\Box= \ , \text{ or } \circ \\ |\boldsymbol{\alpha}|=a_1+b_1+1, d(\boldsymbol{\alpha})=a_1+1 \\ \vdots \\ |\boldsymbol{\varepsilon}|=a_5+b_5+1, d(\boldsymbol{\varepsilon})=a_5+1}} (-1)^{N(\circ)} \zeta^\star(\boldsymbol{\alpha} \Box \boldsymbol{\beta} \Box \boldsymbol{\gamma} \circ \boldsymbol{\delta} \circ \boldsymbol{\varepsilon}_+)\Big].$$

where

$$\begin{aligned}
P_{1,\sigma} &= \mu_{\sigma(1)}^{a_1}\mu_{\sigma(2)}^{a_2}\mu_{\sigma(3)}^{a_3}\xi_{\sigma(1)}^{b_1}\xi_{\sigma(2)}^{b_2}\xi_{\sigma(3)}^{b_3}, \\
P_{2,\sigma} &= \mu_{\sigma(1)}^{a_1}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}\mu_{\sigma(2)}^{a_3}\mu_{\sigma(3)}^{a_4}\xi_{\sigma(1)}^{b_1}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}\xi_{\sigma(2)}^{b_3}\xi_{\sigma(3)}^{b_4}, \\
P_{3,\sigma} &= \mu_{\sigma(1)}^{a_1+a_3}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}\mu_{\sigma(3)}^{a_4}\xi_{\sigma(1)}^{b_1+b_3}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}\xi_{\sigma(3)}^{b_4}, \\
P_{4,\sigma} &= \mu_{\sigma(1)}^{a_1+a_3}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}(\mu_{\sigma(1)}+\mu_{\sigma(3)})^{a_4}\mu_{\sigma(3)}^{a_5} \\
&\quad \times \xi_{\sigma(1)}^{b_1+b_3}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}(\xi_{\sigma(1)}+\xi_{\sigma(3)})^{b_4}\xi_{\sigma(3)}^{b_5}, \\
P_{5,\sigma} &= \mu_{\sigma(1)}^{a_1+a_3+a_5}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}(\mu_{\sigma(1)}+\mu_{\sigma(3)})^{a_4}\xi_{\sigma(1)}^{b_1+b_3+b_5}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}(\xi_{\sigma(1)}+\xi_{\sigma(3)})^{b_4}, \\
P_{6,\sigma} &= \mu_{\sigma(1)}^{a_1}\mu_{\sigma(2)}^{a_2}(\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_3}\mu_{\sigma(3)}^{a_4}\xi_{\sigma(1)}^{b_1}\xi_{\sigma(2)}^{b_2}(\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_3}\xi_{\sigma(3)}^{b_4}, \\
P_{7,\sigma} &= \mu_{\sigma(1)}^{a_1}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}\mu_{\sigma(2)}^{a_3}(\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_4}\mu_{\sigma(3)}^{a_5} \\
&\quad \times \xi_{\sigma(1)}^{b_1}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}\xi_{\sigma(2)}^{b_3}(\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_4}\xi_{\sigma(3)}^{b_5}, \\
P_{8,\sigma} &= \mu_{\sigma(1)}^{a_1}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}(\mu_{\sigma(1)}+\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_3}(\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_4}\mu_{\sigma(3)}^{a_5} \\
&\quad \times \xi_{\sigma(1)}^{b_1}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}(\xi_{\sigma(1)}+\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_3}(\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_4}\xi_{\sigma(3)}^{b_5}, \\
P_{9,\sigma} &= \mu_{\sigma(1)}^{a_1}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}(\mu_{\sigma(1)}+\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_3}(\mu_{\sigma(1)}+\mu_{\sigma(3)})^{a_4}\mu_{\sigma(3)}^{a_5} \\
&\quad \times \xi_{\sigma(1)}^{b_1}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}(\xi_{\sigma(1)}+\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_3}(\xi_{\sigma(1)}+\xi_{\sigma(3)})^{b_4}\xi_{\sigma(3)}^{b_5}, \\
P_{10,\sigma} &= \mu_{\sigma(1)}^{a_1+a_5}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}(\mu_{\sigma(1)}+\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_3}(\mu_{\sigma(1)}+\mu_{\sigma(3)})^{a_4} \\
&\quad \times \xi_{\sigma(1)}^{b_1+b_5}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}(\xi_{\sigma(1)}+\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_3}(\xi_{\sigma(1)}+\xi_{\sigma(3)})^{b_4}, \\
P_{11,\sigma} &= \mu_{\sigma(1)}^{a_1}\mu_{\sigma(2)}^{a_2+a_4}(\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_3}\xi_{\sigma(1)}^{b_1}\xi_{\sigma(2)}^{b_2+b_4}(\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_3}, \\
P_{12,\sigma} &= \mu_{\sigma(1)}^{a_1}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}\mu_{\sigma(2)}^{a_3+a_5}(\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_4} \\
&\quad \times \xi_{\sigma(1)}^{b_1}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}\xi_{\sigma(2)}^{b_3+b_5}(\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_4}, \\
P_{13,\sigma} &= \mu_{\sigma(1)}^{a_1}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2}(\mu_{\sigma(1)}+\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_3}(\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_4}\mu_{\sigma(2)}^{a_5} \\
&\quad \times \xi_{\sigma(1)}^{b_1}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2}(\xi_{\sigma(1)}+\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_3}(\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_4}\xi_{\sigma(2)}^{b_5}, \\
P_{14,\sigma} &= \mu_{\sigma(1)}^{a_1}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2+a_4}(\mu_{\sigma(1)}+\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_3}\mu_{\sigma(2)}^{a_5} \\
&\quad \times \xi_{\sigma(1)}^{b_1}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2+b_4}(\xi_{\sigma(1)}+\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_3}\xi_{\sigma(2)}^{b_5}, \\
P_{15,\sigma} &= \mu_{\sigma(1)}^{a_1+a_5}(\mu_{\sigma(1)}+\mu_{\sigma(2)})^{a_2+a_4}(\mu_{\sigma(1)}+\mu_{\sigma(2)}+\mu_{\sigma(3)})^{a_3} \\
&\quad \times \xi_{\sigma(1)}^{b_1+b_5}(\xi_{\sigma(1)}+\xi_{\sigma(2)})^{b_2+b_4}(\xi_{\sigma(1)}+\xi_{\sigma(2)}+\xi_{\sigma(3)})^{b_3}.
\end{aligned}$$

3.2 Proof of Theorems

In this section, we show a brief sketch of the proofs of Theorems 3.1 and 3.2. The proof of Theorem 3.1 is almost the same as in [18], namely to obtain Theorem 3.1 we calculate the following integral in two ways:

$$\frac{1}{m!n!}\int\limits_{\substack{0<s_1<s_2<1\\0<t_1<t_2<1}}\left(\mu_1\log\frac{s_2(1-s_1)}{s_1(1-s_2)}+\mu_2\log\frac{t_2(1-t_1)}{t_1(1-t_2)}\right)^m$$
$$\times\left(\xi_1\log\frac{s_2}{s_1}+\xi_2\log\frac{t_2}{t_1}\right)^n\frac{ds_1ds_2dt_1dt_2}{(1-s_1)s_2(1-t_1)t_2}.$$

• The first calculations:
Expanding the two factors

$$\left(\mu_1\log\frac{s_2(1-s_1)}{s_1(1-s_2)}+\mu_2\log\frac{t_2(1-t_1)}{t_1(1-t_2)}\right)^m$$

and

$$\left(\xi_1\log\frac{s_2}{s_1}+\xi_2\log\frac{t_2}{t_1}\right)^n,$$

and by an argument similar to [18, eq. (1)], we can see that the above integral is

$$=\sum_{\substack{a_1+a_2=m\\b_1+b_2=n}}\mu_1^{a_1}\mu_2^{a_2}\xi_1^{b_1}\xi_2^{b_2}\sum_{|\boldsymbol{\alpha}|=a_1+b_1+1}\zeta^\star(\alpha_0,\alpha_1,\dots,\alpha_{a_1}+1)$$
$$\times\sum_{|\boldsymbol{\alpha}|=a_2+b_2+1}\zeta^\star(\alpha_0,\alpha_1,\dots,\alpha_{a_2}+1).$$

Then, the sum formula for MZSVs [13, Section 3, equation (1)] (that is the sum of MZSVs with fixed weight k and fixed depth r is equal to $\binom{k-1}{r-1}\zeta(k)$) implies that the above is equal to the left hand side of the statement of Theorem 3.1.
• The second calculations:
We divide the region $0<s_1<s_2<1, 0<t_1<t_2<1$ to six regions and calculate the integral on each regions:

$$\begin{aligned}
&D_1: 0<s_1<s_2<t_1<t_2<1,\\
&D_2: 0<t_1<t_2<s_1<s_2<1,\\
&D_3: 0<s_1<t_1<s_2<t_2<1,\\
&D_4: 0<t_1<s_1<t_2<s_2<1,\\
&D_5: 0<s_1<t_1<t_2<s_2<1,\\
&D_6: 0<t_1<s_1<s_2<t_2<1.
\end{aligned}$$

As the same as in the second calculations in [18], it is sufficient to calculate the integrals on the regions D_1, D_3 and D_5. However, there is one different point from the second calculations in [18] that is we have to use the partial fraction decomposition $\frac{1}{1-t} = \frac{1}{t(1-t)} - \frac{1}{t}$ to rewrite the integrals as the sum of MZSVs. The summation $\sum_{\square = \text{ , or } \circ}$ arises from this partial fraction decomposition. For example, let us see the calculations of the integral on the region D_3. Modifying and expanding two factors, we find that the integral on the region D_3 is

$$
\begin{aligned}
&= \sum_{\substack{a_1+a_2+a_3=m \\ b_1+b_2+b_3=n}} \frac{\mu_1^{a_1}(\mu_1+\mu_2)^{a_2}\mu_2^{a_3}\xi_1^{b_1}(\xi_1+\xi_2)^{b_2}\xi_2^{b_3}}{a_1!a_2!a_3!b_1!b_2!b_3!} \int_{D_3} \frac{ds_1}{1-s_1} \left(\log \frac{t_1(1-s_1)}{s_1(1-t_1)}\right)^{a_1} \left(\log \frac{t_1}{s_1}\right)^{b_1} \\
&\quad \times \frac{dt_1}{1-t_1} \left(\log \frac{s_2(1-t_1)}{t_1(1-s_2)}\right)^{a_2} \left(\log \frac{s_1}{t_1}\right)^{b_2} \frac{ds_2}{s_2} \left(\log \frac{t_2(1-s_2)}{s_2(1-t_2)}\right)^{a_3} \left(\log \frac{t_2}{s_2}\right)^{b_3} \frac{dt_2}{t_2} \\
&= \sum_{\substack{a_1+a_2+a_3=m \\ b_1+b_2+b_3=n}} \frac{\mu_1^{a_1}(\mu_1+\mu_2)^{a_2}\mu_2^{a_3}\xi_1^{b_1}(\xi_1+\xi_2)^{b_2}\xi_2^{b_3}}{a_1!a_2!a_3!b_1!b_2!b_3!} \int_{D_3} \frac{ds_1}{1-s_1} \left(\int_{s_1}^{t_1} \frac{dp}{p(1-p)}\right)^{a_1} \left(\int_{s_1}^{t_1} \frac{dq}{q}\right)^{b_1} \\
&\quad \times \frac{dt_1}{1-t_1} \left(\int_{t_1}^{s_2} \frac{dp}{p(1-p)}\right)^{a_2} \left(\int_{t_1}^{s_2} \frac{dq}{q}\right)^{b_2} \frac{ds_2}{s_2} \left(\int_{s_2}^{t_2} \frac{dp}{p(1-p)}\right)^{a_3} \left(\int_{s_2}^{t_2} \frac{dq}{q}\right)^{b_3} \frac{dt_2}{t_2}.
\end{aligned}
$$

In the above integral, we notice that there are differential forms $\frac{dt}{1-t}$, $\frac{dt}{t(1-t)}$ and $\frac{dt}{t}$. As we know from (3.1), the integral representation of MZSVs can contain the differential form $\frac{dt}{1-t}$ just one time for the smallest integral variable, that is s_1 in this case. Therefore, we have to use the partial fraction decomposition $\frac{1}{1-t_1} = \frac{1}{t_1(1-t_1)} - \frac{1}{t_1}$ and we must NOT use the partial fraction decomposition for $\frac{1}{1-s_1}$ to rewrite the above integral as the sum of MZSVs. In the summation $\sum_{\square = \text{ , or } \circ}$, the term which arises from $\frac{1}{t_1(1-t_1)}$ corresponds to " , ", and the term which arises from $\frac{1}{t_1}$ corresponds to " $\circ$ ". Similarly, we use $\frac{1}{1-t_1} = \frac{1}{t_1(1-t_1)} - \frac{1}{t_1}$ in the calculations of the integral on the regions D_1 and D_5. After using the partial fraction decomposition, the proof will be completed by the same way as in the second calculations in [18].

