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Doctoral Thesis

Quantum Adiabatic/Game-theoretic Control from Continuous-/Discrete-time Perspectives

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Declaration of Authorship

I, Hiroaki MISHIMA, declare that this thesis titled, "Quantum Adiabatic/Game-theoretic Control from Continuous-/Discrete-time Perspectives" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at Nagoya University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
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"Nobody understands quantum mechanics."

Richard Feynman

NAGOYA UNIVERSITY

Abstract

Graduate School of Information Science
Department of Complex Systems Science

Doctor of Information Science

Quantum Adiabatic/Game-theoretic Control from Continuous-/Discrete-time Perspectives

by Hiroaki MISHIMA

Quantum control – technology that enables us to design and to observe a micro-scale world – is recently important in quantum mechanics. There exist many quantum objects we want to control, for example, atoms, qubits, and other microscopic ones, but they are usually sensitive and fragile. It is meaningful to realize stabilization and speed-up of such manipulations of "quanta" for innovation of technology because their interactions with an environment degrade the coherence of systems. In this dissertation, we focus on how to control two kinds of quantum model, a quantum parametric oscillator (QPO) and a qubit system. We approach each model through different ways.

For the QPO as a continuous-time dynamical system, we apply *shortcuts to adiabaticity* (STA). Adiabatic processes, in a classical and quantum realm, are beneficial to control a state of the system but require a long-time operation. By using STA, one can evolve the system toward the desired final state as the same state as the final state of the adiabatic process but in an arbitrary short time. For example, the transitionless tracking (TT) algorithm as one of the methods of STA brings about the preservation of a quantum number of the system by adding a counter-diabatic Hamiltonian to the original adiabatic Hamiltonian. Since adiabatic processes often appear in atomic, molecular and optical physics, there exists a broad scope of application for STA. By applying Husimi's method, we derive a concrete form of a propagator of the QPO with the counter-diabatic Hamiltonian being added. We then calculate two types of transition-probability generating function with two types of measure of adiabaticity. From an analysis of these measures of adiabaticity, we characterize the quantum adiabatic evolution of the QPO with the TT algorithm being applied in terms of Husimi's method.

For the qubit system as a discrete-time dynamical system, we apply *quantum game theory* as a method to control "quantum." We want to achieve the desired final state even under the presence of disturbances, but time is discrete. Note that this scheme is almost the same as an adiabatic control. We propose a way to control the qubit system in discrete time step with a language of game theory. Quantum natures – superposition, entanglement, and non-commutativity of physical quantities – produce non-intuitive effects in terms of the probability. Naively, a quantum game theory is given by regarding quantum amplitudes as the classical probabilities in game theory. A quantum player generates superpositions state of the game by using unitary operations, which implies that the quantum player possesses indecisive decision-makings. Then, game theory and quantum information have common features such as a quantum error correction and quantum algorithms. We analyze a quantum penny flip game with several kinds of modifications as a problem of quantum error correction. We challenge how much we can do without using any ancillae, i.e., without relying on quantum entanglement at all.

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Contents

List of Abbreviations

$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{Z}$	the set of integers
$\mathbb{N}$	the set of natural numbers = $\{1, 2, 3, \cdots\}$
$\mathbb{N}_0$	the set of non-negative integers = $\{0, 1, 2, \cdots\}$
$\mathbb{R}^n$	the n ($\in \mathbb{N}$)-dimensional real space
$\mathbb{S}^2$	2-sphere; the unit sphere in the Euclidean 3-space $\mathbb{R}^3$

ad	**ad**iabatic
cd	**c**ounter-**d**iabatic/**c**ounter-**d**issipative
CPO(s)	**c**lassical **p**arametric **o**scillator(**s**)
diag	**diag**onal
EL	**E**rmakov-**L**ewis
EoM(s)	**e**quation(**s**) **o**f **m**otion
Eq(s).	**eq**uation(**s**)
H	**H**usimi
IE	**I**nverse **E**ngineering
TT	**t**ransitionless **t**racking
LR	**L**ewis-**R**iesenfeld
meas.	**meas**urement
ODE(s)	**o**rdinary **d**ifferential **e**quation(**s**)
op(s).	**op**erator(**s**)
ph	**ph**ase
QPO(s)	**q**uantum **p**arametric **o**scillator(**s**)
Ref(s).	**ref**erence(**s**)
Sol(s).	**sol**ution(**s**)
STA	**s**hortcuts **t**o **a**diabaticity
STI	**s**hortcuts **t**o **i**sothermality
TPGF	**t**ransition-**p**robability **g**enerating **f**unction

Chapter 1

Introduction

Control is one of the means to connect us to the world. From day-to-day, we unconsciously live by controlling objects within reach. The "catch" that anyone would have played once in childhood is a play that is established by the control of the ball accurately. In PC games, we control the monitor screen by typing a keyboard and clicking a mouse correctly. At the same time, lots of small electronic components are precisely controlled inside the PC. In space exploration, we can indirectly control a robot from a room on the earth and come to know a state of the moon, etc.

The purpose of control is to move an object from a given state to the desired state. We usually want to control the object as quickly and/or robustly as possible. The most straightforward control would be "move" such as in the catch. However, we know that there are objects that the same control as the catch cannot be applied. What if we replace the ball with a soap bubble? As soon as we touch it, it will crack. We must examine the nature of the object and change the type of control.

So what if the controlled objects belong to a microscopic world, for example, "quanta" such as atoms, electrons and other microscopic objects? We cannot even see them directly. The quanta are much more sensitive and fragile than the soap bubble of the previous example. The act of "seeing" is to apply electromagnetic wave (i.e., light) to the object and to receive the response. Since the energy of light is enormous for the quanta, even this visual observation disturbs them considerably.

To control the microscopic world (i.e., quantum systems) has become a major issue of modern physics. Physicists are now able to manipulate simple quantum systems such as a few atoms, photons and electrons. This technology would give us numerous applications: chemical reaction through exploiting the quantum nature of electrons, superconductivity via making use of Bose-Einstein condensation with cold atoms, and quantum computation by controlling a few electrons or polarization of photons.

Adiabatic transport provides a powerful way to manipulate quantum objects, such as quantum heat engines with quantum parametric oscillators (QPOs) [1, 2] and a quantum computation [3–6]. In a reciprocating quantum heat engine, the working medium is a quantum system such as spin systems [7] or a QPO [8]. There exists a need for quantum adiabatic transport that can preserve a quantum number of the system over the variation of the external parameters. In an annealing quantum computation, by setting a system in a state that we can readily prepare as the initial state and then slowly changing its Hamiltonian, one could achieve the desired quantum state. The ground state of the system is such a readily prepared state. However, the dephasing effects of the environment may limit the quantum correlations and degrade the power of such an adiabatic computation [9].

The adiabatic transport process is described by the quantum adiabatic theorem [10, 11]. Since, unfortunately, the adiabatic process needs an infinitely long time, it is hard to realize such a process in a laboratory. However, there exists a method for evolving the system toward the desired final state as the same state as the final state of the adiabatic process in an arbitrary short time, which is called shortcuts to adiabaticity (STA) [12]. STA has some kinds of formulation such as the assisted adiabatic passage [13, 14], transitionless tracking algorithm [15], fast-forward method [16–20], Lewis-Riesenfeld invariant-based inverse engineering [21, 22], scale-invariant driving [23], generator of adiabatic transport [24], and quantum brachistochrone [25–27]. There exists research showing that some of them are substantially the same [28]. It is crucial to specify differences and to find common features of STA, in order to construct a unified theory of STA. Toward this goal, in this dissertation, we study STA of the QPO as the simplest example in terms of measure-of-adiabaticity approach.

Before considering another example of quantum control, let us consider relationship between information and physics. There exists a hypothesis that information can inherently describe physical world (i.e., the universe) and therefore it is computable, which is called *digital ontology* [29]. According to the hypothesis, the results of physics are represented by the output of a deterministic or probabilistic computer program and are regarded as mathematically isomorphic to a digital computing device. Every computer can be consistent with the principle of information theory, statistical mechanics, and quantum mechanics. Suppose that we identify a physical state of a two-state system with a "bit." We consider switching a physical state of this system to the other state, which is equivalent to change the bit from a value (e.g., 0) to the other value (e.g., 1). In this sense, every physical state may be regarded as information, and every change of the state may be equivalent to transformation of information that requires the manipulation of one or more bits. Recently, a *quantum version of* digital ontology [30] reinforces the idea of quantum computer [31] that uses information stored in the quantum state, i.e., *quantum information*. This hypothesis is convenient for combining fundamental physics, in particular, quantum mechanics, with quantum Boolean algebra [32] and quantum logic [33, 34]. A unit of quantum information is expressed by a qubit that is a two-state quantum-mechanical system such as a quantum spin and polarization of a single photon. The qubit can be in a spin-up state $|0\rangle$ and a spin-down state $|1\rangle$ just as a single electron and can also be in superposition states of the orthogonal basis, $|0\rangle$ and $|1\rangle$.

Quantum mechanics includes *quasi-probability* which can take a negative or complex value. These features reflect quantum natures. In this sense, quantum mechanics includes an aspect of probability theory. Game theory, which is initially formulated by von Neumann and Morgenstein [35], has been applied to diverse fields from economics to biology [36]. Game theory is also mathematically formulated with a probability which takes a real number less than or equal to unity. Hence, quantum mechanics and game theory have a common feature as probability theory. The theory which replaces the probability in the *conventional* game theory with the quasi-probability and incorporates the quantum natures is called *quantum game theory*. Quantum game theory was proposed independently by Meyer [37] and Eisert et al. [38] at the same year. Although game theory is a mature field of applied mathematics, the quantum game theory is still a young field. Players in quantum game theory, which is called *quantum players*, possess a set of unitary operations as their *quantum strategy*. States of a game are represented by quantum states. Therefore, the possible quantum game flows could be regarded as quantum circuits.

Quantum mechanical protocols are known to be superior to classical ones both in solving specific computational tasks and for cryptographic purposes. Strategy in games is similar to an algorithm. Quantum algorithms such as Shor's algorithm [39] have been shown to be more efficient than classical algorithms.

Let us challenge protecting quantum information (states, data, etc.) from errors. This problem constitutes another quantum control we want to consider in this dissertation. Quantum information processing can be expressed by using quantum mechanics [40]. The game-like situation is drawn as:

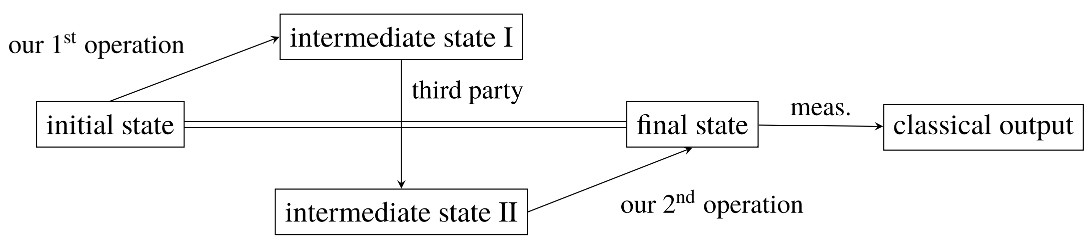

FIGURE 1.1: Schematic picture of a simple quantum error correction as a game flow.

We have an initial quantum state that should be protected. A malicious third party tries to disturb (tamper) this state. The third party triggers an error. For what kind of the third party can we preserve and recover the quantum state? We can regard this situation as a problem of quantum error correction. This game-like situation can be adapted by Meyer's quantum penny flip game [37]. In the penny flip game, one coin has two states, heads or tails, and two players apply alternating operations on the coin. In the original Meyer's game, the first player is allowed to use quantum (i.e., non-commutative) operations, but the second player is still only allowed to use classical (i.e., commutative) operations. In Meyer's game with several kinds of modifications, both players are allowed to use non-commutative operations, with the second player being partially restricted in what operations they use. This consideration gives us an answer whether there exists a method for restoring the quantum state disturbed by another agent. We will apply geometric algebra approach, which is adequate for analysis of two-or-more-player games [41].

This dissertation is structured as follows. In chapter 2, we review the adiabatic theorem, STA, and Husimi's method that enables us to analyze the quantum adiabatic evolution of the QPO in terms of the adiabatic evolution of the corresponding classical system. In chapter 3, we apply Husimi's method to STA of the QPO, in order to analyze STA by using classical description and to elucidate features of STA. Some derivations used in chapter 3 are in Appendix A. In Appendix B, we give calculations of a propagator of a generalized quantum parametric oscillator (GQPO) based on Husimi's method. In chapter 4, we consider the quantum penny flip game as a model of a quantum discrete-time control. We try to modify the game without any ancilla qubits. In chapter 5, we give concluding remarks.

Due to the length of the manuscript and the different topics discussed, the notation is consistent within each chapter, but not necessarily throughout the dissertation.

My, Hiroaki Mishima's, original works are in chapter 3, 4 and appendices, but Sec. 3.5 and Appendix B are unpublished works.

Chapter 2

Adiabaticity in Quantum and Classical Mechanics

2.1 Adiabaticity

The notion of *adiabaticity* appears when we consider slow changes of control parameters of classical and quantum mechanical systems or thermodynamical systems. [1] Here, we consider the only former context. Adiabaticity has played important roles in the history of physics, such as the important contribution to the birth of the old quantum theory. This notion is on the border of statics and dynamics [43], and it relates to an existence of an *invariant* under an *infinitely slow change* of the system. In classical mechanics, an action variable is such an adiabatic invariant. In quantum mechanics, a principal quantum number is such an adiabatic invariant.

2.1.1 Classical adiabatic evolution

In classical realm, we consider a classical system of one degree of freedom, described by a Hamiltonian with a kinetic term and a time-dependent potential term,

$$H_t \equiv H(x, p; \vec{\lambda}_t) := \frac{p^2}{2M} + V(x, p; \vec{\lambda}_t), \tag{2.1}$$

where M, x, p and $\vec{\lambda}_t := (\lambda_t^{(1)}, \cdots, \lambda_t^{(k)})^\top$ are mass, position, momentum, and a set of external parameters such as an angular frequency, respectively. Suppose that H_t varies with time $t \in [t_0, t_\mathrm{f}]$, and $k \in \mathbb{N}$.

Let Γ be a phase space. A classical state corresponds to the point of a classical phase space $(x, p) \in \Gamma$, and a solution of a classical dynamics defines a trajectory in the phase space $\vec{\gamma}_t^{\mathrm{ph}} := \{(x, p; \vec{\lambda}_t)\}_t \subseteq \Gamma$. The term *energy shell* denotes a level set of H_t; that is, the set of all points where H_t takes a particular value, E_t at time t. We then have $E_t = H(x, p; \vec{\lambda}_t)$. By solving this with respect to p as $p = p(x, E_t; \vec{\lambda}_t)$, the volume of phase space enclosed by a trajectory, that is, an energy shell $\Xi := \{(x, p) | H(x, p; \vec{\lambda}_t) = E_t\}$, is given by

$$\mathcal{V}(E_t; \vec{\lambda}_t) := \int_\Gamma \mathrm{d}x\mathrm{d}p\ \Theta[E_t - H(x, p; \vec{\lambda}_t)] = \oint_\Xi \mathrm{d}x\ p(x, E_t; \vec{\lambda}_t). \tag{2.2}$$

[1] In thermodynamics, an adiabatic process is a process with no heat transfer between a thermodynamic system and a heat reservoir.

For a classical parametric oscillator (CPO) (see Sec. 2.1.1.1), the notion of adiabaticity in classical mechanics may be "thermodynamically" understood by regarding an intrinsic oscillatory motion of the CPO as a "thermal motion" and by supplying an external work to the system by varying external parameters (e.g., length of a pendulum) much slowly, without any correlation with the intrinsic oscillatory motion [42]. In this case, the work is supplied to the CPO over the variation of the external parameters without "heat transfer." This is similar to the adiabatic process in thermodynamics.

$\Theta(x)$ is Heaviside step function that satisfies $\Theta(x \geq 0) = 1$ and $\Theta(x < 0) = 0$. The notation $\oint_\Xi \mathrm{d}x$ denotes an integration around an energy shell Ξ, that is, over the complete range of excursion of x from one turning point to the other and back with the opposite momentum.

The *action variable*,

$$S_t \equiv S(x, p; \vec{\lambda}_t) := \frac{1}{2\pi}\mathcal{V}(H_t; \vec{\lambda}_t) \tag{2.3}$$

is proportional to the volume enclosed by the trajectory that runs through the point (x, p).

