Mean values and various analytic properties of multiple zeta-functions

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Abstract

In this thesis, we describe various analytic properties of the Riemann zeta-function and some multiple zeta-functions, and mean square values of the Barnes double zeta-functions and Hurwitz multiple zeta-functions.

Firstly, we discuss the theory of multiple zeta-functions from the historical aspect, and introduce some kinds of multiple zeta-functions.

Secondly, we obtain asymptotic formulas for mean square values of the Barnes double zeta-function $\zeta_2(s,\alpha;v,w) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha+vm+wn)^{-s}$ with respect to $\mathrm{Im}(s)$ as $\mathrm{Im}(s) \to +\infty$. Furthermore, we consider asymptotic formulas for mean square values of the Hurwitz multiple zeta-function $\zeta_r(s,\alpha) = \sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}(\alpha+m_1+\cdots+m_r)^{-s}$ with respect to $\mathrm{Im}(s)$, in $\mathrm{Re}(s) \geq r-1/2$. Also, we found that the straight line $\sigma = r-1/2$ is an analogue of the critical line for $\zeta_r(s,\alpha)$.

Thirdly, we describe approximate functional equations for the Hurwitz and Lerch zeta-functions. The approximate functional equation is one of the asymptotic formulas for approximating the zeta-function by a finite sum. In 2003, R. Garunkštis, A. Laurinčikas, and J. Steuding (in [7]) proved the Riemann-Siegel type of the approximate functional equation for the Lerch zeta-function. We prove another type of approximate functional equations for the Hurwitz and Lerch zeta-functions. Furthermore, we consider the approximate functional equations for the Barnes double zeta-function $\zeta_2(s,\alpha;v,w)$.

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Contents

1 Introduction to zeta-functions and multiple zeta-functions

In this section, we introduce the basic properties of the Riemann zeta-function and Dirichlet L-functions, and the precedent results on mean value theorems. Also, we will introduce several analytic properties of the Euler-Zagier type, and the other types of multiple zeta-functions and L-functions.

1.1 The Riemann zeta-function and Dirichlet L-functions

Let $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$. The Riemann zeta-function $\zeta(s)$ is one of the most important function in analytic number theory, which is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{n^s} + \cdots . \tag{1.1}$$

Also, $\zeta(s)$ can be written by the Euler product

$$\zeta(s) = \prod_{p:\text{prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \quad (\sigma > 1) \tag{1.2}$$

where p runs through all primes. From this representation, $\zeta(s)$ has a deep connection with prime numbers. Also, since the right hand-side of (1.1) is absolutely convergence for $\sigma > 1$, $\zeta(s)$ is a regular function defined by $\sigma > 1$. When $s \to 1 + 0$, since (1.1) diverges and (1.2), we prove that prime numbers exist infinitely. In addition, Euler proved a more precise result

$$\sum_{p \le x} \frac{1}{p} \sim \log\log x \quad (x \to \infty).$$

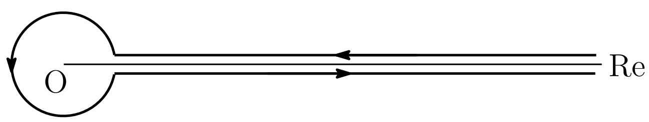
On the other hand as an analytic property, $\zeta(s)$ has the contour integral representation

$$\zeta(s) = \frac{1}{(e^{2\pi i s} - 1)\Gamma(s)} \int_C \frac{z^{s-1}}{e^z - 1} dz, \tag{1.3}$$

where $\Gamma(s)$ is the Gamma function defined by

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt \quad (\mathrm{Re}(s) > 0),$$

and C is the contour integral path that comes from $+\infty$ to ε ; then goes along the circle of radius ε counter clockwise, and finally goes from ε to $+\infty$, as the following figure:



By the contour integral representation (1.3), $\zeta(s)$ can be analytically continued to a meromorphic function on $\mathbb{C}$, and its only pole is a simple pole at $s = 1$ with residue 1. Also $\zeta(s)$ satisfies the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{1.4}$$

Furthermore, let $\pi(x)$ denote the number of prime numbers not exceeding x . It is known that

$$\pi(x) \sim \frac{x}{\log x} \quad (x \to \infty), \tag{1.5}$$

which is called the prime number theorem, as an important property associated with prime numbers and $\zeta(s)$. This theorem was predicted by Legendre and Gauss, proved by Hadamard and de la Vallée Poussin in 1896. The key point of the proof of (1.5) is the fact that

$$\zeta(1+it) \neq 0 \quad (t \in \mathbb{R}). \tag{1.6}$$

Also, the region where $\zeta(\sigma+it)$ does not have zeros is an important study theme, which is called the zero-free region. As an improvement of (1.6), de la Vallée Poussin proved that there exists a constant $A > 0$ such that $\zeta(\sigma+it) \neq 0$ in the region

$$\sigma \geq 1 - \frac{A}{\log(|t|+1)},$$

and furthermore in 1958, Vinogradov and Korobov prove a more precise result. The ultimate conjecture for the zero-free region is $\sigma > 1/2$, which is now called the Riemann hypothesis. In other words, the real part of any zeros of $\zeta(s)$ in the critical region $0 \leq \sigma \leq 1$ is $1/2$.

As an extension on the prime number theorem, Dirichlet attempted to study the prime number distribution in arithmetic progressions, by using the series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

which is now called Dirichlet L-function, and χ is a Dirichlet character. We call χ is a Dirichlet character of modulo q when a mapping $\chi : \mathbb{Z} \to \mathbb{C}$ satisfies all of the following conditions;

(i) $m \equiv n \pmod{q} \;\Rightarrow\; \chi(m) = \chi(n)$,

(ii) $\chi(mn) = \chi(m)\chi(n)$, especially $\chi(1) = 1$,

(iii) $\gcd(n,q) > 1 \;\Rightarrow\; \chi(n) = 0$.

The function $L(s,\chi)$ is known to have many properties similar to $\zeta(s)$; an Euler product

$$L(s,\chi) = \prod_{p:\mathrm{prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \quad (\sigma > 1)$$

and can be holomorphically continued to $\mathbb{C}$, and satisfies a functional equation similar to (1.4). Dirichlet proved that prime numbers are included infinitely in any arithmetic progression where the first term and the common difference are relatively prime, by using $L(s,\chi)$. Furthermore let

$$\pi_{k,q}(x) = \#\{p : \text{prime} \mid p \leq x,\ p \equiv k \pmod{q}\},$$

Dirichlet proved similar to the prime number theorem

$$\pi_{k,q}(x) \sim \frac{1}{\varphi(q)} \frac{x}{\log x},$$

where $\varphi(q)$ is the Euler totient function. Now, this is called Dirichlet's prime number theorem.

1.2 Mean values of the Riemann zeta-function

The order of $|\zeta(\sigma+it)|$ with respect to t is an extremely important problem in the deeper theory of $\zeta(s)$. For example, the simplest result is $\zeta(\sigma+it) = O(1)$ $(\sigma > 1)$, and also,

$$\zeta(\sigma+it) = O(|t|^{1/2-\sigma}) \quad (\sigma < 0)$$

by using the functional equation (1.4) and the Stirling formula for $\Gamma(s)$. In particular, the order of $|\zeta(1/2+it)|$ is the most important, and some results are listed here: The first,

$$\zeta\left(\frac{1}{2}+it\right) = O(t^{1/4+\varepsilon})$$

was proved by the Phragmen-Lindelöf convexity principle. Hardy-Littlewood improved to

$$\zeta\left(\frac{1}{2}+it\right) = O(t^{1/6+\varepsilon}).$$

A precision tool for obtaining such evaluations is an approximate expression of $\zeta(s)$ by a finite sum, which is proved by Hardy-Littlewood, and an expression of the form is as follows: Suppose that $0 \leq \sigma \leq 1,\ x \geq 1,\ y \geq 1$ and $2\pi xy = |t|$, then

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + X(s) \sum_{n \leq y} \frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(|t|^{1/2-\sigma} y^{\sigma-1}), \tag{1.7}$$

where $X(s) = 2\Gamma(1-s)\sin(\pi s/2)(2\pi)^{s-1}$. This is called the approximate functional equation, and the details will be described in Section 3.1.
In 1988, $\zeta(1/2+it) = O(t^{9/56+\varepsilon})$ was proved by Bombieri and Iwaniec, after which many mathematicians gradually improved, and now $\zeta(1/2+it) = O(t^{32/205+\varepsilon})$ has been proved by Huxley in 2005. Furthermore in 2017,

$$\zeta\left(\frac{1}{2}+it\right) = O(t^{13/84+\varepsilon})$$

was proved by Bourgain (in see [6]). What is considered to be true is called the Lindelöf hypothesis, which is the following conjecture;

$$\zeta(\sigma+it)=O(t^{\varepsilon}) \qquad \left(|t|\geq 2,\ \frac{1}{2}\leq\sigma\leq 1\right), \tag{1.8}$$

would hold for any $\varepsilon>0$ and arbitrary σ $(1/2\leq\sigma\leq 1)$. However, it is almost impossible to solve the Lindelöf hypothesis by the current technology.

It is difficult to consider the order of $|\zeta(\sigma+it)|$, so an attempt was made to study the order of the mean values of $|\zeta(\sigma+it)|$ as a compromise. In particular, the study of the order of mean squre value

$$\int_2^T |\zeta(\sigma+it)|^2 dt$$

has been a main stream, and for example we can show

$$\int_2^T |\zeta(\sigma+it)|^2 dt \sim \zeta(2\sigma)T \quad \left(\sigma>\frac{1}{2}\right). \tag{1.9}$$

However, the reason for studying the mean value is not only because it is easy to calculate but also because it can be expected that the prediction on the mean values of higher order powers leads to an equivalent of the Lindelöf hypothesis. Therefore, studying the mean values is extremely important in studying the zeta-function. The result supporting this is the following theorem.

Theorem 1.1 (Theorem 13.2 in Titchmarsh[27])**.** *The fact that, for any $k\in\mathbb{N}$ and any $\varepsilon>0$,*

$$\int_2^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt = O(T^{1+\varepsilon}) \tag{1.10}$$

is equivalent to the Lindelöf hypothesis (1.8).

On the other hand, (1.9) can be regarded as follows. The asymptotic formula

$$\int_2^T |\zeta(\sigma+it)|^2 dt = \zeta(2\sigma)T+O(T^{2-2\sigma}) \qquad (T\to\infty)$$

has been known as a classical result in the critical strip $1/2<\sigma<1$ (see Ivić [10], (8.112)). It is known that (1.10) holds in the case only $k=1$ and $k=2$. In the case $k=1$ that is the mean square value

$$\int_2^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt \sim T\log T$$

was proved by Hardy and Littlewood in 1918. In the case $k=2$, that is mean value of the fourth power

$$\int_2^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \sim \frac{1}{2\pi^2}T\log^4 T \tag{1.11}$$

was proved by Ingham in 1928. His method was to use an analogue of approximate functional equation (1.7) for $\zeta(s)^2$. In the case $k\geq 3$, The same method only produces a formula with a too large error term, so it cannot be used for the study of the mean value theorems. Also, Heath-Brown obtained the result on the mean value of 12th power

$$\int_2^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{12} dt = O(T^{2+\varepsilon}) \tag{1.12}$$

by using the Halász-Montgomery inequality (see Chapter8 in Ivić [10]). However, the result (1.12) is not sufficient for the case $k=6$ in (1.10).

A convenient tool is the Montgomery-Vaughan inequality. The Montgomery-Vaughan inequality is

$$\int_0^T \left|\sum_{n\le N} a_n n^{it}\right|^2 dt = T\sum_{n\le N}|a_n|^2 + O\left(\sum_{n\le N} n|a_n|^2\right)$$

for any complex numbers $a_1,\dots,a_N$. Ramachandra used this inequality to give a slightly more precise result

$$\int_2^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt = \frac{1}{2\pi^2}T\log^4 T + O(T\log^3 T)$$

than (1.11) (see in Chapter4 in Ivić [10]). Heath-Brown (in 1979) also improved to

$$\int_2^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt = \sum_{j=0}^4 a_j T\log^j T + E_2(T),$$

where $a_j (0\le j\le 4)$ is a constant with $a_4 = 1/2\pi^2$ and $E_2(T)$ is an error term satisfying $E_2(T) = O(T^{7/8+\varepsilon})$. Further, this error has been improved to $E_2(T) = O(T^{2/3+\varepsilon})$ by Zavorotnyi.

On the other hand, the situation which has been deepest studied is the case $k=1$ that is the mean square values. In 1920's, the result of the form

$$\int_2^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt = T\log T + (2\gamma - 1 - \log 2\pi)T + E(T)$$

was obtained. $E(T) = O(T^{1/2+\varepsilon})$ was indicated by Ingham. After that, in 1934 Titchmarsh improved to $E(T) = O(T^{5/12+\varepsilon})$, further Balasubramanian to $E(T) = O(T^{1/3+\varepsilon})$ in 1978, Huxley proved $E(T) = O(T^{72/227+\varepsilon})$ in 1994.
As another approach, Atkinson [1] gave a formula that precisely represents $E(T)$ in 1949 (See in [1]). Although it was not considered important for many years, this formula is very useful, because the results of the mean values of 12th power and other results can be obtained relatively easily. Also, an analogue of $E(T)$ in $1/2<\sigma<1$ is E_σ which satisfies

$$\int_2^T |\zeta(\sigma+it)|^2 dt = \zeta(2\sigma)T + (2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma} + E_\sigma(T).$$

In 1989, Matsumoto [14] proved an analogue of the Atkinson formula for E_σ in $1/2<\sigma<3/4$. Also in 1993, Matsumoto and Meurman [19] proved a similar formula in $3/4<\sigma<1$.

1.3 Various multiple zeta-functions and L-functions

In 1950, Tornheim [28] studied the values

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^s n^t (m+n)^u} \tag{1.13}$$

where $s,t,u\in\mathbb{Z}$ is in the region of absolute convergence. Also independently, Mordell [25] considered case $s=t=u$ in (1.13) and multiple series

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{1}{m_1\cdots m_r(m_1+\cdots+m_r+a)} \quad (a>-r). \tag{1.14}$$

On the other hand, Apostol and Vu [2] treated the series of the form

$$\sum_{m=1}^{\infty}\sum_{m<n}\frac{1}{m^s n^t (m+n)^u}. \tag{1.15}$$

Furthermore, Matsumoto [15], [17] has introduced the multiple zeta-functions

$$\zeta_{MT,r}(s_1,\dots,s_r;s_{r+1})=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{1}{m_1^{s_1}\cdots m_r^{s_r}(m_1+\cdots+m_r)^{s_{r+1}}} \tag{1.16}$$

and

$$\zeta_{AV,r}(s_1,\dots,s_r;s_{r+1})=\sum\cdots\sum_{1\le m_1<\cdots<m_r<\infty}\frac{1}{m_1^{s_1}\cdots m_r^{s_r}(m_1+\cdots+m_r)^{s_{r+1}}} \tag{1.17}$$

which generalize (1.13) and (1.15) respectively. (1.16) is called the Mordell-Tornheim type, and (1.17) is called the Apostol-Vu type. Matsumoto [16] proved the meromorphic continuation to $\mathbb{C}^{r+1}$ of these multiple zeta-functions by using the Mellin-Barnes integral formula

$$(1+\lambda)^{-s}=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)}\lambda^z dz \tag{1.18}$$

where $s,\lambda\in\mathbb{C}$ with $\sigma=\mathrm{Re}(s)>0, |\arg\lambda|<\pi, \lambda\neq 0$ and $c\in\mathbb{R}$ with $-\sigma<c<0$, and the path (c) of integration is the vertical line $\mathrm{Re}(z)=c$. By using the Mellin-Barnes formula, we find that (1.16) has the recursive structure as

$$\zeta_{MT,r}\to\zeta_{MT,r-1}\to\cdots\to\zeta_{MT,2}\to\zeta$$

(here $A\to B$ means that A can be expressed as an integral involving B) and we can prove meromorphic continuation by tracing the above recursive sequence in reverse. Also, in the case of Apostol-Vu type (1.17), Matsumoto introduced an auxiliary function

$$\widehat{\zeta}_{AV,j,r}(s_1,\dots,s_j;s_{j+1},\dots,s_r;s_{r+1})=\sum\cdots\sum_{1\le m_1<\cdots<m_r<\infty}\frac{1}{m_1^{s_1}\cdots m_r^{s_r}(m_1+\cdots+m_j)^{s_{r+1}}}.$$

The Mellin-Barnes integral formula implies the recursive sequence

$$\zeta_{AV,r}=\widehat{\zeta}_{AV,r,r}\to\widehat{\zeta}_{AV,r,r-1}\to\cdots\to\widehat{\zeta}_{AV,1,r}$$

along which goes the proof of meromorphic continuation to $\mathbb{C}^{r+1}$.

In [29], Maoxiang Wu introduced the χ-analogue of (1.16) and (1.17). Let $\chi_1, \ldots, \chi_r$ be Dirichlet characters of the same modulus q ($q \geq 2$), and define

$$\begin{aligned} &L_{MT,r}(s_1, \ldots, s_r; s_{r+1}; \chi_1, \ldots, \chi_r) \\ &\qquad = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{\chi_1(m_1) \cdots \chi_r(m_r)}{m_1^{s_1} \cdots m_r^{s_r} (m_1 + \cdots + m_r)^{s_{r+1}}} \end{aligned} \tag{1.19}$$

$$\begin{aligned} &L_{AV,r}(s_1, \ldots, s_r; s_{r+1}; \chi_1, \ldots, \chi_r) \\ &\qquad = \sum \cdots \sum_{1 \leq m_1 < \cdots < m_r < \infty} \frac{\chi_1(m_1) \cdots \chi_r(m_r)}{m_1^{s_1} \cdots m_r^{s_r} (m_1 + \cdots + m_r)^{s_{r+1}}}. \end{aligned} \tag{1.20}$$

Wu proved the meromorphic continuation to $\mathbb{C}^{r+1}$, and studied the distribution of possible singularities of (1.19) and (1.20).
Furthermore, in 2016 the author [22] considered the following analogue of the Mordell-Tornheim multiple zeta-function;

$$\begin{aligned} &\widehat{\zeta}_{MT,j,r}(s_1, \ldots, s_j; s_{j+1}, \ldots, s_{r+1}) \\ &= \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{1}{m_1^{s_1} \cdots m_j^{s_j} (m_1 + \cdots + m_j)^{s_{j+1}} \cdots (m_1 + \cdots + m_r)^{s_{r+1}}} \end{aligned} \tag{1.21}$$

and the Mordell-Tornheim multiple L-function;

$$\begin{aligned} &\widehat{L}_{MT,j,r}(s_1, \ldots, s_j; s_{j+1}, \ldots, s_{r+1}; \chi_1, \ldots, \chi_r) \\ &= \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{\chi_1(m_1) \cdots \chi_r(m_r)}{m_1^{s_1} \cdots m_j^{s_j} (m_1 + \cdots + m_j)^{s_{j+1}} \cdots (m_1 + \cdots + m_r)^{s_{r+1}}} \end{aligned} \tag{1.22}$$

where $\chi_1, \chi_2 \cdots, \chi_r$ are Diriclet characters of the same modulus q (≥ 2). By using the Mellin-Barnes integral formula (1.18), series (1.21), (1.22) has the recursive structures

$$\widehat{\zeta}_{MT,j,r} \longrightarrow \widehat{\zeta}_{MT,j,r-1} \longrightarrow \widehat{\zeta}_{MT,j,r-2} \longrightarrow \cdots \longrightarrow \widehat{\zeta}_{MT,j,j+1} \longrightarrow \widehat{\zeta}_{MT,j,j} = \zeta_{MT,j},$$

$$\widehat{L}_{MT,j,r} \longrightarrow \widehat{L}_{MT,j,r-1} \longrightarrow \widehat{L}_{MT,j,r-2} \longrightarrow \cdots \longrightarrow \widehat{L}_{MT,j,j+1} \longrightarrow \widehat{L}_{MT,j,j} = L_{MT,j}$$

respectively, and we can prove the following two theorems:

