Refined Large N Duality
and
Positivity Conjecture of Refined Chern-Simons Invariants

(Refined Large N 双対性と Refined Chern-Simons 不変量の正整数予想)

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ABSTRACT: In this thesis, we study Chern-Simons theory, topological string theory and its large N duality at refined level. The large N duality between Chern-Simons theory and topological string theory gives us a surprising connection both in physics and in mathematics. In the recent developments, some refinements have been proposed in each theory. We formulate the large N duality of $SU(N)$ refined Chern-Simons theory with a torus knot/link in S^3. By studying refined BPS states in M-theory, we provide an explicit form of low-energy effective action of Type IIA string theory with D4-branes on the Ω-background. This form enables us to relate refined Chern-Simons invariants of a torus knot/link in S^3 to refined BPS invariants in the resolved conifold. Assuming that the extra $U(1)$ global symmetry acts on BPS states trivially, we predict graded dimensions of cohomology classes of the moduli spaces of M2-M5 bound states associated to a torus knot/link in the resolved conifold from the duality. As a result, this formulation can be also interpreted as a positivity conjecture of refined Chern-Simons invariants of torus knots/links. We also discuss about an extension to non-torus knots.

Contents

1 Introduction

General motivation

Quantum field theory is the universal framework of modern physics. The standard model of elementary particles, which is based on quantum field theory, matches precisely with experimental results, and it is a huge success. Moreover, quantum field theory has produced significant mathematical conjectures and inspired mathematics although it cannot be formulated as mathematics yet. However, we cannot say that we fully understand quantum field theory even in physics. In the recent developments, some mysterious phenomena, duality and non-Lagrangian theory, have been discovered from string/M-theory. Duality means an equivalence between two different physical theories. On the other hand, a quantum field theory considered that the classical Lagrangian description does not exist[1] is called non-Lagrangian theory. In many cases there are only physical evidences for these phenomena. Therefore, we would like to verify these phenomena by reducing them to extent mathematics.

Topological quantum field theory and topological string theory are important classes since these theories have a good feature that physical observables become topological invariants. These theories are toy models as physics, but one can apply some techniques and ideas in quantum field theory and string theory to mathematics. Of course, such techniques and ideas are not formulated as rigorous mathematics yet, but we can formulate physical ideas and results as mathematical statements. We would like to understand mathematical structures behind quantum field theory and string theory via such topological theories. In this thesis, we study the relation between physics and knot theory from the viewpoint of "refinements." Our main interests are the large N duality between Chern-Simons theory and topological string theory and refined Chern-Simons theory which is one of non-Lagrangian theory.

Chern-Simons theory and knot theory

There is a long history between physics and knot theory. It would be Gauss who first found out the deep relationship between physics and knot (see e.g. [RN11]). He found a link invariant called Gauss linking number from electromagnetism. Then electromagnetism developed into gauge theory. On the other hand, in knot theory some knot invariants such as the Jones polynomial $J(K;q) \in \mathbb{Z}[q^{\pm 1/2}]$ were discovered, where K denotes a knot and q denotes a physical variable of the polynomial.

The first breakthrough was given by Chern-Simons theory, which is a typical example of topological quantum field theory. Since the seminal paper by Witten [Wit89], quantum knot invariants have been investigated in the context of Chern-Simons theory and topological string theory. One of the remarkable features of Chern-Simons theory is exact solvability, so that we can non-perturbatively compute physical observables. Witten showed that the partition function of Chern-Simons theory with a gauge group G on a three-manifold M gives a topological invariants of M, and the expectation value of the Wilson loop along a knot K with a representation R of G gives a knot invariant. Chern-Simons theory gives us a natural framework of quantum knot invariants since it has the manifest three-dimension symmetry and organizes quantum knot invariants associated with the gauge group G and the

[1] There is a possibility that classical Lagrangian description is not known at present.

representation R. In particular, when $G = SU(2)$ and R is two-dimensional irreducible representation, the expectation value is equal to the Jones polynomial [Wit89]. When $G = SU(N)$ and R is an irreducible representation (i.e. Young diagram λ), the expectation value is equal to the colored HOMFLY-PT polynomial $H_\lambda(K; a, q)$ which has two variables q and a. The colored HOMFLY-PT polynomial is one of the most important quantum knot invariants since this polynomial reduces the Jones polynomial when $a = q^2$. The result that the expectation values become knot polynomials with integer coefficients is interesting from the viewpoint of quantum field theory. The expectation values are defined by using path integral, so there is no a priori reason that the expectation values should be such polynomials with integer coefficients.

Knot homology

The second breakthrough was the discovery of knot homology. Khovanov proposed the categorification of the Jones polynomial [Kho00]. He constructed a bi-graded (co)homology $\mathcal{H}^{i,j}(K)$ and showed that its Euler characteristic is equal to the Jones polynomial

$$J(K; \mathbf{q}) = \sum_{i,j} (-1)^i \mathbf{q}^j \dim \mathcal{H}^{i,j}(K) \in \mathbb{Z}[\mathbf{q}^{\pm 1}], \tag{1.1}$$

where $\mathbf{q}$ denotes a variable used in knot theory notation. This gives not only an explanation of why the coefficients of knot polynomials are integers, but also a refinement of the Jones polynomial since we can consider its Poincaré polynomial

$$Kh(K; \mathbf{q}, \mathbf{t}) = \sum_{i,j} \mathbf{t}^i \mathbf{q}^j \dim \mathcal{H}^{i,j}(K) \in \mathbb{Z}_{\geq 0}[\mathbf{q}^{\pm 1}, \mathbf{t}^{\pm 1}] \tag{1.2}$$

instead of (1.1). The polynomial $Kh(K; \mathbf{q}, \mathbf{t})$, called the Khovanov polynomial, distinguishes some knots which the Jones polynomial cannot distinguish. After that, Khovanov's work was generalized for some quantum knot invariants. The knot homologies of the colored Jones polynomial and the HOMFLY-PT polynomial were constructed [Kho05, KR08a, KR08b]. The procedure of constructing a homology which reproduces a quantum knot invariant is called categorification. Categorification of quantum knot invariants defines homological knot polynomials which have non-negative integer coefficients. However, construction of knot homology is difficult in general[2]. In particular, the construction of the Poincaré polynomial of the colored HOMFLY-PT homology is a crucial open problem in mathematics. Motivated by knot homology, we would like to consider two questions:

1. *What is the physical meaning of knot homology?*
2. *How to compute homological knot polynomials in Chern-Simons theory?*

Large N duality

An answer for question 1 was suggested by Gukov, Schwarz and Vafa [GSV05], who expected that knot homology can be realized as a Hilbert space of BPS states. Here, the Hilbert space

[2] Recently, the HOMFLY-PT homology colored by arbitrary representations seems to be defined in [Cau15, Cau16] although it is formidable to carry out computation via the definition.

was introduced by Labastida, Mariño, Ooguri and Vafa (LMOV) [OV00, LMn01, LMnV00] in the context of the large N duality between Chern-Simons theory and topological string theory. Large N duality in general is an equivalence between a $U(N)$ gauge theory at large N and a string theory. The original idea was due to 't Hooft [tH74], and we have witnessed various successful incarnations of this idea in string theory. Among them, Gopakumar and Vafa proposed in the celebrated paper [GV99] that the large N limit of $U(N)$ Chern-Simons theory[3] on S^3 is equivalent to topological string theory on the resolved conifold. Since Chern-Simons theory is realized on topological branes wrapped on S^3 in the deformed conifold T^*S^3 [Wit95], this duality can be also interpreted as geometric transition at large N in string theory. This proposal has far-reaching consequences both in physics and in mathematics. One of the significant consequences is a striking connection between two seemingly different theories of invariants. On the one hand, Chern-Simons theory gives quantum invariants of three-manifolds and knots as mentioned above. On the other hand, topological string theory on a Calabi-Yau threefold is mathematically formulated as theories of enumerative invariants involving some moduli spaces of curves in the threefold. In particular, LMOV have put forth a remarkable relationship between quantum knot invariants and enumerative integral invariants of the resolved conifold by incorporating a knot in the duality.

The main part of the work of LMOV consists of two conjectures. The first one is an equivalence between the expectation value of Ooguri-Vafa operator in $SU(N)$ Chern-Simons theory on S^3 and the exponent of the free energy of open topological string on the resolved conifold with topological D-branes. Mathematically, this is an equivalence between the generation function of the colored HOMFLY-PT polynomials and the generating function of open Gromov-Witten invariants. The second one is target space formulation of open topological string theory, which predicts that the generating function of open Gromov-Witten invariants can be written by new integer valued invariants called LMOV invariants. It is hard to check the conjectures because all of open Gromov-Witten invariants are very difficult to compute. However, combining the two conjectures and forgetting open Gromov-Witten invariants, we obtain a reformulation of the colored HOMFLY-PT polynomials and an integrality conjecture of the invariants. Physically, the integrality conjecture is just a consistency check, but it reflects an integrality structure of the colored HOMFLY-PT polynomials. The integrality conjecture has been tested in [LMn01, RS01, LMnV00, ZR12, MMM$^+$17], and a proof is proposed by [LP10]. The large N duality was the third breakthrough.

Refined Chern-Simons theory

An answer for question 2 was suggested by Aganagic and Shakirov [AS15]. They proposed a refinement of Chern-Simons theory (refined Chern-Simons theory) from M-theory. Refined Chern-Simons theory is one of non-Lagrangian theories. Its "definition" is very complicated, but the theory leads to refined Chern-Simons invariants $\overline{\mathrm{rCS}}_\lambda(T_{m,n}; a, q, t)$, where the invariants can be defined only for torus knot/link $T_{m,n}$ at present. Refined Chern-Simons invariants are unrelated to homological knot invariants a priori. Nevertheless, after making a change of

[3]The brane setting gives rise to $U(N)$ gauge group instead of $SU(N)$ gauge group. However, the $U(1)$ part merely provides the correction due to the framing number as well as the linking number of a knot/link, which play no role in this thesis. Therefore, the difference between $U(N)$ and $SU(N)$ invariants will be ignored in this thesis.

variables

$$a=-\mathbf{a}^2\mathbf{t}\ ,\quad q^{\frac{1}{2}}=-\mathbf{q}\mathbf{t}\ ,\quad t^{\frac{1}{2}}=\mathbf{q}\ , \tag{1.3}$$

refined Chern-Simons invariants reproduce homological knot invariants of $T_{m,n}$ known in mathematics. Furthermore, refined Chern-Simons invariants enable us to investigate arbitrary colors beyond knot homology. When the color λ is a rectangular Young diagram, it is conjectured that the refined Chern-Simons invariant coincides with the Poincaré polynomial of the corresponding colored HOMFLY-PT homology. However, refined Chern-Simons invariants lead some Laurent polynomials with both positive and negative integer coefficients when the color is not rectangular Young diagram, which is in the areas beyond knot homology. This is a problem because such polynomials cannot be interpreted as homological knot invariants by definition i.e. homological knot invariants must have non-negative integer coefficients. If one trusts refined Chern-Simons theory, the Laurent polynomials seem to imply the colored HOMFLY-PT homology does not exist for the colors. Or, if one assumes that the colored HOMFLY-PT homology exists, refined Chern-Simons theory might be something wrong although the theory captures some knot homologies and satisfies some physical consistency. The problem of negative integer coefficients of refined Chern-Simons invariants has been a mystery because the Poincaré polynomial of the colored HOMFLY-PT homology is not constructed yet.

Main results

Main results are based on the paper [KN17] in which the author and Nawata formulated refined large N duality including a knot as a refinement of the work of LMOV. Refined large N duality gives a striking relation between refined Chern-Simons invariants of a torus knot and graded dimensions of the cohomology groups of the moduli spaces of M2-M5 bound states in the resolved conifold and also extends the proposal of [GSV05] to any colors. As a result, we observe that a positivity can be seen through refined large N duality. The positivity corresponds to a refinement of the integrality in the work of LMOV. Therefore, refined large N duality leads to a reformulation of refined Chern-Simons invariants and its positivity. The statements are completely mathematical conjectures although the derivation depends on physical discussions.

Definition 1. We define refined reformulated invariants f^q_μ and $f^{\bar{t}}_\mu$ via equations

$$\sum_\lambda \overline{\mathrm{rCS}}_\lambda(T_{m,n};a,q,t)\ g_\lambda(q,t)P_\lambda(x;q,t)=\exp\left(\sum_{d=1}^\infty\sum_\mu\frac{1}{d}\frac{f^q_\mu(T_{m,n};a^d,q^d,t^d)}{q^{\frac{d}{2}}-q^{-\frac{d}{2}}}s_\mu(x^d)\right)\ ,$$

$$\sum_\lambda \overline{\mathrm{rCS}}_\lambda(T_{m,n};a,q,t)\ P_{\lambda^T}(-x;t,q)=\exp\left(\sum_{d=1}^\infty\sum_\mu\frac{1}{d}\frac{f^{\bar{t}}_\mu(T_{m,n};a^d,q^d,t^d)}{t^{-\frac{d}{2}}-t^{\frac{d}{2}}}s_\mu(x^d)\right),$$

where Greek letters λ and μ denote Young diagrams, λ^T is the transpose of λ, the summation takes over all Young diagrams, $P_\lambda(x;q,t)$ are the Macdonald functions, $s_\mu(x)$ are the Schur functions and g_λ is the Macdonald norm.

From the above definition, we derive the explicit formulas for f^q and $f^{\bar{t}}$ in terms of refined Chern-Simons invariants:

Proposition 2.

$$\frac{1}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} f^q_\mu(T_{m,n};a,q,t) = \sum_{d,m=1}^{\infty}(-1)^{m-1}\frac{\mu(d)}{d\cdot m}\sum_{\{\vec{k}^{(\alpha)}\}}\sum_{\{\lambda^{(\alpha)}\}}\chi_\mu(C(\sum_{\alpha=1}^{m}(\vec{k}^{(\alpha)})_d)) \times \prod_{\alpha=1}^{m} g_{\lambda^{(\alpha)}}(q^d,t^d)\mathfrak{X}_{\lambda^{(\alpha)}}(\vec{k}^{(\alpha)};q^d,t^d)\overline{\mathrm{rCS}}_{\lambda^{(\alpha)}}(T_{m,n};a^d,q^d,t^d)\ , \tag{1.4}$$

$$\frac{1}{t^{-\frac{d}{2}}-t^{\frac{d}{2}}} f^{\bar{t}}_\mu(T_{m,n};a,q,t) = \sum_{d,m=1}^{\infty}(-1)^{m-1}\frac{\mu(d)}{d\cdot m}\sum_{\{\vec{k}^{(\alpha)}\}}\sum_{\{\lambda^{(\alpha)}\}}\chi_{\mu^T}(C(\sum_{\alpha=1}^{m}(\vec{k}^{(\alpha)})_d)) \times \prod_{\alpha=1}^{m}(-1)^{|\lambda^{(\alpha)}|}\mathfrak{X}_{(\lambda^{(\alpha)})^T}(\vec{k}^{(\alpha)};t^d,q^d)\overline{\mathrm{rCS}}_{\lambda^{(\alpha)}}(T_{m,n};a^d,q^d,t^d)\ , \tag{1.5}$$

where $\mu(d)$ is the Möbius function and the definition of $\chi_\mu(C(\vec{k}))$ and $\mathfrak{X}_\lambda(\vec{k};q,t)$ can be seen in the Appendix A.

The derivations and above formulas and the examples for a few boxes can be seen in the Appendix B.

Then our conjecture is on the form of refined reformulated invariants f^q_μ and $f^{\bar{t}}_\mu$ after inserting refined Chern-Simons invariants:

Conjecture 3. Refined reformulated invariants f^q_μ and $f^{\bar{t}}_\mu$ take the following form with a common part $\widehat{f}_\rho(T_{m,n};a,q,t)$ as

$$f^q_\mu(T_{m,n};a,q,t) = \sum_\rho M_{\mu\rho}(t)\widehat{f}_\rho(T_{m,n};a,q,t)\ ,$$
$$f^{\bar{t}}_\mu(T_{m,n};a,q,t) = \sum_\rho M_{\mu\rho}(q^{-1})\widehat{f}_\rho(T_{m,n};a,q,t)\ ,$$

where $M_{\mu\rho}$ is an invertible symmetric matrix. Moreover, $\widehat{f}_\rho(T_{m,n};a,q,t)$ has an expression

$$\widehat{f}_\rho(T_{m,n};a,q,t) = \sum_{g\geq 0}\sum_{\beta,J_r\in\mathbb{Z}}(-1)^{2J_r}\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^g(t^{-\frac{1}{2}}-t^{\frac{1}{2}})^g\left(\frac{q}{t}\right)^{J_r-\frac{\beta}{2}}a^\beta$$

with *non-negative* integers $\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})\in\mathbb{Z}_{\geq 0}$ up on the a-grading shift by $\pm 1/2$.

We also call $\widehat{f}_\rho(T_{m,n};a,q,t)$ as refined reformulated invariants and the conjecture 3 (and also the next conjecture) as the *positivity conjecture of refined Chern-Simons invariants* since, after making the change of variables (1.3), refined reformulated invariants $\widehat{f}_\rho(T_{m,n})$ can be written in the form

$$\widehat{f}_\rho(T_{m,n}) = \sum_{g\geq 0} \sum_{\beta, J_r\in\mathbb{Z}} \widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})(\mathbf{qt}-\mathbf{q}^{-1}\mathbf{t}^{-1})^g(\mathbf{q}-\mathbf{q}^{-1})^g\mathbf{a}^{2\beta}\mathbf{t}^{2J_r} \ . \tag{1.6}$$

Therefore, refined reformulated invariants are the Laurent polynomials with non-negative integer coefficients in knot variables $(\mathbf{a},\mathbf{q},\mathbf{t})$:

$$\widehat{f}_\rho(T_{m,n}) \in \mathbb{Z}_{\geq 0}[\mathbf{a}^{\pm 1}, \mathbf{t}^{\pm 1}, (\mathbf{qt}-\mathbf{q}^{-1}\mathbf{t}^{-1})(\mathbf{q}-\mathbf{q}^{-1})]. \tag{1.7}$$

Amazingly, refined reformulated invariants $\widehat{f}_\rho(T_{m,n})$ are still Laurent polynomials with non-negative integer coefficients after expanding $(\mathbf{qt}-\mathbf{q}^{-1}\mathbf{t}^{-1})(\mathbf{q}-\mathbf{q}^{-1})$ parts:

Conjecture 4.

$$\widehat{f}_\rho(T_{m,n}) = \sum_{i,j,k\in\mathbb{Z}} \widehat{\mathbf{N}}_{\mu;i,j,k}(T_{m,n})\mathbf{a}^{2i}\mathbf{q}^{2j}\mathbf{t}^{k} \ \in \mathbb{Z}_{\geq 0}[\mathbf{a}^{\pm 2}, \mathbf{q}^{\pm 2}, \mathbf{t}^{\pm 1}]. \tag{1.8}$$

We check the conjectures 3, 4 in many examples (see Appendix C and a `Mathematica` file attached to arXiv page in [KN17]). Refined reformulated invariants can be expressed as appropriate combinations of refined Chern-Simons invariants and vice versa, so these invariants are equivalent to each other. Therefore, the positivity conjecture implies that negative coefficients of refined Chern-Simons invariants does not cause any problem since the positivity should be realized in refined reformulated invariants. Furthermore, the conjecture can be extended to torus links and a certain class of homologically-thin non-torus knots.

Organization of this thesis

In §2, we briefly review Chern-Simons theory, topological string theory and its large N duality. In §3, we explain its realization in M-theory, which induces refinements in each theory. Then we review refined Chern-Simons theory and generating functions of refined Chern-Simons invariants of torus knots. Furthermore, we perform Schwinger's computation for BPS states in M-theory to determine low-energy effective actions of Type IIA string theory in the refined context. We first consider M-theory without probe M5'-branes for illustrative purpose and then proceed to the case with M5'-branes. In §4, the large N duality is proposed for torus knots at the refined level. This relates refined Chern-Simons invariants to graded dimensions of BPS states in the resolved conifold, which results in a positivity conjecture of refined Chern-Simons invariants. The positivity property attributes to trivial action of the extra $U(1)$ symmetry on the BPS states. In §5, the large N duality is generalized to multi-component torus links. In §6, we see that the refined large N duality can be extended to a certain class of non-torus knots. In §7, we discuss related open problems. In Appendix A, we present a brief summary of symmetric functions used in this thesis. Appendix B provides explicit formulas for reformulated invariants in terms of refined Chern-Simons invariants by using the large N duality. Appendix C contains tables for BPS degeneracies associated to torus knots/links in the resolved conifold.

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2 Unrefined Large N duality

In this section we review Chern-Simons theory, topological string theory and its large N duality at unrefined level. We refer the reader to [HKK+03, Mn05, AK06, Ste14] for detailed introductions.

2.1 Chern-Simons theory

Chern-Simons theory is a topological quantum field theory on a three-manifold M with a gauge group G, which is defined by the action

$$S_{\rm CS} = \frac{k}{4\pi}\int_M {\rm Tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A), \tag{2.1}$$

where the Chern-Simons coupling (or level) k is an integer, Tr denotes an invariant bilinear form on the Lie algebra $\mathfrak{g}$ of G and A is a G-gauge connection on the trivial bundle over M. The connection is a $\mathfrak{g}$-valued one form. The Wilson loop along a (framed and oriented) knot K with a representation R of $\mathfrak{g}$ is defined by

$$W_R(K) = {\rm Tr}_R {\rm P}\exp\left(\oint_K A\right), \tag{2.2}$$

where P denotes path-ordering operator and the trace is taken over the representation space of R. Using path integral, we define the Chern-Simons partition function

$$Z_{\rm CS}(M,L,R) = \int \mathcal{D}A \prod_{i=1}^{n} W_R(K_i) e^{iS_{\rm CS}}, \tag{2.3}$$

where $L=(K_1,\cdots,K_n)$ is n-components link in M with representations $R=(R_1,\cdots,R_n)$ and the path integral is a formal functional integral over all equivalence classes of connections modulo gauge transformation. If M does not contain knots or all representations of a link are trivial, we denote the partition function as

$$Z_{\rm CS}(M) = \int \mathcal{D}A e^{iS_{\rm CS}}. \tag{2.4}$$

The lessons [Wit89] are

- $Z_{\rm CS}(M)$ gives a topological invariant of M.
- $Z_{\rm CS}(M,L,R)/Z_{\rm CS}(M)$ gives a link invariant of L.

We can accept these statements, at least formally, since the action and the Wilson loop are metric independent and gauge invariant[4]. Indeed, when $M=S^3$, $G=SU(N)$ and R denotes irreducible representations (i.e. Young diagrams $\lambda_1\cdots\lambda_n$), $Z_{\rm CS}(S^3,L,R)/Z_{\rm CS}(S^3)$ gives the colored HOMFLY-PT polynomial with the variables $q=\exp(\frac{2\pi i}{k+N})$ and $a=q^N$, which is a generalization of the Jones polynomial. The colored HOMFLY-PT polynomial reduces the Jones polynomial when $N=2$ and $R=\square$.

[4]The action is locally gauge invariant, but not globally gauge invariant. However, (2.4) is globally gauge invariant because the action transforms

$$S_{\rm CS}\to S_{\rm CS}+2\pi k z(g)\ \ (z(g)\in\mathbb{Z}),$$

under global gauge transformation $A\to g^{-1}Ag+g^{-1}dg$ (g is a G-valued 0-form) so that exponential of the action is gauge invariant.

Chern-Simons theory as a topological quantum field theory

Chern-Simons theory is an exactly solvable quantum field theory. The key ideas to obtain knot polynomials are cutting/gluing of three-manifold with knots, the canonical quantization on $\Sigma\times\mathbb{R}$ [Wit89, EMSS89, LLR91], and the assumption that the Chern-Simons path integral satisfies the axiom of topological quantum field theory [Ati89]. The canonical quantization reveals that the physical Hilbert space of Chern-Simons theory is identified with the space of conformal blocks of Wess-Zumino-Witten (WZW) model with gauge group G at level k on Σ. As a result, the Chern-Simons partition function (or the path integral) is assumed as follows:

1. The physical Hilbert space $\mathcal{H}_\Sigma$={conformal blocks of WZW model gauge group G at level k on Σ} is assigned to each oriented closed smooth surface Σ.
2. When M is a compact oriented smooth three-manifold with a boundary $\partial M=\Sigma$, the path integral gives a state $Z_{\rm CS}(M)\in\mathcal{H}_\Sigma$.

Note that the manifold M may include knots and/or the boundary Σ intersects knots. In this case Wilson loops and/or lines are inserted into the path integral, and marked paints with representations are assigned to the boundary. Furthermore,

I. For empty set $\emptyset$, $\mathcal{H}_\emptyset=\mathbb{C}$ i.e. when M is a compact oriented smooth three-manifold without boundaries $\partial M=\emptyset$, then $Z_{\rm CS}(M)\in\mathbb{C}$.

II. When $-\Sigma$ is the opposite orientation surface of Σ, $\mathcal{H}_{-\Sigma}$ is the dual Hilbert space: $\mathcal{H}_{-\Sigma}=\mathcal{H}_\Sigma^*$.

III. For disjoint unions $\Sigma\sqcup\Sigma'$, $\mathcal{H}_{\Sigma\sqcup\Sigma'}=\mathcal{H}_\Sigma\otimes\mathcal{H}_{\Sigma'}$. Moreover, for $\partial M=-\Sigma\sqcup\Sigma'$, $Z_{\rm CS}(M)\in\mathcal{H}_\Sigma^*\otimes\mathcal{H}_{\Sigma'}={\rm Hom}(\mathcal{H}_{\Sigma'},\mathcal{H}_\Sigma)$.

IV. For $\partial M_1=-\Sigma_1\sqcup\Sigma_2$, $\partial M_2=-\Sigma_2\sqcup\Sigma_3$ and $M=M_1\cup_{\Sigma_2}M_2$ is the manifold obtained by gluing together the common boundary Σ_2, $Z_{\rm CS}(M)=Z_{\rm CS}(M_2)\circ Z_{\rm CS}(M_1)\in\mathcal{H}_{\Sigma_1}^*\otimes\mathcal{H}_{\Sigma_2}={\rm Hom}(\mathcal{H}_{\Sigma_3},\mathcal{H}_{\Sigma_1})$.

V. Let $I\subset\mathbb{R}$ be an interval, $Z_{\rm CS}(\Sigma\times I)$ is the identity map of $\mathcal{H}_\Sigma$

From III and V, ${\rm Diff}^+(\Sigma)$ acts on the Hilbert space $\mathcal{H}_\Sigma$, where ${\rm Diff}^+(\Sigma)$ is the group of orientation preserving diffeomorphisms of Σ. Moreover, the mapping class group of Σ acts on the Hilbert space [Koh02]. Above assumptions give us an effective computation of the Chern-Simons partition function. In the following, let M be a three-manifold without boundaries but may include knots. We consider a cutting of M along a surface Σ, i.e. we split M into two three-manifolds M_L and M_R with a common boundary Σ, where the orientations of the boundaries are opposite each other. We chose $\partial M_R=\Sigma$ and $\partial M_L=-\Sigma$. Then the Chern-Simons partition functions of M_R and M_L are states $|\Psi_{M_R}\rangle:=Z_{\rm CS}(M_R)\in\mathcal{H}_\Sigma$ and $\langle\Psi_{M_L}|:=Z_{\rm CS}(M_L)\in\mathcal{H}_\Sigma^*$ respectively, so that the partition function of M is

$$Z_{CS}(M)=\langle\Psi_{M_L}|\Psi_{M_R}\rangle\,, \tag{2.5}$$

where the bracket is the natural pairing between $\mathcal{H}_\Sigma$ and $\mathcal{H}_\Sigma^*$.