In the limit of an infinitely slow change of $\vec{\lambda}_t$, the value of S_t remains constant along a trajectory evolving under H_t. The adiabatic evolution is identified with preservation of the action variable S_t along the trajectory in the classical case, which can be shown as follows [44]. When we change $\vec{\lambda}_t$ slowly in such a manner that it is not correlating with a motion cycle of the system, E_t varies over the variation of $\vec{\lambda}_t$ in the Hamiltonian H_t; $\dot{E}_t = \frac{\partial H_t}{\partial \vec{\lambda}_t} \cdot \dot{\vec{\lambda}}_t$, where $\frac{\partial}{\partial \vec{\lambda}_t} := \left(\frac{\partial}{\partial \lambda_t^{(1)}}, \cdots, \frac{\partial}{\partial \lambda_t^{(k)}}\right)^\top$, and the dot denotes time derivative. To be more specific, we impose the condition of $|\dot{\vec{\lambda}}_t| T \ll |\vec{\lambda}_t|$, where T is one cycle period of the system. The value of S_t also varies over the variation of $\vec{\lambda}_t$. By differentiating Eq. (2.3) with respect to time t, we have

$$\frac{\mathrm{d}S_t}{\mathrm{d}t} = \frac{1}{2\pi}\oint_\Xi \mathrm{d}x \left(\frac{\partial p}{\partial E_t}\dot{E}_t + \frac{\partial p}{\partial \vec{\lambda}_t} \cdot \dot{\vec{\lambda}}_t\right) \simeq \frac{\langle \dot{\vec{\lambda}}_t \rangle}{2\pi} \cdot \left[\oint_\Xi \mathrm{d}x \left(\frac{\partial p}{\partial E_t}\frac{\partial H_t}{\partial \vec{\lambda}_t} + \frac{\partial p}{\partial \vec{\lambda}_t}\right)\right]. \tag{2.4}$$

Here, since $|\dot{\vec{\lambda}}_t| \simeq 0$ and $\dot{\vec{\lambda}}_t$ does not depend on a motion cycle of the system as we assumed, we replaced $\dot{\vec{\lambda}}_t$ with $\langle \dot{\vec{\lambda}}_t \rangle$, which is an averaged value during a cycle, and put it outside the integral. By substituting $p = p(x, E_t; \vec{\lambda}_t)$ into the Hamiltonian H_t, we have $H(x, p(x, E_t; \vec{\lambda}_t); \vec{\lambda}_t) = E_t$. Then, by differentiating this by $\vec{\lambda}_t$ and E_t, respectively, we have the following relations,

$$\frac{\partial H_t}{\partial \vec{\lambda}_t} + \frac{\partial H_t}{\partial p}\frac{\partial p}{\partial \vec{\lambda}_t} = \vec{0}, \qquad \frac{\partial H_t}{\partial p}\frac{\partial p}{\partial E_t} = 1. \tag{2.5}$$

By multiplying the former of Eq. (2.5) by $\frac{\partial p}{\partial E_t}$ and by using the latter, we have

$$\frac{\partial p}{\partial E_t}\frac{\partial H_t}{\partial \vec{\lambda}_t} + \frac{\partial p}{\partial E_t}\frac{\partial H_t}{\partial p}\frac{\partial p}{\partial \vec{\lambda}_t} = \frac{\partial p}{\partial E_t}\frac{\partial H_t}{\partial \vec{\lambda}_t} + \frac{\partial p}{\partial \vec{\lambda}_t} = \vec{0}. \tag{2.6}$$

Therefore, if we can neglect $O(|\dot{\vec{\lambda}}_t|^2)$, from Eqs. (2.4) and (2.6), we obtain

$$\frac{\mathrm{d}S_t}{\mathrm{d}t} \simeq 0. \tag{2.7}$$

As a result, we have the following theorem:

Theorem 1 (Classical adiabatic theorem)**.** *When a set of external parameters $\vec{\lambda}_t$ varies slowly enough, the action variable S_t remains constant along the trajectory. In this case, S_t becomes an invariant J_t called an adiabatic invariant.*

2.1.1.1 Example: classical parametric oscillator

Let us consider a Hamiltonian of a classical parametric oscillator (CPO)

$$H_t \equiv H(x, p; \omega_t) = \frac{p^2}{2M} + \frac{M}{2}\omega_t^2 x^2 = E_t, \tag{2.8}$$

where ω_t is the angular frequency at time t. The momentum p is given as

$$p(x, E_t; \omega_t) = \pm\sqrt{2M\left(E_t - \frac{M}{2}\omega_t^2 x^2\right)}. \tag{2.9}$$

Under the condition of $|\dot{\omega}_t|T \ll |\omega_t|$, we can then calculate the adiabatic invariant J_t of the CPO from Eqs. (2.2) and (2.3) as

$$J_t = S_t = \frac{1}{2\pi} \cdot 2 \int_{-\sqrt{2E_t/M\omega_t^2}}^{\sqrt{2E_t/M\omega_t^2}} \mathrm{d}x \sqrt{2M\left(E_t - \frac{M}{2}\omega_t^2 x^2\right)} = \frac{E_t}{\omega_t}. \tag{2.10}$$

2.1.2 Quantum adiabatic evolution

In quantum realm, suppose that a quantum system evolves in time t according to the Schrödinger equation

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = \hat{H}_t^{\mathrm{ad}}|\Psi(t)\rangle, \tag{2.11}$$

with a Hamiltonian $\hat{H}_t^{\mathrm{ad}} \equiv \hat{H}^{\mathrm{ad}}(\vec{\lambda}_t)$, which is Hermitian and may be a function of the time-dependent parameters $\vec{\lambda}_t$. It is difficult to solve the Schrödinger equation Eq. (2.11) in general. A formal solution of Eq. (2.11) is given by the Dyson series [45]. However, by using the quantum adiabatic theorem, we can solve it approximately. This theorem was first stated in Ref. [10] and was proved as a more general one in Ref. [11].

We can obtain the adiabatic approximate solution of Eq. (2.11) as follows [43, 46]. We assume that the spectrum of $\hat{H}_t^{\mathrm{ad}}$ is discrete and non-degenerate for an arbitrary time t. Without any loss of generality, we expand the solution of Eq. (2.11) in terms of the instantaneous eigenstates $|n; \vec{\lambda}_t\rangle$ of $\hat{H}_t^{\mathrm{ad}}$ as

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} c_{n,t} e^{-\frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}\tau E_{n,\tau}^{\mathrm{ad}}} |n; \vec{\lambda}_t\rangle, \tag{2.12}$$

where the factor $e^{-\frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}\tau E_{n,\tau}^{\mathrm{ad}}}$ is a dynamical phase factor. The instantaneous eigenstate $|n; \vec{\lambda}_t\rangle$ satisfies

$$\hat{H}_t^{\mathrm{ad}}|n; \vec{\lambda}_t\rangle = E_{n,t}^{\mathrm{ad}}|n; \vec{\lambda}_t\rangle, \tag{2.13}$$

where $E_{n,t}^{\mathrm{ad}}$ is the eigenenergy. We impose the normalization condition $\langle m; \vec{\lambda}_t|n; \vec{\lambda}_t\rangle = \delta_{m,n}$. By substituting Eq. (2.12) into Eq. (2.11) and then by multiplying the result by $\langle m; \vec{\lambda}_t|$ from the left, we have the following equations for the coefficients $c_{n,t}$:

$$\dot{c}_{m,t} + c_{m,t}\langle m; \vec{\lambda}_t|\frac{\mathrm{d}}{\mathrm{d}t}|m; \vec{\lambda}_t\rangle + \sum_{(m\neq)n=1}^{\infty} c_{n,t} e^{-\frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}\tau (E_{n,\tau}^{\mathrm{ad}} - E_{m,\tau}^{\mathrm{ad}})} \langle m; \vec{\lambda}_t|\frac{\mathrm{d}}{\mathrm{d}t}|n; \vec{\lambda}_t\rangle = 0. \tag{2.14}$$

By differentiating Eq. (2.13) with respect to time t, and then by multiplying $\langle m; \vec{\lambda}_t|$ from the left, we have

$$\langle m; \vec{\lambda}_t|\frac{\mathrm{d}}{\mathrm{d}t}|n; \vec{\lambda}_t\rangle + \frac{\langle m; \vec{\lambda}_t|\frac{\mathrm{d}\hat{H}_t^{\mathrm{ad}}}{\mathrm{d}t}|n; \vec{\lambda}_t\rangle}{E_{m,t}^{\mathrm{ad}} - E_{n,t}^{\mathrm{ad}}} = 0; \quad n \neq m. \tag{2.15}$$

Here we introduce a condition for the adiabatic approximation: The time evolution governed by $\hat{H}_t^{\rm ad}$ is adiabatic if the following condition holds:

$$\left|\langle m;\vec{\lambda}_t|\frac{{\rm d}\hat{H}_t^{\rm ad}}{{\rm d}t}|n;\vec{\lambda}_t\rangle\right|T \ll |E_{m,t}^{\rm ad}-E_{n,t}^{\rm ad}|, \tag{2.16}$$

where T is an intrinsic time scale of the quantum system. From Eqs. (2.15) and (2.16), we then find [2]

$$\hbar\frac{\left|\langle m;\vec{\lambda}_t|\frac{{\rm d}\hat{H}_t^{\rm ad}}{{\rm d}t}|n;\vec{\lambda}_t\rangle\right|}{(E_{m,t}^{\rm ad}-E_{n,t}^{\rm ad})^2} = \hbar\left|\frac{\langle m;\vec{\lambda}_t|\frac{\rm d}{{\rm d}t}|n;\vec{\lambda}_t\rangle}{E_{m,t}^{\rm ad}-E_{n,t}^{\rm ad}}\right| \ll 1; \quad n\neq m. \tag{2.17}$$

Thus, in this approximation, Eq. (2.14) simplifies to

$$\dot{c}_{m,t} + c_{m,t}\langle m;\vec{\lambda}_t|\frac{\rm d}{{\rm d}t}|m;\vec{\lambda}_t\rangle \simeq 0. \tag{2.18}$$

By integrating Eq. (2.18) and imposing an initial condition $c_{m,t_0} = \delta_{m,n}$, we find that $c_{n,t}$ is a phase factor,

$$c_{n,t} \simeq e^{-\int_{t_0}^{t}{\rm d}\tau\langle n;\vec{\lambda}_\tau|\frac{\rm d}{{\rm d}\tau}|n;\vec{\lambda}_\tau\rangle}, \tag{2.19}$$

which is well known as the geometric phase factor [48]. Therefore, we obtain the *adiabatic approximate solution* of the Schrödinger equation Eq. (2.11), where

$$|\Psi(t)\rangle \simeq \sum_{n=0}^{\infty} C_n e^{i\xi_{n,t}}|n;\vec{\lambda}_t\rangle, \tag{2.20}$$

with C_n being a time-*independent* amplitude, and the time-dependent phase angle $\xi_{n,t}\in\mathbb{R}$ is defined as

$$\xi_{n,t} := -\frac{1}{\hbar}\int_{t_0}^{t}{\rm d}\tau E_{n,\tau}^{\rm ad} + i\int_{t_0}^{t}{\rm d}\tau\langle n;\vec{\lambda}_\tau|\frac{\rm d}{{\rm d}\tau}|n;\vec{\lambda}_\tau\rangle. \tag{2.21}$$

The first and second terms of $\xi_{n,t}$ are the dynamical phase and geometric phase, respectively.

Let us prepare the initial state of the system at $t=t_0$ as an eigenstate of $\hat{H}_{t_0}^{\rm ad}$ as $|\Psi(t_0)\rangle = |n;\vec{\lambda}_{t_0}\rangle$. If the Hamiltonian $\hat{H}_t^{\rm ad}$ changes under the condition of Eq. (2.16), the adiabatic approximate solution Eq. (2.20) yields $|\Psi(t)\rangle = e^{i\xi_{n,t}}|n;\vec{\lambda}_t\rangle \propto |n;\vec{\lambda}_t\rangle$ for an arbitrary later time $t\geq t_0$. Namely, the state of the system exhibits no transition. Therefore, we proved the following theorem.

Theorem 2 (Quantum adiabatic theorem)**.** *When we control a set of external parameters $\vec{\lambda}_t$ adiabatically, a quantum system exhibits no transition between non-degenerated eigenstates with different quantum numbers. In other words, although a set of external parameters $\vec{\lambda}_t$ varies, the system remains in an instantaneous eigenstate. By using transition probabilities, this can be represented as*

$$P_{t,t_0}^{m,n} := |\langle m;\vec{\lambda}_t|\hat{U}(t\leftarrow t_0)|n;\vec{\lambda}_{t_0}\rangle|^2 \simeq \delta_{m,n}, \tag{2.22}$$

where $\hat{U}(t\leftarrow t_0)$ is a time-evolution unitary operator.

[2] The condition for the adiabatic approximation Eq. (2.17) has been used, for example, to determine the running time required by an adiabatic quantum algorithm [47].

2.2 Shortcuts to adiabaticity

As we have seen in Sec. 2.1.2, the quantum adiabatic theorem ensures the transitionless process under an infinitely slow change of the parameters of the system. However, from an experimental viewpoint, such a long time process is not favorable. Besides, because the time needed to achieve the desired final state is long, the system may lose its coherence during the process due to surrounding disturbance. These problems may also arise in an adiabatic process in classical mechanics. Hence, it is desirable to have a method for achieving the same final state of the adiabatic process in a finite duration. There exist various theoretical methods proposed as shortcuts to adiabaticity (STA) [12], such as the assisted adiabatic passage [13, 14], transitionless tracking algorithm [15], fast-forward method [16–20], Lewis-Riesenfeld invariant-based inverse engineering [22, 28], generator of adiabatic transport [24, 49], scale-invariant driving [23], and quantum brachistochrone [25–27].

In this section, we review some methods of STA that include Lewis-Riesenfeld invariant-based inverse engineering (Sec. 2.2.1), transitionless tracking algorithm (Sec. 2.2.2), and classical dissipationless driving [23, 24] (Sec. 2.2.3).

2.2.1 Lewis-Riesenfeld invariant-based inverse engineering

We review the Lewis-Riesenfeld (LR) invariant-based inverse engineering based on Ref. [28]. Suppose that a quantum system $|\Psi(t)\rangle$ obeys a time-dependent Hamiltonian $\hat{H}_t^{\mathrm{IE}}$:

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = \hat{H}_t^{\mathrm{IE}}|\Psi(t)\rangle. \tag{2.23}$$

We assume that there exists a Hermitian dynamical invariant $\hat{I}_t^{\mathrm{LR}}$ in the system, which is called the LR invariant [21]. The LR invariant satisfies the following equation

$$\frac{\mathrm{d}\hat{I}_t^{\mathrm{LR}}}{\mathrm{d}t} = \frac{\partial\hat{I}_t^{\mathrm{LR}}}{\partial t} + \frac{1}{i\hbar}[\hat{I}_t^{\mathrm{LR}}, \hat{H}_t^{\mathrm{IE}}] = 0. \tag{2.24}$$

We can express a solution of the Schrödinger equation Eq. (2.23) as

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} C_n e^{i\alpha_{n,t}^{\mathrm{LR}}}|\phi_n; \vec{\lambda}_t\rangle. \tag{2.25}$$

Here, the instantaneous eigenstates $|\phi_n; \vec{\lambda}_t\rangle$ satisfy the eigenvalue equation of $\hat{I}_t^{\mathrm{LR}}$,

$$\hat{I}_t^{\mathrm{LR}}|\phi_n; \vec{\lambda}_t\rangle = \iota_n|\phi_n; \vec{\lambda}_t\rangle, \tag{2.26}$$

where ι_n is the eigenvalue of $\hat{I}_t^{\mathrm{LR}}$. The phase $\alpha_{n,t}^{\mathrm{LR}}$ is defined by

$$\alpha_{n,t}^{\mathrm{LR}} := \frac{1}{\hbar}\int_{t_0}^{t} \mathrm{d}\tau\langle\phi_n; \vec{\lambda}_\tau|i\hbar\frac{\mathrm{d}}{\mathrm{d}\tau} - \hat{H}_\tau^{\mathrm{IE}}|\phi_n; \vec{\lambda}_\tau\rangle, \tag{2.27}$$

which is known as the LR phase.