Theorem 1.2. *For $1 \leq j \leq r$, we have*

(i) *the function $\widehat{\zeta}_{MT,j,r}(s_1, \ldots, s_j; s_{j+1}, \ldots, s_{r+1})$ can be continued meromorphically to the whole $\mathbb{C}^{r+1}$-space,*

(ii) *in the case $j = r-1$, the possible singularities of $\widehat{\zeta}_{MT,r-1,r}$ are located only on the subsets of $\mathbb{C}^{r+1}$ defined by one of the following equations;*

$$\begin{aligned} &s_{r+1} = 1, \\ &s_j + s_r + s_{r+1} = 1 - \ell \quad (1 \leq j \leq r-1,\ \ell \geq -1), \\ &s_{j_1} + s_{j_2} + s_r + s_{r+1} = 2 - \ell \quad (1 \leq j_1 < j_2 \leq r-1,\ \ell \geq -1), \\ &\quad \vdots \\ &s_{j_1} + \cdots + s_{j_{r-2}} + s_r + s_{r+1} = r - 2 - \ell \quad (1 \leq j_1 < \cdots < j_{r-2} \leq r-1,\ \ell \geq -1), \\ &s_1 + \cdots + s_{r-1} + s_r + s_{r+1} = r - 1 - d \quad (d = -1, 0, 1, 3, 5, 7, 9, \ldots). \end{aligned}$$

Also, in the cases $1 \leq j \leq r-2$, possible singularities of $\widehat{\zeta}_{MT,j,r}$ are located only on the subsets of $\mathbb{C}^{r+1}$ defined by one of the following equations;

$$
\begin{aligned}
&s_{r+1} = 1,\\
&s_r + s_{r+1} = 1 - d \quad (d = -1, 0, 1, 3, 5, 7, 9, \ldots),\\
&s_{r-1} + s_r + s_{r+1} = 3 - \ell \quad (\ell \in \mathbb{N}_0),\\
&s_{r-2} + s_{r-1} + s_r + s_{r+1} = 4 - \ell \quad (\ell \in \mathbb{N}_0),\\
&\qquad \vdots\\
&s_{j+2} + s_{j+3} + \cdots + s_r + s_{r+1} = r - j - \ell \quad (\ell \in \mathbb{N}_0),\\
&s_{k_1} + s_{j+1} + \cdots + s_r + s_{r+1} = 1 - \ell' \quad (1 \leq k_1 \leq j,\ \ell' \geqq -(r-j)),\\
&s_{k_1} + s_{k_2} + s_{j+1} + \cdots + s_r + s_{r+1} = 2 - \ell' \quad (1 \leq k_1 < k_2 \leq j,\ \ell' \geqq -(r-j)),\\
&\qquad \vdots\\
&s_{k_1} + \cdots + s_{k_{j-1}} + s_{j+1} + \cdots + s_r + s_{r+1} = j - 1 - \ell'\\
&\qquad\qquad\qquad\qquad\qquad\qquad (1 \leq k_1 < \cdots < k_{j-1} \leq j,\ \ell' \geqq -(r-j)),\\
&s_1 + \cdots + s_j + s_{j+1} + \cdots + s_r + s_{r+1} = j - \ell' \quad (\ell' \geqq -(r-j)).
\end{aligned}
$$

(*iii*) *each of these singularities can be canceled by the corresponding linear factor, and*

(*iv*) $\widehat{\zeta}_{MT,j,r}$ *is of polynomial order with respect to* $|\mathrm{Im}(s_{r+1})|$.

Theorem 1.3. *For $1 \leq j \leq r$, we have*

(*i*) *the function $\widehat{L}_{MT,j,r}(s_1, \ldots, s_j; s_{j+1}, \ldots, s_{r+1}; \chi_1, \ldots \chi_r)$ can be continued meromorphically to the $\mathbb{C}^{r+1}$-space.*

(*ii*) *If none of the characters $\chi_1, \ldots, \chi_r$ are principal, then $\widehat{L}_{MT,j,r}$ is entire. If $\chi_{t_1}, \ldots, \chi_{t_k}$ $(1 \leq t_1 < \cdots < t_k \leq j)$ and $\chi_{r-d_1}, \ldots, \chi_{r-d_h}$ $(1 \leq d_1 < \cdots < d_h \leq r-j)$ are principal character and other characters are non-principal, in the case of $j = r-1$, then possible singularities are located only on the subsets of $\mathbb{C}^{r+1}$ defined by one of the following equation;*

$$
\begin{aligned}
&s_{t_{u(1)}} + s_r + s_{r+1} = 1 - \ell \quad (1 \leq u(1) \leq k,\ \ell \geq -\delta_r),\\
&s_{t_{u(1)}} + s_{t_{u(2)}} + s_r + s_{r+1} = 2 - \ell \quad (1 \leq u(1) < u(2) \leq k,\ \ell \geq -\delta_r),\\
&\qquad \vdots\\
&s_{t_{u(1)}} + \cdots + s_{t_{u(k-1)}} + s_r + s_{r+1} = k - 1 - \ell\\
&\qquad\qquad\qquad\qquad\qquad\qquad (1 \leq u(1) < \cdots < u(k-1) \leq k,\ \ell \geq -\delta_r),\\
&s_{t_1} + \cdots + s_{t_k} + s_r + s_{r+1} = k - \ell \quad (\ell \geqq -\delta_r),
\end{aligned}
\tag{1.23}
$$

where

$$
\delta_r = \begin{cases} 1 & (\chi_r \text{ is principal}),\\ 0 & (\chi_r \text{ is non principal}),\end{cases}
$$

also in the cases of $1 \leq j \leq r-2$*, then possible singularities are located only on the subsets of* $\mathbb{C}^{r+1}$ *defined by one of the following equation;*

$$\begin{aligned}
&s_{r-d_1+1} + s_{r-d_1+2} + \cdots + s_{r+1} = d_1 + 1 - \ell_0 \quad (\ell_0 \in \mathbb{N}_0),\\
&\qquad \vdots\\
&s_{r-d_h+1} + s_{r-d_h+2} + \cdots + s_{r+1} = d_h + 1 - \ell_0 \quad (\ell_0 \in \mathbb{N}_0),\\
&s_{t_{u(1)}} + s_{j+1} + \cdots + s_r + s_{r+1} = 1 - \ell' \quad (1 \leq u(1) \leq k,\ \ell' \geq -\Delta_j),\\
&s_{t_{u(1)}} + s_{t_{u(2)}} + s_{j+1} + \cdots + s_r + s_{r+1} = 2 - \ell'\\
&\qquad\qquad\qquad\qquad\qquad\qquad (1 \leq u(1) < u(2) \leq k,\ \ell' \geq -\Delta_j), \qquad (1.24)\\
&\qquad \vdots\\
&s_{u(1)} + \cdots + s_{u(j-1)} + s_{j+1} + \cdots + s_r + s_{r+1} = j - 1 - \ell'\\
&\qquad\qquad\qquad\qquad\qquad (1 \leq u(1) < \cdots < u(j-1) \leq k,\ \ell' \geq -\Delta_j),\\
&s_1 + \cdots + s_j + s_{j+1} + \cdots + s_r + s_{r+1} = j - \ell' \quad (\ell' \geq -\Delta_j),
\end{aligned}$$

where $\Delta_j = \delta_r + \delta_{r-1} + \cdots + \delta_{r-j}$*. Moreover, if* χ_r *is principal character, then*

$$s_{r+1} = 1$$

is a possible singularity in addition to the above possible singularities (1.23) and (1.24).

(iii) *each of these singularities can be canceled by the corresponding linear factor, and*

(iv) $\widehat{L}_{MT,j,r}$ *is of polynomial order with respect to* $|\mathrm{Im}(s_{r+1})|$*.*

2 Mean values of the Barnes double zeta-functions and the Hurwitz multiple zeta-functions

In the study of order estimation of the Riemann zeta-function, solving Lindelöf hypothesis is an important theme. As one of the relevant matters, asymptotic behavior of mean values has been studied. Furthermore, the theory of the mean values is also noted in the double zeta-functions, and the mean values of the Euler-Zagier type of double zeta-function and Mordell-Tornheim type of double zeta-function were studied. In this section, we prove asymptotic formulas for mean square values of the Barnes double zeta-function with respect to $\mathrm{Im}(s)$ as $\mathrm{Im}(s) \to +\infty$.

2.1 Introduction and the statement of results

The Barnes double zeta-function was first introduced by Barnes [3] in the course of developing his theory of double gamma functions, and the double series of the form as

$$\zeta_2(s, \alpha; v, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(\alpha + vm + wn)^s} \tag{2.1}$$

was introduced and studied in [4]. As a subsequent research, multiple series of similar form as (2.1) was introduced in connection with the theory of multiple gamma functions by Barnes [5].

Let r be a positive integer, $s = \sigma + it$ a complex variable, α a real parameter, and w_j $(j = 1, \ldots, r)$ complex parameters which are located on one of the complex half-plane divided by a straight line through the origin. The Barnes multiple zeta-function $\zeta_r(s, \alpha; w_1, \ldots, w_r)$ is defined by

$$\zeta_r(s, \alpha; w_1, \ldots, w_r) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \frac{1}{(\alpha + w_1 m_1 + \cdots + w_r m_r)^s} \tag{2.2}$$

where the series on the right-hand side is absolutely convergent for $\mathrm{Re}(s) > r$, and is continued meromorphically to the complex s-plane, and its only singularities are the simple poles located at $s = j$ $(j = 1, 2, \ldots, r)$.

In this section, we focus on the case $r = 2$ and $(w_1, w_2) = (v, w)$ for any $v, w > 0$ of (2.2), which is the Barnes double zeta-function (2.1), and study the asymptotic behavior of

$$\int_1^T |\zeta_2(\sigma + it, \alpha; v, w)|^2 dt$$

as $T \to +\infty$.

Let

$$\zeta_2^{[2]}(s_1, s_2, \alpha; v, w) = \sum_{\substack{m_1, n_1, m_2, n_2 \geq 0 \\ vm_1 + wn_1 = vm_2 + wn_2}} \frac{1}{(\alpha + vm_1 + wn_1)^{s_1} (\alpha + vm_2 + wn_2)^{s_2}},$$

which is absolutely convergent for $\mathrm{Re}(s_1 + s_2) > 2$. If v, w are linearly independent over $\mathbb{Q}$, then $vm_1 + wn_1 = vm_2 + wn_2$ is equivalent to $(m_1, n_1) = (m_2, n_2)$, and hence we have

$$\zeta_2^{[2]}(s_1, s_2, \alpha; v, w) = \zeta_2(s_1 + s_2, \alpha; v, w).$$

Theorem 2.1. *For $s = \sigma + it \in \mathbb{C}$ with $\sigma > 2$, we have*

$$\int_1^T |\zeta_2(s, \alpha; v, w)|^2 dt = \zeta_2^{[2]}(\sigma, \sigma, \alpha; v, w)T + O(1)$$

as $T \to +\infty$.

Theorem 2.2. *For $s = \sigma + it \in \mathbb{C}$ with $3/2 < \sigma \leq 2$, we have*

$$\int_1^T |\zeta_2(s, \alpha; v, w)|^2 dt$$
$$= \zeta_2^{[2]}(\sigma, \sigma, \alpha; v, w)T + \begin{cases} O(T^{4-2\sigma} \log T) & (3/2 < \sigma \leq 7/4) \\ O(T^{1/2}) & (7/4 < \sigma \leq 2) \end{cases}$$

as $T \to +\infty$.

Remark 1. We mention here some recent results on mean values of double zeta-functions. Matsumoto-Tsumura [21] treated the Euler double zeta-function

$$\zeta_2(s_1, s_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^{s_1}(m_1 + n_1)^{s_2}}$$

and gave some formulas which imply

$$\int_2^T |\zeta_2(\sigma_1 + it_1, \sigma_2 + it_2)|^2 dt_2 \sim \zeta_2^{[2]}(\sigma_1 + it_1, 2\sigma_2)T \qquad (T \to \infty) \tag{2.3}$$

in some subsets in a region for $\sigma_1 + \sigma_2 > 3/2$, see [21] for the details. Here, $\zeta_2^{[2]}(\sigma_1 + it_1, 2\sigma_2)$ is defined by

$$\zeta_2^{[2]}(s_1, s_2) = \sum_{k=2}^{\infty} \left| \sum_{m=1}^{k-1} \frac{1}{m^{s_1}} \right|^2 \frac{1}{k^{s_2}}$$

which is absolutely convergent for $\mathrm{Re}(s_2) > 1/2$ and $\mathrm{Re}(s_1 + s_2) > 3/2$. Ikeda-Matsuoka-Nagata [9] extended the region of results of Matsumoto-Tsumura [21], and further they gave some asymptotic formulas which imply

$$\int_2^T |\zeta_2(\sigma_1 + it_1, \sigma_2 + it_2)|^2 dt_2 \asymp T \log T \qquad (T \to \infty)$$

on the polygonal line $\{(\sigma_1, \sigma_2) \mid \sigma_1 + \sigma_2 = 3/2 \text{ and } \sigma_2 > 1/2\} \cup \{(\sigma_1, \sigma_2) \mid \sigma_1 \geq 1 \text{ and } \sigma_2 = 1/2\}$. Also, they gave similar results on

$$\int_2^T |\zeta_2(\sigma + it, s_2)|^2 dt, \quad \int_2^T |\zeta_2(\sigma_1 + it, \sigma_2 + it)|^2 dt,$$

see [9] for the details. On the other hand, for the Mordell-Tornheim double zeta-function

$$\zeta_{MT,2}(s_1, s_2; s_3) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3}},$$

Okamoto-Onozuka [26] obtained some results on the mean square values which imply

$$\int_2^T |\zeta_{MT,2}(s_1, s_2; \sigma + it)|^2 dt \sim \zeta_{MT,2}^{[2]}(s_1, s_2; 2\sigma)T \qquad (T \to \infty) \tag{2.4}$$

in some subset in the region for $\sigma_1 + \sigma_2 + \sigma > 3/2$, here $\zeta_{MT,2}^{[2]}(s_1, s_2; 2\sigma)$ is defined by

$$\zeta_{MT,2}^{[2]}(s_1, s_2; s) = \sum_{k=2}^{\infty} \left| \sum_{m=1}^{k-1} \frac{1}{m^{s_1}(k-m)^{s_2}} \right|^2 \frac{1}{k^s},$$

which is absolutely convergent for $2\mathrm{Re}(s_1)+\mathrm{Re}(s) > 1$, $2\mathrm{Re}(s_2)+\mathrm{Re}(s) > 1$ and $2\mathrm{Re}(s_1)+2\mathrm{Re}(s_2)+\mathrm{Re}(s) > 3$. Theorem 2.1 and Theorem 2.2 are the results corresponding to (2.3), (2.4) for the Barnes double zeta-function of the (2.1) version.

2.2 Proof of Theorem 2.1

In this section, we give a proof of Theorem 2.1.

Proof of Theorem 2.1. Let $\sigma + it \in \mathbb{C}$ with $\sigma > 2$. We first calculate $|\zeta_2(s, \alpha; v, w)|^2$. We have

$$\begin{aligned}
&|\zeta_2(s, \alpha; v, w)|^2 = \zeta_2(s, \alpha; v, w)\overline{\zeta_2(s, \alpha; v, w)} \\
&= \sum_{m_1=0}^{\infty}\sum_{n_1=0}^{\infty} \frac{1}{(\alpha + vm_1 + wn_1)^{\sigma+it}} \sum_{m_2=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{1}{(\alpha + vm_2 + wn_2)^{\sigma-it}} \\
&= \sum_{\substack{m_1,n_1,m_2,n_2 \geq 0 \\ vm_1+wn_1 = vm_2+wn_2}} \frac{1}{(\alpha + vm_1 + wn_1)^{\sigma}(\alpha + vm_2 + wn_2)^{\sigma}} \\
&\quad + \sum_{\substack{m_1,n_1,m_2,n_2 \geq 0 \\ vm_1+wn_1 \neq vm_2+wn_2}} \frac{1}{(\alpha + vm_1 + wn_1)^{\sigma}(\alpha + vm_2 + wn_2)^{\sigma}} \left(\frac{\alpha + vm_2 + wn_2}{\alpha + vm_1 + wn_1} \right)^{it} \\
&= \zeta_2^{[2]}(\sigma, \sigma, \alpha; v, w) \\
&\quad + \sum_{\substack{m_1,n_1,m_2,n_2 \geq 0 \\ vm_1+wn_1 \neq vm_2+wn_2}} \frac{1}{(\alpha + vm_1 + wn_1)^{\sigma}(\alpha + vm_2 + wn_2)^{\sigma}} \left(\frac{\alpha + vm_2 + wn_2}{\alpha + vm_1 + wn_1} \right)^{it}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\int_1^T |\zeta_2(s, \alpha; v, w)|^2 dt &= \zeta_2^{[2]}(\sigma, \sigma, \alpha; v, w)(T-1) \\
&\quad + \sum_{\substack{m_1,n_1,m_2,n_2 \geq 0 \\ vm_1+wn_1 \neq vm_2+wn_2}} \frac{1}{(\alpha + vm_1 + wn_1)^{\sigma}(\alpha + vm_2 + wn_2)^{\sigma}} \\
&\qquad \times \frac{e^{iT\log\{(\alpha+vm_2+wn_2)/(\alpha+vm_1+wn_1)\}} - e^{i\log\{(\alpha+vm_2+wn_2)/(\alpha+vm_1+wn_1)\}}}{i\log\{(\alpha + vm_2 + wn_2)/(\alpha + vm_1 + wn_1)\}}.
\end{aligned}$$

The second term on the right-hand side is

$$
\begin{aligned}
&\ll \sum_{\substack{m_1,n_1,m_2,n_2\ge 0\\ vm_1+wn_1<vm_2+wn_2}} \frac{1}{(\alpha+vm_1+wn_1)^\sigma(\alpha+vm_2+wn_2)^\sigma} \\
&\qquad\qquad \times \frac{1}{\log\{(\alpha+vm_2+wn_2)/(\alpha+vm_1+wn_1)\}} \\
&= \left(\sum_{\substack{m_1,n_1,m_2,n_2\ge 0\\ \alpha+vm_1+wn_1<\alpha+vm_2+wn_2<2(\alpha+vm_1+wn_1)}} + \sum_{\substack{m_1,n_1,m_2,n_2\ge 0\\ \alpha+vm_2+wn_2\ge 2(\alpha+vm_1+wn_1)}} \right) \\
&\qquad \frac{1}{(\alpha+vm_1+wn_1)^\sigma(\alpha+vm_2+wn_2)^\sigma \log\{(\alpha+vm_2+wn_2)/(\alpha+vm_1+wn_1)\}}.
\end{aligned}
$$

We denote the right-hand side by V_1+V_2. Then we have

$$
\begin{aligned}
V_2 &\ll \sum_{\substack{m_1,n_1,m_2,n_2\ge 0\\ \alpha+vm_2+wn_2\ge 2(\alpha+vm_1+wn_1)}} \frac{1}{(\alpha+vm_1+wn_1)^\sigma(\alpha+vm_2+wn_2)^\sigma} \\
&\le \sum_{m_1\ge 0}\sum_{n_1\ge 0} \frac{1}{(\alpha+vm_1+wn_1)^\sigma} \sum_{m_2\ge 0}\sum_{n_2\ge 0} \frac{1}{(\alpha+vm_2+wn_2)^\sigma} = O(1).
\end{aligned}
$$

Next we consider the order of V_1. The range of n_2 satisfying the inequalities $\alpha+vm_1+wn_1<\alpha+vm_2+wn_2<2(\alpha+vm_1+wn_1)$ of the condition on the sum V_1 is

$$
\frac{v}{w}(m_1-m_2)+n_1<n_2<\frac{\alpha}{w}+\frac{v}{w}(2m_1-m_2)+2n_1.
$$

Let $\varepsilon=\varepsilon(m_1,m_2,n_1), \delta=\delta(m_1,m_2,n_1)$ be the quantities satisfying $0\le\varepsilon,\delta<1$ and $(v/w)(m_1-m_2)+n_1+\varepsilon\in\mathbb{Z}$ and $\alpha/w+(v/w)(2m_1-m_2)+2n_1-\delta\in\mathbb{Z}$. Then $K=\alpha/w+(v/w)m_1+n_1-\varepsilon-\delta$ is an integer, and n_2 can be rewritten as

$$
n_2=\frac{v}{w}(m_1-m_2)+n_1+\varepsilon+k \quad (\text{for some } k=0,\ 1,\ 2,\ \ldots,\ K).
$$

Since $K\asymp\alpha+vm_1+wn_1\asymp 1+m_1+n_1$, we have

$$
\log\frac{\alpha+vm_2+wn_2}{\alpha+vm_1+wn_1}=\log\left(1+\frac{w\varepsilon+wk}{\alpha+vm_1+wn_1}\right)\asymp\frac{w\varepsilon+wk}{\alpha+vm_1+wn_1},
$$

and hence

$$
\begin{aligned}
V_1 &\ll \sum_{m_1\ge 0}\sum_{n_1\ge 0}\sum_{0\le m_2\ll K}\sum_{0\le k\le K} \frac{1}{(\alpha+vm_1+wn_1)^\sigma} \\
&\qquad\qquad \times \frac{1}{(\alpha+vm_1+wn_1+wk+w\varepsilon)^\sigma} \times \frac{\alpha+vm_1+wn_1}{wk+w\varepsilon} \\
&\ll \sum_{m_1\ge 0}\sum_{n_1\ge 0}\sum_{0\le m_2\ll K} \frac{\log K}{(\alpha+vm_1+wn_1)^{2\sigma-1}} \\
&\ll \sum_{m_1\ge 0}\sum_{n_1\ge 0} \frac{\log(\alpha+vm_1+wn_1)}{(\alpha+vm_1+wn_1)^{2\sigma-2}} \\
&\ll \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\log(2+m+n)}{(1+m+n)^{2\sigma-2}} \ll 1,
\end{aligned}
$$

provided that $\sigma > 2$. Therefore the proof of Theorem 2.1 is complete. $\square$

2.3 The approximation theorem

Let $\sigma_1 > 0$, $x \geq 1$ and $C > 1$. Suppose $s = \sigma + it \in \mathbb{C}$ with $\sigma \geq \sigma_1$ and $|t| \leq 2\pi x/C$. Then

$$\zeta(s) = \sum_{1 \leq n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) \qquad (x \to \infty). \tag{2.5}$$

This asymptotic formula has been proved by Hardy and Littlewood (see Theorem 4.11 in Titchmarsh [27]). Here we prove an analogue of (2.5) for the case of the Barnes double zeta-functions as follows.