To prove Theorem 3.2, we calculate the following integral in two ways:

$$
\begin{aligned}
&\frac{1}{m!n!} \int_{\substack{0<s_1<s_2<1 \\ 0<t_1<t_2<1 \\ 0<u_1<u_2<1}} \left(\mu_1 \log \frac{s_2(1-s_1)}{s_1(1-s_2)} + \mu_2 \log \frac{t_2(1-t_1)}{t_1(1-t_2)} + \mu_3 \log \frac{u_2(1-u_1)}{u_1(1-u_2)}\right)^m \\
&\qquad \times \left(\xi_1 \log \frac{s_2}{s_1} + \xi_2 \log \frac{t_2}{t_1} + \xi_3 \log \frac{u_2}{u_1}\right)^n \frac{ds_1 ds_2 dt_1 dt_2 du_1 du_2}{(1-s_1)s_2(1-t_1)t_2(1-u_1)u_2}.
\end{aligned}
$$

The different point from the proof of Theorem 3.1 is that we have to use the partial fraction decomposition two times for the differential form $\frac{dt}{1-t}$ except for the differential form for the smallest integral variable.

Remark 3.3. (1) Eie, Liaw and Ong ([8], and their papers referred in that book) have given several relations among MZVs by considering various integrands and using the same

method. In the present section, the author can find a suitable integrand based on the idea of [9] for getting relations among MZSVs. However, in the case of other results on Eie, Liaw and Ong, the author can NOT find any suitable integrand to get MZSV analogues.
(2) In [18], the author got a weighted sum formula for MZVs [9, Main Theorem] by specializing parameters and using the Ohno relation [38] for MZVs. However, we cannot go through the same process because the analogue of the Ohno relation for MZSVs has not been known yet.

4 Evaluation of Tornheim's type of double series

4.1 Evaluation of Tornheim's type of double series

For integers $a, b, k_1, k_2, k_3 \geq 1$, let

$$\zeta_{a,b}(k_1, k_2, k_3) := \sum_{m,n>0} \frac{1}{m^{k_1} n^{k_2} (am+bn)^{k_3}},$$

which converges absolutely and gives a real number. Since Tornheim [45] first studied the value $\zeta_{1,1}(k_1, k_2, k_3)$, we call the value $\zeta_{a,b}(k_1, k_2, k_3)$ Tornheim's type of double series (note that the function $\zeta_{a,b}(s_1, s_2, s_3)$ with $s_i \in \mathbb{C}$ can be viewed as a special case of the Shintani zeta function, but we will focus on its special values). In [40], Okamoto examined the values $\zeta_{a,b}(k_1, k_2, k_3)$ in the study of evaluations of special values of the zeta functions of root systems associated with A_2, B_2 and G_2. The goal was to express the special values of the zeta functions of root systems as $\mathbb{Q}$-linear combinations of two products of certain zeta values. As a prototype, we have in mind the analogous story for the parity theorem for multiple zeta values [17, Corollary 8] (see also [47]) and for Tornheim's series [15, Theorem 2] (see also [50]). For example, the identity

$$\zeta_{1,1}(1,1,3) = 4\zeta(5) - 2\zeta(2)\zeta(3)$$

is well-known. Similar studies have been done in many articles [37, 43, 46, 49, 50, 51, 57] (see also [41]). In this section, we will generalize the above expression to the value $\zeta_{a,b}(k_1, k_2, k_3)$ with $k_1 + k_2 + k_3$ odd. As a consequence, we give an affirmative answer to a conjecture about special values of the zeta function of the root system of G_2, which was proposed by Komori, Matsumoto and Tsumura [27, Eq. (7.1)].

We now state our main result. We use the Clausen-type functions defined for a positive integer $k \geq 2$ and $x \in \mathbb{R}$ by

$$\begin{aligned} C_k(x) &:= \mathrm{Re}\ Li_k(e^{2\pi i x}) = \sum_{m>0} \frac{\cos(2\pi m x)}{m^k}, \\ S_k(x) &:= \mathrm{Im}\ Li_k(e^{2\pi i x}) = \sum_{m>0} \frac{\sin(2\pi m x)}{m^k}, \end{aligned} \tag{4.1}$$

where $Li_k(z)$ is the polylogarithm $\sum_{m>0} \frac{z^m}{m^k}$. Note that $C_k(x)$ equals the Riemann zeta value $\zeta(k) := \sum_{m>0} \frac{1}{m^k}$ when $x \in \mathbb{Z}$, and $S_k(x)$ is 0 when $x \in \frac{1}{2}\mathbb{Z}$.

Theorem 4.1. *For positive integers $N, a, b, k, k_1, k_2, k_3$ with $N = \mathrm{lcm}(a, b)$ and $k = k_1 + k_2 + k_3$ odd, the value $\zeta_{a,b}(k_1, k_2, k_3)$ can be expressed as $\mathbb{Q}$-linear combinations of $\pi^{2n} C_{k-2n}(\frac{d}{N})$ and $\pi^{2n+1} S_{k-2n-1}(\frac{d}{N})$ for $0 \leq n \leq \frac{k-3}{2}$ and $d \in \mathbb{Z}/N\mathbb{Z}$.*

Theorem 4.1 will be proved in Section 4.4 by using the generating functions. This leads to a recipe for giving a formula for the $\mathbb{Q}$-linear combination in Theorem 4.1. More precisely, one can deduce an explicit formula from Corollary 4.3 and Propositions 4.4, 4.8

and 4.9, but it might be much more complicated (we do not develop the explicit formulas in this section). As an example of a simple identity, we have

$$\zeta_{1,3}(1,1,3) = \frac{1}{81}\left(367\zeta(5) - 19\pi^2\zeta(3) - 27\pi S_4(\tfrac{1}{3}) - 4\pi^3 S_2(\tfrac{1}{3})\right). \tag{4.2}$$

We apply Theorem 4.1 to prove the conjecture suggested by Komori, Matsumoto and Tsumura [27, Eq. (7.1)]. This will be described in Section 4.5.

It is worth mentioning that since the value $\zeta_{a,b}(k_1,k_2,k_3)$ can be expressed as $\mathbb{Q}$-linear combinations of double polylogarithms

$$Li_{k_1,k_2}(z_1,z_2) = \sum_{0<m<n} \frac{z_1^m z_2^n}{m^{k_1} n^{k_2}}, \tag{4.3}$$

Theorem 4.1 might be proved by the parity theorem for double polylogarithms obtained by Panzer [42] and Nakamura [37], which is illustrated in Remark 4.10. In this thesis, we however do not use their result to prove Theorem 4.1, since we want to keep this section self-contained.

The contents of this section are as follows. In Section 4.2, we give an integral representation of the generating function of the values $\zeta_{a,b}(k_1,k_2,k_3)$ for any integers $a,b \geq 1$. In Section 4.3, the integral is computed. Section 4.4 gives a proof of Theorem 4.1. In Section 4.5, we recall the question [27, Eq. (7.1)] and give an affirmative answer to this.

4.2 Integral representation

In this subsection, we give an integral representation of the generating function of the values $\zeta_{a,b}(k_1,k_2,k_3)$ for any integers $a,b \geq 1$. The integral representation of the value $\zeta_{a,b}(k_1,k_2,k_3)$ was first given by Okamoto [40, Theorem 4.4], following the method used by Zagier (see also [35]). We recall it briefly.

For an integer $k \geq 0$, the Bernoulli polynomial $B_k(x)$ of order k is defined by

$$\sum_{k\geq 0} B_k(x)\frac{t^k}{k!} = \frac{te^{xt}}{e^t - 1}.$$

The polynomial $B_k(x)$ admits the following expression (see [1, Theorem 4.11]): for $k \geq 1$ and $x \in \mathbb{R}$ ($x \in \mathbb{R} - \mathbb{Z}$, if $k = 1$)

$$B_k(x - [x]) = \begin{cases} -2i\dfrac{k!}{(2\pi i)^k}\displaystyle\sum_{m>0}\frac{\sin(2\pi m x)}{m^k} & k \geq 1 : \text{odd}, \\ -2\dfrac{k!}{(2\pi i)^k}\displaystyle\sum_{m>0}\frac{\cos(2\pi m x)}{m^k} & k \geq 2 : \text{even}, \end{cases}$$

where $i = \sqrt{-1}$ and the summation $\sum_{m>0}$ is regarded as $\lim_{N\to\infty}\sum_{N>m>0}$ when $k = 1$ (this ensures convergence). We define the modified (generalized) Clausen function for $k \geq 1$

and $x \in \mathbb{R}$ ($x \in \mathbb{R} - \mathbb{Z}$, if $k = 1$) by

$$Cl_k(x-[x]) = \begin{cases} -\dfrac{k!}{(2\pi i)^{k-1}} \displaystyle\sum_{m>0} \dfrac{\cos(2\pi m x)}{m^k} & k \geq 1 : \text{odd}, \\ -i\dfrac{k!}{(2\pi i)^{k-1}} \displaystyle\sum_{m>0} \dfrac{\sin(2\pi m x)}{m^k} & k \geq 2 : \text{even}. \end{cases}$$

With this, for $k \geq 1$ and $x \in \mathbb{R}$ ($x \in \mathbb{R} - \mathbb{Z}$ if $k = 1$), the polylogarithm $Li_k(e^{2\pi i x})$ can be written in the form

$$Li_k(e^{2\pi i x}) = -\frac{(2\pi i)^{k-1}}{k!} \left(Cl_k(x-[x]) + \pi i B_k(x-[x]) \right).$$

We introduce the formal generating functions. For $x \in \mathbb{R} - \mathbb{Z}$, let

$$\beta(x;t) := \sum_{k>0} \frac{B_k(x-[x])t^k}{k!} \quad \text{and} \quad \gamma(x;t) := \sum_{k>0} \frac{Cl_k(x-[x])t^k}{k!}.$$

Proposition 4.2. *For integers $a, b \geq 1$, we have*

$$\begin{aligned} &\sum_{k_1,k_2,k_3>0} \zeta_{a,b}(k_1,k_2,k_3) t_1^{k_1} t_2^{k_2} t_3^{k_3} \\ &= -\frac{1}{4\pi i} \int_0^1 \big(\gamma(ax; 2\pi i t_1)\beta(bx; 2\pi i t_2) + \beta(ax; 2\pi i t_1)\gamma(bx; 2\pi i t_2) \big) \\ &\times \beta(x; -2\pi i t_3) dx \\ &+ \frac{1}{4\pi^2} \int_0^1 \big(\gamma(ax; 2\pi i t_1)\gamma(bx; 2\pi i t_2) - \pi^2 \beta(ax; 2\pi i t_1)\beta(bx; 2\pi i t_2) \big) \\ &\times \beta(x; -2\pi i t_3) dx, \end{aligned}$$

where the integrals on the right-hand side are defined formally by term-by-term integration.