We want to drive the state of the system with the time-dependent Hamiltonian $\hat{H}_t^{\mathrm{IE}}$ such that the quantum numbers in the initial and final instantaneous eigenstates are the same. We will achieve this goal by determining the appropriate Hamiltonian $\hat{H}_t^{\mathrm{IE}}$. The LR invariant $\hat{I}_t^{\mathrm{LR}}$ is formally written as

$$\hat{I}_t^{\mathrm{LR}} = \sum_{n=0}^{\infty} \iota_n|\phi_n; \vec{\lambda}_t\rangle\langle\phi_n; \vec{\lambda}_t|. \tag{2.28}$$

We introduce the time-evolution unitary operator $\hat{U}^{\mathrm{IE}}_{t,t_0}$ as

$$\hat{U}^{\mathrm{IE}}_{t,t_0} := \sum_{n=0}^{\infty} e^{i\alpha^{\mathrm{LR}}_{n,t}} |\phi_n; \vec{\lambda}_t\rangle\langle\phi_n; \vec{\lambda}_{t_0}|. \tag{2.29}$$

Since the operator $\hat{U}^{\mathrm{IE}}_{t,t_0}$ obeys $i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}^{\mathrm{IE}}_{t,t_0} = \hat{H}^{\mathrm{IE}}_t\hat{U}^{\mathrm{IE}}_{t,t_0}$, we can formally solve it for the Hamiltonian as

$$\hat{H}^{\mathrm{IE}}_t = i\hbar\frac{\mathrm{d}\hat{U}^{\mathrm{IE}}_{t,t_0}}{\mathrm{d}t}\hat{U}^{\mathrm{IE}\dagger}_{t,t_0} =: \hat{F}^{\mathrm{diag}}_t + i\hbar\sum_{n=0}^{\infty}\frac{\mathrm{d}|\phi_n; \vec{\lambda}_t\rangle}{\mathrm{d}t}\langle\phi_n; \vec{\lambda}_t|, \tag{2.30}$$

where $\hat{F}^{\mathrm{diag}}_t$ is a time-dependent diagonal operator in the basis $|\phi_n; \vec{\lambda}_t\rangle$:

$$\hat{F}^{\mathrm{diag}}_t := -\hbar\sum_{n=0}^{\infty}\frac{\mathrm{d}\alpha^{\mathrm{LR}}_{n,t}}{\mathrm{d}t}|\phi_n; \vec{\lambda}_t\rangle\langle\phi_n; \vec{\lambda}_t|. \tag{2.31}$$

Since we can choose the LR phase $\alpha^{\mathrm{LR}}_{n,t}$ freely, we can obtain different $\hat{F}^{\mathrm{diag}}_t$ in $\hat{H}^{\mathrm{IE}}_t$ depending on the choice of $\alpha^{\mathrm{LR}}_{n,t}$ for a given invariant $\hat{I}^{\mathrm{LR}}_t$.

If we impose $[\hat{I}^{\mathrm{LR}}_{t_0}, \hat{H}^{\mathrm{IE}}_{t_0}] = [\hat{I}^{\mathrm{LR}}_{t_\mathrm{f}}, \hat{H}^{\mathrm{IE}}_{t_\mathrm{f}}] = 0$, $\hat{I}^{\mathrm{LR}}_t$ and $\hat{H}^{\mathrm{IE}}_t$ have the simultaneous eigenstate at the endpoints of the time evolution. Therefore, if the eigenstate of $\hat{H}^{\mathrm{IE}}_t$ has the same quantum numbers at the initial and the final states, we can achieve the desired final state with the same quantum number as the initial state.

2.2.1.1 Example: $\hat{H}^{\mathrm{IE}}_t$ for a quantum parametric oscillator

We consider a *usual* quantum parametric oscillator (QPO) described by the following Hamiltonian,

$$\hat{H}^{\mathrm{ad}}_t = \frac{\hat{p}^2}{2M} + \frac{M}{2}\omega_t^2\hat{x}^2, \tag{2.32}$$

where $\hat{x}$ and $\hat{p}$ are position operator and momentum operator, respectively. $\hat{x}$ and $\hat{p}$ satisfy the canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$. For the QPO, we choose $\hat{I}^{\mathrm{LR}}_t = \sum_{n=0}^{\infty}\iota_n|n; \vec{\lambda}_t\rangle\langle n; \vec{\lambda}_t|$ with $\iota_n = \hbar\omega_{t_0}\left(n + \frac{1}{2}\right)$, that is,

$$\hat{I}^{\mathrm{LR}}_t = \frac{\omega_{t_0}}{\omega_t}\hat{H}^{\mathrm{ad}}_t. \tag{2.33}$$

We set the LR phase as

$$\alpha^{\mathrm{LR}}_{n,t} = \xi_{n,t} = -\frac{1}{\hbar}\int_{t_0}^{t}\mathrm{d}\tau E^{\mathrm{ad}}_{n,\tau}, \tag{2.34}$$

by using $\xi_{n,t}$, where the geometric phase $i\int_{t_0}^{t}\mathrm{d}\tau\langle n; \vec{\lambda}_\tau|\frac{\mathrm{d}}{\mathrm{d}\tau}|n; \vec{\lambda}_\tau\rangle$ in $\xi_{n,t}$ in Eq. (2.21) vanishes since $\langle n; \vec{\lambda}_\tau|\frac{\mathrm{d}}{\mathrm{d}\tau}|n; \vec{\lambda}_\tau\rangle = 0$ for the QPO [50]. We then obtain

$$\hat{H}^{\mathrm{IE}}_t = \hat{F}^{\mathrm{diag}}_t - \frac{1}{2}\frac{\dot{\omega}_t}{\omega_t}\frac{\hat{x}\hat{p} + \hat{p}\hat{x}}{2}, \tag{2.35}$$

from Eqs. (2.30) and (2.31), where $\hat{F}^{\mathrm{diag}}_t = \frac{\omega_t}{\omega_{t_0}}\hat{I}^{\mathrm{LR}}_t = \hat{H}^{\mathrm{ad}}_t$. For the QPO, we may adopt a different LR invariant with another quadratic form. Such an invariant yields different inverse engineering of the Hamiltonian (see Ref. [22] and Sec. III in Ref. [28]).

2.2.2 Transitionless tracking algorithm

We review the transitionless tracking (TT) algorithm based on Ref. [15]. Let us consider an arbitrary time-dependent Hamiltonian $\hat{H}_t^{\rm ad}$, with instantaneous eigenstate and eigenenergy given by Eq. (2.13). In the adiabatic approximation, we have already known that the state driven by $\hat{H}_t^{\rm ad}$ is given by Eq. (2.20). In the TT algorithm, we seek a TT Hamiltonian $\hat{H}_t^{\rm TT}$ that satisfies

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = \hat{H}_t^{\rm TT}|\Psi(t)\rangle, \tag{2.36}$$

where

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} C_n e^{i\xi_{n,t}}|n;\vec{\lambda}_t\rangle. \tag{2.37}$$

For this solution, it is obvious that no transition occurs between the different instantaneous eigenstates of $\hat{H}_t^{\rm ad}$ in an arbitrary time duration.

We construct the Hamiltonian $\hat{H}_t^{\rm TT}$ by adding a counter-diabatic Hamiltonian $\hat{H}_t^{\rm cd}$ to $\hat{H}_t^{\rm ad}$ as

$$\hat{H}_t^{\rm TT} := \hat{H}_t^{\rm ad} + \hat{H}_t^{\rm cd}. \tag{2.38}$$

In order to find $\hat{H}_t^{\rm cd}$, we introduce the time-evolution unitary operator $\hat{U}_{t,t_0}^{\rm TT}$ as

$$\hat{U}_{t,t_0}^{\rm TT} := \sum_{n=0}^{\infty} e^{i\xi_{n,t}}|n;\vec{\lambda}_t\rangle\langle n;\vec{\lambda}_{t_0}|. \tag{2.39}$$

$\hat{U}_{t,t_0}^{\rm TT}$ is the solution of $i\hbar\dot{\hat{U}}_{t,t_0}^{\rm TT} = \hat{H}_t^{\rm TT}\hat{U}_{t,t_0}^{\rm TT}$, which can be formally solved for the TT Hamiltonian $\hat{H}_t^{\rm TT} = i\hbar\frac{\mathrm{d}\hat{U}_{t,t_0}^{\rm TT}}{\mathrm{d}t}\hat{U}_{t,t_0}^{\rm TT\dagger}$. Therefore, we can obtain the TT Hamiltonian as

$$\begin{aligned}
\hat{H}_t^{\rm ad} &= \sum_{n=0}^{\infty} E_{n,t}^{\rm ad}|n;\vec{\lambda}_t\rangle\langle n;\vec{\lambda}_t|, &(2.40)\\
\hat{H}_t^{\rm cd} &:= i\hbar\sum_{n=0}^{\infty}(\hat{\mathbb{1}} - |n;\vec{\lambda}_t\rangle\langle n;\vec{\lambda}_t|)\frac{\mathrm{d}|n;\vec{\lambda}_t\rangle}{\mathrm{d}t}\langle n;\vec{\lambda}_t| &(2.41)\\
&= i\hbar\dot{\vec{\lambda}}_t\cdot\sum_{n=0}^{\infty}(\hat{\mathbb{1}} - |n;\vec{\lambda}_t\rangle\langle n;\vec{\lambda}_t|)\frac{\mathrm{d}|n;\vec{\lambda}_t\rangle}{\mathrm{d}\vec{\lambda}_t}\langle n;\vec{\lambda}_t| &(2.42)\\
&= -i\hbar\sum_{\substack{m,n=0\\(m\neq n)}}^{\infty}|m;\vec{\lambda}_t\rangle\frac{\langle m;\vec{\lambda}_t|\frac{\mathrm{d}\hat{H}_t^{\rm ad}}{\mathrm{d}t}|n;\vec{\lambda}_t\rangle}{E_{m,t}^{\rm ad} - E_{n,t}^{\rm ad}}\langle n;\vec{\lambda}_t|, &(2.43)
\end{aligned}$$

where we have used Eq. (2.15) for the last equality.

We can express $\hat{H}_t^{\rm cd}$ in Eq. (2.42) as

$$\hat{H}_t^{\rm cd} =: \dot{\vec{\lambda}}_t\cdot\hat{\vec{\eta}}_t, \tag{2.44}$$

where the operator $\hat{\vec{\eta}}_t$ can be regarded as a generator of adiabatic transport [24, 51]. We let δt be an infinitesimal time displacement. We then have

$$e^{-i\delta t\hat{H}_t^{\rm cd}/\hbar}|n;\vec{\lambda}_t\rangle = \left(1 - \frac{i}{\hbar}\hat{H}_t^{\rm cd}\delta t + O(\delta t^2)\right)|n;\vec{\lambda}_t\rangle$$

$$
\begin{aligned}
&\overset{3}{=} \left(1 + \frac{1}{i\hbar}\delta\vec{\lambda}_t \cdot \hat{\vec{\eta}}_t\right)|n;\vec{\lambda}_t\rangle + O(|\delta\vec{\lambda}_t|^2) \\
&= \left\{1 + \frac{1}{i\hbar}\left[i\hbar\delta\vec{\lambda}_t \cdot \sum_{m=0}^{\infty}(\hat{\mathbb{1}} - |m;\vec{\lambda}_t\rangle\langle m;\vec{\lambda}_t|)\frac{\mathrm{d}|m;\vec{\lambda}_t\rangle}{\mathrm{d}\vec{\lambda}_t}\langle m;\vec{\lambda}_t|\right]\right\}|n;\vec{\lambda}_t\rangle + O(|\delta\vec{\lambda}_t|^2) \\
&= |n;\vec{\lambda}_t\rangle + \left(\frac{\mathrm{d}|n;\vec{\lambda}_t\rangle}{\mathrm{d}\vec{\lambda}_t} - \langle n;\vec{\lambda}_t|\frac{\mathrm{d}|n;\vec{\lambda}_t\rangle}{\mathrm{d}\vec{\lambda}_t}|n;\vec{\lambda}_t\rangle\right)\cdot\delta\vec{\lambda}_t + O(|\delta\vec{\lambda}_t|^2) \\
&\overset{4}{=} |n;\vec{\lambda}_{t+\delta t}\rangle - \langle n;\vec{\lambda}_t|\frac{\mathrm{d}|n;\vec{\lambda}_t\rangle}{\mathrm{d}\vec{\lambda}_t}(|n;\vec{\lambda}_{t+\delta t}\rangle + O(|\delta\vec{\lambda}_t|))\cdot\delta\vec{\lambda}_t + O(|\delta\vec{\lambda}_t|^2) \\
&= \left(1 - \langle n;\vec{\lambda}_t|\frac{\mathrm{d}|n;\vec{\lambda}_t\rangle}{\mathrm{d}\vec{\lambda}_t}\cdot\delta\vec{\lambda}_t\right)|n;\vec{\lambda}_{t+\delta t}\rangle + O(|\delta\vec{\lambda}_t|^2) \\
&= \exp\left(-\langle n;\vec{\lambda}_t|\frac{\mathrm{d}|n;\vec{\lambda}_t\rangle}{\mathrm{d}t}\delta t\right)|n;\vec{\lambda}_{t+\delta t}\rangle + O(\delta t^2). \qquad (2.45)
\end{aligned}
$$

Therefore, we find

$$
\begin{aligned}
e^{-i\delta t\hat{H}_t^{\mathrm{TT}}/\hbar}|n;\vec{\lambda}_t\rangle &= e^{-i\delta t(\hat{H}_t^{\mathrm{ad}}+\hat{H}_t^{\mathrm{cd}})/\hbar}|n;\vec{\lambda}_t\rangle \\
&\overset{5}{=} e^{-i\delta t\hat{H}_t^{\mathrm{ad}}/\hbar}e^{-i\delta t\hat{H}_t^{\mathrm{cd}}/\hbar}|n;\vec{\lambda}_t\rangle(1 + O(\delta t^2)) \qquad (2.46) \\
&= e^{-i\delta tE_{n,t}^{\mathrm{ad}}/\hbar}\exp\left(-\langle n;\vec{\lambda}_t|\frac{\mathrm{d}|n;\vec{\lambda}_t\rangle}{\mathrm{d}t}\delta t\right)|n;\vec{\lambda}_{t+\delta t}\rangle + O(\delta t^2) \\
&= \exp\left[i\left(-\frac{E_{n,t}^{\mathrm{ad}}}{\hbar} + i\langle n;\vec{\lambda}_t|\frac{\mathrm{d}|n;\vec{\lambda}_t\rangle}{\mathrm{d}t}\right)\delta t\right]|n;\vec{\lambda}_{t+\delta t}\rangle + O(\delta t^2) \\
&= e^{i\dot{\xi}_{n,t}\delta t}|n;\vec{\lambda}_{t+\delta t}\rangle + O(\delta t^2). \qquad (2.47)
\end{aligned}
$$

Eq. (2.47) shows that the addition of $\hat{H}_t^{\mathrm{cd}}$ to $\hat{H}_t^{\mathrm{ad}}$ generates the adiabatic evolution.

If we choose $\hat{I}_t^{\mathrm{LR}} = \sum_{n=0}^{\infty} E_{n,t_0}^{\mathrm{ad}}|n;\vec{\lambda}_t\rangle\langle n;\vec{\lambda}_t|$ and $\alpha_{n,t}^{\mathrm{LR}} = \xi_{n,t}$, $\hat{H}_t^{\mathrm{TT}}$ generally coincides with $\hat{H}_t^{\mathrm{IE}}$ in Eq. (2.30) [28]. Therefore, we can understand the TT algorithm in terms of the dynamical invariant (see also Sec. 3.5). For the QPO, we can confirm from the example below that $\hat{H}_t^{\mathrm{IE}}$ in Eq. (2.35) indeed coincides with $\hat{H}_t^{\mathrm{TT}}$ when we choose $\hat{I}_t^{\mathrm{LR}} = \frac{\omega_{t_0}}{\omega_t}\hat{H}_t^{\mathrm{ad}}$ and $\alpha_{n,t}^{\mathrm{LR}} = \xi_{n,t}$.

2.2.2.1 Example: $\hat{H}_t^{\mathrm{cd}}$ for a quantum parametric oscillator

For the QPO given by Eq. (2.32), the following relations hold as

$$\hat{x} = \sqrt{\frac{\hbar}{2M\omega_t}}(\hat{a}_t + \hat{a}_t^\dagger), \qquad (2.48)$$

$$\hat{p} = \frac{1}{i}\sqrt{\frac{\hbar M\omega_t}{2}}(\hat{a}_t + \hat{a}_t^\dagger), \qquad (2.49)$$

$$\hat{a}_t^{\dagger N}|n;\omega_t\rangle = \sqrt{\frac{(n+N)!}{N!}}|n+N;\omega_t\rangle, \qquad (2.50)$$

$$\hat{a}_t^{N}|n;\omega_t\rangle = \sqrt{\frac{n!}{(n-N)!}}|n-N;\omega_t\rangle; \quad n > N \in \mathbb{N}_0, \qquad (2.51)$$

[3] $\delta\vec{\lambda}_t := \dot{\vec{\lambda}}_t\delta t$.