Theorem 2.3. *Let $1 < \sigma_1 < \sigma_2$, $x \geq 1$ and $C > 1$. Suppose $s = \sigma + it \in \mathbb{C}$ with $\sigma_1 < \sigma < \sigma_2$ and $|t| \leq 2\pi x/C$. Then*

$$\begin{aligned}\zeta_2(s, \alpha; v, w) = &\sum_{0 \leq m \leq x} \sum_{0 \leq n \leq x} \frac{1}{(\alpha + vm + wn)^s} \\ &+ \frac{(\alpha + vx)^{2-s} + (\alpha + wx)^{2-s} - (\alpha + vx + wx)^{2-s}}{vw(s-1)(s-2)} + O(x^{1-\sigma}) \end{aligned} \tag{2.6}$$

as $x \to \infty$.

Lemma 2.4 (Lemma 4.10 in [27])**.** *Let $f(\xi)$ be a real function with a continuous and steadily decreasing derivative $f'(\xi)$ in (a, b), and let $f'(b) = c$, $f'(a) = d$. Let $g(\xi)$ be a real positive decreasing function with a continuous derivative $g'(\xi)$, satisfying that $|g'(\xi)|$ is steadily decreasing. Then*

$$\begin{aligned}\sum_{a < n \leq b} g(n) e^{2\pi i f(n)} \quad = \quad &\sum_{\substack{\nu \in \mathbb{Z} \\ c - \varepsilon < \nu < d + \varepsilon}} \int_a^b g(\xi) e^{2\pi i (f(\xi) - \nu\xi)} d\xi \\ &+ O(g(a) \log(d - c + 2)) + O(|g'(a)|) \end{aligned} \tag{2.7}$$

for an arbitrary $\varepsilon \in (0, 1)$.

Proof of Theorem 2.3. Let $N \in \mathbb{N}$ be sufficiently large. Then we have

$$\begin{aligned}&\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(\alpha + vm + wn)^s} \\ &\quad = \left(\sum_{m=0}^{N} \sum_{n=0}^{N} + \sum_{m=0}^{N} \sum_{n=N+1}^{\infty} + \sum_{m=N+1}^{\infty} \sum_{n=0}^{N} + \sum_{m=N+1}^{\infty} \sum_{n=N+1}^{\infty} \right) \frac{1}{(\alpha + vm + wn)^s}. \end{aligned}$$

We denote the second, the third and the fourth term on the right-hand side by A_1, A_2 and A_3, respectively. By the Euler-Maclaurin summation formula (see Equation (2.1.2) in [27]), we have for any $a, b \in \mathbb{Z}$ with $0 < a < b$,

$$\sum_{m=a+1}^{b} \frac{1}{(\alpha+vm+wn)^s} = \frac{(\alpha+vb+wn)^{1-s}-(\alpha+va+wn)^{1-s}}{v(1-s)}$$
$$-vs\int_a^b \frac{x-[x]-1/2}{(\alpha+vx+wn)^{s+1}}dx + \frac{1}{2}\left\{(\alpha+vb+wn)^{-s}-(\alpha+va+wn)^{-s}\right\},$$
$$\sum_{n=a+1}^{b} \frac{1}{(\alpha+vm+wn)^s} = \frac{(\alpha+vm+wb)^{1-s}-(\alpha+vm+wa)^{1-s}}{w(1-s)}$$
$$-ws\int_a^b \frac{x-[x]-1/2}{(\alpha+vm+wx)^{s+1}}dx + \frac{1}{2}\left\{(\alpha+vm+wb)^{-s}-(\alpha+vn+wa)^{-s}\right\}.$$

If we take $a = N$ and let $b \to \infty$, we have

$$\begin{aligned}
&\sum_{n=N+1}^{\infty} \frac{1}{(\alpha+vm+wn)^s} \\
&= \frac{1}{w(s-1)} \cdot \frac{1}{(\alpha+vm+wN)^{s-1}} \\
&\qquad -ws\int_N^\infty \frac{x-[x]-1/2}{(\alpha+vm+wx)^{s+1}}dx - \frac{1}{2}\cdot\frac{1}{(\alpha+vm+wN)^s} \\
&= \frac{1}{w(s-1)} \cdot \frac{1}{(\alpha+vm+wN)^{s-1}} - \frac{1}{2}\cdot\frac{1}{(\alpha+vm+wN)^s} + O\left(N^{-\sigma}\right),
\end{aligned}$$

for $\sigma > 1$, uniformly in $m = 0, 1, \ldots$. Therefore we have

$$\begin{aligned}
A_1 &= \sum_{m=0}^{N}\sum_{n=N+1}^{\infty} \frac{1}{(\alpha+vm+wn)^s} \\
&= \sum_{m=0}^{N}\left\{\frac{1}{w(s-1)}\cdot\frac{1}{(\alpha+vm+wN)^{s-1}} - \frac{1}{2}\cdot\frac{1}{(\alpha+vm+wN)^s} + O\left(N^{-\sigma}\right)\right\} \\
&= \frac{1}{w(s-1)}\sum_{m=1}^{N}\frac{1}{(\alpha+vm+wN)^{s-1}} - \frac{1}{2}\sum_{m=1}^{N}\frac{1}{(\alpha+vm+wN)^s} \\
&\qquad + \frac{1}{w(s-1)}\cdot\frac{1}{(\alpha+wN)^{s-1}} - \frac{1}{2}\cdot\frac{1}{(\alpha+wN)^s} + O(N^{1-\sigma}).
\end{aligned}$$

Applying again the formula (2.1.2) in [27] to the first term and the second term on the right-hand side of the above, we obtain

$$\begin{aligned}
A_1 &= \frac{1}{vw(s-1)(s-2)}\left\{\frac{1}{(\alpha+wN)^{s-2}} - \frac{1}{(\alpha+vN+wN)^{s-2}}\right\} \\
&\quad -\frac{v}{w}\int_0^N \frac{x-[x]-1/2}{(\alpha+vx+wN)^s}dx \\
&\quad -\frac{1}{2w(s-1)}\left\{\frac{1}{(\alpha+wN)^{s-1}} - \frac{1}{(\alpha+vN+wN)^{s-1}}\right\} \\
&\quad -\frac{1}{2v(s-1)}\left\{\frac{1}{(\alpha+wN)^{s-1}} - \frac{1}{(\alpha+vN+wN)^{s-1}}\right\} \\
&\quad +\frac{vs}{2}\int_0^N \frac{x-[x]-1/2}{(\alpha+vx+wN)^{s+1}}dx + \frac{1}{4}\left\{\frac{1}{(\alpha+wN)^{s}} - \frac{1}{(\alpha+vN+wN)^{s}}\right\} \\
&\quad +\frac{1}{w(s-1)}\cdot\frac{1}{(\alpha+wN)^{s-1}} - \frac{1}{2}\cdot\frac{1}{(\alpha+wN)^s} + O(N^{1-\sigma}) \\
&= \frac{(\alpha+wN)^{2-s} - (\alpha+vN+wN)^{2-s}}{vw(s-1)(s-2)} + O(N^{1-\sigma}).
\end{aligned}$$

Applying the same method to A_2 and A_3, we obtain

$$\begin{aligned}
A_2 &= \frac{(\alpha+vN)^{2-s} - (\alpha+vN+wN)^{2-s}}{vw(s-1)(s-2)} + O(N^{1-\sigma}) \qquad (\sigma > 1) \\
A_3 &= \frac{(\alpha+vN+wN)^{2-s}}{vw(s-1)(s-2)} + O(N^{1-\sigma}) \qquad (\sigma > 1).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\zeta_2(s,\alpha;v,w) = &\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{(\alpha+vm+wn)^s} \\
&+\frac{(\alpha+vN)^{2-s} + (\alpha+wN)^{2-s} - (\alpha+vN+wN)^{2-s}}{vw(s-1)(s-2)} + O(N^{1-\sigma}) \quad (2.8)
\end{aligned}$$

for $\sigma > 1$. Next we consider the double sum on the right-hand side of (2.8). First we divide the sum as follows:

$$\begin{aligned}
&\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{(\alpha+vm+wn)^s} \\
&= \left(\sum_{m\le x}\sum_{n\le x} + \sum_{m\le x}\sum_{x<n\le N} + \sum_{n\le x}\sum_{x<m\le N} + \sum_{x<m\le N}\sum_{x<n\le N}\right)\frac{1}{(\alpha+vm+wn)^s}.
\end{aligned}$$

We denote the second, the third and the fourth term on the right-hand side by B_1, B_2 and B_3, respectively. Fix $m \in \mathbb{N}$, set

$$f(\xi) = \frac{t}{2\pi}\log(\alpha+vm+w\xi), \ \ g(\xi) = (\alpha+vm+w\xi)^{-\sigma}$$

and take $(a, b) = (x, N)$ in Lemma 2.4. Then we have

$$(c, d) = \left(\frac{tw}{2\pi(\alpha + vm + wN)}, \frac{tw}{2\pi(\alpha + vm + wx)} \right).$$

We see that

$$|f'(x)| = \left| \frac{tw}{2\pi(\alpha + vm + wx)} \right| \le \frac{1}{2\pi} \left| \frac{2\pi x}{C} \cdot \frac{w}{\alpha + vm + wx} \right| \le \frac{1}{C} < 1.$$

When $\sigma > 0$, the function $g(\xi)$ is decreasing, and hence Lemma 2.4 can be applied. For sufficiently large N, we can take ε such that $c - \varepsilon < 0 < d + \varepsilon < 1$, by which only the term with $\nu = 0$ appears in the sum on the right-hand side of (2.7). We obtain from (2.7) that

$$\sum_{x < n \le N} \frac{e^{it \log(\alpha + vm + wn)}}{(\alpha + vm + wn)^{\sigma}} = \int_x^N (\alpha + vm + w\xi)^{-\sigma + it} d\xi + O((m + x)^{-\sigma}).$$

Taking complex conjugates on the both sides, we have

$$\begin{aligned} \sum_{x < n \le N} \frac{1}{(\alpha + vm + wn)^s} &= \int_x^N (\alpha + vm + w\xi)^{-s} d\xi + O((m + x)^{-\sigma}) \\ &= \frac{(\alpha + vm + wN)^{1-s} - (\alpha + vm + wx)^{1-s}}{w(1 - s)} + O((m + x)^{-\sigma}). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} B_1 &= \sum_{m \le x} \sum_{x < n \le N} \frac{1}{(\alpha + vm + wn)^s} \\ &= \sum_{m \le x} \left\{ \frac{(\alpha + vm + wN)^{1-s} - (\alpha + vm + wx)^{1-s}}{w(1 - s)} + O((m + x)^{-\sigma}) \right\} \\ &= \frac{1}{w(1 - s)} \left\{ \sum_{m \le x} \frac{1}{(\alpha + vm + wN)^{s-1}} - \sum_{m \le x} \frac{1}{(\alpha + vm + wx)^{s-1}} \right\} + O(x^{1-\sigma}). \end{aligned}$$

We denote the first and the second term on the right-hand side by $(1/w(1 - s))(B_{11} - B_{12})$, and apply Lemma 2.4 for B_{11} and B_{12}. For B_{11} set

$$f(\xi) = \frac{t}{2\pi} \log(\alpha + v\xi + wN), \; g(\xi) = (\alpha + v\xi + wN)^{1-\sigma}$$

and on taking $(a, b) = (0, x)$ in Lemma 2.4. We can treat B_{12} similarly, where Lemma 2.4 is applied on replacing the variable ξ by η, on setting

$$f(\eta) = \frac{t}{2\pi} \log(\alpha + v\eta + wx), \; g(x) = (\alpha + v\eta + wx)^{1-\sigma}$$

and $(a, b) = (0, x)$. Then we have

$$\begin{aligned} B_{11} &= \frac{(\alpha + vx + wN)^{2-s} - (\alpha + wN)^{2-s}}{v(2 - s)} + O(N^{1-\sigma}), \\ B_{12} &= \frac{(\alpha + vx + wx)^{2-s} - (\alpha + wx)^{2-s}}{v(2 - s)} + O(x^{1-\sigma}). \end{aligned}$$

Therefore, we obtain

$$B_1 = \frac{(\alpha+vx+wN)^{2-s}-(\alpha+vx+wx)^{2-s}-(\alpha+wN)^{2-s}+(\alpha+wx)^{2-s}}{vw(s-1)(s-2)}$$
$$+O(N^{1-\sigma})+O(x^{1-\sigma}).$$

By the argument similar to the treatment of B_1, we obtain

$$B_2 = \frac{(\alpha+vN+wx)^{2-s}-(\alpha+vx+wx)^{2-s}-(\alpha+vN)^{2-s}+(\alpha+vx)^{2-s}}{vw(s-1)(s-2)}$$
$$+O(N^{1-\sigma})+O(x^{1-\sigma})$$

and

$$B_3 = \frac{(\alpha+vN+wN)^{2-s}-(\alpha+vx+wN)^{2-s}-(\alpha+vN+wx)^{2-s}+(\alpha+vx+wx)^{2-s}}{vw(s-1)(s-2)}$$
$$+O(N^{1-\sigma})+O(x^{1-\sigma}).$$

Summing up the results above, we obtain

$$\begin{aligned}
&\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{(\alpha+vm+wn)^s}\\
&= \sum_{m\le x}\sum_{n\le x}\frac{1}{(\alpha+vm+wn)^s}+\frac{(\alpha+vx)^{2-s}+(\alpha+wx)^{2-s}-(\alpha+vx+wx)^{2-s}}{vw(s-1)(s-2)}\\
&-\frac{(\alpha+vN)^{2-s}+(\alpha+wN)^{2-s}-(\alpha+vN+wN)^{2-s}}{vw(s-1)(s-2)}+O(x^{1-\sigma})+O(N^{1-\sigma}),
\end{aligned}$$

and by (2.8), we conclude that

$$\begin{aligned}
&\zeta_2(s,\alpha;v,w)\\
&= \sum_{m\le x}\sum_{n\le x}\frac{1}{(\alpha+vm+wn)^s}+\frac{(\alpha+vx)^{2-s}+(\alpha+wx)^{2-s}-(\alpha+vx+wx)^{2-s}}{vw(s-1)(s-2)}\\
&+O(x^{1-\sigma})+O(N^{1-\sigma})
\end{aligned}$$

in the region $\sigma>1$. Letting $N\to\infty$, we obtain the proof of Theorem 2.3. □

2.4 Proof of Theorem 2.2

In this section, we prove Theorem 2.2 from Theorem 2.3.