Proof. When $k_1, k_2, k_3 \geq 2$, it follows that

$$\begin{aligned} &\int_0^1 Li_{k_1}\big(e^{2\pi i a x}\big) Li_{k_2}\big(e^{2\pi i b x}\big) \overline{Li_{k_3}\big(e^{2\pi i x}\big)} dx \\ &= \int_0^1 \sum_{m,n,l>0} \frac{e^{2\pi i m a x} e^{2\pi i n b x} e^{-2\pi i l x}}{m^{k_1} n^{k_2} l^{k_3}} dx \\ &= \sum_{m,n,l>0} \frac{1}{m^{k_1} n^{k_2} l^{k_3}} \int_0^1 e^{2\pi i x (am+bn-l)} dx = \zeta_{a,b}(k_1,k_2,k_3), \end{aligned}$$

where $\overline{Li_{k_3}\big(e^{2\pi i x}\big)}$ stands for the complex conjugate of $Li_{k_3}\big(e^{2\pi i x}\big)$. For $k_1, k_2, k_3 \geq 1$, the above equality is justified by replacing the integral $\displaystyle\int_0^1$ with

$$\lim_{\varepsilon \to 0} \sum_{j=1}^{\mathrm{lcm}(a,b)} \int_{\frac{j-1}{\mathrm{lcm}(a,b)}+\varepsilon}^{\frac{j}{\mathrm{lcm}(a,b)}-\varepsilon}, \tag{4.4}$$

where $\mathrm{lcm}(a,b)$ is the least common multiple of a and b (see [40, Theorem 4.4] for the details). Letting $Li(x;t) := \sum_{k>0} Li_k(e^{2\pi i x})t^k$, we therefore obtain

$$\sum_{k_1,k_2,k_3>0} \zeta_{a,b}(k_1,k_2,k_3)t_1^{k_1}t_2^{k_2}t_3^{k_3} = \int_0^1 Li(ax;t_1)Li(bx;t_2)\overline{Li(x;t_3)}dx. \tag{4.5}$$

Furthermore, the generating function of $Li_k(e^{2\pi i x})$ with $x \in \mathbb{R} - \mathbb{Z}$ can be written in the form

$$Li(x;t) = -\frac{1}{2\pi i}\left(\gamma(x;2\pi i t) + \pi i \beta(x;2\pi i t)\right), \tag{4.6}$$

and hence, the right-hand side of (4.5) is equal to

$$\begin{aligned}&\frac{1}{(2\pi i)^3}\int_0^1 \big(\gamma(ax;2\pi i t_1) + \pi i\beta(ax;2\pi i t_1)\big)\\ &\quad\times \big(\gamma(bx;2\pi i t_2) + \pi i\beta(bx;2\pi i t_2)\big)\big(\gamma(x;-2\pi i t_3) - \pi i\beta(x;-2\pi i t_3)\big)dx.\end{aligned} \tag{4.7}$$

We note that, similarly to (4.5), one obtains the relation

$$\int_0^1 Li(ax;t_1)Li(bx;t_2)Li(x;-t_3)dx = 0,$$

and substituting (4.6) to the above identity, one has

$$\begin{aligned}&\int_0^1 \big(\gamma(ax;2\pi i t_1) + \pi i\beta(ax;2\pi i t_1)\big)\big(\gamma(bx;2\pi i t_2) + \pi i\beta(bx;2\pi i t_2)\big)\\ &\quad\times \gamma(x;-2\pi i t_3)dx\\ &= -\pi i\int_0^1 \big(\gamma(ax;2\pi i t_1) + \pi i\beta(ax;2\pi i t_1)\big)\big(\gamma(bx;2\pi i t_2) + \pi i\beta(bx;2\pi i t_2)\big)\beta(x;-2\pi i t_3)dx.\end{aligned}$$

With this, (4.7) is reduced to

$$\begin{aligned}&-\frac{1}{(2\pi i)^2}\int_0^1 \big(\gamma(ax;2\pi i t_1) + \pi i\beta(ax;2\pi i t_1)\big)\big(\gamma(bx;2\pi i t_2) + \pi i\beta(bx;2\pi i t_2)\big)\\ &\quad\times \beta(x;-2\pi i t_3)dx,\end{aligned}$$

which completes the proof. □

The coefficient of t^k in $\gamma(x;2\pi i t)$ (resp. $\beta(x;2\pi i t)$) is a real-valued function, if k is even, and a real-valued function times $i = \sqrt{-1}$, if k is odd. Thus, comparing the coefficient of both sides, we have the following corollary. For simplicity, for integers $a,b \geq 1$ we let

$$F_{a,b}(t_1,t_2,t_3) := \int_0^1 \gamma(ax;t_1)\beta(bx;t_2)\beta(x;-t_3)dx, \tag{4.8}$$

where the integral is defined formally by term-by-term integration and by (4.4).

Corollary 4.3. *One has*

$$\sum_{\substack{k_1,k_2,k_3>0\\ k_1+k_2+k_3:\text{odd}}} \zeta_{a,b}(k_1,k_2,k_3)t_1^{k_1}t_2^{k_2}t_3^{k_3}$$
$$= -\frac{1}{4\pi i}F_{a,b}(2\pi i t_1, 2\pi i t_2, 2\pi i t_3) - \frac{1}{4\pi i}F_{b,a}(2\pi i t_2, 2\pi i t_1, 2\pi i t_3).$$

Remark that, using the same method, one can give an integral expression of the generating function of the Riemann zeta values, which will be used later.

Proposition 4.4. *For integers $a, b \geq 1$, we have*

$$\frac{1}{2\pi i}\int_0^1 \gamma(ax; 2\pi i t_1)\beta(bx; -2\pi i t_2)dx = \sum_{\substack{r,s>0\\ r+s:\text{odd}}} \frac{\gcd(a,b)^{r+s}}{a^s b^r}\zeta(r+s)t_1^r t_2^s. \tag{4.9}$$

Proof. Let $d = \gcd(a,b)$ and set $a = a'd, b = b'd$. It follows that

$$\int_0^1 Li_r(e^{2\pi i a x})\overline{Li_s(e^{2\pi i b x})}dx = \sum_{m,n>0}\frac{1}{m^r n^s}\int_0^1 e^{2\pi i x(am-bn)}dx$$
$$= \sum_{\substack{m,n>0\\ m=\frac{b'}{a'}n}}\frac{1}{m^r n^s} = \left(\frac{a'}{b'}\right)^r \sum_{\substack{n>0\\ a'|n}}\frac{1}{n^{r+s}} = \frac{1}{a'^s b'^r}\zeta(r+s).$$

Hence we have

$$\int_0^1 Li(ax;t_1)\overline{Li(bx;t_2)}dx = \sum_{r,s>0}\frac{\gcd(a,b)^{r+s}}{a^s b^r}\zeta(r+s)t_1^r t_2^s.$$

By the relation $\int_0^1 Li(ax;t_1)Li(bx;-t_2)dx = 0$ $(a,b\geq 1)$ and (4.6), the left-hand side of the above equation can be reduced to

$$\frac{1}{2\pi i}\int_0^1 \big(\gamma(ax;2\pi i t_1) + \pi i\beta(ax;2\pi i t_1)\big)\beta(bx;-2\pi i t_2)dx.$$

Comparing the coefficients of $t_1^r t_2^s$, we complete the proof. □

4.3 Evaluation of integrals

In this subsection, we compute the integral $F_{a,b}(t_1,t_2,t_3)$.

We denote the generating function of the Bernoulli polynomials by $\beta_0(x;t)$:

$$\beta_0(x;t) := \frac{te^{xt}}{e^t - 1} = \sum_{k\geq 0} B_k(x)\frac{t^k}{k!}.$$

For integers $b, c \geq 1$, we set

$$\alpha_b(t_1,t_2) := \beta_0(0;t_1)\beta_0(0;-t_2)\frac{e^{bt_1-t_2}-1}{bt_1-t_2},$$
$$\widetilde{\alpha}_{b,c}(t_1,t_2) := -t_1e^{-ct_1}\beta_0(0;-t_2)\frac{e^{bt_1-t_2}-1}{bt_1-t_2},$$

which are elements in the formal power series ring $\mathbb{Q}[[t_1,t_2]]$.

Lemma 4.5. *For any integers $b, d \geq 1$, we have*

$$e^{-dt_1}\alpha_b(t_1,t_2) = \alpha_b(t_1,t_2) + \sum_{c=1}^{d}\widetilde{\alpha}_{b,c}(t_1,t_2).$$

Proof. By the relation $B_k(x) = B_k(x+1) - kx^{k-1}$ for $k \in \mathbb{Z}_{\geq 0}$ (see [1, Proposition 4.9 (2)]), we have $\beta_0(x;t) = \beta_0(x+1;t) - te^{xt}$. Using this formula with $x = -d, -d+1, \ldots, 1$ repeatedly, one gets

$$\beta_0(-d;t) = \beta_0(-d+1;t) - te^{-dt} = \cdots = \beta_0(0;t) - t\sum_{c=1}^{d}e^{-ct}.$$

Hence, we obtain

$$\begin{aligned} e^{-dt_1}\alpha_b(t_1,t_2) &= \beta_0(-d;t_1)\beta_0(0;-t_2)\frac{e^{bt_1-t_2}-1}{bt_1-t_2} \\ &= \alpha_b(t_1,t_2) - t_1\sum_{c=1}^{d}e^{-ct_1}\beta_0(0;-t_2)\frac{e^{bt_1-t_2}-1}{bt_1-t_2} \\ &= \alpha_b(t_1,t_2) + \sum_{c=1}^{d}\widetilde{\alpha}_{b,c}(t_1,t_2), \end{aligned}$$

which completes the proof. □

Remark 4.6. Let us denote by $A_b(r,s)$ (resp. $\widetilde{A}_{b,c}(r,s)$) the coefficient of $t_1^rt_2^s$ in $\alpha_b(t_1,t_2)$ (resp. in $\widetilde{\alpha}_{b,c}(t_1,t_2)$). Then, we have

$$A_b(r,s) = \sum_{\substack{p_1+q_1=r \\ p_2+q_2=s \\ p_1,p_2,q_1,q_2\geq 0}} \frac{(-1)^{q_2+p_2}b^{p_1}B_{q_1}B_{q_2}}{p_1!p_2!q_1!q_2!(p_1+p_2+1)}$$

and

$$\widetilde{A}_{b,c}(r,s) = \sum_{\substack{p_1+q_1=r \\ p_2+q_2=s \\ p_1,p_2,q_2\geq 0 \\ q_1\geq 1}} \frac{(-1)^{q_1+q_2+p_2}c^{q_1-1}b^{p_1}B_{q_2}}{p_1!(q_1-1)!p_2!q_2!(p_1+p_2+1)},$$

where $B_k = B_k(1) = (-1)^kB_k(0)$ is the k-th Bernoulli number. We note that since $\widetilde{\alpha}_{b,c}(t_1,t_2) \in t_1\mathbb{Q}[[t_1,t_2]]$, we have $\widetilde{A}_{b,c}(0,s) = 0$ for any $s \in \mathbb{Z}_{\geq 0}$.

Lemma 4.7. *Let b, d be positive integers with $d \in \{0, 1, \ldots, b-1\}$. Then, for $x \in (\frac{d}{b}, \frac{d+1}{b})$, we have*

$$\beta(bx; t_1)\beta(x; -t_2) = e^{-dt_1}\alpha_b(t_1, t_2)\beta_0(x; bt_1 - t_2) - \beta(bx; t_1) - \beta(x; -t_2) - 1,$$

where we recall $\beta(x; t) = \sum_{k>0} \frac{B_k(x - [x])}{k!} t^k$.