Figure 1. The left figure shows (1,0) and (0,1) cycles on the surface of solid torus $\mathcal{T}$. The right figure shows torus knot $T_{2,3}$.

In the following we consider $G = SU(N)$, a solid torus $\mathcal{T} = D \times S^1$ with $\Sigma = T^2$, and the boundary does not cut knots but each $\mathcal{T}$ may include knots. We chose (0,1) cycle of the torus to be contractible in the interior and (1,0) cycle to be non-contractible (see Figure 1). The mapping class group of the torus is $SL(2,\mathbb{Z})$, where the generators s and t satisfy

$$s^4 = 1, \ (st)^3 = s^2. \tag{2.6}$$

In this case we can explicitly write down a basis, the pairing, some operators of the Hilbert space. The Hilbert space $\mathcal{H}_{T^2}$ has a basis $|\lambda\rangle$ which can be obtained from the path integral over the gauge fields on solid torus containing a knot inserted along (1,0) cycle with the $SU(N)$ irreducible representation λ (i.e. a Young diagram), and the pairing is

$$\langle\lambda|\mu\rangle = \delta_{\lambda,\mu}. \tag{2.7}$$

The $SL(2,\mathbb{Z})$ action on the Hilbert space is

$$S_{\lambda\mu} = S_{\emptyset\emptyset} s_\lambda(q^{\rho_1}, \cdots, q^{\rho_N}) s_\mu(q^{\rho_1+\lambda_1}, \cdots, q^{\rho_N+\lambda_N}), \tag{2.8}$$

$$T_{\lambda\mu} = \delta_{\lambda,\mu} q^{\frac{1}{2}(||\mu||^2+||\mu^T||^2-N|\mu|+\frac{|\mu|}{N})}, \tag{2.9}$$

where s_λ is the Schur polynomial (see Appendix A), $q = e^{\frac{2\pi i}{k+N}}$, ρ is the Weyl vector such that $\rho_i = (N+1)/2 - i$ $(1 \le i \le N)$ and $||\lambda||^2 = \sum_i \lambda_i^2$. The normalization factor of the S-matrix equals to the partition function on S^3, $S_{\emptyset\emptyset} = Z_{\rm CS}(S^3)$[5], since S^3 is obtained by gluing two solid tori without knots by using s transformation $S^3 = \mathcal{T} \cup_s \mathcal{T}$. We call (2.8) as the modular S-matrix and (2.9) as the modular T-matrix. Note that the power-sum symmetric function in this case is

$$p_n(q^{\rho_1+\lambda_1}, \cdots, q^{\rho_N+\lambda_N}) = \frac{a^{\frac{n}{2}} - a^{-\frac{n}{2}}}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} + a^{\frac{n}{2}} \sum_{i=1}^{\infty} (q^{n\lambda_i} - 1) q^{n(\frac{1}{2}-i)}, \tag{2.10}$$

where we set $a = q^N$. Moreover, the knot operator $\mathcal{W}_\lambda(T_{m,n})$ is given by [LLR91] which creates the torus knot $T_{m,n}$ on the surface. A torus knot $T_{m,n}$ is a special kind of knot which can be put on the torus. Each torus knot can be specified by coprime integers m and n which correspond to (m,n) cycle of the torus. For instance, the trefoil is the torus knot $T_{2,3}$ (see Figure 1). The knot operator can be defined as

$$\begin{aligned} \mathcal{W}_\lambda(T_{1,0}) |\emptyset\rangle &= |\lambda\rangle, \\ \mathcal{W}_\lambda(T_{1,0}) \mathcal{W}_\mu(T_{1,0}) &= \sum_\nu N_{\lambda\mu}^\nu \mathcal{W}_\nu(T_{1,0}), \\ \mathcal{W}_\lambda(T_{m,n}) &= \mathcal{K}_{m,n} \mathcal{W}_\lambda(T_{1,0}) \mathcal{K}_{m,n}^{-1}, \end{aligned} \tag{2.11}$$

[5] We do not need to know the factor because the factor will be canceled out.

where $N_{\lambda\mu}^{\nu}$ are Verlinde coefficients [Ver88] and $\mathcal{K}_{m,n}$ is an element of $SL(2,\mathbb{Z})$ which takes $(1,0)$ cycle to (m,n) cycle such that $(1,0)\mathcal{K}_{m,n}=(m,n)$ i.e.

$$\mathcal{K}=\begin{pmatrix} m & n \\ a & b \end{pmatrix}, \quad mb-na=1. \tag{2.12}$$

These operators enable us to compute the unreduced colored HOMFLY-PT polynomials:

$$\overline{H}_{\lambda}(T_{m,n}) \equiv \frac{Z_{\text{CS}}(S^3, T_{n,n}, \lambda)}{Z_{\text{CS}}(S^3)} = \frac{\langle\emptyset|S\mathcal{W}_{\lambda}(T_{n,m})|\emptyset\rangle}{\langle\emptyset|S|\emptyset\rangle}. \tag{2.13}$$

We will comment on the precisely meaning of the symbol "≡" later.

For example the colored HOMFLY-PT polynomials for unknot and the Hopf link can be easily obtained by using knot operator. In the above explanation, S^3 is obtained by gluing two solid tori without knots. If we insert a knot along (1,0) cycle into one of solid tori and glue them, we get S^3 including unknot. Since $\mathcal{W}_{\lambda}(T_{1,0})\,|\emptyset\rangle=|\lambda\rangle$, The colored HOMFLY-PT polynomial for unknot is

$$\overline{H}_{\lambda}(\bigcirc) = \frac{\langle\emptyset|S|\lambda\rangle}{\langle\emptyset|S|\emptyset\rangle} = \frac{S_{\emptyset\lambda}}{S_{\emptyset\emptyset}} = s_{\lambda}(q^{\rho_1},\cdots,q^{\rho_N}). \tag{2.14}$$

Similarly, if we insert knots along (1,0) cycle into each solid torus, we get S^3 including the Hopf link

$$\overline{H}_{\lambda\mu}(\text{Hopf}) = \frac{S_{\lambda\mu}}{S_{\emptyset\emptyset}} = s_{\lambda}(q^{\rho_1},\cdots,q^{\rho_N})s_{\mu}(q^{\rho_1+\lambda_1},\cdots,q^{\rho_N+\lambda_N}). \tag{2.15}$$

Furthermore, the general expression for torus knot is given by [Ste10]:

$$\begin{aligned}\overline{H}_{\lambda}(T_{m,n}) &= q^{mn\kappa(\lambda)/2}a^{n(m-1)|\lambda|/2}\sum_{|\nu|=m|\lambda|} c^{\nu}_{\lambda,m}q^{-\frac{n}{2m}\kappa(\nu)}s_{\nu}(q^{\rho_1},\cdots,q^{\rho_N}),\\ c^{\nu}_{\lambda,m} &= \sum_{\vec{k}}\frac{1}{z_{\vec{k}}}\chi_{\lambda}(C(\vec{k}))\chi_{\nu}(C(m\vec{k})),\end{aligned} \tag{2.16}$$

where $\kappa(\lambda)=\sum_i\lambda_i(\lambda_i+1-2i)$ and $\chi_{\lambda}(C(\vec{k}))$ are the characters of the permutation group (see Appendix A). This result is nothing but Rosso-Jones formula [RJ93].

Note that there is a difference of conventions between physics and knot theory at unrefined level. In knot theory, the *reduced* colored HOMFLY-PT polynomial is well studied, and it is Laurent polynomial in $\mathbb{Z}[\mathbf{a}^{\pm},\mathbf{q}^{\pm}]$, where "*reduced*" means that the polynomials are normalized to one for unknot with all coloring. Indeed, (2.14), (2.15) and (2.16) are not polynomials[6] in terms of a and q, and they become polynomials after dividing by unknot

$$H_{\lambda}(K) = \frac{\overline{H}_{\lambda}(K)}{\overline{H}_{\lambda}(\bigcirc)} \in \mathbb{Z}[a^{\pm\frac{1}{2}},q^{\pm\frac{1}{2}}]. \tag{2.17}$$

As a summary, a change of the variables

$$a=\mathbf{a}^2,\ q=\mathbf{q}^2 \tag{2.18}$$

[6] Note that the Schur polynomial is a polynomial in terms of $x=(x_1,\cdots,x_N)$.

relates the conventions between physics and knot theory at unrefined level.

Now we explain the meaning of the symbol "≡". In general quantum knot invariants have formal variables, and the $SU(N)$ quantum invariant $\overline{J}_{SU(N),\lambda}(K;q)$ has one variable q with fixed N. On the other hand, the Chern-Simons partition function is a function of Chern-Simons coupling k and the rank N of the gauge group $SU(N)$, where we fix these values in our mind. Therefore, the right hand side of (2.13) is equal to the $SU(N)$ quantum invariant $\overline{J}_{SU(N),\lambda}(K;q)$ with the relation $q = \exp(\frac{2\pi i}{k+N})$. In other words, replacing $\exp(\frac{2\pi i}{k+N})$ with the formal variable q, we regard the Chern-Simons partition function as the $SU(N)$ quantum invariants. However, the dependence of N is still remaining. It is known that the $SU(N)$ quantum invariants $\overline{J}_{SU(N),\lambda}(K;q)$ become the colored HOMFLY-PT polynomials $\overline{H}_\lambda(K;a,q)$ when we replace q^N with the formal variable a and forget the dependence of N. Therefore, the symbol "≡" means that we regard the Chen-Simons partition function as the colored HOMFLY-PT polynomial with formal variables by performing above replacements.

Finally, we have reviewed the computation of the HOMFLY-PT polynomials only for torus knots. If one wants to compute the invariants for non-torus knots, one needs another algorithm. Fortunately, there are many algorithms (see e.g. [NRZ13] for a Chern-Simons theoretic method or [Mor93, LZ10] for a mathematical method).

2.2 Topological string theory

Topological string theories are defined by topological sigma models, called A-model and B-model, coupled to two dimensional gravity. From the viewpoint of worldsheet, we consider holomorphic maps from genus g Riemann surface Σ_g to a target space and perform an integral over the moduli space of Σ_g as a path integral. In the following we consider closed topological string of the A-model, where a Calabi-Yau threefold X is chosen as the target space. In this setup one can compute the free energy $F_g(\tau)$ contributed from Σ_g, which is equivalent to the generating function of Gromov-Witten invariants $\mathrm{GW}_{g,\beta}$ (see [HKK⁺03])

$$F_g(\tau) = \sum_{\beta \in H_2(X,\mathbb{Z})} \mathrm{GW}_{g,\beta}\, e^{-\tau\cdot\beta}, \tag{2.19}$$

where τ denotes Kahler parameters. Therefore, the full free energy is defined by

$$F^{\mathrm{GW}}_{\mathrm{closed}} = \sum_{g=0}^{\infty} g_{\mathrm{st}}^{2g-2} F_g(\tau), \tag{2.20}$$

where g_{st} is the string coupling constant. Gromov-Witten invariants "count" the number of the holomorphic curves in the Calabi-Yau threefold, but they are rational numbers. This fact does not intuitively match counting holomorphic maps.

On the other hand, Gopakumar and Vafa (GV) proposed a target space interpretation of the free energy, which defines new integer invariants. GV formulation is as follows. The free energy of topological string is encoded in F-term in Type IIA string theory. Then, instead of evaluating each genus amplitude in Type IIA string theory, the whole effective action can be obtained by summing up contributions of BPS states in M-theory. They arise from M2-branes wrapped on a holomorphic curve Σ in the Calabi-Yau manifold X. This calculation can be considered as a supersymmetric version of Schwinger's computation of a one-loop effective action due to a charged particle in a constant magnetic field. As a result, they conjectured that the closed string free energy can be rewritten by new integers $n_{g,\beta}$, called GV invariants, as

$$F^{\rm GV}_{\rm closed} = \sum_{d=1}^{\infty} \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{g=0}^{\infty} \frac{n_{g,\beta}}{d} (q^{\frac{d}{2}} - q^{-\frac{d}{2}})^{2g-2} e^{-d\beta\cdot\tau}, \tag{2.21}$$

where $q = e^{ig_{\rm st}}$. We will see the refined version of GV formalism in §3.2.1.

In open string case, we consider holomorphic maps from genus g Riemann surface with h boundaries $\Sigma_{g,h}$ to X, where boundaries are mapped to a Lagrangian submanifold $\mathcal{L}$ in X. We wrap the topological D-branes on the Lagrangian submanifold which is assumed having one non-trivial H_1. The free energy is analogically defined by using open Gromov-Witten invariants $\mathrm{OGW}_{g,n,\beta}$

$$\begin{aligned} F^{\rm OGW}_{\rm open} &= \sum_{g=0}^{\infty} \sum_{h} \sum_{\vec{n}} \frac{i^h}{h!} g_{\rm st}^{2g-2+h} F_{g,\vec{n}}(\tau) \prod_{i=1}^{h} p_{n_i}(x_i^{n_i}), \\ F_{g,\vec{n}}(\tau) &= \sum_{\beta \in H_2(X,\mathbb{Z})} \mathrm{OGW}_{g,\vec{n},\beta}\ e^{-\tau\cdot\beta}, \end{aligned} \tag{2.22}$$

where x_i denotes eigenvalue of the holonomy matrix around H_1 and $\vec{n} = (n_1, \cdots, n_h)$ denotes the winding numbers of each hole. As in the closed string case, LMOV conjectured that the partition function of open topological string can be rewritten by using integer invariants $\widehat{N}_{\rho,g,\beta}$, called LMOV invariants[7], as

$$F^{\rm LMOV}_{\rm open} = \sum_{d=1}^{\infty} \sum_{\lambda} \frac{1}{d} \frac{f_\lambda(a^d, q^d)}{q^{\frac{d}{2}} - q^{-\frac{d}{2}}} s_\lambda(x^d), \tag{2.23}$$

$$f_\lambda(a,q) = \sum_{\sigma,\rho} \sum_{g=0}^{\infty} \sum_{\beta \in H_2(X,\mathbb{Z})} C_{\lambda\sigma\rho} B_\sigma(q) \widehat{N}_{\rho,g,\beta} (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{2g} a^\beta, \tag{2.24}$$

where $q = e^{ig_{\rm st}}$, $a = e^{-\tau}$, $C_{\mu\sigma\rho}$ are the Clebsch-Gordon coefficients of the permutation group $\mathfrak{S}_h$ (see Appendix A)

$$C_{\mu\sigma\rho} = \sum_{\vec{k}} \frac{1}{z_{\vec{k}}} \chi_\mu(C(\vec{k})) \chi_\sigma(C(\vec{k})) \chi_\rho(C(\vec{k})) \tag{2.25}$$

and $B_\sigma(q)$ is

$$B_\sigma(q) = \begin{cases} (-q)^d q^{-\frac{|\sigma|-1}{2}} & \sigma : \text{hook rep for } \wedge^d V \\ 0 & \sigma : \text{otherwise} \end{cases}. \tag{2.26}$$

We will see the refined version of LMOV formalism in §3.2.2.

[7]It is also conjectured that there are another integer invariants $N_{\rho,g,\beta}$, called OV invariants, and the $f_\mu(a,q)$ can be written as

$$f_\mu(a,q) = \sum_{s,\beta} N_{\rho,s,\beta} q^s a^\beta. \tag{2.24'}$$

2.3 Large N duality

Now, we see the formulation of the large N duality. There is a slight difference in open/closed string but the duality implies an equivalence of partition functions. We will see the closed string case at first. In closed string case the duality is formulated as an equation between the partition function of $SU(N)$ Chern-Simons theory on S^3 and the partition function (i.e. exp of the free energy) of topological closed string on resolved conifold

$$Z_{\mathrm{CS}}(S^3) \text{ at large } N = \exp(F_{\mathrm{closed}}). \tag{2.27}$$

In this case both sides are exactly computed in physics and in mathematics, and this equality is shown (see [GV99, OV00]).

On the other hand, the duality of open string case is formulated as an equation between the expectation value of Ooguri-Vafa operator of Chern-Simons theory on S^3 including a knot K and the partition function of topological open string on resolved conifold with topological D-branes $\mathcal{L}_K$

$$\lim_{N\to\infty} Z_{\mathrm{OV}}(K;x) = \exp(F_{\mathrm{open}}^{\mathrm{OGW}}), \tag{2.28}$$

where the expectation value is

$$Z_{\mathrm{OV}}(K;x) = \sum_{\lambda} \overline{J}_{SU(N),\lambda}(K;q) s_\lambda(x). \tag{2.29}$$

At the large N limit, we interpret that the $SU(N)$ quantum invariant is replaced with the colored HOMFLY-PT polynomial, so that

$$\lim_{N\to\infty} Z_{\mathrm{OV}}(K;x) = \sum_{\lambda} \overline{H}_\lambda(K;a,q) s_\lambda(x). \tag{2.30}$$

We cannot verify this claim (2.28) in almost all cases since the computation of the free energy of open topological string (i.e. the generating function of open Gromov-Witten invariants) is very difficult.

The verifiable claim is as follows. Assuming $F_{\mathrm{open}}^{\mathrm{OGW}} = F_{\mathrm{open}}^{\mathrm{LMOV}}$ and (2.28), we define $f_\lambda(a,q)$ via the equation

$$\sum_{\lambda} \overline{H}_\lambda(K) s_\lambda(x) = \exp\left(\sum_{d=1}^{\infty} \sum_{\lambda} \frac{1}{d} \frac{f_\lambda(K;a^d,q^d)}{q^{\frac{d}{2}} - q^{-\frac{d}{2}}} s_\lambda(x^d) \right). \tag{2.31}$$

The explicit formula for $f_\lambda(a,q)$ is shown in [LMn02] as

$$\begin{aligned} \frac{1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} f_\mu(K;a,q) = \sum_{d,m=1}^{\infty} (-1)^{m-1} \frac{\mu(d)}{d\cdot m} \sum_{\{\vec{k}^{(\alpha)}\}} \sum_{\{\lambda^{(\alpha)}\}} \chi_\mu(C(\sum_{\alpha=1}^{m} (\vec{k}^{(\alpha)})_d)) \\ \times \prod_{\alpha=1}^{m} \frac{\chi_{\lambda^{(\alpha)}}(\vec{k}^{(\alpha)})}{z_{\vec{k}^{(\alpha)}}} \overline{H}_{\lambda^{(\alpha)}}(K;a^d,q^d)\ . \end{aligned} \tag{2.32}$$

For example,

$$
\begin{aligned}
\frac{f_{\square}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} =&\overline{H}_{\square},\\
\frac{f_{\tiny\yng(2)}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} =&\overline{H}_{\tiny\yng(2)}-\frac{1}{2}\overline{H}_{\square}^{2}-\frac{1}{2}\overline{H}_{\square}^{(2)},\\
\frac{f_{\tiny\yng(1,1)}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} =&\overline{H}_{\tiny\yng(1,1)}-\frac{1}{2}\overline{H}_{\square}^{2}+\frac{1}{2}\overline{H}_{\square}^{(2)},\\
\frac{f_{\tiny\yng(3)}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} =&\overline{H}_{\tiny\yng(3)}-\overline{H}_{\tiny\yng(2)}\overline{H}_{\square}+\frac{1}{3}\overline{H}_{\square}^{3}-\frac{1}{3}\overline{H}_{\square}^{(3)},\\
\frac{f_{\tiny\yng(2,1)}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} =&\overline{H}_{\tiny\yng(2,1)}-\left[\overline{H}_{\tiny\yng(2)}+\overline{H}_{\tiny\yng(1,1)}\right]\overline{H}_{\square}+\frac{2}{3}\overline{H}_{\square}^{3}+\frac{1}{3}\overline{H}_{\square}^{(3)},\\
\frac{f_{\tiny\yng(1,1,1)}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} =&\overline{H}_{\tiny\yng(1,1,1)}-\overline{H}_{\tiny\yng(1,1)}\overline{H}_{\square}+\frac{1}{3}\overline{H}_{\square}^{3}-\frac{1}{3}\overline{H}_{\square}^{(3)},
\end{aligned}
$$

where $\overline{H}_{\lambda}^{(k)}=\overline{H}_{\lambda}(K;a^k,q^k)$. Then we can verify the integrality structure (2.24) (and (2.24')). It is convenient to introduce $\widehat{f}_{\rho}(a,q)$ and an invertible symmetric matrices $M_{\mu\rho}$

$$
\begin{aligned}
f_{\lambda}(K;a,q) &= \sum_{\rho} M_{\lambda\rho}(q)\widehat{f}_{\rho}(K;a,q),\\
M_{\mu\rho}(q) &= \sum_{\sigma} C_{\mu\sigma\rho}B_{\sigma}(q),
\end{aligned}
\tag{2.33}
$$

then one can verify

$$
\widehat{f}_{\rho}(K;a,q)=\sum_{g=0}^{\infty}\sum_{\beta}\widehat{N}_{\lambda,g,Q}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{2g}a^{\beta}. \tag{2.34}
$$

We call both f and $\widehat{f}$ reformulated invariants. This claim has been tested in [LMn01, RS01, LMnV00, ZR12, MMM$^+$17] and a proof is proposed by [LP10]. Note that the above story can be extended to the link case [LMnV00, LMn02].

Unknot is a special and important example because the colored HOMFLY-PT polynomials for unknot are given by Schur polynomial itself, so that the Cauchy formula (A.3) enables us to compute all coloring i.e. the total free energy is

$$
\sum_{\lambda}\overline{H}_{\lambda}(\bigcirc)s_{\lambda}(x)=\exp\left(\sum_{d=1}^{\infty}\frac{1}{d}\frac{a^{\frac{d}{2}}-a^{-\frac{d}{2}}}{q^{\frac{d}{2}}-q^{-\frac{d}{2}}}s_{\square}(x^d)\right). \tag{2.35}
$$

Therefore, we can read $f_{\square}(\bigcirc;a,q)=\widehat{f}_{\square}(\bigcirc;a,q)=a^{\frac{1}{2}}-a^{-\frac{1}{2}}$ and $\widehat{N}_{\square,0,1/2}=-\widehat{N}_{\square,0,1/2}=1$.

3 Realizations in M-theory and refinements

Chern-Simons theory is realized on topological branes wrapped on S^3 in the deformed conifold T^*S^3 [Wit95]. In addition to this, the key idea of GV/LMOV formalism is to go to Type IIA/M-theory. Therefore, the large N duality is encoded in Type IIA/M-theory with branes. It is summarized by the setup of table.1

	q-brane setting					$\bar{t}$-brane setting				
space-time	S^1	$\times$	TN_4	$\times$	T^*S^3	S^1	$\times$	TN_4	$\times$	T^*S^3
N M5-branes	S^1	$\times$	D_q	$\times$	S^3	S^1	$\times$	D_q	$\times$	S^3
M M5'-branes	S^1	$\times$	D_q	$\times$	$\mathcal{N}_K$	S^1	$\times$	$D_{\bar{t}}$	$\times$	$\mathcal{N}_K$

Table 1. Brane settings on deformed conifold side

Here, S^1 is the M-theory circle, D_q is a two-dimensional cigar and $D_{\bar{t}}$ is a two-dimensional base of the Taub-NUT space TN_4. Moreover, writing the local complex coordinates z_1 for D_q and z_2 for $D_{\bar{t}}$, we turn on the Ω-background by the action

$$(z_1, z_2) \to (qz_1, t^{-1}z_2) \ . \tag{3.1}$$

We wrap the N M5-branes on the zero section (a special Lagrangian submanifold) S^3 of the cotangent bundle T^*S^3. Then Chern-Simons theory with $U(N)$ gauge group are realized on S^3 [Wit95]. The M probe M5'-branes are located at another Lagrangian submanifold, which is the co-normal bundle $\mathcal{N}_K \subset T^*S^3$ to a knot $K \subset S^3$ where the knot K is realized as the intersection of the two stacks of the M5-branes $K = S^3 \cap \mathcal{N}_K$ [OV00]. As in [AS12b], we consider two distinct probe M5'-branes; one spanning on D_q and the other on $D_{\bar{t}}$. Although both the configurations preserve four supercharges, the q and $\bar{t}$ branes become topological branes and anti-branes, respectively, in topological string theory.

The proposal in [GV99] is that, at large N, the geometry undergoes the transition where S^3 shrinks and $S^2 = \mathbb{CP}^1$ is blown up. As S^3 shrinks, in order for the probe M5'-branes to avoid the singularity, the conormal bundle $\mathcal{N}_K$ is lifted to the fiber direction, and it no longer touches S^3. We refer the reader to [DSV13] for detailed treatment. As a result, the Calabi-Yau threefold becomes the resolved conifold $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$ where the N M5-branes turn into B-field flux supporting $\mathbb{CP}^1$ and the M M5'-branes are situated on a Lagrangian submanifold $\mathcal{L}_K \subset X$ keeping the information of the knot K.

	q-brane setting					$\bar{t}$-brane setting				
space-time	S^1	$\times$	TN_4	$\times$	X	S^1	$\times$	TN_4	$\times$	X
M M5'-branes	S^1	$\times$	D_q	$\times$	$\mathcal{L}_K$	S^1	$\times$	$D_{\bar{t}}$	$\times$	$\mathcal{L}_K$

Table 2. Brane settings after geometric transition

Now, we are ready to discuss refinements. M-theory is the physical origin of refinements and induces refinements in each theory.

3.1 Deformed conifold side

The work of Witten [Wit95, Wit11] provides a way to interpret Chern-Simons partition functions as an index counting BPS states in M-theory. From the viewpoint of M-theory, BPS states arise from M2-branes bridging between the M5-branes and the M5'-branes, and they propagate only along the M5-branes. In low energy, one can suppress $S^3 \subset T^*S^3$ so that these states propagate along $\mathbb{R} \times D_q \subset \mathbb{R} \times TN_4$ when S^1 is large. Thus, $SU(N)$ Chern-Simons partition functions $Z_{\rm CS}$ can be identified with an index $\mathrm{Tr}(-1)^F q^{J_1-J_2}$ of the three-dimensional theory on $S^1 \times D_q$ with $\mathcal{N}=2$ supersymmetry

$$Z_{\rm CS}(S^3, L, R; q) = \mathrm{Tr}(-1)^F q^{J_1-J_2}, \tag{3.2}$$

where J_1 is the Cartan generator of the $SO(3)$ Lorentz symmetry of $S^1 \times D_q$ and J_2 is the generator of the $U(1)_2$ $\mathcal{R}$-symmetry in $\mathcal{N}=2$ supersymmetry. In fact, J_1 and J_2 can be considered as generators of the rotations around the z_1 and z_2 plane, respectively.

