博士論文

A freeness criterion for spherical twists
(球面捻りに対する自由性判定法)

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A FREENESS CRITERION FOR SPHERICAL TWISTS

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Abstract. We introduce the notion of a complete collection of spherical objects in a triangulated category $\mathscr{D}$. We then show that the subgroup of the autoequivalence group $\mathrm{Auteq}(\mathscr{D})$ generated by the spherical twists along spherical objects in an essential and null-triangular collection admitting a complete partition of type $(m_1, \dots, m_\alpha)$ is isomorphic to $\mathbf{Z}^{m_1} * \cdots * \mathbf{Z}^{m_\alpha}$.

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1. Introduction

Dehn twists and spherical twists. Inspired by Kontsevich's homological mirror symmetry conjecture [Kon], Seidel and Thomas [ST] introduced *spherical objects* of triangulated categories and special kinds of autoequivalences of triangulated categories called *spherical twists* along spherical objects. They can be thought of as an algebraic analogue of Dehn twists along simple closed curves in a surface, or more generally, along Lagrangian spheres in a symplectic manifold.

As its origin implies, spherical twists share many properties with Dehn twists. For a simple closed curve c on a compact oriented surface S, τ_c will denote the

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Dehn twist along c. We will also use the same symbol τ_c to denote its isotopy class considered as an element of the mapping class group $\pi_0(\mathrm{Diff}^+(\mathrm{S}, \partial\mathrm{S}))$. For a pair of simple closed curves c_1, c_2, we denote by $i(c_1, c_2)$ the minimum of the geometric intersection number between the isotopy classes of c_1 and c_2. It is then well-known that if $i(c_1, c_2) = 0$ then τ_{c_1} and τ_{c_2} commute, i.e., $\tau_{c_1}\tau_{c_2} = \tau_{c_2}\tau_{c_1}$, and if $i(c_1, c_2) = 1$ then τ_{c_1} and τ_{c_2} satisfy the braid relation, i.e., $\tau_{c_1}\tau_{c_2}\tau_{c_1} = \tau_{c_2}\tau_{c_1}\tau_{c_2}$. Seidel and Thomas [ST] showed that similar properties hold for spherical twists by interpreting the dimension of the morphism space between two spherical objects as the intersection number (Proposition 4.1).

A result for $i(c_1, c_2) \geq 2$ is also classical and rediscovered by several authors, for example, by Ishida [Ish, Theorem 1.2]. It says that if $i(c_1, c_2) \geq 2$ then there are no relations between τ_{c_1} and τ_{c_2}, i.e., the subgroup of the mapping class group generated by τ_{c_1} and τ_{c_2} is isomorphic to the free group of rank 2. The corresponding result for spherical twists was obtained by Keating [Kea, Theorem 1.2] by adopting Ishida's proof in the categorical setting.

Freeness criterion for Dehn twists. In general, it is difficult to describe what the subgroup generated by Dehn twists along more than two simple closed curves. Even for the case $i(c_1, c_2) = 1$, τ_{c_1} and τ_{c_2} can have more relations other than the braid relation. However, it was noticed by Humphries [Hum, Theorem 2.1] that there are special sorts of collections of simple closed curves such that the subgroup generated by the Dehn twists along simple closed curves in the collection can be completely described.

Consider a collection $C = \{c_1, \ldots, c_m\}$ of essential simple closed curves on a compact oriented surface S. Humphries [Hum] introduced the notion of a *complete partition* for a partition $C_1, \ldots, C_\alpha$ of C (Definition 2.1). Then he showed that if a collection $C = \{c_1, \ldots, c_m\}$ has a complete partition and $c_1, \ldots, c_m$ do not bound a disk, then the subgroup of the mapping class group generated by $\tau_{c_1}, \ldots, \tau_{c_m}$ is isomorphic to $\mathbf{Z}^{m_1} * \cdots * \mathbf{Z}^{m_\alpha}$ where $*$ denotes the free product and m_μ is the number of elements in C_μ (Theorem 2.2).

There is a related result for spherical twists by Licata [Lic, Theorem 1.1]. For the homotopy categories of projective modules over the zigzag algebras associated to complete graphs with specific gradings, he showed that the subgroup generated by the spherical twists along the indecomposable projective modules is isomorphic to the free group of rank n where n is the number of vertices of the complete graph one started with.

Freeness criterion for spherical twists. In this paper, we will prove a theorem for spherical twists which can be considered as a categorical analogue of Humphries' theorem. For the proof, we shall translate and reformulate Humphries' argument into our categorical setting.

Let us state our main theorem precisely. Let $\mathscr{D}$ be an *enhanced triangulated category* with a *dg enhancement* $\mathscr{A}$. The notion of a complete partition for collections of simple closed curves is directly translated into that for collections of spherical objects (Definition 4.2). As additional assumptions, we will introduce the notions of an *essential* and a *null-triangular* collection of spherical objects (Definition 4.3). Due to a technical difficulty, we will also impose a formality assumption on a collection $\{E_1, \ldots, E_m\}$ of spherical objects throughout the paper, more precisely, that the dg algebra $\mathrm{Hom}_{\mathscr{A}}(E_1 \oplus \cdots \oplus E_m)$ is formal. Our main theorem then can be stated as follows.

Theorem (Theorem 4.4)**.** *Let $\mathscr{D}$ be an enhanced triangulated category with a dg enhancement $\mathscr{A}$. Let $C = \{E_1, \ldots, E_m\}$ be an essential and null-triangular collection of $d_{>1}$-spherical objects of $\mathscr{D}$. Assume that the collection C admits a complete partition of type $(m_1, \ldots, m_\alpha)$. We also assume that the dg algebra* $\mathrm{End}_{\mathscr{A}}(E_1 \oplus \cdots \oplus E_m)$ *is formal. Then the subgroup of* $\mathrm{Auteq}(\mathscr{D})$ *generated by the spherical twists $T_{E_1}, \ldots, T_{E_m}$ is isomorphic to $\mathbf{Z}^{m_1} * \cdots * \mathbf{Z}^{m_\alpha}$.*

Convention. All categories considered in this paper are assumed to be small and categories and functors are assumed to be $\mathbf{k}$-linear for a fixed algebraically closed field $\mathbf{k}$. The composition of two morphisms $\phi \in \mathrm{Hom}_{\mathscr{A}}(E_1, E_2)$ and $\psi \in \mathrm{Hom}_{\mathscr{A}}(E_2, E_3)$ will be denoted by $\psi \circ \phi \in \mathrm{Hom}_{\mathscr{A}}(E_1, E_3)$. Gradings for dg categories or graded vector spaces are $\mathbf{Z}$-gradings. The shift functor of a triangulated category is denoted by $[1]$ and its p times iteration by $[p]$. For a graded vector space V, we also denote its grading shift by $V[p]$. In particular, we can write $V = \bigoplus_{p \in \mathbf{Z}} V^p[-p]$ where V^p is the degree p part of V regarded as a graded vector space concentrated in degree 0.

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2. Freeness criterion for Dehn twists

In this section, we briefly review Humphries' freeness criterion for Dehn twists [Hum]. This will serve as a guideline for our categorical analogue of his result.

Let S be a compact oriented surface, with or without boundary and $C = \{c_1, \dots, c_m\}$ be a collection of pairwise non-isotopic essential simple closed curves on S. By a *partition* of C, we mean disjoint subsets $C_1, \dots, C_\alpha$ of C such that $\cup_{\mu=1}^{\alpha} C_\mu = C$.

Definition 2.1. A partition $C_1, \dots, C_\alpha$ of C is a *complete partition* if the following two conditions are satisfied:

(P_1) $i(c_i, c_j) = 0$ if $c_i, c_j \in C_\mu$ with $i \neq j$ for some μ;

(P_2) $i(c_i, c_j) \geq 2$ if $c_i \in C_\mu, c_j \in C_\nu$ with $\mu \neq \nu$.

Let $C_1, \dots, C_\alpha$ be a complete partition of a collection C and m_μ be the number of elements in C_μ. Renumbering the labels if necessary, we can assume $m_1 \leq \cdots \leq m_\alpha$. In this case, we say that the complete partition $C_1, \dots, C_\alpha$ is of *type* $(m_1, \dots, m_\alpha)$.

Theorem 2.2 (Humphries [Hum])**.** *Let $C = \{c_1, \dots, c_m\}$ be a collection of essential simple closed curves. Assume that the collection C admits a complete partition of type $(m_1, \dots, m_\alpha)$, and that no component of $S \setminus \cup_{i=1}^{m} c_i$ is a disk. Then the subgroup of $\pi_0(\mathrm{Diff}^+(\mathrm{S}, \partial\mathrm{S}))$ generated by the Dehn twists $\tau_{c_1}, \dots, \tau_{c_m}$ is isomorphic to $\mathbf{Z}^{m_1} * \cdots * \mathbf{Z}^{m_\alpha}$.*

Remark 2.3. The assumption that no component of $S \setminus \cup_{i=1}^{m} c_i$ is a disk is necessary especially for $m \geq 3$. It excludes cases such as $C = \{c_1, c_2, c_3 = \tau_{c_2} c_1\}$ which is often a complete collection but does not satisfy the theorem as $\tau_{c_3} \simeq \tau_{c_2} \tau_{c_1} \tau_{c_2}^{-1}$.

Here we shall illustrate the main idea of Humphries' proof via a simple example. This will help the reader to understand the idea of the categorical proof of our main theorem developed in later sections.

Consider the surface S with genus one and two boundary components. Let c_1 and c_2 be curves in S depicted as the red and blue curves in the left hand side of Figure 1. As $i(c_1, c_2) = 2$, the collection $C = \{c_1, c_2\}$ is a complete collection with a complete partition $C_1 = \{c_1\}, C_2 = \{c_2\}$.

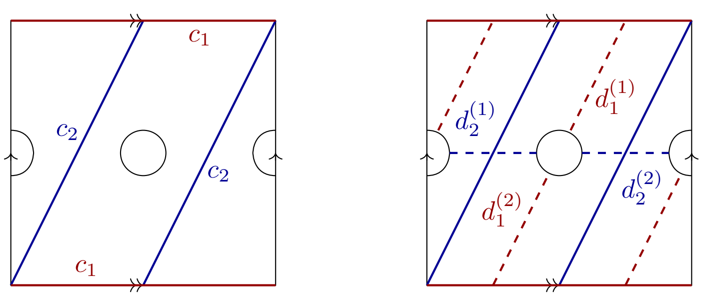

FIGURE 1

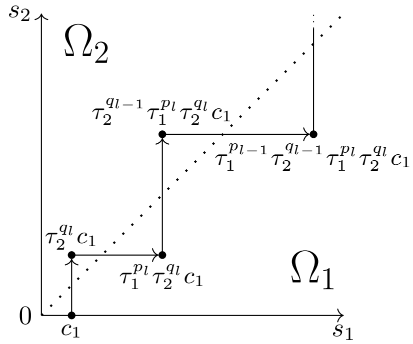

FIGURE 2

Now, since $S \setminus \cup_{i=1}^{2} c_i$ consists of two annuli, we can draw two arcs $d_i^{(1)}$ and $d_i^{(2)}$ on S, for each i, which start and end at the punctures and traverse c_i exactly once (see the right hand side of Figure 1). Using these arcs, we define

$$s_k(c) = \min\{i(c, d_k^{(1)}), i(c, d_k^{(2)})\}. \tag{2.1}$$

for a simple closed curve c.

The main ingredient in Humphries' proof is a set of inequalities involving the intersection numbers i and s_k's. Let us write $\tau_i = \tau_{c_i}$ for simplicity. For a given $k = 1, 2$, the inequalities can be written as

$$s_k(\tau_j^p c) = s_k(c) \tag{2.2}$$

for all $j \neq k$ and $p \in \mathbf{Z}$, and

$$s_k(\tau_k^p c) \geq i(c_k, c_j)s_j(c) - s_k(c) \tag{2.3}$$

for all $j \neq k$ and $p \in \mathbf{Z} \setminus \{0\}$.

Humphries' proof is completed by a ping-pong argument. First, define two sets Ω_i $(i = 1, 2)$ by

$$\Omega_i = \{c \mid s_i(c) > s_j(c) \text{ for all } j \neq i\}.$$

Note that they are disjoint and non-empty since $c_i \in \Omega_i$. Moreover, using the inequalities (2.2), (2.3) and the completeness assumption, one can easily show that if $c \in \Omega_i$ then τ_j^p $(j \neq i,\ p \in \mathbf{Z} \setminus \{0\})$ sends c into Ω_j. Now suppose there exists a non-trivial relation $\tau_1^{p_1}\tau_2^{q_1} \cdots \tau_1^{p_l}\tau_2^{q_l} \simeq \mathrm{id}$. Then $\tau_2^{q_1} \cdots \tau_1^{p_l}\tau_2^{q_l}c_1 \simeq c_1$. On the other hand, the above argument shows that $\tau_2^{q_1} \cdots \tau_1^{p_l}\tau_2^{q_l}c_1 \in \Omega_2$ while $c_1 \in \Omega_1$ which is a contradiction (see Figure 2).

Outline. Let us explain the outline of the rest of this paper. In Section 3, we briefly recall necessary notions from category theory such as dg categories and A_∞

categories. In Section 4, after a quick review of the theory of spherical objects and twists, we start to formulate the main theorem, a freeness criterion for spherical twists. It will be done by adjusting Humphries' notion of complete collections to our categorical setting. We also give a categorical interpretation of the no disk assumption in Theorem 2.2. The proof of our main theorem is given in Sections 5 and 6. Section 5 is devoted to the construction of objects which play the role of the arcs $d_1^{(j)}, d_2^{(j)}$ in Humphries' proof. More precisely, for a collection $\{E_1, \dots, E_m\}$ of spherical objects, we will construct a collection $\{S_1, \dots, S_m\}$ which, in some sense, can be considered as being orthogonal to the original collection. In Section 6, we first define some numbers ι and σ_k's which correspond to the intersection numbers i and s_k's in Humphries' argument. The collection $\{S_1, \dots, S_m\}$ constructed in Section 5 will be used in the definition of the numbers σ_k's. After that, we state and prove a set of inequalities which have the same form as the inequalities (2.2) and (2.3). Finally, in Section 7, we follow Humphries' ping-pong argument to complete the proof of the main theorem.

3. Preliminaries

Dg categories. Let us recall some notations and terminologies which we will use later. For details, we refer to [Kel] and [AL, Sections 2, 3 and 4].

A category $(\mathscr{A}, d_{\mathscr{A}})$ is called a *dg category* if every morphism space $\mathrm{Hom}_{\mathscr{A}}(E,F)$ is a dg $\mathbf{k}$-module with the differential $d_{\mathscr{A}}$ of degree 1 and the composition map $\mathrm{Hom}_{\mathscr{A}}(E_1,E_2) \otimes \mathrm{Hom}_{\mathscr{A}}(E_2,E_3) \to \mathrm{Hom}_{\mathscr{A}}(E_1,E_3)$ is a morphism of dg $\mathbf{k}$-modules. For a dg category $\mathscr{A}$, the *homotopy category* $H^0(\mathscr{A})$ of $\mathscr{A}$ has the same set of objects as that of $\mathscr{A}$ and the morphism spaces are given by $\mathrm{Hom}_{H^0(\mathscr{A})}(E,F) = H^0(\mathrm{Hom}_{\mathscr{A}}(E,F), d_{\mathscr{A}})$.

A *dg functor* between two dg categories $\mathscr{A}$ and $\mathscr{B}$ is a functor $\mathscr{F} : \mathscr{A} \to \mathscr{B}$ such that the maps $\mathscr{F}_{E,F} : \mathrm{Hom}_{\mathscr{A}}(E,F) \to \mathrm{Hom}_{\mathscr{B}}(\mathscr{F}E, \mathscr{F}F)$ are morphisms of dg $\mathbf{k}$-modules compatible with the composition maps and the units. A dg functor $\mathscr{F} : \mathscr{A} \to \mathscr{B}$ induces a functor $H^0(\mathscr{F}) : H^0(\mathscr{A}) \to H^0(\mathscr{B})$ between homotopy categories. We call a dg functor $\mathscr{F} : \mathscr{A} \to \mathscr{B}$ *quasi-fully faithful* if the induced functor $H^0(\mathscr{F})$ is fully faithful, and *quasi-essentially surjective* if $H^0(\mathscr{F})$ is essentially surjective. A dg functor $\mathscr{F} : \mathscr{A} \to \mathscr{B}$ is called a *quasi-equivalence* if it is quasi-fully faithful and quasi-essentially surjective.

For a pair of dg functors $\mathscr{F}, \mathscr{G} : \mathscr{A} \to \mathscr{B}$, there is also the notion of *dg natural transformations* from $\mathscr{F}$ to $\mathscr{G}$ which, together with dg functors from $\mathscr{A}$ to $\mathscr{B}$, form the dg category $\mathrm{dgFun}(\mathscr{A}, \mathscr{B})$ of dg functors. For details, see for instance [Kel].

Dg modules. Given two dg categories $\mathscr{A}$ and $\mathscr{B}$, denote by $\mathrm{dgFun}(\mathscr{A},\mathscr{B})$ the dg category of dg functors from $\mathscr{A}$ to $\mathscr{B}$. Let $\mathrm{dgMod}(\mathbf{k})$ be the dg category of dg $\mathbf{k}$-modules. The dg category of *(left) dg $\mathscr{A}$-modules* is defined by $\mathrm{dgMod}(\mathscr{A}) = \mathrm{dgFun}(\mathscr{A}^{\mathrm{op}}, \mathrm{dgMod}(\mathbf{k}))$. The homotopy category $H^0(\mathrm{dgMod}(\mathscr{A}))$ has a natural structure of a triangulated category.