Proof. Since $bx - [bx] = bx - d$ when $x \in (\frac{d}{b}, \frac{d+1}{b})$, one has

$$\begin{aligned}
&\big(\beta(bx; t_1) + 1\big)\big(\beta(x; -t_2) + 1\big) = \frac{t_1 e^{(bx-d)t_1}}{e^{t_1} - 1} \frac{-t_2 e^{-xt_2}}{e^{-t_2} - 1} \\
&= e^{-dt_1} \frac{t_1}{e^{t_1} - 1} \frac{-t_2}{e^{-t_2} - 1} e^{(bt_1 - t_2)x} \\
&= e^{-dt_1} \beta_0(0; t_1)\beta_0(0; -t_2) \frac{e^{bt_1 - t_2} - 1}{bt_1 - t_2} \frac{(bt_1 - t_2)e^{(bt_1 - t_2)x}}{e^{bt_1 - t_2} - 1} \\
&= e^{-dt_1}\alpha_b(t_1, t_2)\beta_0(x; bt_1 - t_2),
\end{aligned}$$

from which the statement follows. ∎

Proposition 4.8. *For any integers $a, b \geq 1$, we have*

$$\begin{aligned}
F_{a,b}(t_1, t_2, t_3) = \alpha_b(t_2, t_3) &\int_0^1 \gamma(ax; t_1)\beta_0(x; bt_2 - t_3)dx \\
&+ \sum_{c=1}^{b-1} \widetilde{\alpha}_{b,c}(t_2, t_3) \int_{\frac{c}{b}}^1 \gamma(ax; t_1)\beta_0(x; bt_2 - t_3)dx \\
&- \int_0^1 \gamma(ax; t_1)\big(\beta(bx; t_2) + \beta(x; -t_3)\big)dx.
\end{aligned} \tag{4.10}$$

Proof. Splitting the integral $\int_0^1 = \sum_{d=0}^{b-1} \int_{\frac{d}{b}}^{\frac{d+1}{b}}$ in the definition of $F_{a,b}$ (see Eq. (4.8)) and then using Lemma 4.7, we have

$$\begin{aligned}
&F_{a,b}(t_1, t_2, t_3) \\
&= \sum_{d=0}^{b-1} \int_{\frac{d}{b}}^{\frac{d+1}{b}} \gamma(ax; t_1)\beta(bx; t_2)\beta(x; -t_3)dx \\
&= \sum_{d=0}^{b-1} e^{-dt_2}\alpha_b(t_2, t_3) \int_{\frac{d}{b}}^{\frac{d+1}{b}} \gamma(ax; t_1)\beta_0(x; bt_2 - t_3)dx \\
&\quad - \sum_{d=0}^{b-1} \int_{\frac{d}{b}}^{\frac{d+1}{b}} \gamma(ax; t_1)\big(\beta(bx; t_2) + \beta(x; -t_3) + 1\big)dx
\end{aligned}$$

$$= \sum_{d=0}^{b-1} \left(\alpha_b(t_2,t_3) + \sum_{c=1}^{d} \widetilde{\alpha}_{b,c}(t_2,t_3) \right) \int_{\frac{d}{b}}^{\frac{d+1}{b}} \gamma(ax;t_1)\beta_0(x;bt_2-t_3)dx$$
$$- \int_0^1 \gamma(ax;t_1)\big(\beta(bx;t_2)+\beta(x;-t_3)+1\big)dx,$$

where for the last equality we have used Lemma 4.5. Since $\int_0^1 Li(ax;t)dx = 0$ holds, we have

$$\int_0^1 \gamma(ax;t_1)dx = 0. \tag{4.11}$$

Hence, the statement follows from and the interchange of order of summation $\sum_{d=1}^{b-1}\sum_{c=1}^{d} = \sum_{c=1}^{b-1}\sum_{d=c}^{b-1}$. ☐

We now deal with the integral of the second term on the right-hand side of (4.10).

Proposition 4.9. *For any integers $a, b \geq 1$ and $c \in \{0, 1, \ldots, b-1\}$, we have*

$$\frac{1}{2\pi i}\int_{\frac{c}{b}}^{1} \gamma(ax;2\pi i t_1)\beta_0(x;2\pi i(bt_2-t_3))dx$$
$$= -i \sum_{\substack{s\geq 1\\ p,q\geq 0\\ p+s:\text{odd}}} \frac{(-1)^s(2\pi i)^{q-1}}{q!a^s} S_{p+s+1}(\tfrac{ac}{b})B_q(\tfrac{c}{b})t_1^{p+1}(bt_2-t_3)^{q+s-1}$$
$$+ \sum_{\substack{s\geq 1\\ p,q\geq 0\\ p+s:\text{even}}} \frac{(-1)^s(2\pi i)^{q-1}}{q!a^s} \big(\zeta(p+s+1)B_q - C_{p+s+1}(\tfrac{ac}{b})B_q(\tfrac{c}{b})\big)$$
$$\times t_1^{p+1}(bt_2-t_3)^{q+s-1},$$

where $S_n(x)$ and $C_n(x)$ are defined in (4.1).

Proof. For an integer $s \geq 1$, we let

$$\gamma_s(x;t) = \sum_{k\geq s} \frac{Cl_k(x-[x])}{k!}t^k.$$

It is easily seen that for any integer $s \geq 2$ we have

$$\frac{d}{dx}\gamma_s(ax;t) = at\gamma_{s-1}(ax;t) \quad \text{and} \quad \frac{d}{dx}\beta_0(x;t) = t\beta_0(x;t).$$

By repeated use of the integration by parts and noting that $\gamma_1(x;t) = \gamma(x;t)$, we have

$$\begin{aligned}
&\int_{\frac{c}{b}}^{1} \gamma(ax; 2\pi i t_1)\beta_0(x; 2\pi i(bt_2 - t_3))dx \\
&= \sum_{s\geq 2} \frac{(-2\pi i(bt_2 - t_3))^{s-2}}{(2\pi i a t_1)^{s-1}} \left[\gamma_s(ax; 2\pi i t_1)\beta_0(x; 2\pi i(bt_2 - t_3)\right]_{\frac{c}{b}}^{1} \\
&= \sum_{\substack{s\geq 2\\ p\geq s\\ q\geq 0}} \frac{(-1)^s(2\pi i)^{p+q-1}}{p!q!a^{s-1}} \left[Cl_p(ax - [ax])B_q(x)\right]_{\frac{c}{b}}^{1} t_1^{p-s+1}(bt_2 - t_3)^{q+s-2} \\
&= \sum_{\substack{s\geq 1\\ p,q\geq 0}} \frac{(-1)^{s+1}(2\pi i)^{p+q+s}}{(p+s+1)!q!a^s} \left[Cl_{p+s+1}(ax - [ax])B_q(x)\right]_{\frac{c}{b}}^{1} t_1^{p+1}(bt_2 - t_3)^{q+s-1}.
\end{aligned}$$

By definition, for any $x \in \mathbb{Q}$ and $k \geq 2$ we have

$$Cl_k(x - [x]) = \begin{cases} -\dfrac{k!}{(2\pi i)^{k-1}} C_k(x) & k : \text{odd}, \\ -i\dfrac{k!}{(2\pi i)^{k-1}} S_k(x) & k : \text{even}, \end{cases}$$

and hence, the above last line is computed as follows:

$$\begin{aligned}
i &\sum_{\substack{s\geq 1\\ p,q\geq 0\\ p+s:\text{odd}}} \frac{(-1)^s(2\pi i)^q}{q!a^s} \left(S_{p+s+1}(a)B_q(1) - S_{p+s+1}(\tfrac{ac}{b})B_q(\tfrac{c}{b})\right) \\
&\times t_1^{p+1}(bt_2 - t_3)^{q+s-1} \\
&+ \sum_{\substack{s\geq 1\\ p,q\geq 0\\ p+s:\text{even}}} \frac{(-1)^s(2\pi i)^q}{q!a^s} \left(C_{p+s+1}(a)B_q(1) - C_{p+s+1}(\tfrac{ac}{b})B_q(\tfrac{c}{b})\right) \\
&\times t_1^{p+1}(bt_2 - t_3)^{q+s-1},
\end{aligned}$$

which completes the proof. □

4.4 Proof of Theorem 4.1

We can now complete the proof of Theorem 4.1 as follows.

Proof of Theorem 4.1. We compute the real part of the coefficient of $t_1^{k_1}t_2^{k_2}t_3^{k_3}$ in the generating function $\frac{1}{2\pi i}F_{a,b}(2\pi i t_1, 2\pi i t_2, 2\pi i t_3)$ for positive integers k, k_1, k_2, k_3 with $k =$

$k_1+k_2+k_3$ odd. By (4.10) with $t_j \to 2\pi i t_j$, we have

$$\begin{aligned}
&\frac{1}{2\pi i}F_{a,b}(2\pi i t_1, 2\pi i t_2, 2\pi i t_3)\\
&= \alpha_b(2\pi i t_2, 2\pi i t_3) \times \frac{1}{2\pi i}\int_0^1 \gamma(ax; 2\pi i t_1)\beta_0(x; -2\pi i(t_3 - bt_2))dx \qquad (4.12)\\
&\quad + \sum_{c=1}^{b-1} \widetilde{\alpha}_{b,c}(2\pi i t_2, 2\pi i t_3) \times \frac{1}{2\pi i}\int_{\frac{c}{b}}^1 \gamma(ax; 2\pi i t_1)\beta_0(x; 2\pi i(bt_2 - t_3))dx \qquad (4.13)\\
&\quad - \frac{1}{2\pi i}\int_0^1 \gamma(ax; 2\pi i t_1)\big(\beta(bx; -2\pi i(-t_2)) + \beta(x; -2\pi i t_3)\big)dx. \qquad (4.14)
\end{aligned}$$

By (4.9), the coefficient of $t_1^{k_1}t_2^{k_2}t_3^{k_3}$ in the last term (4.14) is a rational multiple of $\zeta(k)$. For the first term (4.12), using (4.9) and (4.11), we have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_0^1 \gamma(ax; 2\pi i t_1)\beta_0(x; -2\pi i(t_3 - bt_2))dx\\
&\in \sum_{\substack{k_1,k_2,k_3>0\\ k_1+k_2+k_3:\text{odd}}} \mathbb{Q}\zeta(k_1+k_2+k_3)t_1^{k_1}t_2^{k_2}t_3^{k_3},
\end{aligned}$$

where $\sum a_r t^r \in \sum V_r t^r$ means $a_r \in V_r$ for all r. We also have

$$\alpha_b(2\pi i t_1, 2\pi i t_2) \in \sum_{r,s\geq 0} \mathbb{Q}(2\pi i)^{r+s}t_1^r t_2^s.$$

Hence the real part of the coefficient of $t_1^{k_1}t_2^{k_2}t_3^{k_3}$ in (4.12) can be expressed as $\mathbb{Q}$-linear combinations of $\pi^{2n}\zeta(k-2n)$ with $0 \leq n \leq \frac{k-3}{2}$. For the second term (4.13), using Proposition 4.9 (see also Remark 4.6), we have

$$\begin{aligned}
&\widetilde{\alpha}_{b,c}(2\pi i t_2, 2\pi i t_3) \times \frac{1}{2\pi i}\int_{\frac{c}{b}}^1 \gamma(ax; 2\pi i t_1)\beta_0(x; 2\pi i(bt_2 - t_3))dx\\
&= -i \sum_{\substack{n_2\geq 1\\ n_3\geq 0}} \sum_{\substack{s\geq 1\\ p,q\geq 0\\ p+s:\text{odd}}} \frac{(-1)^s \widetilde{A}_{b,c}(n_2,n_3)}{q!a^s}(2\pi i)^{n_2+n_3+q-1} S_{p+s+1}(\tfrac{ac}{b})B_q(\tfrac{c}{b})\\
&\quad \times t_1^{p+1}(bt_2 - t_3)^{q+s-1}t_2^{n_2}t_3^{n_3}\\
&\quad + \sum_{\substack{n_2\geq 1\\ n_3\geq 0}} \sum_{\substack{s\geq 1\\ p,q\geq 0\\ p+s:\text{even}}} \frac{(-1)^s \widetilde{A}_{b,c}(n_2,n_3)}{q!a^s}(2\pi i)^{n_2+n_3+q-1}\\
&\quad \times \big(\zeta(p+s+1)B_q - C_{p+s+1}(\tfrac{ac}{b})B_q(\tfrac{c}{b})\big)\, t_1^{p+1}(bt_2 - t_3)^{q+s-1}t_2^{n_2}t_3^{n_3},
\end{aligned} \qquad (4.15)$$

where we note that in the above both summations, $p+s+1$ runs over integers greater than 1. Since for any $x \in \mathbb{Q}$ and $k \geq 0$ we have $B_k(x) \in \mathbb{Q}$, the real part of the coefficient of $t_1^{k_1}t_2^{k_2}t_3^{k_3}$ in the first term (resp. the second term) on the right-hand side of (4.15) is a $\mathbb{Q}$-linear combination of $\pi^{2n+1}S_{k-2n-1}(\frac{ac}{b})$ with $0 \le n \le \frac{k-3}{2}$ (resp. $\pi^{2n}C_{k-2n}(\frac{ac}{b})$ and $\pi^{2n}\zeta(k-2n)$ with $0 \le n \le \frac{k-3}{2}$). We therefore find that the real part of the coefficient of $t_1^{k_1}t_2^{k_2}t_3^{k_3}$ in the generating function $\frac{1}{2\pi i}F_{a,b}(2\pi i t_1, 2\pi i t_2, 2\pi i t_3)$ can be expressed as $\mathbb{Q}$-linear combinations of $\pi^{2n+1}S_{k-2n-1}(\frac{ac}{b})$ and $\pi^{2n}C_{k-2n}(\frac{ac}{b})$ with $0 \le n \le \frac{k-3}{2}$ and $c \in \mathbb{Z}/b\mathbb{Z}$. Thus by Corollary 4.3 we complete the proof. □