In [AS15], it is argued that an extra $U(1)_R$ symmetry exists when K is a torus knot $T_{m,n}$ (more generally Seifert three-manifolds and Seifert knots). Then, using the charge S_R of this $U(1)_R$ symmetry, Aganagic and Shakirov defined a refined partition function of a torus knot by a refined index [8]

$$Z^{\rm ref}_{\rm CS}(S^3, T_{m,n}, R; q, t) := \mathrm{Tr}_{\mathcal{H}_R(T_{m,n})} (-1)^F \, q^{J_1-S_R} \, t^{S_R-J_2} \; . \tag{3.3}$$

Hence, refined Chern-Simons theory is defined via the 3d/3d correspondence and non-Lagrangian theory. Furthermore, they deduced a refinement of modular S and T matrices from (3.3):

$$\begin{aligned} S^{\rm ref}_{\lambda\mu} &= S_{\emptyset\emptyset} P_\lambda(t^{\rho_1}, \cdots, t^{\rho_N}; q, t) P_\mu(t^{\rho_1} q^{\lambda_1}, \cdots, t^{\rho_N} q^{\lambda_N}; q, t), \\ T^{\rm ref}_{\lambda\mu} &= \delta_{\lambda,\mu} t^{\frac{1}{2}(||\mu^T||^2 - N|\mu|)} q^{\frac{1}{2}(-||\mu||^2 + \frac{|\mu|}{N})}, \end{aligned} \tag{3.4}$$

where $P_\lambda(x; q, t)$ are the Macdonald functions (see Appendix A). Once we know refined modular matrices, we can define refined knot invariants, called refined Chern-Simons invariants, as an analogy to (2.13). We define the knot operator

$$\begin{aligned} \mathcal{W}_\lambda(T_{1,0}) \left|\emptyset\right\rangle &= \left|\lambda\right\rangle, \\ \mathcal{W}_\lambda(T_{1,0}) \mathcal{W}_\mu(T_{1,0}) &= \sum_\nu N^\nu_{\lambda\mu} \mathcal{W}_\nu(T_{1,0}), \\ \mathcal{W}_\lambda(T_{m,n}) &= \mathcal{K}_{m,n} \mathcal{W}_\lambda(T_{1,0}) \mathcal{K}^{-1}_{m,n}, \end{aligned} \tag{3.5}$$

where $\mathcal{K}_{m,n}$ is the same as before and $N^\nu_{\lambda\mu}$ are refined Verlinde coefficients (see [AS15]). Therefore, we define refined Chern-Simons invariants

$$\overline{\mathrm{rCS}}_{\lambda, SU(N)}(T_{m,n}; q, t) = \frac{\langle \emptyset | S^{\rm ref} \mathcal{W}_\lambda(T_{m,n}) | \emptyset \rangle}{\langle \emptyset | S^{\rm ref} | \emptyset \rangle}, \tag{3.6}$$

where the refined Hilbert space of torus has an orthogonal basis and the inner product

$$\langle \lambda | \mu \rangle = G_\lambda(q, t) \delta_{\lambda,\mu}, \tag{3.7}$$

[8] Charges of J_1, J_2 and S_R are normalized to be half-integers. In §3.2.2, we shall provide more explanation about the refined index.

where $G_\lambda(q,t)$ is defined in [AS15, (5.10)]. In principle, fixing N, one can directly calculate $\overline{\mathrm{rCS}}_{\lambda,SU(N)}(T_{m,n};q,t)$, but the calculation is hard.

It is proven in [GN15] that there exists a unique rational function $\overline{\mathrm{rCS}}_\lambda(T_{m,n};a,q,t)$ which is the stable large N limit of refined Chern-Simons invariants with $SU(N)$ gauge group of a torus knot in the following sense:

$$\overline{\mathrm{rCS}}_\lambda(T_{m,n};a=t^N,q,t)=\overline{\mathrm{rCS}}_{SU(N),\lambda}(T_{m,n};q,t)\ .$$

The stable limit $\overline{\mathrm{rCS}}_\lambda(T_{m,n};a,q,t)$ is also called *DAHA-superpolynomial* [Che13].

For instance, the refined Chern-Simons invariants of the unknot is

$$\overline{\mathrm{rCS}}_\lambda(\bigcirc;a,q,t)=\left(\frac{t}{a}\right)^{\frac{|\lambda|}{2}}\prod_{x\in\lambda}\frac{t^{l'(x)}-aq^{a'(x)}}{1-q^{a(x)}t^{l(x)+1}} \tag{3.8}$$

so that, in the specialization $a=t^N$, it becomes the Macdonald function

$$\overline{\mathrm{rCS}}_\lambda(\bigcirc;a=t^N,q,t)=P_\lambda(t^{\rho_1},\cdots,t^{\rho_N};q,t)=\overline{\mathrm{rCS}}_{SU(N),\lambda}(\bigcirc;q,t)\ ,$$

where ρ is the Weyl vector of $\mathfrak{sl}(N)$. The refine Chern-Simons invariants of Hopf link is

$$\overline{\mathrm{rCS}}_{\lambda\mu}(\mathrm{Hopf};a,q,t)=P_\lambda(t^{\rho_1},\cdots,t^{\rho_N};q,t)P_\mu(t^{\rho_1}q^{\lambda_1},\cdots,t^{\rho_N}q^{\lambda_N};q,t). \tag{3.9}$$

The general form of refined Chern-Simons invariant for torus knots such as (2.16) is not known, but refined Chern-Simons invariants of torus knot $T_{2,2p+1}$ with symmetric and anti-symmetric representations are given[9] [FGS13, FGSA12]:

$$\begin{aligned}&\overline{\mathrm{rCS}}_{[r]}(T_{2,2p+1};a,q,t)\\&=\left(\frac{q}{t}\right)^{-rp/2}a^{r/2}t^{-r/2}\frac{(t;q)_r}{(a;q)_r}\sum_{\ell=0}^{r}\frac{(t;q)_\ell(q;q)_r(a;q)_{r+\ell}(t^{-1}a;q)_{r-\ell}}{(q;q)_\ell(t;q)_r(t;q)_{r+\ell}(q;q)_{r-\ell}}\frac{(1-tq^{2\ell})}{(1-tq^{r+\ell})}\\&\qquad\times a^{-r}q^{\frac{r-\ell}{2}}t^{\frac{3r-\ell}{2}}\left[(-1)^{r-\ell}a^{\frac{r}{2}}q^{\frac{r^2-\ell^2}{2}}t^{-\frac{\ell}{2}}\right]^{2p+1},\end{aligned} \tag{3.10}$$

$$\begin{aligned}&\overline{\mathrm{rCS}}_{[1^r]}(T_{2,2p+1};a,q,t)\\&=\left(\frac{q}{t}\right)^{-rp/2}(-1)^ra^{-r/2}t^{-r/2}\frac{(t;t)_r}{(a^{-1};t)_r}\sum_{\ell=0}^{r}\frac{(q;t)_\ell(t;t)_{r+\ell}(a^{-1};t)_{r+\ell}(q^{-1}a^{-1};t)_{r-\ell}}{(t;t)_\ell(qt;t)_{r+\ell}(t;t)_{r+\ell}(t;t)_{r-\ell}}\\&\qquad\times\frac{(1-qt^{2\ell})}{(1-q)}a^rq^{r-\frac{\ell}{2}}t^{r-\frac{\ell}{2}}\left[(-1)^\ell a^{\frac{r}{2}}q^{\frac{\ell}{2}}t^{\frac{\ell^2-r^2}{2}}\right]^{2p+1},\end{aligned} \tag{3.11}$$

where $(x;q)_n$ is the q-Pochhammer symbol

$$(x;q)_n=\prod_{k=0}^{n-1}(1-xq^k)=(1-x)(1-xq)(1-xq^2)\dots(1-xq^{n-1}). \tag{3.12}$$

Other examples are listed in [DBMM+13]. The invariants are effectively computed by Γ-factor and the stable limit of refined modular S and T matrices [Sha13]. When the color

[9] We changed a prefactor by $(q/t)^{-rp/2}$.

λ is labelled by a rectangular Young diagram, it is conjectured that the reduced invariant[10] $\text{rCS}_\lambda(T_{m,n}; a, q, t)$ with the change of variables

$$a = -\mathbf{a}^2\mathbf{t} \ , \quad q^{\frac{1}{2}} = -\mathbf{q}\mathbf{t} \ , \quad t^{\frac{1}{2}} = \mathbf{q} \ , \tag{3.13}$$

coincides with the Poincaré polynomial of the corresponding HOMFLY-PT homology. However, when the color λ is not rectangular Young diagram, refined Chern-Simons invariants have both positive and negative integer coefficients in $(\mathbf{a}, \mathbf{q}, \mathbf{t})$ variables. For example

$$\text{rCS}_{\yng(2,1)}(T_{2,3}; a, q, t) =$$

$$\frac{a^6}{q^3t^8}(-a^3q^6(q^5t^5 + 2q^4t^4 + q^3(-t^4 + 2t^3 + t^2) - q^2(t-2)t^2 - q(t-2)t + 1) + a^2q^3(q^8t^7(t+1) + q^7t^6(2t+3) + q^6(-t^7 + 2t^6 + 5t^5 + t^4) + q^5(-2t^6 + t^5 + 6t^4 + t^3) + q^4(-2t^5 + t^4 + 6t^3 + t^2) + q^3t^2(-2t^2 + t + 5) + q^2t(-2t^2 + 2t + 3) + q(-t^2 + 2t + 1) + 1) - aq(q^{10}t^{10} + q^9(3t+1)t^8 + q^8(-t^2 + 5t + 3)t^7 + q^7(-3t^8 + 6t^7 + 5t^6 + t^5) + q^6(-4t^7 + 6t^6 + 6t^5 + t^4) + q^5(-5t^6 + 5t^5 + 6t^4 + t^3) + q^4(-5t^2 + 6t + 5)t^3 + q^3(-4t^2 + 6t + 3)t^2 + q^2(-3t^3 + 5t^2 + t) - q(t-3)t + 1) + t(q^{10}t^{10} + 2q^9t^9 + q^8(-t^9 + 4t^8 + t^7) - 2q^7(t^2 - 2t - 1)t^6 + q^6(-3t^7 + 4t^6 + 2t^5 + t^4) + q^5(-3t^2 + 4t + 2)t^4 + q^4(-3t^2 + 4t + 2)t^3 + q^3(-3t^4 + 4t^3 + t^2) - 2q^2(t-2)t^2 - q(t-2)t + 1))$$

$$= \frac{\mathbf{a}^{12}}{\mathbf{q}^{20}}(\mathbf{a}^6\mathbf{q}^{10}\mathbf{t}^{15}(\mathbf{q}^{20}\mathbf{t}^{10} + 2\mathbf{q}^{16}\mathbf{t}^8 - (\mathbf{q}^2-2)\mathbf{q}^8\mathbf{t}^4 + (-\mathbf{q}^8 + 2\mathbf{q}^6 + \mathbf{q}^4)\mathbf{q}^6\mathbf{t}^6 - (\mathbf{q}^2-2)\mathbf{q}^4\mathbf{t}^2 + 1) + \mathbf{a}^4\mathbf{q}^4\mathbf{t}^8((\mathbf{q}^2+1)\mathbf{q}^{30}\mathbf{t}^{16} + (2\mathbf{q}^2+3)\mathbf{q}^{26}\mathbf{t}^{14} + (-\mathbf{q}^6 + 2\mathbf{q}^4 + 5\mathbf{q}^2 + 1)\mathbf{q}^{20}\mathbf{t}^{12} + (-2\mathbf{q}^{12} + \mathbf{q}^{10} + 6\mathbf{q}^8 + \mathbf{q}^6)\mathbf{q}^{10}\mathbf{t}^{10} + (-2\mathbf{q}^4 + \mathbf{q}^2 + 5)\mathbf{q}^{10}\mathbf{t}^6 + (-2\mathbf{q}^{10} + \mathbf{q}^8 + 6\mathbf{q}^6 + \mathbf{q}^4)\mathbf{q}^8\mathbf{t}^8 + (-2\mathbf{q}^4 + 2\mathbf{q}^2 + 3)\mathbf{q}^6\mathbf{t}^4 + (-\mathbf{q}^4 + 2\mathbf{q}^2 + 1)\mathbf{q}^2\mathbf{t}^2 + 1) + \mathbf{a}^2\mathbf{t}^3(\mathbf{q}^{40}\mathbf{t}^{20} + (3\mathbf{q}^2+1)\mathbf{q}^{34}\mathbf{t}^{18} + (-\mathbf{q}^4 + 5\mathbf{q}^2 + 3)\mathbf{q}^{30}\mathbf{t}^{16} + (-3\mathbf{q}^6 + 6\mathbf{q}^4 + 5\mathbf{q}^2 + 1)\mathbf{q}^{24}\mathbf{t}^{14} + (-4\mathbf{q}^6 + 6\mathbf{q}^4 + 6\mathbf{q}^2 + 1)\mathbf{q}^{20}\mathbf{t}^{12} + (-5\mathbf{q}^4 + 6\mathbf{q}^2 + 5)\mathbf{q}^{14}\mathbf{t}^8 + (-5\mathbf{q}^{12} + 5\mathbf{q}^{10} + 6\mathbf{q}^8 + \mathbf{q}^6)\mathbf{q}^{10}\mathbf{t}^{10} + (-4\mathbf{q}^4 + 6\mathbf{q}^2 + 3)\mathbf{q}^{10}\mathbf{t}^6 + (-3\mathbf{q}^6 + 5\mathbf{q}^4 + \mathbf{q}^2)\mathbf{q}^4\mathbf{t}^4 - (\mathbf{q}^2-3)\mathbf{q}^4\mathbf{t}^2 + 1) + \mathbf{q}^{40}\mathbf{t}^{20} + 2\mathbf{q}^{36}\mathbf{t}^{18} - 2(\mathbf{q}^4 - 2\mathbf{q}^2 - 1)\mathbf{q}^{26}\mathbf{t}^{14} + (-3\mathbf{q}^6 + 4\mathbf{q}^4 + 2\mathbf{q}^2 + 1)\mathbf{q}^{20}\mathbf{t}^{12} + (-3\mathbf{q}^4 + 4\mathbf{q}^2 + 2)\mathbf{q}^{18}\mathbf{t}^{10} + (-\mathbf{q}^{18} + 4\mathbf{q}^{16} + \mathbf{q}^{14})\mathbf{q}^{16}\mathbf{t}^{16} + (-3\mathbf{q}^4 + 4\mathbf{q}^2 + 2)\mathbf{q}^{14}\mathbf{t}^8 - 2(\mathbf{q}^2-2)\mathbf{q}^8\mathbf{t}^4 + (-3\mathbf{q}^8 + 4\mathbf{q}^6 + \mathbf{q}^4)\mathbf{q}^6\mathbf{t}^6 - (\mathbf{q}^2-2)\mathbf{q}^4\mathbf{t}^2 + 1).$$

It turns out that refined Chern-Simons invariants have surprisingly rich properties. Especially, it is proven in [Che16] that the reduced invariants satisfy the following properties:

- mirror/transposition symmetry

$$\text{rCS}_{\lambda^T}(T_{m,n}; a, q, t) = \text{rCS}_\lambda(T_{m,n}; a, t^{-1}, q^{-1}) \ . \tag{3.14}$$

- refined exponential growth property

$$\begin{aligned} \text{rCS}_{\sum_{i=1}^\ell \lambda_i\omega_i}(T_{m,n}; a, q=1, t) &= \prod_{i=1}^{\ell} \Big[\text{rCS}_{\omega_i}(T_{m,n}; a, q=1, t)\Big]^{\lambda_i} \ , \\ \text{rCS}_{\sum_{i=1}^\ell \lambda_i\omega_i}(T_{m,n}; a, q, t=1) &= \prod_{i=1}^{\ell} \Big[\text{rCS}_{\omega_i}(T_{m,n}; a, q, t=1)\Big]^{\lambda_i} \ , \end{aligned} \tag{3.15}$$

  where ω_i are the fundamental weights of $\mathfrak{sl}(N)$.

[10] We denote the *unreduced* invariants by $\overline{\text{rCS}}_\lambda(T_{m,n}; a, q, t)$ and the *reduced* invariants by $\text{rCS}_\lambda(T_{m,n}; a, q, t)$ where they are related by

$$\overline{\text{rCS}}_\lambda(T_{m,n}; a, q, t) = \overline{\text{rCS}}_\lambda(\bigcirc; a, q, t) \ \text{rCS}_\lambda(T_{m,n}; a, q, t) \ .$$

Indeed, the partition function of refined topological string theory on the deformed conifold can be determined by taking into account of BPS particles arising from annulus M2-branes stretched between S^3 and $\mathcal{N}_{T_{m,n}}$. For the q-brane setting in the table 1, the M2-brane stretched between S^3 and $\mathcal{N}_{T_{m,n}}$ gives rise to two bifundamental $\mathcal{N}=2$ chiral multiplets, Φ and $\widetilde{\Phi}$, on $S^1 \times D_q$ charged with $(\mathbf{N}, \overline{\mathbf{M}})$ and $(\overline{\mathbf{N}}, \mathbf{M})$, respectively, under $U(N) \times U(M)$. The chiral multiplet Φ has charges $(0,0;-\frac{1}{2})$ under $(J_1, J_2; S_R)$ and $\widetilde{\Phi}$ is neutral. The three-dimensional $\mathcal{N}=2$ index can be evaluated by counting "single-letter index" [IY11, KW11] where the chiral multiplet Φ contributes to the single-letter index by $(q^{1/2}t^{-1/2}-1)/(1-q)$ and the other $\widetilde{\Phi}$ yields $(1-q^{1/2}t^{1/2})/(1-q)$. From the viewpoint of the three-dimensional theory on $S^1 \times D_q$, the M M5'-branes give rise to $U(M)$ flavor symmetry, and a torus knot $T_{m,n}$ in S^3 contributes to the single-letter index via the 3d/3d correspondence. Therefore, by putting the single-letter index invariant under $U(N) \times U(M)$ into the plethystic exponent, the refined index takes the form

$$Z^q_{\text{def},SU(N)} = \exp\left(\sum_{d>0}\sum_{\mu} \frac{1}{d}\frac{t^{\frac{d}{2}}-t^{-\frac{d}{2}}}{q^{\frac{d}{2}}-q^{-\frac{d}{2}}}\ \text{Ind}_{T_{m,n},\mu}(q^d,t^d)\ s_\mu(x^d)\right) ,$$

where $\text{Ind}_{T_{m,n},\mu}(q,t)$ is a contribution of the knot $T_{m,n}$ to the single-particle index charged under the flavor representation μ. This expression can be expanded by the basis of holonomies of the $U(M)$ group, which are the Macdonald functions $P_\lambda(x;q,t)$ in refined Chern-Simons theory

$$Z^q_{\text{def},SU(N)} = \sum_{\lambda} g_\lambda(q,t)\ \overline{\text{rCS}}_{SU(N),\lambda}(T_{m,n};q,t)\ P_\lambda(x;q,t) , \tag{3.16}$$

where the function g_λ can be determined by using the unknot invariants and it turns out to be the Macdonald norm defined in Appendix A. In the unrefined limit $q=t$, it reduces to the generating function of $SU(N)$ quantum invariants $\overline{J}_{SU(N),\lambda}(T_{m,n};q)$ first obtained in [OV00]

$$Z_{\text{def},SU(N)} = \sum_{\lambda} \overline{J}_{SU(N),\lambda}(T_{m,n};q)\ s_\lambda(x) ,$$

where $s_\lambda(x)$ are the Schur functions.

On the other hand, the $\bar{t}$-branes intersect with the q-branes at a point in D_q in the table 1 so that annulus M2-branes stretched between S^3 and $\mathcal{N}_{T_{m,n}}$ bring about only a single bifundamental fermionic particle. Since it contributes to the refined index by -1, the index is of the form

$$Z^{\bar{t}}_{\text{def},SU(N)} = \exp\left(\sum_{d>0}\sum_{\mu} \frac{(-1)}{d}\ \text{Ind}_{T_{m,n},\mu}(q^d,t^d)\ s_\mu(x^d)\right) .$$

For the $\bar{t}$-brane setting, the expansion in terms of the basis of holonomies of the $U(M)$ group is more subtle. Since the $\bar{t}$-branes are topological anti-branes, colors for the holonomy need to be transposed from the ordinary branes. In addition, the role of the equivariant parameters should be exchanged $(q,t) \leftrightarrow (t^{-1},q^{-1})$ due to (3.1) for the holonomy of the $\bar{t}$-branes. Thus, the partition function for the $\bar{t}$-branes takes the form

$$Z^{\bar{t}}_{\text{def},SU(N)} = \sum_{\lambda} \overline{\text{rCS}}_{SU(N),\lambda}(T_{m,n};q,t)\ P_{\lambda^T}(-x;t,q) , \tag{3.17}$$

where we use the property $P_{\lambda^T}(-x;t^{-1},q^{-1}) = P_{\lambda^T}(-x;t,q)$ of the Macdonald functions presented in (A.2).

3.2 Resolved conifold side

In the seminal papers [GV98b, GV98a], Gopakumar and Vafa proposed that an effective action of Type IIA string theory compactified on a Calabi-Yau threefold can be determined by considering effects of BPS particles arising M2-branes in the anti-selfdual graviphoton background. Moreover, the form of the effective action has been explicitly evaluated by applying Schwinger computations. Subsequently, Labastida, Mariño, Ooguri and Vafa (LMOV) [OV00, LMn01, LMnV00, LMn02] carried out similar analyses in the presence of D4-branes, which can be regarded as an open-string analogue of the GV formula. At the unrefined level, thorough analysis and elaborate explanation for these formulas have been presented in [Mn05, DW16]. In this section, we will find an explicit form of the effective action of Type IIA string theory with D4-branes in the *refined* case.

3.2.1 Without M5-branes

To this end, let us first review an effective action of Type IIA string theory compactified on a Calabi-Yau manifold X without D4-branes. In this subsection, we assume that the Calabi-Yau threefold X is general, and we do not necessarily restrict ourselves to the resolved conifold. The effective action (free energy) of Type IIA string theory has F-terms that admit genus expansion

$$F_{\text{closed}} = \log Z^{\text{res}}_{\text{closed}} = -i\sum_{g\geq 0}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{F}_g(\mathcal{X}_\Lambda)(\mathcal{W}_{AB}\mathcal{W}^{AB})^g ,$$

where $\mathcal{F}_g$ are holomorphic functions of chiral superfields $\mathcal{X}_\Lambda$ ($\Lambda = 0,\cdots,b_2(X)$) associated to vector multiplets and $\mathcal{W}_{AB}$ is a chiral superfield whose bottom component is anti-selfdual graviphoton field strength

$$\mathcal{W}_{AB} = \frac{1}{2}T_{AB} - R_{ABCD}\theta\sigma^{CD}\theta + \cdots .$$

Since the graviphoton field strength takes the value $T = g_{\text{st}}(\mathrm{d}x_1\wedge\mathrm{d}x_2 - \mathrm{d}x_3\wedge\mathrm{d}x_4)$, the genus g contribution is proportional to g_{st}^{2g-2}.

Gopakumar and Vafa proposed that, instead of evaluating each genus amplitude $\mathcal{F}_g$ in Type IIA string theory, the whole effective action F_{closed} can be obtained by summing up contributions of BPS states in M-theory. They arise from M2-branes wrapped on a holomorphic curve Σ in the Calabi-Yau manifold X. This calculation can be considered as a supersymmetric version of Schwinger's computation of one-loop effective action due to a charged particle in a constant magnetic field.

To see the spin content of BPS states, let us recall the five-dimensional $\mathcal{N} = 1$ supersymmetry algebra consisting of eight supercharges with $\mathrm{Sp}(1)_R = SU(2)_R$ $\mathcal{R}$-symmetry. Since the graviphoton field breaks the five-dimensional Lorentz group SO(1,4), it is convenient to rewrite the algebra in terms of four-dimensional notation. Then, the supercharges can be organized as $Q^I_\alpha, Q^I_{\dot\alpha}$ where $I = 1,2$ are the $SU(2)_R$ indices, and α and $\dot\alpha$ are negative and positive chirality of the rotational group $\mathrm{SO}(4) = SU(2)_\ell \times SU(2)_r$. With this notation, the supersymmetry algebra is given by

$$\begin{aligned}
\{Q^I_\alpha, Q^J_\beta\} &= \varepsilon_{\alpha\beta}\varepsilon^{IJ}(H+\zeta)\ , \\
\{Q^I_{\dot\alpha}, Q^J_{\dot\beta}\} &= \varepsilon_{\dot\alpha\dot\beta}\varepsilon^{IJ}(H-\zeta)\ , \\
\{Q^I_\alpha, Q^J_{\dot\beta}\} &= -i\Gamma^\mu_{\alpha\dot\beta}\varepsilon^{IJ}P_\mu\ .
\end{aligned}$$

where ζ is the *real* five-dimensional central charge. Thus, short left-handed BPS multiplets take the form,

$$\Big((0,0;\tfrac{1}{2})\oplus(\tfrac{1}{2},0;0)\Big)\otimes(J_\ell, J_r; S_R)\ , \tag{3.18}$$

which represent BPS particles of mass $m=\zeta$ at rest. The first spin content is indeed the half-hypermultiplet representation and the second is an arbitrary finite dimensional representation of $SU(2)_\ell \times SU(2)_r \times SU(2)_R$. It is easy to see that the unrefined index

$$\mathrm{Tr}(-1)^{2(J_\ell+J_r)}q^{2J_\ell}e^{-\beta H}$$

receives contributions only from the short left-handed BPS multiplets. Hence, a one-loop calculation involving small fluctuations around a BPS particle trajectory takes the form

$$\begin{aligned}
F_{\text{closed}} &= -\int_0^\infty \frac{ds}{s}\frac{\mathrm{Tr}_{\mathcal{BPS}}(-1)^{2(J_\ell+J_r)}q^{2sJ_\ell}e^{-sm}}{(q^{\frac{s}{2}}-q^{-\frac{s}{2}})^2}\ , \\
&= -\sum_{d>0}\sum_{\beta\in H_2(X,\mathbb{Z})}\frac{1}{d}\frac{\mathrm{Tr}_{\mathcal{H}(\beta)}(-1)^{2(J_\ell+J_r)}q^{2dJ_\ell}}{(q^{\frac{d}{2}}-q^{-\frac{d}{2}})^2}e^{-d\beta\cdot\tau}\ ,
\end{aligned} \tag{3.19}$$

where we identify the parameters by $q=e^{ig_{\text{st}}}$ and m is the central charge of BPS particles. Let us closely look at the meaning of the GV formula (3.19).

- The denominator is obtained by Schwinger computation for the one-loop determinant of BPS particle with the half-hypermultiplet representation in the anti-selfdual graviphoton background of the form

$$T^- = \frac{1}{2}\begin{pmatrix} 0 & g_{\text{st}} & & \\ -g_{\text{st}} & 0 & & \\ & & 0 & -g_{\text{st}} \\ & & g_{\text{st}} & 0 \end{pmatrix}\ . \tag{3.20}$$

  The two-dimensional contribution (the upper block) can be evaluated by summing up all the Landau levels $(\frac{1}{2}+n)g_{\text{st}}$ for $n\in\mathbb{Z}_{\geq 0}$:

$$\sum_{n\geq 0}e^{-ig_{\text{st}}(1+2n)/2} = \frac{1}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\ .$$

  Including an identical factor for the lower block, the Schwinger calculation provides the denominator.