[4] $|n;\vec{\lambda}_{t+\delta t}\rangle = \sum_{m=0}^{\infty}\frac{1}{m!}\left(\delta\vec{\lambda}_t\cdot\frac{\mathrm{d}}{\mathrm{d}\vec{\lambda}_t}\right)^m|n;\vec{\lambda}_t\rangle = |n;\vec{\lambda}_t\rangle + \delta\vec{\lambda}_t\cdot\frac{\mathrm{d}|n;\vec{\lambda}_t\rangle}{\mathrm{d}\vec{\lambda}_t} + \frac{1}{2}\left(\delta\vec{\lambda}_t\cdot\frac{\mathrm{d}}{\mathrm{d}\vec{\lambda}_t}\right)^2|n;\vec{\lambda}_t\rangle + \cdots$.

[5] We here used the Zassenhaus formula [52]: $e^{t(\hat{A}+\hat{B})} = e^{t\hat{A}}e^{t\hat{B}}e^{-t^2[\hat{A},\hat{B}]/2}e^{t^3([\hat{A},[\hat{A},\hat{B}]]-2[\hat{B},[\hat{B},\hat{A}]])/6}\ldots$.

where $[\hat{a}_t, \hat{a}_t^\dagger] = 1$. According to the TT algorithm, by using Eq. (2.43), the counter-diabatic Hamiltonian $\hat{H}_t^{\rm cd}$ for the QPO [Eq. (2.32)] is calculated as [50, 53]

$$
\begin{aligned}
\hat{H}_t^{\rm cd} &= -i\hbar \sum_{\substack{m,n=0\\(m\neq n)}}^{\infty} |m;\omega_t\rangle \frac{\langle m;\omega_t|\frac{\mathrm{d}\hat{H}_t^{\rm ad}}{\mathrm{d}t}|n;\omega_t\rangle}{E_{m,t}^{\rm ad} - E_{n,t}^{\rm ad}} \langle n;\omega_t| \\
&= -i\hbar \sum_{\substack{m,n=0\\(m\neq n)}}^{\infty} |m;\omega_t\rangle \frac{\langle m;\omega_t|\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\hat{p}^2}{2M} + \frac{M}{2}\omega_t^2\hat{x}^2\right)\right]|n;\omega_t\rangle}{E_{m,t}^{\rm ad} - E_{n,t}^{\rm ad}} \langle n;\omega_t| \\
&= -iM\dot{\omega}_t \sum_{\substack{m,n=0\\(m\neq n)}}^{\infty} |m;\omega_t\rangle \frac{\langle m;\omega_t|\hat{x}^2|n;\omega_t\rangle}{m-n} \langle n;\omega_t| \\
&= -\frac{i\hbar}{2}\frac{\dot{\omega}_t}{\omega_t} \sum_{\substack{m,n=0\\(m\neq n)}}^{\infty} |m;\omega_t\rangle \frac{\langle m;\omega_t|(\hat{a}_t + \hat{a}_t^\dagger)^2|n;\omega_t\rangle}{m-n} \langle n;\omega_t| \\
&= -\frac{i\hbar}{2}\frac{\dot{\omega}_t}{\omega_t} \sum_{\substack{m,n=0\\(m\neq n)}}^{\infty} |m;\omega_t\rangle \frac{\langle m;\omega_t|\hat{a}_t^2 + \hat{a}_t^{\dagger 2} + 2\hat{a}_t^\dagger\hat{a}_t + \hat{\mathbb{1}}|n;\omega_t\rangle}{m-n} \langle n;\omega_t| \\
&= -\frac{i\hbar}{2}\frac{\dot{\omega}_t}{\omega_t} \sum_{\substack{m,n=0\\(m\neq n)}}^{\infty} |m;\omega_t\rangle \frac{\sqrt{(m+1)(m+2)}\delta_{m+2,n} + \sqrt{(n+1)(n+2)}\delta_{m,n+2} + (2n+1)\delta_{m,n}}{m-n} \langle n;\omega_t| \\
&= \frac{i\hbar}{2}\frac{\dot{\omega}_t}{\omega_t} \sum_{n=0}^{\infty} \frac{\sqrt{(n-1)n}|n-2;\omega_t\rangle - \sqrt{(n+1)(n+2)}|n+2;\omega_t\rangle}{2} \langle n;\omega_t| \\
&= \frac{i\hbar}{2}\frac{\dot{\omega}_t}{\omega_t}\frac{\hat{a}_t^2 - \hat{a}_t^{\dagger 2}}{2} \sum_{n=0}^{\infty} |n;\omega_t\rangle\langle n;\omega_t| \\
&= \frac{i\hbar}{2}\frac{\dot{\omega}_t}{\omega_t}\frac{\hat{a}_t^2 - \hat{a}_t^{\dagger 2}}{2} = -\frac{1}{2}\frac{\dot{\omega}_t}{\omega_t}\frac{\hat{x}\hat{p} + \hat{p}\hat{x}}{2}.
\end{aligned}
\tag{2.52}
$$

We can confirm that the second term in Eq. (2.35) obtained by using the LR invariant-based inverse engineering coincides with the counter-diabatic term in Eq. (2.52).

2.2.3 Classical dissipationless driving

We here review the *classical dissipationless driving* based on Refs. [23, 24]. The goal of this method is to derive a counter-dissipative Hamiltonian such that an adiabatic invariant of a classical system is conserved in an arbitrary time duration. This method can be regarded as the classical analog of the TT algorithm.

Let us consider a classical Hamiltonian whose time-dependence originates from a set of external parameters, i.e.,

$$
H_t^{\rm ad} \equiv H^{\rm ad}(x, p; \vec{\lambda}_t) := \frac{p^2}{2M} + V(x; \vec{\lambda}_t). \tag{2.53}
$$

Under an infinitely slow change of $\vec{\lambda}_t$, the action variable $S_t = \frac{1}{2\pi}\mathcal{V}(H_t^{\rm ad}; \vec{\lambda}_t)$ in Eq. (2.3) remains constant as the adiabatic invariant J_t. We now seek a counter-dissipative Hamiltonian with a set of counter-dissipative generators $\vec{\eta}_t \equiv \vec{\eta}(x, p; \vec{\lambda}_t)$ for the original system described by Eq. (2.53):

$$
H_t^{\rm cd} \equiv H^{\rm cd}(x, p; \vec{\lambda}_t) = \dot{\vec{\lambda}}_t \cdot \vec{\eta}_t, \tag{2.54}
$$

such that the following action variable is conserved in an arbitrary time duration: [6]

$$J_t \equiv J(x, p; \vec{\lambda}_t) := \frac{1}{2\pi}\mathcal{V}[H^{\mathrm{ad}}(x, p; \vec{\lambda}_t); \vec{\lambda}_t]. \tag{2.55}$$

We note that Eq. (2.54) is a classical version of the counter-diabatic Hamiltonian in Eq. (2.42). This action variable is characterized by the phase space volume $\mathcal{V}(\bar{E}_t; \vec{\lambda}_t)$ enclosed by the *adiabatic energy shell* [20, 23, 24] $\bar{\Xi} := \{(x, p)|H^{\mathrm{ad}}(x, p; \vec{\lambda}_t) = \bar{E}_t\}$ in the phase space Γ. The adiabatic energy shell $\bar{\Xi}$ is the level set of H_t^{ad} with the value $\bar{E}_t$, enclosing the volume $\mathcal{V}(\bar{E}_t; \vec{\lambda}_t)$. The *adiabatic energy* $\bar{E}_t$ is defined as the energy such that $\mathcal{V}(\bar{E}_t; \vec{\lambda}_t)$ is exactly conserved.

We next consider the micro-canonical average of a quantity $A(x, p; \vec{\lambda}_t)$ as

$$\langle A(x, p; \vec{\lambda}_t)\rangle_{E_t, \vec{\lambda}_t} := \left(\frac{\partial \mathcal{V}(E_t; \vec{\lambda}_t)}{\partial E_t}\right)^{-1} \int_\Gamma \mathrm{d}x\mathrm{d}p\, \delta[E_t - H^{\mathrm{ad}}(x, p; \vec{\lambda}_t)]A(x, p; \vec{\lambda}_t). \tag{2.56}$$

By solving $\mathcal{V}(E_t; \vec{\lambda}_t)$ with respect to E_t, we have $E_t = E(\mathcal{V}; \vec{\lambda}_t)$. By substituting $\mathcal{V}(E_t; \vec{\lambda}_t)$ into $E(\mathcal{V}; \vec{\lambda}_t)$ again, we obtain $E(\mathcal{V}; \vec{\lambda}_t) = E(\mathcal{V}(E_t; \vec{\lambda}_t); \vec{\lambda}_t)$. By differentiating $\mathcal{V}(E_t; \vec{\lambda}_t)$ [Eq. (2.2)] with respect to $\vec{\lambda}_t$ and using Eq. (2.56), we have

$$\begin{aligned}
\frac{\partial \mathcal{V}(E_t; \vec{\lambda}_t)}{\partial \vec{\lambda}_t} &= \int_\Gamma \mathrm{d}x\mathrm{d}p \frac{\partial}{\partial \vec{\lambda}_t}\Theta[E_t - H^{\mathrm{ad}}(x, p; \vec{\lambda}_t)] \\
&= -\int_\Gamma \mathrm{d}x\mathrm{d}p\, \delta[E_t - H^{\mathrm{ad}}(x, p; \vec{\lambda}_t)]\frac{\partial H^{\mathrm{ad}}(x, p; \vec{\lambda}_t)}{\partial \vec{\lambda}_t} \\
&= -\frac{\partial \mathcal{V}(E_t; \vec{\lambda}_t)}{\partial E_t}\left\langle \frac{\partial H^{\mathrm{ad}}(x, p; \vec{\lambda}_t)}{\partial \vec{\lambda}_t}\right\rangle_{E_t, \vec{\lambda}_t}.
\end{aligned} \tag{2.57}$$

By using the cyclic chain rule [7]

$$\left.\frac{\partial \mathcal{V}(E_t; \vec{\lambda}_t)}{\partial \vec{\lambda}_t}\right|_{E_t} \cdot \left.\frac{\partial \vec{\lambda}_t(E_t, \mathcal{V})}{\partial E_t}\right|_{\mathcal{V}} \left.\frac{\partial E(\mathcal{V}; \vec{\lambda}_t)}{\partial \mathcal{V}}\right|_{\vec{\lambda}_t} = -1, \tag{2.58}$$

we have

$$\frac{\partial E(\mathcal{V}; \vec{\lambda}_t)}{\partial \vec{\lambda}_t} = -\frac{\partial \mathcal{V}(E_t; \vec{\lambda}_t)}{\partial \vec{\lambda}_t}\left(\frac{\partial \mathcal{V}(E_t; \vec{\lambda}_t)}{\partial E_t}\right)^{-1}. \tag{2.59}$$

From Eqs. (2.57) and (2.59), we obtain

$$\frac{\partial E(\mathcal{V}; \vec{\lambda}_t)}{\partial \vec{\lambda}_t} = -\left(\frac{\partial \mathcal{V}}{\partial E_t}\right)^{-1}\frac{\partial \mathcal{V}}{\partial \vec{\lambda}_t} = \left\langle \frac{\partial H_t^{\mathrm{ad}}}{\partial \vec{\lambda}_t}\right\rangle_{E_t, \vec{\lambda}_t}. \tag{2.60}$$

We here define the Poisson bracket as

$$\{\vec{A}, \vec{B}\}^{\mathrm{P}} := \frac{\partial \vec{A}}{\partial x} \cdot \frac{\partial \vec{B}}{\partial p} - \frac{\partial \vec{A}}{\partial p} \cdot \frac{\partial \vec{B}}{\partial x}. \tag{2.61}$$

[6] From Eq. (2.2), we may define $\mathcal{V}[H^{\mathrm{ad}}(x, p; \vec{\lambda}_t); \vec{\lambda}_t] := \int_\Gamma \mathrm{d}x'\mathrm{d}p'\Theta[H^{\mathrm{ad}}(x, p; \vec{\lambda}_t) - H^{\mathrm{ad}}(x', p'; \vec{\lambda}_t)]$.

[7] The rule relates with partial derivatives of three interdependent variables. In each factor, the variable (vector variable) in the numerator is set to be an implicit function of the other two. The variable (vector variable) in the subscript is being held constant (constant vector).

We then introduce the following conditions for η_t [54] [8]

$$\{\vec{\eta}_t, H_t^{\mathrm{ad}}\}^{\mathrm{P}} = \frac{\partial H_t^{\mathrm{ad}}}{\partial \vec{\lambda}_t} - \left\langle \frac{\partial H_t^{\mathrm{ad}}}{\partial \vec{\lambda}_t} \right\rangle_{H_t^{\mathrm{ad}}, \vec{\lambda}_t}, \tag{2.62}$$

$$\langle \vec{\eta}_t \rangle_{E_t, \vec{\lambda}_t} = \vec{0}. \tag{2.63}$$

By calculating the left-hand side of Eq. (2.62), we have

$$\{\vec{\eta}_t, H_t^{\mathrm{ad}}\}^{\mathrm{P}} = 2\pi \left(\frac{\partial \mathcal{V}(H_t^{\mathrm{ad}}; \vec{\lambda}_t)}{\partial H_t^{\mathrm{ad}}} \right)^{-1} \{\vec{\eta}_t, J_t\}^{\mathrm{P}}, \tag{2.64}$$

where J_t is the action variable as already defined in Eq. (2.55). By calculating the right-hand side of Eq. (2.62) and by using Eq. (2.60), we have [9]

$$\frac{\partial H_t^{\mathrm{ad}}}{\partial \vec{\lambda}_t} - \left\langle \frac{\partial H_t^{\mathrm{ad}}}{\partial \vec{\lambda}_t} \right\rangle_{H_t^{\mathrm{ad}}, \vec{\lambda}_t} = 2\pi \left(\frac{\partial \mathcal{V}(H_t^{\mathrm{ad}}; \vec{\lambda}_t)}{\partial H_t^{\mathrm{ad}}} \right)^{-1} \frac{\partial J(x, p; \vec{\lambda}_t)}{\partial \vec{\lambda}_t}. \tag{2.65}$$

From Eqs. (2.64) and (2.65), we obtain the simpler form of Eq. (2.62) as

$$\{\vec{\eta}_t, J_t\}^{\mathrm{P}} = \frac{\partial J_t}{\partial \vec{\lambda}_t}. \tag{2.66}$$

As discussed in Ref. [24], $\vec{\eta}_t$ in Eq. (2.54) can be a generator of the infinitesimal transformation

$$z \to z + \mathrm{d}z, \tag{2.67}$$

where $z := (x, p)$. Since $\dot{z} = \{z, H_t^{\mathrm{cd}}\}^{\mathrm{P}}$ and using Eq. (2.54), we find

$$\mathrm{d}z = \{z, \vec{\eta}_t\}^{\mathrm{P}} \cdot \mathrm{d}\vec{\lambda}_t. \tag{2.68}$$

Eq. (2.68) provides a rule for converting a small change of the parameters $\mathrm{d}\vec{\lambda}_t$ into a small displacement in the phase space, $\mathrm{d}z$. In order to achieve the classical dissipationless driving under Eq. (2.68), the adiabatic energy shell $\bar{\Xi} = \{(x, p) | H^{\mathrm{ad}}(x, p; \vec{\lambda}_t) = \bar{E}_t\}$ should be mapped onto the adiabatic energy shell $\bar{\Xi}' := \{(x, p) | H^{\mathrm{ad}}(x, p; \vec{\lambda}_t + \mathrm{d}\vec{\lambda}_t) = \bar{E}_t + \mathrm{d}\bar{E}_t\}$ such that $\mathcal{V}(\bar{E}_t; \vec{\lambda}_t) = \mathcal{V}(\bar{E}_t + \mathrm{d}\bar{E}; \vec{\lambda}_t + \mathrm{d}\vec{\lambda}_t)$ holds. By using Eqs. (2.66) and (2.68), we can show that this mapping is ensured as

$$\begin{aligned} J(x + \mathrm{d}x, p + \mathrm{d}p; \vec{\lambda}_t + \mathrm{d}\vec{\lambda}_t) &= J(x, p; \vec{\lambda}_t) + \frac{\partial J(x, p; \vec{\lambda}_t)}{\partial x} \mathrm{d}x + \frac{\partial J(x, p; \vec{\lambda}_t)}{\partial p} \mathrm{d}p + \frac{\partial J(x, p; \vec{\lambda}_t)}{\partial \vec{\lambda}_t} \cdot \mathrm{d}\vec{\lambda}_t \\ &= J(x, p; \vec{\lambda}_t) + \left(\{J(x, p; \vec{\lambda}_t), \vec{\eta}_t\}^{\mathrm{P}} + \frac{\partial J(x, p; \vec{\lambda}_t)}{\partial \vec{\lambda}_t} \right) \cdot \mathrm{d}\vec{\lambda}_t \\ &= J(x, p; \vec{\lambda}_t). \end{aligned} \tag{2.69}$$

[8] From Eq. (2.60), we may have

$$\begin{aligned} \left\langle \frac{\partial H_t^{\mathrm{ad}}}{\partial \vec{\lambda}_t} \right\rangle_{H_t^{\mathrm{ad}}, \vec{\lambda}_t} &= -\left(\frac{\partial \mathcal{V}(H_t^{\mathrm{ad}}; \vec{\lambda}_t)}{\partial H_t^{\mathrm{ad}}} \right)^{-1} \int_\Gamma \mathrm{d}x' \mathrm{d}p' \delta[H_t^{\mathrm{ad}} - H^{\mathrm{ad}}(x', p'; \vec{\lambda}_t)] \frac{\partial H^{\mathrm{ad}}(x', p'; \vec{\lambda}_t)}{\partial \vec{\lambda}_t} \\ &= -\left(\frac{\partial \mathcal{V}(H_t^{\mathrm{ad}}; \vec{\lambda}_t)}{\partial H_t^{\mathrm{ad}}} \right)^{-1} \frac{\partial \mathcal{V}(H_t^{\mathrm{ad}}; \vec{\lambda}_t)}{\partial \vec{\lambda}_t}. \end{aligned}$$

[9] Here we used $\frac{\partial f}{\partial \vec{\lambda}_t} = \frac{\partial f}{\partial \cdot} \frac{\partial g}{\partial \vec{\lambda}_t} + \frac{\partial f}{\partial \vec{\lambda}_t}$ for a function $f(g(\,\cdot\,; \vec{\lambda}_t); \vec{\lambda}_t)$, where f contains g and $\vec{\lambda}_t$ as the arguments, and g also contains $\vec{\lambda}_t$ as the argument. We then used the following calculation: $\frac{\partial \mathcal{V}}{\partial \vec{\lambda}_t} = -\left\langle \frac{\partial H_t^{\mathrm{ad}}}{\partial \vec{\lambda}_t} \right\rangle_{H_t^{\mathrm{ad}}, \vec{\lambda}_t} \frac{\partial \mathcal{V}}{\partial H_t^{\mathrm{ad}}}$.