Proof of Theorem 2.2. Setting $C=2\pi$ and $x=t$ in (2.6), we easily see that the second term on the right-hand side is $O(t^{-\sigma})$, hence we have

$$\zeta_2(s,\alpha;v,w)=\sum_{m\le t}\sum_{n\le t}\frac{1}{(\alpha+vm+wn)^s}+O(t^{1-\sigma}). \tag{2.9}$$

We denote the first term on the right-hand side by $\Sigma(s)$. Then

$$\begin{aligned}
&\int_1^T |\Sigma(s)|^2 dt \\
&\quad = \int_1^T \sum_{m_1\le t}\sum_{n_1\le t}\frac{1}{(\alpha+vm_1+wn_1)^{\sigma+it}}\sum_{m_2\le t}\sum_{n_2\le t}\frac{1}{(\alpha+vm_2+wn_2)^{\sigma-it}}dt.
\end{aligned}$$

Now we change the order of summation and integration. First we note that $1\le m_1,n_1,m_2,n_2\le T$. Let us fix one such (m_1,n_1,m_2,n_2). Then from the condition $m_1\le t,\ n_1\le t,\ m_2\le t,\ n_2\le t$, we find that the range of t is $M=\max\{m_1,n_1,m_2,n_2\}\le t\le T$. Therefore

$$\begin{aligned}
&\int_1^T |\Sigma(s)|^2 dt \\
&\quad = \sum_{m_1,n_1\le T}\frac{1}{(\alpha+vm_1+wn_1)^{\sigma}}\sum_{m_2,n_2\le T}\frac{1}{(\alpha+vm_2+wn_2)^{\sigma}} \\
&\qquad\qquad \times\int_M^T\left(\frac{\alpha+vm_1+wn_1}{\alpha+vm_2+wn_2}\right)^{it}dt \\
&\quad = \sum_{\substack{0\le m_1,n_1,m_2,n_2\le T\\ vm_1+wn_1=vm_2+wn_2}}\frac{1}{(\alpha+vm_1+wn_1)^{\sigma}(\alpha+vm_2+wn_2)^{\sigma}}\times(T-M) \\
&\quad + \sum_{\substack{m_1,n_1,m_2,n_2\ge 0\\ vm_1+wn_1\ne vm_2+wn_2}}\frac{1}{(\alpha+vm_1+wn_1)^{\sigma}(\alpha+vm_2+wn_2)^{\sigma}} \\
&\quad \times\frac{e^{iT\log\{(\alpha+vm_2+wn_2)/(\alpha+vm_1+wn_1)\}}-e^{iM\log\{(\alpha+vm_2+wn_2)/(\alpha+vm_1+wn_1)\}}}{i\log\{(\alpha+vm_2+wn_2)/(\alpha+vm_1+wn_1)\}}.
\end{aligned}$$

We denote the first and the second term on the right-hand side by S_1 and S_2 respectively. As for S_1, we have

$$\begin{aligned}
S_1 &= T\left\{\zeta_2^{[2]}(\sigma,\sigma,\alpha,v,w)-(U_1+U_2+U_3+U_4)\right\} \\
&\qquad - \sum_{\substack{0\le m_1,n_1,m_2,n_2\le T\\ vm_1+wn_1=vm_2+wn_2}}\frac{M(m_1,n_1,m_2,n_2)}{(\alpha+vm_1+wn_1)^{\sigma}(\alpha+vm_2+wn_2)^{\sigma}},
\end{aligned}$$

where

$$
\begin{aligned}
U_1 &= \left(\sum_{\substack{m_1>T\\ 0\le n_1,m_2,n_2\le T\\ vm_1+wn_1=vm_2+wn_2}} + \sum_{\substack{n_1>T\\ 0\le m_1,m_2,n_2\le T\\ vm_1+wn_1=vm_2+wn_2}} \right.\\
&\quad \left. + \sum_{\substack{m_2>T\\ 0\le m_1,n_1,n_2\le T\\ vm_1+wn_1=vm_2+wn_2}} + \sum_{\substack{n_2>T\\ 0\le m_1,n_1,m_2\le T\\ vm_1+wn_1=vm_2+wn_2}} \right) \frac{1}{(\alpha+vm_1+wn_1)^\sigma(\alpha+vm_2+wn_2)^\sigma},\\
U_2 &= \left(\sum_{\substack{m_1,m_2>T\\ 0\le n_1,n_2\le T\\ vm_1+wn_1=vm_2+wn_2}} + \sum_{\substack{n_1,n_2>T\\ 0\le m_1,m_2\le T\\ vm_1+wn_1=vm_2+wn_2}} \right) \frac{1}{(\alpha+vm_1+wn_1)^\sigma(\alpha+vm_2+wn_2)^\sigma},\\
U_3 &= \left(\sum_{\substack{m_1,n_2>T\\ 0\le n_1,m_2\le T\\ vm_1+wn_1=vm_2+wn_2}} + \sum_{\substack{n_1,m_2>T\\ 0\le m_1,n_2\le T\\ vm_1+wn_1=vm_2+wn_2}} \right) \frac{1}{(\alpha+vm_1+wn_1)^\sigma(\alpha+vm_2+wn_2)^\sigma},\\
U_4 &= \left(\sum_{\substack{m_1,m_2,n_1>T\\ 0\le n_2\le T\\ vm_1+wn_1=vm_2+wn_2}} + \sum_{\substack{m_1,m_2,n_2>T\\ 0\le n_1\le T\\ vm_1+wn_1=vm_2+wn_2}} \right.\\
&\quad \left. + \sum_{\substack{m_1,n_1,n_2>T\\ 0\le m_2\le T\\ vm_1+wn_1=vm_2+wn_2}} + \sum_{\substack{m_2,n_1,n_2>T\\ 0\le m_1\le T\\ vm_1+wn_1=vm_2+wn_2}} \right) \frac{1}{(\alpha+vm_1+wn_1)^\sigma(\alpha+vm_2+wn_2)^\sigma}.
\end{aligned}
$$

We can estimate U_1 as follows. Since $\alpha + vm + wn \asymp 1 + m + n$ we have

$$
U_1 \ll \sum_{\substack{k>T\\ 0\le l,m,n\le T\\ k+l\asymp m+n}} \frac{1}{(1+k+l)^\sigma(1+m+n)^\sigma}.
$$

Setting $j = k + l$, since $T + 1 < j \asymp m + n \le 2T \ll T$ and $m \ll j$, we obtain

$$
\begin{aligned}
U_1 &\ll \sum_{T+1\le j\ll T} \sum_{T<k\le j} \sum_{0\le m\ll j} \frac{1}{(1+j)^{2\sigma}}\\
&\ll \sum_{T+1\le j\ll T} \sum_{T<j\le j} \frac{1}{(1+j)^{2\sigma-1}} = \sum_{T+1\le j\ll T} \frac{j-T}{(1+j)^{2\sigma-1}}\\
&\ll \sum_{T+1\le j\ll T} \frac{1}{(1+j)^{2\sigma-2}} \asymp T^{3-2\sigma} \quad \left(\sigma \ge \frac{3}{2}\right).
\end{aligned}
$$

Similarly we obtain $U_2,\ U_3,\ U_4 \ll T^{3-2\sigma}$. The sum involving $M(m_1,n_1,m_2,n_2)$ in S_1 is estimated, since $M(k,l,m,n) \ll k+l+m+n$ in this case, as

$$\begin{aligned}
&\ll \sum_{\substack{0\le k,l,m,n\le T\\ k+l\asymp m+n}} \frac{k+l+m+n}{(1+k+l)^\sigma(1+m+n)^\sigma} \ll \sum_{0\le j\le 2T}\sum_{0\le k\le j}\sum_{0\le m\ll j}\frac{1}{(1+j)^{2\sigma-1}}\\
&\ll \sum_{0\le j\le 2T}\frac{1}{(1+j)^{2\sigma-3}} \ll \begin{cases} T^{4-2\sigma} & (3/2\le\sigma<2)\,,\\ \log T & (\sigma=2).\end{cases}
\end{aligned}$$

Therefore, we have

$$S_1 = \zeta_2^{[2]}(\sigma,\sigma,\alpha,v,w)T + \begin{cases} O(T^{4-2\sigma}) & (3/2\le\sigma<2)\,,\\ O(\log T) & (\sigma=2).\end{cases} \tag{2.10}$$

Next, as for S_2, we have

$$\begin{aligned}
S_2 \ll &\sum_{\substack{0\le m_1,n_1,m_2,n_2\le T\\ vm_1+wn_1<vm_2+wn_2}} \frac{1}{(\alpha+vm_1+wn_1)^\sigma(\alpha+vm_2+wn_2)^\sigma}\\
&\qquad\times\frac{1}{\log\{(\alpha+vm_2+wn_2)/(\alpha+vm_1+wn_1)\}}\\
= &\left(\sum_{\substack{0\le m_1,n_1,m_2,n_2\le T\\ \alpha+vm_1+wn_1<\alpha+vm_2+wn_2<2(\alpha+vm_1+wn_1)}} + \sum_{\substack{0\le m_1,n_1,m_2,n_2\le T\\ \alpha+vm_2+wn_2\ge 2(\alpha+vm_1+wn_1)}}\right)\\
&\frac{1}{(\alpha+vm_1+wn_1)^\sigma(\alpha+vm_2+wn_2)^\sigma\log\{(\alpha+vm_2+wn_2)/(\alpha+vm_1+wn_1)\}}.
\end{aligned}$$

We denote the first and the second term on the right-hand side by W_1 and W_2, respectively. As for W_2, we have

$$\begin{aligned}
W_2 &\ll \sum_{\substack{0\le m_1,n_1,m_2,n_2\le T\\ \alpha+vm_2+wn_2\ge 2(\alpha+vm_1+wn_1)}} \frac{1}{(\alpha+vm_1+wn_1)^\sigma(\alpha+vm_2+wn_2)^\sigma}\\
&\ll \left\{\sum_{0\le m,n\le T}\frac{1}{(1+m+n)^\sigma}\right\}^2\\
&= \left\{\sum_{0\le j\le 2T}\sum_{m=0}^{j}\frac{1}{(1+j)^\sigma}\right\}^2 = \left\{\sum_{0\le j\le 2T}\frac{1}{(1+j)^{\sigma-1}}\right\}^2\\
&\ll \begin{cases} T^{4-2\sigma} & (1<\sigma<2)\,,\\ (\log T)^2 & (\sigma=2).\end{cases}
\end{aligned}$$

Next we consider the order of W_1.The range of n_2 in the inequalities $\alpha+vm_1+wn_1 < \alpha+vm_2+wn_2 < 2(\alpha+vm_1+wn_1)$ of the summation condition on W_1 is

$$\frac{v}{w}(m_1-m_2)+n_1 < n_2 < \min\left\{\frac{\alpha}{w}+\frac{v}{w}(2m_1-m_2)+2n_1,\ [T]+1\right\}.$$

Let $\varepsilon = \varepsilon(m_1, m_2, n_1), \delta = \delta(m_1, m_2, n_1)$ be constants satisfying $0 \leq \varepsilon, \delta < 1$ and $(v/w)(m_1 - m_2) + n_1 + \varepsilon \in \mathbb{Z}$ and $\alpha/w + (v/w)(2m_1 - m_2) + 2n_1 - \delta \in \mathbb{Z}$. Then

$$K = \min\left\{\frac{\alpha}{w} + \frac{v}{w} \cdot m_1 + n_1 - \varepsilon - \delta,\ [T] - \frac{v}{w}(m_1 - m_2) - n_1 - \varepsilon\right\}$$

is an integer, and n_2 can be rewritten as

$$n_2 = \frac{v}{w}(m_1 - m_2) + n_1 + \varepsilon + k \quad (\text{for some } k = 0,\ 1,\ 2,\ \ldots,\ K).$$

Since

$$\log \frac{\alpha + vm_2 + wn_2}{\alpha + vm_1 + wn_1} = \log\left(1 + \frac{wk + w\varepsilon}{\alpha + vm_1 + wn_1}\right) \asymp \frac{wk + w\varepsilon}{\alpha + vm_1 + wn_1},$$

we obtain

$$\begin{aligned}
W_1 &\ll \sum_{0 \leq m_1 \leq T} \sum_{0 \leq n_1 \leq T} \sum_{0 \leq m_2 \ll K} \sum_{0 \leq k \leq K} \frac{1}{(\alpha + vm_1 + wn_1)^{\sigma}} \\
&\qquad\qquad \times \frac{1}{(\alpha + vm_1 + wn_1 + wk + w\varepsilon)^{\sigma}} \times \frac{\alpha + vm_1 + wn_1}{wk + w\varepsilon} \\
&\ll \sum_{0 \leq m_1 \leq T} \sum_{0 \leq n_1 \leq T} \sum_{0 \leq m_2 \ll K} \frac{\log K}{(\alpha + vm_1 + wn_1)^{2\sigma - 1}} \\
&\ll \sum_{0 \leq m_1 \leq T} \sum_{0 \leq n_1 \leq T} \frac{\log(\alpha + vm_1 + wn_1)}{(\alpha + vm_1 + wn_1)^{2\sigma - 2}} \\
&\ll \sum_{0 \leq m \leq T} \sum_{0 \leq n \leq T} \frac{\log(2 + m + n)}{(1 + m + n)^{2\sigma - 2}} \\
&\ll \int_0^T \int_0^T \frac{\log(2 + x + y)}{(1 + x + y)^{2\sigma - 2}} dxdy \\
&= \begin{cases} O(T^{4 - 2\sigma} \log T) & (1 < \sigma < 3/2, 3/2 < \sigma < 2)\,, \\ O(T(\log T)^2) & (\sigma = 3/2)\,, \\ O((\log T)^2) & (\sigma = 2). \end{cases}
\end{aligned}$$

Then, we have

$$\begin{aligned}
S_2 &= \begin{cases} O(T^{4 - 2\sigma}) & (1 < \sigma < 2) \\ O((\log T)^2) & (\sigma = 2) \end{cases} + \begin{cases} O(T^{4 - 2\sigma} \log T) & (1 < \sigma < 3/2, 3/2 < \sigma < 2) \\ O(T(\log T)^2) & (\sigma = 3/2) \\ O((\log T)^2) & (\sigma = 2) \end{cases} \\
&= \begin{cases} O(T^{4 - 2\sigma} \log T) & (1 < \sigma < 3/2, 3/2 < \sigma < 2)\,, \\ O(T(\log T)^2) & (\sigma = 3/2)\,, \\ O((\log T)^2) & (\sigma = 2). \end{cases}
\end{aligned}$$

By (2.10), we have

$$
\begin{aligned}
&\int_1^T |\Sigma(s)|^2 dt \\
&= \zeta_2^{[2]}(\sigma,\sigma;\alpha,v,w)T + \begin{cases} O(T^{4-2\sigma}\log T) & (1<\sigma<3/2, 3/2<\sigma<2)\,, \\ O(T(\log T)^2) & (\sigma=3/2)\,, \\ O((\log T)^2) & (\sigma=2). \end{cases}
\end{aligned}
$$

Furthermore, we obtain from (2.9) that

$$
\begin{aligned}
&\int_1^T |\zeta_2(\sigma+it,\alpha;v,w)|^2 dt \\
&= \int_1^T |\Sigma(s)+O(t^{1-\sigma})|^2 dt \\
&= \int_1^T |\Sigma(s)|^2 dt + O\left(\int_1^T |\Sigma(s)|t^{1-\sigma} dt\right) + O\left(\int_1^T t^{2-2\sigma} dt\right).
\end{aligned}
$$

We see that the third term on the right-hand side is estimated as

$$
\begin{cases} O(T^{3-2\sigma}) & (1<\sigma<3/2), \\ O(\log T) & (\sigma=3/2), \\ O(1) & (3/2<\sigma\le 2). \end{cases} \tag{2.11}
$$

Also, using the Cauchy-Schwarz inequality for the second term on the right-hand side, we see that

$$
\begin{aligned}
\int_1^T |\Sigma(s)|t^{1-\sigma} dt &\le \left(\int_1^T |\Sigma(s)|^2 dt\right)^{1/2} \left(\int_1^T t^{2-2\sigma} dt\right)^{1/2} \\
&= \begin{cases} O(T^{7/2-2\sigma}(\log T)^{1/2}) & (1<\sigma<3/2)\,, \\ O(T^{1/2}(\log T)^{3/2}) & (\sigma=3/2)\,, \\ O(T^{1/2}) & (3/2<\sigma\le 2). \end{cases}
\end{aligned} \tag{2.12}
$$

Therefore, since (2.11) and (2.12) we have

$$
\int_1^T |\zeta_2(s,\alpha;v,w)|^2 dt = \zeta_2^{[2]}(\sigma,\sigma,\alpha;v,w)T + \begin{cases} O(T^{4-2\sigma}\log T) & (3/2<\sigma\le 7/4)\,, \\ O(T^{1/2}) & (7/4<\sigma\le 2), \end{cases}
$$

and hence the proof of Theorem 2.2 is complete. □

2.5 Hurwitz double and triple zeta-functions

Let $\alpha>0$. Hurwitz multiple zeta-function is defined by

$$
\zeta_r(s,\alpha) = \sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty} \frac{1}{(\alpha+m_1+\cdots+m_r)^s} \quad (\mathrm{Re}(s)>r).
$$

This function $\zeta_r(s,\alpha)$ can be expressed in terms of simple series as follows;

$$\zeta_r(s,\alpha)=\sum_{j=0}^{r-1}p_{r,j}(\alpha)\zeta(s-j,\alpha),$$
$$p_{r,j}(\alpha)=\frac{1}{(r-1)!}\sum_{l=j}^{r-1}(-1)^{r+1-j}\binom{l}{j}S(r,l+1)\alpha^{l-j},$$

where $S(r,l+1)$ is the Stirling number of the 1st kind, and in the case $r=2, r=3$ they are

$$\zeta_2(s,\alpha)=(1-\alpha)\zeta(s,\alpha)+\zeta(s-1,\alpha),$$
$$\zeta_3(s,\alpha)=\frac{1}{2}(1-\alpha)(2-\alpha)\zeta(s,\alpha)+\frac{1}{2}(3-2\alpha)\zeta(s-1,\alpha)+\frac{1}{2}\zeta(s-2,\alpha)$$

respectively. For example, in the case $r=2$, it can be shown simply as follows.

$$\begin{aligned}\zeta_2(s,\alpha)&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(\alpha+m+n)^s}=\sum_{k=0}^{\infty}\frac{1}{(\alpha+k)^s}\sum_{\substack{m,n\geq 0\\ m+n=k}}1\\ &=\sum_{k=0}^{\infty}\frac{k+1}{(\alpha+k)^s}=\sum_{k=0}^{\infty}\frac{(\alpha+k)+(1-\alpha)}{(\alpha+k)^s}\\ &=(1-\alpha)\zeta(s,\alpha)+\zeta(s-1,\alpha)\quad(\sigma>2),\end{aligned}$$

and then it is possible to extend the result to $\mathbb{C}$ by analytic continuation.
In the case $r=2$, using the above equation we have

$$\begin{aligned}\int_1^T|\zeta_2(s,\alpha)|^2dt&=\int_1^T|(1-\alpha)\zeta(s,\alpha)+\zeta(s-1,\alpha)|^2dt\\ &=(1-\alpha)^2\int_1^T|\zeta(s,\alpha)|^2dt+\int_1^T|\zeta(s-1,\alpha)|^2dt\\ &\quad+2(1-\alpha)\mathrm{Re}\left(\int_1^T\zeta(s,\alpha)\overline{\zeta(s-1,\alpha)}dt\right).\end{aligned}$$

Evaluating the integral of the third term, we have

$$\int_1^T\zeta(s,\alpha)\overline{\zeta(s-1,\alpha)}dt=\zeta(2\sigma-1,\alpha)T+\begin{cases}O(T^{1/2}) & (3/2<\sigma\leq 2),\\ O((T\log T)^{1/2}) & (\sigma=3/2),\end{cases}$$

and using the results on mean values of the Hurwitz zeta-function,

$$\int_1^T|\zeta(\sigma+it,\alpha)|^2dt=\zeta(2\sigma,\alpha)T+O(T^{2-2\sigma})$$

and

$$\int_1^T|\zeta(\sigma+it,\alpha)|^2dt=T\log T+\left(\gamma(\alpha)+\frac{\gamma}{\alpha}-1-\log 2\pi\right)T+O((T\log T)^{1/2})$$

where γ is Euler constant and

$$\gamma(\alpha) := \lim_{N\to\infty}\left(\sum_{n=0}^{N}\frac{1}{n+\alpha} - \log N\right),$$

we obtain the following Theorem 2.5 and Theorem 2.6.

Theorem 2.5. (i) *In the case $3/2 < \sigma \le 2$, we have*

$$\begin{aligned}&\int_1^T |\zeta_2(\sigma+it,\alpha)|^2 dt\\ &= \{(1-\alpha)^2\zeta(2\sigma,\alpha) + 2(1-\alpha)\zeta(2\sigma-1,\alpha) + \zeta(2\sigma-2,\alpha)\}T\\ &\qquad +O(T^{1/2}) + O(T^{4-2\sigma}),\end{aligned}$$

as $T \to \infty$.

(ii) *In the case $\sigma = 3/2$, we have*

$$\begin{aligned}&\int_1^T \left|\zeta_2\left(\frac{3}{2}+it,\alpha\right)\right|^2 dt\\ &= T\log T + \left\{(1-\alpha)^2\zeta(3,\alpha) + 2(1-\alpha)\zeta(2,\alpha) + \gamma(\alpha) + \frac{\gamma}{\alpha} - 1 - \log 2\pi\right\}T\\ &\qquad +O((T\log T)^{1/2}).\end{aligned}$$

as $T \to \infty$.

Let

$$\zeta_3^{[2]}(s_1,s_2,\alpha;w_1,w_2,w_3) = \sum_{\substack{m_1,m_2,m_3,n_1,n_2,n_3\ge 0\\ w_1m_1+w_2m_2+w_3m_3=w_1n_1+w_2n_2+w_3n_3}} \frac{1}{(\alpha+w_1m_1+w_2m_2+w_3m_3)^{s_1}(\alpha+w_1n_1+w_2n_2+w_3n_3)^{s_2}},$$

which is absolutely convergent for $\mathrm{Re}(s_1+s_2) > 2$. If w_1, w_2, w_3 are linearly independent over $\mathbb{Q}$, then $w_1m_1+w_2m_2+w_3m_3 = w_1n_1+w_2n_2+w_3n_3$ is equivalent to $(m_1,m_2,m_3) = (n_1,n_2,n_3)$, and hence we have

$$\zeta_2^{[2]}(s_1,s_2,\alpha;w_1,w_2,w_3) = \zeta_2(s_1+s_2,\alpha;w_1,w_2,w_3).$$

Theorem 2.6. (i) *In the case $5/2 < \sigma \le 3$, we have*

$$\int_1^T |\zeta_3(\sigma+it,\alpha)|^2 dt = \zeta_3^{[2]}(\sigma,\sigma,\alpha;1,1)T + O(T^{1/2}) + O(T^{4-2\sigma}).$$

as $T \to \infty$.