- The unrefined index in the numerator receives contributions from other massive BPS states (3.18), which can be understood as fermion zero modes on M2-branes. The detailed analysis for the BPS spectra will follow below.

- The central charge of a BPS state is given by the area of a holomorphic curve Σ on which the M2-brane is wrapped and the Kaluza-Klein momentum. If we take a basis C_I $(I = 1, \cdots, b_2(X))$ of $H_2(X, \mathbb{Z})$, then the homology class of the curve is expressed by $[\Sigma] = \sum_I \beta_I C_I$ with $\beta_I \in \mathbb{Z}$. By denoting the complexified Kahler parameter of the 2-cycle C_I by τ^I, the area of the M2-brane is equal to $\tau \cdot \beta = \sum_I \tau^I \beta_I$. Then, the central charge of the BPS state is given by $m = \tau \cdot \beta + 2\pi i n$ where n is the Kaluza-Klein momentum of the M2-brane along the M-theory circle.

- From the first to the second line, we perform a Poisson resummation $\sum_{n\in\mathbb{Z}} e^{-2\pi i n s} = \sum_{d\in\mathbb{Z}} \delta(s-d)$, which re-expresses the sum over the Kaluza-Klein momenta as a sum over winding numbers.

In refined topological string theory, the graviphoton field is no longer anti-selfdual, and it rather takes the form

$$T = \epsilon_1 dx_1 \wedge dx_2 - \epsilon_2 dx_3 \wedge dx_4 \ , \tag{3.21}$$

which introduces the Ω-background (3.1). Furthermore, by using the $SU(2)_R$ $\mathcal{R}$-symmetry, we define a refined index [Nek04] as

$$\mathrm{Tr}(-1)^{2(J_\ell + J_r)} q_\ell^{2J_\ell} q_r^{2(J_r - S_R)} e^{-\beta H} \ .$$

Then, one can confirm that only left-handed multiplets (3.18) again contribute to the index while the long multiplets and the right-handed multiplets do not. To write the refined index in terms of the equivariant parameters $q = e^{i\epsilon_1}$ and $t = e^{i\epsilon_2}$, we introduce $J_1 = J_\ell + J_r$ and $J_2 = J_r - J_\ell$ so that it takes the form

$$\mathrm{Tr}(-1)^F q^{J_1 - S_R} t^{S_R - J_2} e^{-\beta H} \ ,$$

where we define $q_\ell = (qt)^{1/2}$ and $q_r = (q/t)^{1/2}$. Then, the free energy at the refined level can be written as

$$\begin{aligned} F^{\mathrm{ref}}_{\mathrm{closed}} &= \int_0^\infty \frac{ds}{s} \frac{\mathrm{Tr}_{\mathcal{BPS}}(-1)^{2(J_\ell+J_r)} q^{s(J_1-S_R)} t^{s(S_R-J_2)} e^{-sm}}{(q^{\frac{s}{2}} - q^{-\frac{s}{2}})(t^{-\frac{s}{2}} - t^{\frac{s}{2}})} \ , \\ &= \sum_{d>0} \sum_{\beta \in H_2(X,\mathbb{Z})} \frac{1}{d} \frac{\mathrm{Tr}_{\mathcal{H}(\beta)}(-1)^{2(J_\ell+J_r)} q^{d(J_1-S_R)} t^{d(S_R-J_2)}}{(q^{\frac{d}{2}} - q^{-\frac{d}{2}})(t^{-\frac{d}{2}} - t^{\frac{d}{2}})} e^{-d\beta\cdot\tau} \ . \end{aligned}$$

Since we now have different equivariant parameters the upper and lower block in (3.20), the denominator is resolved as $(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2 \to (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(t^{\frac{1}{2}} - t^{-\frac{1}{2}})$.

Now let us study other massive BPS states that contribute to the refined index. A propagating M2-brane wrapped on a holomorphic curve $\Sigma_g \subset X$ of genus g generates a 5d particle that preserves a half of supersymmetry [BBS95]. For the low energy description, we need to take into account fermion zero-modes on Σ_g where half of them transform under $SU(2)_\ell \times SU(2)_r$ as $(\frac{1}{2}, 0)$ and the other half transform as $(0, \frac{1}{2})$. The $(\frac{1}{2}, 0)$ fermionic zero modes can be interpreted as one forms on Σ_g and therefore there are zero-modes consisting of $2g$ copies of the $(\frac{1}{2}, 0)$ representation on the curve Σ_g of genus g. In other words, the $(\frac{1}{2}, 0)$ zero modes on Σ_g can be interpreted as a cohomology class of the Jacobian of Σ_g where the $SU(2)_\ell$ action can be understood as the natural Lefschetz $SU(2)$ action on the cohomology

$\mathcal{H}_g := H^*(\text{Jac}(\Sigma_g))$ of the Jacobian. Quantization of this system gives g copies of the spin content $(0,0;\frac{1}{2}) \oplus (\frac{1}{2},0;0)$ under $SU(2)_\ell \times SU(2)_r \times SU(2)_R$ so that the contribution to the index is $(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^g (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^g$.

On the other hand, the $(0,\frac{1}{2})$ fermion zero modes are related by supersymmetry to infinitesimal deformations of Σ_g as a holomorphic curve in X. Let us denote the moduli space $\mathcal{M}_{g,\beta}$ that parametrizes the holomorphic deformations of Σ_g under the homology class $\beta \in H_2(X,\mathbb{Z})$ inside the Calabi-Yau threefold X. Then, if $\mathcal{M}_{g,\beta}$ and Σ_g are both smooth, the total moduli space $\widehat{\mathcal{M}}_{\text{closed}}$ for the BPS states in this configuration is

$$\begin{array}{ccc} \text{Jac}(\Sigma_g) & \longrightarrow & \widehat{\mathcal{M}}_{\text{closed}} \\ & & \downarrow \pi \\ & & \mathcal{M}_{g,\beta} \end{array} \quad . \tag{3.22}$$

Since the zero-modes of $(0,\frac{1}{2})$ fermions are one forms on $\mathcal{M}_{g,\beta}$, the space of total BPS states in this situation is the de Rham cohomology $H^*(\mathcal{M}_{g,\beta};\mathcal{H}_g)$ of $\mathcal{M}_{g,\beta}$ with values in $\mathcal{H}_g$ where the $SU(2)_r$ action is the natural Lefschetz $SU(2)$ action on $H^*(\mathcal{M}_{g,\beta})$. However, the assumption that Σ_g is always smooth is almost never satisfied and there are usually singular fibers in (3.22), which makes it difficult to give a rigorous mathematical definition of GV invariants. (However, important progress has been made recently in mathematics [PT10, MT16].)

In physics, one can formally count the number of BPS states. For this purpose, we decompose the zero-modes of $(0,\frac{1}{2})$ fermions into the spectrum $H^*(\mathcal{M}_{g,\beta}) \cong \bigoplus A_{g,\beta,J_r,S_R}$ with respect to J_r and S_R spins so that the total BPS spectrum takes the form

$$\mathcal{H}(\beta) \cong \bigoplus_{g,J_r,S_R} \mathcal{H}_g \otimes A_{g,\beta,J_r,S_R} \ .$$

As a result, denoting the number of states with fixed charges by

$$\widehat{N}_{g,\beta,J_r,S_R} := \dim A_{g,\beta,J_r,S_R} \ ,$$

the refined free energy takes the form

$$F^{\text{ref}}_{\text{closed}} = \sum_{d>0} \sum_{\text{charges}} \frac{1}{d}(-1)^{2J_r} \widehat{N}_{g,\beta,J_r,S_R} (q^{\frac{d}{2}} - q^{-\frac{d}{2}})^{g-1} (t^{-\frac{d}{2}} - t^{\frac{d}{2}})^{g-1} e^{-d\beta\cdot\tau} \sum_{j_r=-J_r}^{J_r} \sum_{s_R=-S_R}^{S_R} \left(\frac{q}{t}\right)^{d(j_r-s_R)} ,$$

where charges are summed over $g \geq 0$, $\beta \in H_2(X,\mathbb{Z})$, and $J_r, S_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. In particular, the integral numbers for unrefined BPS states

$$n_{g,\beta} = \sum_{J_r,S_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}} (-1)^{2J_r}(2J_r+1)(2S_R+1)\widehat{N}_{g,\beta,J_r,S_R}$$

are GV invariants, and $n_{g,\beta} = (-1)^{\dim_{\mathbb{C}} \mathcal{M}_{g,\beta}} \chi(\mathcal{M}_{g,\beta})$ if $\mathcal{M}_{g,\beta}$ is smooth. We refer the reader to [AS12b, CKK14, CDDP15, NO14, GHKPK17, references therein] for recent developments on refined closed BPS invariants.

3.2.2 With M5-branes

Now let us include M5'-branes on $S^1\times\mathbb{R}^2\times\mathcal{L}\subset S^1\times\mathbb{R}^4\times X$ like the table 2 where $\mathcal{L}$ is a special Lagrangian submanifold of a Calabi-Yau threefold X. In general, a half of supersymmetry is preserved even if we include a number of M5'-branes unless their supports $\mathcal{L}_i\subset X$ are special Lagrangian [BBS95]. By shrinking the M-theory circle, we have Type IIA string theory with D4-branes. Then, its low energy effective action has terms that are supported on the world-volume $\mathbb{R}^2\subset\mathbb{R}^4$ of the D4-branes so that it takes the following form

$$F_{\text{open}}=\sum_{g,h\geq 0}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{d}^4\theta\;\delta(x^2)\delta(\theta^2)\;\mathcal{F}_{g,h}(\mathcal{X}_\Lambda;\mathcal{V}_\sigma)(\mathcal{W}^2)^g\mathcal{W}_{\|}^{h-1}\;, \tag{3.23}$$

where $\mathcal{V}_\sigma$ ($\sigma=1,\ldots,b_1(\mathcal{L})$) are chiral superfields associated to the moduli of $\mathcal{L}$. Here $\mathcal{W}_\|$ is the "parallel" component of the graviphoton superfield $\mathcal{W}_{AB}$. More precisely, for the anti-selfdual graviphoton background (3.20), $T^-_{12}=T_\|/2$ if the D4-branes are located on the D_q plane and $T^-_{34}=T_\|/2$ if they are put on the $D_{\bar{t}}$ plane where $T_\|$ is the bottom component of the superfield $\mathcal{W}_\|$. At the unrefined level, each integral in (3.23) is proportional to $g_s^{2g+h-2}=g_s^{-\chi}$, which is natural from the viewpoint of string perturbation theory since the Euler characteristic of a Riemann surface of genus g with h holes is equal to $\chi=2-2g-h$.

Even in the presence of M5'-branes, one can apply the same idea that F_{open} can be determined by analyzing contributions of BPS states in M-theory [OV00, LMn01, LMnV00, LMn02]. A relevant BPS state arises from an M2-brane wrapped on $\Sigma\subset X$ and generally attached to the M5'-branes on $\mathcal{L}$ so that they propagate only along the M5'-branes. From a low energy point of view, these states propagate along $S^1\times\mathbb{R}^2\subset S^1\times\mathbb{R}^4$.

To evaluate contributions to the Type IIA effective action (free energy) supported on $\mathbb{R}^2$, let us investigate quantum numbers of BPS states in three-dimensional $\mathcal{N}=2$ supersymmetric theory. The three-dimensional $\mathcal{N}=2$ supersymmetry algebra is given by

$$\{Q_\alpha,\overline{Q}_\beta\}=-i\sigma^\mu_{\alpha\beta}P_\mu+i\epsilon_{\alpha\beta}\zeta\;,$$
$$\{Q_\alpha,Q_\beta\}=0=\{\overline{Q}_\alpha,\overline{Q}_\beta\}\;.$$

For the theory on the q-branes, the supercharges Q are complex spinors in the spin-$\frac{1}{2}$ representation of the rotation group $SO(3)\cong SU(2)_1$, which is the diagonal subgroup of $SU(2)_\ell\times SU(2)_r$. In addition, a $U(1)_2$ $\mathcal{R}$-symmetry, which is actually the $U(1)$ subgroup of the anti-diagonal subgroup $SU(2)_2=\{(x,x^{-1})\in SU(2)_\ell\times SU(2)_r\}$ in five-dimension, rotates Q_α and $\overline{Q}_\alpha$. (See Table 3.) For the theory on the $\bar{t}$-branes, the roles of $SU(2)_1$ and $SU(2)_2$ are exchanged so that the rotational group is identified with $SU(2)_2$ and the $\mathcal{R}$-symmetry is $U(1)_1\subset SU(2)_1$.

The three-dimensional unrefined index defined by

$$\mathrm{Tr}(-1)^F q^{J_1-J_2}e^{-\beta H} \tag{3.24}$$

counts states annihilated by the supercharges Q_+ and $\overline{Q}_-$. Then, only the left short multiplets, which represent BPS particles of mass $M=\zeta$ at rest,

$$\big((0;-\tfrac{1}{2})\oplus(-\tfrac{1}{2};0)\big)\otimes(J_1;J_2)\;, \tag{3.25}$$

contribute to the index. The proposal of LMOV is that the free energy (3.23) in the anti-selfdual graviphoton background (3.20) takes the form

$$F_{\rm open} = \pm \int_0^\infty \frac{ds}{s} \frac{{\rm Tr}_{\mathcal{BPS}}(-1)^F q^{s(J_1-J_2)} e^{-sm}}{(q^{\frac{s}{2}} - q^{-\frac{s}{2}})} \ ,$$

$$= \pm \sum_{d>0} \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{\vec{k}} \frac{1}{d} \frac{{\rm Tr}_{\mathcal{H}(\beta,\vec{k})}(-1)^F q^{d(J_1-J_2)}}{(q^{\frac{d}{2}} - q^{-\frac{d}{2}})} e^{-d\beta\cdot\tau} p_{\vec{k}}(x) \ .$$

- Since BPS particles propagate only along the M5'-branes, the Schwinger computation is here performed only on $\mathbb{R}^2 \subset \mathbb{R}^4$ spanned by the M5'-branes so that the denominator originates from either the upper (q-brane) or the lower block ($\bar{t}$-brane) of (3.20), depending on the M5'-brane configurations.

- As in the closed case §3.2.1, the unrefined index in the numerator receives contributions from other massive BPS states (3.25), which can be understood as fermion zero modes on M2-branes attached to the M5'-branes. The detailed analysis for the BPS spectra will be given below.

- The central charge of a BPS state is expressed by the area of the M2-brane as well as the momentum of the Kaluza-Klein modes. The area of a holomorphic curve $\Sigma \subset X$ whose boundary is on $\mathcal{L}$ is determined by its relative homology class in $H_2(X,\mathcal{L};\mathbb{Z})$. A Lagrangian subvariety $\mathcal{L}$ in the table.2 for a knot is topologically homeomorphic to $S^1 \times \mathbb{R}^2$ [HLJ82, AV00], which simplifies the relative homology as

$$H_2(X,\mathcal{L};\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \oplus H_1(\mathcal{L};\mathbb{Z}) \ .$$

  Hence, the homology class $[\Sigma] \in H_2(X,\mathcal{L};\mathbb{Z})$ is expressed by $\beta = (\beta_1, \cdots, \beta_{b_2(X)}) \in H_2(X;\mathbb{Z})$ as in the closed string as well as the winding numbers $w = (w_1, \cdots, w_h) \in (H_1(\mathcal{L};\mathbb{Z}))^h \cong \mathbb{Z}^h$ of boundary components $\partial\Sigma \cong (S^1)^h$. Since Σ is oriented, one can assume that w_i are all non-negative integers. To express the contributions from the boundary components, we define a vector $\vec{k}$ as follows: the i-th entry of $\vec{k}$ is the number of w_i's that take the value i. Then, when we wrap M M5'-branes on $S^1 \times \mathbb{R}^2 \times \mathcal{L}$, we can write

$$e^{-m} = e^{-\tau\cdot\beta - 2\pi i n} \ p_{\vec{k}}(x) \ ,$$

  where the fugacities x parametrize the Cartan subgroup of the $U(M)$-valued moduli of $\mathcal{L}$ and $p_{\vec{k}}(x)$ is defined in Appendix §A. Let us note that it is easy to transform from the winding basis $p_{\vec{k}}(x)$ to the representation basis $s_\mu(x)$ for the moduli of $\mathcal{L}$ by using (A.1).

Now, let us consider the case where X is the resolved conifold and $\mathcal{L}$ is the configuration $\mathcal{L}_{T_{m,n}}$ for a torus knot as in the table.2. As we have discussed in §3.1, this configuration preserves the extra $U(1)_R$ global symmetry, which can be actually interpreted as the $U(1)$ subgroup of the $SU(2)_R$ $\mathcal{R}$-symmetry in five dimension with eight supercharges. In this case, the three-dimensional $\mathcal{N}=2$ supercharges have the quantum numbers under this symmetry shown in Table 3. Since $J_1 - S_R$ and $S_R - J_2$ commute with the supercharges Q_+ and $\overline{Q}_-$, one can refine the index by

$$\mathrm{Tr}\,(-1)^F\, q^{J_1-S_R}\, t^{S_R-J_2} e^{-\beta H}\ .$$

Thus, one can turn on the Ω-background (3.21) in the presence of M5'-branes supported on $S^1\times\mathbb{R}^2\times\mathcal{L}_{T_{m,n}}$ by using the refined index. First, we shall consider the q-brane setting in the table.2 at the refined level. In the refined graviphoton background (3.21), the Schwinger computation provides the following form of the free energy:

	$2J_1$	$2J_2$	$2S_R$
Q_+	$+1$	$+1$	$+1$
Q_-	-1	$+1$	$+1$
$\overline{Q}_+$	$+1$	-1	-1
$\overline{Q}_-$	-1	-1	-1

Table 3. Charges for 3d $\mathcal{N}=2$ supersymmetry on the q-brane.

$$F^q_{\mathrm{ref}} = \sum_{d>0}\sum_{\beta\in H_2(X,\mathbb{Z})}\sum_{\mu}\frac{1}{d}\frac{\mathrm{Tr}_{\mathcal{H}(\beta,\mu)}(-1)^F q^{d(J_1-S_R)}t^{d(S_R-J_2)}}{q^{\frac{d}{2}}-q^{-\frac{d}{2}}}e^{-d\beta\cdot\tau}s_\mu(x^d)\ .$$

The BPS states that contribute to the refined index are fermion zero modes on the M2-brane wrapped on a holomorphic curve $\Sigma_{g,h}\subset X$ whose boundary is on $\mathcal{L}$. Since the presence of the M5'-branes breaks $SU(2)_\ell\times SU(2)_r\times SU(2)_R$ to $SU(2)_1\times U(1)_2\times U(1)_R$, it is not so straightforward to study quantum numbers of BPS states as in §3.2.1. However, as before, fermion zero modes on the M2-brane can be associated to cohomology classes of the moduli space

$$\begin{array}{ccc} \mathrm{Jac}(\Sigma_{g,h}) & \longrightarrow & \widehat{\mathcal{M}}_{\mathrm{open}} \\ & & \downarrow{\scriptstyle\pi} \\ & & \mathcal{M}_{g,h,\beta} \end{array}\ ,$$

where the moduli space $\mathcal{M}_{g,h,\beta}$ parametrizes deformations of $\Sigma_{g,h}\subset X$ that preserve a half of supersymmetry. More precisely, as emphasized in [LMnV00], the fermion zero modes are cohomology classes in $H^*(\widehat{\mathcal{M}}_{\mathrm{open}})\cong H^*(\mathrm{Jac}(\Sigma_{g,h}))\otimes H^*(\mathcal{M}_{g,h,\beta})$ of the moduli space *mod out by the action (Sprecht module) of the permutation group* $\mathfrak{S}_h$ which exchanges h distinguished holes of $\Sigma_{g,h}$. The Jacobian $\mathrm{Jac}(\Sigma_{g,h})$ of a curve of genus g with h holes is topologically $(T^2)^g\times(S^1)^{h-1}$ where $\mathfrak{S}_h$ does not act on the cohomology $H^*((T^2)^g)$ of the Jacobian of the "bulk" Riemann surface. Therefore, the contribution to the refined index from $H^*((T^2)^g)$ is $(q^{\frac12}-q^{-\frac12})^g(t^{-\frac12}-t^{\frac12})^g$ as in §3.2.1.

The projection of $H^*((S^1)^{h-1})\otimes H^*(\mathcal{M}_{g,h,\beta})$ onto the invariant subspace can be done by using the Schur functor $\mathbf{S}_\mu$. More explicitly, the invariant subspace in BPS states $\mathcal{H}(\beta,\mu)$ with charge $\beta\in H_2(X,\mathbb{Z})$ and μ (a representation of $U(M)$) can be written as

$$\mathrm{Inv}\Big(H^*((S^1)^{h-1})\otimes H^*(\mathcal{M}_{g,h,\beta})\Big)_{\mathcal{H}(\beta,\mu)} = \bigoplus_{\sigma,\rho} C_{\mu\sigma\rho}\mathbf{S}_\sigma(H^*((S^1)^{h-1}))\otimes\mathbf{S}_\rho(H^*(\mathcal{M}_{g,h,\beta}))\ , \quad (3.26)$$

where the Clebsch-Gordon coefficients $C_{\mu\sigma\rho}$ of the permutation group $\mathfrak{S}_h$ (see Appendix A) are

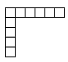
$$C_{\mu\sigma\rho} = \sum_{\vec{k}} \frac{1}{z_{\vec{k}}} \chi_\mu(C(\vec{k}))\chi_\sigma(C(\vec{k}))\chi_\rho(C(\vec{k})) \ . \tag{3.27}$$

Indeed, they are symmetric under the permutations of (μ, σ, ρ).

A cohomology class of the Jacobian of $\Sigma_{g,h}$ can be interpreted as differential forms on $\Sigma_{g,h}$. In particular, the boundary part $H^*((S^1)^{h-1})$ is spanned by one-forms $d\theta_i$, $(i = 1, \cdots, h)$, which are Poincaré dual to the holes in the curve $\Sigma_{g,h}$, and they are subject to the linear constraint $\sum_i d\theta_i = 0$. Moreover, one can consider the differential form $d\theta_i$ as the fermion zero modes ψ_i on a rigid curve with charges

$$(J_1; J_2, S_R) = (\tfrac{1}{2}; -\tfrac{1}{2}, \tfrac{1}{2}) \ . \tag{3.28}$$

As explained in [LMnV00, Mn05], the BPS spectra $\mathbf{S}_\sigma(H^*((S^1)^{h-1}))$ can be obtained by acting the fermion zero modes ψ_i on the vacuum $|0\rangle$. The defining representation V of $\mathfrak{S}_h$ can be constructed by acting one fermion ψ_i on the vacuum $|0\rangle$, and its dimension is $h-1$ due to $\sum_i \psi_i = 0$. The rest of the spectra are generated by taking the wedge products $\wedge^d V$ of the defining representation. Assigning the Young diagram with h boxes of one row to the trivial representation $|0\rangle$, the irreducible representations $\wedge^d V$ of $\mathfrak{S}_h$ are called *hook representations* since their Young tableau are of the form with $(h-d)$-boxes in the first row

 .

By assuming that the vacuum $|0\rangle$ is neutral, the state $\wedge^d V$ has charges

$$(J_1; J_2, S_R) = (\tfrac{d}{2}; -\tfrac{d}{2}, \tfrac{d}{2}) \ ,$$

because it is essentially obtained by acting d-wedge products of the fermions with charge (3.28). Hence, the contribution from the state $\wedge^d V$ to the refined index is $(-t)^d$. As a result, the refined index only over the BPS states $\mathbf{S}_\sigma(H^*((S^1)^{h-1}))$

$$B_\sigma := \mathrm{Tr}_{\mathbf{S}_\sigma(H^*((S^1)^{h-1}))} (-1)^F q^{J_1 - S_R} t^{S_R - J_2}$$

is summarized as

$$B_\sigma(t) = \begin{cases} (-t)^d t^{-\frac{|\sigma|-1}{2}} & \sigma : \text{hook rep for } \wedge^d V \\ 0 & \sigma : \text{otherwise} \end{cases} \ . \tag{3.29}$$

In fact, we can normalize B_σ by $t^{-\frac{|\sigma|-1}{2}}$ so that they satisfy

$$B_\sigma(t^{-1}) = (-1)^{|\sigma|-1} B_{\sigma^T}(t) \ . \tag{3.30}$$

As can be easily seen, the BPS states $\mathcal{H}_{\sigma,g} \cong H^*((T^2)^g) \otimes \mathbf{S}_\sigma(H^*((S^1)^{h-1}))$ obey $J_1 + J_2 = 0$, corresponding to the $(\frac{1}{2}, 0)$ fermions under $SU(2)_\ell \times SU(2)_r$ in five dimension. On the other hand, the BPS states $\mathbf{S}_\rho(H^*(\mathcal{M}_{g,h,\beta}))$ satisfy $J_1 - J_2 = 0$, analogous to the $(0, \frac{1}{2})$ fermions in five dimension. Although they can contribute to the unrefined index (3.24) only by signs, the refined index receives non-trivial contributions. Defining $J_r := \frac{1}{2}(J_1+J_2)$, one can decompose $\mathbf{S}_\rho(H^*(\mathcal{M}_{g,h,\beta}))$ into the spectrum $\bigoplus A_{\rho,g,\beta,J_r,S_R}$ with respect to the charges J_r and S_R so that the total BPS states are

$$\mathcal{H}(\beta,\mu) \cong \bigoplus_{\sigma,\rho,g,J_r,S_R} C_{\mu\sigma\rho}\, \mathcal{H}_{\sigma,g} \otimes A_{\rho,g,\beta,J_r,S_R} \ .$$

As a result, writing the number of BPS states with fixed charges by

$$\widehat{N}_{\rho,g,\beta,J_r,S_R} := \dim A_{\rho,g,\beta,J_r,S_R} \ , \tag{3.31}$$

the refined free energy takes the form

$$\begin{aligned} F^q_{\rm ref} &= \sum_{d>0}\sum_{\mu} \frac{1}{d}\frac{f^q_\mu(a^d,q^d,t^d)}{q^{\frac{d}{2}}-q^{-\frac{d}{2}}} s_\mu(x^d) \ , \\ f^q_\mu(a,q,t) &= \sum_{\text{charges}} (-1)^{2J_r} C_{\mu\sigma\rho} B_\sigma(t) \widehat{N}_{\rho,g,\beta,J_r,S_R} (q^{\frac12}-q^{-\frac12})^g (t^{-\frac12}-t^{\frac12})^g \left(\frac{q}{t}\right)^{J_r-S_R-\frac{\beta}{2}} a^\beta \ , \end{aligned} \tag{3.32}$$

where the charges are summed over $g \geq 0$, $\beta \in H_2(X,\mathbb{Z})$, $J_r, S_R \in \frac{1}{2}\mathbb{Z}$, and all representations σ, ρ of $U(M)$. Note that, to see the relation to refined Chern-Simons invariants in the next section, here we define the parameter by

$$a := e^{-\tau}\sqrt{\frac{q}{t}} \ .$$

In fact, the integral numbers for unrefined BPS states

$$\widehat{N}_{\mu,g,\beta} = \sum_{J_r,S_R\in\frac12\mathbb{Z}} (-1)^{2J_r} \widehat{N}_{\mu,g,\beta,J_r,S_R} \tag{3.33}$$

are called LMOV invariants. Moreover, Mariño and Vafa proposed the multi-covering formula that relates the LMOV invariants $\widehat{N}_{\rho,g,\beta}$ to open Gromov-Witten invariants in the presence of D4-branes supported on $\mathbb{R}^2\times\mathcal{L}$ [MnV02]. In the case of the framed unknot, the multi-covering formula has been proven based on localization method and combinatorics [LLZ03].