Hence, the term H_t^{cd} with $\vec{\eta}_t$ satisfying Eq. (2.66) provides precisely the classical dissipationless driving.

Let us check the invariance of the action variable J_t with the generator $\vec{\eta}_t$. We consider a point $z = (x, p)$ evolving under Hamilton equation

$$\dot{z} = \{z, H_t^{\mathrm{TT}}\}^{\mathrm{P}}, \tag{2.70}$$

where $H_t^{\mathrm{TT}} := H_t^{\mathrm{ad}} + H_t^{\mathrm{cd}}$. From Eqs. (2.55), (2.66) and (2.70), we find

$$\frac{\mathrm{d}J(x, p; \vec{\lambda}_t)}{\mathrm{d}t} = \{J(x, p; \vec{\lambda}_t), H_t^{\mathrm{TT}}\}^{\mathrm{P}} + \frac{\partial J(x, p; \vec{\lambda}_t)}{\partial \vec{\lambda}_t} \cdot \dot{\vec{\lambda}}_t = 0. \tag{2.71}$$

Therefore, owing to the counter-dissipative Hamiltonian H_t^{cd} [Eq. (2.54)], the action variable J_t is exactly conserved as the adiabatic invariant.

2.2.3.1 Example: scale-invariant classical system

Suppose that $\vec{\lambda}_t =: (\lambda_t^{(1)}, \lambda_t^{(2)})^\top$. Let us consider the following Hamiltonian

$$H^{\mathrm{ad}}(x, p; \lambda_t^{(1)}, \lambda_t^{(2)}) = \frac{p^2}{2M} + \frac{1}{\lambda_t^{(1)2}} V^{\mathrm{sc}}\left(\frac{x - \lambda_t^{(2)}}{\lambda_t^{(1)}}\right). \tag{2.72}$$

This Hamiltonian Eq. (2.72) satisfies the following scale-invariant conditions:

$$H^{\mathrm{ad}}(x + a, p; \lambda_t^{(1)}, \lambda_t^{(2)} + a) = H^{\mathrm{ad}}(x, p; \lambda_t^{(1)}, \lambda_t^{(2)}), \tag{2.73}$$

$$b^2 H^{\mathrm{ad}}\left(bx, \frac{p}{b}; b\lambda_t^{(1)}, b\lambda_t^{(2)}\right) = H^{\mathrm{ad}}(x, p; \lambda_t^{(1)}, \lambda_t^{(2)}), \tag{2.74}$$

$$\mathcal{V}(E_t; \lambda_t^{(1)}, \lambda_t^{(2)}) = \mathcal{V}(\lambda_t^{(1)2} E_t; 1, 0), \tag{2.75}$$

with $a \in \mathbb{R}$ and $b > 0$.

Let us next consider the following canonical transformation

$$(x, p) \to \left(x + \mathrm{d}\lambda_t^{(2)} + \frac{x - \lambda_t^{(2)}}{\lambda_t^{(1)}} \mathrm{d}\lambda_t^{(1)}, p - \frac{p}{\lambda_t^{(1)}} \mathrm{d}\lambda_t^{(1)}\right), \tag{2.76}$$

where the change $\lambda_t^{(1)} \to \lambda_t^{(1)} + \mathrm{d}\lambda_t^{(1)}$ squeezes the adiabatic energy shell Ξ, while the change $\lambda_t^{(2)} \to \lambda_t^{(2)} + \mathrm{d}\lambda_t^{(2)}$ translates it in the phase space Γ. We can find that the transformation Eq. (2.76) satisfies the condition Eq. (2.69) by using Eqs. (2.73)–(2.75) and by neglecting $O(\mathrm{d}\lambda_t^{(1)2})$. [10] By

[10] The detailed calculation is as bellows:

$$
\begin{aligned}
& J(x + \mathrm{d}x, p + \mathrm{d}p; \lambda_t^{(1)} + \mathrm{d}\lambda_t^{(1)}, \lambda_t^{(2)} + \mathrm{d}\lambda_t^{(2)}) \\
\overset{\text{Eq. (2.76)}}{=} & J\left(x + \mathrm{d}\lambda_t^{(2)} + \frac{x - \lambda_t^{(2)}}{\lambda_t^{(1)}}\mathrm{d}\lambda_t^{(1)}, p - \frac{p}{\lambda_t^{(1)}}\mathrm{d}\lambda_t^{(1)}; \lambda_t^{(1)} + \mathrm{d}\lambda_t^{(1)}, \lambda_t^{(2)} + \mathrm{d}\lambda_t^{(2)}\right) \\
\overset{\text{Eq. (2.55)}}{=} & \frac{1}{2\pi}\mathcal{V}\left[H^{\mathrm{ad}}\left(x + \mathrm{d}\lambda_t^{(2)} + \frac{x - \lambda_t^{(2)}}{\lambda_t^{(1)}}\mathrm{d}\lambda_t^{(1)}, p - \frac{p}{\lambda_t^{(1)}}\mathrm{d}\lambda_t^{(1)}; \lambda_t^{(1)} + \mathrm{d}\lambda_t^{(1)}, \lambda_t^{(2)} + \mathrm{d}\lambda_t^{(2)}\right); \lambda_t^{(1)} + \mathrm{d}\lambda_t^{(1)}, \lambda_t^{(2)} + \mathrm{d}\lambda_t^{(2)}\right] \\
\overset{\text{Eq. (2.73)}}{=} & \frac{1}{2\pi}\mathcal{V}\left[H^{\mathrm{ad}}\left(x + \frac{x - \lambda_t^{(2)}}{\lambda_t^{(1)}}\mathrm{d}\lambda_t^{(1)}, \left(1 - \frac{\mathrm{d}\lambda_t^{(1)}}{\lambda_t^{(1)}}\right)p; \lambda_t^{(1)} + \mathrm{d}\lambda_t^{(1)}, \lambda_t^{(2)}\right); \lambda_t^{(1)} + \mathrm{d}\lambda_t^{(1)}, \lambda_t^{(2)} + \mathrm{d}\lambda_t^{(2)}\right] \\
\overset{\text{Taylor expansion}}{=} & \frac{1}{2\pi}\mathcal{V}\left[H^{\mathrm{ad}}\left(\left(1 + \frac{\mathrm{d}\lambda_t^{(1)}}{\lambda_t^{(1)}}\right)x - \frac{\mathrm{d}\lambda_t^{(1)}}{\lambda_t^{(1)}}\lambda_t^{(2)}, \frac{p}{1 + \frac{\mathrm{d}\lambda_t^{(1)}}{\lambda_t^{(1)}}} + O(\mathrm{d}\lambda_t^{(1)2}); \left(1 + \frac{\mathrm{d}\lambda_t^{(1)}}{\lambda_t^{(1)}}\right)\lambda_t^{(1)}, \left(1 + \frac{\mathrm{d}\lambda_t^{(1)}}{\lambda_t^{(1)}}\right)\lambda_t^{(2)} - \frac{\mathrm{d}\lambda_t^{(1)}}{\lambda_t^{(1)}}\lambda_t^{(2)}\right); \lambda_t^{(1)} + \mathrm{d}\lambda_t^{(1)}, \lambda_t^{(2)} + \mathrm{d}\lambda_t^{(2)}\right]
\end{aligned}
$$

using Eqs. (2.68) and (2.76), we find that the following equations for η_t:

$$\begin{cases} \dfrac{\partial \eta_t^{(1)}}{\partial x} = \dfrac{p}{\lambda_t^{(1)}}, \quad \dfrac{\partial \eta_t^{(2)}}{\partial x} = 0, & (2.77) \\ \dfrac{\partial \eta_t^{(1)}}{\partial p} = \dfrac{x - \lambda_t^{(2)}}{\lambda_t^{(1)}}, \quad \dfrac{\partial \eta_t^{(2)}}{\partial p} = 1. & (2.78) \end{cases}$$

By solving Eqs. (2.77) and (2.78), we obtain the generator for the infinitesimal canonical transformation Eq. (2.76) as

$$\vec{\eta}_t =: (\eta_t^{(1)}, \eta_t^{(2)})^\top = \left(\frac{x - \lambda_t^{(2)}}{\lambda_t^{(1)}} p, p \right)^\top. \tag{2.79}$$

By substituting Eq. (2.79) into Eq. (2.54), we obtain

$$H_t^{\mathrm{cd}} = \frac{\dot{\lambda}_t^{(1)}}{\lambda_t^{(1)}} (x - \lambda_t^{(2)}) p + \dot{\lambda}_t^{(2)} p. \tag{2.80}$$

Upon quantization with the Weyl (symmetric) ordering rule, this result agrees with the *quantum* counter-diabatic Hamiltonian for the scale-invariant system.

For a CPO with $\lambda_t^{(1)} = \frac{1}{\sqrt{\omega_t}}$ and $\lambda_t^{(2)} = 0$ as the most straightforward scale-invariant system, we can easily have

$$H_t^{\mathrm{cd}} = -\frac{1}{2} \frac{\dot{\omega}_t}{\omega_t} xp. \tag{2.81}$$

By quantizing Eq. (2.81) with the Weyl (symmetric) ordering rule, we obtain the counter-diabatic term for a QPO [Eq. (2.52)].

2.3 Quantum-classical correspondence in adiabatic theorems

Although there is a conceptual difference between classical and quantum adiabatic theorems, a method of classical mechanics is applicable for analyzing the adiabatic evolution of a QPO. This is called Husimi's method [55].

$$\overset{\text{Eq. (2.73)}}{=} \frac{1}{2\pi} \mathcal{V}\left[H^{\mathrm{ad}}\left(\left(1 + \frac{\mathrm{d}\lambda_t^{(1)}}{\lambda_t^{(1)}}\right) x, \frac{p}{1 + \frac{\mathrm{d}\lambda_t^{(1)}}{\lambda_t^{(1)}}}; \left(1 + \frac{\mathrm{d}\lambda_t^{(1)}}{\lambda_t^{(1)}}\right) \lambda_t^{(1)}, \left(1 + \frac{\mathrm{d}\lambda_t^{(1)}}{\lambda_t^{(1)}}\right) \lambda_t^{(2)} \right); \lambda_t^{(1)} + \mathrm{d}\lambda_t^{(1)}, \lambda_t^{(2)} + \mathrm{d}\lambda_t^{(2)} \right] + \mathcal{O}(\mathrm{d}\lambda_t^{(1)2})$$

$$\overset{\text{Eq. (2.74)}}{=} \frac{1}{2\pi} \mathcal{V}\left[\frac{\lambda_t^{(1)2}}{(\lambda_t^{(1)} + \mathrm{d}\lambda_t^{(1)})^2} H^{\mathrm{ad}}(x, p; \lambda_t^{(1)}, \lambda_t^{(2)}); \lambda_t^{(1)} + \mathrm{d}\lambda_t^{(1)}, \lambda_t^{(2)} + \mathrm{d}\lambda_t^{(2)} \right] + \mathcal{O}(\mathrm{d}\lambda_t^{(1)2})$$

$$\overset{\text{Eq. (2.75)}}{=} \frac{1}{2\pi} \mathcal{V}[\lambda_t^{(1)2} H^{\mathrm{ad}}(x, p; \lambda_t^{(1)}, \lambda_t^{(2)}); 1, 0] + \mathcal{O}(\mathrm{d}\lambda_t^{(1)2})$$

$$\overset{\text{Eq. (2.75)}}{=} \frac{1}{2\pi} \mathcal{V}[H^{\mathrm{ad}}(x, p; \lambda_t^{(1)}, \lambda_t^{(2)}); \lambda_t^{(1)}, \lambda_t^{(2)}] + \mathcal{O}(\mathrm{d}\lambda_t^{(1)2})$$

$$\overset{\text{Eq. (2.55)}}{=} J(x, p; \lambda_t^{(1)}, \lambda_t^{(2)}) + \mathcal{O}(\mathrm{d}\lambda_t^{(1)2}).$$

2.3.1 Husimi's method

Let us consider a *usual* QPO described by the following Hamiltonian,

$$\hat{H}_t^{\mathrm{ad}} = \frac{\hat{p}^2}{2M} + \frac{M}{2}\omega_t^2 \hat{x}^2. \tag{2.82}$$

For this Hamiltonian, the x-representation of the wave function

$$\langle x|\Psi(t)\rangle = \int_{\mathbb{R}} \mathrm{d}x_0 U^{\mathrm{H}}_{t,t_0}(x|x_0)\langle x_0|\Psi(t_0)\rangle, \qquad t \in [t_0, \infty), \tag{2.83}$$

satisfies the Schröginger equation

$$i\hbar\frac{\partial}{\partial t}\langle x|\Psi(t)\rangle = \langle x|\hat{H}_t^{\mathrm{ad}}|\Psi(t)\rangle, \tag{2.84}$$

where $U^{\mathrm{H}}_{t,t_0}(x|x_0)$ is the propagator. Here, we assume the following Gaussian form of the propagator as the specific ansatz [55]:

$$U^{\mathrm{H}}_{t,t_0}(x|x_0) = \sqrt{\frac{M}{2\pi i\hbar\mu_t}}e^{i(\alpha_t x^2 + \beta_t x x_0 + \gamma_t x_0^2)/\hbar}, \tag{2.85}$$

where the coefficients μ_t, α_t, β_t, and γ_t are time-dependent real-valued functions. By substituting Eq. (2.85) into Eq. (2.84), we have

$$0 = \left[\left(\dot{\alpha}_t + \frac{2}{M}\alpha_t^2 + \frac{M}{2}\omega_t^2\right)x^2 + \left(\dot{\beta}_t + \frac{2}{M}\alpha_t\beta_t\right)x x_0 + \left(\dot{\gamma}_t + \frac{\beta_t^2}{2M}\right)x_0^2 + \frac{i\hbar}{2}\left(\frac{\dot{\mu}_t}{\mu_t} - \frac{2}{M}\alpha_t\right)\right]U^{\mathrm{H}}_{t,t_0}(x|x_0). \tag{2.86}$$

We then find that four coupled ordinary differential equations (ODEs) for the time-dependent coefficients α_t, β_t, γ_t, and μ_t:

$$\begin{cases} \dfrac{\dot{\mu}_t}{\mu_t} - \dfrac{2}{M}\alpha_t = 0, & (2.87) \\ \dot{\alpha}_t + \dfrac{2}{M}\alpha_t^2 + \dfrac{M}{2}\omega_t^2 = 0, & (2.88) \\ \dot{\beta}_t + \dfrac{2}{M}\alpha_t\beta_t = 0, & (2.89) \\ \dot{\gamma}_t + \dfrac{\beta_t^2}{2M} = 0. & (2.90) \end{cases}$$