(ii) *In the case $\sigma = 5/2$, we have*

$$\int_1^T \left|\zeta_3\left(\frac{5}{2}+it,\alpha\right)\right|^2 dt$$
$$= \frac{1}{4}T\log T + \frac{1}{4}\Big\{(1-\alpha)^2(2-\alpha)^2\zeta(5,\alpha) + 2(1-\alpha)(2-\alpha)(3-2\alpha)\zeta(4,\alpha)$$
$$+(6\alpha^2-18\alpha+13)\zeta(3,\alpha) + 2(3-2\alpha)\zeta(2,\alpha) + \gamma(\alpha) + \frac{\gamma}{\alpha} - 1 - \log 2\pi\Big\}T$$
$$+O((T\log T)^{1/2}).$$

as $T \to \infty$.

2.6 Hurwitz mulitple zeta-functions

The function $\zeta_r(s,\alpha)$ satisfies

$$\zeta_r(s,\alpha) = \sum_{j=0}^{r-1} p_{r,j}(\alpha)\zeta(s-j,\alpha),$$

and so

$$\int_1^T |\zeta_r(s,\alpha)|^2 dt = \sum_{j=0}^{r-1} p_{r,j}(\alpha)^2 \int_1^T |\zeta(s-j,\alpha)|^2 dt$$
$$+2\sum_{0\le k<l\le r-1} p_{r,k}(\alpha)p_{r,l}(\alpha)\cdot \mathrm{Re}\left(\int_1^T \zeta(s-k,\alpha)\overline{\zeta(s-l,\alpha)}dt\right).$$

Consider the evaluation of the each term of above equation. In particular, the main term in the case $\sigma = r-1/2$ is

$$\int_1^T |\zeta(s-r+1,\alpha)|^2 dt \sim T\log T.$$

Then we obtain the following result:

Theorem 2.7. (i) *In the case $\sigma > r-1/2$, as $T\to\infty$*

$$\int_1^T |\zeta_r(\sigma+it,\alpha)|^2 dt \asymp T.$$

(ii) *In the case $\sigma = r-1/2$, as $T\to\infty$*

$$\int_1^T \left|\zeta_r\left(r-\frac{1}{2}+it,\alpha\right)\right|^2 dt = \frac{1}{\{(r-1)!\}^2}T\log T + O(T^{1/2}\log T).$$

Remark 2. Recall that the mean square value of $\zeta(s)$ on the critical line $\sigma = 1/2$, which is asymptotically $T\log T$. From the results of Theorem 2.7 (ii), it can be expected that for r-ple zeta-function the line $\sigma = r-1/2$ would be an analogue of the critical line.

3 Approximate functional equations for the Hurwitz and Lerch zeta-functions

As mentioned in Section 1, the approximate functional equation (1.7) is effective in studying order and mean values, so it is an important theme to study approximate functional equations for other type of zeta-functions. In 2003, R. Garunkštis, A. Laurinčikas, and J. Steuding (in [7]) proved the Riemann-Siegel type of the approximate functional equation for the Lerch zeta-function $\zeta_L(s,\alpha,\lambda)$. In this section, we prove another type of approximate functional equations for the Hurwitz and Lerch zeta-functions. R. Garunkštis, A. Laurinčikas, and J. Steuding (in [8]) obtained the results on the mean square values of $\zeta_L(\sigma+it,\alpha,\lambda)$ with respect to t. We obtain the main term of the mean square values of $\zeta_L(1/2+it,\alpha,\lambda)$ using a simpler method than their method in [8].

3.1 Introduction and the statement of results

Let $s=\sigma+it$ be a complex variable, and let $0<\alpha\le 1, 0<\lambda\le 1$ be real parameters. The Hurwitz zeta-function $\zeta_H(s,\alpha)$ and the Lerch zeta-function $\zeta_L(s,\alpha,\lambda)$ are defined by

$$\zeta_H(s,\alpha)=\sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^s}, \tag{3.1}$$

$$\zeta_L(s,\alpha,\lambda)=\sum_{n=0}^{\infty}\frac{e^{2\pi in\lambda}}{(n+\alpha)^s}, \tag{3.2}$$

respectively. These series are absolutey convergent for $\sigma>1$. Also, if $0<\lambda<1$, then the series (3.2) is convergent even for $\sigma>0$.

As a classical asymptotic formula for the Riemann zeta-function, the following was proved by Hardy and Littlewood (§4 in [27]); we suppose that $\sigma_0>0$, $x\ge 1$, then

$$\zeta(s)=\sum_{n\le x}\frac{1}{n^s}-\frac{x^{1-s}}{1-s}+O(x^{-\sigma})$$

uniformly for $\sigma\ge\sigma_0$, $|t|<2\pi x/C$, where $C>1$ is a constant. Also, Hardy and Littlewood proved the following asymptotic formula (§4 in [27]); we suppose that $0\le\sigma\le 1$, $x\ge 1$, $y\ge 1$ and $2\pi xy=|t|$, then

$$\zeta(s)=\sum_{n\le x}\frac{1}{n^s}+X(s)\sum_{n\le y}\frac{1}{n^{1-s}}+O(x^{-\sigma})+O(|t|^{1/2-\sigma}y^{\sigma-1}), \tag{3.3}$$

where $X(s)=2\Gamma(1-s)\sin(\pi s/2)(2\pi)^{s-1}$ and note that $\zeta(s)=X(s)\zeta(1-s)$ holds. This is called the approximate functional equation.

Further, there is a Riemann-Siegel type of the approximate functional equation for $\zeta(s)$; suppose that $0\le\sigma\le 1, x=\sqrt{t/2\pi}$, and $N<Ct$ with a sufficiently small constant C. Then

$$\zeta(s)=\sum_{n\le x}\frac{1}{n^s}+X(s)\sum_{n\le x}\frac{1}{n^{1-s}}+(-1)^{[x]-1}e^{\pi i(1-s)/2}(2\pi t)^{s/2-1/2}e^{it/2-i\pi/8}$$
$$\times\Gamma(1-s)\left(S_N+O\left(\left(\frac{CN}{t}\right)^{N/6}\right)+O(e^{-Ct})\right), \tag{3.4}$$

where

$$S_N = \sum_{n=0}^{N-1} \sum_{\nu \le n/2} \frac{n! i^{\nu-n}}{\nu!(n-2\nu)!2^n} \left(\frac{2}{\pi}\right)^{n/2-\nu} a_n \psi^{(n-2\nu)}\left(\sqrt{\frac{2t}{\pi}} - 2[x]\right),$$

with a_n defined by

$$\exp\left((s-1)\log\left(1+\frac{z}{\sqrt{t}}\right) - iz\sqrt{t} + \frac{1}{2}iz^2\right) = \sum_{n=0}^{\infty} a_n z^n,$$

with $a_0 = 1, a_n \ll t^{-n/2+[n/3]}$. R. Garunkštis, A. Laurinčikas, and J. Steuding proved an analogue of (3.4) for the Lerch zeta-function as follows;

Theorem 3.1 (R. Garunkštis, A. Laurinčikas, and J. Steuding [7]). *Suppose that* $0 < \alpha \le 1, 0 < \lambda < 1$ *and* $0 \le \sigma \le 1$. *Suppose that* $t \ge 1, x = \sqrt{t/2\pi}, N = [x], M = [x-\alpha]$ *and* $\beta = N - M$. *Then*

$$\zeta_L(s,\alpha,\lambda) = \sum_{m=0}^{M} \frac{e^{2\pi i m\lambda}}{(m+\alpha)^s} + \left(\frac{2\pi}{t}\right)^{\sigma-1/2+it} e^{it+\pi i/4-2\pi i\{\lambda\}\alpha} \sum_{n=0}^{N} \frac{e^{-2\pi i\alpha n}}{(n+\lambda)^{1-s}}$$
$$+ \left(\frac{2\pi}{t}\right)^{\sigma/2} e^{\pi i f(\lambda,\alpha,\sigma,t)} \phi(2x - 2N + \beta - \{\lambda\} - \alpha) + O(t^{(\sigma-2)/2}), \quad (3.5)$$

where

$$f(\lambda,\alpha,\sigma,t) = -\frac{t}{2\pi}\log\frac{t}{2\pi e} - \frac{7}{8} + \frac{1}{2}(\alpha^2 - \{\lambda\}^2)$$
$$-\alpha\beta + 2x(\beta + \{\lambda\} - \alpha) - \frac{1}{2}(N+M) - \{\lambda\}(\beta+\alpha).$$

We prove an analogue of the approximate functional equation (3.3) for (3.1) and (3.2) (in Theorem 3.2), and gave another proof of the mean square formula for $\zeta_L(1/2+it,\alpha,\lambda)$ with respect to t (in Theorem 3.3).

Theorem 3.2. *Let* $0 < \alpha \le 1$ *and* $0 < \lambda < 1$. *Suppose that* $0 \le \sigma \le 1,\ x \ge 1,\ y \ge 1$ *and* $2\pi xy = |t|$. *Then*

$$\zeta_L(s,\alpha,\lambda) = \sum_{0 \le n \le x} \frac{e^{2\pi i n\lambda}}{(n+\alpha)^s}$$
$$+\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left\{e^{\{(1-s)/2-2\alpha\lambda\}\pi i} \sum_{0 \le n \le y} \frac{e^{2\pi i n(1-\alpha)}}{(n+\lambda)^{1-s}}\right.$$
$$\left. + e^{\{-(1-s)/2+2\alpha(1-\lambda)\}\pi i} \sum_{0 \le n \le y} \frac{e^{2\pi i n\alpha}}{(n+1-\lambda)^{1-s}}\right\}$$
$$+O(x^{-\sigma}) + O(|t|^{1/2-\sigma} y^{\sigma-1}). \quad (3.6)$$

Also, in the case $\lambda = 1$ that is $\zeta_H(s,\alpha)$ it follows that

$$\begin{aligned}\zeta_H(s,\alpha) = &\sum_{0\le n\le x} \frac{1}{(n+\alpha)^s} \\ &+\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left\{ e^{\frac{\pi i}{2}(1-s)} \sum_{1\le n\le y} \frac{e^{2\pi i n(1-\alpha)}}{n^{1-s}} + e^{-\frac{\pi i}{2}(1-s)} \sum_{1\le n\le y} \frac{e^{2\pi i n\alpha}}{n^{1-s}} \right\} \\ &+O(x^{-\sigma}) + O(|t|^{1-\sigma} y^{\sigma-1}). \end{aligned} \tag{3.7}$$

Remark 3. Theorem 3.2 can be proved by the method similar to the proof of Theorem 3.1, but results of Theorem 3.2 have advantage of choosing parameters x and y freely, only under the condition $2\pi xy = |t|$ as compared with the result of Theorem 3.1. Also for approximate functional equations (3.6) and (3.7), $\zeta_L(s,\alpha,\lambda)$ is a generalization of $\zeta_H(s,\alpha)$, but (3.6) in Theorem 3.2 does not include (3.7).

Theorem 3.3. *Let* $0<\alpha\le 1,\ 0<\lambda\le 1$. *Then,*

$$\int_1^T \left|\zeta_L\left(\frac{1}{2}+it,\alpha,\lambda\right)\right|^2 dt = T\log\frac{T}{2\pi} + \begin{cases} O(T(\log T)^{1/2}) & (0<\alpha<1), \\ O(T(\log T)^{3/4}) & (\alpha=1), \end{cases} \tag{3.8}$$

as $T\to\infty$.

Remark 4. The result of Theorem 3.3 has larger error term than the result already proved by R. Garunkštis, A. Laurinčikas and J. Steuding [8], and they proved using Theorem 3.1 (see [8]). However, the main term on the right-hand side of (3.8) can be obtained more simply than the method of [8] by using Theorem 3.2. We will describe the proof of Theorem 3.3 in Section 3.3.

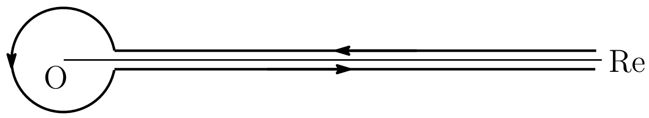
3.2 Proof of Theorem 3.2

In this section, we prove Theorem 3.2. The basic tool of the proof is the same as the approximate functional equation for the Riemann zeta-function (3.3), that is the saddle point method.

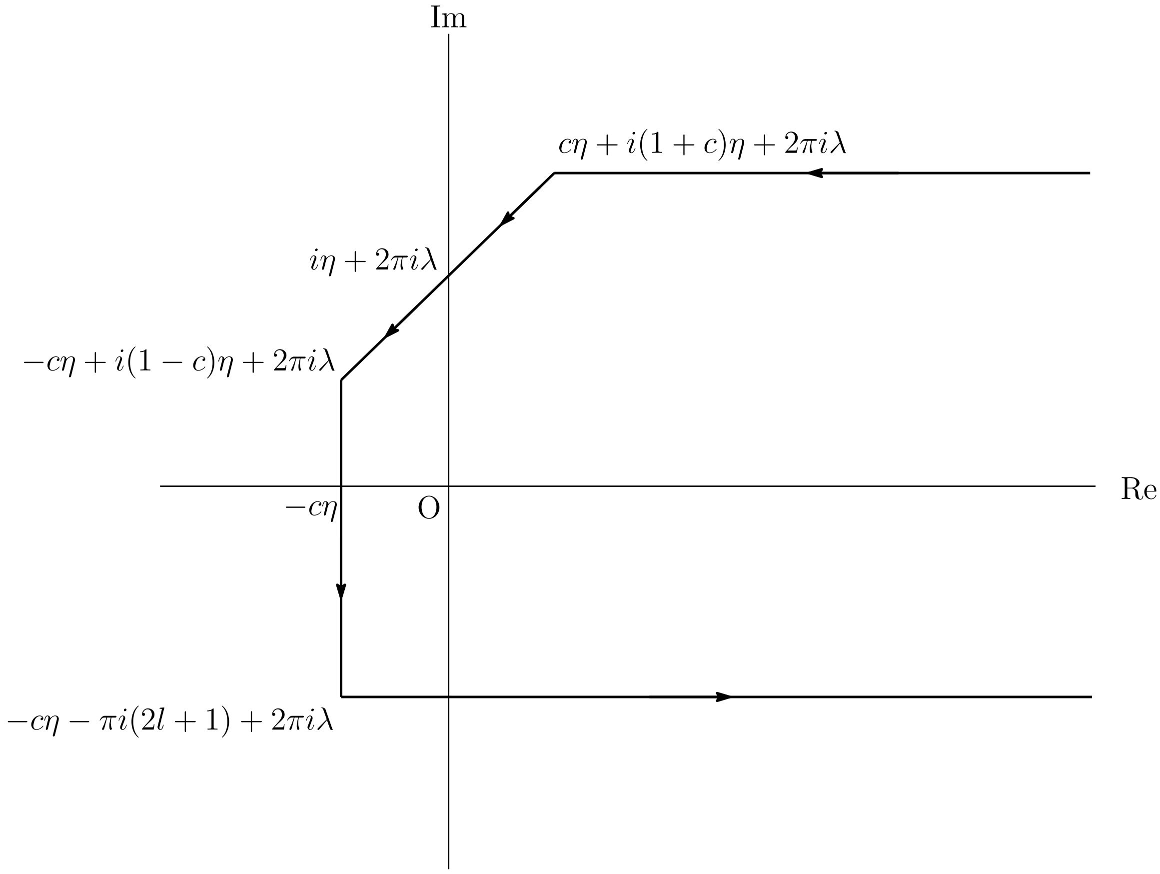
Proof of Theorem 3.2. Let $M\in\mathbb{N}$ be sufficiently large. We have

$$\begin{aligned}\zeta_L(s,\alpha,\lambda) &= \sum_{n=0}^{M} \frac{e^{2\pi i n\lambda}}{(n+\alpha)^s} + \sum_{n=M+1}^{\infty} \frac{e^{2\pi i n\lambda}}{(n+\alpha)^s} \\ &= \sum_{n=0}^{M} \frac{e^{2\pi i n\lambda}}{(n+\alpha)^s} + \frac{e^{2\pi i\lambda M}}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-(M+\alpha)t}}{e^{t-2\pi i\lambda}-1} dt \\ &= \sum_{n=0}^{M} \frac{e^{2\pi i n\lambda}}{(n+\alpha)^s} + \frac{e^{2\pi i\lambda M}\Gamma(1-s)}{2\pi i e^{\pi i s}} \int_C \frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1} dz, \end{aligned} \tag{3.9}$$

where C is the contour integral path that comes from $+\infty$ to ε along the real axis, then goes along the circle of radius ε counter clockwise, and finally goes from ε to $+\infty$.



Let $t > 0$ and $x \le y$, so that $1 \le x \le \sqrt{t/2\pi}$. Let $\sigma \le 1, M = [x], N = [y], \eta = 2\pi y$. We deform the contour integral path C to the combination of the straight lines C_1, C_2, C_3, C_4 joining ∞, $c\eta + i(1+c)\eta + 2\pi i\lambda$, $-c\eta + i(1-c)\eta + 2\pi i\lambda$, $-c\eta - \pi i(2l+1) + 2\pi i\lambda$, ∞, where c is an absolute constant, $0 < c \le 1/2$.



We calculate the residue of the integrand of (3.9). Since

$$
\begin{aligned}
&\lim_{z \to 2\pi i(\lambda+n)} \{z - 2\pi i(\lambda+n)\} \cdot \frac{z^{s-1} e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda} - 1} \\
&= \lim_{z \to 2\pi i(\lambda+n)} \left(\frac{e^{z-2\pi i\lambda} - 1}{z - 2\pi i(\lambda+n)} \right)^{-1} e^{-(M+\alpha)z} \cdot z^{s-1} = e^{-2\pi i(M+\alpha)(\lambda+n)} (2\pi i(n+\lambda))^{s-1},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\operatorname*{Res}_{z=2\pi i(\lambda+n)} \frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1} \\
&\quad = e^{-2\pi i(M+\alpha)(\lambda+n)}(2\pi(n+\lambda)i)^{s-1} \\
&\quad = \begin{cases} e^{-2\pi i(M+\alpha)(\lambda+n)}(2\pi(n+\lambda)e^{\pi i/2})^{s-1} & (n \geq 0) \\ e^{2\pi i(M+\alpha)(|n|-\lambda)}(2\pi(|n|-\lambda)e^{3\pi i/2})^{s-1} & (n \leq -1) \end{cases} \\
&\quad = \begin{cases} -\dfrac{e^{\pi is}}{(2\pi)^{1-s}} \cdot e^{\{(1-s)/2-2(M+\alpha)\lambda\}\pi i} \cdot \dfrac{e^{2\pi in(1-\alpha)}}{(n+\lambda)^{1-s}} & (n \geq 0) \\ -\dfrac{e^{\pi is}}{(2\pi)^{1-s}} \cdot e^{-\{(1-s)/2+2(M+\alpha)(1-\lambda)\}\pi i} \cdot \dfrac{e^{2\pi i(-n)\alpha}}{(|n|-\lambda)^{1-s}} & (n \leq -1) \end{cases}
\end{aligned}
$$

and we have

$$
\begin{aligned}
&\sum_{n=-N+1}^{N} \operatorname*{Res}_{z=2\pi in} \frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1} \\
&= -\frac{e^{\pi is}}{(2\pi)^{s-1}} \left\{ e^{\{(1-s)/2-2(M+\alpha)\lambda\}\pi i} \sum_{n=0}^{N} \frac{e^{2\pi in(1-\alpha)}}{(n+\lambda)^{1-s}} \right. \\
&\qquad\qquad \left. + e^{-\{(1-s)/2+2(M+\alpha)(1-\lambda)\}\pi i} \sum_{n=0}^{N} \frac{e^{2\pi in\alpha}}{(n+1-\lambda)^{1-s}} \right\}.
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
\zeta_L(s,\alpha,\lambda) &= \sum_{n=0}^{M} \frac{e^{2\pi in\lambda}}{(n+\alpha)^s} \\
&\quad + \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left\{ e^{\{(1-s)/2-2\alpha\lambda\}\pi i} \sum_{n=0}^{N} \frac{e^{2\pi in(1-\alpha)}}{(n+\lambda)^{1-s}} \right. \\
&\qquad\qquad \left. + e^{-\{(1-s)/2+2\alpha(1-\lambda)\}\pi i} \sum_{n=0}^{N} \frac{e^{2\pi in\alpha}}{(n+1-\lambda)^{1-s}} \right\} \\
&\quad + \frac{e^{2\pi i\lambda M}\Gamma(1-s)}{2\pi i e^{\pi is}} \left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} \right) \frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1} dz. \qquad (3.10)
\end{aligned}
$$

From here, we consider the order of integral terms on the right-hand side of (3.10).