Let $\mathrm{Ac}(\mathscr{A})$ be the full dg subcategory of $\mathrm{dgMod}(\mathscr{A})$ consisting of *acyclic* dg $\mathscr{A}$-modules. Its homotopy category $H^0(\mathrm{Ac}(\mathscr{A}))$ is a localizing subcategory of $H^0(\mathrm{dgMod}(\mathscr{A}))$. The *derived category* $D(\mathscr{A})$ of $\mathscr{A}$ is defined to be the Verdier quotient $H^0(\mathrm{dgMod}(\mathscr{A}))/H^0(\mathrm{Ac}(\mathscr{A}))$. Note that $D(\mathscr{A})$ is closed under countable direct sums because $H^0(\mathrm{dgMod}(\mathscr{A}))$ is closed under countable direct sums and $H^0(\mathrm{Ac}(\mathscr{A}))$ is localizing [BN, Lemma 1.5].

For an object F of a dg category $\mathscr{A}$, $h^F_{\mathscr{A}}(-) = \mathrm{Hom}_{\mathscr{A}}(-, F)$ can naturally be regarded as a dg $\mathscr{A}$-module. We call a dg $\mathscr{A}$-module isomorphic to $h^F_{\mathscr{A}}$ *representable*. The dg functor $h_{\mathscr{A}} : \mathscr{A} \to \mathrm{dgMod}(\mathscr{A})$ which sends F to $h^F_{\mathscr{A}}$ is called the *Yoneda dg functor*. In a similar way to the classical proof of the Yoneda lemma, one can show, for an object $F \in \mathrm{Ob}\,\mathscr{A}$ and a dg $\mathscr{A}$-module M, that $\mathrm{Hom}_{\mathrm{dgMod}(\mathscr{A})}(h^F_{\mathscr{A}}, M) \cong M(F)$ as dg $\mathbf{k}$-modules. In particular, the dg Yoneda functor $h_{\mathscr{A}}$ is quasi-fully faithful.

A dg category $\mathscr{A}$ is said to be *pretriangulated* if the essential image of the functor $H^0(h_{\mathscr{A}}) : H^0(\mathscr{A}) \to H^0(\mathrm{dgMod}(\mathscr{A}))$ is a full triangulated subcategory of $H^0(\mathrm{dgMod}(\mathscr{A}))$. Whenever $\mathscr{A}$ is pretriangulated dg category, we equip its homotopy category $H^0(\mathscr{A})$ with the structure of a triangulated category inherited from that of $H^0(\mathrm{dgMod}(\mathscr{A}))$.

Enhancements. Let $\mathscr{D}$ be a $\mathbf{k}$-linear triangulated category. A *dg enhancement* [BK] of $\mathscr{D}$ is a pair $(\mathscr{A}, \epsilon)$ of a pretriangulated dg category $\mathscr{A}$ and an equivalence $\epsilon : H^0(\mathscr{A}) \to \mathscr{D}$ of triangulated categories. If a $\mathbf{k}$-linear triangulated category $\mathscr{D}$ admits a dg enhancement $(\mathscr{A}, \epsilon)$, we say $\mathscr{D}$ is an *enhanced triangulated category*. Although it is not standard, we will consider enhanced triangulated categories up to quasi-equivalences of dg enhancements (cf. [AL, Section 4.1]). In other words, we regard an enhanced triangulated category as an isomorphism class of the objects of the homotopy category $\mathrm{Ho}(\mathrm{dgCat})$ of dg categories [Tab].

For example, the derived category $D(\mathscr{A})$ of a dg category $\mathscr{A}$ is an enhanced triangulated category. We can choose a dg enhancement for $D(\mathscr{A})$ as follows. A dg $\mathscr{A}$-module is called *free* if it is isomorphic to a direct sum of shifts of representable dg $\mathscr{A}$-modules. Moreover, a dg $\mathscr{A}$-module M is called *semi-free* if it admits a filtration $0 = F_0 \subset F_1 \subset \cdots \subset M$ such that every quotient F_{i+1}/F_i is free. Let

$\mathrm{SF}(\mathscr{A})$ be the full subcategory of $\mathrm{dgMod}(\mathscr{A})$ consisting of semi-free dg $\mathscr{A}$-modules. Then $D(\mathscr{A}) \simeq H^0(\mathrm{SF}(\mathscr{A}))$ [Dri, Section C.8].

For an enhanced triangulated category $\mathscr{D}$, we can define *functorial cones* in $\mathscr{D}$. Consider two exact functors $\mathscr{F}, \mathscr{F}' : \mathscr{D} \to \mathscr{D}$ and a natural transformation $\nu : \mathscr{F} \to \mathscr{F}'$. Assume that they lift to dg functors $\widetilde{\mathscr{F}}, \widetilde{\mathscr{F}'} : \mathscr{A} \to \mathscr{A}$ and a dg natural transformation $\widetilde{\nu} : \widetilde{\mathscr{F}} \to \widetilde{\mathscr{F}'}$. Then, as the dg category $\mathrm{dgFun}(\mathscr{A}, \mathscr{A})$ is pretriangulated and thus has functorial cones, we have a dg functor $\mathrm{Cone}(\widetilde{\nu}) : \mathscr{A} \to \mathscr{A}$ which gives rise to the exact functor $\mathrm{Cone}(\nu) : \mathscr{D} \to \mathscr{D}$. In particular, the cone fits into the exact triangle

$$\mathscr{F} \xrightarrow{\nu} \mathscr{F}' \to \mathrm{Cone}(\nu) \to \mathscr{F}[1].$$

Homotopy colimits. Since the derived category $D(\mathscr{A})$ of a dg category $\mathscr{A}$ is a triangulated category with countable direct sums, we can define the *homotopy colimit* [BN] $\operatorname{hocolim} E^{(n)} \in \operatorname{Ob} D(\mathscr{A})$ of a sequence of morphisms

$$E^{(0)} \xrightarrow{\phi^{(0)}} E^{(1)} \xrightarrow{\phi^{(1)}} \cdots \xrightarrow{\phi^{(n-1)}} E^{(n)} \xrightarrow{\phi^{(n)}} \cdots \tag{3.1}$$

in $D(\mathscr{A})$. It is defined by the exact triangle

$$\bigoplus_{n=0}^{\infty} E^{(n)} \xrightarrow{\mathrm{id}-\sigma} \bigoplus_{n=0}^{\infty} E^{(n)} \to \operatorname{hocolim} E^{(n)} \to \bigoplus_{n=0}^{\infty} E^{(n)}[1]$$

where the nth component of the morphism σ is given by the morphism $\phi^{(n)}$.

If a dg $\mathscr{A}$-module F is *compact*, i.e., if the functor $\mathrm{Hom}_{D(\mathscr{A})}(F, -)$ commutes with arbitrary direct sums, we have

$$\mathrm{Hom}_{D(\mathscr{A})}(F, \operatorname{hocolim} E_i^{(n)}) \cong \operatorname{colim} \mathrm{Hom}_{D(\mathscr{A})}(F, E_i^{(n)})$$

where the colimit in the right hand side is taken with respect to the sequence of morphisms

$$\mathrm{Hom}_{D(\mathscr{A})}(F, E^{(0)}) \to \mathrm{Hom}_{D(\mathscr{A})}(F, E^{(1)}) \to \cdots \to \mathrm{Hom}_{D(\mathscr{A})}(F, E^{(n)}) \to \cdots$$

obtained by applying the functor $\mathrm{Hom}_{D(\mathscr{A})}(F, -)$ to the sequence (3.1).

A_∞ **categories.** In the rest of this section, we recall the notion of A_∞ categories introduced by Fukaya [Fuk]. For details, see for exmaple [FOOO] or [Sei, Chapter I].

An A_∞ *category* $(\mathscr{A}, m_{\mathscr{A}})$ consists of a set $\operatorname{Ob} \mathscr{A}$ of objects, a graded $\mathbf{k}$-vector space $\mathrm{Hom}_{\mathscr{A}}(E, F)$ for every pair of objects $E, F \in \operatorname{Ob} \mathscr{A}$ and a set of linear maps $\{m_{\mathscr{A}}^n\}_{n \geq 1}$

$$m_{\mathscr{A}}^n : \mathrm{Hom}_{\mathscr{A}}(E_0, E_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{A}}(E_{n-1}, E_n) \to \mathrm{Hom}_{\mathscr{A}}(E_0, E_n)$$

of degree $2-n$ satisfying the A_∞ *relations*

$$\sum_{j+k=n+1}\sum_{i=0}^{k-1}(-1)^{\dagger_i}m^k_{\mathscr{A}}(\phi_1,\dots,\phi_i,m^j_{\mathscr{A}}(\phi_{i+1},\dots,\phi_{i+j}),\phi_{i+j+1},\dots,\phi_n)=0 \quad (3.2)$$

for all $n\geq 1$ where $\dagger_i=\deg\phi_1+\cdots\deg\phi_i+i$. An A_∞ category $\mathscr{A}$ is called *strictly unital* if there is a morphism $e_E\in\mathrm{Hom}^0_{\mathscr{A}}(E,E)$ for each $E\in\mathrm{Ob}\,\mathscr{A}$ such that $m^2_{\mathscr{A}}(e_E,\phi)=(-1)^{\deg\phi}m^2_{\mathscr{A}}(\phi,e_F)=\phi$ for all $\phi\in\mathrm{Hom}_{\mathscr{A}}(E,F)$ and $m^n_{\mathscr{A}}(\cdots,e_E,\cdots)=0$ for every $n\neq 2$.

The first three of the A_∞ relations (3.2) give $m^1_{\mathscr{A}}m^1_{\mathscr{A}}=0$, the Leibniz rule with respect to $m^1_{\mathscr{A}}$ and $m^2_{\mathscr{A}}$, and the associativity of $m^2_{\mathscr{A}}$ up to a homotopy given by $m^1_{\mathscr{A}}$ and $m^3_{\mathscr{A}}$. In particular, the morphism space $\mathrm{Hom}_{\mathscr{A}}(E,F)$ becomes a dg $\mathbf{k}$-module with the differential $m^1_{\mathscr{A}}$. Thus we can associate to A_∞ category the *cohomology category* $H^0(\mathscr{A})$ whose objects are the same as that of $\mathscr{A}$ and $\mathrm{Hom}_{H^0(\mathscr{A})}(E,F)=H^0(\mathrm{Hom}_{\mathscr{A}}(E,F),m^1_{\mathscr{A}})$. If $\mathscr{A}$ is strictly unital then its cohomology category $H^0(\mathscr{A})$ is a category in the classical sense.

An A_∞ category $\mathscr{A}$ is called *minimal* if $m^1_{\mathscr{A}}=0$. Note that a strictly unital A_∞ category $\mathscr{A}$ with $m^n_{\mathscr{A}}=0$ for every $n\geq 3$ can be regarded as a dg category by defining the differential by $d_{\mathscr{A}}(\phi)=(-1)^{\deg\phi}m^1_{\mathscr{A}}(\phi)$ and the composition by $\psi\circ\phi=(-1)^{\deg\phi(\deg\psi+1)}m^2_{\mathscr{A}}(\phi,\psi)$.

An A_∞ *functor* $\mathscr{F}:\mathscr{A}\to\mathscr{B}$ between two A_∞ categories $\mathscr{A}$ and $\mathscr{B}$ consists of a map $\mathscr{F}:\mathrm{Ob}\,\mathscr{A}\to\mathrm{Ob}\,\mathscr{B}$ and a set of linear maps $\{\mathscr{F}^n\}_{n\geq 1}$

$$\mathscr{F}^n:\mathrm{Hom}_{\mathscr{A}}(E_0,E_1)\otimes\cdots\otimes\mathrm{Hom}_{\mathscr{A}}(E_{n-1},E_n)\to\mathrm{Hom}_{\mathscr{B}}(\mathscr{F}E_0,\mathscr{F}E_n)$$

of degree $1-n$ satisfying the A_∞ *relations*

$$\begin{aligned}&\sum_{l\geq 1}\sum_{i_1+\cdots+i_l=n}m^l_{\mathscr{B}}(\mathscr{F}^{i_1}(\phi_1,\dots,\phi_{i_1}),\dots,\mathscr{F}^{i_l}(\phi_{n-i_l+1},\dots,\phi_n))\\&=\sum_{j+k=n+1}\sum_{i=0}^{k-1}(-1)^{\dagger_i}\mathscr{F}^k(\phi_1,\dots,\phi_i,m^j_{\mathscr{A}}(\phi_{i+1},\dots,\phi_{i+j}),\phi_{i+j+1},\dots,\phi_n)\end{aligned} \quad (3.3)$$

for all $n\geq 1$. An A_∞ functor $\mathscr{F}:\mathscr{A}\to\mathscr{B}$ between strictly unital A_∞ categories $\mathscr{A}$ and $\mathscr{B}$ is called *strictly unital* if $\mathscr{F}^1(e_E)=e_{\mathscr{F}E}$ for all $E\in\mathrm{Ob}\,\mathscr{A}$ and $\mathscr{F}^n(\cdots,e_E,\cdots)=0$ for every $n\geq 2$.

The A_∞ relation (3.3) for $n=1$ says that the map $\mathscr{F}^1:\mathrm{Hom}_{\mathscr{A}}(E,F)\to\mathrm{Hom}_{\mathscr{B}}(\mathscr{F}E,\mathscr{F}F)$ is a morphism of dg $\mathbf{k}$-modules. In particular, an A_∞ functor $\mathscr{F}:\mathscr{A}\to\mathscr{B}$ induces a functor $H^0(\mathscr{F}):H^0(\mathscr{A})\to H^0(\mathscr{B})$ between cohomology categories. It is a functor in the classical sense if $\mathscr{F}$ is strictly unital. We call an A_∞ functor $\mathscr{F}:\mathscr{A}\to\mathscr{B}$ *quasi-isomorphism* if the induced functor $H^0(\mathscr{F})$ is an isomorphism of categories.

Twisted complexes. The *additive enlargement* of an A_∞ category $\mathscr{A}$ is the A_∞ category $\Sigma\mathscr{A}$ whose objects are formal expressions $E = E_1[p_1] \oplus \cdots \oplus E_n[p_n]$ where $E_i \in \operatorname{Ob}\mathscr{A}$ and $p_i \in \mathbf{Z}$, whose morphism spaces are given by linearly extending $\operatorname{Hom}^r_{\Sigma\mathscr{A}}(E[p], F[q]) = \operatorname{Hom}^{r-p+q}_{\mathscr{A}}(E, F)$, and whose A_∞ products $\{m^n_{\Sigma\mathscr{A}}\}_{n\geq 1}$ are naturally induced from those of $\mathscr{A}$.

A *twisted complex* over an A_∞ category $\mathscr{A}$ is a pair (E, δ_E) of an object $E \in \operatorname{Ob}\Sigma\mathscr{A}$ and a morphism $\delta_E \in \operatorname{Hom}^1_{\Sigma\mathscr{A}}(E, E)$ satisfying

$$\sum_{n=1}^{\infty} m^n_{\Sigma\mathscr{A}}(\delta_E, \ldots, \delta_E) = 0. \tag{3.4}$$

In what follows, we will only consider *one-sided* twisted complexes, i.e., a twisted complex (E, δ_E) which admits a decomposition $E = E_1 \oplus \cdots \oplus E_n$ where $E_i \in \operatorname{Ob}\Sigma\mathscr{A}$ such that the corresponding decomposition $\delta_E^{ij} \in \operatorname{Hom}^1_{\Sigma\mathscr{A}}(E_i, E_j)$ of δ_E vanishes for every $i \geq j$. Note that the equation (3.4) makes sense for a one-sided twisted complex (E, δ_E) as the summation in the equation is finite in such a case.

The A_∞ category $\operatorname{Tw}\mathscr{A}$ of twisted complexes over $\mathscr{A}$ has twisted complexes over $\mathscr{A}$ as its objects and the morphism spaces are given by $\operatorname{Hom}_{\operatorname{Tw}\mathscr{A}}((E, \delta_E), (F, \delta_F)) = \operatorname{Hom}_{\Sigma\mathscr{A}}(E, F)$. The A_∞ products $\{m^n_{\operatorname{Tw}\mathscr{A}}\}_{n\geq 1}$of $\operatorname{Tw}\mathscr{A}$ are defined as follows

$$\begin{aligned} m^n_{\operatorname{Tw}\mathscr{A}}(\phi_1, \ldots, \phi_n) = \sum_{i_0, \ldots, i_n \geq 0} & m^{n+i_0+\cdots+i_n}_{\Sigma\mathscr{A}}(\overbrace{\delta_{E_0}, \ldots, \delta_{E_0}}^{i_0}, \phi_1, \\ & \underbrace{\delta_{E_1}, \ldots, \delta_{E_1}}_{i_1}, \phi_2, \ldots, \phi_n, \underbrace{\delta_{E_n}, \ldots, \delta_{E_n}}_{i_n}) \end{aligned}$$

where $\phi_i \in \operatorname{Hom}_{\operatorname{Tw}\mathscr{A}}((E_{i-1}, \delta_{E_{i-1}}), (E_i, \delta_{E_i}))$.

It is known that the cohomology category $H^0(\operatorname{Tw}\mathscr{A})$ has a natural structure of a triangulated category (cf. [Sei, Proposition 3.29]). In particular, for a closed morphism $\phi \in \operatorname{Hom}^0_{\operatorname{Tw}\mathscr{A}}((E, \delta_E), (F, \delta_F))$, the mapping cone $\operatorname{Cone}(\phi)$ of ϕ is a twisted complex given by

$$\operatorname{Cone}(\phi) = \left(E[1] \oplus F, \begin{pmatrix} -\delta_E & \phi \\ 0 & \delta_F \end{pmatrix} \right).$$

Together with naturally defined morphisms $F \to \operatorname{Cone}(\phi)$ and $\operatorname{Cone}(\phi) \to E[1]$, this gives an exact triangle in $H^0(\operatorname{Tw}\mathscr{A})$.