From Eq. (2.87), we have

$$\alpha_t = \frac{M}{2}\frac{\dot{\mu}_t}{\mu_t}. \tag{2.91}$$

By substituting Eq. (2.91) into Eq. (2.88), we obtain the equation of motion (EoM) of a CPO as

$$\ddot{\mu}_t + \omega_t^2\mu_t = 0. \tag{2.92}$$

By substituting Eq. (2.91) into Eq. (2.89), we have

$$\dot{\beta}_t + \frac{\dot{\mu}_t}{\mu_t}\beta_t = 0, \qquad \therefore\ \beta_t = \frac{B_1}{\mu_t}, \tag{2.93}$$

where B_1 is an integral constant. By substituting Eq. (2.93) into Eq. (2.90), we have

$$\dot{\gamma}_t + \frac{B_1}{2M\mu_t^2} = 0, \qquad \therefore \gamma_t = B_2 - \frac{B_1}{2M}\int_{t_0}^{t}\frac{\mathrm{d}\tau}{\mu_\tau^2}, \tag{2.94}$$

where B_2 is an integral constant. On the other hand, it is known that the short-time asymptotic form of the propagator is

$$U^{\mathrm{H}}_{t,t_0}(x|x_0)|_{t\simeq t_0} \simeq \sqrt{\frac{M}{2\pi i\hbar(t-t_0)}}\exp\left[\frac{i}{\hbar}\frac{M}{2}\frac{(x-x_0)^2}{t-t_0} + O(t-t_0)\right], \tag{2.95}$$

which satisfies

$$U^{\mathrm{H}}_{t,t_0}(x|x_0)|_{t\simeq t_0} \xrightarrow{t\to t_0+0} \delta(x-x_0). \tag{2.96}$$

From Eqs. (2.85) and (2.95), we can deduce the initial condition imposed on μ_t:

$$\mu_t|_{t\simeq t_0} \simeq t - t_0 + O^2(t-t_0) \xrightarrow{t\to t_0+0} \mu_{t_0} = 0, \tag{2.97}$$

$$\dot{\mu}_t|_{t\simeq t_0} \simeq 1 + O(t-t_0) \xrightarrow{t\to t_0+0} \dot{\mu}_{t_0} = 1. \tag{2.98}$$

Hence, when we have the solution μ_t of Eq. (2.92) under the initial condition Eqs. (2.97) and (2.98), we obtain the solutions of Eqs. (2.87)–(2.90) in succession. As a result, we obtain $U^{\mathrm{H}}_{t,t_0}(x|x_0)$.

Let ν_t now be a second solution of Eq. (2.92) with the initial values

$$\nu_t \xrightarrow{t\to t_0+0} \nu_{t_0} = 1, \tag{2.99}$$

$$\dot{\nu}_t \xrightarrow{t\to t_0+0} \dot{\nu}_{t_0} = 0. \tag{2.100}$$

We consider the Wronskian $W_t := \dot{\mu}_t\nu_t - \mu_t\dot{\nu}_t$. From EoMs of μ_t and ν_t, we have $\dot{W}_t = 0$. With the two initial conditions [Eqs. (2.97)–(2.100)], we find

$$W_t = \dot{\mu}_t\nu_t - \mu_t\dot{\nu}_t = 1. \tag{2.101}$$

From this, we have

$$\frac{\nu_t}{\mu_t} = -\int_{t_0}^{t}\frac{\mathrm{d}\tau}{\mu_\tau^2}. \tag{2.102}$$

Here, by comparing Eq. (2.85) with Eq. (2.95) in relation to the coefficients of x, we have to set the integral constant of β_t in Eq. (2.93) as $B_1 = -M$. We then have

$$\beta_t = -\frac{M}{\mu_t}. \tag{2.103}$$

From Eq. (2.94), we have

$$\gamma_t = \frac{M}{2}\frac{\nu_t}{\mu_t} + B_2. \tag{2.104}$$

By comparing Eq. (2.85) with Eq. (2.95) in relation to the coefficients of x_0^2, we have to set the integral constant of γ_t in Eq. (2.104) as $B_2 = 0$, then we have

$$\gamma_t = \frac{M}{2}\frac{\nu_t}{\mu_t}. \tag{2.105}$$

Summarizing the above results, using the two solutions μ_t and ν_t of Eq. (2.92) that satisfy the initial conditions (2.97)–(2.100), we can write down the propagator of the QPO as

$$U^{\mathrm{H}}_{t,t_0}(x|x_0) = \sqrt{\frac{M}{2\pi i\hbar\mu_t}}\exp\left[\frac{iM}{2\hbar\mu_t}(\dot{\mu}_t x^2 - 2xx_0 + \nu_t x_0^2)\right]. \tag{2.106}$$

2.3.2 Husimi's measure of adiabaticity

In a usual QPO, we consider the transition probability $P^{(\mathrm{H})m,n}_{t,t_0}$ from an initial state $|n;\omega_{t_0}\rangle$ at initial time t_0 to a certain state $|m;\omega_t\rangle$ at time $t\in[t_0,t_\mathrm{f}]$ is

$$P^{(\mathrm{H})m,n}_{t,t_0} := \left|\iint_{\mathbb{R}^2}\mathrm{d}x\mathrm{d}x_0\langle m;\omega_t|x\rangle U^{\mathrm{H}}_{t,t_0}(x|x_0)\langle x_0|n;\omega_{t_0}\rangle\right|^2, \tag{2.107}$$

where $U^{\mathrm{H}}_{t,t_0}(x|x_0)$ is the propagator. By using the unitary operator [11]

$$\hat{U}^{\mathrm{H}}_{t,t_0} := \iint_{\mathbb{R}^2}\mathrm{d}x\mathrm{d}x_0|x\rangle U^{\mathrm{H}}_{t,t_0}(x|x_0)\langle x_0|, \tag{2.108}$$

we can also write $P^{(\mathrm{H})m,n}_{t,t_0} = |\langle m;\omega_t|\hat{U}^{\mathrm{H}}_{t,t_0}|n;\omega_{t_0}\rangle|^2$. The energy eigenfunction of the QPO [Eq. (2.82)] for position x is given as

$$\langle x|n;\omega_t\rangle = \frac{1}{\sqrt{2^n n!}}\left(\frac{M\omega_t}{\pi\hbar}\right)^{1/4}\mathrm{H}_n\left(\sqrt{\frac{M\omega_t}{\hbar}}x\right)\exp\left(-\frac{M\omega_t}{2\hbar}x^2\right), \tag{2.109}$$

where the n-th-degree Hermite polynomials $\mathrm{H}_n(\cdot)$ are defined as

$$\mathrm{H}_n(x) := (-1)^n e^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{-x^2}. \tag{2.110}$$

By using Mehler's formula:

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{z}{2}\right)^n\mathrm{H}_n(x)\mathrm{H}_n(y) = \frac{1}{\sqrt{1-z^2}}\exp\left[-\frac{z^2(x^2+y^2)-2zxy}{1-z^2}\right], \tag{2.111}$$

and the energy eigenfunction Eq. (2.109), we obtain

$$\sum_{n=0}^{\infty}z^n\langle n;\omega_t|x\rangle\langle y|n;\omega_t\rangle = \sqrt{\frac{m\omega_t}{\pi\hbar(1-z^2)}}\exp\left[-\frac{m\omega_t}{2\hbar}\frac{(1+z^2)(x^2+y^2)-4zxy}{1-z^2}\right]. \tag{2.112}$$

By using Eq. (2.112), we can calculate the transition-probability generating function (TPGF) as

$$\begin{aligned}\mathcal{P}^{(\mathrm{H})u,v}_{t,t_0} &:= \sum_{n,m=0}^{\infty}u^n v^m P^{(\mathrm{H})m,n}_{t,t_0}\\ &= \sum_{n,m=0}^{\infty}u^n v^m\left|\iint_{\mathbb{R}^2}\mathrm{d}x\mathrm{d}x_0\langle m;\omega_t|x\rangle U^{\mathrm{H}}_{t,t_0}(x|x_0)\langle x_0|n;\omega_{t_0}\rangle\right|^2\\ &= \iiiint_{\mathbb{R}^4}\mathrm{d}x\mathrm{d}x_0\mathrm{d}x'\mathrm{d}x_0' U^{\mathrm{H}*}_{t,t_0}(x|x_0)U^{\mathrm{H}}_{t,t_0}(x'|x_0')\end{aligned}$$

[11] Eq. (2.108) may be represented as an exponential operator, $\hat{U}^{\mathrm{H}}_{t,t_0} = \overleftarrow{\mathcal{T}}\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}\mathrm{d}\tau\hat{H}^{\mathrm{ad}}_{\tau}\right)$, where $\overleftarrow{\mathcal{T}}$ denotes a time ordered product.

$$\times \sum_{m=0}^{\infty} v^m \langle m; \omega_t | x \rangle \langle x' | m; \omega_t \rangle \sum_{n=0}^{\infty} u^n \langle n; \omega_{t_0} | x_0 \rangle \langle x_0' | n; \omega_{t_0} \rangle$$
$$= \frac{2}{\mu_t} \left(\frac{M}{2\pi\hbar} \right)^2 \sqrt{\frac{\omega_t \omega_{t_0}}{(1-u^2)(1-v^2)}} \int_{\mathbb{R}^4} \mathrm{d}\vec{x} \exp\left(-\frac{M}{2\hbar} \vec{x} \cdot A \vec{x} \right)$$
$$= \frac{2}{\mu_t} \sqrt{\frac{\omega_t \omega_{t_0}}{(1-u^2)(1-v^2) \det A}}, \tag{2.113}$$

where we defined

$$\vec{x} := \begin{pmatrix} x \\ x_0 \\ x' \\ x_0' \end{pmatrix}, \quad A := \begin{pmatrix} \frac{1+v^2}{1-v^2}\omega_t + i\frac{\dot{\mu}_t}{\mu_t} & -\frac{i}{\mu_t} & -\frac{2v}{1-v^2}\omega_t & 0 \\ -\frac{i}{\mu_t} & \frac{1+u^2}{1-u^2}\omega_{t_0} + i\frac{\nu_t}{\mu_t} & 0 & -\frac{2u}{1-u^2}\omega_{t_0} \\ -\frac{2v}{1-v^2}\omega_t & 0 & \frac{1+v^2}{1-v^2}\omega_t - i\frac{\dot{\mu}_t}{\mu_t} & \frac{i}{\mu_t} \\ 0 & -\frac{2u}{1-u^2}\omega_{t_0} & \frac{i}{\mu_t} & \frac{1+u^2}{1-u^2}\omega_{t_0} - i\frac{\nu_t}{\mu_t} \end{pmatrix}, \tag{2.114}$$

and used the following formula of the Gaussian integral:

$$\int_{\mathbb{R}^n} \mathrm{d}\vec{x} e^{-a\vec{x}\cdot A\vec{x}} = \sqrt{\frac{(\pi/a)^n}{\det A}}, \tag{2.115}$$

that holds for $a > 0$, $\vec{x} \in \mathbb{R}^n$, and the n-by-n symmetric matrix A. By using the Wronskian given by Eq. (2.101), we obtain

$$\det A = \frac{1}{\mu_t^2} \frac{2\omega_t \omega_{t_0}}{(1-u^2)(1-v^2)} \left[Q_t^* (1-u^2)(1-v^2) + (1+u^2)(1+v^2) - 4uv \right]. \tag{2.116}$$

Therefore, we have

$$\mathcal{P}_{t,t_0}^{(\mathrm{H})u,v} = \sqrt{\frac{2}{Q_t^*(1-u^2)(1-v^2) + (1+u^2)(1+v^2) - 4uv}}. \tag{2.117}$$

We here introduced the Husimi's measure of adiabaticity Q_t^* as

$$Q_t^* := \omega_{t_0} \frac{E_t^{(\mu)}}{\omega_t} + \omega_{t_0}^{-1} \frac{E_t^{(\nu)}}{\omega_t}, \tag{2.118}$$

where

$$E_t^{(\mu)} := \frac{1}{2}(\dot{\mu}_t^2 + \omega_t^2 \mu_t^2), \tag{2.119}$$
$$E_t^{(\nu)} := \frac{1}{2}(\dot{\nu}_t^2 + \omega_t^2 \nu_t^2), \tag{2.120}$$

are the classical energies of μ_t and ν_t, respectively. The time-dependent variables, μ_t and ν_t, obey the EoMs for the CPO with linearly independent initial conditions:

$$\ddot{\mu}_t + \omega_t^2 \mu_t = 0, \qquad (\mu_{t_0}, \dot{\mu}_{t_0}) = (0, 1), \tag{2.121}$$
$$\ddot{\nu}_t + \omega_t^2 \nu_t = 0, \qquad (\nu_{t_0}, \dot{\nu}_{t_0}) = (1, 0). \tag{2.122}$$

Q_t^* is a linear combination of these two energies of the CPO divided by the common angular frequency, $\frac{E_t^{(\mu)}}{\omega_t}$ and $\frac{E_t^{(\nu)}}{\omega_t}$. During an adiabatic process with the slowly changing angular frequency $\dot{\omega}_t \simeq 0$, these quantities are conserved as the *adiabatic invariants* as $\frac{1}{2\omega_{t_0}} = J^{(\mu)}$ and $\frac{\omega_{t_0}}{2} = J^{(\nu)}$, respectively, which leads to $Q_t^* = 1$. They are equivalent to the areas of the ellipses enclosed by the trajectories of the CPO on the classical phase space with $E_{t_0}^{(\mu)} = \frac{1}{2}$ and $E_{t_0}^{(\nu)} = \frac{\omega_{t_0}^2}{2}$ determined from the initial conditions in Eqs. (2.121) and (2.122).

If $Q_t^* = 1$, the TPGF becomes $\mathcal{P}_{t,t_0}^{(\mathrm{H})u,v}|_{Q_t^*=1} = \frac{1}{1-uv} = \sum_{n,m=0}^{\infty} u^n v^m \delta_{m,n}$, which means $P_{t,t_0}^{(\mathrm{H})m,n} = \delta_{m,n}$. In this case, the time evolution is adiabatic (transitionless). This fact can also be confirmed as below. By taking the first derivative of Eq. (2.117) with respect to v and putting $v = 1$, we obtain

$$\sum_{n=0}^{\infty} u^n \sum_{m=0}^{\infty} m P_{t,t_0}^{(\mathrm{H})m,n} = \left.\frac{\partial \mathcal{P}_{t,t_0}^{(\mathrm{H})u,v}}{\partial v}\right|_{v=1} = \frac{Q_t^*(1+u) - (1-u)}{2(1-u)^2}. \tag{2.123}$$

By expanding the right-hand side of Eq. (2.123) with respect to u, we can show the following relation between Q_t^* and the mean quantum number $\langle m \rangle_{n,t}^{\mathrm{H}} := \sum_{m=0}^{\infty} m P_{t,t_0}^{(\mathrm{H})m,n}$ [55, 56]:

$$Q_t^* = \frac{\langle m \rangle_{n,t}^{\mathrm{H}} + \frac{1}{2}}{n + \frac{1}{2}}. \tag{2.124}$$

Therefore, we find $\langle m \rangle_{n,t}^{\mathrm{H}} = n$ when $Q_t^* = 1$, from which we conclude $P_{t,t_0}^{(\mathrm{H})m,n} = \delta_{m,n}$.

Chapter 3

Adiabaticity and Invariants in a Continuous-time Control

This chapter is mainly based on the contents from the published paper [57]. One exception is Sec. 3.5, which is based on the unpublished work [58].

3.1 Introductory remarks for chapter 3

In the previous chapter 2, we reviewed the quantum and classical adiabatic evolution, the shortcuts to adiabaticity (STA), and the quantum-classical correspondence in a quantum parametric oscillator (QPO) based on Husimi's method. In this chapter 3, we shall discuss the relation between STA and Husimi's method.

Suppose that we can vary external parameters of a system to control it. An adiabatic process is the dynamics of the system with the external parameters changing slowly enough compared to the intrinsic time scale of the system. An adiabatic invariant is a quantity that is conserved in the limit of infinitely slow change of the control parameter. Adiabatic invariants appear in both classical and quantum mechanics. A classical example of an adiabatic invariant is the volume $\mathcal{V}$ enclosed by a trajectory $\vec{\gamma}_t^{\mathrm{ph}}$ in a two-dimensional phase space Γ. A quantum analog of the adiabatic invariant is the principal quantum number, which labels different energy levels. Ideally, a quantum system exhibits no transition between energy levels during an adiabatic process, whereas in a realistic process carried out for a finite duration, the adiabatic invariant is not conserved and transition occurs in a quantum realm.