First, we consider the integral path C_4. Let $z = u + iv = re^{i\theta}$ then $|z^{s-1}| = r^{\sigma-1}$, and since $\theta \geq 5\pi/4,\ r \gg \eta,\ |e^{z-2\pi i\lambda} - 1| \gg 1$, we have

$$
\begin{aligned}
\int_{C_4} \frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1} dz &= \int_{C_4} \frac{(re^{i\theta})^{\sigma+it-1}e^{-(M+\alpha)(u+iv)}}{e^{z-2\pi i\lambda}-1} dz \\
&\ll \eta^{\sigma-1}e^{-5\pi t/4} \int_{c\eta}^{\infty} e^{-(M+\alpha)u} du \\
&= \eta^{\sigma-1}(M+\alpha)^{-1}e^{(M+\alpha)c\eta-5\pi t/4} \\
&\ll e^{(c-5\pi/4)t}. \qquad (3.11)
\end{aligned}
$$

Secondly, we consider the order of integral on C_3 of (3.10). Noting

$$\arctan\varphi = \int_0^{\varphi} \frac{d\mu}{1+\mu^2} > \int_0^{\varphi} \frac{d\mu}{(1+\mu)^2} = \frac{\varphi}{1+\varphi}$$

for $\varphi > 0$, we can write

$$\theta = \arg z = \frac{\pi}{2} + \arctan\frac{c}{1-c} = \frac{\pi}{2} + c + A(c)$$

on C_3, where $A(c)$ is a constant depending on c. Then we have

$$\begin{aligned} |z^{s-1}e^{-(M+\alpha)z}| &= r^{\sigma}e^{-t\theta+(M+\alpha)c\eta} \\ &\ll \eta^{\sigma-1}e^{-(\pi/2+c+A(c))t+(M+\alpha)\eta} \\ &\ll \eta^{\sigma-1}e^{-(\pi/2+A(c)}t. \end{aligned}$$

Therefore, since $|e^{z-2\pi i\lambda}-1| \gg 1$, we have

$$\int_{C_3} \frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1}dz \ll \eta^{\sigma}e^{-(\pi/2+A(c))t}. \tag{3.12}$$

Thirdly, since $|e^{z-2\pi i\lambda}-1| \gg e^u$ on C_1, we have

$$\frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1} \ll \eta^{\sigma-1}\exp\left(-t\arctan\frac{(1+c)\eta+2\pi\lambda}{u} - (M+\alpha+1)u\right).$$

Since $M+\alpha+1 \geq x = t/\eta$, the term $(M+\alpha+1)u$ on the right-hand side of the above may be replaced by tu/η. Also, since

$$\frac{d}{du}\left(\arctan\frac{(1+c)\eta+2\pi\lambda}{u} + \frac{u}{\eta}\right) = -\frac{(1+c)\eta+2\pi\lambda}{u^2+((1+c)\eta+2\pi\lambda)^2} + \frac{1}{\eta} > 0$$

and

$$\arctan\varphi = \int_0^{\varphi}\frac{d\mu}{1+\mu^2} < \int_0^{\varphi} d\mu = \varphi.$$

for $0 < \varphi < \pi/2$, we have

$$\begin{aligned} \arctan\frac{(1+c)\eta+2\pi i}{u} + \frac{u}{\eta} &\geq \arctan\left(\frac{1+c}{c} + \frac{2\pi\lambda}{\eta}\right) + c \\ &= \frac{\pi}{2} - \arctan\frac{c}{1+c+2\pi c\lambda/\eta} + c > \frac{\pi}{2} + B(c) \end{aligned}$$

in $u \geq c\eta$, where $B(c) = (\eta + c\eta + 2\pi\lambda c)C/\{\eta + (2\pi\lambda+\eta)c\}$. Then we have

$$\frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1} \ll \eta^{\sigma-1}\exp\left(-\left(\frac{\pi}{2}+B(c)\right)t\right).$$

Since

$$\frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1} \ll \begin{cases} \eta^{\sigma-1}\exp\left(-\left(\frac{\pi}{2}+B(c)\right)t\right) & (c\eta \leq u \leq \pi\eta), \\ \eta^{\sigma-1}\exp\left(-xu\right) & (u \geq \pi\eta), \end{cases}$$

we obtain

$$
\begin{aligned}
\int_{C_1} \frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1} &\ll \eta^{\sigma-1}\left\{\int_{c\eta}^{\pi\eta} e^{-(\pi/2+B(c))t}du + \int_{\pi\eta}^{\infty} e^{-xu}du\right\} \\
&\ll \eta^{\sigma}e^{-(\pi/2+B(c))t} + \eta^{\sigma-1}e^{-\pi\eta x} \\
&\ll \eta^{\sigma}e^{-(\pi/2+B(c))t}. \qquad (3.13)
\end{aligned}
$$

Finally, we describe the evaluation of the integral on C_2. Rewriting $z = i(\eta+2\pi\lambda) + \xi e^{\pi i/4}$ (where $\xi \in \mathbb{R}$ and $|\xi| \le \sqrt{2}c\eta$), we have

$$
\begin{aligned}
z^{s-1} &= \exp\left\{(s-1)\left(\log\left(i(\eta+2\pi\lambda)+\xi e^{\pi i/4}\right)\right)\right\} \\
&= \exp\left\{(s-1)\left(\frac{\pi i}{2} + \log\left(\eta+2\pi\lambda+\xi e^{-\pi i/4}\right)\right)\right\} \\
&= \exp\left\{(s-1)\left(\frac{\pi i}{2} + \log(\eta+2\pi\lambda) + \frac{\xi}{\eta+2\pi\lambda}e^{-\pi i/4}\right.\right. \\
&\qquad\qquad \left.\left. - \frac{\xi^2}{2(\eta+2\pi\lambda)^2}e^{-\pi i/2} + O\left(\frac{\xi^3}{\eta^3}\right)\right)\right\} \\
&\ll (\eta+2\pi\lambda)^{\sigma-1}\exp\left\{\left(-\frac{\pi}{2} + \frac{\xi}{\sqrt{2}(\eta+2\pi\lambda)} - \frac{\xi^2}{2(\eta+2\pi\lambda)^2} + O\left(\frac{\xi^3}{\eta^3}\right)\right)t\right\}
\end{aligned}
$$

as $\eta \to \infty$. Also, since

$$
\frac{e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1} = \frac{e^{-(M+\alpha-x)z}}{e^{z-2\pi i\lambda}-1}\cdot e^{-xz}
$$

and

$$
\frac{e^{-(M+\alpha-x)z}}{e^{z-2\pi i\lambda}-1} \ll \begin{cases} e^{(x-M-\alpha-1)u} & \left(u > \dfrac{\pi}{2}\right) \\ e^{(x-M-\alpha)u} & \left(u < -\dfrac{\pi}{2}\right), \end{cases}
$$

we have

$$
\frac{e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1} \ll |e^{-xz}| = e^{-\xi t/\sqrt{2}\eta} \qquad \left(|u| > \frac{\pi}{2}\right).
$$

Hence

$$
\begin{aligned}
&\int_{C_2\cap\{z\,|\,|u|>\pi/2\}} \frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda}-1}dz \\
&\ll \int_{C_2\cap\{z\,|\,|u|>\pi/2\}} (\eta+2\pi\lambda)^{\sigma-1} \\
&\quad \times \exp\left\{\left(-\frac{\pi}{2} + \frac{\xi}{\sqrt{2}(\eta+2\pi\lambda)} - \frac{\xi^2}{2(\eta+2\pi\lambda)^2} + O\left(\frac{\xi^3}{\eta^3}\right)\right)t\right\}\exp\left(-\frac{\xi t}{\sqrt{2}\eta}\right)d\xi \\
&\ll \int_{-\sqrt{2}c\eta}^{\sqrt{2}c\eta} (\eta+2\pi\lambda)^{\sigma-1}e^{-\pi t/2}\exp\left\{\left(-\frac{\xi^2}{2(\eta+2\pi\lambda)^2} + O\left(\frac{\xi^3}{\eta^3}\right)\right)t\right\}d\xi \\
&\ll \int_{-\infty}^{\infty} (\eta+2\pi\lambda)^{\sigma-1}e^{-\pi t/2}\exp\left\{\left(-\frac{\xi^2}{2(\eta+2\pi\lambda)^2} + O\left(\frac{\xi^3}{\eta^3}\right)\right)t\right\}d\xi \\
&\ll \eta^{\sigma-1}e^{-\pi t/2}\int_{-\infty}^{\infty}\exp\left\{-\frac{D(c)\xi^2 t}{\eta^2}\right\}d\xi \\
&\ll \eta^{\sigma}t^{-1/2}e^{\pi t/2}, \qquad (3.14)
\end{aligned}
$$

where $D(c)$ is a constant depending on c. The argument can also be applied to the part $|u| \leq \pi/2$ if $|e^{z-2\pi i\lambda}| > A$. If not, that is the case when the contour goes too near to the pole at $z = 2\pi iN + 2\pi i\lambda$, we take an arc of the circle $|z - 2\pi iN - 2\pi i\lambda| = \pi/2$. On this arc we can write to $z = 2\pi iN + 2\pi i\lambda + (\pi/2)e^{i\beta}$, and

$$\begin{aligned}
\log\left(z^{s-1}\right) &= (s-1)\log\left(2\pi iN + 2\pi i\lambda + \frac{\pi}{2}e^{i\beta}\right) \\
&= (s-1)\log e^{\pi i/2}\left(2\pi N + 2\pi\lambda + \frac{\pi}{2}\cdot\frac{e^{i\beta}}{i}\right) \\
&= (s-1)\left\{\frac{\pi i}{2} + \log(2\pi(N+\lambda)) + \log\left(1 + \frac{e^{i\beta}}{4(N+\lambda)i}\right)\right\} \\
&= -\frac{\pi t}{2} + (s-1)\log(2\pi(N+\lambda)) + \frac{te^{i\beta}}{4(N+\lambda)} + O(1).
\end{aligned}$$

On the last line of the above calculations, we used $N^2 \gg t$ which follows from the assumption $x \leq y$. Then

$$\begin{aligned}
&z^{s-1}e^{-(M+\alpha)z} \\
&\quad = \exp\left(-\frac{\pi t}{2} + (s-1)\log(2\pi(N+\lambda)) + \frac{te^{i\beta}}{4(N+\lambda)} - \frac{\pi}{2}(M+\alpha)e^{i\beta} + O(1)\right),
\end{aligned}$$

and since

$$\frac{te^{i\beta}}{4(N+\lambda)} - \frac{\pi}{2}(M+\alpha)e^{i\beta} = \frac{2\pi xy - 2\pi([x]+\alpha)([y]+\lambda)}{4(N+\lambda)}e^{i\beta} = O(1)$$

we have

$$\begin{aligned}
z^{s-1}e^{-(M+\alpha)z} &\ll \exp\left(-\frac{\pi t}{2} + (s-1)\log(2\pi(N+\lambda)) + O(1)\right) \\
&\ll N^{\sigma-1}e^{-\pi t/2}.
\end{aligned}$$

Hence, the integral on the small semicircle can be evaluated as $O(\eta^{\sigma-1}e^{-\pi t/2})$. Therefore together with (4.8), we have

$$\int_{C_2} \frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda} - 1}dz \ll \eta^{\sigma}t^{-1/2}e^{-\pi t/2} + \eta^{\sigma-1}e^{-\pi t/2}. \tag{3.15}$$

Now, evaluation of all the integrals was done. Using the results (3.11), (3.12), (3.13), (3.15) and $e^{2\pi i(\lambda N - s/2)}\Gamma(1-s) \ll t^{1/2-\sigma}e^{\pi t/2}$, we see that the integral term of (3.10) is

$$\begin{aligned}
&\ll t^{1/2-\sigma}e^{\pi t/2}\{\eta^{\sigma}e^{-(\pi/2+B(c))t} + \eta^{\sigma}t^{-1/2}e^{-\pi t/2} + \eta^{\sigma-1}e^{\pi t/2} \\
&\qquad\qquad\qquad + \eta^{\sigma}e^{-(\pi/2+A(c))t} + e^{(c-5\pi/4)t}\} \\
&\ll t^{1/2}\left(\frac{\eta}{t}\right)^{\sigma}e^{-(A(c)+B(c))t} + \left(\frac{\eta}{t}\right)^{\sigma} + t^{-1/2}\left(\frac{\eta}{t}\right)^{\sigma-1} + t^{1/2-\sigma}e^{(c-3\pi/4)t} \\
&\ll e^{-\delta t} + x^{-\sigma} + t^{-1/2}x^{1-\sigma} \ll x^{-\sigma},
\end{aligned}$$

where δ is a small positive real number. Therefore we have

$$\begin{aligned}\zeta_L(s,\alpha,\lambda) &= \sum_{0\le n\le x}\frac{e^{2\pi i n\lambda}}{(n+\alpha)^s}\\ &+\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left\{e^{\{(1-s)/2-2\alpha\lambda\}\pi i}\sum_{0\le n\le y}\frac{e^{2\pi i n(1-\alpha)}}{(n+\lambda)^{1-s}}\right.\\ &\qquad\left.+e^{\{-(1-s)/2+2\alpha(1-\lambda)\}\pi i}\sum_{0\le n\le y}\frac{e^{2\pi i n\alpha}}{(n+1-\lambda)^{1-s}}\right\}\\ &+O(x^{-\sigma}),\end{aligned}\tag{3.16}$$

that is, Theorem 3.2 in the case of $x\le y$ has been proved.

To prove Theorem 3.2 in the case $x\ge y$, we use the following functional equation of the Lerch zeta-function;

$$\begin{aligned}\zeta_L(s,\alpha,\lambda) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\{&e^{\{(1-s)/2-2\alpha\lambda\}\pi i}\zeta_L(1-s,\lambda,1-\alpha)\\ &+e^{\{-(1-s)/2+2\alpha(1-\lambda)\}\pi i}\zeta_L(1-s,1-\lambda,\alpha)\}.\end{aligned}\tag{3.17}$$

Applying (3.16) to $\zeta_L(1-s,\lambda,1-\alpha)$ and $\zeta_L(1-s,1-\lambda,\alpha)$, and substitute these into (3.17), we have

$$\begin{aligned}&\zeta_L(s,\alpha,\lambda)\\ &=\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left[e^{\{(1-s)/2-2\alpha\lambda\}\pi i}\left\{\sum_{0\le n\le x}\frac{e^{2\pi i n\lambda}}{(n+\lambda)^{1-s}}\right.\right.\\ &\quad\left.+\frac{\Gamma(s)}{(2\pi)^s}\left(e^{\{s/2-2\lambda(1-\lambda)\}\pi i}\sum_{0\le n\le y}\frac{e^{2\pi i n(1-\lambda)}}{(n+1-\alpha)^s}+e^{\{-s/2+2\alpha\lambda\}\pi i}\sum_{0\le n\le y}\frac{e^{2\pi i n\lambda}}{(n+\alpha)^s}\right)\right\}\\ &\qquad\qquad+e^{\{-(1-s)/2+2\alpha(1-\lambda)\}\pi i}\left\{\sum_{0\le n\le x}\frac{e^{2\pi i n\alpha}}{(n+1-\lambda)^{1-s}}\right.\\ &\quad+\frac{\Gamma(s)}{(2\pi)^s}\left(e^{\{(s/2-2(1-\lambda)\alpha\}\pi i}\sum_{0\le n\le y}\frac{e^{2\pi i n\lambda}}{(n+\alpha)^s}\right.\\ &\qquad\qquad\left.\left.\left.+e^{\{-s/2+2(1-\lambda)(1-\alpha)\}\pi i}\sum_{0\le n\le y}\frac{e^{2\pi i n(1-\lambda)}}{(n+\alpha)^s}\right)\right\}\right]\\ &\quad+O(\Gamma(\sigma-1)(2\pi)^{-\sigma}(e^{\pi t/2}+e^{-\pi t/2})x^{\sigma-1})\\ &=\sum_{0\le n\le y}\frac{e^{2\pi i n\lambda}}{(n+\alpha)^s}+\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left\{e^{\{(1-s)/2-2\alpha\lambda\}\pi i}\sum_{0\le n\le x}\frac{e^{2\pi i n(1-\alpha)}}{(n+\lambda)^{1-s}}\right.\\ &\qquad\qquad\left.+e^{\{-(1-s)/2+2\alpha(1-\lambda)\}\pi i}\sum_{0\le n\le x}\frac{e^{2\pi i n\alpha}}{(n+1-\lambda)^{1-s}}\right\}\\ &\quad+O(t^{1/2-\sigma}x^{\sigma-1}).\end{aligned}$$

Interchanging x and y, we obtain the theorem with $x \geq y$. Combining this equation with (3.16), we obtain the proof of (3.6).

The proof of (3.7) is similar. However, the four integral path C_1, C_2, C_3 and C_4 are different from the proof of (3.6), that is, as follows; The straight lines C_1, C_2, C_3, C_4 joining ∞, $c\eta + i\eta(1+c)$, $-c\eta + i\eta(1-c)$, $-c\eta - (2L+1)\pi i$, ∞, where c is an absolute constant, $0 < c \leq 1/2$. Also, in the proof for the case $x \geq y$, we use the functional equation

$$\zeta_H(s,\alpha) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\{e^{(1-s)\pi i/2}\zeta_L(1-s,1,1-\alpha) + e^{-(1-s)\pi i/2}\zeta_L(1-s,1,\alpha)\},$$

but this equation is not included in the functional equation (3.17). Noticing these points, we can prove (3.7) by a similar method. This completes the proof of Theorem 3.2. □

3.3 Proof of Theorem 3.3

In this section, using Theorem 3.2, we give the proof of Theorem 3.3.