4. Freeness criterion for spherical twists

To state the main theorem precisely, we begin by recalling the definitions and basic properties of a spherical object and the twist along it.

Let $\mathscr{D}$ be an enhanced triangulated category. For $E, F \in \operatorname{Ob}\mathscr{D}$, we write $\operatorname{Hom}^p_{\mathscr{D}}(E, F) = \operatorname{Hom}_{\mathscr{D}}(E, F[p])$ and denote by $\operatorname{Hom}^\bullet_{\mathscr{D}}(E, F)$ the graded $\mathbf{k}$-vector spaces

$$\operatorname{Hom}^\bullet_{\mathscr{D}}(E,F)=\bigoplus_{p\in\mathbf{Z}}\operatorname{Hom}^p_{\mathscr{D}}(E,F)[-p].$$

Thus the degree p part of $\operatorname{Hom}^\bullet_{\mathscr{D}}(E,F)$ is $\operatorname{Hom}^p_{\mathscr{D}}(E,F)$.

An object $E\in\operatorname{Ob}\mathscr{D}$ is said to be *d-spherical* ($d>0$) if it satisfies the following two conditions:

(S_d) $\operatorname{Hom}^p_{\mathscr{D}}(E,E)\cong\mathbf{k}$ if and only if $p=0,d$, and vanishes otherwise;

(CY_d) there is a functorial isomorphism $\operatorname{Hom}^\bullet_{\mathscr{D}}(E,-)\cong\operatorname{Hom}^\bullet_{\mathscr{D}}(-,E[d])^\vee$.

Here $(-)^\vee$ is the graded $\mathbf{k}$-dual. Whenever a spherical object E is considered, we always assume that E is of *finite type*, i.e., $\dim\operatorname{Hom}^\bullet_{\mathscr{D}}(E,F)=\dim\operatorname{Hom}^\bullet_{\mathscr{D}}(F,E)<\infty$ for every $F\in\operatorname{Ob}\mathscr{D}$.

Let $E\in\operatorname{Ob}\mathscr{D}$ be an object of finite type. The exact functor $E\otimes\operatorname{Hom}^\bullet_{\mathscr{D}}(E,-):\mathscr{D}\to\mathscr{D}$ is well-defined and has a canonical dg lift [AL, Section 2.2]. Then taking the cone of the natural transformation $E\otimes\operatorname{Hom}^\bullet_{\mathscr{D}}(E,-)\to\operatorname{Id}_{\mathscr{D}}$ induced by the evaluation morphism, we obtain the *twist functor* $T_E:\mathscr{D}\to\mathscr{D}$ associated to E which fits into the exact triangle

$$E\otimes\operatorname{Hom}^\bullet_{\mathscr{D}}(E,-)\to\operatorname{Id}_{\mathscr{D}}\to T_E\to E\otimes\operatorname{Hom}^\bullet_{\mathscr{D}}(E,-)[1].$$

It was proved by Seidel and Thomas [ST, Proposition 2.10] that if $E\in\operatorname{Ob}\mathscr{D}$ is a d-spherical object then the associated twist functor T_E is an exact autoequivalence of $\mathscr{D}$. In this case, we call $T_E\in\operatorname{Auteq}(\mathscr{D})$ the *spherical twist* along E.

The following is one of the fundamental results for spherical twists which reveals the similarity between Dehn twists and spherical twists [ST, Propositions 2.12 and 2.13].

Proposition 4.1 (Seidel-Thomas [ST])**.** *Let $E_1,E_2\in\operatorname{Ob}\mathscr{D}$ be spherical objects.*

(1) $\dim\operatorname{Hom}^\bullet_{\mathscr{D}}(E_1,E_2)=0$ *then* $T_{E_1}T_{E_2}\cong T_{E_2}T_{E_1}$*;*

(2) $\dim\operatorname{Hom}^\bullet_{\mathscr{D}}(E_1,E_2)=1$ *then* $T_{E_1}T_{E_2}T_{E_1}\cong T_{E_2}T_{E_1}T_{E_2}$*.*

The notion of a complete partition for a collection of spherical objects can be directly translated from that for a collection of simple closed curves. We also introduce several definitions which will be used in the formulation of the main theorem.

Definition 4.2. Let $C=\{E_1,\ldots,E_m\}$ be a collection of d-spherical objects of $\mathscr{D}$. A *complete partition* of C is a partition $C_1,\ldots,C_\alpha$ of C satisfying the following properties:

(P_1) $\dim\operatorname{Hom}^\bullet_{\mathscr{D}}(E_i,E_j)=0$ if $E_i,E_j\in C_\mu$ with $i\neq j$ for some μ;

(P_2) $\dim\operatorname{Hom}^\bullet_{\mathscr{D}}(E_i,E_j)\geq 2$ if $E_i\in C_\mu,E_j\in C_\nu$ with $\mu\neq\nu$.

Let $C_1, \dots, C_\alpha$ be a complete partition of a collection C and m_μ be the number of elements in C_μ. Rearranging the labels if necessary, we can assume $m_1 \le \dots \le m_\alpha$. In this case, we say that the complete partition $C_1, \dots, C_\alpha$ is of *type* $(m_1, \dots, m_\alpha)$.

Definition 4.3. A collection $C = \{E_1, \dots, E_m\}$ of d-spherical objects of $\mathscr{D}$ is called

(E) *essential* if $E_i \ncong E_j$ in $\mathscr{D}$ up to shifts for every $i \neq j$;

(N) *null-triangular* if the composition map $\mathrm{Hom}^\bullet_{\mathscr{D}}(E_i, E_j) \otimes \mathrm{Hom}^\bullet_{\mathscr{D}}(E_j, E_k) \to \mathrm{Hom}^\bullet_{\mathscr{D}}(E_i, E_k)$ vanishes for every $i \neq j \neq k \neq i$.

The following is the main theorem of this paper.

Theorem 4.4. *Let $\mathscr{D}$ be an enhanced triangulated category with a dg enhancement $\mathscr{A}$. Let $C = \{E_1, \dots, E_m\}$ be an essential and null-triangular collection of $d_{>1}$-spherical objects of $\mathscr{D}$. Assume that the collection C admits a complete partition of type $(m_1, \dots, m_\alpha)$. We also assume that the dg algebra $\mathrm{End}_{\mathscr{A}}(E_1 \oplus \dots \oplus E_m)$ is formal. Then the subgroup of $\mathrm{Auteq}(\mathscr{D})$ generated by the spherical twists $T_{E_1}, \dots, T_{E_m}$ is isomorphic to $\mathbf{Z}^{m_1} * \dots * \mathbf{Z}^{m_\alpha}$.*

Remark 4.5. See Definition A.2 for the definition of the formality. The formality assumption extremely simplifies calculations in the proofs of Propositions 5.1 and 6.1. However, it seems it is a somewhat superfluous assumption and can be removed or relaxed in the future.

Remark 4.6. The condition that the collection C is null-triangular can be considered as a counterpart of the no disk assumption in Theorem 2.2. It excludes counterexamples like $C = \{E_1, E_2, E_3 = T_{E_2}E_1\}$ in which case we happen to have an unwanted relation $T_{E_3} \cong T_{E_2} T_{E_1} T_{E_2}^{-1}$ [ST, Lemma 2.11]. Indeed, applying Lemma B.1 to the 3-periodic long exact sequence

$$\cdots \to \mathrm{Hom}^\bullet_{\mathscr{D}}(E_1, E_2) \otimes \mathrm{Hom}^\bullet_{\mathscr{D}}(E_2, E_1) \to \mathrm{Hom}^\bullet_{\mathscr{D}}(E_1, E_1) \to \mathrm{Hom}^\bullet_{\mathscr{D}}(E_1, E_3) \to \cdots$$

we obtain $\dim \mathrm{Hom}^\bullet_{\mathscr{D}}(E_1, E_3) = (\dim \mathrm{Hom}^\bullet_{\mathscr{D}}(E_1, E_2))^2$. On the other hand, from another 3-periodic long exact sequence

$$\cdots \to \mathrm{Hom}^\bullet_{\mathscr{D}}(E_3, E_2) \otimes \mathrm{Hom}^\bullet_{\mathscr{D}}(E_2, E_1) \xrightarrow{\mu} \mathrm{Hom}^\bullet_{\mathscr{D}}(E_3, E_1) \to \mathrm{Hom}^\bullet_{\mathscr{D}}(E_3, E_3) \to \cdots$$

we have

$$\begin{aligned} \dim \mathrm{Im}\, \mu &= \frac{1}{2}((\dim \mathrm{Hom}^\bullet_{\mathscr{D}}(E_1, E_2))^2 + \dim \mathrm{Hom}^\bullet_{\mathscr{D}}(E_3, E_1) - 2) \\ &= (\dim \mathrm{Hom}^\bullet_{\mathscr{D}}(E_1, E_2))^2 - 1. \end{aligned}$$

In particular, if $\dim \mathrm{Hom}^\bullet_{\mathscr{D}}(E_1, E_2) \ge 2$, the composition map μ does not vanish. Hence the collection $C = \{E_1, E_2, E_3 = T_{E_2}E_1\}$ cannot be both complete and null-triangular.

5. Orthogonal collection

The first step towards the proof of Theorem 4.4 is to find appropriate counterparts of s_k's defined by the equation (2.1), which played an essential role in the proof of Theorem 2.2, in our categorical setting. For that, we first have to construct a set of objects $S_1, \ldots, S_m$ from given spherical objects $E_1, \ldots, E_m$ which has similar properties with the arcs $d_1^{(j)}, d_2^{(j)}$ constructed from the curves c_1, c_2 in Section 2. Note that a similar construction was already appeared in an algebraic context [Ric, Section 5].

Without loss of generality, we assume that $\mathscr{D} = H^0(\mathscr{A})$ for a pretriangulated dg category $\mathscr{A}$. Fix a collection $C = \{E_1, \ldots, E_m\}$ of spherical objects of $\mathscr{D}$. Composing the exact functor $H^0(h_{\mathscr{A}}) : \mathscr{D} = H^0(\mathscr{A}) \to H^0(\mathrm{dgMod}(\mathscr{A}))$ with the quotient functor $H^0(\mathrm{dgMod}(\mathscr{A})) \to D(\mathscr{A})$, we obtain an exact functor $\mathscr{Y} : \mathscr{D} \to D(\mathscr{A})$. Abusing notation, we will always use the same symbol E to denote the image of $E \in \mathrm{Ob}\,\mathscr{D}$ under the functor $\mathscr{Y}$ which is an object of $D(\mathscr{A})$. Note that, since every $E \in \mathrm{Ob}\,\mathscr{D}$ regarded as an object of $D(\mathscr{A})$ via $\mathscr{Y}$ is h-projective and the functor $H^0(h_{\mathscr{A}})$ is fully faithful, we have $\mathrm{Hom}_{D(\mathscr{A})}(E, F) \cong \mathrm{Hom}_{H^0(\mathrm{dgMod}(\mathscr{A}))}(E, F) \cong \mathrm{Hom}_{\mathscr{D}}(E, F)$ for every $F \in \mathrm{Ob}\,\mathscr{D}$.

Proposition 5.1. *Let $C = \{E_1, \ldots, E_m\}$ be an essential collection of d-spherical objects of $\mathscr{D}$. Moreover, assume that $\mathrm{End}_{\mathscr{A}}(E_1 \oplus \cdots \oplus E_m)$ is formal. Then there exists a collection $\{S_1, \ldots, S_m\}$ of objects of $D(\mathscr{A})$ such that*

$$\dim \mathrm{Hom}^p_{D(\mathscr{A})}(E_j, S_i) = \delta_{ij}\delta_{p0}$$

for all $i, j = 1, \ldots, m$ and $p \in \mathbf{Z}$.

Remark 5.2. Note that $S_1, \ldots, S_m$ are objects of the derived category $D(\mathscr{A})$ rather than the category $\mathscr{D}$ in which the spherical objects $E_1, \ldots, E_m$ live. It is because the category $\mathscr{D}$ is, in general, too small to carry a collection orthogonal to a given collection. More precisely, it stem from the fact that the category $\mathscr{D}$ does not always have countable direct sums.

Remark 5.3. As is mentioned in Remark 4.5, it seems the formality assumption is unnecessary, or can be replaced by another reasonable assumption. Indeed the proposition is still valid under some other formulations. For instance, one can drop the formality assumption and impose a new assumption that $\mathrm{Hom}^p_{\mathscr{D}}(E_i, E_j) = 0$ for every $p \leq 0$ and $i \neq j$ as in [Ric, Section 5].

In order to prove Proposition 5.1, we need the following two technical lemmas.

Lemma 5.4. *Let $(\mathscr{A}, m_{\mathscr{A}})$ be a minimal A_∞ category and $E_1, \dots, E_m$ be objects of $\mathscr{A}$ such that*

$$m^k_{\mathscr{A}} : \operatorname{Hom}_{\mathscr{A}}(E_{i_1}, E_{i_2}) \otimes \cdots \otimes \operatorname{Hom}_{\mathscr{A}}(E_{i_{k-1}}, E_{i_k}) \to \operatorname{Hom}_{\mathscr{A}}(E_{i_1}, E_{i_k})$$

vanish for all $i_1, \dots, i_k$ and $k \geq 3$. Consider an object $Z \in \operatorname{Ob} \Sigma\mathscr{A}$ which is a direct sum of shifts of $E_1, \dots, E_m$ and a twisted complex $(X, \delta_X) \in \operatorname{Ob} \operatorname{Tw} \mathscr{A}$ such that the underlying object of X is $X_1 \oplus \cdots \oplus X_n$ where each X_i is a direct sum of shifts of $E_1, \dots, E_m$ and the differential δ_X can be decomposed as $\delta^{ij}_X \in \operatorname{Hom}^1_{\Sigma\mathscr{A}}(X_i, X_j)$ with $i < j$. Let $\phi \in \operatorname{Hom}^0_{\operatorname{Tw}\mathscr{A}}((Z, 0), (X, \delta_X))$ be a closed morphism homotopic to a morphism given by $(\phi^1, 0, \dots, 0)$ for some morphism $\phi^1 \in \operatorname{Hom}^0_{\Sigma\mathscr{A}}(Z, X_1)$.

Now, for some $k = 1, \dots, m$ and $p \in \mathbf{Z}$, assume that for every closed morphism $\beta \in \operatorname{Hom}^p_{\operatorname{Tw}\mathscr{A}}((E_k, 0), (X, \delta_X))$ there exist morphisms $\eta \in \operatorname{Hom}^p_{\Sigma\mathscr{A}}(E_k, Z)$ and $\xi \in \operatorname{Hom}^{p-1}_{\operatorname{Tw}\mathscr{A}}((E_k, 0), (X, \delta_X))$ such that $\beta = m^2_{\operatorname{Tw}\mathscr{A}}(\eta, \phi) + m^1_{\operatorname{Tw}\mathscr{A}}(\xi)$. Then every closed morphism in $\operatorname{Hom}^p_{\operatorname{Tw}\mathscr{A}}((E_k, 0), (Z[1] \oplus X, \delta_{\operatorname{Cone}(\phi)}))$ is homotopic to a closed morphism $(\alpha, 0)$ where $\alpha \in \operatorname{Hom}^{p+1}_{\Sigma\mathscr{A}}(E_k, Z)$.

Proof. By the assumption, we can assume from the beginning that the morphism $\phi \in \operatorname{Hom}^0_{\operatorname{Tw}\mathscr{A}}((Z, 0), (X, \delta_X))$ has the unique component given by a morphism $\phi^1 \in \operatorname{Hom}^0_{\Sigma\mathscr{A}}(Z, X_1)$.

Every morphism in $\operatorname{Hom}^p_{\operatorname{Tw}\mathscr{A}}((E_k, 0), (Z[1] \oplus X, \delta_{\operatorname{Cone}(\phi)}))$ can be written as (α, β) where $\alpha \in \operatorname{Hom}^{p+1}_{\Sigma\mathscr{A}}(E_k, Z)$ and $\beta \in \operatorname{Hom}^p_{\Sigma\mathscr{A}}(E_k, X)$. Let us write the component of β to X_i by $\beta^i \in \operatorname{Hom}^p_{\Sigma\mathscr{A}}(E_k, X_i)$. Then the condition for this morphism to be closed is

$$\begin{aligned} 0 &= m^2_{\Sigma\mathscr{A}}(\alpha, \phi^1), \\ 0 &= \sum_{j=1}^{i-1} m^2_{\Sigma\mathscr{A}}(\beta^j, \delta^{ji}_X) + (\text{terms involving higher } m^k_{\mathscr{A}}\text{'s}) \\ &= \sum_{j=1}^{i-1} m^2_{\Sigma\mathscr{A}}(\beta^j, \delta^{ji}_X) \end{aligned} \tag{5.1}$$

for all $i = 2, \dots, n$. Note that the condition can be divided in this way because ϕ is given by a morphism in $\operatorname{Hom}^0_{\Sigma\mathscr{A}}(Z, X_1)$ and the differential δ_X does not contain a component to X_1.