The quantum adiabatic theorem, which was discussed in Sec. 2.1.2, is summarized as follows. Suppose that the system obeys a Hamiltonian $\hat{H}_t^{\mathrm{ad}} = \hat{H}^{\mathrm{ad}}(\vec{\lambda}_t)$, which is a function of external time-dependent parameters $\vec{\lambda}_t$. The instantaneous eigenstate $|n; \vec{\lambda}_t\rangle$ satisfies $\hat{H}_t^{\mathrm{ad}}|n; \vec{\lambda}_t\rangle = E_{n,t}^{\mathrm{ad}}|n; \vec{\lambda}_t\rangle$. The quantum adiabatic theorem ensures that the solution of the time-dependent Schrödinger equation is approximated with the instantaneous eigenstate if the initial state is an instantaneous eigenstate and the parameters $\vec{\lambda}_t$ vary slowly enough. Under this adiabatic approximation, the solution $|\Psi(t)\rangle$ of the Schrödinger equation, $i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = \hat{H}_t^{\mathrm{ad}}|\Psi(t)\rangle$, is given by [Eq. (2.20)]

$$|\Psi(t)\rangle \simeq \sum_{n=0}^{\infty} C_n e^{i\xi_{nt}}|n; \vec{\lambda}_t\rangle, \tag{3.1}$$

with the time-independent coefficients C_n and the time-dependent phases $\xi_{n,t}$, which is given in Eq. (2.21).

We discussed the propagator of the QPO in Sec. 2.3. For the QPO, the propagator can be expressed in terms of the solutions of the corresponding classical parametric oscillator (CPO). From the propagator, one can define a transition-probability generating function (TPGF) for probabilities between arbitrary two states. Husimi found that the TPGF of the QPO is characterized by a parameter Q_t^*, which is called Husimi's measure of adiabaticity. The value of Q_t^* is unity if and only if no transitions occur between arbitrary instantaneous eigenstates. Besides, Q_t^* is a function of two adiabatic invariants of the CPO [Eq. (2.118)]. Each adiabatic invariant is defined in terms of each solution of the CPO [Eqs. (2.121) and (2.122)].

Among the various methods in STA, the transitionless tracking (TT) algorithm introduces a counter-diabatic Hamiltonian $\hat{H}_t^{\mathrm{cd}}$ Eqs. (2.41)–(2.43) for canceling the deviation from exact tracking along instantaneous eigenstates of the original adiabatic Hamiltonian $\hat{H}_t^{\mathrm{ad}}$ [15]. In this method, it is assumed that the system obeys the total Hamiltonian $\hat{H}_t^{\mathrm{TT}} = \hat{H}_t^{\mathrm{ad}} + \hat{H}_t^{\mathrm{cd}}$, which we call the TT Hamiltonian. Then, the adiabatic approximate solution Eq. (3.1) becomes an exact solution of the Schrödinger equation for $\hat{H}_t^{\mathrm{TT}}$. For a case in which the original Hamiltonian $\hat{H}_t^{\mathrm{ad}}$ is a QPO, the counter-diabatic Hamiltonian $\hat{H}_t^{\mathrm{cd}}$ has been calculated explicitly [50] (Eq. (2.52)).

On the other hand, Husimi showed that the TPGF of the QPO without the counter-diabatic term is a function of an adiabatic invariant. Then, it is natural to ask what type of parameter characterizes the TPGF of the QPO with the counter-diabatic term. For answering this question, it is necessary to calculate the TPGF of the QPO with the counter-diabatic term by applying Husimi's method.

Here, we characterize the QPO with the counter-diabatic term by using a TPGF with a new parameter. By introducing an instantaneous eigenstate of the TT Hamiltonian, we apply Husimi's method to the QPO with the counter-diabatic term to obtain the propagator expressed with independent solutions of the corresponding CPO (Sec. 3.2). By using this propagator, we obtain the TPGF with the time-dependent parameter [Eqs. (3.15) and (3.21)], from which the adiabatic process in an arbitrary short time achieved by the TT algorithm is easily characterized (Sec. 3.3). We obtain this parameter by solving the equations of the CPO by using the phase-amplitude method [59]. We illustrate this result by exhibiting some trajectories of the solutions of the CPO of a specific case, which visualize the effect of the counter-diabatic term of the QPO on the classical phase space (Sec. 3.4). We also introduce the extended Husimi's measure of adiabaticity to characterize the TT algorithm in terms of dynamical-invariant perspectives (Sec. 3.5).

3.2 Propagator of a QPO with the counter-diabatic Hamiltonian

For the convenience of later arguments, we shall rewrite the TT Hamiltonian of a QPO [Eq. (3.2)] and derive a propagator of the QPO with a counter-diabatic term concretely by applying Husimi's method.

3.2.1 Bosonic operator and instantaneous eigenstates for the TT Hamiltonian

Let $\omega_t, M, \hat{x}$, and $\hat{p}$ be, the angular frequency at time t, mass, position operator, and momentum operator, respectively, where $\hat{x}$ and $\hat{p}$ satisfy the canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$. For the QPO, the TT Hamiltonian $\hat{H}_t^{\mathrm{TT}}$ is given by [50]

$$\hat{H}_t^{\mathrm{TT}} = \underbrace{\frac{\hat{p}^2}{2M} + \frac{M}{2}\omega_t^2\hat{x}^2}_{\hat{H}_t^{\mathrm{ad}}} \underbrace{-\frac{1}{2}\frac{\dot{\omega}_t}{\omega_t}\frac{\hat{x}\hat{p}+\hat{p}\hat{x}}{2}}_{\hat{H}_t^{\mathrm{cd}}}, \tag{3.2}$$

where $\hat{H}_t^{\mathrm{ad}}$ and $\hat{H}_t^{\mathrm{cd}}$ denote the adiabatic Hamiltonian and the counter-diabatic one, respectively. We denote the time derivative by a dot.

We rewrite $\hat{H}_t^{\mathrm{TT}}$ of the QPO in Eq. (3.2) with the instantaneous (Bosonic) ladder operator $\hat{b}_t$ as

$$\hat{H}_t^{\mathrm{TT}} = \hbar\Omega_t\left(\hat{b}_t^\dagger\hat{b}_t + \frac{1}{2}\right), \tag{3.3}$$

where

$$\Omega_t := \sqrt{\omega_t^2 - \left(\frac{1}{2}\frac{\dot{\omega}_t}{\omega_t}\right)^2}, \tag{3.4}$$

$$\hat{b}_t := \sqrt{\frac{M\Omega_t}{2\hbar}}\left(\zeta_t \hat{x} + \frac{i\hat{p}}{M\Omega_t}\right), \quad \text{with} \quad \zeta_t := 1 + \frac{1}{2i\Omega_t}\frac{\dot{\omega}_t}{\omega_t}. \tag{3.5}$$

Since $\hat{b}_t$ satisfies the Bosonic commutation relation $[\hat{b}_t, \hat{b}_t^\dagger] = 1$, $\hat{H}_t^{\mathrm{TT}}$ can be regarded as the Hamiltonian of a certain type of harmonic oscillator with the energy-level interval $\hbar\Omega_t$. We assume $\Omega_t > 0$ to avoid trap inversion. We adopt the Schrödinger picture to interpret these operators at time t. It is to be noted that the Bosonic operators, $\hat{b}_t$ and $\hat{b}_t^\dagger$, defined at different times do not have simple commutation relations.

Let $|n;\Omega_t\rangle$ be an instantaneous n-th excited energy eigenstate of $\hat{H}_t^{\mathrm{TT}}$ in Eq. (3.3) that satisfies $\sqrt{n!}|n;\Omega_t\rangle = \hat{b}_t^{\dagger n}|0;\Omega_t\rangle$ and $\hat{b}_t^\dagger \hat{b}_t|n;\Omega_t\rangle = n|n;\Omega_t\rangle$, where the vacuum state $|0;\Omega_t\rangle$ is defined as $\hat{b}_t|0;\Omega_t\rangle = 0$. The instantaneous n-th excited energy eigenfunction in the position representation is given as (see Appendix A.1)

$$\langle x|n;\Omega_t\rangle = \frac{1}{\sqrt{2^n n!}}\left(\frac{M\Omega_t}{\pi\hbar}\right)^{1/4} \mathrm{H}_n\left(\sqrt{\frac{M\Omega_t}{\hbar}}x\right)\exp\left(-\frac{\zeta_t M\Omega_t}{2\hbar}x^2\right), \tag{3.6}$$

where $\mathrm{H}_n(\cdot)$ are the n-th-degree Hermite polynomials [Eq. (2.110)].

3.2.2 Propagator based on Husimi's method

We calculate the probability for the transition from an initial state $|n;\Omega_{t_0}\rangle$ at initial time t_0 to a certain state $|m;\Omega_t\rangle$ at time $t \in [t_0, t_{\mathrm{f}}]$

$$P_{t,t_0}^{m,n} := \left|\iint_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}x_0 \langle m;\Omega_t|x\rangle U_{t,t_0}^{\mathrm{TT}}(x|x_0)\langle x_0|n;\Omega_{t_0}\rangle\right|^2, \tag{3.7}$$

where $U_{t,t_0}^{\mathrm{TT}}(x|x_0)$ is the propagator associated to the Hamiltonian $\hat{H}_t^{\mathrm{TT}}$ in Eq. (3.3). If we define the unitary time-evolution operator

$$\hat{U}_{t,t_0}^{\mathrm{TT}} := \iint_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}x_0 |x\rangle U_{t,t_0}^{\mathrm{TT}}(x|x_0)\langle x_0|, \tag{3.8}$$

we can also write $P_{t,t_0}^{m,n} = |\langle m;\Omega_t|\hat{U}_{t,t_0}^{\mathrm{TT}}|n;\Omega_{t_0}\rangle|^2$. The TT algorithm usually imposes the boundary condition $\dot{\omega}_{t_0} = \dot{\omega}_{t_{\mathrm{f}}} = 0$ ($\hat{H}_{t_0}^{\mathrm{cd}} = \hat{H}_{t_{\mathrm{f}}}^{\mathrm{cd}} = 0$) at the initial and final times $t = t_0$ and t_{f}, respectively, such that the instantaneous eigenstates of the original Hamiltonian, $\hat{H}_t^{\mathrm{ad}}$, and the TT Hamiltonian, $\hat{H}_t^{\mathrm{TT}}$, coincide with these times. However, we first consider the transition probability of Eq. (3.7) without a boundary condition on ω_t and later impose this boundary condition.

By applying Husimi's method [55], we obtain the concrete propagator as (see Appendix A.2 for derivation and also see Appendix B for more generalized cases)

$$U_{t,t_0}^{\mathrm{TT}}(x|x_0) = \sqrt{\frac{M}{2\pi i\hbar\mu_t}}\exp\left[\frac{iM}{2\hbar}\left\{\left(\frac{\dot{\mu}_t}{\mu_t} + \frac{1}{2}\frac{\dot{\omega}_t}{\omega_t}\right)x^2 - \frac{2xx_0}{\mu_t} + \left(\frac{\nu_t}{\mu_t} - \frac{1}{2}\frac{\dot{\omega}_{t_0}}{\omega_{t_0}}\right)x_0^2\right\}\right], \tag{3.9}$$

where μ_t and ν_t are the solutions of the equations of motion (EoMs) of the CPO with different initial conditions at $t = t_0$:

$$\ddot{\mu}_t + \tilde{\Omega}_t^2\mu_t = 0, \qquad (\mu_{t_0}, \dot{\mu}_{t_0}) = (0, 1), \tag{3.10}$$

$$\ddot{\nu}_t + \tilde{\Omega}_t^2\nu_t = 0, \qquad (\nu_{t_0}, \dot{\nu}_{t_0}) = (1, 0), \tag{3.11}$$

where

$$\tilde{\Omega}_t := \sqrt{\Omega_t^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{\omega}_t}{\omega_t}} = \sqrt{\omega_t^2 - \frac{3}{4}\frac{\dot{\omega}_t^2}{\omega_t^2} + \frac{1}{2}\frac{\ddot{\omega}_t}{\omega_t}}. \tag{3.12}$$

Here it should be noted that the angular frequency $\tilde{\Omega}_t$ is not equal to ω_t of the original oscillator [Eq. (2.82)] nor to Ω_t of the transitionless-tracked oscillator [Eq. (3.3)], in general. We can confirm that the solutions μ_t and ν_t satisfying Eqs. (3.10) and (3.11) are linearly independent by verifying that the Wronskian W_t is a non-zero constant at an arbitrary time t:

$$W_t := \dot{\mu}_t \nu_t - \mu_t \dot{\nu}_t = 1. \tag{3.13}$$

Note that μ_t has the dimension of time, whereas ν_t is dimensionless.

3.3 Measure of adiabaticity characterizing the transitionless tracking algorithm

In this section 3.3, we derive a measure of adiabaticity of the QPO with the counter-diabatic term by applying Husimi's method.

3.3.1 Transition-probability generating function

Although the concrete expression for the transition probabilities in Eq. (3.7) is complicated (Eqs. (A.58) and (A.59)), its generating function,

$$\mathcal{P}_{t,t_0}^{u,v} := \sum_{n,m=0}^{\infty} u^n v^m P_{t,t_0}^{m,n}, \tag{3.14}$$

becomes a rather simple expression (see Appendix A.3 for detailed calculation):

$$\mathcal{P}_{t,t_0}^{u,v} = \sqrt{\frac{2}{Q_t^{\mathrm{TT}}(1-u^2)(1-v^2) + (1+u^2)(1+v^2) - 4uv}}, \tag{3.15}$$

where Q_t^{TT} is a time-dependent parameter defined as

$$Q_t^{\mathrm{TT}} := \Omega_{t_0}\frac{\mathcal{E}_t^{(\mu)}}{\Omega_t} + \Omega_{t_0}^{-1}\frac{\mathcal{E}_t^{(\nu)}}{\Omega_t} + \frac{\dot{\omega}_t}{\omega_t}\frac{\Omega_{t_0}^2\dot{\mu}_t\mu_t + \dot{\nu}_t\nu_t + \frac{1}{2}\frac{\dot{\omega}_t}{\omega_t}(\Omega_{t_0}^2\mu_t^2 + \nu_t^2)}{\Omega_t\Omega_{t_0}}. \tag{3.16}$$

Here, we have introduced the two classical "energies" as (see just after Eq. (3.23) for more detail)

$$\mathcal{E}_t^{(\mu)} := \frac{1}{2}(\dot{\mu}_t^2 + \Omega_t^2\mu_t^2), \tag{3.17}$$

$$\mathcal{E}_t^{(\nu)} := \frac{1}{2}(\dot{\nu}_t^2 + \Omega_t^2\nu_t^2). \tag{3.18}$$

Note that the angular frequency Ω_t in the definition of the "energy" is different from the angular frequency $\tilde{\Omega}_t$ in the EoMs (3.10) and (3.11) for μ_t and ν_t in general. We can simplify this parameter Q_t^{TT} by imposing $\dot{\omega}_{t_0} = 0$ ($\hat{H}_{t_0}^{\mathrm{cd}} = 0$). In this case, the initial state is $|n;\Omega_{t_0}\rangle = |n;\omega_{t_0}\rangle$. Then, the solutions of μ_t and ν_t in Eqs. (3.10) and (3.11) are

$$\mu_t = \frac{1}{\sqrt{\omega_t \omega_{t_0}}} \sin\left(\int_{t_0}^{t} \omega_\tau \mathrm{d}\tau\right); \qquad \dot{\omega}_{t_0} = 0, \tag{3.19}$$

$$\nu_t = \sqrt{\frac{\omega_{t_0}}{\omega_t}} \cos\left(\int_{t_0}^{t} \omega_\tau \mathrm{d}\tau\right); \qquad \dot{\omega}_{t_0} = 0, \tag{3.20}$$

respectively; [12] these solutions will be derived in Sec. 3.3.3. Then, by substituting Eqs. (3.19) and (3.20) into Eq. (3.16), we find that the third term in Eq. (3.16) vanishes and that the former two terms are reduced to

$$Q_t^{\mathrm{TT}} = \frac{\omega_t}{\Omega_t}; \qquad \dot{\omega}_{t_0} = 0. \tag{3.21}$$

The key to obtaining this simple form is the explicit solutions Eqs. (3.19) and (3.20), despite the time dependence of ω_t. The TPGF in Eq. (3.15) with the simple time-dependent parameter Eq. (3.21) is the main result of the chapter 3.