Next let us consider the $\bar{t}$-brane setting in the table.2. The form of the free energy is

$$F^{\bar t}_{\rm ref} = \sum_{d>0}\sum_{\beta\in H_2(X,\mathbb{Z})}\sum_{\mu} \frac{1}{d}\frac{\operatorname{Tr}_{\mathcal{H}(\beta,\mu)}(-1)^F q^{d(J_1-S_R)} t^{d(S_R-J_2)}}{t^{-\frac{d}{2}}-t^{\frac{d}{2}}} e^{-d\beta\cdot\tau} s_\mu(x^d) \ ,$$

where the Schwinger computation on the $D_{\bar t}$ plane yields the denominator. Although the BPS spectra in the $\bar{t}$-brane setting are essentially the same as those in the q-brane setting, their J_1 and J_2 charges are exchanged. Hence, the fermion zero modes ψ_i for $H^*((S^1)^{h-1})$ have charges $(J_2; J_1, S_R) = (\frac12; -\frac12, \frac12)$ so that the refined index over the BPS states $\mathbf{S}_\sigma(H^*((S^1)^{h-1}))$ is given by $B_\sigma(q^{-1})$. The remaining part is exactly the same as the q-brane setting so that the free energy for the $\bar{t}$-branes is

$$\begin{aligned} F^{\bar t}_{\rm ref} &= \sum_{d>0}\sum_{\mu} \frac{1}{d}\frac{f^{\bar t}_\mu(a^d,q^d,t^d)}{t^{-\frac{d}{2}}-t^{\frac{d}{2}}} s_\mu(x^d) \ , \\ f^{\bar t}_\mu(a,q,t) &= \sum_{\text{charges}} (-1)^{2J_r} C_{\mu\sigma\rho} B_\sigma(q^{-1}) \widehat{N}_{\rho,g,\beta,J_r,S_R} (q^{\frac12}-q^{-\frac12})^g (t^{-\frac12}-t^{\frac12})^g \left(\frac{q}{t}\right)^{J_r-S_R-\frac{\beta}{2}} a^\beta \ . \end{aligned} \tag{3.34}$$

It is easy to see that the free energy for the q-branes and that for the $\bar{t}$-branes are related by

$$F^{\bar{t}}_{\rm ref}(a,q,t) = F^{q}_{\rm ref}(a,t^{-1},q^{-1}) \ , \tag{3.35}$$

which can be expected from the equivariant action (3.1) on $\mathbb{C}^2$.

Finally, let us extract the common part of $f^q_\mu(a,q,t)$ and $f^{\bar{t}}_\mu(a,q,t)$ as

$$\widehat{f}_\rho(a,q,t) = \sum_{\rm charges} (-1)^{2J_r} \widehat{N}_{\rho,g,\beta,J_r,S_R} (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^g (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^g \left(\frac{q}{t}\right)^{J_r - S_R - \frac{\beta}{2}} a^\beta \quad , \tag{3.36}$$

which is invariant under the exchange $(q,t) \leftrightarrow (t^{-1},q^{-1})$. If we can define an invertible symmetric matrix

$$M_{\mu\rho}(t) := \sum_\sigma C_{\mu\sigma\rho} B_\sigma(t) \ ,$$

then we obtain concise expressions

$$\begin{aligned} f^q_\mu(a,q,t) &= \sum_\rho M_{\mu\rho}(t) \widehat{f}_\rho(a,q,t) \ , \\ f^{\bar{t}}_\mu(a,q,t) &= \sum_\rho M_{\mu\rho}(q^{-1}) \widehat{f}_\rho(a,q,t) \ . \end{aligned} \tag{3.37}$$

4 Large N duality for torus knots

As the number of M5-branes goes to infinity, the three-sphere S^3 in the deformed conifold T^*S^3 shrinks and the deformed conifold is transformed into the resolved conifold $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{CP}^1$. At large N, the N M5-branes turn into a flux supporting $\mathbb{CP}^1$ in the resolved conifold. Thus, large N duality implies that a generating function of Chern-Simons invariants of a knot at large N is equal to a topological string amplitude with M M5'-branes associated to the knot in the resolved conifold. The work of LMOV not only determines the form of low-energy effective actions of Type IIA string theory with D4-branes on the resolved conifold but also provides its connection to Chern-Simons invariants of a knot at large N. In other words, colored HOMFLY-PT polynomials are related to LMOV invariants. In this section, we shall put forth the large N duality for torus knots in the refined context.

On the deformed conifold side, we have reviewed generating functions of refined Chern-Simons invariants in §3.1. At large N, we substitute the stable limit $\overline{\mathrm{rCS}}_\lambda(T_{m,n};a,q,t)$ [AS15, GN15] for SU(N) invariants $\overline{\mathrm{rCS}}_{\mathrm{SU}(N),\lambda}(T_{m,n};q,t)$ in (3.16) and (3.17). Then, by using the forms of the refined free energy on the resolved conifold determined in §3.2.2, the equivalences of the partition functions for both the q-brane and $\bar{t}$-brane setting can be recapitulated as

$$\sum_\lambda \overline{\mathrm{rCS}}_\lambda(T_{m,n};a,q,t)\; g_\lambda(q,t)P_\lambda(x;q,t)=\exp\left(\sum_{d=1}^\infty\sum_\mu \frac{1}{d}\frac{f^q_\mu(T_{m,n};a^d,q^d,t^d)}{q^{\frac{d}{2}}-q^{-\frac{d}{2}}}s_\mu(x^d)\right)\;, \tag{4.1}$$

$$\sum_\lambda \overline{\mathrm{rCS}}_\lambda(T_{m,n};a,q,t)\; P_{\lambda^T}(-x;t,q)=\exp\left(\sum_{d=1}^\infty\sum_\mu \frac{1}{d}\frac{f^{\bar{t}}_\mu(T_{m,n};a^d,q^d,t^d)}{t^{-\frac{d}{2}}-t^{\frac{d}{2}}}s_\mu(x^d)\right)\;. \tag{4.2}$$

These identities determine the refined indices f^q_μ, $f^{\bar{t}}_\mu$ and $\widehat{f}_\rho$ on the resolved conifold in terms of refined Chern-Simons invariants $\overline{\mathrm{rCS}}_\lambda$. Therefore, we call f^q_μ, $f^{\bar{t}}_\mu$ and $\widehat{f}_\rho$ refined reformulated invariants. We shall present general formulas in Appendix B.

In the case of the unknot $K=\bigcirc$, the formulas above become the Cauchy formulas (A.4) so that only two BPS numbers are non-vanishing as in the unrefined case, *i.e.* the refined reformulated invariants $f^q_\mu(\bigcirc)$, $f^{\bar{t}}_\mu(\bigcirc)$ colored by non-trivial representations ($\mu\neq\square$) of the unknot vanish. Geometric picture is drawn in [OV00, Figure 3] where $\mathcal{L}$ is $S^1\times\mathbb{R}^2$ with S^1 the equator of $\mathbb{CP}^1$ in the resolved conifold and the two BPS states correspond to the M2-branes covering the upper and lower hemisphere of $\mathbb{CP}^1$.

The refined reformulated invariants can be explicitly evaluated by using refined Chern-Simons invariants of torus knots obtained in [AS15, DBMM+13, Che13, FGSA12, Sha13]. In all the examples we have checked, the refined reformulated invariants $f^q_\mu(T_{m,n})$ and $f^{\bar{t}}_\mu(T_{m,n})$ obey the relation (3.37). Moreover, after making change of variables

$$a=-\mathbf{a}^2\mathbf{t}\;,\quad q^{\frac12}=-\mathbf{q}\mathbf{t}\;,\quad t^{\frac12}=\mathbf{q}\;, \tag{4.3}$$

the reformulated invariants $\widehat{f}_\rho(T_{m,n})$ can be written in the form

$$\widehat{f}_\rho(T_{m,n})=\sum_{g\geq0}\sum_{\beta,F\in\mathbb{Z}}\widehat{\mathbf{N}}_{\rho,g,\beta,F}(T_{m,n})(\mathbf{q}\mathbf{t}-\mathbf{q}^{-1}\mathbf{t}^{-1})^g(\mathbf{q}-\mathbf{q}^{-1})^g\mathbf{a}^{2\beta}\mathbf{t}^F\;, \tag{4.4}$$

up on the a-grading shift by $\pm\frac12$. Surprisingly, we observe that the numbers $\widehat{\mathbf{N}}_{\rho,g,\beta,F}(T_{m,n})$ are always *non-negative* integers for any ρ,g,β,F. Let us emphasize that this is not obvious.

Even if we assume that $\widehat{f}_\rho(T_{m,n})$ takes the form (3.36), we have

$$\widehat{\mathbf{N}}_{\rho,g,\beta,F}(T_{m,n}) = \sum_{2(J_r-S_R)=F} (-1)^{2S_R}\widehat{N}_{\rho,g,\beta,J_r,S_R}(T_{m,n}) \ ,$$

which could be negative. Since $\widehat{N}_{\rho,g,\beta,J_r,S_R}(T_{m,n})$ are non-negative integers by definition (3.31), the positivity of $\widehat{\mathbf{N}}_{\rho,g,\beta,F}(T_{m,n})$ strongly suggests that the extra $U(1)_R$ global symmetry S_R acts trivially on the BPS states $\mathbf{S}_\rho(H^*(\mathcal{M}_{g,h,\beta}))$ so that F is indeed equal to $2J_r$. The same phenomenon has been found for refinement of analytically continued WRT invariants of Lens spaces $L(p,1)$ defined by the 3d/3d correspondence [GPV16, GPPV17].

Now let us formulate the conjecture of refined large N duality for torus knots, which is the main claim of this thesis.

The extra $U(1)_R$ global symmetry S_R acts trivially on the BPS states $\mathbf{S}_\rho(H^*(\mathcal{M}_{g,h,\beta}))$ in the resolved conifold. Thus, the refined reformulated invariants $f^q_\mu(T_{m,n})$ and $f^{\bar t}_\mu(T_{m,n})$, expressed in terms of refined Chern-Simons invariants of a torus knot $T_{m,n}$ via the geometric transition (4.1) and (4.2) (or more explicitly (B.1) and (B.2)), can be written

$$\begin{aligned} f^q_\mu(T_{m,n};a,q,t) &= \sum_\rho M_{\mu\rho}(t)\widehat{f}_\rho(T_{m,n};a,q,t) \ , \\ f^{\bar t}_\mu(T_{m,n};a,q,t) &= \sum_\rho M_{\mu\rho}(q^{-1})\widehat{f}_\rho(T_{m,n};a,q,t) \ , \end{aligned} \tag{4.5}$$

where, upon the a-grading shift by $\pm\frac{1}{2}$, $\widehat{f}_\rho(T_{m,n})$ takes the form

$$\widehat{f}_\rho(T_{m,n};a,q,t) = \sum_{\text{charges}} (-1)^{2J_r}\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})(q^{\frac12}-q^{-\frac12})^g(t^{-\frac12}-t^{\frac12})^g\left(\frac{q}{t}\right)^{J_r-\frac{\beta}{2}}a^\beta \ , \tag{4.6}$$

with *non-negative* integers $\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})\in\mathbb{Z}_{\geq0}$. Furthermore, for ρ,g,β fixed, the $2J_r$ charges of non-zero (hence positive) integers $\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})$ are either all even or all odd so that no cancellation occurs in the unrefined limit (3.33) and therefore the LMOV invariant is

$$\widehat{N}_{\rho,g,\beta}(T_{m,n}) = \pm\sum_{J_r\in\frac12\mathbb{Z}}\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n}) \ . \tag{4.7}$$

The conjecture on the trivial action of S_R implies that the numbers $\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})$ indeed yield complete information about BPS degeneracies in M-theory on the resolved conifold with the M5'-branes associated to a torus knot $T_{m,n}$. From geometric point of view, they are graded dimensions of the cohomology classes $\mathbf{S}_\rho(H^*(\mathcal{M}_{g,h,\beta}))$ of the moduli space of M2-M5' bound states. In Appendix C, we present some examples of $\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})$ for the trefoil $T_{2,3}$ and the $T_{2,5}$ knot. In addition, a `Mathematica` file attached to arXiv page contains more information about reformulated invariants in [KN17]. In all the examples, one can confirm that, for ρ,g,β fixed, $2J_r$ charges of non-trivial $\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})$ are either all even or all odd. In addition, the property (4.7) is manifest if we compare them with tables given in [LMnV00].

It is known that refined Chern-Simons invariants $\overline{\mathrm{rCS}}_\lambda$ of a torus knot colored by non-rectangular Young diagrams generally contain both positive and negative coefficients even after the change of variables (4.3). (For instance, see [Che13, §3.4].) However, this formulation lends itself to the natural interpretation of refined Chern-Simons invariants as a generating function of BPS states, providing non-negative integers $\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})$ for any color ρ. Thus, this can be interpreted as a *positivity conjecture* of refined Chern-Simons invariants of a torus knot.

Furthermore, we notice several interesting features of refined reformulated invariants. First, instead of taking the genus expansion of M2-branes (4.6), we find that the naive change of variables (4.3) for $\widehat{f}_\rho(T_{m,n})$ always yields a Laurent polynomial with non-negative integral coefficients

$$\widehat{f}_\rho(T_{m,n}; a = -\mathbf{a}^2\mathbf{t}, q = \mathbf{q}^2\mathbf{t}^2, t = \mathbf{q}^2) = \sum_{i,j,k} \widehat{\mathbf{N}}_{\mu;i,j,k}(T_{m,n})\mathbf{a}^{2i}\mathbf{q}^{2j}\mathbf{t}^k \ ,$$

with $\widehat{\mathbf{N}}_{\mu;i,j,k} \in \mathbb{Z}_{\geq 0}$. Hence, this evidence also indicates that there exists underlying cohomology classes of some moduli spaces for $\widehat{\mathbf{N}}_{\mu;i,j,k}$. Below some examples are given:

$$\widehat{f}_{\square}(T_{2,3}) = \frac{\mathbf{a}^2}{\mathbf{q}^2}(\mathbf{a}^2\mathbf{t}+1)(\mathbf{a}^2\mathbf{q}^2\mathbf{t}^3+\mathbf{q}^4\mathbf{t}^2+1)$$

$$\widehat{f}_{\square\square}(T_{2,3}) = \frac{\mathbf{a}^2}{\mathbf{q}^4}\left(\mathbf{a}^2\mathbf{t}+1\right)\left(\mathbf{a}^2\mathbf{t}^3+1\right)\left(\mathbf{q}^4\mathbf{t}^2+1\right)\left(\mathbf{a}^2\mathbf{t}+\mathbf{q}^2\right)\left(\mathbf{a}^2\mathbf{q}^2\mathbf{t}^3+1\right)$$

$$\widehat{f}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(T_{2,3}) = \frac{\mathbf{a}^2}{\mathbf{q}^6}\left(\mathbf{a}^2\mathbf{t}+1\right)\left(\mathbf{a}^2\mathbf{t}+\mathbf{q}^2\right)\left(\mathbf{a}^2\mathbf{q}^2\mathbf{t}^3+1\right)\left(\mathbf{q}^8\left(\mathbf{a}^2\mathbf{t}^7+\mathbf{t}^4\right)+\mathbf{q}^4\mathbf{t}^2\left(\mathbf{a}^2\left(\mathbf{t}^3+\mathbf{t}\right)+2\right)+\mathbf{a}^2\mathbf{t}^3+1\right) \ .$$

Second, we also observe a positivity property when we make the same substitution as (4.3) for $f^q_{[r]}(T_{m,n})$ and $f^{\bar t}_{[r]}(T_{m,n})$ colored by symmetric representations $\lambda = [r]$:

$$f^q_{[r]}(T_{m,n}; -\mathbf{a}^2\mathbf{t}, \mathbf{q}^2\mathbf{t}^2, \mathbf{q}^2) = \pm\mathbf{q}^\bullet \sum_{i,j,k} \mathbf{N}^q_{[r];i,j,k}(T_{m,n})\mathbf{a}^{2i}\mathbf{q}^{2j}\mathbf{t}^k \ ,$$
$$f^{\bar t}_{[r]}(T_{m,n}; -\mathbf{a}^2\mathbf{t}, \mathbf{q}^2\mathbf{t}^2, \mathbf{q}^2) = \pm\mathbf{q}^\bullet \sum_{i,j,k} \mathbf{N}^{\bar t}_{[r];i,j,k}(T_{m,n})\mathbf{a}^{2i}\mathbf{q}^{2j}\mathbf{t}^k \ ,$$

where $\mathbf{N}^q_{[r];i,j,k}(T_{m,n}), \mathbf{N}^{\bar t}_{[r];i,j,k}(T_{m,n}) \in \mathbb{Z}_{\geq 0}$. For instance, we have

$$f^q_{\square\square}(T_{2,3}) = -\frac{\mathbf{a}^2\mathbf{t}^2}{\mathbf{q}}(\mathbf{a}^2\mathbf{t}+1)(\mathbf{q}^8(\mathbf{a}^4\mathbf{t}^8+\mathbf{a}^2\mathbf{t}^5)+\mathbf{q}^6(\mathbf{a}^6\mathbf{t}^9+\mathbf{a}^4\mathbf{t}^6+\mathbf{a}^2\mathbf{t}^5+\mathbf{t}^2)+\mathbf{a}^2\mathbf{q}^4\mathbf{t}^3(\mathbf{a}^2(\mathbf{t}^3+\mathbf{t})+2)+\mathbf{q}^2(\mathbf{a}^6\mathbf{t}^5+\mathbf{a}^4\mathbf{t}^4+\mathbf{a}^2\mathbf{t}+1)+\mathbf{a}^2\mathbf{t}(\mathbf{a}^2\mathbf{t}+1)) \ ,$$

$$f^{\bar t}_{\square\square}(T_{2,3}) = -\frac{\mathbf{a}^2}{\mathbf{q}^7\mathbf{t}}(\mathbf{a}^2\mathbf{t}+1)(\mathbf{a}^2\mathbf{q}^8\mathbf{t}^5(\mathbf{a}^2\mathbf{t}+1)+\mathbf{q}^6\mathbf{t}^2(\mathbf{a}^6\mathbf{t}^5+\mathbf{a}^4\mathbf{t}^4+\mathbf{a}^2\mathbf{t}+1)+\mathbf{a}^2\mathbf{q}^4\mathbf{t}^3(\mathbf{a}^2(\mathbf{t}^3+\mathbf{t})+2)+\mathbf{q}^2(\mathbf{a}^6\mathbf{t}^7+\mathbf{a}^4\mathbf{t}^4+\mathbf{a}^2\mathbf{t}^3+1)+\mathbf{a}^4\mathbf{t}^4+\mathbf{a}^2\mathbf{t}) \ .$$

Let us conclude this section by mentioning the implication of the symmetry (3.35) of the free energy in the resolved conifold. As the right hand sides of (4.1) and (4.2) are interchanged by $(q,t) \leftrightarrow (t^{-1},q^{-1})$, so are the left hand sides. The property of Macdonald functions $P_\lambda(-x;t,q) = (-1)^{|\lambda|}P_\lambda(x;t^{-1},q^{-1})$ implies

$$g_\lambda(q,t)\ \overline{\mathrm{rCS}}_\lambda(T_{m,n};a,q,t) = (-1)^{|\lambda|}\ \overline{\mathrm{rCS}}_{\lambda^T}(T_{m,n};a,t^{-1},q^{-1}) \ ,$$

which is the *unreduced* version of the mirror/transposition symmetry (3.14) of the refined Chern-Simons invariants. This explanation was first provided in [AS12b, §3.1].

5 Large N duality for torus links

In this section, we generalize refined large N duality to torus links with L components. For each component of a torus link, we introduce M_i M5'-branes supported on the conormal bundle $\mathcal{L}_i$ of the component in the deformed conifold T^*S^3. These M5'-branes still remain after the geometric transition. Since a half of supersymmetry is preserved if the supports of M5'-branes are special Lagrangian submanifolds in a Calabi-Yau as explained in §3.2.2, it is straightforward to extend the analysis in the previous sections to torus links. To avoid repetitious explanation, we shall present only essential results in this section.

For a torus link $T_{m,n}$ with $\gcd(m,n) = L$, we need to introduce the fugacities x_i ($i = 1, \cdots, L$) that parametrize the Cartan subgroup of the $U(M_i)$-valued moduli of $\mathcal{L}_i$ both on the deformed conifold and on the resolved conifold. In the resolved conifold, we consider a holomorphic curve $\Sigma_{g,h}$ with $h = \sum_{i=1}^{L} h_i$ boundaries where h_i boundaries end on $\mathcal{L}_i$. Therefore, we have to project the space $H^*((S^1)^{h-1}) \otimes H^*(\mathcal{M}_{g,h,\beta})$ on the invariant subspace of the relevant symmetry $\mathfrak{S}_{h_1} \times \cdots \times \mathfrak{S}_{h_L}$. Eventually, the BPS spectrum for a torus link analogous to (3.26) is

$$\begin{aligned}&\mathrm{Inv}\Big(H^*((S^1)^{h-1}) \otimes H^*(\mathcal{M}_{g,h,\beta})\Big)_{\mathcal{H}(\beta,\mu_1,\cdots,\mu_L)} = \\ &\qquad \bigoplus_{\{\sigma_i\}\,\{\rho_i\}} C_{\mu_1\,\sigma_1\rho_1} \cdots C_{\mu_L\,\sigma_L\rho_L} \mathbf{S}_{\sigma_1,\cdots,\sigma_L}(H^*((S^1)^{h-1})) \otimes \mathbf{S}_{\rho_1,\cdots,\rho_L}(H^*(\mathcal{M}_{g,h,\beta}))\ .\end{aligned}$$

For the q-branes, the fermion zero modes $\psi_1^{(i)}, \cdots, \psi_{h_i}^{(i)}$ coming from the boundaries ending on $\mathcal{L}_i$ contribute to the refined index by $(t^{\frac12} - t^{-\frac12})B_{\sigma_i}(t)$. The factor $(t^{\frac12} - t^{-\frac12})$, which is absent in the case of a torus knot, stems from the fact that we do not impose the linear constraint $\sum_{j=1}^{h_i} \psi_j^{(i)} = 0$ on each boundary. Hence, we have

$$\mathrm{Tr}_{\mathbf{S}_{\sigma_1,\cdots,\sigma_L}(H^*((S^1)^{h-1}))}\,(-1)^F\, q^{J_1-S_R}\, t^{S_R-J_2} = (t^{\frac12} - t^{-\frac12})^{L-1} \prod_{i=1}^{L} B_{\sigma_i}(t) \tag{5.1}$$

where the linear constraint $\sum_{i=1}^{L}\sum_{j=1}^{h_i} \psi_j^{(i)} = 0$ on the total boundary fermion zero modes yields the factor $(t^{\frac12} - t^{-\frac12})^{-1}$.

As in the case of torus knots, we conjecture that *the extra $U(1)_R$ global symmetry S_R acts trivially on the BPS states $\mathbf{S}_{\rho_1,\cdots,\rho_L}(H^*(\mathcal{M}_{g,h,\beta}))$*. Thus, we decompose the BPS states

$$\mathbf{S}_{\rho_1,\cdots,\rho_L}(H^*(\mathcal{M}_{g,h,\beta})) \cong \bigoplus_{J_r} A_{\rho_1,\cdots,\rho_L,g,\beta,J_r}$$

with respect to only J_r charges but not S_R charges and define

$$\widehat{N}_{\rho_1,\cdots,\rho_L,g,\beta,J_r} := \dim A_{\rho_1,\cdots,\rho_L,g,\beta,J_r}\ .$$

The free energy of the $\bar{t}$-branes can be obtained from that of the q-branes by exchanging

$(q,t) \leftrightarrow (t^{-1},q^{-1})$. Therefore, we can write them in the forms

$$F^q_{\rm ref} = \sum_{d>0}\sum_{\{\mu_i\}} \frac{1}{d}\frac{(t^{\frac{d}{2}}-t^{-\frac{d}{2}})^{L-1}}{q^{\frac{d}{2}}-q^{-\frac{d}{2}}} f^q_{\mu_1,\cdots,\mu_L}(T_{m,n};a^d,q^d,t^d)\prod_{i=1}^L s_{\mu_i}(x_i^d) \ ,$$

$$F^{\bar t}_{\rm ref} = \sum_{d>0}\sum_{\{\mu_i\}} \frac{1}{d}\frac{(q^{-\frac{d}{2}}-q^{\frac{d}{2}})^{L-1}}{t^{-\frac{d}{2}}-t^{\frac{d}{2}}} f^{\bar t}_{\mu_1,\cdots,\mu_L}(T_{m,n};a^d,q^d,t^d)\prod_{i=1}^L s_{\mu_i}(x_i^d) \ ,$$

where the refined reformulated invariants take the forms

$$\begin{aligned} f^q_{\mu_1,\cdots,\mu_L}(T_{m,n};a,q,t) &= \sum_{\rho_1,\cdots,\rho_L} M_{\mu_1\rho_1}(t)\cdots M_{\mu_L\rho_L}(t)\widehat{f}_{\rho_1,\cdots,\rho_L}(T_{m,n};a,q,t) \ , \\ f^{\bar t}_{\mu_1,\cdots,\mu_L}(T_{m,n};a,q,t) &= \sum_{\rho_1,\cdots,\rho_L} M_{\mu_1\rho_1}(q^{-1})\cdots M_{\mu_L\rho_L}(q^{-1})\widehat{f}_{\rho_1,\cdots,\rho_L}(T_{m,n};a,q,t) \ , \end{aligned} \tag{5.2}$$

and $\widehat{f}_{\rho_1,\cdots,\rho_L}(T_{m,n})$ are of the form

$$\widehat{f}_{\rho_1,\cdots,\rho_L}(T_{m,n};a,q,t) = \sum_{\rm charges} (-1)^{2J_r}\widehat{N}_{\rho_1,\cdots,\rho_L,g,\beta,J_r}(T_{m,n})(q^{\frac12}-q^{-\frac12})^g(t^{-\frac12}-t^{\frac12})^g\left(\frac{q}{t}\right)^{J_r-\frac{\beta}{2}}a^\beta \tag{5.3}$$

with *non-negative* integers $\widehat{N}_{\rho_1,\cdots,\rho_L,g,\beta,J_r}(T_{m,n}) \in \mathbb{Z}_{\geq 0}$. Here we factor out $(t^{\frac12}-t^{-\frac12})^{L-1}$ in (5.1) from the definition of refined reformulated invariants since it depends only on the number L of link components.