The bottom condition in (5.1) shows that $\beta \in \operatorname{Hom}^p_{\operatorname{Tw}\mathscr{A}}((E_k, 0), (X, \delta_X))$ is closed. Therefore by the assumption, there exist morphisms $\eta \in \operatorname{Hom}^p_{\Sigma\mathscr{A}}(E_k, Z)$ and $\xi \in \operatorname{Hom}^{p-1}_{\operatorname{Tw}\mathscr{A}}((E_k, 0), (X, \delta_X))$ such that

$$\beta = m^2_{\operatorname{Tw}\mathscr{A}}(\eta, \phi) + m^1_{\operatorname{Tw}\mathscr{A}}(\xi).$$

Then for the morphism $(\eta, \xi) \in \mathrm{Hom}^{p-1}_{\mathrm{Tw}\,\mathscr{A}}((E_k, 0), (Z[1] \oplus X, \delta_{\mathrm{Cone}(\phi)}))$ we have

$$m^1_{\mathrm{Tw}\,\mathscr{A}}(\eta, \xi) = (0, m^2_{\mathrm{Tw}\,\mathscr{A}}(\eta, \phi) + m^1_{\mathrm{Tw}\,\mathscr{A}}(\xi)) = (0, \beta).$$

Hence the morphism (α, β) is homotopic to the morphism $(\alpha, 0)$. □

Lemma 5.5. *Let $(\mathscr{A}, m_{\mathscr{A}})$ be a minimal A_∞ category. Consider objects $E, Z \in \mathrm{Ob}\,\Sigma\mathscr{A}$ and $(X, \delta_X) \in \mathrm{Ob}\,\mathrm{Tw}\,\mathscr{A}$. Let $\phi \in \mathrm{Hom}^0_{\mathrm{Tw}\,\mathscr{A}}((Z, 0), (X, \delta_X))$ and $\epsilon \in \mathrm{Hom}^p_{\mathrm{Tw}\,\mathscr{A}}((E, 0), (X, \delta_X))$ be closed morphisms.*

Assume that the morphism $(0, \epsilon) \in \mathrm{Hom}^p_{\mathrm{Tw}\,\mathscr{A}}((E, 0), (Z[1] \oplus X, \delta_{\mathrm{Cone}(\phi)}))$ is homotopic to a morphism $(\alpha, 0)$ for some $\alpha \in \mathrm{Hom}^{p+1}_{\Sigma\mathscr{A}}(E_k, Z)$. Then there exist morphisms $\eta \in \mathrm{Hom}^p_{\Sigma\mathscr{A}}(E, Z)$ and $\xi \in \mathrm{Hom}^{p-1}_{\mathrm{Tw}\,\mathscr{A}}((E, 0), (X, \delta_X))$ such that $\epsilon = m^2_{\mathrm{Tw}\,\mathscr{A}}(\eta, \phi) + m^1_{\mathrm{Tw}\,\mathscr{A}}(\xi)$.

Proof. By the assumption, there is a morphism $(\eta, \xi) \in \mathrm{Hom}^{p-1}_{\mathrm{Tw}\,\mathscr{A}}((E, 0), (Z[1] \oplus X, \delta_{\mathrm{Cone}(\phi)}))$ where $\eta \in \mathrm{Hom}^p_{\Sigma\mathscr{A}}(E, Z)$ and $\xi \in \mathrm{Hom}^{p-1}_{\mathrm{Tw}\,\mathscr{A}}((E, 0), (Y, \delta_Y))$, such that

$$\begin{aligned} -\alpha &= m^1_{\Sigma\mathscr{A}}(\eta) = 0, \\ \epsilon &= m^2_{\mathrm{Tw}\,\mathscr{A}}(\eta, \phi) + m^1_{\mathrm{Tw}\,\mathscr{A}}(\xi). \end{aligned}$$

In particular, the bottom condition completes the proof. □

Proof of Proposition 5.1. For each i, we shall construct an object $S_i \in \mathrm{Ob}\,D(\mathscr{A})$ as the homotopy colimit of a sequence

$$E_i^{(0)} \overset{\phi_i^{(0)}}{\to} E_i^{(1)} \overset{\phi_i^{(1)}}{\to} \cdots \overset{\phi_i^{(n-1)}}{\to} E_i^{(n)} \overset{\phi_i^{(n)}}{\to} \cdots. \tag{5.2}$$

In Steps 1 and 2 below, we will inductively construct the sequence (5.2) and then prove the equality $\dim \mathrm{Hom}^p_{D(\mathscr{A})}(E_j, S_i) = \delta_{ij}\delta_{p0}$ in Step 3.

Step 1. First of all, let $E_i^{(0)} = E_i$ and

$$Z_i^{(0)} = E_i \otimes \mathrm{Hom}^\bullet_\circ(E_i, E_i^{(0)}) \oplus \bigoplus_{j \neq i} E_j \otimes \mathrm{Hom}^\bullet_{\mathscr{D}}(E_j, E_i^{(0)})$$

where $\mathrm{Hom}^\bullet_\circ(E_i, E_i^{(0)}) = \mathrm{Hom}^d_{\mathscr{D}}(E_i, E_i^{(0)})[-d]$. We then define an object $E_i^{(1)}$ and a morphism $\phi_i^{(0)} : E_i^{(0)} \to E_i^{(1)}$ by taking the cone of the evaluation morphism $Z_i^{(0)} \to E_i^{(0)}$ so that they fit into the exact triangle

$$Z_i^{(0)} \to E_i^{(0)} \overset{\phi_i^{(0)}}{\to} E_i^{(1)} \to Z_i^{(0)}[1].$$

By definition, the map the map $\mathrm{Hom}^p_{\mathscr{D}}(E_j, \phi_i^{(0)})$ is zero unless $j = i$ and $p = 0$. Moreover, by the essentialness, the map $\mathrm{Hom}^0_{\mathscr{D}}(E_i, \phi_i^{(0)})$ has a one-dimensional image which is spanned by $\phi_i^{(0)} \circ \mathrm{id}_{E_i}$ (see Lemma B.3 (2)).

Step 2. Suppose we have constructed a sequence

$$E_i^{(0)} \overset{\phi_i^{(0)}}{\to} E_i^{(1)} \overset{\phi_i^{(1)}}{\to} \cdots \overset{\phi_i^{(n-1)}}{\to} E_i^{(n)}$$

where, for each $l = 1, \dots, n$, the object $E_i^{(l)}$ is defined as the cone of the evaluation morphism $Z_i^{(l-1)} \to E_i^{(l-1)}$ from the object of the form

$$Z_i^{(l-1)} = E_i \otimes \mathrm{Hom}_\circ^\bullet(E_i, E_i^{(l-1)}) \oplus \bigoplus_{j \neq i} E_j \otimes \mathrm{Hom}_{\mathscr{D}}^\bullet(E_j, E_i^{(l-1)}).$$

Here $\mathrm{Hom}_\circ^\bullet(E_i, E_i^{(l-1)}) = \mathrm{Hom}_\circ^0(E_i, E_i^{(l-1)}) \oplus \bigoplus_{p \neq 0} \mathrm{Hom}_{\mathscr{D}}^p(E_i, E_i^{(l-1)})[-p]$ where $\mathrm{Hom}_\circ^0(E_i, E_i^{(l-1)})$ is a subspace of $\mathrm{Hom}_{\mathscr{D}}^0(E_i, E_i^{(l-1)})$ defined inductively. Each object $E_i^{(l)}$ thus fits into the exact triangle

$$Z_i^{(l-1)} \to E_i^{(l-1)} \xrightarrow{\phi_i^{(l-1)}} E_i^{(l)} \to Z_i^{(l-1)}[1].$$

We impose the following additional induction hypothesis:

($\mathrm{I}_1^{(n)}$) the map $\mathrm{Hom}_{\mathscr{D}}^p(E_j, \phi_i^{(l-1)}) : \mathrm{Hom}_{\mathscr{D}}^p(E_j, E_i^{(l-1)}) \to \mathrm{Hom}_{\mathscr{D}}^p(E_j, E_i^{(l)})$ is zero unless $j = i$ and $p = 0$, for every $l = 1, \dots, n$;

($\mathrm{I}_2^{(n)}$) the map $\mathrm{Hom}_{\mathscr{D}}^0(E_i, \phi_i^{(l-1)}) : \mathrm{Hom}_{\mathscr{D}}^0(E_i, E_i^{(l-1)}) \to \mathrm{Hom}_{\mathscr{D}}^0(E_i, E_i^{(l)})$ has a one-dimensional image which is spanned by $\phi_i^{(l-1)} \circ \cdots \circ \phi_i^{(0)} \circ \mathrm{id}_{E_i}$ for every $l = 1, \dots, n$;

($\mathrm{I}_3^{(n)}$) the object $E_i^{(l)}$ can be represented as a twisted complex whose underlying object is

$$Z_i^{(l-1)}[1] \oplus \cdots \oplus Z_i^{(0)}[1] \oplus E_i^{(0)}$$

and the differential is of the form

$$\begin{pmatrix} 0 & * & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & * \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Consider the object

$$Y_i^{(n)} = E_i \otimes \mathrm{Hom}_{\mathscr{D}}^{\bullet \neq 0}(E_i, E_i^{(n)}) \oplus \bigoplus_{j \neq i} E_j \otimes \mathrm{Hom}_{\mathscr{D}}^\bullet(E_j, E_i^{(n)})$$

where $\mathrm{Hom}_{\mathscr{D}}^{\bullet \neq 0}(E_i, E_i^{(n)}) = \bigoplus_{p \neq 0} \mathrm{Hom}_{\mathscr{D}}^p(E_i, E_i^{(n)})[-p]$.

Claim 5.a. *Under the induction hypotheses ($\mathrm{I}_1^{(n)}$), ($\mathrm{I}_2^{(n)}$) and ($\mathrm{I}_3^{(n)}$), the composition map $\mu_i^{(n)} : \mathrm{Hom}_{\mathscr{D}}^0(E_i, Y_i^{(n)}) \to \mathrm{Hom}_{\mathscr{D}}^0(E_i, E_i^{(n)})$, i.e., the map from*

$$\mathrm{Hom}_{\mathscr{D}}^d(E_i, E_i) \otimes \mathrm{Hom}_{\mathscr{D}}^{-d}(E_i, E_i^{(n)}) \oplus \bigoplus_{j \neq i, p \in \mathbf{Z}} \mathrm{Hom}_{\mathscr{D}}^{-p}(E_i, E_j) \otimes \mathrm{Hom}_{\mathscr{D}}^p(E_j, E_i^{(n)})$$

to $\mathrm{Hom}_{\mathscr{D}}^0(E_i, E_i^{(n)})$ given by the composition of morphisms, does not contain the morphism $\phi_i^{(n-1)} \circ \cdots \circ \phi_i^{(0)} \circ \mathrm{id}_{E_i}$.

Proof. In order to prove the claim, we should know what the compositions of morphisms are. In general, it is not so easy to compute the compositions of morphisms explicitly in a triangulated category because of the homotopy information lurking in it. Thus, instead of computing the compositions of morphisms directly in our triangulated category $\mathscr{D} = H^0(\mathscr{A})$, we will compute them in an A_∞ category $\mathrm{Tw}\,\widetilde{\mathscr{A}}$ quasi-isomorphic to the dg category $\mathscr{A}$.

Let us make it more precise. First, by perturbing the dg structure of the dg category $\mathscr{A}$ as in Appendix A, one can construct a minimal strictly unital A_∞ category $\widetilde{\mathscr{A}}$ with the same objects as those of $\mathscr{A}$ which is quasi-isomorphic to $\mathscr{A}$. Furthermore, by the assumption that $\mathrm{End}_{\mathscr{A}}(E_1 \oplus \cdots \oplus E_m)$ is formal, one sees that the A_∞ products

$$m^k_{\widetilde{\mathscr{A}}} : \mathrm{Hom}_{\widetilde{\mathscr{A}}}(E_{i_0}, E_{i_1}) \otimes \cdots \otimes \mathrm{Hom}_{\widetilde{\mathscr{A}}}(E_{i_{k-1}}, E_{i_k}) \to \mathrm{Hom}_{\widetilde{\mathscr{A}}}(E_{i_0}, E_{i_k}) \tag{5.3}$$

vanish for all $i_0, \ldots, i_k$ and $k \geq 3$.

By the induction hypotheses $(\mathrm{I}_1^{(n)})$ and $(\mathrm{I}_3^{(n)})$ for $l = n$ and Lemma 5.4, we see that a morphism in $\mathrm{Hom}^p_{\mathscr{D}}(E_j, E_i^{(n)})$ can be represented as a morphism

$$(\phi, 0, \ldots, 0) : (E_j, 0) \to (Z_i^{(n-1)}[1] \oplus \cdots \oplus Z_i^{(0)}[1] \oplus E_i^{(0)}, \delta_{E_i^{(n)}}) \tag{5.4}$$

between twisted complexes. Then one sees that the composition, i.e., $m^2_{\mathrm{Tw}\,\widetilde{\mathscr{A}}}$, of a morphism $\psi \in \mathrm{Hom}^{-p}_{\mathscr{D}}(E_i, E_j)$ with the morphism (5.4) in $\mathrm{Hom}^p_{\mathscr{D}}(E_j, E_i^{(n)})$ can be represented as a morphism between twisted complexes of the form

$$(*, 0, \ldots, 0) : (E_i, 0) \to (Z_i^{(n-1)}[1] \oplus \cdots \oplus Z_i^{(0)}[1] \oplus E_i^{(0)}, \delta_{E_i^{(n)}}) \tag{5.5}$$

between twisted complexes. Indeed, as the higher A_∞ products (5.3) vanish, the composition of those two morphisms is

$$\begin{aligned} m^2_{\mathrm{Tw}\,\widetilde{\mathscr{A}}}(\psi, (\phi, 0, \ldots, 0)) &= (m^2_{\Sigma\widetilde{\mathscr{A}}}(\psi, \phi), 0, \ldots, 0) + (\text{terms involving higher } m^k_{\widetilde{\mathscr{A}}}\text{'s}) \\ &= (m^2_{\Sigma\widetilde{\mathscr{A}}}(\psi, \phi), 0, \ldots, 0). \end{aligned}$$

A similar argument shows that the composition of a morphism in $\mathrm{Hom}^d_{\mathscr{D}}(E_i, E_i)$ with a morphism in $\mathrm{Hom}^{-d}_{\mathscr{D}}(E_i, E_i^{(n)})$ can be represented as a morphism also of the form (5.5). This implies that every morphism in the image of the map $\mu_i^{(n)}$ is represented by a morphism of the form (5.5).

By definition, the morphism $\phi_i^{(n-1)} \circ \cdots \circ \phi_i^{(0)} \circ \mathrm{id}_{E_i} \in \mathrm{Hom}^0_{\mathscr{D}}(E_i, E_i^{(n)})$ is represented by the morphism

$$(0, \ldots, 0, \mathrm{id}_{E_i}) : (E_i, 0) \to (Z_i^{(n-1)}[1] \oplus \cdots \oplus Z_i^{(0)}[1] \oplus E_i^{(0)}, \delta_{E_i^{(n)}})$$

between twisted complexes but, by the induction hypothesis $(\mathrm{I}_2^{(n)})$ for $l = n$ and Lemma 5.5, any morphisms of the form (5.5) cannot be homotopic to $\phi_i^{(n-1)} \circ$

$\cdots \circ \phi_i^{(0)} \circ \mathrm{id}_{E_i}$. This shows that the image of the map $\mu_i^{(n)}$ does not contain the morphism $\phi_i^{(n-1)} \circ \cdots \circ \phi_i^{(0)} \circ \mathrm{id}_{E_i}$. □

Consider a map from $\mathrm{Hom}^0_{\mathrm{Tw}\,\widetilde{\mathscr{A}}}((E_i, 0), (Z_i^{(n-1)}[1] \oplus \cdots \oplus Z_i^{(0)}[1] \oplus E_i^{(0)}, \delta_{E_i^{(n)}}))$ to $\mathrm{Hom}^0_{\mathrm{Tw}\,\widetilde{\mathscr{A}}}((E_i, 0), (Z_i^{(n-2)}[1] \oplus \cdots \oplus Z_i^{(0)}[1] \oplus E_i^{(0)}, \delta_{E_i^{(n-1)}}))$ which sends $(\alpha_1, \dots, \alpha_n)$ to $(\alpha_2, \dots, \alpha_n)$. Denote by $K_i^{(n)}$ the kernel of this map. We define $\mathrm{Hom}^0_\circ(E_i, E_i^{(n)})$ to be a subspace of $\mathrm{Hom}^0(E_i, E_i^{(n)})$ whose elements consist of morphisms which can be represented by closed morphisms in $K_i^{(n)}$. Note that $\mathrm{Hom}^0_\circ(E_i, E_i^{(n)})$ is a direct complement of the one-dimensional subspace spanned by $\phi_i^{(n-1)} \circ \cdots \circ \phi_i^{(0)} \circ \mathrm{id}_{E_i}$ in $\mathrm{Hom}^0_{\mathscr{D}}(E_i, E_i^{(n)})$ and contains the image of the map $\mu_i^{(n)}$.

Now we define

$$\begin{aligned} Z_i^{(n)} &= E_i \otimes \mathrm{Hom}^\bullet_\circ(E_i, E_i^{(n)}) \oplus \bigoplus_{j \neq i} E_j \otimes \mathrm{Hom}^\bullet_{\mathscr{D}}(E_j, E_i^{(n)}) \\ &= E_i \otimes \mathrm{Hom}^0_\circ(E_i, E_i^{(n)}) \oplus Y_i^{(n)} \end{aligned}$$

where $\mathrm{Hom}^\bullet_\circ(E_i, E_i^{(n)}) = \mathrm{Hom}^0_\circ(E_i, E_i^{(n)}) \oplus \bigoplus_{p \neq 0} \mathrm{Hom}^p_{\mathscr{D}}(E_i, E_i^{(n)})[-p]$. An object $E_i^{(n+1)}$ and a morphism $\phi_i^{(n)} : E_i^{(n)} \to E_i^{(n+1)}$ are then obtained by taking the cone of the evaluation morphism $Z_i^{(n)} \to E_i^{(n)}$ thus fitting into the exact triangle

$$Z_i^{(n)} \to E_i^{(n)} \xrightarrow{\phi_i^{(n)}} E_i^{(n+1)} \to Z_i^{(n)}[1].$$

Claim 5.b. *Under the induction hypotheses* $(\mathrm{I}_1^{(n)})$, $(\mathrm{I}_2^{(n)})$ *and* $(\mathrm{I}_3^{(n)})$, *the objects* $Z_i^{(n)}$, $E_i^{(n+1)}$ *and the morphism* $\phi_i^{(n)}$ *defined above satisfy the conditions* $(\mathrm{I}_1^{(n+1)})$, $(\mathrm{I}_2^{(n+1)})$ *and* $(\mathrm{I}_3^{(n+1)})$.