Proof of Theorem 3.3. Let

$$x = \frac{t}{2\pi\sqrt{\log t}}, \quad y = \sqrt{\log t}$$

and we assume $t > 0$ satisfies $x \geq 1$ and $y \geq 1$. Use the Stirling formula

$$\Gamma(1-s)e^{\{(1-s)/2 - 2\alpha\lambda\}\pi i} \ll 1,\ \Gamma(1-s)e^{\{-(1-s)/2 - 2\alpha(1-\lambda)\}\pi i} \ll 1.$$

Then if $0 < \lambda < 1$, using (3.6) we have

$$\begin{aligned}\zeta_L\left(\frac{1}{2}+it,\alpha,\lambda\right) = &\sum_{0\leq n\leq x}\frac{e^{2\pi i n\lambda}}{(n+\alpha)^{1/2+it}} \\ &+O\left(\sum_{0\leq n\leq y}\frac{e^{2\pi i n(1-\alpha)}}{(n+\lambda)^{1/2-it}} + \sum_{0\leq n\leq y}\frac{e^{2\pi i n\alpha}}{(n+1-\lambda)^{1/2-it}}\right) \\ &+O(t^{-1/2}(\log t)^{1/4}) + O((\log t)^{-1/4}), \end{aligned} \tag{3.18}$$

and if $\lambda = 1$, using (3.7) we have

$$\begin{aligned}\zeta_H\left(\frac{1}{2}+it,\alpha\right) = &\sum_{0\leq n\leq x}\frac{1}{(n+\alpha)^{1/2+it}} + O\left(\sum_{1\leq n\leq y}\frac{e^{2\pi i n(1-\alpha)}}{n^{1/2-it}} + \sum_{1\leq n\leq y}\frac{e^{2\pi i n\alpha}}{n^{1/2-it}}\right) \\ &+O(t^{-1/2}(\log t)^{1/4}) + O((\log t)^{-1/4}). \end{aligned} \tag{3.19}$$

(i) In the case $0 < \lambda < 1$ and $0 < \alpha < 1$, since

$$\sum_{n=0}^{\infty}\frac{e^{2\pi i n(1-\alpha)}}{(n+\lambda)^{1/2}}, \quad \sum_{n=0}^{\infty}\frac{e^{2\pi i n\alpha}}{(n+1-\lambda)^{1/2}}$$

are convergent, and $t^{-1/2}(\log t)^{1/4} = o(1)$, $(\log t)^{-1/4} = o(1)$, we have

$$\zeta_L\left(\frac{1}{2}+it,\alpha,\lambda\right) = \sum_{0\leq n\leq x}\frac{e^{2\pi i n\lambda}}{(n+\alpha)^{1/2+it}} + O(1).$$

(ii) In the case $0 < \lambda < 1$ and $\alpha = 1$, the second term on right-hand side of (3.6) is

$$\ll \int_0^y \frac{1}{(u+\lambda)^{1/2}} du = O(\sqrt{y}) = O((\log t)^{1/4}),$$

so we have

$$\zeta_L\left(\frac{1}{2}+it, 1, \lambda\right) = \sum_{1 \le n \le x} \frac{e^{2\pi i n\lambda}}{n^{1/2+it}} + O((\log t)^{1/4}).$$

(iii) In the case $\lambda = 1$ and $0 < \alpha < 1$, consider similarly as in the case of (i) to obtain

$$\zeta_L\left(\frac{1}{2}+it, \alpha, 1\right) = \sum_{0 \le n \le x} \frac{1}{(n+\alpha)^{1/2+it}} + O(1).$$

(iv) In the case $\lambda = 1$ and $\alpha = 1$, since $\zeta_L(s, 1, 1) = \zeta(s)$ we obtain

$$\zeta_L\left(\frac{1}{2}+it, 1, 1\right) = \sum_{1 \le n \le x} \frac{1}{n^{1/2+it}} + O((\log t)^{1/4})$$

(see Chap. VII in [27]).

Let

$$\Sigma(\alpha, \lambda) = \sum_{0 \le n \le x} \frac{e^{2\pi i n\lambda}}{(n+\alpha)^{1/2+it}},$$

and calculate as

$$\begin{aligned} |\Sigma(\alpha,\lambda)|^2 &= \sum\sum_{0 \le m,n \le x} \frac{e^{2\pi i(m-n)\lambda}}{(m+\alpha)^{1/2}(n+\alpha)^{1/2}} \left(\frac{n+\alpha}{m+\alpha}\right)^{it} \\ &= \sum_{0 \le n \le x} \frac{1}{n+\alpha} + \sum\sum_{\substack{0 \le m,n \le x \\ m \ne n}} \frac{e^{2\pi i(m-n)\lambda}}{(m+\alpha)^{1/2}(n+\alpha)^{1/2}} \left(\frac{n+\alpha}{m+\alpha}\right)^{it}. \end{aligned}$$

Also $T_1 = T_1(m,n)$ is a function in m, n satisfying

$$\max\{m,n\} = \frac{T_1}{2\pi\sqrt{\log T_1}}.$$

Let $X = T/2\pi\sqrt{\log T}$, then

$$\begin{aligned} \int_1^T |\Sigma(\alpha,\lambda)|^2 dt &= \sum_{0 \le n \le X} \frac{1}{n+\alpha}\{T - T_1(n,n)\} \\ &+ O\left(\sum\sum_{0 \le m < n \le X} \frac{e^{2\pi i(m-n)\lambda}}{(m+\alpha)^{1/2}(n+\alpha)^{1/2}} \left(\log \frac{n+\alpha}{m+\alpha}\right)^{-1}\right). \end{aligned} \tag{3.20}$$

Here, since

$$n\sqrt{\log n} = \frac{T_1}{2\pi\sqrt{\log T_1}} \left(\log \frac{T_1}{2\pi\sqrt{\log T_1}}\right)^{1/2} \sim \frac{1}{2\pi} T_1(n,n)$$

and

$$\sum_{0 \le m < n \le X} \sum \frac{e^{2\pi i(m-n)\lambda}}{(m+\alpha)^{1/2}(n+\alpha)^{1/2}} \left(\log \frac{n+\alpha}{m+\alpha} \right)^{-1} \ll X \log X \ll T(\log T)^{1/2}$$

(see Lemma 3 in [8] or Lemma 2.6 in [13]), (3.20) can be rewritten as

$$\int_1^T |\Sigma(\alpha, \lambda)|^2 dt = T \log \frac{T}{2\pi} + O(T(\log T)^{1/2}). \tag{3.21}$$

Therefore from (i), (ii), (iii), (iv) and (3.21) , and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\int_1^T \left| \zeta_L \left(\frac{1}{2} + it, \alpha, \lambda \right) \right|^2 dt \\
&= \int_1^T |\Sigma(\alpha, \lambda)|^2 dt + \begin{cases} O(T^{1/2}(\log T)^{1/4})) + O(T) & (0 < \alpha < 1), \\ O(T(\log T)^{3/4}) + O(T(\log T)^{1/2}) & (\alpha = 1) \end{cases} \\
&= T \log \frac{T}{2\pi} + \begin{cases} O(T(\log T)^{1/2}) & (0 < \alpha < 1), \\ O(T(\log T)^{3/4}) & (\alpha = 1). \end{cases}
\end{aligned}$$

Thus we obtain the proof of Theorem 3.3. $\square$

4 Approximate functional equations for the Barnes double zeta-function

In this section, we give result on the approximate functional equations for the Barnes double zeta-function (2.1).

4.1 Statement of results

Let $s = \sigma + it$ be a complex variable, and let $\alpha > 0$ and $v, w > 0$ are real parameters.

We prove an analogue of the approximate functional equation (3.3) for (2.1) (in Theorem 4.1). In the following theorem, the results are different when the complex parameters v, w are linearly independent, are different from the results when v, w are linearly dependent over $\mathbb{Q}$.

Theorem 4.1. *Suppose that* $0 \le \sigma \le 2, x = x(t) \ge 1, y = y(t) \ge x(t)$ *and* $2\pi xy = |t|$. *Let* L, M, N *are non-negative integer as satisfying* $N = [x/(v+w)]$ *and* $\max\{L/v,\ M/w\} < y < \min\{(L+1)/v,\ (M+1)/w\}$.

(*i*) *If* v, w *are linearly independent over* $\mathbb{Q}$*;*

$$\begin{aligned}
&\zeta_2(s,\alpha;v,w)\\
&= \sum\sum_{0\le m,n\le N} \frac{1}{(\alpha+vm+wn)^s} + \frac{1}{w^s}\sum_{m=0}^{N}\zeta_H^*(s,\alpha_{v,m}) + \frac{1}{v^s}\sum_{n=0}^{N}\zeta_H^*(s,\alpha_{w,n})\\
&\quad -\frac{\Gamma(1-s)}{(2\pi i)^{1-s}e^{\pi i s}}\left\{\frac{1}{v^s}\sum_{0<|n|<L}\frac{e^{-2\pi i n(\alpha+wN)/v}}{(e^{2\pi i n w/v}-1)n^{1-s}} + \frac{1}{w^s}\sum_{0<|n|<M}\frac{e^{-2\pi i n(\alpha+vN)/w}}{(e^{2\pi i n v/w}-1)n^{1-s}}\right\}\\
&\quad +O(x^{-\sigma})
\end{aligned} \tag{4.1}$$

(*ii*) *If* v, w *are linearly dependent over* $\mathbb{Q}$*, exist* $p, q \in \mathbb{N}$ *such as* $pv = qw$ *and* $(p,\ q) = 1$. *Then we have*

$$\begin{aligned}
&\zeta_2(s,\alpha;v,w)\\
&= \sum\sum_{0\le m,n\le N} \frac{1}{(\alpha+vm+wn)^s} + \frac{1}{w^s}\sum_{m=0}^{N}\zeta_H^*(s,\alpha_{v,m}) + \frac{1}{v^s}\sum_{n=0}^{N}\zeta_H^*(s,\alpha_{w,n})\\
&\quad -\frac{\Gamma(1-s)}{(2\pi i)^{1-s}e^{\pi i s}}\left\{\frac{1}{v^s}\sum_{\substack{0<|n|<L\\ q\nmid n}}\frac{e^{-2\pi i n(\alpha+wN)/v}}{(e^{2\pi i n w/v}-1)n^{1-s}} + \frac{1}{w^s}\sum_{\substack{0<|n|<M\\ p\nmid n}}\frac{e^{-2\pi i n(\alpha+vN)/w}}{(e^{2\pi i n v/w}-1)n^{1-s}}\right\}\\
&\quad -2\pi i\sum_{0<|n|<M}\left\{\frac{q}{pv^2}(s-1)e^{-2q\pi i n\alpha/v}\left(\frac{2q\pi i n}{v}\right)^{s-2}\right.\\
&\qquad\left. -\left(\frac{\alpha q}{pv^2}+\frac{qN}{pv}+\frac{N}{v}+\frac{2pq-p-q}{2v}-\frac{p+2q}{2v^2}\right)e^{-2q\pi i n\alpha/v}\left(\frac{2q\pi i n}{v}\right)^{s-1}\right\}\\
&\quad +O(x^{-\sigma}),
\end{aligned} \tag{4.2}$$

where

$$\zeta_H^*(s,\alpha_{v,m}) := \zeta_H(s,\alpha_{v,m}) - \sum_{n=0}^{N+n_{v,m}} \frac{1}{(n+\alpha_{v,m})^s},$$

$$\alpha_{v,m} := \begin{cases} \left\{\frac{vm+\alpha}{w}\right\} & \left(\frac{vm+\alpha}{w} \notin \mathbb{N}\right), \\ 1 & \left(\frac{vm+\alpha}{w} \in \mathbb{N}\right), \end{cases}$$

$$n_{v,m} := \begin{cases} \left[\frac{vm+\alpha}{w}\right] - 1 & \left(\frac{vm+\alpha}{w} \geq 1\right), \\ 0 & \left(0 < \frac{vm+\alpha}{w} < 1\right). \end{cases}$$

The definitions of $\zeta_H^*(s,\alpha_{w,n})$ *and* $\alpha_{w,n}$ *are similar.*

4.2 Proof of theorem 4.1

In this section, we give the proof of Theorem 4.1.

Proof of Theorem 4.1.

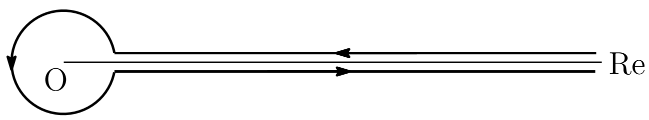
Let $N \in \mathbb{N}$ be sufficiently large. Then we consider

$$\begin{aligned} &\zeta_2(s,\alpha;v,w) \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{(\alpha+vm+wn)^s} \\ &= \left(\sum_{m=0}^{N}\sum_{n=0}^{N} + \sum_{m=0}^{N}\sum_{n=N+1}^{\infty} + \sum_{m=N+1}^{\infty}\sum_{n=0}^{N} + \sum_{m=N+1}^{\infty}\sum_{n=N+1}^{\infty}\right) \frac{1}{(\alpha+vm+wn)^s}. \end{aligned}$$

Transform the fourth term on the right hand-side of the above equation to the contour integral to obtain

$$\sum_{m=N+1}^{\infty}\sum_{n=N+1}^{\infty} \frac{1}{(\alpha+vm+wn)^s} = \frac{\Gamma(1-s)}{2\pi i e^{\pi i s}} \int_C \frac{z^{s-1}e^{-(\alpha+vN+wN)z}}{(e^{vz}-1)(e^{wz}-1)}dz \tag{4.3}$$

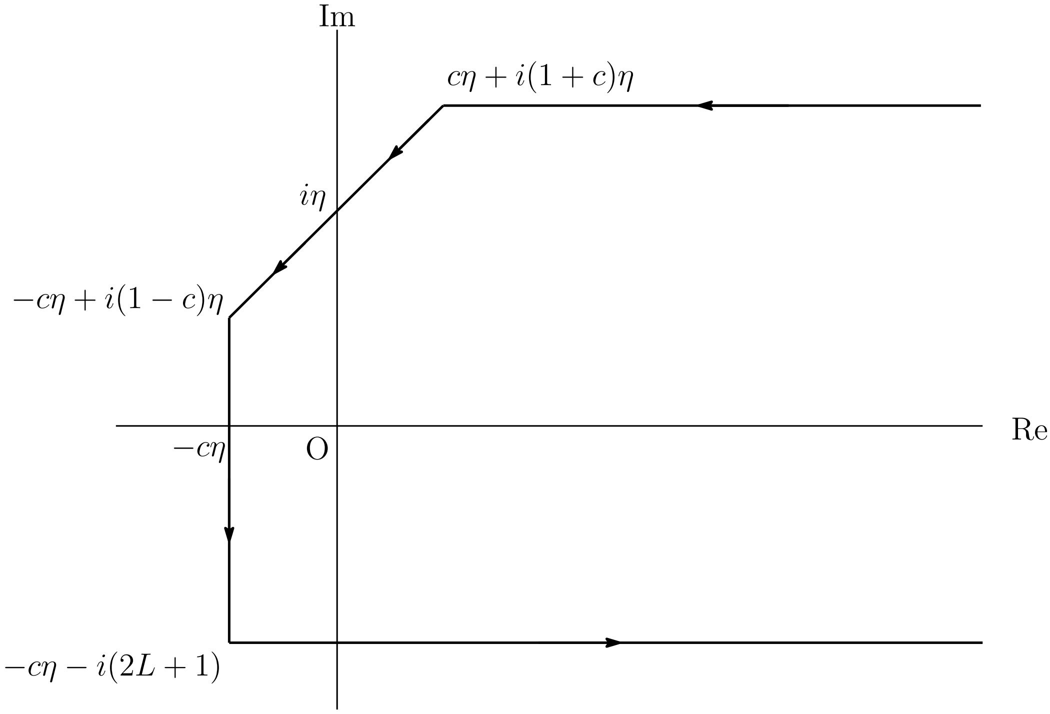
where C is the contour integral path that comes from $+\infty$ to ε along the real axis, then continues along the circle of radius ε counter clockwise, and finally goes from ε to $+\infty$.



Let $\sigma \leq 2, t > 0$ and $1 \leq x < y$, so that $1 \leq x \leq \sqrt{t/2\pi}$. Let L, M, N be non-negative integers satisfying

$$N = \left[\frac{x}{v+w}\right], \quad \max\left\{\frac{L}{v}, \frac{M}{w}\right\} < y < \min\left\{\frac{L+1}{v}, \frac{M+1}{w}\right\},$$

and let $\eta = 2\pi y$. We deform the contour integral C to the straight lines C_1, C_2, C_3, C_4 joining $\infty, c\eta + i\eta(1+c), -c\eta + i(1-c)\eta, -c\eta - (2L+1)\pi i, \infty$ where c is an absolute constant $0 < c \leq 1/2$.



Next we consider the residue of the integrand of (4.3)

$$f(z) = \frac{z^{s-1}e^{-(\alpha+vN+wN)z}}{(e^{vz}-1)(e^{wz}-1)}.$$

(i) In the case when v, w are linear by independent on $\mathbb{Q}$, $f(z)$ has simple poles at

$$z = \frac{2\pi in}{v},\ \frac{2\pi in}{w}\ (n = \pm 1,\ \pm 2,\ \cdots).$$

Also, we assume $v \in \mathbb{Q}$, then

$$\begin{aligned}
\lim_{z\to 2\pi in/v}\left(z - \frac{2\pi in}{v}\right) f(z) &= \lim_{z\to 2\pi in/v}\left(z - \frac{2\pi in}{v}\right)\frac{z^{s-1}e^{-(\alpha+vN+wN)z}}{(e^{vz}-1)(e^{wz}-1)} \\
&= \lim_{z\to 2\pi in/v}\left(\frac{e^{vz}-e^{2\pi in}}{z-2\pi in/v}\right)^{-1}\frac{z^{s-1}e^{-(\alpha+vN+wN)z}}{e^{wz}-1} \\
&= \frac{1}{v}\left(\frac{2\pi in}{v}\right)^{s-1}\frac{e^{-(\alpha+wN)2\pi in/v}}{e^{2\pi inw/v}-1},
\end{aligned}$$

therefore, we have

$$\begin{aligned}
\operatorname*{Res}_{z=2\pi in/v} f(z) &= \frac{1}{v}\left(\frac{2\pi in}{v}\right)^{s-1}\frac{e^{-(\alpha+wN)2\pi in/v}}{e^{2\pi inw/v}-1} \\
&= \begin{cases} e^{-2\pi in\alpha}(2\pi n)^{s-1}e^{\pi(s-1)/2} & (n > 0) \\ e^{2\pi in\alpha}(-2\pi n)^{s-1}e^{3\pi(s-1)/2} & (n < 0) \end{cases}
\end{aligned}$$

and we obtain

$$\begin{aligned}
&\zeta_2(s,\alpha;v,w)\\
&=\sum\sum_{0\le m,n\le N}\frac{1}{(\alpha+vm+wn)^s}+\frac{1}{w^s}\sum_{m=0}^{N}\zeta_H^*(s,\alpha_{v,m})+\frac{1}{v^s}\sum_{n=0}^{N}\zeta_H^*(s,\alpha_{w,n})\\
&\quad-\frac{\Gamma(1-s)}{(2\pi i)^{1-s}e^{\pi is}}\left\{\frac{1}{v^s}\sum_{0<|n|\le L}\frac{e^{2\pi in(\alpha+wN)/v}}{(e^{2\pi inw/v-1})n^{1-s}}+\frac{1}{w^s}\sum_{0<|n|\le M}\frac{e^{-2piin(\alpha+vN)/w}}{(e^{2\pi inv/w-1})n^{1-s}}\right\}\\
&\quad+\frac{1}{\Gamma(s)(e^{2\pi is}-1)}\left(\int_{C_1}+\int_{C_2}+\int_{C_3}+\int_{C_4}\right)\frac{z^{s-1}e^{-(\alpha+vN+wN)z}}{(e^{vz}-1)(e^{wz}-1)}dz. \qquad (4.4)
\end{aligned}$$

From here, we consider the order of the integral term on the right-hand side of (4.10). First, we consider it on the integral path C_4. Let $z=u+iu'=re^{i\theta}$ then $|z^{s-1}|=r^{\sigma-1}e^{-t\theta}$. Since $\theta\le(5/4)\pi, r\asymp u, |e^{vz}-1|\gg 1$ and $|e^{wz}-1|\gg 1$ we have

$$\begin{aligned}
\int_{C_4}f(z)dz &= \int_{C_4}\frac{z^{s-1}e^{-(\alpha+vN+wN)z}}{(e^{vz}-1)(e^{wz}-1)}dz\\
&\ll e^{-(5/4)\pi t}\int_{-c\eta}^{\infty}u^{\sigma-1}e^{-(\alpha+vN+wN)u}du\\
&\ll e^{-(5/4)\pi t}(\eta^{-\sigma}+\eta^{-\sigma}e^{c\eta})\\
&\ll e^{-(5/4)\pi t}(\alpha+vN+wN)^{-\sigma}(1+e^{c\eta})\\
&\ll e^{-(5/4)\pi t}(\alpha+vN+wN)^{-\sigma}e^{c\eta}\\
&\ll x^{-\sigma}e^{c\eta-(5/4)\pi t}\ll x^{-\sigma}e^{(c-(5/4)\pi)t} \qquad (4.5)
\end{aligned}$$