As a result, the large N duality of refined Chern-Simons theory with a torus link $T_{m,n}$ with L components can be summarized as

$$\begin{aligned} &\sum_{\lambda_i}\overline{\rm rCS}_{\lambda_1,\cdots,\lambda_L}(T_{m,n};a,q,t)\prod_{i=1}^L g_{\lambda_i}(q,t)P_{\lambda_i}(x_i;q,t) = \\ &\qquad \exp\left(\sum_{d>0}\sum_{\{\mu_i\}}\frac{1}{d}\frac{(t^{\frac{d}{2}}-t^{-\frac{d}{2}})^{L-1}}{q^{\frac{d}{2}}-q^{-\frac{d}{2}}} f^q_{\mu_1,\cdots,\mu_L}(T_{m,n};a^d,q^d,t^d)\prod_{i=1}^L s_{\mu_i}(x_i^d)\right) \ , \end{aligned} \tag{5.4}$$

$$\begin{aligned} &\sum_{\lambda_i}\overline{\rm rCS}_{\lambda_1,\cdots,\lambda_L}(T_{m,n};a,q,t)\prod_{i=1}^L P_{\lambda_i^T}(-x_i;t,q) = \\ &\qquad \exp\left(\sum_{d>0}\sum_{\{\mu_i\}}\frac{1}{d}\frac{(q^{-\frac{d}{2}}-q^{\frac{d}{2}})^{L-1}}{t^{-\frac{d}{2}}-t^{\frac{d}{2}}} f^{\bar t}_{\mu_1,\cdots,\mu_L}(T_{m,n};a^d,q^d,t^d)\prod_{i=1}^L s_{\mu_i}(x_i^d)\right) \ , \end{aligned} \tag{5.5}$$

where the reformulated invariants are of the form (5.2) with (5.3). These identities enable us to express the reformulated invariants in terms of refined Chern-Simons invariants of a torus link, which are presented in (B.1) and (B.2). Thus, the large N duality provides a rather non-trivial connection of refined Chern-Simons invariants $\overline{\rm rCS}_{\lambda_1,\cdots,\lambda_L}(T_{m,n};a,q,t)$ of a torus link to enumerative invariants $\widehat{N}_{\rho_1,\cdots,\rho_L,g,\beta,J_r}(T_{m,n}) \in \mathbb{Z}_{\geq 0}$ in the resolved conifold.

In Appendix C, we present some examples of $\widehat{N}_{\rho_1,\rho_2,g,\beta,J_r}$ for the Hopf link $T_{2,2}$ and the $T_{2,4}$ link obtained by using the results in [DBMM+13, GNS+16]. As in the case of torus knots, one can verify that $2J_r$ charges of non-trivial $\widehat{N}_{\rho_1,\rho_2,g,\beta,J_r}$ are either all even or all odd with ρ_1, ρ_2, g, β fixed. Therefore, we conjecture that this is true for any torus link with L components so that the LMOV invariant is

$$\widehat{N}_{\rho_1,\cdots,\rho_L,g,\beta}(T_{m,n}) = \pm \sum_{J_r \in \frac{1}{2}\mathbb{Z}} \widehat{N}_{\rho_1,\cdots,\rho_L,g,\beta,J_r}(T_{m,n}) \ .$$

6 Extension to non-torus knots

Let us consider an extension of the above formulation to non-torus knots. In the case of non-torus knots, one option is to consider a generating function of Poincaré polynomials of colored HOMFLY-PT homology, which was first examined in [GKS15] in the case of symmetric representations. In addition, various structural properties of the HOMFLY-PT homology are conjectured [GS12, GGS13, GNS+16, Wed16] when colors are specified by rectangular Young diagrams. Using these properties, conjectural formulas for Poincaré polynomials of colored HOMFLY-PT homology have been obtained for a certain class of non-torus knots [GNS+16, references therein]. In [GGS13, GNS+16], two homological gradings called $\mathbf{t_r}$- and $\mathbf{t_c}$-gradings have been introduced. In the case of torus knots, after the change of variables (4.3), refined Chern-Simons invariants expressed in terms of the $(\mathbf{a}, \mathbf{q}, \mathbf{t})$ variables yields $\mathbf{t_c}$-gradings.

As we have seen in the previous sections, the variables (a, q, t) for the equivariant parameters (3.1) are suitable for the formulations of large N duality in refined topological string theory. Hence, for a straightforward extension of refined large N duality to non-torus knots, we consider generating functions of the refined version of HOMFLY-PT polynomials, which we denote by $\overline{\mathscr{P}}_\lambda(K; a, q, t)$. In the case of rectangular Young diagrams, they can be obtained by re-writing Poincaré polynomials of colored HOMFLY-PT homology with $\mathbf{t_c}$-grading in term of the (a, q, t) variables by using the change of variables (4.3). Then, the natural extension of refined large N duality (4.1) and (4.2) to non-torus knots is

$$\sum_\lambda \overline{\mathscr{P}}_\lambda(K; a, q, t)\; g_\lambda(q,t) P_\lambda(x; q, t) = \exp\left(\sum_{d=1}^{\infty}\sum_\mu \frac{1}{d}\frac{f^q_\mu(K; a^d, q^d, t^d)}{q^{\frac{d}{2}} - q^{-\frac{d}{2}}} s_\mu(x^d)\right) \ , \tag{6.1}$$

$$\sum_\lambda \overline{\mathscr{P}}_\lambda(K; a, q, t)\; P_{\lambda^T}(-x; t, q) = \exp\left(\sum_{d=1}^{\infty}\sum_\mu \frac{1}{d}\frac{f^{\bar{t}}_\mu(K; a^d, q^d, t^d)}{t^{-\frac{d}{2}} - t^{\frac{d}{2}}} s_\mu(x^d)\right) \ . \tag{6.2}$$

Recently, the Poincaré polynomials of HOMFLY-PT homology colored by (anti-)symmetric representations have been obtained in closed forms for the $(2s-1, 1, 2t-1)$-pretzel knots [GNS+16, §5.3] as well as the knots $\mathbf{6_2}$ and $\mathbf{6_3}$ [NO15, §2.3]. Using these data, one can compute reformulated invariants $f^q_\mu(K; a, q, t)$ and $f^{\bar{t}}_\mu(K; a, q, t)$ up to two boxes. Remarkably, it turns out that reformulated invariants of these knots can be brought into the form (4.6) with (4.5). Some of the resulting BPS degeneracies are tabulated in Appendix C and more data are included in the `Mathematica` file.

As an example, let us look at the uncolored BPS degeneracies of the figure-eight presented in Table 18. Unlike the case (4.7) of torus knots, there are (boson-fermion) cancellations by sign in the unrefined limit although they reduce to the corresponding LMOV invariants [LMnV00, Figure 7]. It was observed in [LMnV00, LMn02] that, for any knot K, the LMOV invariants $\widehat{N}_{\rho,g,\beta}(K)$ have the same parity of their a-gradings β, *i.e.* $(-1)^\beta \widehat{N}_{\rho,g,\beta}$ become all non-negative integers for any μ, g, β up to appropriate grading shifts. However, as we see in this example, there are cancellations behind for non-torus knots, which becomes manifest only at the refined level. This property can be seen in other examples of the $(2s-1, 1, 2t-1)$-pretzel knots as well as the knots $\mathbf{6_2}$ and $\mathbf{6_3}$ up to two boxes.

To go beyond two boxes, we need a ⊞-colored refined invariant, whose definition is not available yet. Nevertheless, we can seek the ⊞-colored refined invariant of the figure-eight which satisfies the mirror symmetry (3.14) and the exponential growth property (3.15). Since the figure-eight knot is amphichiral (a knot which is the same as its mirror image), we also impose the condition

$$\overline{\mathscr{P}}_\lambda(\mathbf{4_1};a,q,t) = \overline{\mathscr{P}}_\lambda(\mathbf{4_1};a^{-1},q^{-1},t^{-1}) \ . \tag{6.3}$$

Like refined Chern-Simons invariants, we however allow that the refined invariant (6.4) can have both positive and negative coefficients even after the change of variables (4.3) so that it cannot be interpreted as a Poincaré polynomial. With this condition, we find the *reduced* ⊞-colored refined invariant of the figure-eight:

$$\begin{aligned}\mathscr{P}_{\boxplus}(\mathbf{4_1};a,q,t) &= \frac{a^3q^{\frac72}}{t^{\frac72}} + a^2\left[-\frac{q^{\frac72}}{t^{\frac32}} - \frac{q^{\frac52}}{t^{\frac32}} - \frac{q^{\frac52}}{t^{\frac52}} + \frac{q^{\frac32}}{t^{\frac32}} - \frac{q^{\frac32}}{t^{\frac52}} - \frac{q^{\frac32}}{t^{\frac72}} - \frac{q^{\frac72}}{t^{\frac12}} + \frac{q^3}{t} + \frac{q^2}{t^2} - \frac{q^{\frac12}}{t^{\frac72}} + \frac{q}{t^3}\right] \\
&+ a\left[q^{\frac72}t^{\frac32} + \frac{1}{q^{\frac32}t^{\frac72}} + 3q^{\frac52}t^{\frac12} + \frac{q^{\frac52}}{t^{\frac12}} - q^{\frac32}t^{\frac12} + \frac{5q^{\frac32}}{t^{\frac12}} - q^3t - 3q^2 + \frac{5q^{\frac12}}{t^{\frac32}} + \frac{q^{\frac12}}{t^{\frac52}} - \frac{1}{q^{\frac12}t^{\frac32}} + \frac{3}{q^{\frac12}t^{\frac52}} - \frac{1}{qt^3} - \frac{4q}{t} - \frac{3}{t^2}\right] \\
&- 2q^{\frac52}t^{\frac52} + 2q^2t^2 + q^{\frac32}\left(t^{\frac12} - 5t^{\frac32}\right) + 4qt + q^{\frac12}\left(t^{\frac32} - 8t^{\frac12}\right) + 7 + \frac{1}{q^{\frac12}}\left(\frac{1}{t^{\frac32}} - \frac{8}{t^{\frac12}}\right) + \frac{4}{qt} + \frac{1}{q^{\frac32}}\left(\frac{1}{t^{\frac12}} - \frac{5}{t^{\frac32}}\right) + \frac{2}{q^2t^2} - \frac{2}{q^{\frac52}t^{\frac52}} \\
&+ a^{-1}\left[q^{\frac32}t^{\frac72} + \frac{1}{q^{\frac72}t^{\frac32}} + \frac{5t^{\frac12}}{q^{\frac32}} + \frac{t^{\frac12}}{q^{\frac52}} - \frac{1}{q^{\frac32}t^{\frac12}} + \frac{3}{q^{\frac52}t^{\frac12}} - \frac{1}{q^3t} - \frac{3}{q^2} + 3q^{\frac12}t^{\frac52} + \frac{t^{\frac52}}{q^{\frac12}} - q^{\frac12}t^{\frac32} + \frac{5t^{\frac32}}{q^{\frac12}} - qt^3 - \frac{4t}{q} - 3t^2\right] \\
&+ a^{-2}\left[-\frac{t^{\frac72}}{q^{\frac32}} - \frac{t^{\frac52}}{q^{\frac32}} - \frac{t^{\frac52}}{q^{\frac52}} + \frac{t^{\frac32}}{q^{\frac32}} - \frac{t^{\frac32}}{q^{\frac52}} - \frac{t^{\frac32}}{q^{\frac72}} - \frac{t^{\frac12}}{q^{\frac72}} + \frac{t}{q^3} + \frac{t^2}{q^2} - \frac{t^{\frac72}}{q^{\frac12}} + \frac{t^3}{q}\right] + \frac{t^{\frac72}}{a^3q^{\frac72}} \ .\end{aligned} \tag{6.4}$$

Surprisingly, the reformulated invariants computed by using this datum also can be written in the form of (4.6) with (4.5), and the corresponding BPS degeneracies are presented in Appendix C. Because of the properties (6.3) and (A.2), the left hand sides of (6.1) and (6.2) stay invariant under the change of variables $(a,q,t) \leftrightarrow (a^{-1},q^{-1},t^{-1})$ for an amphichiral knot K. Then, the right hand sides are also invariant if

$$\widehat{f}_\rho(K;a^{-1},q^{-1},t^{-1}) = (-1)^{|\mu|}\ \widehat{f}_{\mu^T}(K;a,q,t)$$

due to (3.30), which is equivalent to the condition

$$\widehat{N}_{\rho,g,\beta,J_r}(K) = \widehat{N}_{\rho^T,g,-\beta,-J_r}(K) \ ,$$

to the BPS degeneracies (up to overall sign). This property can be seen for amphichiral knots like the figure-eight and the knot $\mathbf{6_3}$ in Appendix C.

Let us mention the cases in which situations are different. It is known that the $(2s-1,1,2t-1)$-pretzel knots as well as the knots $\mathbf{6_2}$ and $\mathbf{6_3}$ are homologically-thin and their HOMFLY-PT homology colored by rectangular Young diagrams is subject to the exponential growth property. On the other hand, the *thick* HOMFLY-PT homology has more complicated properties and it is *not* endowed with the exponential growth property. This was shown in [GS12, Appendix B] by explicitly obtaining the Poincaré polynomials of the ▭-colored HOMFLY-PT homology with $\mathbf{t_r}$-grading of the knot $\mathbf{9_{42}}$, which is a non-torus homological-thick knot with the fewest crossing. It is a straightforward exercise to obtain both ▭-colored and ⊟-colored HOMFLY-PT homology with $\mathbf{t_c}$-grading for the knot $\mathbf{9_{42}}$ and the corresponding refined invariants. It turns out that the reformulated invariants of the knot $\mathbf{9_{42}}$ obtained by these data *cannot* be expressed in the desired form (4.6) with (4.5).

Moreover, the HOMFLY-PT homology of some non-torus links including the Whitehead link has been obtained in [GNS+16]. However, the straightforward extension of (5.4) and

(5.5) to any non-torus links fails to provide the desired form of the reformulated invariants even in the fundamental representation.

7 Discussion

In this thesis, we have formulated large N duality of refined Chern-Simons theory with a torus knot/link. By assuming that the extra $U(1)_R$ global symmetry acts trivially on the BPS states coming from deformations of M2-branes, this formulation gives a striking relation between refined Chern-Simons invariants of a torus knot/link and graded dimensions of cohomology classes of moduli spaces of M2-M5 bound states in the resolved conifold. Therefore, this leads to the *positivity conjecture of refined Chern-Simons invariants* of a torus knot/link. Conversely, one can obtain complete information about BPS spectra in M-theory on the resolved conifold with M5'-branes supported on $\mathbb{R}^3 \times \mathcal{L}_{T_{m,n}}$ by using the geometric transition. It is also worth mentioning that, for M-theory on any toric Calabi-Yau threefold with M5-branes, its free energy on the Ω-background takes the forms (3.32) and (3.34) if the extra $U(1)_R$ global symmetry is preserved. It is important to understand when the extra $U(1)_R$ global symmetry acts on the space of BPS states trivially, which is important assumption in this thesis.

As we have seen in §6, the refined large N duality can be extended to a certain class of homologically-thin non-torus knots. However, we checked that this does not work for homologically-thick non-torus knots as well as any non-torus links. These results are still at the level of observation and the underlying structure needs to be investigated.

Giving a mathematical definition of refined LMOV invariants $\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})$ discussed in this thesis is a challenging, but important open problem. Refined GV invariants have been discussed in the literature [CKK14, CDDP15, NO14, GHKPK17, references therein] as refined Pandharipande-Thomas (Donaldson-Thomas) invariants for toric Calabi-Yau threefolds. However, mathematical understanding of their open analogues are still immature although the Poincaré polynomials of uncolored HOMFLY-PT homology of torus knots have been related to motivic Donaldson-Thomas invariants in [DHS12]. Actually, upon the reduction on the cigar of the Taub-NUT in the table.2, BPS states can be understood as D6-D4-D2-D0 bound states. Hence, it is an important task to give a mathematical definition of D6-D4-D2-D0 bound states for refine Chern-Simons invariants discussed in this thesis. It would be also intriguing to find a connection to (a certain variant of) "P=W conjecture" [CDDP15, DDP17, references therein].

Another direction to pursue is to find large N duality of refined Chern-Simons theory with different gauge groups. In fact, SO$(2N)$ refined Chern-Simons theory has been proposed [AS12a], generalizing Kauffman polynomials. In addition, large N duality for Kauffman polynomials has been put forward by incorporating orientifolds in the resolved conifold [Mn10]. Consequently, this leads to an integrality conjecture involving both colored Kauffman and HOMFLY-PT polynomials. It is natural to ask whether a positivity property can be seen when the conjecture of [Mn10] is refined.

A Symmetric functions

In this appendix, we review basics of symmetric functions relevant to this thesis. For more detail, we refer the reader to [Mac98].

Let $x = (x_1, x_2, \cdots)$ be an infinite number of the variables, $\lambda = (\lambda_1, \lambda_2, \cdots)$ be a Young diagram (i.e. non-negative integers such that $\lambda_i \geq \lambda_{i+1}$ and $|\lambda| = \sum_i \lambda_i < \infty$) and $\vec{k} = (k_1, k_2, \cdots)$ be a vector with a infinite number of entries, almost zero, and whose nonzero entries are positive integers. The Young diagram λ and the vector $\vec{k}$ are in one-to-one correspondence with the relation $k_i = m_i(\lambda)$, where $m_i(\lambda)$ is a multiplicity of i in λ.

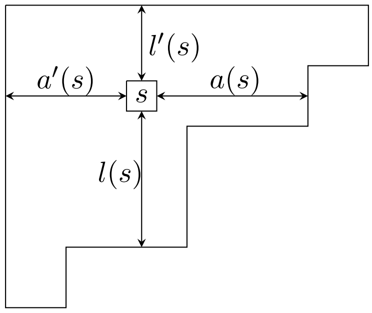


Figure 2. Arm, leg, co-arm and co-leg

First, we define the power-sum symmetric functions by $p_d(x) = \sum_{i=1}^{\infty} x_i^d$. It is convenient to denote their products by $p_\lambda(x) = \prod_i p_{\lambda_i}(x)$ and $p_{\vec{k}} = \prod_i p_i^{k_i}$. These are bases of the ring of symmetric functions. Schur and Macdonald functions[11] can be defined by introducing an inner product on the ring of symmetric functions.

Schur functions

The Schur functions $s_\lambda(x)$ are uniquely defined by orthogonality and normalization conditions:

$$\langle s_\lambda, s_\mu \rangle = 0 \ , \qquad \text{if} \ \ \lambda \neq \mu \ ,$$
$$s_\lambda(x) = w_\lambda(x) + \sum_{\mu < \lambda} u_{\lambda\mu} w_\mu(x) \ , \qquad u_{\lambda\mu} \in \mathbb{Q} \ ,$$

where $w_\lambda(x)$ is the monomial symmetric function, $<$ is dominance partial ordering ($\lambda \geq \mu \Leftrightarrow |\lambda| = |\mu|$ and $\sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i$ for all i), and the inner product is defined by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \ , \qquad z_\lambda = \prod_{i \geq 1} i^{m_i} m_i! \ ,$$

where $m_i = m_i(\lambda)$ is a multiplicity of i in λ.

The relation between Schur and power sum symmetric functions is known as Frobenius formula:

$$s_\lambda(x) = \sum_{\vec{k}} \frac{\chi_\lambda(C(\vec{k}))}{z_{\vec{k}}} p_{\vec{k}}(x) \ , \qquad p_{\vec{k}}(x) = \sum_\lambda \chi_\lambda(C(\vec{k})) s_\lambda(x) \ , \tag{A.1}$$

where $\chi_\lambda(C(\vec{k}))$ is the character of the representation λ of the permutation group $\mathfrak{S}_h$ evaluated at the conjugacy class $C(\vec{k})$ where $h = \sum_j j k_j$. The Frobenius formula is used for the computation of Clebsch-Gordon coefficients (3.27) of the permutation group $\mathfrak{S}_h$.

[11]When these functions have an finite number of the variables, we call Schur and Macdonald polynomials.

Macdonald functions

The Macdonald functions $P_\lambda(x;q,t)$ are uniquely defined by orthogonality and normalization conditions:

$$\begin{aligned} &\langle P_\lambda, P_\mu\rangle_{q,t} = 0 \ , && \text{if} \quad \lambda \neq \mu, \\ &P_\lambda(x;q,t) = w_\lambda(x) + \textstyle\sum_{\mu<\lambda} u_{\lambda\mu}(q,t)\, w_\mu(x) \ , && u_{\lambda\mu}(q,t) \in \mathbb{Q}(q,t) \ , \end{aligned}$$

where the inner product is defined by

$$\langle p_\lambda, p_\mu\rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i\geq 1} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} \ , \qquad z_\lambda = \prod_{i\geq 1} i^{m_i} m_i! \ .$$

At the $q = t$ specialization, the Macdonald functions reduce to the Schur functions. From the definition one can show

$$\frac{(q/t)^{|\lambda|}}{g_\lambda(q,t)} := \langle P_\lambda, P_\lambda\rangle_{q,t} = \prod_{s\in\lambda} \frac{1-q^{a(s)+1}t^{l(s)}}{1-q^{a(s)}t^{l(s)+1}} \ ,$$

where an arm length $a(s) = \lambda_i - j$ and a leg length $l(s) = \lambda_j^T - i$ for each box $s = (i,j)$ in λ are depicted in Figure 2.

We denote by $\mathfrak{X}_\lambda(\vec{k};q,t)$ coefficients in the expansion of a Macdonald function $P_\lambda(x;q,t)$ with respect to $p_{\vec{k}}(x)$:

$$P_\lambda(x;q,t) = \sum_{\vec{k}} \mathfrak{X}_\lambda(\vec{k};q,t) p_{\vec{k}}(x) \ .$$

Since the rational functions $\mathfrak{X}_\lambda(\vec{k};q,t)$ are invariant under the exchange $(q,t) \leftrightarrow (q^{-1},t^{-1})$, the Macdonald functions have the following property

$$P_\lambda(x;q,t) = P_\lambda\left(x;q^{-1},t^{-1}\right) \ . \tag{A.2}$$

Note that at the $q = t$ specialization

$$\mathfrak{X}_\lambda(\vec{k};q,q) = \frac{\chi_\lambda(C(\vec{k}))}{z_{\vec{k}}} \ .$$

In the following, we list some Macdonald functions expressed in terms of the power-sum functions:

$$\begin{aligned} P_{\square} =& p_1 \\ P_{\square\square} =& \frac{(1-t)(1+q)}{(1-tq)}\frac{p_1^2}{2} + \frac{(1+t)(1-q)}{(1-tq)}\frac{p_2}{2}, \\ P_{\substack{\square\\\square}} =& \frac{p_1^2}{2} - \frac{p_2}{2} \\ P_{\square\square\square} =& \frac{(1+q)(1-q^3)(1-t)^2}{(1-q)(1-tq)(1-tq^2)}\frac{p_1^3}{6} + \frac{(1-q)(1-t^2)(1-q^3)}{(1-q)(1-tq)(1-tq^2)}\frac{p_1p_2}{2} \\ &+ \frac{(1-q)(1-q^2)(1-t^3)}{(1-t)(1-tq)(1-tq^2)}\frac{p_3}{3} \end{aligned}$$

$$P_{\tiny\boxplus} = \frac{(1-t)(2qt+q+t+2)}{1-qt^2}\frac{p_1^3}{6} + \frac{(1+t)(t-q)}{1-qt^2}\frac{p_1p_2}{2} - \frac{(1-q)(1-t^3)}{(1-t)(1-qt^2)}\frac{p_3}{3}$$
$$P_{\tiny\square\!\square\!\square} = \frac{p_1^3}{6} - \frac{p_2p_1}{2} + \frac{p_3}{3}.$$

For instance, we can read off $\mathfrak{X}_{\tiny\boxplus}(\vec{k}=(3,0,0);q,t) = \frac{(1-t)(2qt+q+t+2)}{6(1-qt^2)}$ because of $p_{\vec{k}=(3,0,0)} = p_1^3$.