Proof. The condition $(\mathrm{I}_1^{(n+1)})$ is by definition. The condition $(\mathrm{I}_2^{(n+1)})$ holds since

$$\mathrm{Ker}\,\mathrm{Hom}^0_{\mathscr{D}}(E_i, \phi_i^{(n)}) = \mathrm{Hom}^0_\circ(E_i, E_i^{(n)}) + \mathrm{Im}\,\mu_i^{(n)} = \mathrm{Hom}^0_\circ(E_i, E_i^{(n)})$$

and hence the map $\mathrm{Hom}^0_{\mathscr{D}}(E_i, \phi_i^{(n)})$ has a one-dimensional image which is spanned by $\phi_i^{(n)} \circ \cdots \circ \phi_i^{(0)} \circ \mathrm{id}_{E_i}$. The condition $(\mathrm{I}_3^{(n+1)})$ follows from Lemma 5.4. □

Step 3. For every i, define an object $S_i \in \mathrm{Ob}\, D(\mathscr{A})$ to be the homotopy colimit of the sequence (5.2) constructed in Step 2. Since each $E_i \in \mathrm{Ob}\, D(\mathscr{A})$ is a representable, in particular compact, dg $\mathscr{A}$-module, we have an isomorphism

$$\begin{aligned} \mathrm{Hom}^p_{D(\mathscr{A})}(E_j, S_i) &= \mathrm{Hom}^p_{D(\mathscr{A})}(E_j, \mathrm{hocolim}\, E_i^{(n)}) \\ &\cong \mathrm{colim}\, \mathrm{Hom}^p_{D(\mathscr{A})}(E_j, E_i^{(n)}) \\ &\cong \mathrm{colim}\, \mathrm{Hom}^p_{\mathscr{D}}(E_j, E_i^{(n)}) \end{aligned}$$

as remarked in Section 3.

By the condition $(\mathrm{I}_1^{(n)})$, the maps in the sequence

$$\mathrm{Hom}^p_{\mathscr{D}}(E_j, E_i^{(0)}) \to \mathrm{Hom}^p_{\mathscr{D}}(E_j, E_i^{(1)}) \to \cdots \to \mathrm{Hom}^p_{\mathscr{D}}(E_j, E_i^{(n)}) \to \cdots$$

obtained by applying the functor $\mathrm{Hom}^p_{\mathscr{D}}(E_j, -)$ to the sequence (5.2) are all zeros unless $j = i$ and $p = 0$. This shows that

$$\mathrm{Hom}^p_{D(\mathscr{A})}(E_j, S_i) \cong \operatorname{colim} \mathrm{Hom}^p_{\mathscr{D}}(E_j, E_i^{(n)}) = 0$$

for all the cases except the case $j = i$ and $p = 0$.

For the case $j = i$ and $p = 0$, we consider the sequence

$$\mathrm{Hom}^0_{\mathscr{D}}(E_i, E_i^{(0)}) \to \mathrm{Hom}^0_{\mathscr{D}}(E_i, E_i^{(1)}) \to \cdots \to \mathrm{Hom}^0_{\mathscr{D}}(E_i, E_i^{(n)}) \to \cdots$$

obtained by applying the functor $\mathrm{Hom}^0_{\mathscr{D}}(E_i, -)$ to the sequence (5.2). Then by the condition $(\mathrm{I}_2^{(n)})$, we see that

$$\mathrm{Hom}^0_{D(\mathscr{A})}(E_i, S_i) \cong \operatorname{colim} \mathrm{Hom}^0_{\mathscr{D}}(E_i, E_i^{(n)}) \cong \mathrm{Hom}^0_{\mathscr{D}}(E_i, E_i^{(0)})$$

and therefore $\dim \mathrm{Hom}^0_{D(\mathscr{A})}(E_i, S_i) = 1$. □

6. Geometric inequalities

Let us define some categorically defined numbers ι and σ_k's which corresponds to the intersection numbers i and s_k's in Section 2 respectively.

Again let $C = \{E_1, \ldots, E_m\}$ be an essential collection of d-spherical objects of an enhanced triangulated category $\mathscr{D} = H^0(\mathscr{A})$ such that $\mathrm{End}_{\mathscr{A}}(E_1 \oplus \cdots \oplus E_m)$ is formal. By comparing Proposition 4.1 with the corresponding result for Dehn twists, we find that the number ι should be defined as

$$\iota(E, F) = \dim \mathrm{Hom}^{\bullet}_{\mathscr{D}}(E, F)$$

for $E, F \in \mathrm{Ob}\,\mathscr{D}$. On the other hand, we have constructed in Proposition 5.1 a collection $\{S_1, \ldots, S_m\}$ of objects in $D(\mathscr{A})$ orthogonal to the collection $C = \{E_1, \ldots, E_m\}$. It enables us to define the number σ_k, in a way similar to the definition of s_k, as

$$\sigma_k(F) = \dim \mathrm{Hom}^{\bullet}_{D(\mathscr{A})}(F, S_k)$$

for $F \in \mathrm{Ob}\,\mathscr{D}$ and $k = 1, \ldots, m$. In the rest of this section, we will show that they satisfy inequalities of the same form as (2.2) and (2.3).

For notational simplicity, let us denote by $T_k = T_{E_k}$ the spherical twist along E_k. Moreover, we define Ω_C to be the set of objects in the smallest triangulated category of $\mathscr{D}$ containing $E_1, \ldots, E_m$. For example, every object obtained by iteratively applying $T_1, \ldots, T_m$ to E_i is contained in Ω_C.

Proposition 6.1. *Let $C = \{E_1, \dots, E_m\}$ be an essential and null-triangular collection of d-spherical objects of $\mathscr{D}$. Assume also that $\operatorname{End}_{\mathscr{A}}(E_1 \oplus \cdots \oplus E_m)$ is formal. Then for any $F \in \Omega_C$ and $k = 1, \dots, m$, the following inequalities hold:*

(1) $\sigma_k(T_j^p F) = \sigma_k(F)$ *for all* $j \neq k$ *and* $p \in \mathbf{Z}$;

(2) $\sigma_k(T_k^p F) \geq \iota(E_k, E_j)\sigma_j(F) - \sigma_k(F)$ *for all* $j \neq k$ *and* $p \in \mathbf{Z} \setminus \{0\}$.

Remark 6.2. Observe that the number $\sigma_k(F)$ might be infinite for some $F \in \operatorname{Ob} \mathscr{D}$. However, by Proposition 5.1, it is finite for the objects in Ω_C, hence the inequalities in the statement make sense.

Also note that the proposition asserts that the inequalities hold for any choice of a collection $\{S_1, \dots, S_m\}$ obtained in Proposition 5.1.

Proof of Proposition 6.1. (1) By the definition of the twist functor, we have an exact triangle

$$E_j \otimes \operatorname{Hom}^\bullet_{\mathscr{D}}(E_j, F) \to F \to T_j F \to E_j \otimes \operatorname{Hom}^\bullet_{\mathscr{D}}(E_j, F)[1].$$

Applying the functor $\operatorname{Hom}_{D(\mathscr{A})}(-, S_k)$ to this triangle, we obtain a 3-periodic long exact sequence

$$\cdots \to \operatorname{Hom}^\bullet(T_j F, S_k) \to \operatorname{Hom}^\bullet(F, S_k) \to \operatorname{Hom}^\bullet(E_j \otimes \operatorname{Hom}^\bullet_{\mathscr{D}}(E_j, F), S_k) \to \cdots \tag{6.1}$$

of $\mathbf{k}$-vector spaces.

Then, as $j \neq k$, we have $\operatorname{Hom}^\bullet_{D(\mathscr{A})}(E_j \otimes \operatorname{Hom}^\bullet_{\mathscr{D}}(E_j, F), S_k) = 0$ by Proposition 5.1. From the long exact sequence (6.1), we get an isomorphism

$$\operatorname{Hom}^\bullet_{D(\mathscr{A})}(T_j F, S_k) \cong \operatorname{Hom}^\bullet_{D(\mathscr{A})}(F, S_k)$$

and hence $\sigma_k(T_j F) = \sigma_k(F)$. It then follows inductively that $\sigma_k(T_j^p F) = \sigma_k(F)$ for every $p \in \mathbf{Z}$.

(2) In Step 1, we describe the iterated twist $T_k^p F$ in terms of a certain twisted complex $C_k^p F$ following [Kea, Definition 7.1]. Using this expression, we prove the inequality (6.6). Then, in Step 2, we complete the proof by showing the inequality (6.7).

Step 1. Note that we only need to show the assertion for $p \in \mathbf{Z}_{>0}$ by symmetry.

Let us perturb the dg structure of the dg category $\mathscr{S} = \operatorname{SF}(\mathscr{A})$, as in Step 2 in the proof of Proposition 5.1 (also see Appendix A), to obtain a minimal strictly unital A_∞ category $\widetilde{\mathscr{S}}$ with the same objects as those of $\mathscr{S}$ which is quasi-isomorphic to $\mathscr{S}$ and whose A_∞ products

$$m^k_{\widetilde{\mathscr{S}}} \colon \operatorname{Hom}_{\widetilde{\mathscr{S}}}(E_{i_0}, E_{i_1}) \otimes \cdots \otimes \operatorname{Hom}_{\widetilde{\mathscr{S}}}(E_{i_{k-1}}, E_{i_k}) \to \operatorname{Hom}_{\widetilde{\mathscr{S}}}(E_{i_0}, E_{i_k}) \tag{6.2}$$

vanish for all $i_0, \dots, i_k$ and $k \geq 3$.

Consider an indecomposable object $F \in \Omega_C$ which is not isomorphic to E_k up to shifts. For every $p \in \mathbf{Z}_{>0}$, we can define a twisted complex $C_k^p F$ so that $T_k^p F$ fits into the exact triangle

$$C_k^p F \xrightarrow{\epsilon} F \to T_k^p F \to C_k^p F[1]. \tag{6.3}$$

According to [Kea, Definition 7.1], the twisted complex $C_k^p F$ can be explicitly written as

$$\begin{gathered}(E_k \otimes (\mathrm{Hom}^d_{\widetilde{\mathscr{F}}}(E_k, E_k))^{\otimes(p-1)} \otimes \mathrm{Hom}_{\widetilde{\mathscr{F}}}(E_k, F)[p-1]) \\ \oplus(E_k \otimes (\mathrm{Hom}^d_{\widetilde{\mathscr{F}}}(E_k, E_k))^{\otimes(p-2)} \otimes \mathrm{Hom}_{\widetilde{\mathscr{F}}}(E_k, F)[p-2]) \\ \cdots \\ \oplus(E_k \otimes \mathrm{Hom}^d_{\widetilde{\mathscr{F}}}(E_k, E_k) \otimes \mathrm{Hom}_{\widetilde{\mathscr{F}}}(E_k, F)[1]) \\ \oplus(E_k \otimes \mathrm{Hom}_{\widetilde{\mathscr{F}}}(E_k, F))\end{gathered}$$

with the differential acting on $E_k \otimes (\mathrm{Hom}^d_{\widetilde{\mathscr{F}}}(E_k, E_k))^{\otimes l} \otimes \mathrm{Hom}_{\widetilde{\mathscr{F}}}(E_k, F)[l]$ by

$$-\,\mathrm{ev} \otimes \mathrm{id}^{\otimes l} \pm \sum_{s \geq 2} \mathrm{id}_{E_k} \otimes \mathrm{id}^{\otimes(l-1-s)} \otimes m^s_{\widetilde{\mathscr{F}}}. \tag{6.4}$$

Moreover, the morphism $\epsilon : C_k^p F \to F$ in the exact triangle (6.3) is given by a morphism ${}^t(0, \ldots, 0, \mathrm{ev})$ between twisted complexes where $\mathrm{ev} : E_k \otimes \mathrm{Hom}_{\widetilde{\mathscr{F}}}(E_k, F) \to F$ is the evaluation morphism.

In our case, the terms involving $m^s_{\widetilde{\mathscr{F}}}$ $(s \geq 3)$ in (6.4) vanish by the formality assumption, and the term involving $m^2_{\widetilde{\mathscr{F}}}$ in (6.4) vanishes since F is not isomorphic to E_k up to shifts (see Lemma B.2). Therefore, fixing a basis x_k of $\mathrm{Hom}^d_{\widetilde{\mathscr{F}}}(E_k, E_k)$, we can simply write $C_k^p F$ as

$$\begin{gathered}(E_k \otimes \mathrm{Hom}_{\widetilde{\mathscr{F}}}(E_k, F)[(p-1)(1-d)]) \\ \oplus(E_k \otimes \mathrm{Hom}_{\widetilde{\mathscr{F}}}(E_k, F)[(p-2)(1-d)]) \\ \cdots \\ \oplus(E_k \otimes \mathrm{Hom}_{\widetilde{\mathscr{F}}}(E_k, F)[1-d]) \\ \oplus(E_k \otimes \mathrm{Hom}_{\widetilde{\mathscr{F}}}(E_k, F))\end{gathered}$$

with the differential

$$\begin{pmatrix} 0 & -x_k \otimes \mathrm{id} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & -x_k \otimes \mathrm{id} \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Moreover, by defining another twisted complex $B^p_k F$ to be

$$E_k[(p-1)(1-d)] \oplus E_k[(p-2)(1-d)] \oplus \cdots \oplus E_k[1-d] \oplus E_k$$

with the differential

$$\begin{pmatrix} 0 & -x_k & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & -x_k \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix},$$

we can also write $C^p_k F$ as $B^p_k F \otimes \mathrm{Hom}_{\mathscr{F}}(E_k, F)$.

Now, applying the functor $\mathrm{Hom}_{D(\mathscr{A})}(-, S_k)$ to the exact triangle (6.3), we obtain a 3-periodic long exact sequence

$$\cdots \to \mathrm{Hom}^\bullet(T^p_k F, S_k) \to \mathrm{Hom}^\bullet(F, S_k) \xrightarrow{\eta^{(k)}_{T^p_k F}} \mathrm{Hom}^\bullet(C^p_k F, S_k) \to \cdots \tag{6.5}$$

of $\mathbf{k}$-vector spaces. Then applying Lemma B.1 to the long exact sequence (6.5), we get

$$\sigma_k(T^p_k F) = \dim \mathrm{Hom}^\bullet_{D(\mathscr{A})}(C^p_k F, S_k) + 2 \dim \mathrm{Ker}\, \eta^{(k)}_{T^p_k F} - \sigma_k(F).$$

Claim 6.a. *For every $p \in \mathbf{Z}_{>0}$ and an indecomposable object $F \in \Omega_C$ which is not isomorphic to E_k up to shifts, we have the following inequalities:*

1. $\dim \mathrm{Hom}^\bullet_{D(\mathscr{A})}(C^p_k F, S_k) \geq \iota(E_k, F)$*;*
2. $\dim \mathrm{Ker}\, \eta^{(k)}_{T^p_k F} = \dim \mathrm{Ker}\, \rho^{(k)}_F$ *where* $\rho^{(k)}_F = \eta^{(k)}_{T_k F}$*.*

To prove Claim 6.a, we first show the following claim.

Claim 6.b. *Every closed and non-exact morphism in $\mathrm{Hom}^0_{\mathrm{Tw}\,\mathscr{F}}(E_k, S_k)$ does not factor through morphisms in $\mathrm{Hom}_{\mathscr{F}}(E_k, E_j)$ $(j \neq k)$.*

Proof. Assume that there is a closed and non-exact morphism in $\mathrm{Hom}^0_{\mathrm{Tw}\,\mathscr{F}}(E_k, S_k)$ which factors through a morphism in $\mathrm{Hom}_{\mathscr{F}}(E_k, E_j)$ $(j \neq k)$. Then it implies that, for some i, the morphism $\phi^{(i-1)}_k \circ \cdots \circ \phi^{(0)}_k \circ \mathrm{id}_{E_k} \in \mathrm{Hom}^0_{\mathscr{F}}(E_k, E^{(i)}_k)$ factors through a morphism in $\mathrm{Hom}_{\mathscr{F}}(E_k, E_j)$ $(j \neq k)$. In particular, it implies that $\phi^{(i-1)}_k \circ \cdots \circ \phi^{(0)}_k \circ \mathrm{id}_{E_k}$ is exact. This contradicts the construction in the proof of Proposition 5.1. □

Proof of Claim 6.a. (1) Since $C^p_k F = B^p_k F \otimes \mathrm{Hom}_{\mathscr{F}}(E_k, F)$, we have

$$\dim \mathrm{Hom}^\bullet_{D(\mathscr{A})}(C^p_k F, S_k) = \dim \mathrm{Hom}^\bullet_{D(\mathscr{A})}(B^p_k F, S_k) \cdot \iota(E_k, F).$$

Thus the inequality follows if we show that $\mathrm{Hom}^0_{D(\mathscr{A})}(B^p_k F, S_k) \neq 0$.