Secondly, we consider the order of the integral on C_3 of (4.10). Noticing

$$\arctan\varphi=\int_0^{\varphi}\frac{d\mu}{1+\mu^2}>\int_0^{\varphi}\frac{d\mu}{(1+\mu)^2}=\frac{\varphi}{1+\varphi},$$

at $\varphi>0$, we have

$$\theta=\arg z=\frac{\pi}{2}+\arctan\frac{c}{1-c}=\frac{\pi}{2}+c+A(c)$$

on C_3, where $A(c)$ is a constant depending on c. Then we have

$$\begin{aligned}
|z^{s-1}e^{-(\alpha+vN+wN)z}| &\ll \eta^{\sigma-1}e^{-(\pi/2+c+A(c))t}e^{(\alpha+vN+wN)c\eta}\\
&\ll \eta^{\sigma-1}e^{-(\pi/2+A(c))t}.
\end{aligned}$$

Also, since $|e^{vz}-1|\gg 1$, $|e^{wz}-1|\gg 1$ we have

$$\begin{aligned}
\int_{C_3}f(z)dz &= \int_{C_3}\frac{z^{s-1}e^{-(\alpha+vN+wN)z}}{(e^{vz}-1)(e^{wz}-1)}dz\\
&\ll \int_{-\eta}^{\eta}\eta^{\sigma-1}e^{-(\pi/2+A(c))t}dz\ll\eta^{\sigma}e^{-(\pi/2+A(c))t} \qquad (4.6)
\end{aligned}$$

Thirdly, since $|e^{vz}-1| \gg e^{vu}$ and $|e^{wz}-1| \gg e^{wu}$ on C_1, we have

$$\begin{aligned}&\frac{z^{s-1}e^{-(\alpha+vN+wN)z}}{(e^{vz}-1)(e^{wz}-1)}\\ &\qquad \ll \eta^{\sigma-1}\exp\left(-t\arctan\frac{(1+c)\eta}{u}-(\alpha+(N+1)(v+w))u\right).\end{aligned}$$

Since $N+1 \geq x/(v+w) = t/(v+w)\eta$ are included in the fractional part of the right hand-side of the above $-(\alpha+(N+1)(v+w))u$ may be replaced with tu/η. Also, since

$$\frac{d}{du}\left(\arctan\frac{(1+c)\eta}{u}+\frac{u}{\eta}\right) = -\frac{(1+c)\eta}{u^2+(1+c)^2\eta^2}+\frac{1}{\eta} > 0$$

and

$$\arctan\varphi = \int_0^{\varphi}\frac{d\mu}{1+\mu^2} < \int_0^{\varphi} d\mu = \varphi,$$

we have

$$\begin{aligned}\arctan\frac{(1+c)\eta}{u}+\frac{u}{\eta} &\geq \arctan\frac{1+c}{c}+c = \frac{\pi}{2}-\arctan\frac{c}{1+c}+c\\ &> \frac{\pi}{2}+B(c)\end{aligned}$$

in $u \leq \pi\eta$, and let $B(c) = c^2/(1+c)^2$. Then we have

$$\frac{z^{s-1}e^{-(\alpha+vN+wN)z}}{(e^{vz}-1)(e^{wz}-1)} \ll \eta^{\sigma-1}\exp\left(-\left(\frac{\pi}{2}+B(c)\right)t\right).$$

Also,

$$\frac{z^{s-1}e^{-(\alpha+vN+wN)z}}{(e^{vz}-1)(e^{wz}-1)} \ll \eta^{\sigma-1}\exp\left(-(\alpha+vx+wx)u\right)$$

in $u \geq \pi\eta$. Therefore, we obtain

$$\begin{aligned}&\int_{C_1}\frac{z^{s-1}e^{-(\alpha+vN+wN)z}}{(e^{vz}-1)(e^{wz}-1)}dz\\ &\qquad \ll \eta^{\sigma-1}\left\{\int_{c\eta}^{\pi\eta}e^{-(\pi/2+B(c))t}du+\int_{\pi\eta}^{\infty}e^{-(\alpha+vx+wx)u}du\right\}\\ &\qquad \ll \eta^{\sigma}e^{-(\pi/2+B(c))t}+\eta^{\sigma-1}e^{-(\alpha+vx+wx)\pi\eta}\\ &\qquad \ll \eta^{\sigma}e^{-(\pi/2+B(c))t}.\end{aligned} \tag{4.7}$$

Finally, we describe the integral evaluation on C_2. Since, it can be rewritten that $z = i\eta+\xi e^{\pi i/4}$ (where $\xi \in \mathbb{R}$ and $|\xi| \leq \sqrt{2}c\eta$),we have

$$\begin{aligned}z^{s-1} &= \exp\left\{(s-1)\left(\frac{\pi}{2}+\log\left(\eta+\xi e^{-\pi i/4}\right)\right)\right\}\\ &= \exp\left\{(s-1)\left(\frac{\pi}{2}+\log\eta+\frac{\xi}{\eta}e^{-\pi i/4}-\frac{\xi^2}{2\eta^2}e^{-\pi i/2}+O\left(\frac{\xi^3}{\eta^3}\right)\right)\right\}\\ &\ll \eta^{\sigma-1}\exp\left\{\left(-\frac{\pi}{2}+\frac{\xi}{\sqrt{2}\eta}-\frac{\xi^2}{2\eta^2}+O\left(\frac{\xi^3}{\eta^3}\right)\right)t\right\} \quad (\xi \to \infty).\end{aligned}$$

as $\eta \to \infty$. Also, since

$$\begin{aligned}\frac{e^{-(\alpha+vN+wN)z}}{(e^{vz}-1)(e^{wz}-1)} &= \frac{e^{-(\alpha+vN+wN)z+(\alpha+vx+wx)z}}{(e^{vz}-1)(e^{wz}-1)} \cdot e^{-(\alpha+vx+wx)z} \\ &= \frac{e^{(v+w)(x-N)z}}{(e^{vz}-1)(e^{wz}-1)} \cdot e^{-(\alpha+vx+wx)z}\end{aligned}$$

and

$$\frac{e^{(v+w)(x-N)z}}{(e^{vz}-1)(e^{wz}-1)} \ll \begin{cases} e^{(v+w)(x-N-1)u} & \left(u > \dfrac{\pi}{2}\right) \\ e^{(v+w)(x-N-1)u} & \left(u < -\dfrac{\pi}{2}\right) \end{cases},$$

we have

$$\frac{e^{(v+w)(x-N)z}}{(e^{vz}-1)(e^{wz}-1)} \ll |e^{-(\alpha+vx+wx)\xi/\sqrt{2}}| \quad \left(|u| > \frac{\pi}{2}\right).$$

Hence

$$\begin{aligned}&\int_{C_2\cap\{z\,|\,|u|>\pi/2\}} \frac{z^{s-1}e^{-(\alpha+vN+wN)z}}{(e^{vz}-1)(e^{wz}-1)}dz \\ &\ll \eta^{\sigma-1}e^{-\pi t/2}\int_{-\sqrt{2}c\eta}^{\sqrt{2}c\eta} \exp\left\{\left(\frac{\xi}{\sqrt{2}\eta}(1-v-w) - \frac{\xi^2}{2\eta^2} + O\left(\frac{\xi^3}{\eta^3}\right)\right)t\right\}d\xi \\ &\ll \eta^{\sigma-1}e^{-\pi t/2}\int_{-\infty}^{\infty} \exp\left\{-\frac{D(c)\xi^2 t}{\eta^2}\right\}d\xi \\ &\ll \eta^{\sigma}t^{-1/2}e^{\pi t/2}, \qquad (4.8)\end{aligned}$$

where $D(c)$ is a constant depending on c. The argument can also be applied to the part $|u| \le \pi/2$ if $|e^{z-2\pi i\lambda}| > A$. If not, that is the case when the contour goes too near to the pole at $z = 2L\pi i/v$ (or $2M\pi i/w$), we take an arc of the circle $|z - 2L\pi i/v| = \varepsilon$ (or $|z - 2M\pi i/w| = \varepsilon$) . On this arc we can write

$$z = \frac{2L\pi i}{v} + \varepsilon e^{i\beta} \quad \text{or} \quad z = \frac{2M\pi i}{w} + \varepsilon e^{i\beta},$$

where ε is a positive real number less than the distance between any two poles, that is,

$$0 < \varepsilon < \min_{k,l}\left\{\left|\frac{2k\pi i}{v} - \frac{2l\pi i}{w}\right| \;\middle|\; 0 < \frac{2k\pi}{v}, \frac{2l\pi}{w} < \eta\right\}.$$

Then,

$$\begin{aligned}\log\left(z^{s-1}\right) &= (s-1)\log\left(\frac{2L\pi i}{v} + \varepsilon e^{i\beta}\right) \\ &= (s-1)\log e^{\pi i/2}\left(\frac{2L\pi}{v} + \frac{\varepsilon e^{i\beta}}{i}\right) \\ &= (\sigma+it-1)\left\{\frac{\pi i}{2} + \log\frac{2L\pi}{v} + \log\left(1 + \frac{v\varepsilon e^{i\beta}}{2L\pi i}\right)\right\} \\ &= -\frac{\pi t}{2} + (s-1)\log\frac{2L\pi}{v} + \frac{v\varepsilon e^{i\beta}}{2L\pi i} + O(1).\end{aligned}$$

On the last line of the above calculations, we used $N^2 \gg t$ which follows from the assumption $x \leq y$. Then

$$\begin{aligned} & z^{s-1} e^{-(\alpha+vN+wN)z} \\ &= \exp\left(-\frac{\pi t}{2} + (s-1)\log\frac{2L\pi}{v} + \frac{v\varepsilon e^{i\beta}}{2L\pi i} + O(1)\right) \\ &\qquad\qquad \times \exp\left(-(\alpha+vN+wN)\left(\frac{2L\pi}{v} + \varepsilon e^{i\beta}\right)\right) \\ &= \exp\left(-\frac{\pi t}{2} + (s-1)\log\frac{2L\pi}{v} + \left(\frac{vt}{2\pi L} - (\alpha+vN+wN)\right)\varepsilon e^{i\beta} + O(1)\right), \end{aligned}$$

and since

$$\begin{aligned} & \left(\frac{vt}{2\pi L} - (\alpha+vN+wN)\right)\varepsilon e^{i\beta} \\ &= \frac{vt - (\alpha+vN+wN)2\pi L}{2\pi L}\varepsilon e^{i\beta} \\ &= \frac{2\pi xyv - 2\alpha\pi L - (v+w)[x/(v+w)]2\pi L}{2\pi L}\varepsilon e^{i\beta} \\ &= -\alpha\varepsilon e^{i\beta} + \frac{2\pi xyv - (v+w)[x/(v+w)]2\pi L}{2\pi L}\varepsilon e^{i\beta} \\ &\asymp -\alpha\varepsilon e^{i\beta} + \frac{2\pi xyv - 2\pi(v+w)[x/(v+w)]y}{2\pi L}\varepsilon e^{i\beta} = O(1) \end{aligned}$$

we have

$$\begin{aligned} z^{s-1} e^{-(\alpha+vN+wN)z} &= \exp\left(-\frac{\pi t}{2} + (s-1)\log\frac{2L\pi}{v} + O(1)\right) \\ &\ll \left(\frac{L}{v}\right)^{\sigma-1} e^{-\pi t/2} = O(\eta^{\sigma-1}e^{-\pi t/2}). \end{aligned}$$

In the case when the path is running around the pole $z = 2k\pi i/w + \varepsilon e^{i\beta}$, use a similar method to obtain

$$z^{s-1} e^{-(\alpha+vN+wN)z} = O(\eta^{\sigma-1}e^{-\pi t/2}).$$

Therefore together with (4.8), we have

$$\int_{C_2} \frac{z^{s-1} e^{-(\alpha+vN+wN)z}}{(e^{vz}-1)(e^{wz}-1)} dz \ll \eta^\sigma t^{-1/2} e^{-\pi t/2} + \eta^{\sigma-1} e^{-\pi t/2}. \tag{4.9}$$

Since, the evaluation of all integrals was obtained, using the evaluation formulas (4.5), (4.6), (4.7), (4.9) and $\Gamma(1-s) \ll t^{1/2-\sigma}e^{\pi t/2}$, we find that the evaluation of the integral term of (4.10) is

$$\begin{aligned} &\ll t^{1/2-\sigma}e^{\pi t/2}\{\eta^\sigma e^{-(\pi/2+B(c))t} + \eta^\sigma t^{-1/2}e^{-\pi t/2} + \eta^{\sigma-1}e^{\pi t/2} \\ &\qquad\qquad + \eta^\sigma e^{t(\pi/2+A(c))} + x^{-\sigma}e^{(c-5\pi/4)t}\} \\ &\ll t^{1/2}\left(\frac{\eta}{t}\right)^\sigma e^{-(A(c)+B(c))t} + \left(\frac{\eta}{t}\right)^\sigma + t^{-1/2}\left(\frac{\eta}{t}\right)^{\sigma-1} + t^{1/2-\sigma}x^{-\sigma}e^{(c-5\pi/4)t} \\ &\ll e^{-\delta t} + x^{-\sigma} + t^{-1/2}x^{1-\sigma} \ll x^{-\sigma}, \end{aligned}$$

where δ is a small positive real number. Therefore we obtain (4.1).

(ii) In the case when v, w are linear dependent on $\mathbb{Q}$, that is, there exist $p, q \in \mathbb{N}$ such that $pv = qw$ and $(p,q) = 1$, $f(z)$ has simple poles at

$$z = \frac{2\pi in}{v} \ \ (n \in \mathbb{Z} \setminus \{0\},\ q \not| n), \quad \frac{2\pi in}{w} \ \ (n \in \mathbb{Z} \setminus \{0\},\ p \not| n).$$

On the other hand here for $w = pv/q$,

$$\begin{aligned}
&\lim_{z \to 2q\pi in/v} \frac{d}{dz} \left\{ \left(z - \frac{2q\pi in}{v} \right)^2 \frac{z^{s-1} e^{-(\alpha + vN + pvN/q)z}}{(e^{vz}-1)(e^{pvz/q}-1)} \right\} \\
&= \lim_{z_0 \to 0} \frac{d}{dz_0} \left\{ z_0^2 \left(z_0 + \frac{2q\pi in}{v} \right)^{s-1} \frac{e^{-(\alpha+vN+pvN/q)z_0} e^{2q\pi in\alpha/v}}{(e^{vz_0}-1)(e^{pvz_0/q}-1)} \right\} \\
&= \lim_{z_0 \to 0} \frac{d}{dz_0} \left\{ z_0^2 \left(z_0 + \frac{2q\pi in}{v} \right)^{s-1} \frac{e^{-(\alpha+vN+pvN/q)z_0} e^{2q\pi in\alpha/v}}{\left(vz_0 + \frac{v^2}{2!} z_0^2 + O(z_0^2)\right) \left(\frac{pv}{q} z_0 + \frac{1}{2!} \left(\frac{pv}{q}\right)^2 z_0^2 + O(z_0^2) \right)} \right\} \\
&= \frac{q}{pv^2} (s-1) e^{-2q\pi in\alpha/v} \left(\frac{2q\pi in}{v} \right)^{s-2} \\
&\qquad - \left(\frac{\alpha q}{pv^2} + \frac{qN}{pv} + \frac{N}{v} + \frac{2pq-p-q}{2v} - \frac{p+2q}{2v^2} \right) e^{-2q\pi in\alpha/v} \left(\frac{2q\pi in}{v} \right)^{s-1}.
\end{aligned}$$

Therefore $f(z)$ has double poles at

$$z = \frac{2\pi iqk}{v} = \frac{2\pi ipk}{w} \ \ (k \in \mathbb{Z} \setminus \{0\}).$$

Then, we calculate the following residue sum;

$$\begin{aligned}
\sum_{0<|n|\le M} \operatorname{Res} f(z) &= \sum_{\substack{0<|n|\le L \\ q \not| n}} \operatorname*{Res}_{z=2\pi in/v} f(z) + \sum_{\substack{0<|n|\le K \\ p \not| n}} \operatorname*{Res}_{z=2\pi in/w} f(z) \\
&\quad + \sum_{0<|n|\le K} \operatorname*{Res}_{z=2\pi iqn/v} f(z).
\end{aligned}$$

We have

$$\begin{aligned}
\operatorname*{Res}_{z=2\pi in/v} f(z) &= \frac{1}{v} \left(\frac{2\pi in}{v} \right)^{s-1} \frac{e^{-(\alpha+wN)2\pi in/v}}{e^{2\pi inw/v}-1} \\
&= \begin{cases} e^{-2\pi in\alpha} (2\pi n)^{s-1} e^{\pi(s-1)/2} & (n>0) \\ e^{2\pi in\alpha} (-2\pi n)^{s-1} e^{3\pi(s-1)/2} & (n<0) \end{cases} \\
\operatorname*{Res}_{z=2\pi iqn/v} f(z) &= \frac{q}{pv^2} (s-1) e^{-2q\pi in\alpha/v} \left(\frac{2q\pi in}{v} \right)^{s-2} \\
&\quad - \left(\frac{\alpha q}{pv^2} + \frac{qN}{pv} + \frac{N}{v} + \frac{2pq-p-q}{2v} - \frac{p+2q}{2v^2} \right) e^{-2q\pi in\alpha/v} \left(\frac{2q\pi in}{v} \right)^{s-1}
\end{aligned}$$

and we obtain

$$
\begin{aligned}
&\zeta_2(s,\alpha;v,w)\\
&=\sum\sum_{0\le m,n\le N}\frac{1}{(\alpha+vm+wn)^s}+\frac{1}{w^s}\sum_{m=0}^{N}\zeta_H^*(s,\alpha_{v,m})+\frac{1}{v^s}\sum_{n=0}^{N}\zeta_H^*(s,\alpha_{w,n})\\
&\quad-\frac{\Gamma(1-s)}{(2\pi i)^{1-s}e^{\pi is}}\left\{\frac{1}{v^s}\sum_{\substack{0<|n|<L\\ q\nmid n}}\frac{e^{-2\pi in(\alpha+wN)/v}}{(e^{2\pi inw/v}-1)n^{1-s}}+\frac{1}{w^s}\sum_{\substack{0<|n|<M\\ p\nmid n}}\frac{e^{-2\pi in(\alpha+vN)/w}}{(e^{2\pi inv/w}-1)n^{1-s}}\right\}\\
&\quad-2\pi i\sum_{0<|n|<M}\left\{\frac{q}{pv^2}(s-1)e^{-2q\pi in\alpha/v}\left(\frac{2q\pi in}{v}\right)^{s-2}\right.\\
&\qquad\left.-\left(\frac{\alpha q}{pv^2}+\frac{qN}{pv}+\frac{N}{v}+\frac{2pq-p-q}{2v}-\frac{p+2q}{2v^2}\right)e^{-2q\pi in\alpha/v}\left(\frac{2q\pi in}{v}\right)^{s-1}\right\}\\
&\quad+\frac{1}{\Gamma(s)(e^{2\pi is}-1)}\left(\int_{C_1}+\int_{C_2}+\int_{C_3}+\int_{C_4}\right)\frac{z^{s-1}e^{-(\alpha+vN+wN)z}}{(e^{vz}-1)(e^{wz}-1)}dz.
\end{aligned}
$$

Furthermore, four integrals in the last term of the above are evaluated to lead the same result by the similar method as in (i).

Hence the proof of Theorem 4.1 is complete. $\square$

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