Remark 4.10. As mentioned in Section 4.1, the value $\zeta_{a,b}(k_1,k_2,k_3)$ is expressible as $\mathbb{Q}$-linear combinations of double polylogarithms $Li_{r,s}(z_1,z_2)$ defined in (4.3), where the expression is obtained from the partial fractional decomposition

$$\frac{1}{x^r y^s} = \sum_{\substack{p+q=r+s \\ p,q\ge 1}} \frac{1}{(x+y)^p}\left(\binom{p-1}{s-1}\frac{1}{x^q} + \binom{p-1}{r-1}\frac{1}{y^q}\right) \qquad (r,s \in \mathbb{Z}_{\ge 1})$$

and the orthogonality relation

$$\frac{1}{N}\sum_{n\in\mathbb{Z}/N\mathbb{Z}} \mu_N^{dn} = \begin{cases} 1 & N \mid d \\ 0 & N \nmid d \end{cases},$$

where $\mu_N = e^{2\pi i/N}$ and $d \in \mathbb{Z}$. For example, one can check

$$\zeta_{1,3}(1,1,3) = \sum_{u\in\mathbb{Z}/3\mathbb{Z}} Li_{1,4}(\mu_3^{-u}, \mu_3^u) + \sum_{u\in\mathbb{Z}/3\mathbb{Z}} Li_{1,4}(\mu_3^u, 1). \tag{4.16}$$

From this, Theorem 4.1 might be proved by the parity theorem for double polylogarithms examined in [42, Eq. (3.2)]. Although we do not proceed this in general, let us illustrate an example. As a special case of [42, Eq. (3.2)], one obtains

$$\begin{aligned} &Li_{1,4}(z_1,z_2) + Li_{1,4}(z_1^{-1}, z_2^{-1}) \\ &= \sum_{n=1}^{5}(-1)^{n+1}Li_n(z_1)\mathcal{B}_{5-n}(z_1z_2) - Li_1(z_1)\mathcal{B}_4(z_2) \\ &\quad + \sum_{n=4}^{5}\binom{n-1}{3}Li_n(z_2^{-1})\mathcal{B}_{5-n}(z_1z_2) - Li_5(z_1z_2), \end{aligned}$$

where for each integer $k \ge 0$ we set $\mathcal{B}_k(z) = \frac{(2\pi i)^k}{k!}B_k\left(\frac{1}{2} + \frac{\log(-z)}{2\pi i}\right)$. We note that $Li_k(\mu_3^u) = C_k(\frac{u}{3}) + iS_k(\frac{u}{3})$ and $\mathcal{B}_k(\mu_3) = \frac{(2\pi i)^k}{k!}B_k(\frac{1}{3})$ since $\log(-\mu_3) = -\frac{\pi i}{3}$. With this, the above formula gives

$$\begin{aligned}
&\mathrm{Re}\ \left(Li_{1,4}(\mu_3^{-1},\mu_3)+Li_{1,4}(\mu_3^{-2},\mu_3^2)\right)\\
&=\frac{1}{243}\left(-843\zeta(5)+36\pi^2\zeta(3)+4\pi^4\log 3\right),\\
&\mathrm{Re}\ \left(Li_{1,4}(\mu_3,1)+Li_{1,4}(\mu_3^2,1)\right)\\
&=\frac{1}{243}\left(972\zeta(5)-12\pi^2\zeta(3)-4\pi^4\log 3-81\pi S_4(\tfrac{1}{3})-12\pi^3 S_2(\tfrac{1}{3})\right),\\
&2Li_{1,4}(1,1)=4\zeta(5)-\frac{1}{3}\pi^2\zeta(3).
\end{aligned}$$

where we have used $C_k(\frac{1}{3})=C_k(\frac{2}{3})=\frac{1-3^{k-1}}{2\cdot 3^{k-1}}\zeta(k)$ for $k\geq 2$ and $C_1(\frac{1}{3})=C_1(\frac{2}{3})=-\frac{1}{2}\log 3$. Substituting the above formulas to (4.16), one gets (4.2). We have checked Theorem 4.1 for $(a,b)=(1,3)$ and $(2,3)$ in this direction.

4.5 The zeta function of the root system G_2

In this subsection, we give an affirmative answer to the question posed by Komori, Matsumoto and Tsumura [27, Eq. (7.1)].

The zeta-function associated with the exceptional Lie algebra G_2 is defined for complex variables $\mathbf{s}=(s_1,s_2,\ldots,s_6)\in\mathbb{C}^6$ by

$$\zeta(\mathbf{s};G_2):=\sum_{m,n>0}\frac{1}{m^{s_1}n^{s_2}(m+n)^{s_3}(m+2n)^{s_4}(m+3n)^{s_5}(2m+3n)^{s_6}}.$$

The function $\zeta(\mathbf{s};G_2)$ was first introduced by Komori, Matsumoto and Tsumura (see [25, 27]), where they developed its analytic properties and functional relations. They also examined explicit evaluations of the special values of $\zeta(\mathbf{k};G_2)$ at $\mathbf{k}\in\mathbb{Z}_{>0}^6$ (see [54] for $\mathbf{k}\in\mathbb{Z}_{\geq 0}^6$), where we note that the series $\zeta(\mathbf{k};G_2)$ converges absolutely for $\mathbf{k}\in\mathbb{Z}_{>0}^6$. For example, they showed

$$\zeta(2,1,1,1,1,1;G_2)=-\frac{109}{1296}\zeta(7)+\frac{1}{18}\zeta(2)\zeta(5).$$

Komori, Matsumoto and Tsumura [27, Eq. (7.1)] suggested a conjecture that the value $\zeta(k_1,\ldots,k_6;G_2)$ with $k_1+\cdots+k_6$ odd lies in the polynomial ring over $\mathbb{Q}$ generated by $\zeta(k)$ $(k\in\mathbb{Z}_{\geq 2})$ and $L(k,\chi_3)$ $(k\in\mathbb{Z}_{\geq 1})$, where $L(s,\chi_3)$ is the Dirichlet L-function associated with the character χ_3 defined by

$$L(s,\chi_3)=\sum_{m>0}\frac{\chi_3(m)}{m^s}$$

and the character χ_3 is determined by $\chi_3(n)=1$ if $n\equiv 1 \mod 3$, $\chi_3(n)=-1$ if $n\equiv 2 \mod 3$ and $\chi_3(n)=0$ if $n\equiv 0 \mod 3$. We remark that Okamoto [40] showed that the value $\zeta(k_1,\ldots,k_6;G_2)$ with $k_1+\cdots+k_6$ odd can be written in terms of

$$\zeta(s), L(s,\chi_3), S_r(\tfrac{d}{N}), C_r(\tfrac{d}{N})$$

for $N = 4, 12$ and $0 < d < N$, $(d, N) = 1$ (see also [27, §7]). The following theorem gives an affirmative answer to the question.

Theorem 4.11. *For any integers $k, k_1, \ldots, k_6 \geq 1$ with $k = k_1 + \cdots + k_6$ odd, the value $\zeta(k_1, \ldots, k_6; G_2)$ can be expressed as $\mathbb{Q}$-linear combinations of $\zeta(2n)\zeta(k-2n)$ $(0 \leq n \leq \frac{k-3}{2})$ and $L(2n+1, \chi_3)L(k-2n-1, \chi_3)$ $(0 \leq n \leq \frac{k-3}{2})$, where $\zeta(0) = -\frac{1}{2}$.*

Proof. In [40, Theorem 2.3], Okamoto proved that for any integers $l_1, \ldots, l_6 \geq 1$, the value $\zeta(l_1, \ldots, l_6; G_2)$ can be expressed as $\mathbb{Q}$-linear combinations of $\zeta_{a,b}(n_1, n_2, n_3)$ with $(a, b) = (1, 1), (1, 2), (1, 3), (2, 3)$, $n_1 + n_2 + n_3 = l_1 + \cdots + l_6$ and $n_1, n_2, n_3 \in \mathbb{Z}_{>0}$. As a consequence, it follows from Theorem 4.1 that the value $\zeta(k_1, \ldots, k_6; G_2)$ can be written as $\mathbb{Q}$-linear combinations of $\pi^{2n}C_{k-2n}(\frac{d}{6})$ and $\pi^{2n+1}S_{k-2n-1}(\frac{d}{6})$ with $0 \leq n \leq \frac{k-3}{2}$ and $d \in \mathbb{Z}/6\mathbb{Z}$. Now consider the values $C_k(\frac{d}{6})$ and $S_k(\frac{d}{6})$. They are expressible as $\mathbb{Q}$-linear combinations of

$$\zeta_l^{(d)}(k) = \sum_{\substack{m>0 \\ m \equiv d \mod l}} \frac{1}{m^k} \quad (d \in \mathbb{Z}/l\mathbb{Z}).$$

For $k \geq 2$, using the identities $\zeta(k) = \sum_{d \in \mathbb{Z}/l\mathbb{Z}} \zeta_l^{(d)}(k)$ and $\zeta_l^{(0)}(k) \in \mathbb{Q}\zeta(k)$, we have $C_k(\frac{1}{2}) = \zeta_2^{(0)}(k) - \zeta_2^{(1)}(k) \in \mathbb{Q}\zeta(k)$ and $C_k(\frac{1}{3}) = C_k(\frac{2}{3}) = \zeta_3^{(0)}(k) - \frac{1}{2}\left(\zeta_3^{(1)}(k) + \zeta_3^{(2)}(k)\right) \in \mathbb{Q}\zeta(k)$. Furthermore, using the identity $\zeta_{al}^{(ad)}(k) = a^{-k}\zeta_l^{(d)}(k)$, we have

$$\begin{aligned} &C_k(\frac{1}{6}) = C_k(\frac{5}{6}) \\ &= \zeta_6^{(0)}(k) - \zeta_6^{(3)}(k) + \frac{1}{2}\left(\zeta_6^{(1)}(k) + \zeta_6^{(5)}(k)\right) - \frac{1}{2}\left(\zeta_6^{(2)}(k) + \zeta_6^{(4)}(k)\right) \in \mathbb{Q}\zeta(k). \end{aligned}$$

Thus, $C_k(\frac{d}{6}) \in \mathbb{Q}\zeta(k)$ holds for any $d \in \mathbb{Z}/6\mathbb{Z}$ and $k \geq 2$. Likewise, it is easily seen that $S_k(\frac{d}{6}) \in \mathbb{Q}\sqrt{3}L(k, \chi_3)$ holds. Then the result follows from the well-known formula: $\zeta(2n) \in \mathbb{Q}\pi^{2n}$, $L(2n+1, \chi_3) \in \mathbb{Q}\sqrt{3}\pi^{2n+1}$ for any $n \in \mathbb{Z}_{\geq 0}$ (see [1, Theorem 9.6]). ☐

Let us illustrate an example of the formula for $\zeta(k_1, \ldots, k_6; G_2)$. Applying the partial fractional decomposition repeatedly to the form $(m+n)^{-k_3}(m+2n)^{-k_4}(m+3n)^{-k_5}(2m+3n)^{-k_6}$, we get

$$\begin{aligned} &\zeta(1, 1, 1, 1, 1, 2; G_2) \\ &= \frac{1}{2}\zeta_{1,1}(5, 1, 1) - 16\zeta_{1,2}(5, 1, 1) + \frac{9}{2}\zeta_{1,3}(5, 1, 1) + 9\zeta_{2,3}(4, 1, 2) + 18\zeta_{2,3}(5, 1, 1). \end{aligned}$$

Then, by Theorem 4.1 (actually we use Corollary 4.3 together with Propositions 4.4, 4.8 and 4.9), we have

$$\begin{aligned} \zeta(1, 1, 1, 1, 1, 2; G_2) &= \frac{2507}{1296}\zeta(7) - \frac{505}{648}\pi^2\zeta(5) + \frac{9}{4}\pi S_6(\tfrac{1}{3}) \\ &= \frac{2507}{1296}\zeta(7) - \frac{505}{108}\zeta(2)\zeta(5) + \frac{3}{8}L(1, \chi_3)L(6, \chi_3), \end{aligned}$$

where $L(1, \chi_3) = \frac{\pi}{3\sqrt{3}}$.