For every $l = 1, \dots, p-1$, there is an exact triangle

$$E_k[l(1-d)-1] \to B_k^l F \to B_k^{l+1} F \to E_k[l(1-d)].$$

Applying $\mathrm{Hom}_{D(\mathscr{A})}(-, S_k)$ to this triangle, we obtain an exact sequence

$$\cdots \to \mathrm{Hom}^0(B_k^{l+1}F, S_k) \to \mathrm{Hom}^0(B_k^l F, S_k) \to \mathrm{Hom}^1(E_k[l(1-d)], S_k) \to \cdots .$$

By Proposition 5.1, we have $\mathrm{Hom}^1_{D(\mathscr{A})}(E_k[l(1-d)], S_k) = 0$ and hence the map $\mathrm{Hom}^0_{D(\mathscr{A})}(B_k^{l+1}F, S_k) \to \mathrm{Hom}^0_{D(\mathscr{A})}(B_k^l F, S_k)$ is surjective. Therefore we obtain a sequence of surjective maps

$$\mathrm{Hom}^0(B_k^p F, S_k) \to \mathrm{Hom}^0(B_k^{p-1}F, S_k) \to \cdots \to \mathrm{Hom}^0(B_k^1 F, S_k) = \mathrm{Hom}^0(E_k, S_k).$$

Again by Proposition 5.1, $\mathrm{Hom}^0_{D(\mathscr{A})}(E_k, S_k) \neq 0$. Thus $\mathrm{Hom}^0_{D(\mathscr{A})}(B_k^p F, S_k) \neq 0$ as desired.

(2) Let $\phi \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(F, S_k)$ be a closed morphism such that $m^2_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\mathrm{ev}, \phi)$ becomes exact, where $\mathrm{ev} : E_k \otimes \mathrm{Hom}_{\widetilde{\mathscr{S}}}(E_k, F) \to F$ is the evaluation morphism. Then there exists a morphism $\psi^0 \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(E_k \otimes \mathrm{Hom}_{\widetilde{\mathscr{S}}}(E_k, F), S_k)$ such that $m^1_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\psi^0) = m^2_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\mathrm{ev}, \phi)$.

Recall that $\epsilon \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(C_k^p F, F)$ is a morphism given by ${}^t(0, \dots, 0, \mathrm{ev})$. Since the higher A_∞ products (6.2) vanish, we have

$$m^2_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\epsilon, \phi) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m^2_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\mathrm{ev}, \phi) \end{pmatrix}.$$

On the other hand, the closed morphism $m^2_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(x_k \otimes \mathrm{id}, \psi^0) \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(E_k \otimes \mathrm{Hom}_{\widetilde{\mathscr{S}}}(E_k, F), S_k)$ is an exact morphism by Claim 6.b. Thus we find a morphism $\psi^1 \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(E_k \otimes \mathrm{Hom}_{\widetilde{\mathscr{S}}}(E_k, F), S_k)$ such that $m^1_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\psi^1) = m^2_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(x_k \otimes \mathrm{id}, \psi^0)$. Repeating this process, we obtain a sequence of morphisms $\psi^1, \dots, \psi^{p-1} \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(E_k \otimes \mathrm{Hom}_{\widetilde{\mathscr{S}}}(E_k, F), S_k)$ satisfying $m^1_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\psi^i) = m^2_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(x_k \otimes \mathrm{id}, \psi^{i-1})$ for all $i = 1, \dots, p-1$. Therefore if we define a morphism $\psi \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(C_k^p F, S_k)$

by ${}^t(\psi^{p-1}, \dots, \psi^1, \psi^0)$, it satisfies

$$m^1_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\psi) = \begin{pmatrix} m^1_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\psi^{p-1}) - m^2_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(x_k \otimes \mathrm{id}, \psi^{p-2}) \\ \vdots \\ m^1_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\psi^1) - m^2_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(x_k \otimes \mathrm{id}, \psi^0) \\ m^1_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\psi^0) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m^2_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\mathrm{ev}, \phi) \end{pmatrix} = m^2_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\epsilon, \phi).$$

In particular, $m^2_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\epsilon, \phi)$ is exact. □

Consequently, we have

$$\sigma_k(T_k^p F) \geq \iota(E_k, F) + 2 \dim \operatorname{Ker} \rho_F^{(k)} - \sigma_k(F) \tag{6.6}$$

for every indecomposable object $F \in \Omega_C$ not isomorphic to a shift of E_k and $p \in \mathbf{Z}_{>0}$. Note that the inequality (6.6) is also true for a shift of E_k and thus for every object in Ω_C.

Step 2. It is left to prove the inequality

$$\iota(E_k, F) + 2 \dim \operatorname{Ker} \rho_F^{(k)} \geq \iota(E_k, E_j)\sigma_j(F) \tag{6.7}$$

holds for every $F \in \Omega_C$ and $j \neq k$. In the rest of the proof, we fix $j \neq k$.

Consider an object $F \in \Omega_C$. We will regard it as a twisted complex over $\widetilde{\mathscr{S}}$ whose underlying object is a finite direct sum of shifts of $E_1, \dots, E_m$. Moreover, fixing a direct sum decomposition of the underlying object of F, we will write it as

$$F = \bigoplus_{i=1}^{m} \bigoplus_{r=1}^{r_i} E_{i,r}$$

where $r_i \in \mathbf{Z}_{\geq 0}$ and each $E_{i,r}$ is a shift of E_i. By $\iota_{i,r} \in \mathrm{Hom}^0_{\Sigma\widetilde{\mathscr{S}}}(E_{i,r}, F)$ and $\pi_{i,r} \in \mathrm{Hom}^0_{\Sigma\widetilde{\mathscr{S}}}(F, E_{i,r})$, we denote the inclusion and the projection morphism respectively.

The differential of this twisted complex will be denoted by $\delta_F \in \mathrm{Hom}^1_{\Sigma\widetilde{\mathscr{S}}}(F, F)$ and the component of δ_F from $E_{i,r}$ to $E_{i',r'}$ by $\delta_F|_{E_{i,r} \to E_{i',r'}} \in \mathrm{Hom}^1_{\widetilde{\mathscr{S}}}(E_{i,r}, E_{i',r'})$. Similarly, we will denote the component of $\alpha \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(F, S_l)$ from $E_{i,r}$ by $\alpha|_{E_{i,r} \to S_l} \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(E_{i,r}, S_l)$ and the component of $\phi \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(E_l, F)$ to $E_{i,r}$ by $\phi|_{E_l \to E_{i,r}} \in \mathrm{Hom}_{\widetilde{\mathscr{S}}}(E_l, E_{i,r})$.

Claim 6.c. *Any closed morphism $\alpha \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(F, S_l)$ such that every component $\alpha_{i,r} = \alpha|_{E_{i,r} \to S_l} \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(E_{i,r}, S_l)$ is exact is an exact morphism.*

Proof. For each i and r, choose a morphism $\psi^{(1)}_{i,r} \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(E_{i,r}, S_l)$ such that $m^1_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(\psi^{(1)}_{i,r}) = \alpha_{i,r}$. Let $\psi^{(1)} \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(F, S_l)$ be the morphism whose components are given by $\psi^{(1)}_{i,r}$. Then every component of $m^1_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(\psi^{(1)}) = m^2_{\Sigma\widetilde{\mathscr{F}}}(\delta_F, \psi^{(1)})$ is again exact. Indeed, in $m^2_{\Sigma\widetilde{\mathscr{F}}}(\delta_F, \psi^{(1)})$, the terms of the form $m^2_{\Sigma\widetilde{\mathscr{F}}}(\mathrm{id}_{E_i}, \psi^{(1)}_{i,r})$ up to a constant cancel out because of the closedness of the morphism ϕ and the other terms are exact by Claim 6.b.

Let $\beta^{(1)} = \alpha - m^1_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(\psi^{(1)})$ and $\beta^{(1)}_{i,r} = \beta^{(1)}|_{E_{i,r}\to S_l} \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(E_{i,r}, S_l)$. Take a morphism $\psi^{(2)}_{i,r} \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(E_{i,r}, S_l)$ so that $m^1_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(\psi^{(2)}_{i,r}) = \beta^{(1)}_{i,r}$ for each i and r. Then, for the morphism $\psi^{(2)} \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(F, S_l)$ whose components are given by $\psi^{(2)}_{i,r}$, one sees that every component of $m^1_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(\psi^{(2)})$ is exact in exactly the same way as before.

We perform the same processes successively. Then we obtain a sequence of morphisms $\psi^{(1)}, \ldots, \psi^{(n)} \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(F, S_l)$ such that

$$m^1_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(\psi^{(s+1)}|_{E_{i,r}\to S_l}) = (\alpha - m^1_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(\psi^{(1)}) - \cdots - m^1_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(\psi^{(s)}))|_{E_{i,r}\to S_l}$$

for every i, r and $s = 1, \ldots, n-1$. Since F is a finite direct sum and δ_F is one-sided, this process must stop. Namely, for a sufficiently large n, we have

$$\alpha - m^1_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(\psi^{(1)}) - \cdots - m^1_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(\psi^{(n)}) = 0.$$

This shows that the morphism $\alpha \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(F, S_l)$ is exact. $\square$

Fix a basis $x_i \in \mathrm{Hom}^d_{\widetilde{\mathscr{F}}}(E_i, E_i)$ for each $i = 1, \ldots, m$. For a direct summand $E_{i,r}$ of F, we consider the following condition:

(H) there exist $c_1, \ldots, c_{r_i} \in \mathbf{k}$ such that the morphism

$$\xi = \sum_{s=1}^{r_i} c_s m^2_{\Sigma\widetilde{\mathscr{F}}}(x_i, \iota_{i,s}) \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(E_i, F)$$

satisfies $m^1_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(\xi) = m^2_{\Sigma\widetilde{\mathscr{F}}}(x_i, \iota_{i,r})$.

If a direct summand $E_{i,r}$ satisfies the condition (H), we fix a morphism ξ in the condition (H). Let us denote it by $\xi^H_{i,r} \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(E_i, F)$.

Claim 6.d. *For a direct summand $E_{i,r}$ which does not satisfy the condition* (H)*, there exists a closed morphism $\alpha \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(F, S_i)$ such that the component $\alpha|_{E_{i,r}\to S_i} \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(E_{i,r}, S_i)$ is closed and non-exact.*

Proof. Fix a closed and non-exact morphism $\alpha^0 \in \mathrm{Hom}^0_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(E_i, S_i)$. If there is no direct summand $E_{i,s}$ in F such that $\delta_F|_{E_{i,s}\to E_{i,r}} = \mathrm{id}_{E_i}$ up to a constant, we put $\alpha' = m^2_{\Sigma\widetilde{\mathscr{F}}}(\pi_{i,r}, \alpha^0)$. Then every component of $m^1_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(\alpha')$ is exact by Claim 6.b, and as in the proof of Claim 6.c, we find a morphism $\psi \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{F}}}(F, S_i)$ such that $m^1_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\alpha') = m^1_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\psi)$ and $\psi|_{E_{i,r}\to S_i} = 0$. Then $\alpha = \alpha' - \psi$ satisfies the desired properties.

In general, there is a zigzag of identity morphisms in F

$$E_{i,r} \xleftarrow{A_1} E_i^{\oplus s_1} \xrightarrow{B_1} E_i^{\oplus s_1'} \xleftarrow{A_2} E_i^{\oplus s_2} \xrightarrow{B_2} \cdots \xleftarrow{A_n} E_i^{\oplus s_n} \xrightarrow{B_n} E_i^{\oplus s_n'}$$

where each E_i is one of $E_{i,1}, \dots, E_{i,r_i}$, A_l is an $s_l \times s_{l-1}'$ matrix ($s_0' = 1$) and B_l is an $s_l \times s_l'$ matrix. Entries of the matrices are regarded as constant multiples of the identity morphism id_{E_i}. Now since $E_{i,r}$ does not satisfy the condition (H), there is no solution of the system of linear equations

$$\begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} A_1 & B_1 & 0 & \cdots & 0 \\ 0 & A_2 & B_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_n & B_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}.$$

This in particular implies that

$$\mathrm{rk} \begin{pmatrix} A_1 & B_1 & 0 & \cdots & 0 \\ 0 & A_2 & B_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_n & B_n \end{pmatrix} = \mathrm{rk} \begin{pmatrix} 0 & B_1 & 0 & \cdots & 0 \\ 0 & A_2 & B_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_n & B_n \end{pmatrix}.$$

Thus there are $w_1, \dots, w_n$ satisfying $B_1 w_1 = A_1, A_2 w_1 + B_2 w_2 = 0, \dots, A_n w_{n-1} + B_n w_n = 0$. Joining the morphism $\alpha^0 \in \mathrm{Hom}^0_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(E_i, S_i)$ to $E_i^{\oplus s_l}$ according to w_l, we obtain a morphism $\beta \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(F, S_i)$. Then, for $\alpha' = m^2_{\Sigma\widetilde{\mathscr{S}}}(\pi_{i,r}, \alpha^0) - \beta$, every component of $m^1_{\Sigma\widetilde{\mathscr{S}}}(\alpha')$ is exact. As before, there exists a morphism $\psi \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(F, S_i)$ such that $m^1_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\alpha') = m^1_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\psi)$ and $\psi|_{E_{i,r}\to S_i} = 0$. Then define $\alpha = \alpha' - \psi$. $\square$

Let $\sigma = \sigma_j(F) = \dim \mathrm{Hom}^\bullet_{D(\mathscr{A})}(F, S_j)$. For every direct summand $E_{j,r}$ which does not satisfy the condition (H), collect all the closed and non-exact morphisms $\alpha_r^{(1)}, \dots, \alpha_r^{(s_r)} \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(F, S_j)$ constructed as in the proof of Claim 6.d. Assume that there is a morphism $\beta \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(F, S_j)$ such that

$$m^1_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\beta) = \sum_{r=1}^{r_j} \sum_{s=1}^{s_r} c_r^{(s)} \alpha_r^{(s)}$$

for some $c_1^{(1)}, \dots, c_1^{(s_1)}, \dots, c_{r_j}^{(1)}, \dots, c_{r_j}^{(s_{r_j})} \in \mathbf{k}$. Let us fix a basis of the solution space of the linear equation $\sum_{r=1}^{r_j} \sum_{s=1}^{s_r} b_r^{(s)} c_r^{(s)} = 0$. Then, for each element $(b_1^{(1)}, \dots, b_1^{(s_1)}, \dots, b_{r_j}^{(1)}, \dots, b_{r_j}^{(s_{r_j})})$ in the basis, we assign the closed morphism $\sum_{r=1}^{r_j} \sum_{s=1}^{s_r} b_r^{(s)} \alpha_r^{(s)}$. In this way, we can extract $s_1 + \cdots + s_{r_j} - 1$ closed morphisms from the $s_1 + \cdots + s_{r_j}$ closed morphisms $\alpha_1^{(1)}, \ldots, \alpha_1^{(s_1)}, \ldots, \alpha_{r_j}^{(1)}, \ldots, \alpha_{r_j}^{(s_{r_j})}$. Performing the same procedure to the new closed morphisms inductively, we eventually obtain closed morphisms $\alpha_1, \ldots, \alpha_\sigma \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(F, S_j)$ such that their classes form a basis of $\mathrm{Hom}^\bullet_{D(\mathscr{A})}(F, S_j)$, and if all the closed and non-exact components of α_μ are $a_{\mu,l}\alpha^0 \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(E_{j,s_{\mu,l}}, S_j)$ for $a_{\mu,l} \in \mathbf{k}$, $\alpha^0 \in \mathrm{Hom}^0_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(E_j, S_j)$ a closed and non-exact morphism and $l = 1, \ldots, n_\mu$ then

$$\widehat{\iota}_\mu = \sum_{l=1}^{n_\mu} a_{\mu,l}\iota_{j,s_{\mu,l}} \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(E_j, F)$$

satisfies $m^1_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(m^2_{\Sigma\widetilde{\mathscr{S}}}(x_j, \widehat{\iota}_\mu)) = 0$. In what follows, we will fix such morphisms $\widehat{\iota}_1, \ldots, \widehat{\iota}_\sigma$ and regard them as a basis of $\mathrm{Hom}^\bullet_{D(\mathscr{A})}(F, S_j)$.

To an element $\phi_1 \otimes \widehat{\iota}_1 + \cdots + \phi_\sigma \otimes \widehat{\iota}_\sigma \in \mathrm{Hom}^\bullet_{\mathscr{D}}(E_k, E_j) \otimes \mathrm{Hom}^\bullet_{D(\mathscr{A})}(F, S_j)$, we associate $\phi = m^2_{\Sigma\widetilde{\mathscr{S}}}(\phi_1, \widehat{\iota}_1) + \cdots + m^2_{\Sigma\widetilde{\mathscr{S}}}(\phi_\sigma, \widehat{\iota}_\sigma) \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(E_k, F)$. As the collection $C = \{E_1, \ldots, E_m\}$ is null-triangular, there exist direct summands $E_{k,s_1}, \ldots, E_{k,s_n}$ such that

$$m^1_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\phi) = \sum_{l=1}^{n} m^2_{\Sigma\widetilde{\mathscr{S}}}(m^1_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\phi)|_{E_k \to E_{k,s_l}}, \iota_{k,s_l}) \tag{6.8}$$

and each term is non-zero. We also allow $n = 0$ in which case the morphism ϕ is closed. Note that each $m^1_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\phi)|_{E_k \to E_{k,s_l}}$ in the equation (6.8) is a multiple of $x_k \in \mathrm{Hom}^d_{\widetilde{\mathscr{S}}}(E_k, E_k)$ a fixed basis.

Without loss of generality, we assume that $E_{k,1}, \ldots, E_{k,q}$ does not satisfy the condition (H) while $E_{k,q+1}, \ldots, E_{k,r_k}$ satisfy the condition (H). For an element $\phi_1 \otimes \widehat{\iota}_1 + \cdots + \phi_\sigma \otimes \widehat{\iota}_\sigma \in \mathrm{Hom}^\bullet_{\mathscr{D}}(E_k, E_j) \otimes \mathrm{Hom}^\bullet_{D(\mathscr{A})}(F, S_j)$ such that the morphism ϕ as in the previous paragraph satisfies $m^1_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\phi)|_{E_k \to E_{k,l}} = a_l x_k$ for $l = 1, \ldots, q$, assign $(a_1, \ldots, a_q) \in \mathbf{k}^q$. This assignment defines a linear map $\gamma : \mathrm{Hom}^\bullet_{\mathscr{D}}(E_k, E_j) \otimes \mathrm{Hom}^\bullet_{D(\mathscr{A})}(F, S_j) \to \mathbf{k}^q$.