Cauchy formulas

The Cauchy formulas play a very important role in this thesis. The Cauchy formulas for Schur functions read off:

$$\sum_\lambda s_\lambda(x)s_\lambda(y) = \exp\left(\sum_{d>0}\frac{1}{d}p_d(x)p_d(y)\right) , \quad \sum_\lambda s_\lambda(x)s_{\lambda^T}(y) \quad = \exp\left(\sum_{d>0}\frac{(-1)^{d-1}}{d}p_d(x)p_d(y)\right) . \tag{A.3}$$

The analogues of Macdonald functions are

$$\sum_\lambda g_\lambda(q,t)P_\lambda(x;q,t)P_\lambda(y;q,t) = \exp\left(\sum_{d>0}\frac{1}{d}\frac{t^{\frac{d}{2}}-t^{-\frac{d}{2}}}{q^{\frac{d}{2}}-q^{-\frac{d}{2}}}p_d(x)p_d(y)\right) ,$$
$$\sum_\lambda P_\lambda(x;q,t)P_{\lambda^T}(y;t,q) = \exp\left(\sum_{d>0}\frac{(-1)^{d-1}}{d}p_d(x)p_d(y)\right) . \tag{A.4}$$

B Explicit formulas of refined reformulated invariants

In this appendix, we will derive explicit formulas for refined reformulated invariants of a torus link $T_{m,n}$ with L components in terms of its refined Chern-Simons invariants from (5.4) and (5.5) by following [LMn02]. To this end, we define the plethystic exponential and its inverse

$$\mathrm{Exp}(F) := \exp\left(\sum_{d=1}^\infty \frac{\psi_d}{d}\right)\circ F , \qquad \mathrm{Log}(F) := \sum_{d=1}^\infty \frac{\mu(d)}{d}\log(\psi_d\circ F) ,$$

where an operator ψ_d is defined by $\psi_d \circ F(a,q,t;x) := F(a^d,q^d,t^d;x^d)$ and $\mu(d)$ is the Möbius function. If one sets

$$F := \frac{(t^{\frac12}-t^{-\frac12})^{L-1}}{q^{\frac12}-q^{-\frac12}}\sum_{\{\mu_i\}} f^q_{\mu_1,\cdots,\mu_L}(T_{m,n};a,q,t)\prod_{i=1}^L s_{\mu_i}(x_i) ,$$

then the right hand side of (5.4) can be written as $\mathrm{Exp}(F)$. Thus, one can manipulate the identity (5.4) as

$$F = \mathrm{Log}\left(\sum_{\{\lambda_i\}}\overline{\mathrm{rCS}}_{\lambda_1\cdots\lambda_L}(T_{m,n};a,q,t)\prod_{i=1}^L g_{\lambda_i}(q,t)P_{\lambda_i}(x_i;q,t)\right)$$
$$= \sum_{d=1}^\infty\frac{\mu(d)}{d}\log\left(\sum_{\{\lambda_i\}}\overline{\mathrm{rCS}}^{(d)}_{\lambda_1\cdots\lambda_L}\prod_{i=1}^L g_{\lambda_i}(q^d,t^d)P_{\lambda_i}(x_i^d;q^d,t^d)\right)$$

$$
\begin{aligned}
&= \sum_{d=1}^{\infty} \frac{\mu(d)}{d} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \prod_{\alpha=1}^{m} \sum_{\{\lambda_i^{(\alpha)}\}} \overline{\mathrm{rCS}}^{(d)}_{\lambda_1^{(\alpha)} \cdots \lambda_L^{(\alpha)}} \prod_{i=1}^{L} g_{\lambda_i^{(\alpha)}}(q^d, t^d) P_{\lambda_i^{(\alpha)}}(x_i^d; q^d, t^d) \\
&= \sum_{d=1}^{\infty} \frac{\mu(d)}{d} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \prod_{\alpha=1}^{m} \sum_{\{\lambda_i^{(\alpha)}\}} \overline{\mathrm{rCS}}^{(d)}_{\lambda_1^{(\alpha)} \cdots \lambda_L^{(\alpha)}} \prod_{i=1}^{L} g_{\lambda_i^{(\alpha)}}(q^d, t^d) \sum_{\vec{k}_i^{(\alpha)}} \mathfrak{X}_{\lambda_i^{(\alpha)}}(\vec{k}_i^{(\alpha)}; q^d, t^d) p_{\vec{k}_i^{(\alpha)}}(x_i^d) \ ,
\end{aligned}
$$

where $\overline{\mathrm{rCS}}^{(d)}_{\lambda} = \overline{\mathrm{rCS}}_{\lambda}(T_{m,n}; a^d, q^d, t^d)$ and other notations are given in Appendix A. To compare with the coefficient of $\prod_{i=1}^{L} s_{\mu_i}(x_i)$, we introduce $\vec{k}_d$ for $\vec{k} = (k_1, k_2, \cdots)$ as $(\vec{k}_d)_{di} = (\vec{k})_i$, *i.e.*

$$
\vec{k}_d = (0, \cdots, 0, k_1, 0, \cdots, 0, k_2, 0, \cdots) \ ,
$$

where k_1 is d-th entry, k_2 is $2d$-th entry and so on. Then, the properties $p_{\vec{k}}(x^d) = p_{\vec{k}_d}(x)$ and $p_{\vec{k}}(x) p_{\vec{k}'}(x) = p_{\vec{k}+\vec{k}'}(x)$ of the power sum functions tell us

$$
\prod_{\alpha=1}^{m} p_{\vec{k}_i^{(\alpha)}}(x_i^d) = p_{\sum_{\alpha=1}^{m} (\vec{k}_i^{(\alpha)})_d}(x_i) = \sum_{\{\mu_i\}} \chi_{\mu_i}(C(\sum_{\alpha=1}^{m} (\vec{k}_i^{(\alpha)})_d)) s_{\mu_i}(x_i) \ .
$$

Using these results, we obtain the explicit formula for f^q in terms of refined Chern-Simons invariants:

$$
\begin{aligned}
&\frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{L-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} f^q_{\mu_1, \cdots, \mu_L}(T_{m,n}; a, q, t) = \qquad\qquad \text{(B.1)} \\
&\sum_{d,m=1}^{\infty} (-1)^{m-1} \frac{\mu(d)}{d \cdot m} \sum_{\{\vec{k}_i^{(\alpha)}\}} \sum_{\{\lambda_i^{(\alpha)}\}} \prod_{i=1}^{L} \chi_{\mu_i}(C(\sum_{\alpha=1}^{m} (\vec{k}_i^{(\alpha)})_d)) \prod_{\alpha=1}^{m} g_{\lambda_i^{(\alpha)}}(q^d, t^d) \mathfrak{X}_{\lambda_i^{(\alpha)}}(\vec{k}_i^{(\alpha)}; q^d, t^d) \overline{\mathrm{rCS}}^{(d)}_{\lambda_1^{(\alpha)} \cdots \lambda_L^{(\alpha)}} \ .
\end{aligned}
$$

Similarly, we can obtain the explicit formula for $f^{\bar{t}}$:

$$
\begin{aligned}
&\frac{(q^{-\frac{d}{2}} - q^{\frac{d}{2}})^{L-1}}{t^{-\frac{d}{2}} - t^{\frac{d}{2}}} f^{\bar{t}}_{\mu_1, \cdots, \mu_L}(T_{m,n}; a, q, t) = \qquad\qquad \text{(B.2)} \\
&\sum_{d,m=1}^{\infty} (-1)^{m-1} \frac{\mu(d)}{d \cdot m} \sum_{\{\vec{k}_i^{(\alpha)}\}} \sum_{\{\lambda_i^{(\alpha)}\}} \prod_{i=1}^{L} \chi_{\mu_i^T}(C(\sum_{\alpha=1}^{m} (\vec{k}_i^{(\alpha)})_d)) \prod_{\alpha=1}^{m} (-1)^{|\lambda_i^{(\alpha)}|} \mathfrak{X}_{(\lambda_i^{(\alpha)})^T}(\vec{k}_i^{(\alpha)}; t^d, q^d) \overline{\mathrm{rCS}}^{(d)}_{\lambda_1^{(\alpha)} \cdots \lambda_L^{(\alpha)}} \ .
\end{aligned}
$$

In the following, we provide reformulated invariants of a torus knot colored by Young diagrams with a few boxes for the q-branes from (B.1):

$$
\begin{aligned}
\frac{f^q_{\square}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} &= \overline{\mathrm{rCS}}_{\square}, \\
\frac{t^{\frac{1}{2}}}{q^{\frac{1}{2}}} \frac{f^q_{\square\square}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} &= \frac{qt-1}{q^2-1} \overline{\mathrm{rCS}}_{\square\square} - \frac{t-1}{2(q-1)} (\overline{\mathrm{rCS}}_{\square})^2 - \frac{t+1}{2(q+1)} \overline{\mathrm{rCS}}^{(2)}_{\square} \ , \\
\frac{t^{\frac{1}{2}}}{q^{\frac{1}{2}}} \frac{f^q_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} &= \frac{t-q}{q^2-1} \overline{\mathrm{rCS}}_{\square\square} + \frac{t^2-1}{qt-1} \overline{\mathrm{rCS}}_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} - \frac{t-1}{2(q-1)} (\overline{\mathrm{rCS}}_{\square})^2 + \frac{t+1}{2(q+1)} \overline{\mathrm{rCS}}^{(2)}_{\square} \ , \\
\frac{t}{q} \frac{f^q_{\square\square\square}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} &= \frac{(qt-1)\left(q^2t-1\right)}{(q^2-1)\left(q^3-1\right)} \overline{\mathrm{rCS}}_{\square\square\square} - \frac{(t-1)(qt-1)}{(q-1)^2(q+1)} \overline{\mathrm{rCS}}_{\square\square} \overline{\mathrm{rCS}}_{\square}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(t-1)^2}{3(q-1)^2}(\overline{\mathrm{rCS}}_{\yng(1)})^3-\frac{t^2+t+1}{3\left(q^2+q+1\right)}\overline{\mathrm{rCS}}^{(3)}_{\yng(1)}\ ,\\
\frac{t}{q}\frac{f^q_{\yng(2,1)}}{t^{\frac12}-t^{-\frac12}} =&-\frac{(q-t)(qt-1)}{(q-1)(q^3-1)}\overline{\mathrm{rCS}}_{\yng(3)}+\frac{(t-1)\left(qt^2-1\right)}{(q-1)\left(q^2t-1\right)}\overline{\mathrm{rCS}}_{\yng(2,1)}\\
&-\left[\frac{(t-1)^2}{(q-1)^2}\overline{\mathrm{rCS}}_{\yng(2)}+\frac{(t-1)^2(t+1)}{(q-1)(qt-1)}\overline{\mathrm{rCS}}_{\yng(1,1)}\right]\overline{\mathrm{rCS}}_{\yng(1)}\\
&+\frac{2(t-1)^2}{3(q-1)^2}(\overline{\mathrm{rCS}}_{\yng(1)})^3+\frac{t^2+t+1}{3\left(q^2+q+1\right)}\overline{\mathrm{rCS}}^{(3)}_{\yng(1)}\ ,\\
\frac{t}{q}\frac{f^q_{\yng(1,1,1)}}{t^{\frac12}-t^{-\frac12}} =&\frac{(q-t)\left(q^2-t\right)}{(q^2-1)(q^3-1)}\overline{\mathrm{rCS}}_{\yng(3)}-\frac{(t^2-1)(q-t)}{(q-1)\left(q^2t-1\right)}\overline{\mathrm{rCS}}_{\yng(2,1)}+\frac{(t^2-1)(t^3-1)}{(qt-1)\left(qt^2-1\right)}\overline{\mathrm{rCS}}_{\yng(1,1,1)}\\
&+\left[\frac{(t-1)(q-t)}{(q-1)^2(q+1)}\overline{\mathrm{rCS}}_{\yng(2)}-\frac{(t-1)^2(t+1)}{(q-1)(qt-1)}\overline{\mathrm{rCS}}_{\yng(1,1)}\right]\overline{\mathrm{rCS}}_{\yng(1)}\\
&+\frac{(t-1)^2}{3(q-1)^2}(\overline{\mathrm{rCS}}_{\yng(1)})^3-\frac{t^2+t+1}{3\left(q^2+q+1\right)}\overline{\mathrm{rCS}}^{(3)}_{\yng(1)}\ .
\end{aligned}
$$

In addition, we present reformulated invariants of a torus knot colored by Young diagrams with a few boxes for the $\bar{t}$-branes from (B.2):

$$
\begin{aligned}
\frac{f^{\bar t}_{\yng(1)}}{t^{\frac12}-t^{-\frac12}} =&\overline{\mathrm{rCS}}_{\yng(1)},\\
\frac{-f^{\bar t}_{\yng(2)}}{t^{\frac12}-t^{-\frac12}} =&\overline{\mathrm{rCS}}_{\yng(1,1)}+\frac12\overline{\mathrm{rCS}}^{(2)}_{\yng(1)}-\frac12(\overline{\mathrm{rCS}}_{\yng(1)})^2\ ,\\
\frac{-f^{\bar t}_{\yng(1,1)}}{t^{\frac12}-t^{-\frac12}} =&\overline{\mathrm{rCS}}_{\yng(2)}+\frac{q-t}{qt-1}\overline{\mathrm{rCS}}_{\yng(1,1)}-\frac12\overline{\mathrm{rCS}}^2_{\yng(1)}-\frac12\overline{\mathrm{rCS}}^{(2)}_{\yng(1)},\\
\frac{f^{\bar t}_{\yng(3)}}{t^{\frac12}-t^{-\frac12}} =&\overline{\mathrm{rCS}}_{\yng(1,1,1)}-\overline{\mathrm{rCS}}_{\yng(1)}\overline{\mathrm{rCS}}_{\yng(1,1)}+\frac13(\overline{\mathrm{rCS}}_{\yng(1)})^3-\frac13\overline{\mathrm{rCS}}^{(3)}_{\yng(1)}\ ,\\
\frac{f^{\bar t}_{\yng(2,1)}}{t^{\frac12}-t^{-\frac12}} =&\overline{\mathrm{rCS}}_{\yng(2,1)}+\frac{(t+1)(q-t)}{qt^2-1}\overline{\mathrm{rCS}}_{\yng(1,1,1)}-\left[\overline{\mathrm{rCS}}_{\yng(2)}+\frac{(q-1)(t+1)}{qt-1}\overline{\mathrm{rCS}}_{\yng(1,1)}\right]\overline{\mathrm{rCS}}_{\yng(1)}\\
&+\frac23(\overline{\mathrm{rCS}}_{\yng(1)})^3+\frac13\overline{\mathrm{rCS}}^{(3)}_{\yng(1)},\\
\frac{f^{\bar t}_{\yng(1,1,1)}}{t^{\frac12}-t^{-\frac12}} =&\overline{\mathrm{rCS}}_{\yng(3)}+\frac{(q+1)(q-t)}{q^2t-1}\overline{\mathrm{rCS}}_{\yng(2,1)}+\frac{(q-t)\left(q-t^2\right)}{(qt-1)\left(qt^2-1\right)}\overline{\mathrm{rCS}}_{\yng(1,1,1)}\\
&+\left[-\overline{\mathrm{rCS}}_{\yng(2)}+\frac{(q-t)}{qt-1}\overline{\mathrm{rCS}}_{\yng(1,1)}\right]\overline{\mathrm{rCS}}_{\yng(1)}+\frac13(\overline{\mathrm{rCS}}_{\yng(1)})^3-\frac13\overline{\mathrm{rCS}}^{(3)}_{\yng(1)}\ .
\end{aligned}
$$

C Tables of BPS degeneracies

In this appendix, we shall list tables of non-negative integral invariants $\widehat{N}_{\rho_1,\cdots,\rho_L,g,\beta,J_r}$ of some torus knots/links as well as non-torus knots. The prescription to compute these integral invariants is explained in §4 for torus knots, §5 for torus links, and §6 for non-torus knots, respectively. We also note that the `Mathematica` file includes more data.

One can verify that $2J_r$ charges of non-trivial $\widehat{N}_{\rho,g,\beta,J_r}$ are either all even or all odd with ρ, g, β fixed for torus knots. (The situation for torus links is the same.) However, this is not true for $\widehat{N}_{\rho,g,\beta,J_r}$ of non-torus knots. In addition, one can see $\widehat{N}_{\rho,g,\beta,J_r}(K) = \widehat{N}_{\rho^T,g,-\beta,-J_r}(K)$ for the amphichiral knot $\mathbf{4_1}$ and $\mathbf{6_3}$. (Strictly speaking, when the number of boxes of a Young diagram ρ is odd, one needs to shift β and $2J_r$ by $\frac{1}{2}$ to see this symmetry.)

Torus knots

β	1			2			3
$g\backslash 2J_r$	0	1	2	1	2	3	4
0	1	0	1	1	0	2	1
1	0	1	0	0	1	0	0

Table 4. $\widehat{N}_{[1],g,\beta,J_r}(T_{2,3})$ for the trefoil

β	1			2					3							4					5		
$g\backslash 2J_r$	0	1	2	1	2	3	4	5	2	3	4	5	6	7	8	5	6	7	8	9	8	9	10
0	1	0	1	2	0	4	0	2	1	0	5	0	5	0	1	2	0	4	0	2	1	0	1
1	0	1	0	0	3	0	3	0	0	2	0	6	0	2	0	0	3	0	3	0	0	1	0
2	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	1	0	0	0	0	0

Table 5. $\widehat{N}_{[2],g,\beta,J_r}(T_{2,3})$ for the trefoil

β	1					2							3									4							5				
$g\backslash 2J_r$	0	1	2	3	4	1	2	3	4	5	6	7	2	3	4	5	6	7	8	9	10	5	6	7	8	9	10	11	8	9	10	11	12
0	1	0	2	0	1	2	0	7	0	5	0	2	1	0	8	0	9	0	5	0	1	3	0	7	0	4	0	2	2	0	1	0	1
1	0	2	0	2	0	0	5	0	10	0	5	0	0	3	0	14	0	12	0	3	0	0	6	0	9	0	5	0	0	2	0	2	0
2	0	0	1	0	0	0	0	4	0	4	0	0	0	0	3	0	8	0	3	0	0	0	0	4	0	4	0	0	0	0	1	0	0
3	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0

Table 6. $\widehat{N}_{[1,1],g,\beta,J_r}(T_{2,3})$ for the trefoil

β	1		
$g\backslash 2J_r$	0	1	2
0	1	0	1
1	0	1	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0

β	2						
$g\backslash 2J_r$	1	2	3	4	5	6	7
0	3	0	8	0	5	0	2
1	0	6	0	10	0	5	0
2	0	0	4	0	4	0	0
3	0	0	0	1	0	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0

β	3										
$g\backslash 2J_r$	2	3	4	5	6	7	8	9	10	11	12
0	3	0	18	0	23	0	14	0	5	0	1
1	0	9	0	37	0	38	0	18	0	4	0
2	0	0	10	0	29	0	22	0	6	0	0
3	0	0	0	5	0	10	0	4	0	0	0
4	0	0	0	0	1	0	1	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0

β	4												
$g\backslash 2J_r$	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	0	16	0	38	0	34	0	19	0	6	0	2
1	0	4	0	44	0	82	0	65	0	28	0	9	0
2	0	0	6	0	48	0	72	0	45	0	16	0	0
3	0	0	0	4	0	26	0	30	0	14	0	0	0
4	0	0	0	0	1	0	7	0	6	0	0	0	0
5	0	0	0	0	0	0	1	0	0	0	0	0	0

β	5												
$g\backslash 2J_r$	6	7	8	9	10	11	12	13	14	15	16	17	18
0	5	0	27	0	37	0	27	0	12	0	5	0	1
1	0	17	0	69	0	83	0	53	0	24	0	5	0
2	0	0	23	0	74	0	77	0	43	0	10	0	0
3	0	0	0	16	0	42	0	35	0	10	0	0	0
4	0	0	0	0	6	0	12	0	5	0	0	0	0
5	0	0	0	0	0	1	0	1	0	0	0	0	0

β	6										
$g\backslash 2J_r$	9	10	11	12	13	14	15	16	17	18	19
0	7	0	18	0	17	0	10	0	4	0	2
1	0	20	0	43	0	38	0	21	0	9	0
2	0	0	24	0	45	0	36	0	16	0	0
3	0	0	0	16	0	25	0	14	0	0	0
4	0	0	0	0	6	0	6	0	0	0	0
5	0	0	0	0	0	1	0	0	0	0	0

β	7								
$g\backslash 2J_r$	12	13	14	15	16	17	18	19	20
0	3	0	4	0	3	0	1	0	1
1	0	7	0	9	0	6	0	4	0
2	0	0	7	0	9	0	6	0	0
3	0	0	0	4	0	4	0	0	0
4	0	0	0	0	1	0	0	0	0
5	0	0	0	0	0	0	0	0	0

Table 7. $\widehat{N}_{[3],g,\beta,J_r}(T_{2,3})$ for the trefoil

β	1						
$g\backslash 2J_r$	0	1	2	3	4	5	6
0	2	0	5	0	3	0	1
1	0	5	0	7	0	3	0
2	0	0	4	0	3	0	0
3	0	0	0	1	0	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0

β	2										
$g\backslash 2J_r$	1	2	3	4	5	6	7	8	9	10	11
0	8	0	30	0	33	0	19	0	7	0	2
1	0	26	0	71	0	64	0	31	0	9	0
2	0	0	33	0	67	0	48	0	16	0	0
3	0	0	0	21	0	31	0	14	0	0	0
4	0	0	0	0	7	0	6	0	0	0	0
5	0	0	0	0	0	1	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0

β	3														
$g\backslash 2J_r$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	10	0	61	0	104	0	85	0	46	0	19	0	6	0	1
1	0	39	0	188	0	269	0	203	0	102	0	35	0	6	0
2	0	0	63	0	244	0	304	0	203	0	81	0	15	0	0
3	0	0	0	55	0	175	0	184	0	94	0	20	0	0	0
4	0	0	0	0	28	0	72	0	56	0	15	0	0	0	0
5	0	0	0	0	0	8	0	15	0	6	0	0	0	0	0
6	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

β	4																
$g\backslash 2J_r$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	4	0	52	0	142	0	158	0	110	0	57	0	25	0	8	0	2
1	0	18	0	190	0	434	0	454	0	308	0	155	0	55	0	13	0
2	0	0	34	0	299	0	601	0	591	0	369	0	150	0	36	0	0
3	0	0	0	35	0	269	0	484	0	422	0	210	0	55	0	0	0
4	0	0	0	0	21	0	150	0	233	0	160	0	50	0	0	0	0
5	0	0	0	0	0	7	0	51	0	63	0	27	0	0	0	0	0
6	0	0	0	0	0	0	1	0	10	0	8	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0

β	5																
$g\backslash 2J_r$	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
0	16	0	89	0	141	0	122	0	76	0	39	0	17	0	6	0	1
1	0	66	0	301	0	446	0	387	0	244	0	119	0	42	0	7	0
2	0	0	119	0	469	0	665	0	559	0	321	0	121	0	21	0	0
3	0	0	0	125	0	440	0	579	0	428	0	185	0	35	0	0	0
4	0	0	0	0	84	0	261	0	293	0	160	0	35	0	0	0	0
5	0	0	0	0	0	36	0	93	0	76	0	21	0	0	0	0	0
6	0	0	0	0	0	0	9	0	17	0	7	0	0	0	0	0	0
7	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0

β	6														
$g\backslash 2J_r$	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
0	21	0	59	0	63	0	47	0	27	0	14	0	6	0	2
1	0	75	0	190	0	214	0	163	0	97	0	44	0	13	0
2	0	0	124	0	300	0	338	0	249	0	125	0	36	0	0
3	0	0	0	126	0	287	0	299	0	180	0	55	0	0	0
4	0	0	0	0	84	0	169	0	140	0	50	0	0	0	0
5	0	0	0	0	0	36	0	56	0	27	0	0	0	0	0
6	0	0	0	0	0	0	9	0	8	0	0	0	0	0	0
7	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0

β	7												
$g\backslash 2J_r$	12	13	14	15	16	17	18	19	20	21	22	23	24
0	9	0	12	0	11	0	7	0	4	0	2	0	1
1	0	26	0	42	0	39	0	27	0	15	0	6	0
2	0	0	39	0	67	0	63	0	40	0	15	0	0
3	0	0	0	36	0	61	0	50	0	20	0	0	0
4	0	0	0	0	21	0	30	0	15	0	0	0	0
5	0	0	0	0	0	7	0	6	0	0	0	0	0
6	0	0	0	0	0	0	1	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 8. $\widehat{N}_{[2,1],g,\beta,J_r}(T_{2,3})$ for the trefoil

β	1								
$g\backslash 2J_r$	0	1	2	3	4	5	6	7	8
0	1	0	4	0	4	0	2	0	1
1	0	4	0	10	0	8	0	4	0
2	0	0	6	0	10	0	6	0	0
3	0	0	0	4	0	4	0	0	0
4	0	0	0	0	1	0	0	0	0
5	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0

β	2												
$g\backslash 2J_r$	1	2	3	4	5	6	7	8	9	10	11	12	13
0	5	0	23	0	34	0	25	0	14	0	5	0	2
1	0	22	0	78	0	96	0	68	0	31	0	11	0
2	0	0	40	0	112	0	119	0	70	0	25	0	0
3	0	0	0	39	0	87	0	74	0	30	0	0	0
4	0	0	0	0	22	0	37	0	20	0	0	0	0
5	0	0	0	0	0	7	0	7	0	0	0	0	0
6	0	0	0	0	0	0	1	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0

β	3																
$g\backslash 2J_r$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	7	0	46	0	94	0	91	0	62	0	32	0	14	0	5	0	1
1	0	34	0	184	0	321	0	303	0	197	0	98	0	36	0	7	0
2	0	0	71	0	323	0	506	0	447	0	266	0	106	0	21	0	0
3	0	0	0	84	0	328	0	461	0	358	0	165	0	35	0	0	0
4	0	0	0	0	62	0	208	0	248	0	145	0	35	0	0	0	0
5	0	0	0	0	0	29	0	80	0	70	0	21	0	0	0	0	0
6	0	0	0	0	0	0	8	0	16	0	7	0	0	0	0	0	0
7	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

β	4																		
$g\backslash 2J_r$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
0	3	0	39	0	117	0	148	0	120	0	76	0	38	0	19	0	6	0	2
1	0	16	0	173	0	449	0	552	0	452	0	281	0	146	0	52	0	15	0
2	0	0	37	0	345	0	812	0	970	0	772	0	453	0	180	0	49	0	0
3	0	0	0	49	0	410	0	894	0	1007	0	723	0	330	0	91	0	0	0
4	0	0	0	0	41	0	319	0	635	0	627	0	350	0	105	0	0	0	0
5	0	0	0	0	0	22	0	164	0	282	0	216	0	77	0	0	0	0	0
6	0	0	0	0	0	0	7	0	53	0	72	0	35	0	0	0	0	0	0
7	0	0	0	0	0	0	0	1	0	10	0	9	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0

β	5																		
$g\backslash 2J_r$	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	12	0	68	0	119	0	114	0	85	0	49	0	27	0	12	0	5	0	1
1	0	57	0	279	0	468	0	474	0	353	0	218	0	107	0	42	0	8	0
2	0	0	124	0	547	0	907	0	920	0	680	0	374	0	148	0	28	0	0
3	0	0	0	165	0	673	0	1080	0	1043	0	675	0	286	0	56	0	0	0
4	0	0	0	0	148	0	555	0	820	0	680	0	330	0	70	0	0	0	0
5	0	0	0	0	0	91	0	303	0	377	0	230	0	56	0	0	0	0	0
6	0	0	0	0	0	0	37	0	102	0	92	0	28	0	0	0	0	0	0
7	0	0	0	0	0	0	0	9	0	18	0	8	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0

β	6																
$g\backslash 2J_r$	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
0	15	0	45	0	52	0	45	0	30	0	17	0	10	0	4	0	2
1	0	63	0	177	0	227	0	201	0	142	0	86	0	39	0	15	0
2	0	0	128	0	355	0	471	0	426	0	291	0	144	0	49	0	0
3	0	0	0	166	0	451	0	589	0	494	0	275	0	91	0	0	0
4	0	0	0	0	148	0	379	0	446	0	300	0	105	0	0	0	0
5	0	0	0	0	0	91	0	204	0	189	0	77	0	0	0	0	0
6	0	0	0	0	0	0	37	0	64	0	35	0	0	0	0	0	0
7	0	0	0	0	0	0	0	9	0	9	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0

β	7														
$g\backslash 2J_r$	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
0	6	0	9	0	9	0	7	0	4	0	3	0	1	0	1
1	0	22	0	39	0	42	0	34	0	24	0	12	0	7	0
2	0	0	41	0	82	0	93	0	76	0	45	0	21	0	0
3	0	0	0	50	0	104	0	114	0	80	0	35	0	0	0
4	0	0	0	0	41	0	81	0	75	0	35	0	0	0	0
5	0	0	0	0	0	22	0	36	0	21	0	0	0	0	0
6	0	0	0	0	0	0	7	0	7	0	0	0	0	0	0
7	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 9. $\widehat{N}_{[1,1,1],g,\beta,J_r}(T_{2,3})$ for the trefoil

β	1					2					3		
$g\backslash 2J_r$	-1	0	1	2	3	0	1	2	3	4	3	4	5
0	1	0	1	0	1	1	0	2	0	2	1	0	1
1	0	2	0	2	0	0	2	0	3	0	0	1	0
2	0	0	1	0	0	0	0	1	0	0	0	0	0

Table 10. $\widehat{N}_{[1],g,\beta,J_r}(T_{2,5})$

Torus links

β	-1	0
$g\backslash 2J_r$	-2	-1
0	1	1

Table 11. $\widehat{N}_{[1],[1],g,\beta,J_r}(T_{2,2})$ for the Hopf link

β	-1	0
$g\backslash 2J_r$	-3	-2
0	1	1

Table 12. $\widehat{N}_{[2],[1],g,\beta,J_r}(T_{2,2})$ for the Hopf link

β	-2			-1			0
$g\backslash 2J_r$	-7	-6	-5	-6	-5	-4	-3
0	1	0	2	1	0	3	1
1	0	1	0	0	1	0	0

Table 13. $\widehat{N}_{[2],[2],g,\beta,J_r}(T_{2,2})$ for the Hopf link

β	-2	-1
$g\backslash 2J_r$	-5	-4
0	1	1

Table 14. $\widehat{N}_{[2],[1,1],g,\beta,J_r}(T_{2,2})$ for the Hopf link

β	-1			0			1
$g\backslash 2J_r$	-4	-3	-2	-3	-2	-1	0
0	1	0	2	1	0	3	1
1	0	1	0	0	1	0	0