5 Parity result for general multiple series

5.1 Preliminaries and the main result

Let $\mathbb{N}$ be the set of positive integers, $\mathbb{N}_0$ be the set of non negative integers, $\mathbb{Z}$ be the set of integers, $\mathbb{R}$ be the set of real numbers and $\mathbb{C}$ be the set of complex numbers. For a positive integer m, we define $[m] := \{1, 2, \dots, m\}$. Let ℓ and r be positive integers and $\mathcal{M}(\ell, r; \mathbb{N}_0)$ be the set of all $\ell \times r$ matrix $X = (a_{ij})$ with no zero row vectors, no zero column vectors and non negative integer entries. Then for $X = (a_{ij}) \in \mathcal{M}(\ell, r; \mathbb{N}_0), \mathbf{h} = (h_1, \dots, h_r) \in \mathbb{N}^r, \mathbf{k} = (k_1, \dots, k_\ell) \in \mathbb{N}^\ell$ and $\mathbf{y} = (y_1, \dots, y_r) \in \mathbb{R}^r$, we define

$$\begin{aligned}
&\zeta_{r,\ell}(\mathbf{h}, \mathbf{k}, \mathbf{y}, X) \\
&:= \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{e^{2\pi\sqrt{-1}(m_1 y_1 + \cdots + m_r y_r)}}{m_1^{h_1} \dots m_r^{h_r} (a_{11} m_1 + \cdots + a_{1r} m_r)^{k_1} \dots (a_{\ell 1} m_1 + \cdots + a_{\ell r} m_r)^{k_\ell}} \\
&= \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{j \in [r]} \frac{e^{2\pi\sqrt{-1} m_j y_j}}{m_j^{h_j}} \prod_{i \in [\ell]} \frac{1}{(a_{i1} m_1 + \cdots + a_{ir} m_r)^{k_i}}.
\end{aligned}$$

By the assumption on X, $\zeta_{r,\ell}(\mathbf{h}, \mathbf{k}, \mathbf{y}, X)$ converges absolutely. For the above $\mathbf{h}$ and $\mathbf{k}$, we define the weight of them respectively as follows.

$$\mathrm{wt}(\mathbf{h}) := h_1 + \cdots + h_r, \quad \mathrm{wt}(\mathbf{k}) := k_1 + \cdots + k_\ell.$$

In general, for a tuple of numbers $\mathbf{h} = (h_1, \dots, h_r)$ and a non empty subset $J = \{j_1, \dots, j_m \mid j_1 < j_2 < \cdots < j_m\} \subset [r]$, we define

$$\mathbf{h}_J := (h_j)_{j \in J} := (h_{j_1}, h_{j_2}, \dots, h_{j_m}), \quad \mathrm{wt}(\mathbf{h}_J) := \sum_{j \in J} h_j.$$

In order to state the main theorem, we introduce notations. Let m be a positive number. We assume that the real vector space $\mathbb{R}^m$ is equipped with the normal inner product $\langle \cdot, \cdot \rangle$. We denote $\vec{f}$ the part of $\mathbb{R}^m$ of $f \in \mathbb{R}^m \times \mathbb{C}$ and $\dot{f}$ the part of $\mathbb{C}$ of $f \in \mathbb{R}^m \times \mathbb{C}$. Namely, we express $f \in \mathbb{R}^m \times \mathbb{C}$ as $f = (\vec{f}, \dot{f})$. We define $\langle V \rangle := \sum_{\mathbf{v} \in V} \mathbb{Z}\mathbf{v}$ for $V \subset \mathbb{R}^m$, and $\vec{W} := \{\vec{f} \mid f = (\vec{f}, \dot{f}) \in W\}$ for $W \subset \mathbb{R}^m \times \mathbb{C}$. Let Λ be a subset of $(\mathbb{Z}^m \setminus \{\vec{0}\}) \times \mathbb{C}$ with satisfying $|\Lambda| < \infty$ and $\mathrm{rank}\langle \vec{\Lambda} \rangle = m$. For $H \subset \Lambda$ with $\mathrm{rank}\langle \vec{H} \rangle = m - 1$, we define $\mathfrak{H}_h := \sum_{g \in H} \mathbb{R}\vec{g}$. For Λ, let $\mathscr{B}(\Lambda)$ be the set of all subsets $B \subset \Lambda$ such that $\vec{B}$ forms a basis of $\mathbb{R}^m$. For $B = \{f_1, \dots, f_m\} \in \mathscr{B}(\Lambda)$, let $\vec{B}^* = \{\vec{f}_1^B, \dots, \vec{f}_m^B\}$ be the dual basis of $\vec{B} = \{\vec{f}_1, \dots, \vec{f}_m\}$ in $\mathbb{R}^m$.

For real numbers, we define a multi dimensional generalization of fractional part $\{\cdot\}$ which was introduced in [24] first. Let $\mathscr{R}(\Lambda)$ be the set of all subsets $R = \{g_1, \dots, g_{m-1}\} \subset \Lambda$ with satisfying that $\vec{R} = \{\vec{g}_1, \dots, \vec{g}_{m-1}\}$ is a linearly independent set. Here, we fix one vector

$$\rho \in \mathbb{R}^m \setminus \bigcup_{R \in \mathscr{R}} \mathfrak{H}_R$$

with satisfying that $\langle\rho,\vec{f}^B\rangle\neq 0$ for all $B\in\mathscr{B}(\Lambda)$ and $f\in B$. According to ρ, for $\mathbf{y}\in\mathbb{R}^m, B\in\mathscr{B}(\Lambda)$ and $f\in B$, we define the multi dimensional fractional part as follows.

$$\{\mathbf{y}\}_{B,f}:=\begin{cases}\{\langle\mathbf{y},\vec{f}^B\rangle\} & ,\langle\rho,\vec{f}^B\rangle>0,\\ 1-\{-\langle\mathbf{y},\vec{f}^B\rangle\} & ,\langle\rho,\vec{f}^B\rangle<0.\end{cases}$$

Theorem 5.1. *For $X\in\mathcal{M}(\ell,r;\mathbb{N}_0)$ and a non empty subset $J\subset[r]$, we define*

$$I(J):=\{i\in[\ell]\mid a_{ij}\neq 0\quad(^{\exists}j\in J)\},$$
$$\bar{J}:=[r]\setminus J,\quad \bar{I}(J):=[\ell]\setminus I(J),$$

and we put $m=|J|$ and $n=|I(J)|$. From the definition of $I(J)$ and the assumption of X, we can see that for $i\in\bar{I}(J)$ there exists $j\in\bar{J}$ such that $a_{ij}\neq 0$. In particular, $\sum_{j\in\bar{J}}a_{ij}m_j\neq 0$ for any $i\in\bar{I}(J)$ and $m_j>0$. Then for $X\in\mathcal{M}(\ell,r;\mathbb{N}_0),\mathbf{h}=(h_1,\ldots,h_r)\in\mathbb{N}^r,\mathbf{k}=(k_1,\ldots,k_\ell)\in\mathbb{N}^\ell$ with satisfying that

$$\sum_{\substack{m_1=1\\ \sum_{j\in J}a_{ij}m_j-\sum_{j\in[r]\setminus J}a_{ij}m_j\neq 0\ \ (i\in[\ell])}}^{\infty}\cdots\sum_{m_r=1}^{\infty}\prod_{j\in[r]}\frac{1}{m_j^{h_j}}\prod_{i\in[\ell]}\frac{1}{|\sum_{j\in J}a_{ij}m_j-\sum_{j\in[r]\setminus J}a_{ij}m_j|^{k_j}}$$

converges for any subset $J\subset[r]$,we have

$$\zeta_{r,\ell}(\mathbf{h},\mathbf{k},\mathbf{y},X)=(-1)^{\mathrm{wt}(\mathbf{h})+\mathrm{wt}(\mathbf{k})-r}\zeta_{r,\ell}(\mathbf{h},\mathbf{k},-\mathbf{y},X)+\sum_{\emptyset\neq J\subset[r]}T_{r,\ell,J}(\mathbf{h},\mathbf{k},\mathbf{y},X),$$

where

$$T_{r,\ell,J}(\mathbf{h},\mathbf{k},\mathbf{y},X)$$
$$:=(-1)^{\mathrm{wt}(\mathbf{h}_{\bar{J}})+\mathrm{wt}(\mathbf{k}_{\bar{I}(J)})-r+n}$$
$$\times\sum_{\substack{m_j=1\\ j\in\bar{J}}}^{\infty}\prod_{j\in\bar{J}}\frac{e^{-2\pi\sqrt{-1}y_jm_j}}{m_j^{h_j}}\prod_{i\in\bar{I}(J)}\frac{1}{(\sum_{j\in\bar{J}}a_{ij}m_j)^{k_i}}D(\mathbf{h}_J,\mathbf{k}_{I(J)},\mathbf{y}_J;\Lambda(J))\prod_{\substack{j\in J\\ i\in I(J)}}\frac{1}{h_j!k_i!}.$$

Moreover, $D(\mathbf{h}_J,\mathbf{k}_{I(J)},\mathbf{y}_J;\Lambda(J))$ is the coefficient of the Taylor expansion of

$$G(\mathbf{t}_{I_{\Lambda(J)}},\mathbf{y}_J;\Lambda(J))$$
$$=\sum_{B\in\mathscr{B}(\Lambda(J))}\left(\prod_{k\in\bar{I}_B}\frac{-t_k}{\dot{f}_k-\sum_{j\in I_B}\dot{f}_j\langle\vec{f}_k,\vec{f}_j^B\rangle-(t_k-\sum_{j\in I_B}t_j\langle\vec{f}_k,\vec{f}_j^B\rangle)}\right)$$
$$\times\frac{1}{|\mathbb{Z}^m/\langle\vec{B}\rangle|}\sum_{\mathbf{w}\in\mathbb{Z}^m/\langle\vec{B}\rangle}\left(\prod_{j\in I_B}\frac{2\pi\sqrt{-1}t_j\exp(2\pi\sqrt{-1}(t_j-\dot{f}_j)\{\mathbf{y}_J+\mathbf{w}\}_{B,f_j})}{\exp(2\pi\sqrt{-1}t_j)-1}\right)$$

around the origin in $\mathbf{t}_{I_{\Lambda(J)}} = (t_j)_{j \in I_{\Lambda(J)}}$. *Namely,*

$$G(\mathbf{t}_{I_{\Lambda(J)}}, \mathbf{y}_J; \Lambda(J)) = \sum_{\substack{\mathbf{h}_J \in \mathbb{N}_0^m \\ \mathbf{k}_{I(J)} \in \mathbb{N}_0^n}} D(\mathbf{h}_J, \mathbf{k}_{I(J)}, \mathbf{y}_J; \Lambda(J)) \prod_{\substack{j \in J \\ i \in I(J)}} \frac{t_j^{h_j} t_{r+i}^{k_i}}{h_j! k_i!},$$

where δ_{jk} *is the Kronecker's delta symbol,*

$$\begin{aligned} f_j &:= ((\delta_{jk})_{k \in J}, 0) \quad (j \in J), \\ f_{r+i} &:= ((a_{ij})_{j \in J}, -\sum_{j \in \bar{J}} a_{ij} m_j) \quad (i \in I(J)), \\ \Lambda(J) &:= \{f_j, f_{r+i} \mid j \in J, i \in I(J)\}, \\ \mathscr{B}(\Lambda(J)) &:= \{B = \{f_{j_1}, \ldots, f_{j_m}\} \subset \Lambda(J) \mid \vec{B} = \{\vec{f}_{j_1}, \ldots, \vec{f}_{j_m}\} \textit{ is a basis of } \mathbb{R}^m\}, \\ I_{\Lambda(J)} &:= \{j \mid f_j \in \Lambda(J)\}, \\ I_B &:= \{j \mid f_j \in B\}, \\ \bar{I}_B &:= \Lambda(J) \setminus I_B. \end{aligned}$$

In particular, we can give the expression of the real part of $\zeta_{r,\ell}(\mathbf{h}, \mathbf{k}, \mathbf{y}, X)$ in terms of k-tuple series with $k \leq r-1$ when $\mathrm{wt}(\mathbf{h}) + \mathrm{wt}(\mathbf{k})$ and r are of different parity.