For $(a_1, \ldots, a_q) \in \mathbf{k}^q$, consider the following condition:

(I) there exist $c_1, \ldots, c_{r_k} \in \mathbf{k}$ such that the morphism

$$\xi = \sum_{s=1}^{r_k} c_s m^2_{\Sigma\widetilde{\mathscr{S}}}(x_k, \iota_{k,s}) \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(E_k, F)$$

satisfies $m^1_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(\xi) = \sum_{l=1}^q a_l m^2_{\Sigma\widetilde{\mathscr{S}}}(x_i, \iota_{k,l})$.

Let I be a subspace of $\mathbf{k}^q$ whose elements consist of all those elements $(a_1, \ldots, a_q) \in \mathbf{k}^q$ satisfying the condition (I). Fix a basis $v_1, \ldots, v_u$ of I. Moreover, for each v_l, we also fix a morphism ξ in the condition (I) and denote it by $\xi^I_{v_l}$. Then for $(a_1, \ldots, a_q) \in I$, we define a morphism $\xi^I_{a_1,\ldots,a_q} \in \mathrm{Hom}_{\mathrm{Tw}\,\widetilde{\mathscr{S}}}(E_k, F)$ as a linear combination of $\xi^I_{v_1}, \ldots, \xi^I_{v_u}$ so that

$$m^1_{\mathrm{Tw}\,\widehat{\mathscr{S}}}(\xi^I_{a_1,\ldots,a_q}) = \sum_{l=1}^{q} a_l m^2_{\Sigma\widehat{\mathscr{S}}}(x_k, \iota_{k,l}).$$

Recall that, for each of $E_{k,q+1}, \ldots, E_{k,r_k}$, say $E_{k,l}$, we also have fixed a morphism $\xi^H_{k,l} \in \mathrm{Hom}_{\mathrm{Tw}\,\widehat{\mathscr{S}}}(E_k, F)$ satisfying

$$m^1_{\mathrm{Tw}\,\widehat{\mathscr{S}}}(\xi^H_{k,l}) = m^2_{\Sigma\widehat{\mathscr{S}}}(x_i, \iota_{k,l}).$$

We define a linear map $\overline{\gamma} : \mathrm{Hom}^\bullet_{\mathscr{D}}(E_k, E_j) \otimes \mathrm{Hom}^\bullet_{D(\mathscr{A})}(F, S_j) \to \mathbf{k}^q/I$ as the composition of the linear map $\gamma : \mathrm{Hom}^\bullet_{\mathscr{D}}(E_k, E_j) \otimes \mathrm{Hom}^\bullet_{D(\mathscr{A})}(F, S_j) \to \mathbf{k}^q$ and the projection map $\mathbf{k}^q \to \mathbf{k}^q/I$. Let $V = \mathrm{Ker}\,\overline{\gamma}$. For an element $\phi_1 \otimes \widehat{\iota}_1 + \cdots + \phi_\sigma \otimes \widehat{\iota}_\sigma \in V$, we can make the morphism $\phi = m^2_{\Sigma\widehat{\mathscr{S}}}(\phi_1, \widehat{\iota}_1) + \cdots + m^2_{\Sigma\widehat{\mathscr{S}}}(\phi_\sigma, \widehat{\iota}_\sigma) \in \mathrm{Hom}_{\mathrm{Tw}\,\widehat{\mathscr{S}}}(E_k, F)$ closed as follows. First of all, suppose $\gamma(\phi_1 \otimes \widehat{\iota}_1 + \cdots + \phi_\sigma \otimes \widehat{\iota}_\sigma) = (a_1, \ldots, a_q) \in I$ and $m^1_{\mathrm{Tw}\,\widehat{\mathscr{S}}}(\phi)|_{E_k \to E_{k,l}} = b_l x_k$ for $l = q+1, \ldots, r_k$, i.e.,

$$m^1_{\mathrm{Tw}\,\widehat{\mathscr{S}}}(\phi) = \sum_{l=1}^{q} a_l m^2_{\Sigma\widehat{\mathscr{S}}}(x_k, \iota_{k,l}) + \sum_{l=q+1}^{r_k} b_l m^2_{\Sigma\widehat{\mathscr{S}}}(x_k, \iota_{k,l}).$$

Then one sees that

$$\phi - \xi^I_{a_1,\ldots,a_q} - \sum_{l=q+1}^{r_k} b_l \xi^H_{k,l} \in \mathrm{Hom}_{\mathrm{Tw}\,\widehat{\mathscr{S}}}(E_k, F) \tag{6.9}$$

is a closed morphism. Then sending $\phi_1 \otimes \widehat{\iota}_1 + \cdots + \phi_\sigma \otimes \widehat{\iota}_\sigma \in V$ to the class of the closed morphism (6.9), we obtain a linear map $\lambda : V \to \mathrm{Hom}^\bullet_{D(\mathscr{A})}(E_k, F)$. Note that the linear map λ depends on the choices of the morphisms $\xi^H_{k,l}$'s and $\xi^I_{a_1,\ldots,a_q}$'s we have fixed.

Now, relabeling if necessary, we assume that $e_1, \ldots, e_w$ form a basis of $\mathbf{k}^q/I$ where $e_1, \ldots, e_q$ is the class of the standard basis of $\mathbf{k}^q$. For the direct summands $E_{k,1}, \ldots, E_{k,w}$, let us fix morphisms $\alpha_{k,1}, \ldots, \alpha_{k,w} \in \mathrm{Hom}^\bullet_{D(\mathscr{A})}(F, S_k)$ constructed in Claim 6.d. For $l = 1, \ldots, w$, if there is no closed and non-exact morphism $\epsilon \in \mathrm{Hom}_{\mathrm{Tw}\,\widehat{\mathscr{S}}}(E_k, F)$ such that $\epsilon|_{E_k \to E_{k,l}} = \mathrm{id}_{E_k}$, the morphism $\alpha_{k,l}$ is contained in $\mathrm{Ker}\,\rho^{(k)}_F$. Otherwise, we fix a closed and non-exact morphism $\epsilon_{k,l} \in \mathrm{Hom}_{\mathrm{Tw}\,\widehat{\mathscr{S}}}(E_k, F)$ such that $\epsilon_{k,l}|_{E_k \to E_{k,l}} = \mathrm{id}_{E_k}$. We will also denote its class by the same symbol $\epsilon_{k,l} \in \mathrm{Hom}^\bullet_{D(\mathscr{A})}(E_k, F)$. Let W be a subspace of $\mathrm{Hom}^\bullet_{D(\mathscr{A})}(E_k, F)$ spanned by all such morphisms. Then we have $W \cap \mathrm{Im}\,\lambda = 0$.

As above we can define a linear map $\delta : \mathbf{k}^q/I \to W \oplus \mathrm{Ker}\,\rho^{(k)}_F$ by sending $[a_1, \ldots, a_w, 0, \ldots, 0] \in \mathbf{k}^q/I$ to an element in $W \oplus \mathrm{Ker}\,\rho^{(k)}_F$ according to the algorithm given above. We then define a linear map $\kappa_1 : \mathrm{Hom}^\bullet_{\mathscr{D}}(E_k, E_j) \otimes \mathrm{Hom}^\bullet_{D(\mathscr{A})}(F, S_j) \to W \oplus \mathrm{Ker}\,\rho^{(k)}_F$ by $\kappa_1 = \delta \circ \overline{\gamma}$. Then $\mathrm{Ker}\,\kappa_1 = V$ and thus we have

$$\mathrm{codim}\, V = \dim \mathrm{Im}\,\kappa_1 \le \dim W + \dim \mathrm{Ker}\,\rho^{(k)}_F.$$

Moreover, since $W \cap \operatorname{Im}\lambda = 0$, it follows that

$$\begin{aligned}\iota(E_k, E_j)\sigma_j(F) - \dim \operatorname{Ker}\lambda &= \dim V + \operatorname{codim} V - \dim \operatorname{Ker}\lambda \\ &= \dim \operatorname{Im}\lambda + \operatorname{codim} V \\ &\leq \dim \operatorname{Im}\lambda + \dim W + \dim \operatorname{Ker}\rho_F^{(k)} \\ &\leq \iota(E_k, F) + \dim \operatorname{Ker}\rho_F^{(k)}.\end{aligned} \tag{6.10}$$

Finally, we shall show that $\dim \operatorname{Ker}\lambda \leq \dim \operatorname{Ker}\rho_F^{(k)}$. For an element $\phi_1 \otimes \widehat{\iota}_1 + \cdots + \phi_\sigma \otimes \widehat{\iota}_\sigma \in \operatorname{Ker}\lambda$, there is a morphism $\psi \in \operatorname{Hom}_{\operatorname{Tw}\widehat{\mathscr{F}}}(E_k, F)$ such that $m^1_{\operatorname{Tw}\widehat{\mathscr{F}}}(\psi) = \lambda(\phi_1 \otimes \widehat{\iota}_1 + \cdots + \phi_\sigma \otimes \widehat{\iota}_\sigma)$. We can take such a morphism ψ so that it satisfies $\psi|_{E_k \to E_{k,l}} = 0$ for all $l = q+1, \ldots, r_k$ using morphisms $\eta^H_{k,q+1}, \ldots, \eta^H_{k,r_k}$ obtained by replacing x_k in the definition of $\xi^H_{k,q+1}, \ldots, \xi^H_{k,r_k}$ by id_{E_k}. Similarly, we can even assume that such a morphism ψ satisfies $\psi|_{E_k \to E_{k,l}} = 0$ for all $l = w+1, \ldots, r_k$ using morphisms $\eta^I_{a_1,\ldots,a_q}$ obtained by replacing x_k in the definition of $\xi^I_{a_1,\ldots,a_q}$ by id_{E_k}. For such a morphism ψ, suppose $\psi|_{E_k \to E_{k,l}} = a_l$ for $l = 1, \ldots, w$. We then assign to ψ an element $[a_1, \ldots, a_w, 0, \ldots, 0] \in \mathbf{k}^q/I$. As before, we see that the morphism $a_1\alpha_{k,1} + \cdots + a_w\alpha_{k,w}$ is contained in $\operatorname{Ker}\rho_F^{(k)}$ and such an assignment gives us a linear map $\kappa_2 : \operatorname{Ker}\lambda \to \operatorname{Ker}\rho_F^{(k)}$ which is injective. This shows that $\dim \operatorname{Ker}\lambda \leq \dim \operatorname{Ker}\rho_F^{(k)}$.

Combining this with the inequality (6.10), we conclude the inequality (6.7). □

7. Ping-pong argument

Let $C = \{E_1, \ldots, E_m\}$ be an essential collection of d-spherical objects of $\mathscr{D}$ such that $\operatorname{End}_{\mathscr{A}}(E_1 \oplus \cdots \oplus E_m)$ is formal, and $C_1, \ldots, C_\alpha$ be a partition of C. For $\mu = 1, \ldots, \alpha$ and $F \in \operatorname{Ob}\mathscr{D}$, define

$$\widehat{\sigma}_\mu(F) = \max\{\sigma_k(F) \mid E_k \in C_\mu\}.$$

Clearly, by Proposition 5.1, $\widehat{\sigma}_\mu(E_j) = 0$ if $E_j \notin C_\mu$ and $\widehat{\sigma}_\mu(E_j) = 1$ if $E_j \in C_\mu$. Now we define subsets $\Omega_1, \ldots, \Omega_\alpha$ of the set $\Omega = \Omega_C$ by

$$\Omega_\mu = \{F \in \Omega \mid \widehat{\sigma}_\mu(F) > \widehat{\sigma}_\nu(F) \text{ for all } \nu \neq \mu\}.$$

Note that $\Omega_\mu \cap \Omega_\nu = \varnothing$ for every $\mu \neq \nu$ and each Ω_μ is non-empty since $C_\mu \subset \Omega_\mu$.

The following lemma allows us to apply the ping-pong lemma. The proof is a direct translation of that of Humphries [Hum, Section 2].

Proposition 7.1. *Let $C = \{E_1, \ldots, E_m\}$ be an essential null-triangular collection of d-spherical objects of $\mathscr{D}$ such that $\operatorname{End}_{\mathscr{A}}(E_1 \oplus \cdots \oplus E_m)$ is formal. Suppose that $C_1, \ldots, C_\alpha$ is a complete partition of C. Let G_μ be the subgroup of* $\operatorname{Auteq}(\mathscr{D})$

generated by the spherical twists along the objects in C_μ. Then, for all $F \in \Omega_\nu$ and $T \in G_\mu \setminus \{\mathrm{Id}_{\mathscr{D}}\}$ with $\mu \neq \nu$, we have $TF \in \Omega_\mu$.

Proof. Without lose of generality, we can assume that $C_\mu = \{E_1, \dots, E_n\}$. As $\dim \mathrm{Hom}^\bullet_{\mathscr{D}}(E_i, E_j) = 0$ for every distinct pair of $i, j = 1, \dots, n$, the spherical twists $T_1, \dots, T_n$ commute with each other by Proposition 4.1. Thus every element $T \in G_\mu$ can be uniquely written as $T = T_1^{p_1} \cdots T_n^{p_n}$ for some $p_1, \dots, p_n \in \mathbf{Z}$. Now let $F \in \Omega_\nu$ with $\nu \neq \mu$ and $T = T_1^{p_1} \cdots T_n^{p_n} \in G_\mu \setminus \{\mathrm{Id}_{\mathscr{D}}\}$. Since $T \neq \mathrm{Id}_{\mathscr{D}}$, at least one of $p_1, \dots, p_n$, say p_{i_0}, is non-zero. Take j_0 so that $E_{j_0} \in C_\nu$ and $\widehat{\sigma}_\nu(F) = \sigma_{j_0}(F)$. Then, applying Proposition 6.1, we see that

$$\begin{aligned}
\widehat{\sigma}_\mu(TF) &= \max\{\sigma_1(TF), \dots, \sigma_n(TF)\} \\
&= \max\{\sigma_1(T_1^{p_1}F), \dots, \sigma_n(T_l^{p_n}F)\} \\
&\geq \sigma_{i_0}(T_{i_0}^{p_{i_0}}F) \\
&\geq \iota(E_{i_0}, E_{j_0})\sigma_{j_0}(F) - \sigma_{i_0}(F) \\
&\geq 2\widehat{\sigma}_\nu(F) - \widehat{\sigma}_\mu(F) \\
&> \widehat{\sigma}_\nu(F) = \widehat{\sigma}_\nu(TF)
\end{aligned}$$

where the assumption that the partition $C_1, \dots, C_\alpha$ is complete is used in the fifth line and that $F \in \Omega_\nu$ is used in the sixth line. On the other hand, as $F \in \Omega_\nu$, we have $\widehat{\sigma}_\mu(TF) > \widehat{\sigma}_\nu(F) > \widehat{\sigma}_\lambda(F) = \widehat{\sigma}_\lambda(TF)$ for every $\lambda \neq \mu, \nu$. Consequently, we have shown that $\widehat{\sigma}_\mu(TF) > \widehat{\sigma}_\lambda(TF)$ for all $\lambda \neq \mu$ which means that $TF \in \Omega_\mu$. □

Lemma 7.2. *Let $C = \{E_1, \dots, E_m\}$ be an essential null-triangular collection of $d_{>1}$-spherical objects of $\mathscr{D}$ such that $\mathrm{End}_{\mathscr{A}}(E_1 \oplus \cdots \oplus E_m)$ is formal. Then*

(1) *every spherical twist T_i has an infinite order;*
(2) *$T_i \ncong T_j$ for every $i \neq j$.*

Proof. (1) Follows immediately from $T_i E_i \cong E_i[1-d]$ and $d > 1$.

(2) If $\iota(E_i, E_j) = 0$ then $T_i E_i \cong E_i[1-d]$ while $T_j E_i = E_i$. Since $d > 1$, this shows that $T_i \ncong T_j$.