Table 15. $\widehat{N}_{[1],[1],g,\beta,J_r}(T_{2,4})$

β	-1					0					1		
$g\backslash 2J_r$	-7	-6	-5	-4	-3	-6	-5	-4	-3	-2	-3	-2	-1
0	1	0	2	0	3	1	0	3	0	5	1	0	2
1	0	2	0	3	0	0	2	0	4	0	0	1	0
2	0	0	1	0	0	0	0	1	0	0	0	0	0

Table 16. $\widehat{N}_{[2],[1],g,\beta,J_r}(T_{2,4})$

β	-1			0			1
$g\backslash 2J_r$	-5	-4	-3	-4	-3	-2	-1
0	1	0	1	1	0	2	1
1	0	1	0	0	1	0	0

Table 17. $\widehat{N}_{[1,1],[1],g,\beta,J_r}(T_{2,4})$

Non-torus knots

β	-2	-1			0			1
$g\backslash 2J_r$	-3	-2	-1	0	-1	0	1	2
0	1	2	1	1	1	1	2	1
1	0	0	1	0	0	1	0	0

Table 18. $\widehat{N}_{[1],g,\beta,J_r}(\mathbf{4_1})$ for the figure-eight knot

β	-3			-2					-1							0					1					2			3
$g\backslash 2J_r$	-8	-7	-6	-7	-6	-5	-4	-3	-6	-5	-4	-3	-2	-1	0	-3	-2	-1	0	1	0	1	2	3	4	3	4	5	6
0	1	0	1	2	0	4	1	2	1	0	5	2	5	1	1	2	1	4	2	3	1	1	3	0	1	1	0	2	1
1	0	1	0	0	3	0	3	0	0	2	0	6	1	2	0	0	3	1	3	0	0	1	0	1	0	0	1	0	0
2	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0

Table 19. $\widehat{N}_{[2],g,\beta,J_r}(\mathbf{4_1})$ for the figure-eight knot

β	-3	-2			-1					0					1							2					3		
$g\backslash 2J_r$	-6	-5	-4	-3	-4	-3	-2	-1	0	-1	0	1	2	3	0	1	2	3	4	5	6	3	4	5	6	7	6	7	8
0	1	2	0	1	1	0	3	1	1	3	2	4	1	2	1	1	5	2	5	0	1	2	1	4	0	2	1	0	1
1	0	0	1	0	0	1	0	1	0	0	3	1	3	0	0	2	1	6	0	2	0	0	3	0	3	0	0	1	0
2	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	1	0	0	0	0	0

Table 20. $\widehat{N}_{[1,1],g,\beta,J_r}(\mathbf{4_1})$ for the figure-eight knot

β	-4								
$g\backslash 2J_r$	-17	-16	-15	-14	-13	-12	-11	-10	-9
0	1	0	1	0	3	0	4	0	3
1	0	4	0	6	0	9	0	7	0
2	0	0	6	0	9	0	7	0	0
3	0	0	0	4	0	4	0	0	0
4	0	0	0	0	1	0	0	0	0
5	0	0	0	0	0	0	0	0	0

β	-3										
$g\backslash 2J_r$	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6
0	2	0	4	0	10	1	17	2	18	3	7
1	0	9	0	21	0	38	2	43	3	20	0
2	0	0	16	0	36	0	45	1	24	0	0
3	0	0	0	14	0	25	0	16	0	0	0
4	0	0	0	0	6	0	6	0	0	0	0
5	0	0	0	0	0	1	0	0	0	0	0

β	-2												
$g\backslash 2J_r$	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3
0	1	0	5	0	12	2	27	6	37	11	28	5	5
1	0	5	0	24	0	53	5	83	12	69	9	17	0
2	0	0	10	0	43	0	77	4	74	5	23	0	0
3	0	0	0	10	0	35	0	42	1	16	0	0	0
4	0	0	0	0	5	0	12	0	6	0	0	0	0
5	0	0	0	0	0	1	0	1	0	0	0	0	0

β	-1												
$g\backslash 2J_r$	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
0	2	0	6	1	19	6	34	14	41	14	19	2	1
1	0	9	0	28	3	65	14	82	20	46	5	4	0
2	0	0	16	0	45	3	72	9	48	4	6	0	0
3	0	0	0	14	0	30	1	26	1	4	0	0	0
4	0	0	0	0	6	0	7	0	1	0	0	0	0
5	0	0	0	0	0	1	0	0	0	0	0	0	0

β	0										
$g\backslash 2J_r$	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1
0	1	0	5	2	14	7	26	13	26	6	6
1	0	4	0	18	5	38	13	41	9	13	0
2	0	0	6	0	22	4	29	5	11	0	0
3	0	0	0	4	0	10	1	5	0	0	0
4	0	0	0	0	1	0	1	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0

β	1								
$g\backslash 2J_r$	-4	-3	-2	-1	0	1	2	3	4
0	2	1	6	4	15	6	11	1	2
1	0	5	2	12	4	13	1	3	0
2	0	0	4	1	5	0	1	0	0
3	0	0	0	1	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0

β	2						
$g\backslash 2J_r$	1	2	3	4	5	6	7
0	3	2	8	2	6	0	2
1	0	4	1	6	0	3	0
2	0	0	1	0	1	0	0
3	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0

β	3						
$g\backslash 2J_r$	4	5	6	7	8	9	10
0	2	1	6	0	5	0	1
1	0	3	0	6	0	2	0
2	0	0	1	0	1	0	0
3	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0

β	4					5		
$g\backslash 2J_r$	7	8	9	10	11	10	11	12
0	2	0	4	0	2	1	0	1
1	0	3	0	3	0	0	1	0
2	0	0	1	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0

Table 21. $\widehat{N}_{[3],g,\beta,J_r}(\mathbf{4_1})$ for the figure-eight knot

β	-4							-3								
$g\backslash 2J_r$	-15	-14	-13	-12	-11	-10	-9	-14	-13	-12	-11	-10	-9	-8	-7	-6
0	1	0	2	0	4	0	4	2	0	6	0	14	1	19	2	8
1	0	3	0	6	0	6	0	0	7	0	20	0	31	1	18	0
2	0	0	3	0	4	0	0	0	0	9	0	20	0	15	0	0
3	0	0	0	1	0	0	0	0	0	0	5	0	6	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0

β	-2										
$g\backslash 2J_r$	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3
0	1	0	6	0	17	2	33	7	29	5	7
1	0	4	0	21	0	47	3	53	5	17	0
2	0	0	6	0	25	0	36	1	15	0	0
3	0	0	0	4	0	11	0	6	0	0	0
4	0	0	0	0	1	0	1	0	0	0	0

β	-1										
$g\backslash 2J_r$	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
0	2	0	8	1	25	8	40	14	28	5	6
1	0	7	0	25	2	52	10	46	7	13	0
2	0	0	9	0	24	1	27	2	9	0	0
3	0	0	0	5	0	7	0	2	0	0	0
4	0	0	0	0	1	0	0	0	0	0	0

β	0										
$g\backslash 2J_r$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3
0	1	0	7	3	25	13	41	15	26	4	6
1	0	3	0	17	5	42	13	41	6	13	0
2	0	0	3	0	12	2	18	2	9	0	0
3	0	0	0	1	0	2	0	2	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0

β	1										
$g\backslash 2J_r$	-2	-1	0	1	2	3	4	5	6	7	8
0	6	4	26	15	41	13	25	3	7	0	1
1	0	13	6	41	13	42	5	17	0	3	0
2	0	0	9	2	18	2	12	0	3	0	0
3	0	0	0	2	0	2	0	1	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0

β	2										
$g\backslash 2J_r$	1	2	3	4	5	6	7	8	9	10	11
0	6	5	28	14	40	8	25	1	8	0	2
1	0	13	7	46	10	52	2	25	0	7	0
2	0	0	9	2	27	1	24	0	9	0	0
3	0	0	0	2	0	7	0	5	0	0	0
4	0	0	0	0	0	0	1	0	0	0	0

β	3										
$g\backslash 2J_r$	4	5	6	7	8	9	10	11	12	13	14
0	7	5	29	7	33	2	17	0	6	0	1
1	0	17	5	53	3	47	0	21	0	4	0
2	0	0	15	1	36	0	25	0	6	0	0
3	0	0	0	6	0	11	0	4	0	0	0
4	0	0	0	0	1	0	1	0	0	0	0

β	4									5						
$g\backslash 2J_r$	7	8	9	10	11	12	13	14	15	10	11	12	13	14	15	16
0	8	2	19	1	14	0	6	0	2	4	0	4	0	2	0	1
1	0	18	1	31	0	20	0	7	0	0	6	0	6	0	3	0
2	0	0	15	0	20	0	9	0	0	0	0	4	0	3	0	0
3	0	0	0	6	0	5	0	0	0	0	0	0	1	0	0	0
4	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0

Table 22. $\widehat{N}_{[2,1],g,\beta,J_r}(\mathbf{4_1})$ for the figure-eight knot

β	-4			-3					-2						
$g\backslash 2J_r$	-11	-10	-9	-10	-9	-8	-7	-6	-9	-8	-7	-6	-5	-4	-3
0	1	0	1	2	0	4	0	2	1	0	5	0	6	1	2
1	0	1	0	0	3	0	3	0	0	2	0	6	0	3	0
2	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

β	-1							0								
$g\backslash 2J_r$	-6	-5	-4	-3	-2	-1	0	-3	-2	-1	0	1	2	3	4	5
0	2	0	6	2	8	2	3	2	1	11	6	15	4	6	1	2
1	0	3	0	6	1	4	0	0	3	1	13	4	12	2	5	0
2	0	0	1	0	1	0	0	0	0	1	0	5	1	4	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

β	1										
$g\backslash 2J_r$	0	1	2	3	4	5	6	7	8	9	10
0	6	6	26	13	26	7	14	2	5	0	1
1	0	13	9	41	13	38	5	18	0	4	0
2	0	0	11	5	29	4	22	0	6	0	0
3	0	0	0	5	1	10	0	4	0	0	0
4	0	0	0	0	1	0	1	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0

β	2												
$g\backslash 2J_r$	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	2	19	14	41	14	34	6	19	1	6	0	2
1	0	4	5	46	20	82	14	65	3	28	0	9	0
2	0	0	6	4	48	9	72	3	45	0	16	0	0
3	0	0	0	4	1	26	1	30	0	14	0	0	0
4	0	0	0	0	1	0	7	0	6	0	0	0	0
5	0	0	0	0	0	0	0	1	0	0	0	0	0

β	3												
$g\backslash 2J_r$	4	5	6	7	8	9	10	11	12	13	14	15	16
0	5	5	28	11	37	6	27	2	12	0	5	0	1
1	0	17	9	69	12	83	5	53	0	24	0	5	0
2	0	0	23	5	74	4	77	0	43	0	10	0	0
3	0	0	0	16	1	42	0	35	0	10	0	0	0
4	0	0	0	0	6	0	12	0	5	0	0	0	0
5	0	0	0	0	0	1	0	1	0	0	0	0	0

β	4										
$g\backslash 2J_r$	7	8	9	10	11	12	13	14	15	16	17
0	7	3	18	2	17	1	10	0	4	0	2
1	0	20	3	43	2	38	0	21	0	9	0
2	0	0	24	1	45	0	36	0	16	0	0
3	0	0	0	16	0	25	0	14	0	0	0
4	0	0	0	0	6	0	6	0	0	0	0
5	0	0	0	0	0	1	0	0	0	0	0

β	5								
$g\backslash 2J_r$	10	11	12	13	14	15	16	17	18
0	3	0	4	0	3	0	1	0	1
1	0	7	0	9	0	6	0	4	0
2	0	0	7	0	9	0	6	0	0
3	0	0	0	4	0	4	0	0	0
4	0	0	0	0	1	0	0	0	0
5	0	0	0	0	0	0	0	0	0

Table 23. $\widehat{N}_{[1,1,1],g,\beta,J_r}(\mathbf{4_1})$ for the figure-eight knot

β	0			1				2			3
$g\backslash 2J_r$	-1	0	1	0	1	2	3	2	3	4	5
0	1	1	1	1	2	2	1	1	1	2	1
1	0	1	0	0	1	1	0	0	1	0	0

Table 24. $\widehat{N}_{[1],g,\beta,J_r}(\mathbf{5_2})$

β	1			2						3								
$g\backslash 2J_r$	0	1	2	1	2	3	4	5	6	2	3	4	5	6	7	8	9	10
0	1	1	1	2	4	5	4	3	1	1	5	9	13	12	7	4	2	1
1	0	1	0	0	3	4	4	2	0	0	2	9	13	13	9	5	3	0
2	0	0	0	0	0	1	1	0	0	0	0	1	5	5	4	3	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

β	4										5										
$g\backslash 2J_r$	4	5	6	7	8	9	10	11	12	13	6	7	8	9	10	11	12	13	14	15	16
0	2	7	14	19	14	14	6	5	1	2	2	5	14	11	18	6	9	2	5	0	1
1	0	5	15	20	28	14	17	3	7	0	0	6	9	29	12	29	5	18	0	4	0
2	0	0	4	12	9	17	3	9	0	0	0	0	7	5	24	4	22	0	6	0	0
3	0	0	0	1	5	1	5	0	0	0	0	0	0	4	1	10	0	4	0	0	0
4	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0

β	6									7						
$g\backslash 2J_r$	9	10	11	12	13	14	15	16	17	12	13	14	15	16	17	18
0	4	3	10	2	7	1	4	0	2	2	0	2	0	1	0	1
1	0	10	3	19	2	15	0	7	0	0	4	0	4	0	3	0
2	0	0	10	1	16	0	9	0	0	0	0	3	0	3	0	0
3	0	0	0	5	0	5	0	0	0	0	0	0	1	0	0	0
4	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0

Table 25. $\widehat{N}_{[2],g,\beta,J_r}(\mathbf{5_2})$

β	1				
$g\backslash 2J_r$	0	1	2	3	4
0	1	1	2	1	1
1	0	2	1	2	0
2	0	0	1	0	0
3	0	0	0	0	0
4	0	0	0	0	0
5	0	0	0	0	0

β	2							
$g\backslash 2J_r$	1	2	3	4	5	6	7	8
0	2	5	10	9	7	4	3	1
1	0	5	9	13	9	7	3	0
2	0	0	4	5	5	3	0	0
3	0	0	0	1	1	0	0	0
4	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0

β	3										
$g\backslash 2J_r$	2	3	4	5	6	7	8	9	10	11	12
0	1	7	16	22	21	17	13	7	4	2	1
1	0	3	17	29	35	32	22	14	7	4	0
2	0	0	3	15	20	21	16	9	6	0	0
3	0	0	0	1	6	6	5	4	0	0	0
4	0	0	0	0	0	1	1	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0

β	4											
$g\backslash 2J_r$	4	5	6	7	8	9	10	11	12	13	14	15
0	3	10	21	30	26	24	14	14	6	5	1	2
1	0	9	26	42	58	41	49	21	24	4	9	0
2	0	0	10	28	32	53	25	39	6	16	0	0
3	0	0	0	5	17	11	26	4	14	0	0	0
4	0	0	0	0	1	6	1	6	0	0	0	0
5	0	0	0	0	0	0	1	0	0	0	0	0

β	5												
$g\backslash 2J_r$	6	7	8	9	10	11	12	13	14	15	16	17	18
0	2	7	20	17	24	11	18	6	9	2	5	0	1
1	0	8	16	48	28	62	20	44	7	24	0	5	0
2	0	0	13	14	57	20	67	9	43	0	10	0	0
3	0	0	0	11	6	37	5	35	0	10	0	0	0
4	0	0	0	0	5	1	12	0	5	0	0	0	0
5	0	0	0	0	0	1	0	1	0	0	0	0	0

β	6										
$g\backslash 2J_r$	9	10	11	12	13	14	15	16	17	18	19
0	5	4	13	3	10	2	7	1	4	0	2
1	0	15	6	33	6	32	3	21	0	9	0
2	0	0	20	4	41	3	36	0	16	0	0
3	0	0	0	15	1	25	0	14	0	0	0
4	0	0	0	0	6	0	6	0	0	0	0
5	0	0	0	0	0	1	0	0	0	0	0

β	7								
$g\backslash 2J_r$	12	13	14	15	16	17	18	19	20
0	3	0	2	0	2	0	1	0	1
1	0	6	0	8	0	6	0	4	0
2	0	0	7	0	9	0	6	0	0
3	0	0	0	4	0	4	0	0	0
4	0	0	0	0	1	0	0	0	0
5	0	0	0	0	0	0	0	0	0

Table 26. $\widehat{N}_{[1,1],g,\beta,J_r}(\mathbf{5_2})$

β	-3	-2			-1				0			1
$g\backslash 2J_r$	-5	-4	-3	-2	-3	-2	-1	0	-1	0	1	2
0	1	2	1	1	1	2	3	1	1	2	2	1
1	0	0	1	0	0	1	1	0	0	1	0	0

Table 27. $\widehat{N}_{[1],g,\beta,J_r}(\mathbf{6_1})$

β	-5						
$g\backslash 2J_r$	-16	-15	-14	-13	-12	-11	-10
0	1	0	1	0	2	0	2
1	0	3	0	4	0	4	0
2	0	0	3	0	3	0	0
3	0	0	0	1	0	0	0
4	0	0	0	0	0	0	0

β	-4								
$g\backslash 2J_r$	-15	-14	-13	-12	-11	-10	-9	-8	-7
0	2	0	4	1	7	2	10	3	4
1	0	7	0	15	2	19	3	10	0
2	0	0	9	0	16	1	10	0	0
3	0	0	0	5	0	5	0	0	0
4	0	0	0	0	1	0	0	0	0

β	-3										
$g\backslash 2J_r$	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4
0	1	0	5	2	9	6	18	11	15	5	2
1	0	4	0	18	5	29	12	29	9	6	0
2	0	0	6	0	22	4	24	5	7	0	0
3	0	0	0	4	0	10	1	4	0	0	0
4	0	0	0	0	1	0	1	0	0	0	0

β	-2									
$g\backslash 2J_r$	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2
0	2	1	5	6	14	14	21	15	8	2
1	0	7	3	17	14	28	20	16	5	0
2	0	0	9	3	17	9	12	4	0	0
3	0	0	0	5	1	5	1	0	0	0
4	0	0	0	0	1	0	0	0	0	0

β	-1								
$g\backslash 2J_r$	-8	-7	-6	-5	-4	-3	-2	-1	0
0	1	2	4	7	13	15	11	7	1
1	0	3	5	9	13	14	10	2	0
2	0	0	3	4	5	5	1	0	0
3	0	0	0	1	1	0	0	0	0
4	0	0	0	0	0	0	0	0	0

β	0							1					2			3
$g\backslash 2J_r$	-4	-3	-2	-1	0	1	2	0	1	2	3	4	3	4	5	6
0	1	3	5	6	8	3	1	1	3	3	2	1	1	1	2	1
1	0	2	4	5	3	1	0	0	1	1	1	0	0	1	0	0
2	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 28. $\widehat{N}_{[2],g,\beta,J_r}(\mathbf{6_1})$

β	-5				
$g\backslash 2J_r$	-14	-13	-12	-11	-10
0	1	0	1	0	2
1	0	2	0	2	0
2	0	0	1	0	0
3	0	0	0	0	0

β	-4						
$g\backslash 2J_r$	-13	-12	-11	-10	-9	-8	-7
0	2	0	4	1	7	2	3
1	0	5	0	9	1	6	0
2	0	0	4	0	4	0	0
3	0	0	0	1	0	0	0

β	-3								
$g\backslash 2J_r$	-12	-11	-10	-9	-8	-7	-6	-5	-4
0	1	0	5	2	9	6	10	3	1
1	0	3	0	12	3	14	4	3	0
2	0	0	3	0	8	1	3	0	0
3	0	0	0	1	0	1	0	0	0

β	-2							
$g\backslash 2J_r$	-9	-8	-7	-6	-5	-4	-3	-2
0	2	1	5	6	11	8	5	2
1	0	5	2	10	7	7	3	0
2	0	0	4	1	4	1	0	0
3	0	0	0	1	0	0	0	0

β	-1							
$g\backslash 2J_r$	-6	-5	-4	-3	-2	-1	0	1
0	1	2	4	7	8	8	2	2
1	0	2	3	4	6	2	3	0
2	0	0	1	1	0	1	0	0
3	0	0	0	0	0	0	0	0

β	0						
$g\backslash 2J_r$	-2	-1	0	1	2	3	4
0	2	5	10	6	7	2	1
1	0	3	4	8	3	2	0
2	0	0	1	1	1	0	0
3	0	0	0	0	0	0	0

β	1							2					3		
$g\backslash 2J_r$	0	1	2	3	4	5	6	3	4	5	6	7	6	7	8
0	1	4	6	8	5	2	1	2	3	4	1	2	1	0	1
1	0	2	5	6	3	2	0	0	3	1	3	0	0	1	0
2	0	0	1	1	1	0	0	0	0	1	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 29. $\widehat{N}_{[1,1],g,\beta,J_r}(\mathbf{6_1})$

β	-2			-1					0					1		
$g\backslash 2J_r$	-4	-3	-2	-3	-2	-1	0	1	-2	-1	0	1	2	1	2	3
0	1	1	1	2	2	4	1	1	1	1	4	2	2	1	1	1
1	0	1	0	0	3	1	2	0	0	2	1	3	0	0	1	0
2	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0

Table 30. $\widehat{N}_{[1],g,\beta,J_r}(\mathbf{6_3})$

β	-3								
$g\backslash 2J_r$	-12	-11	-10	-9	-8	-7	-6	-5	-4
0	1	1	1	2	4	4	4	3	2
1	0	4	3	6	6	10	6	6	0
2	0	0	6	3	9	4	7	0	0
3	0	0	0	4	1	4	0	0	0
4	0	0	0	0	1	0	0	0	0
5	0	0	0	0	0	0	0	0	0

β	-2										
$g\backslash 2J_r$	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
0	2	2	4	6	12	13	19	16	15	5	3
1	0	9	7	21	20	40	29	40	14	12	0
2	0	0	16	9	36	20	44	14	19	0	0
3	0	0	0	14	5	25	6	15	0	0	0
4	0	0	0	0	6	1	6	0	0	0	0
5	0	0	0	0	0	1	0	0	0	0	0

β	-1												
$g\backslash 2J_r$	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2
0	1	1	5	6	13	15	31	28	36	19	17	3	2
1	0	5	4	24	21	53	42	80	41	51	9	8	0
2	0	0	10	6	43	25	73	32	61	10	13	0	0
3	0	0	0	10	4	35	11	38	5	11	0	0	0
4	0	0	0	0	5	1	12	1	5	0	0	0	0
5	0	0	0	0	0	1	0	1	0	0	0	0	0

β	0												
$g\backslash 2J_r$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
0	2	2	6	7	21	20	38	27	33	11	9	2	2
1	0	9	7	27	22	63	39	74	25	31	5	7	0
2	0	0	16	9	42	21	62	18	37	4	9	0	0
3	0	0	0	14	5	27	6	20	1	5	0	0	0
4	0	0	0	0	6	1	6	0	1	0	0	0	0
5	0	0	0	0	0	1	0	0	0	0	0	0	0

β	1												
$g\backslash 2J_r$	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
0	1	1	5	5	18	17	28	15	15	6	5	1	1
1	0	4	3	17	12	44	24	41	14	18	3	4	0
2	0	0	6	3	20	8	34	9	22	3	6	0	0
3	0	0	0	4	1	9	1	10	1	4	0	0	0
4	0	0	0	0	1	0	1	0	1	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0

β	2								
$g\backslash 2J_r$	1	2	3	4	5	6	7	8	9
0	3	4	10	9	11	6	4	2	2
1	0	9	8	21	12	15	5	7	0
2	0	0	10	5	16	4	9	0	0
3	0	0	0	5	1	5	0	0	0
4	0	0	0	0	1	0	0	0	0
5	0	0	0	0	0	0	0	0	0

β	3						
$g\backslash 2J_r$	4	5	6	7	8	9	10
0	1	2	3	2	1	1	1
1	0	3	3	4	2	3	0
2	0	0	3	1	3	0	0
3	0	0	0	1	0	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0

Table 31. $\widehat{N}_{[2],g,\beta,J_r}(\mathbf{6_3})$

β	-3						
$g\backslash 2J_r$	-10	-9	-8	-7	-6	-5	-4
0	1	1	1	2	3	2	1
1	0	3	2	4	3	3	0
2	0	0	3	1	3	0	0
3	0	0	0	1	0	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0

β	-2								
$g\backslash 2J_r$	-9	-8	-7	-6	-5	-4	-3	-2	-1
0	2	2	4	6	11	9	10	4	3
1	0	7	5	15	12	21	8	9	0
2	0	0	9	4	16	5	10	0	0
3	0	0	0	5	1	5	0	0	0
4	0	0	0	0	1	0	0	0	0
5	0	0	0	0	0	0	0	0	0

β	-1												
$g\backslash 2J_r$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4
0	1	1	5	6	15	15	28	17	18	5	5	1	1
1	0	4	3	18	14	41	24	44	12	17	3	4	0
2	0	0	6	3	22	9	34	8	20	3	6	0	0
3	0	0	0	4	1	10	1	9	1	4	0	0	0
4	0	0	0	0	1	0	1	0	1	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0

β	0												
$g\backslash 2J_r$	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0	2	2	9	11	33	27	38	20	21	7	6	2	2
1	0	7	5	31	25	74	39	63	22	27	7	9	0
2	0	0	9	4	37	18	62	21	42	9	16	0	0
3	0	0	0	5	1	20	6	27	5	14	0	0	0
4	0	0	0	0	1	0	6	1	6	0	0	0	0
5	0	0	0	0	0	0	0	1	0	0	0	0	0

β	1												
$g\backslash 2J_r$	-2	-1	0	1	2	3	4	5	6	7	8	9	10
0	2	3	17	19	36	28	31	15	13	6	5	1	1
1	0	8	9	51	41	80	42	53	21	24	4	5	0
2	0	0	13	10	61	32	73	25	43	6	10	0	0
3	0	0	0	11	5	38	11	35	4	10	0	0	0
4	0	0	0	0	5	1	12	1	5	0	0	0	0
5	0	0	0	0	0	1	0	1	0	0	0	0	0

β	2										
$g\backslash 2J_r$	1	2	3	4	5	6	7	8	9	10	11
0	3	5	15	16	19	13	12	6	4	2	2
1	0	12	14	40	29	40	20	21	7	9	0
2	0	0	19	14	44	20	36	9	16	0	0
3	0	0	0	15	6	25	5	14	0	0	0
4	0	0	0	0	6	1	6	0	0	0	0
5	0	0	0	0	0	1	0	0	0	0	0

β	3								
$g\backslash 2J_r$	4	5	6	7	8	9	10	11	12
0	2	3	4	4	4	2	1	1	1
1	0	6	6	10	6	6	3	4	0
2	0	0	7	4	9	3	6	0	0
3	0	0	0	4	1	4	0	0	0
4	0	0	0	0	1	0	0	0	0
5	0	0	0	0	0	0	0	0	0

Table 32. $\widehat{N}_{[1,1],g,\beta,J_r}(\mathbf{6_3})$

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