Corollary 5.2. *For* $\mathbf{h}, \mathbf{k}, \mathbf{y}$ *and* X *with satisfying the same assumption as in Theorem 5.1, we have*

$$\mathrm{Re}(\zeta_{r,\ell}(\mathbf{h}, \mathbf{k}, \mathbf{y}, X)) = \frac{1}{2} \sum_{\emptyset \neq J \subset [r]} T_{r,\ell,J}(\mathbf{h}, \mathbf{k}, \mathbf{y}, X).$$

5.2 Example

As mentioned in Section 1, the parity result for multiple zeta values is known. Actually, the parity result for Mordell–Tornheim type of multiple zeta values which is defined by

$$\zeta_{MT,r}(k_1, \ldots, k_r; k_{r+1}) = \sum_{m_1, \ldots, m_r > 0} \frac{1}{m_1^{k_1} \ldots m_r^{k_r} (m_1 + \cdots + m_r)^{k_{r+1}}}$$

is also known. This result was proved by Tsumura [50]. Let us deduce the parity result for Mordell–Tornheim type of multiple zeta values. Put $\ell = 1, X = (\underbrace{1, \ldots, 1}_{r})$ and $y_1 = \cdots = y_r = 0$. We can see that $k_1, \ldots, k_{r+1}$ and X satisfy the condition of absolute convergence in Theorem 5.1 by the result of Matsumoto–Tsumura [32, Lemma 4.2]. From now on, we evaluate $T_{r,1,J}(\mathbf{k}, k_{r+1}, \mathbf{0}, X)$. In this case, we fix $\rho = (1, 2, \ldots, m)$ for $J \subset [r]$ with

$|J| = m$ and we have

$$\begin{aligned} f_j &= ((\delta_{jk})_{k\in J}, 0) \quad (j \in J), \\ f_{r+1} &= ((1)_{j\in J}, -\sum_{j\in \bar{J}} m_j), \\ \Lambda(J) &= \{f_j, f_{r+1} \mid j \in J\}, \\ I_{\Lambda(J)} &= J \cup \{r+1\}. \end{aligned}$$

Then all $B \in \mathscr{B}(\Lambda(J))$ belong to

$$\{B \mid f_{r+1} \notin B\} \text{ or } \{B \mid f_{r+1} \in B\}.$$

The former set contains only one element that is $B = \{f_j \mid j \in J\}$. Hence we have

$$\bar{I}_B = \{r+1\}, I_B = J, \vec{f}_j^B = \vec{f}_j,$$

and $\{\mathbf{0}\}_{B,f_j} = 0$ for any $j \in J$ since $\langle \rho, \vec{f}_j^B \rangle > 0$. For B which is an element of the latter set, B corresponds to $i \in J$. Then we can see that

$$\bar{I}_B = \{i\}, I_B = I_{\Lambda(J)} \setminus \{i\}, \vec{f}_j^B = \vec{f}_j - \vec{f}_i \ (j \in J \setminus \{i\}), \vec{f}_{r+1}^B = \vec{f}_i,$$

$$\langle \vec{f}_i, \vec{f}_j^B \rangle = \begin{cases} \langle \vec{f}_i, \vec{f}_j - \vec{f}_i \rangle = -1 & , j \in J \setminus \{i\}, \\ 1 & , j = r+1, \end{cases}$$

and

$$\{\mathbf{0}\}_{B,f_j} = \begin{cases} 0 & , i < j \in (J \cup \{r+1\}) \setminus \{i\}, \\ 1 & , i > j \in (J \cup \{r+1\}) \setminus \{i\}. \end{cases}$$

Moreover, $\mathbb{Z}^m / \langle \vec{B} \rangle = \{\mathbf{0}\}$ for any $B \in \mathscr{B}(\Lambda(J))$. Therefore we have

$$\begin{aligned} &G(\mathbf{t}_{I_{\Lambda(J)}}, \mathbf{0}; \Lambda(J)) \\ &= \frac{-t_{r+1}}{\dot{f}_{r+1} - \sum_{j\in J} \dot{f}_j \langle \vec{f}_{r+1}, \vec{f}_j^B \rangle - (t_{r+1} - \sum_{j\in J} t_j \langle \vec{f}_{r+1}, \vec{f}_j^B \rangle)} \prod_{j\in J} \frac{2\pi\sqrt{-1}t_j}{\exp(2\pi\sqrt{-1}t_j) - 1} \\ &\quad + \sum_{i\in J} \frac{-t_i}{\dot{f}_i - \sum_{j\in (J\cup\{r+1\})\setminus\{i\}} \dot{f}_j \langle \vec{f}_i, \vec{f}_j^B \rangle - (t_i - \sum_{j\in (J\cup\{r+1\})\setminus\{i\}} t_j \langle \vec{f}_i, \vec{f}_j^B \rangle)} \\ &\quad \times \prod_{\substack{j\in (J\cup\{r+1\})\setminus\{i\} \\ i<j}} \frac{2\pi\sqrt{-1}t_j}{\exp(2\pi\sqrt{-1}t_j) - 1} \prod_{\substack{j\in (J\cup\{r+1\})\setminus\{i\} \\ i>j}} \frac{2\pi\sqrt{-1}t_j \exp(2\pi\sqrt{-1}t_j)}{\exp(2\pi\sqrt{-1}t_j) - 1} \\ &= \frac{1}{2\pi\sqrt{-1}} \frac{1}{\sum_{j\in \bar{J}} m_j - (\sum_{j\in J} t_j - t_{r+1})} \prod_{j\in I_{\Lambda(J)}} \frac{2\pi\sqrt{-1}t_j}{\exp(2\pi\sqrt{-1}t_j) - 1} \\ &\quad \times \left\{ \exp(2\pi\sqrt{-1}t_{r+1}) - 1 - \sum_{i\in J} (\exp(2\pi\sqrt{-1}t_i) - 1) \prod_{\substack{j\in J\setminus\{i\} \\ i>j}} \exp(2\pi\sqrt{-1}t_j) \right\}. \end{aligned}$$

Noting that

$$\sum_{i\in J}(\exp(2\pi\sqrt{-1}t_i)-1)\prod_{\substack{j\in J\setminus\{i\}\\ i>j}}\exp(2\pi\sqrt{-1}t_j)$$
$$=\sum_{i\in J}\left(\prod_{\substack{j\in J\\ i\geq j}}\exp(2\pi\sqrt{-1}t_j)-\prod_{\substack{j\in J\setminus\{i\}\\ i>j}}\exp(2\pi\sqrt{-1}t_j)\right)$$
$$=\exp(2\pi\sqrt{-1}\sum_{j\in J}t_j)-1,$$

we have

$$G(\mathbf{t}_{I_{\Lambda(J)}},\mathbf{0};\Lambda(J))$$
$$=\frac{-\exp(2\pi\sqrt{-1}t_{r+1})}{2\pi\sqrt{-1}}\frac{\exp(2\pi\sqrt{-1}(\sum_{j\in J}t_j-t_{r+1}))-1}{\sum_{j\in\bar{J}}m_j-(\sum_{j\in J}t_j-t_{r+1})}\prod_{j\in I_{\Lambda(J)}}\frac{2\pi\sqrt{-1}t_j}{\exp(2\pi\sqrt{-1}t_j)-1}.$$

When $J\neq[r]$, we can see that $D(\mathbf{k}_J,k_{r+1}\mathbf{0};\Lambda(J))$ is a polynomial of powers of $\pi\sqrt{-1}$ and powers of $(\sum_{j\in\bar{J}}m_j)^{-1}$. When $J=[r]$, we can see that $D(\mathbf{k}_J,k_{r+1}\mathbf{0};\Lambda(J))$ is a polynomial of powers of $\pi\sqrt{-1}$. By these observations and by taking real part, we obtain the parity result for Mordell–Tornheim type multiple zeta values.

5.3 Proof of Theorem 5.1

The proof of Theorem 5.1 is not just an extension of the proof of Theorem 4.1. The method is essentially new. Let N be a positive integer. First, we calculate the finite sum

$$\zeta_{N,r,\ell}(\mathbf{h},\mathbf{k},\mathbf{y},X)$$
$$:=\sum_{m_1=1}^{N}\cdots\sum_{m_r=1}^{N}\frac{e^{2\pi\sqrt{-1}(m_1y_1+\cdots+m_ry_r)}}{m_1^{h_1}\dots m_r^{h_r}(a_{11}m_1+\cdots+a_{1r}m_r)^{k_1}\dots(a_{\ell 1}m_1+\cdots+a_{\ell r}m_r)^{k_\ell}}$$
$$=\sum_{m_1=1}^{N}\cdots\sum_{m_r=1}^{N}\prod_{j\in[r]}\frac{e^{2\pi\sqrt{-1}m_jy_j}}{m_j}\prod_{i\in[\ell]}\frac{1}{(a_{i1}m_1+\cdots+a_{ir}m_r)^{k_i}},\tag{5.1}$$

and then we take the limit $N\to\infty$. We define the constant a as follows.

$$a:=\max\{a_{i1}+\cdots+a_{ir}\mid 1\leq i\leq\ell\}.$$

Then we have $a_{i1}m_1+\cdots+a_{ir}m_r\leq aN$ for $m_1,\dots,m_r\leq N$ and any $i\in[\ell]$. From now on, we will evaluate the above sum.

First, from the equation

$$\int_0^1 e^{2\pi\sqrt{-1}kx}\,dx=\begin{cases}1 & ,k=0,\\ 0 & ,k\in\mathbb{Z}_{\neq 0},\end{cases}$$

we have

$$\frac{1}{c^k} = \sum_{n=-N}^{N}{}^{*} \frac{\delta_{nc}}{n^k} = \sum_{n=-N}^{N}{}^{*} \frac{1}{n^k} \int_0^1 e^{2\pi\sqrt{-1}(c-n)x}\, dx \tag{5.2}$$

for non zero integer c and positive integer N with $N \geq |c|$, where $\sum^*$ means that the dummy variable skips zero. Substituting the equation (5.2) with $c = a_{i1}m_1 + \cdots + a_{ir}m_r$ and $k = k_i$ for $i \in [\ell]$ into (5.1) , we obtain

$$\begin{aligned}
&\zeta_{N,r,\ell}(\mathbf{h},\mathbf{k},\mathbf{y},X)\\
&= \sum_{\substack{m_j=1\\ j\in[r]}}^{N} \prod_{j\in[r]} \frac{e^{2\pi\sqrt{-1}m_j y_j}}{m_j^{h_j}} \left(\prod_{i\in[\ell]} \sum_{n_i=-aN}^{aN}{}^{*} \frac{1}{n_i^{k_i}} \int_0^1 \exp(2\pi\sqrt{-1}(\sum_{j\in[r]} a_{ij}m_j - n_i)x_i) dx_i \right)\\
&= \int_0^1 \cdots \int_0^1 \sum_{\substack{m_j=1\\ j\in[r]}}^{N} \left(\prod_{j\in[r]} \frac{e^{2\pi\sqrt{-1}m_j y_j}}{m_j^{h_j}} \right) \sum_{\substack{n_i=-aN\\ i\in[\ell]}}^{aN}{}^{*} \left(\prod_{i\in[\ell]} \frac{e^{-2\pi\sqrt{-1}n_i x_i}}{n_i^{k_i}} \right)\\
&\quad \times \exp(2\pi\sqrt{-1}\sum_{i\in[\ell]}\sum_{j\in[r]} a_{ij}m_j x_i) \prod_{i\in[\ell]} dx_i\\
&= \int_0^1 \cdots \int_0^1 \left(\prod_{j\in[r]} \sum_{m_j=1}^{N} \frac{e^{2\pi\sqrt{-1}(y_j+\sum_{i\in[\ell]} a_{ij}x_i)m_j}}{m_j^{h_j}} \right) \left(\prod_{i\in[\ell]} \sum_{n_i=-aN}^{aN}{}^{*} \frac{e^{-2\pi\sqrt{-1}n_i x_i}}{n_i^{k_i}} \right) \prod_{i\in[\ell]} dx_i.
\end{aligned} \tag{5.3}$$