If $\iota(E_i, E_j) > 0$, assume by contrary that $T_i \cong T_j$. Then, by Proposition 6.1,

$$0 = \sigma_i(E_j) = \sigma_i(T_j E_j) = \sigma_i(T_i E_j) \geq \iota(E_i, E_j) - \sigma_i(E_j) = \iota(E_i, E_j) > 0$$

which is impossible. □

Proof of Theorem 4.4. By Lemma 7.2, the subgroup G_μ of $\mathrm{Auteq}(\mathscr{D})$ generated by the spherical twists along the objects in C_μ is isomorphic to $\mathbf{Z}^{m_\mu}$. The assertion thus follows by combining Proposition 7.1 with Lemma B.4 known as the *ping-pong lemma*. □

8. Example

We can construct a trivial example as follows. First we fix numbers $d > 1$, $m > 0$ and $m_1, \ldots, m_\alpha > 0$ with $m_1 + \cdots + m_\alpha = m$. We then define a minimal dg category $\mathscr{A}$ whose objects consist of

$$\{E_1, \ldots, E_m\} = \{E_1^{(1)}, \ldots, E_{m_1}^{(1)}, \ldots, E_1^{(\alpha)}, \ldots, E_{m_\alpha}^{(\alpha)}\}$$

by requiring that the morphism spaces satisfy the following conditions:

- (S_d) $\mathrm{Hom}^p_{\mathscr{A}}(E_i^{(\mu)}, E_i^{(\mu)}) \cong \mathbf{k}$ if and only if $p = 0, d$ and vanishes otherwise;
- (P_1) $\mathrm{Hom}_{\mathscr{A}}(E_i^{(\mu)}, E_j^{(\mu)}) = 0$ for every $i \neq j$ and μ;
- (P_2) $\mathrm{Hom}_{\mathscr{A}}(E_i^{(\mu)}, E_j^{(\nu)})$ is two dimensional, with a basis $\phi_{ij}^{(\mu\nu)}$ and $\psi_{ij}^{(\mu\nu)}$, for every i, j and $\mu \neq \nu$;
- (CY_d) $\deg \phi_{ji}^{(\nu\mu)} = d - \deg \phi_{ij}^{(\mu\nu)}$, $\deg \psi_{ji}^{(\nu\mu)} = d - \deg \psi_{ij}^{(\mu\nu)}$ and $\psi_{ji}^{(\nu\mu)} \circ \phi_{ij}^{(\mu\nu)} = \phi_{ji}^{(\nu\mu)} \circ \psi_{ij}^{(\mu\nu)} = 0$, $\phi_{ji}^{(\nu\mu)} \circ \phi_{ij}^{(\mu\nu)} = \psi_{ji}^{(\nu\mu)} \circ \psi_{ij}^{(\mu\nu)} = x_i^{(\mu)}$ where $x_i^{(\mu)}$ is a fixed basis of $\mathrm{Hom}^d_{\mathscr{A}}(E_i^{(\mu)}, E_i^{(\mu)})$;
- (N) $\phi_{jk}^{(\nu\lambda)} \circ \phi_{ij}^{(\mu\nu)} = \psi_{jk}^{(\nu\lambda)} \circ \phi_{ij}^{(\mu\nu)} = \phi_{jk}^{(\nu\lambda)} \circ \psi_{ij}^{(\mu\nu)} = \psi_{jk}^{(\nu\lambda)} \circ \psi_{ij}^{(\mu\nu)} = 0$ for every $\mu \neq \nu \neq \lambda \neq \mu$.

Let $\mathscr{D} = H^0(\mathrm{Tw}^\pi \mathscr{A})$ where $\mathrm{Tw}^\pi \mathscr{A}$ denotes the split-closure of the dg category $\mathrm{Tw}\,\mathscr{A}$ of twisted complexes over $\mathscr{A}$ (cf. [Sei, Section 4c]). Then, by the construction, the collection $C = \{E_1, \ldots, E_m\}$ of d-spherical objects in $\mathscr{D}$ is essential, null-triangular and admits a complete partition $C_1 = \{E_1^{(1)}, \ldots, E_{m_1}^{(1)}\}, \ldots, C_\alpha = \{E_1^{(\alpha)}, \ldots, E_{m_\alpha}^{(\alpha)}\}$. Therefore, by Theorem 4.4, we conclude that the subgroup of $\mathrm{Auteq}(\mathscr{D})$ generated by the spherical twists along the spherical objects in C is isomorphic to $\mathbf{Z}^{m_1} * \cdots * \mathbf{Z}^{m_\alpha}$.

Let $E = E_1 \oplus \cdots \oplus E_m$ and $A = \mathrm{End}_{\mathscr{A}}(E) = \mathrm{End}^\bullet_{\mathscr{D}}(E)$. We shall regard the graded algebra A as a dg algebra with the zero differential. Then, from the dg functor $\mathscr{F} = \mathrm{Hom}_{\mathscr{A}}(E, -) : \mathrm{Tw}^\pi \mathscr{A} \to \mathrm{dgMod}(A)$, we obtain the exact functor $H^0(\mathscr{F}) : \mathscr{D} \to H^0(\mathrm{dgMod}(A))$. Furthermore, composing it with the quotient functor $H^0(\mathrm{dgMod}(A)) \to D(A)$, we get an exact functor from $\mathscr{D}$ to $D(A)$ which induces an exact equivalence between $\mathscr{D}$ and the perfect derived category $D_{\mathrm{per}}(A)$ (cf. [HK, Theorem 1.11]). Note that, in this situation, each spherical object $E_i \in \mathrm{Ob}\,\mathscr{D}$ corresponds to the projective A-module $P_i = \mathrm{Hom}_{\mathscr{A}}(E, E_i) \in \mathrm{Ob}\, D_{\mathrm{per}}(A)$.

Appendix A. Formality

In this appendix, we discuss the formality of A_∞ algebras. The facts presented here will be used in Sections 5 and 6.

Let (A, m) be an A_∞ algebra. Its cohomology $H^\bullet(A)$ can be regarded as a cochain complex with the zero differential. We denote it by $(H^\bullet(A), 0)$. Now let us take cochain maps $f^1 : (H^\bullet(A), 0) \to (A, m^1)$ and $g^1 : (A, m^1) \to (H^\bullet(A), 0)$ such that $g^1 \circ f^1 = \mathrm{id}_{H^\bullet(A)}$ and $f^1 \circ g^1 - \mathrm{id}_A = m^1 \circ h^1 + h^1 \circ m^1$ for a linear map $h^1 : A \to A$ of degree -1. Then for every $n \geq 1$, we can define linear maps $\widetilde{m}^n : (H^\bullet(A))^{\otimes n} \to H^\bullet(A)$ of degree $2-n$ and $f^n : (H^\bullet(A))^{\otimes n} \to A$ of degree $1-n$ recursively by setting $\widetilde{m}^1 = 0$ and

$$f^n(a_1, \dots, a_n) = \sum_{l \geq 2} \sum_{i_1 + \dots + i_l = n} h^1(m^l(f^{i_1}(a_1, \dots, a_{i_1}), \dots, f^{i_l}(a_{n-i_l+1}, \dots, a_n))),$$

$$\widetilde{m}^n(a_1, \dots, a_n) = \sum_{l \geq 2} \sum_{i_1 + \dots + i_l = n} g^1(m^l(f^{i_1}(a_1, \dots, a_{i_1}), \dots, f^{i_l}(a_{n-i_l+1}, \dots, a_n)))$$

for every $n \geq 2$. These maps satisfy the relevant A_∞ relations. In particular, $\{\widetilde{m}^n\}_{n \geq 1}$ defines a minimal A_∞ structure on the cohomology $H^\bullet(A)$ and $\{f^n\}_{n \geq 1}$ gives a quasi-isomorphism between the A_∞ algebras $(H^\bullet(A), \widetilde{m})$ and (A, m). To summarize, we have the following.

Theorem A.1 (Kadeishvili [Kad])**.** *Let (A, m) be an A_∞ algebra. Then there exist a minimal A_∞ structure $\widetilde{m}$ on the cohomology $H^\bullet(A)$ and a quasi-isomorphism $f : (H^\bullet(A), \widetilde{m}) \to (A, m)$ of A_∞ algebras.*

Definition A.2. An A_∞ algebra $(H^\bullet(A), \widetilde{m})$ in Theorem A.1 is called a *minimal model* of the A_∞ algebra A. An A_∞ algebra A is called *formal* if its minimal model $(H^\bullet(A), \widetilde{m})$ can be chosen to satisfy $\widetilde{m}^n = 0$ for every $n \neq 2$.

The same idea applies to A_∞ categories. Let $(\mathscr{A}, m)$ be an A_∞ category. For every pair of objects E and F, we take cochain maps $\mathscr{F}^1 : (\mathrm{Hom}_{H^\bullet(\mathscr{A})}(E, F), 0) \to (\mathrm{Hom}_{\mathscr{A}}(E, F), m^1)$ and $\mathscr{G}^1 : (\mathrm{Hom}_{\mathscr{A}}(E, F), m^1) \to (\mathrm{Hom}_{H^\bullet(\mathscr{A})}(E, F), 0)$ such that $\mathscr{G}^1 \circ \mathscr{F}^1 = \mathrm{id}$ and $\mathscr{F}^1 \circ \mathscr{G}^1 - \mathrm{id} = m^1 \circ \mathscr{H}^1 + \mathscr{H}^1 \circ m^1$ for a linear map $\mathscr{H}^1 : \mathrm{Hom}_{\mathscr{A}}(E, F) \to \mathrm{Hom}_{\mathscr{A}}(E, F)$ of degree -1. Then, as before, one can transfer the A_∞ structure m on $\mathscr{A}$ to a minimal A_∞ structure $\widetilde{m}$ on the cohomology category $H^\bullet(\mathscr{A})$ so that they are quasi-isomorphic via an A_∞ functor $\mathscr{F} : H^\bullet(\mathscr{A}) \to \mathscr{A}$ extending $\mathscr{F}^1$ (cf. [Sei, Section 1i]).

We use the above argument in the following situation. Consider an enhanced triangulated category $\mathscr{D}$ with a dg enhancement $\mathscr{A}$ and a collection $\{E_1, \dots, E_m\}$ of objects of $\mathscr{D}$. In what follows, we will regard the dg category $\mathscr{A}$ as an A_∞ category with an A_∞ structure m such that $m^n = 0$ for all $n \geq 3$. Assume that the dg algebra $A = \mathrm{End}_{\mathscr{A}}(E_1 \oplus \dots \oplus E_m)$ is formal. Then by the definition of the formality, there are cochain maps $f^1 : (H^\bullet(A), 0) \to (A, m^1)$ and $g^1 : (A, m^1) \to (H^\bullet(A), 0)$ such that $g^1 \circ f^1 = \mathrm{id}_{H^\bullet(A)}$ and $f^1 \circ g^1 - \mathrm{id}_A = m^1 \circ h^1 + h^1 \circ m^1$ for a linear map $h^1 : A \to A$ of degree -1, and such that the minimal A_∞ structure $\widetilde{m}$ on the cohomology $H^\bullet(A)$ transferred by these maps satisfies $\widetilde{m}^n = 0$ for every $n \neq 2$.

Now for each pair of objects $E, F \in \operatorname{Ob}\mathscr{D} = \operatorname{Ob}\mathscr{A}$, let us take cochain maps $\mathscr{F}^1 : (\operatorname{Hom}_{H^\bullet(\mathscr{A})}(E,F), 0) \to (\operatorname{Hom}_{\mathscr{A}}(E,F), m^1)$ and $\mathscr{G}^1 : (\operatorname{Hom}_{\mathscr{A}}(E,F), m^1) \to (\operatorname{Hom}_{H^\bullet(\mathscr{A})}(E,F), 0)$ such that $\mathscr{G}^1 \circ \mathscr{F}^1 = \mathrm{id}$ and $\mathscr{F}^1 \circ \mathscr{G}^1 - \mathrm{id} = m^1 \circ \mathscr{H}^1 + \mathscr{H}^1 \circ m^1$ for a linear map $\mathscr{H}^1 : \operatorname{Hom}_{\mathscr{A}}(E,F) \to \operatorname{Hom}_{\mathscr{A}}(E,F)$ of degree -1 which extend the above maps f^1, g^1 and h^1. Then the transferred minimal A_∞ structure $\widetilde{m}$ on the cohomology category $H^\bullet(\mathscr{A})$ has the property that the A_∞ products

$$\widetilde{m}^n : \operatorname{Hom}_{H^\bullet(\mathscr{A})}(E_{i_0}, E_{i_1}) \otimes \cdots \otimes \operatorname{Hom}_{H^\bullet(\mathscr{A})}(E_{i_{n-1}}, E_{i_n}) \to \operatorname{Hom}_{H^\bullet(\mathscr{A})}(E_{i_0}, E_{i_n})$$

vanish for all $i_0, \ldots, i_n$ and $n \geq 3$. Moreover, since the A_∞ categories $(H^\bullet(\mathscr{A}), \widetilde{m})$ and $(\mathscr{A}, m)$ are quasi-isomorphic, we have an equivalence $\mathscr{D} \simeq H^0(H^\bullet(\mathscr{A}), \widetilde{m})$ of triangulated categories.

Appendix B. Elementary lemmas

Lemma B.1. *For a 3-periodic long exact sequence*

$$\cdots \to V_3 \xrightarrow{\xi} V_1 \xrightarrow{\phi} V_2 \xrightarrow{\psi} V_3 \xrightarrow{\xi} V_1 \to \cdots$$

of $\mathbf{k}$*-vector spaces, we have* $\dim V_1 + \dim V_2 = \dim V_3 + 2\dim \operatorname{Im}\phi$.

Proof. Successively applying the rank-nullity theorem, we get

$$\begin{aligned}\dim V_1 + \dim V_2 &= \dim \operatorname{Im}\phi + \dim \operatorname{Ker}\phi + \dim \operatorname{Im}\psi + \dim \operatorname{Ker}\psi \\ &= \dim \operatorname{Im}\xi + \dim \operatorname{Im}\psi + 2\dim \operatorname{Im}\phi \\ &= \dim V_3 - \dim \operatorname{Ker}\xi + \dim \operatorname{Im}\psi + 2\dim \operatorname{Im}\phi \\ &= \dim V_3 + 2\dim \operatorname{Im}\phi.\end{aligned}$$

This completes the proof. □

Lemma B.2. *Let E be a d-spherical object and F be an indecomposable object of $\mathscr{D}$. Assume that the composition maps $\operatorname{Hom}^d_{\mathscr{D}}(E,E) \otimes \operatorname{Hom}^\bullet_{\mathscr{D}}(E,F) \to \operatorname{Hom}^{\bullet+d}_{\mathscr{D}}(E,F)$ or $\operatorname{Hom}^\bullet_{\mathscr{D}}(F,E) \otimes \operatorname{Hom}^d_{\mathscr{D}}(E,E) \to \operatorname{Hom}^{\bullet+d}_{\mathscr{D}}(F,E)$ does not vanish. Then $E \cong F$ in $\mathscr{D}$ up to shifts.*

Proof. The proof below can be found in [Kea, Corollary 4.9].

We shall only show the assertion under the assumption that the composition map $\operatorname{Hom}^d_{\mathscr{D}}(E,E) \otimes \operatorname{Hom}^\bullet_{\mathscr{D}}(E,F) \to \operatorname{Hom}^{\bullet+d}_{\mathscr{D}}(E,F)$ does not vanish. By the assumption, there is a morphism $\phi \in \operatorname{Hom}^p_{\mathscr{D}}(E,F)$ such that $\phi \circ x \neq 0$ where x is a basis of $\operatorname{Hom}^d_{\mathscr{D}}(E,E)$. By the condition (CY_d), we find a morphism $\psi \in \operatorname{Hom}^{-p}_{\mathscr{D}}(F,E)$ such that $\psi \circ \phi \circ x = x$ or equivalently $(\psi \circ \phi - \mathrm{id}_E) \circ x = 0$. As the zero morphism $0 \in \operatorname{Hom}^0_{\mathscr{D}}(E,E)$ is the only morphism which is killed by x, it follows that $\psi \circ \phi = \mathrm{id}_E$. Since F is indecomposable, the assertion follows. □

Lemma B.3. *Let $C = \{E_1, \ldots, E_m\}$ be a collection of d-spherical objects of $\mathscr{D}$. Then the following are equivalent:*

(1) *the collection C is essential, i.e., if $i \neq j$ then $E_i \ncong E_j$ in $\mathscr{D}$ up to shifts;*
(2) *the composition map $\mathrm{Hom}^\bullet_{\mathscr{D}}(E_i, E_j) \otimes \mathrm{Hom}^\bullet_{\mathscr{D}}(E_j, E_i) \to \mathrm{Hom}^\bullet_{\mathscr{D}}(E_i, E_i)$ does not hit the identity morphism id_{E_i} for every $i \neq j$;*
(3) *the composition maps $\mathrm{Hom}^d_{\mathscr{D}}(E_i, E_i) \otimes \mathrm{Hom}^\bullet_{\mathscr{D}}(E_i, E_j) \to \mathrm{Hom}^{\bullet+d}_{\mathscr{D}}(E_i, E_j)$ and $\mathrm{Hom}^\bullet_{\mathscr{D}}(E_i, E_j) \otimes \mathrm{Hom}^d_{\mathscr{D}}(E_j, E_j) \to \mathrm{Hom}^{\bullet+d}_{\mathscr{D}}(E_i, E_j)$ vanish for all $i \neq j$.*

Proof. This is also due to [Kea, Section 4.4].

(2) $\Rightarrow$ (1). Easy.

(3) $\Rightarrow$ (2). Easy.

(1) $\Rightarrow$ (3). Apply Lemma B.2. □

Lemma B.4 (Ping-pong Lemma)**.** *Let G be a group generated by its non-trivial subgroups $G_1, \ldots, G_\alpha$ and acting on a set Ω. Assume that the order of at least one of G_μ is greater than 2 and that there are disjoint non-empty subsets $\Omega_1, \ldots, \Omega_\alpha$ of Ω such that $g\Omega_\nu \subset \Omega_\mu$ for every $g \in G_\mu \setminus \{1\}$ with $\mu \neq \nu$. Then $G = G_1 * \cdots * G_\alpha$.*

Proof. We give the following classical proof for the sake of completeness (cf. [LS, Proposition 12.2]).

Relabeling if necessary, we can assume the order of G_1 is greater than 2. Take two distinct elements $g_1, g_2 \in G_1 \setminus \{1\}$. As $g_2^{-1} g_1 \in G_1 \setminus \{1\}$, we have $g_2^{-1} g_1 \Omega_\mu \subset \Omega_1$ for any $\mu \neq 1$ and hence $g_2^{-1} g_1 \Omega_\mu \cap \Omega_\mu = \varnothing$. This in particular means that $g_1 \Omega_\mu \cap g_2 \Omega_\mu = \varnothing$. Thus we have shown that $g\Omega_\mu \subsetneq \Omega_1$ for every $g \in G_1 \setminus \{1\}$ and $\mu \neq 1$.

Now assume $gh = 1$ for some $g \in G_1 \setminus \{1\}$ and $h \in G \setminus \{1\}$ which is a freely reduced word whose first and last word does not lie in G_1. One can see that every relation of G, if there exists, can always be brought into this form. Then the above argument shows that $\Omega_1 = gh\Omega_1 \subsetneq \Omega_1$ which is a contradiction. □

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