Here, substituting the equation

$$\sum_{m=1}^{N} \frac{e^{2\pi\sqrt{-1}my}}{m^h} = \sum_{m=-N}^{N}{}^{*} \frac{e^{2\pi\sqrt{-1}my}}{m^h} + (-1)^{h-1} \sum_{m=1}^{N} \frac{e^{-2\pi\sqrt{-1}my}}{m^h}$$

into the equation (5.3), and expanding the product on j, we have

$$\zeta_{N,r,\ell}(\mathbf{h},\mathbf{k},\mathbf{y},X) = \sum_{J\subset[r]} T_{N,r,\ell,J}(\mathbf{h},\mathbf{k},\mathbf{y},X),$$

where J runs over all subsets of $[r]$, and we define

$$\begin{aligned}
&T_{N,r,\ell,J}(\mathbf{h},\mathbf{k},\mathbf{y},X)\\
&:= \int_0^1 \cdots \int_0^1 \left(\prod_{j\in\bar{J}} (-1)^{h_j-1} \sum_{m_j=1}^{N} \frac{e^{-2\pi\sqrt{-1}(y_j+\sum_{i\in[\ell]} a_{ij}x_i)m_j}}{m_j^{h_j}} \right)\\
&\quad \times \left(\prod_{j\in J} \sum_{m_j=-N}^{N}{}^{*} \frac{e^{2\pi\sqrt{-1}(y_j+\sum_{i\in[\ell]} a_{ij}x_i)m_j}}{m_j^{h_j}} \right) \left(\prod_{i\in[\ell]} \sum_{n_i=-aN}^{aN}{}^{*} \frac{e^{-2\pi\sqrt{-1}n_i x_i}}{n_i^{k_i}} \right) \prod_{i\in[\ell]} dx_i,
\end{aligned} \tag{5.4}$$

where $\bar{J} := [r] \setminus J$. From now on, we evaluate $T_{N,r,\ell,J}(\mathbf{h},\mathbf{k},\mathbf{y},X)$ for each J.

5.3.1 Evaluation of $T_{N,r,\ell,J}(\mathbf{h},\mathbf{k},\mathbf{y},X)$

First, we evaluate $T_{N,r,\ell,\emptyset}(\mathbf{h},\mathbf{k},\mathbf{y},X)$. In (5.4) with $J=\emptyset$, changing variables $x_i'=1-x_i$ and $n_i'=-n_i$ for all $i\in[\ell]$, we have

$$\begin{aligned}
&T_{N,r,\ell,\emptyset}(\mathbf{h},\mathbf{k},\mathbf{y},X)\\
&=\int_0^1\cdots\int_0^1\left(\prod_{j\in[r]}(-1)^{h_j-1}\sum_{m_j=1}^{N}\frac{e^{-2\pi\sqrt{-1}(y_j+\sum_{i\in[\ell]}a_{ij}(1-x_i'))m_j}}{m_j^{h_j}}\right)\\
&\qquad\times\left(\prod_{i\in[\ell]}\sum_{n_i'=-aN}^{aN}{}^{*}\frac{e^{2\pi\sqrt{-1}n_i'(1-x_i')}}{(-n_i')^{k_i}}\right)\prod_{i\in[\ell]}dx_i'\\
&=(-1)^{\mathrm{wt}(\mathbf{h})-r+\mathrm{wt}(\mathbf{k})}\int_0^1\cdots\int_0^1\left(\prod_{j\in[r]}\sum_{m_j=1}^{N}\frac{e^{2\pi\sqrt{-1}(-y_j+\sum_{i\in[\ell]}a_{ij}x_i')m_j}}{m_j^{h_j}}\right)\\
&\qquad\times\left(\prod_{i\in[\ell]}\sum_{n_i'=-aN}^{aN}{}^{*}\frac{e^{-2\pi\sqrt{-1}n_i'x_i'}}{n_i'^{k_i}}\right)\prod_{i\in[\ell]}dx_i'.
\end{aligned}$$

From the equation (5.3) with $y_i\to-y_i$, we obtain

$$\begin{aligned}
T_{N,r,\ell,\emptyset}(\mathbf{h},\mathbf{k},\mathbf{y},X)&=(-1)^{\mathrm{wt}(\mathbf{h})-r+\mathrm{wt}(\mathbf{k})}\zeta_{N,r,\ell}(\mathbf{h},\mathbf{k},-\mathbf{y},X)\\
&\xrightarrow{N\to\infty}(-1)^{\mathrm{wt}(\mathbf{h})-r+\mathrm{wt}(\mathbf{k})}\zeta_{r,\ell}(\mathbf{h},\mathbf{k},-\mathbf{y},X),
\end{aligned}\tag{5.5}$$

where $-\mathbf{y}=(-y_1,-y_2,\ldots,-y_r)$.

Next, in order to evaluate $T_{N,r,\ell,J}(\mathbf{h},\mathbf{k},\mathbf{y},X)$ for a non empty subset $J\subset[r]$, we follow the process reverse to (5.3). Namely, we integrate on x_i's again as follows.

$$\begin{aligned}
&T_{N,r,\ell,J}(\mathbf{h},\mathbf{k},\mathbf{y},X)\\
&=\left(\prod_{j\in\bar J}(-1)^{h_j-1}\right)\sum_{\substack{m_j=1\\ j\in\bar J}}^{N}\prod_{j\in\bar J}\frac{e^{-2\pi\sqrt{-1}y_jm_j}}{m_j^{h_j}}\sum_{\substack{m_j=-N\\ j\in J}}^{N}{}^{*}\prod_{j\in J}\frac{e^{2\pi\sqrt{-1}y_jm_j}}{m_j^{h_j}}\\
&\quad\times\prod_{i\in[\ell]}\left(\sum_{n_i=-aN}^{aN}{}^{*}\frac{1}{n_i^{k_i}}\int_0^1\exp(2\pi\sqrt{-1}(\sum_{j\in J}a_{ij}m_j-\sum_{j\in\bar J}a_{ij}m_j-n_i)x_i)dx_i\right)\\
&=\left(\prod_{j\in\bar J}(-1)^{h_j-1}\right)\sum_{\substack{m_j=1\\ j\in\bar J}}^{N}\prod_{j\in\bar J}\frac{e^{-2\pi\sqrt{-1}y_jm_j}}{m_j^{h_j}}\sum_{\substack{m_j=-N\\ j\in J}}^{N}{}^{*}\prod_{j\in J}\frac{e^{2\pi\sqrt{-1}y_jm_j}}{m_j^{h_j}}\\
&\quad\times\prod_{i\in[\ell]}\sum_{\substack{n_i=-aN\\ \sum_{j\in J}a_{ij}m_j-\sum_{j\in\bar J}a_{ij}m_j-n_i=0}}^{aN}{}^{*}\frac{1}{n_i^{k_i}}
\end{aligned}$$

$$= \left(\prod_{j\in\bar{J}}(-1)^{h_j-1}\right) \sum_{\substack{m_j=1\\ j\in\bar{J}}}^{N} \prod_{j\in\bar{J}} \frac{e^{-2\pi\sqrt{-1}y_j m_j}}{m_j^{h_j}}$$

$$\times \sideset{}{^*}\sum_{\substack{m_j=-N\ (j\in J)\\ \sum_{j\in J} a_{ij}m_j - \sum_{j\in\bar{J}} a_{ij}m_j \neq 0\ (i\in[\ell])}}^{N} \prod_{j\in J} \frac{e^{2\pi\sqrt{-1}y_j m_j}}{m_j^{h_j}} \prod_{i\in[\ell]} \frac{1}{(\sum_{j\in J} a_{ij}m_j - \sum_{j\in\bar{J}} a_{ij}m_j)^{k_i}}. \tag{5.6}$$

Here, we remark that (5.6) absolutely converges when $N\to\infty$ by the assumption of X. Define

$$I(J) := \{i\in[\ell] \mid a_{ij}\neq 0 \quad (^{\exists}j\in J)\}, \quad \bar{I}(J) := [\ell]\setminus I(J).$$

Let m be the number of elements of J. Moreover, defining

$$Z_N(\mathbf{h}_J, \mathbf{k}_{I(J)}, \mathbf{y}_J; \Lambda(J)) := \sum_{\substack{\mathbf{m}_J\in\mathbb{Z}^m\\ |m_j|\leq N\ (j\in J)\\ f_j(\mathbf{m}_J)\neq 0\ (j\in J)\\ f_{r+i}(\mathbf{m}_J)\neq 0\ (i\in I(J))}} e^{2\pi\sqrt{-1}\langle \mathbf{y}_J, \mathbf{m}_J\rangle} \prod_{j\in J} \frac{1}{f_j(\mathbf{m}_J)^{h_j}} \prod_{i\in I(J)} \frac{1}{f_{r+i}(\mathbf{m}_J)^{k_i}},$$

for

$$\begin{aligned}
\mathbf{m}_J &= (m_j)_{j\in J} \in \mathbb{Z}^m,\\
f_j &= (\vec{f}_j, \dot{f}_j) = ((\delta_{jk})_{k\in J}, 0) \in \mathbb{R}^m\times\mathbb{C} \quad (j\in J),\\
f_{r+i} &= ((a_{ij})_{j\in J}, -\sum_{j\in\bar{J}} a_{ij}m_j) \in \mathbb{R}^m\times\mathbb{C} \quad (i\in I(J)),\\
f(\mathbf{m}_J) &= \langle \vec{f}, \mathbf{m}_J\rangle + \dot{f}, \quad (\vec{f}\in\mathbb{R}^m, \dot{f}\in\mathbb{C}),\\
\Lambda(J) &:= \{f_j, f_{r+i} \mid j\in J, i\in I(J)\},
\end{aligned}$$

we see that $T_{N,r,\ell,J}(\mathbf{h}, \mathbf{k}, \mathbf{y}, X)$ can be written as

$$(-1)^{\mathrm{wt}(\mathbf{h}_{\bar{J}})-(r-m)+\mathrm{wt}(\mathbf{k}_{\bar{I}(J)})} \sum_{\substack{m_j=1\\ j\in\bar{J}}}^{N} \prod_{j\in\bar{J}} \frac{e^{-2\pi\sqrt{-1}y_j m_j}}{m_j^{h_j}} \prod_{i\in\bar{I}(J)} \frac{1}{(\sum_{j\in\bar{J}} a_{ij}m_j)^{k_i}}$$
$$\times Z_N(\mathbf{h}_J, \mathbf{k}_{I(J)}, \mathbf{y}_J; \Lambda(J)).$$

From the absolute convergence of (5.6) and the result of Komori–Matsumoto–Tsumura [26, Theorem 2.5]

$$\lim_{N\to\infty} Z_N(\mathbf{h}_J, \mathbf{k}_{I(J)}, \mathbf{y}_J; \Lambda(J)) = (-1)^{|\Lambda(J)|} \left(\prod_{\substack{j\in J\\ i\in I(J)}} \frac{1}{h_j! k_i!}\right) D(\mathbf{h}_J, \mathbf{k}_{I(J)}, \mathbf{y}_J; \Lambda(J)),$$

taking the limit $N \to \infty$, we have

$$T_{N,r,\ell,J}(\mathbf{h}, \mathbf{k}, \mathbf{y}, X) \to T_{r,\ell,J}(\mathbf{h}, \mathbf{k}, \mathbf{y}, X). \tag{5.7}$$

Note that we can use the result of Komori–Matsumoto–Tsumura for any $\mathbf{y}_J \in \mathbb{R}^m$ since the set

$$\{f \in \Lambda(J) \mid \operatorname{rank}\langle \vec{\Lambda}(J) \setminus \{\vec{f}\}\rangle \neq \operatorname{rank}\langle \vec{\Lambda}(J)\rangle\}$$

is empty in our situation. Combining Eqs. (5.5) and (5.7), we can obtain Theorem 5.1.

Remark 5.3. Theorem 5.1 can give the parity result for the Dirichlet series which contains all dummy variables as monomial factors in the denominator. Moreover, the author believes that we can obtain the parity result for the other Dirichlet series by differentiating partially on y_i.

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