As in the case of Eq. (2.124), we can show the following relation between Q_t^{TT} and the mean quantum number $\langle m\rangle_{n,t}^{\mathrm{TT}} := \sum_{m=0}^{\infty} m P_{t,t_0}^{m,n}$ [55, 56]:

$$Q_t^{\mathrm{TT}} = \frac{\langle m\rangle_{n,t}^{\mathrm{TT}} + \frac{1}{2}}{n + \frac{1}{2}}. \tag{3.22}$$

When $Q_t^{\mathrm{TT}} = 1$, we find $\langle m\rangle_{n,t}^{\mathrm{TT}} = n$ and can show $\mathcal{P}_{t,t_0}^{u,v}|_{Q_t^{\mathrm{TT}}=1} = \frac{1}{1-uv}$, from which $P_{t,t_0}^{m,n} = \delta_{m,n}$ (no transition) follows. Indeed, by letting $|n;\Omega_{t_\mathrm{f}}\rangle = |n;\omega_{t_\mathrm{f}}\rangle$ be the final state under imposing $\dot{\omega}_{t_\mathrm{f}} = 0$ ($\hat{H}_{t_\mathrm{f}}^{\mathrm{cd}} = 0$), we find that our new parameter Q_t^{TT} in Eq. (3.21) at $t = t_\mathrm{f}$ is unity. This implies $P_{t_\mathrm{f},t_0}^{m,n} = \delta_{m,n}$, and the transitionless tracking is achieved.

3.3.2 Comparison between Q_t^* and Q_t^{TT}

While Husimi calculated the TPGF in (2.117) for the oscillator without the counter-diabatic term, we calculated the one in Eq. (3.15) for the oscillator with the counter-diabatic term. The results in Eqs. (2.117) and (3.15) look almost identical. Only the difference is the expression of the parameter Q_t^{TT}; while Husimi's definition in Eq. (2.118), our definition is Eq. (3.16) or Eq. (3.21). The Husimi's parameter Q_t^*, Eq. (2.118), is a linear combination of the two energies of the CPO divided by the common angular frequency, $\frac{E_t^{(\mu)}}{\omega_t}$ and $\frac{E_t^{(\nu)}}{\omega_t}$. During an adiabatic process with the slowly changing angular frequency $\dot{\omega}_t \simeq 0$, these quantities are conserved as the *adiabatic invariants* as $J^{(\mu)} := \frac{1}{2\omega_{t_0}}$ and $J^{(\nu)} := \frac{\omega_{t_0}}{2}$, respectively. They are equivalent to the areas of the ellipses enclosed by the trajectories of the CPO on the classical phase space with $E_{t_0}^{(\mu)} = \frac{1}{2}$ and $E_{t_0}^{(\nu)} = \frac{\omega_{t_0}^2}{2}$ determined from

[12] Despite the time dependence of ω_t, the Heisenberg equation for $\hat{x}(t)$ and $\hat{p}(t)$ of this system can also be solved. The equation is given as

$$\begin{cases} \dfrac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t} = \dfrac{1}{i\hbar}[\hat{x}(t), \hat{H}_t^{\mathrm{TT}}] = \dfrac{\hat{p}(t)}{M} - \dfrac{\hat{x}(t)}{2}\dfrac{\dot{\omega}_t}{\omega_t}, \\ \dfrac{\mathrm{d}\hat{p}(t)}{\mathrm{d}t} = \dfrac{1}{i\hbar}[\hat{p}(t), \hat{H}_t^{\mathrm{TT}}] = \dfrac{\hat{p}(t)}{2}\dfrac{\dot{\omega}_t}{\omega_t} - M\omega_t^2 \hat{x}(t). \end{cases}$$

Based on the method developed in Ref. [60], we can obtain the solutions as

$$\hat{x}(t) = \sqrt{\frac{\omega_{t_0}}{\omega_t}}\left[\hat{x}(t_0)\cos\left(\int_{t_0}^{t}\omega_\tau \mathrm{d}\tau\right) + \frac{\hat{p}(t_0)}{M\omega_{t_0}}\sin\left(\int_{t_0}^{t}\omega_\tau \mathrm{d}\tau\right)\right],$$

$$\hat{p}(t) = \sqrt{\frac{\omega_t}{\omega_{t_0}}}\left[\hat{p}(t_0)\cos\left(\int_{t_0}^{t}\omega_\tau \mathrm{d}\tau\right) - M\omega_{t_0}\hat{x}(t_0)\sin\left(\int_{t_0}^{t}\omega_\tau \mathrm{d}\tau\right)\right].$$

the initial conditions in Eqs. (2.121) and (2.122). In contrast to Eqs. (3.10) and (3.11), which have the explicit solutions given by Eqs. (3.19) and (3.20), respectively, we may not obtain explicit solutions for Eqs. (2.121) and (2.122). Therefore, we can not obtain a simple form as in Eq. (3.21) for this usual QPO. However, $Q_t^* \simeq 1$ holds during the adiabatic process owing to the existence of these adiabatic invariants. This is an expression of the quantum adiabatic theorem as it implies $P_{t,t_0}^{m,n} = \delta_{m,n}$ for any t.

For a comparison with Husimi's measure of adiabaticity, it is convenient to express Q_t^{TT} in Eq. (3.21) as

$$Q_t^{\mathrm{TT}} = \omega_{t_0} \frac{\mathcal{E}_t^{(\mu)}}{\Omega_t} + \omega_{t_0}^{-1} \frac{\mathcal{E}_t^{(\nu)}}{\Omega_t}; \qquad \dot{\omega}_{t_0} = 0. \tag{3.23}$$

It should be noted that these "energies," $\mathcal{E}_t^{(\mu)}$ and $\mathcal{E}_t^{(\nu)}$, are defined with the angular frequency Ω_t that appeared in the QPO with the TT Hamiltonian in Eq. (3.4), but μ_t and ν_t are solutions of the equations of the CPO in Eqs. (3.10) and (3.11) with the angular frequency $\tilde{\Omega}_t$. We can rewrite $\mathcal{E}_t^{(\mu)}$ and $\mathcal{E}_t^{(\nu)}$ as (see Sec. 3.3.3)

$$\mathcal{E}_t^{(\mu)} = \frac{\omega_t}{2\omega_{t_0}} - \left(\dot{\mu}_t + \frac{\mu_t}{2}\frac{\dot{\omega}_t}{\omega_t}\right)\frac{\mu_t}{2}\frac{\dot{\omega}_t}{\omega_t}; \qquad \dot{\omega}_{t_0} = 0, \tag{3.24}$$

$$\mathcal{E}_t^{(\nu)} = \frac{\omega_{t_0}\omega_t}{2} - \left(\dot{\nu}_t + \frac{\nu_t}{2}\frac{\dot{\omega}_t}{\omega_t}\right)\frac{\nu_t}{2}\frac{\dot{\omega}_t}{\omega_t}; \qquad \dot{\omega}_{t_0} = 0, \tag{3.25}$$

respectively. Because $\mathcal{E}_{t_0}^{(\mu)} = \frac{1}{2}$ and $\mathcal{E}_{t_0}^{(\nu)} = \frac{\omega_{t_0}^2}{2}$ for $\dot{\omega}_{t_0} = \dot{\omega}_{t_\mathrm{f}} = 0$ readily follow from the initial conditions in Eqs. (3.10) and (3.11), by using Eqs. (3.24) and (3.25), we can also show that $\mathcal{E}_{t_\mathrm{f}}^{(\mu)} = \frac{\omega_{t_\mathrm{f}}}{2\omega_{t_0}}$ and $\mathcal{E}_{t_\mathrm{f}}^{(\nu)} = \frac{\omega_{t_0}\omega_{t_\mathrm{f}}}{2}$. Therefore, we obtain $\frac{\mathcal{E}_{t_0}^{(\mu)}}{\omega_{t_0}} = \frac{\mathcal{E}_{t_\mathrm{f}}^{(\mu)}}{\omega_{t_\mathrm{f}}} = J^{(\mu)}$ and $\frac{\mathcal{E}_{t_0}^{(\nu)}}{\omega_{t_0}} = \frac{\mathcal{E}_{t_\mathrm{f}}^{(\nu)}}{\omega_{t_\mathrm{f}}} = J^{(\nu)}$, i.e., at both the initial and final times, the values of these "energies" divided by the common angular frequency Ω_t agree with the adiabatic invariants $J^{(\mu)}$ and $J^{(\nu)}$. This explains the reason for $Q_{t_\mathrm{f}}^{\mathrm{TT}} = 1$ at the final time t_f in a manner comparable to Husimi's measure of adiabaticity. During the intermediate times, however, $Q_t^{\mathrm{TT}} = 1$ does not hold in general, because the exact solution Eq. (3.1) for the TT Hamiltonian may be *diabatic* (non-adiabatic) with respect to the instantaneous eigenstate of this Hamiltonian [50]. This behavior may be similar to that of the system with the fast-forward method [20] being applied, where the system is allowed to deviate from the original adiabatic path and returns to it only at the end of the process. We note that in our case with the TT algorithm the state vector itself always tracks the original adiabatic path given by Eq. (3.1).

3.3.3 Derivation of key equations

In this subsection, we derive the key equations Eqs. (3.19), (3.20), (3.24), and (3.25).

We first derive Eqs. (3.19) and (3.20) as the solutions of Eqs. (3.10) and (3.11), respectively, by using the phase-amplitude method [59]. We define the time-dependent function ρ_t in terms of the two linearly independent solutions μ_t and ν_t satisfying Eqs. (3.10) and (3.11), respectively, as

$$\rho_t := \sqrt{\frac{\Omega_{t_0}^2 \mu_t^2 + \nu_t^2}{\Omega_{t_0}}}. \tag{3.26}$$

We can then rewrite the Wronskian in Eq. (3.13) by eliminating either μ_t or ν_t as

$$W_t = \begin{cases} -\dfrac{\rho_t}{\sqrt{\Omega_{t_0}^{-1}\rho_t^2 - \mu_t^2}}(\dot{\rho}_t\mu_t - \rho_t\dot{\mu}_t) =: W_t^{(\mu)}, & (3.27) \\ \dfrac{\rho_t}{\sqrt{\Omega_{t_0}\rho_t^2 - \nu_t^2}}(\dot{\rho}_t\nu_t - \rho_t\dot{\nu}_t) =: W_t^{(\nu)}, & (3.28) \end{cases}$$

where we note that $W_t = W_t^{(\mu)} = W_t^{(\nu)} = 1$ holds for an arbitrary time t. The time-evolution equation of ρ_t in Eq. (3.26) is obtained by differentiating the Wronskian of Eqs. (3.27) and (3.28) with respect to time t and by using Eqs. (3.10), (3.11), and (3.13) (see Appendix A.4):

$$\ddot{\rho}_t + \tilde{\Omega}_t^2\rho_t = \frac{W_t^2}{\rho_t^3}, \tag{3.29}$$

which is called the Ermakov equation [61–63]. On the other hand, by integrating Eqs. (3.27) and (3.28), we obtain

$$\mu_t = \frac{\rho_t}{\sqrt{\Omega_{t_0}}}\sin\theta_t, \tag{3.30}$$

$$\nu_t = \sqrt{\Omega_{t_0}}\rho_t\cos\theta_t, \tag{3.31}$$

where

$$\theta_t := \int_{t_0}^{t}\frac{W_\tau^{(\mu)}}{\rho_\tau^2}\mathrm{d}\tau = \int_{t_0}^{t}\frac{W_\tau^{(\nu)}}{\rho_\tau^2}\mathrm{d}\tau \tag{3.32}$$

is a phase function (see Appendix A.5). This description of the coordinate variables μ_t and ν_t in terms of ρ_t and θ_t is called the phase-amplitude method [59]. The Wronskian W_t is then given by $W_t = \rho_t^2\dot{\theta}_t$ by differentiating θ_t with respect to time t. From this Wronskian represented by ρ_t and θ_t, we can derive a general expression of the Ermakov equation based on the phase-amplitude method as (see Appendix A.5)

$$\ddot{\rho}_t + f_t^2\rho_t = \frac{W_t^2}{\rho_t^3}, \tag{3.33}$$

where

$$f_t := \sqrt{\dot{\theta}_t^2 - \frac{3}{4}\frac{\ddot{\theta}_t^2}{\dot{\theta}_t^2} + \frac{1}{2}\frac{\dddot{\theta}_t}{\dot{\theta}_t}}. \tag{3.34}$$

Since ρ_t obeys Eq. (3.29), by comparing the two expressions, we find $\tilde{\Omega}_t = f_t$ from Eq. (3.12). If $\dot{\omega}_{t_0} = 0$, we can self-evidently identify the time differential of the phase function θ_t with the angular frequency ω_t as

$$\dot{\theta}_t = \omega_t; \qquad \dot{\omega}_{t_0} = 0. \tag{3.35}$$

The Wronskian W_t is rewritten by

$$W_t = \rho_t^2\omega_t; \qquad \dot{\omega}_{t_0} = 0. \tag{3.36}$$

Noting $W_t = 1$, we obtain the explicit solution of the Ermakov equation [Eq. (3.29)] as

$$\rho_t = \frac{1}{\sqrt{\omega_t}}; \qquad \dot{\omega}_{t_0} = 0. \tag{3.37}$$

Here, the condition of $\dot{\omega}_{t_0} = 0$ in Eqs. (3.35)–(3.37) is necessary for the following reason. From the definition of ρ_t in Eq. (3.26), we find $\rho_{t_0} = \sqrt{\frac{\Omega_{t_0}^2\mu_{t_0}^2+\nu_{t_0}^2}{\Omega_{t_0}}} = \frac{1}{\sqrt{\Omega_{t_0}}}$ by using Eqs. (3.10) and (3.11). For this expression to be consistent with $\frac{1}{\sqrt{\omega_{t_0}}}$, we must require $\dot{\omega}_{t_0} = 0$. By substituting Eq. (3.37) into Eqs. (3.30) and (3.31), we obtain Eqs. (3.19) and (3.20), respectively.

We next derive Eqs. (3.24) and (3.25). We define the Ermakov-Lewis (EL) invariant [21, 64] as

$$I_t^{\mathrm{EL}} := \begin{cases} \dfrac{\Omega_{t_0}}{2}\left[(\dot{\rho}_t\mu_t - \rho_t\dot{\mu}_t)^2 + W_t^{(\mu)2}\left(\dfrac{\mu_t}{\rho_t}\right)^2\right] =: I_t^{\mathrm{EL}(\mu)}, & (3.38)\\ \dfrac{1}{2\Omega_{t_0}}\left[(\dot{\rho}_t\nu_t - \rho_t\dot{\nu}_t)^2 + W_t^{(\nu)2}\left(\dfrac{\nu_t}{\rho_t}\right)^2\right] =: I_t^{\mathrm{EL}(\nu)}. & (3.39)\end{cases}$$

It can be shown [65] that the Ermakov-Lewis invariant can be related to the Wronskian, Eqs. (3.27) and (3.28), as

$$I_t^{\mathrm{EL}} = I_t^{\mathrm{EL}(\mu)} = I_t^{\mathrm{EL}(\nu)} = \frac{W_t^2}{2}. \tag{3.40}$$

Then, from Eqs. (3.36) and (3.38)–(3.40), we rewrite the Wronskian W_t in terms of the coordinate variables μ_t and ν_t and angular frequency ω_t as (see Appendix A.6)

$$W_t^{(\mu)} = \frac{2\omega_{t_0}}{\omega_t}\left[\mathcal{E}_t^{(\mu)} + \left(\dot{\mu}_t + \frac{\mu_t}{2}\frac{\dot{\omega}_t}{\omega_t}\right)\frac{\mu_t}{2}\frac{\dot{\omega}_t}{\omega_t}\right]; \qquad \dot{\omega}_{t_0} = 0, \tag{3.41}$$

$$W_t^{(\nu)} = \frac{2}{\omega_t\omega_{t_0}}\left[\mathcal{E}_t^{(\nu)} + \left(\dot{\nu}_t + \frac{\nu_t}{2}\frac{\dot{\omega}_t}{\omega_t}\right)\frac{\nu_t}{2}\frac{\dot{\omega}_t}{\omega_t}\right]; \qquad \dot{\omega}_{t_0} = 0. \tag{3.42}$$

By noting $W_t = W_t^{(\mu)} = W_t^{(\nu)} = 1$, we obtain Eqs. (3.24) and (3.25) from Eqs. (3.41) and (3.42), respectively.

3.4 Example: cubic-function angular frequency

Here we illustrate the effect of the counter-diabatic term on the trajectory in the classical phase space. We consider a specific case where the angular frequency of the QPO is a cubic function of time $t \in [t_0, t_{\mathrm{f}}]$,

$$\omega_t = \omega_0 + (\omega_{\mathrm{f}} - \omega_0)\left[1 + 2\frac{(t_{\mathrm{f}} - t_0)(t_{\mathrm{f}} - t)}{t_0^2 + t_{\mathrm{f}}^2}\right]\left(\frac{t - t_0}{t_{\mathrm{f}} - t_0}\right)^2, \tag{3.43}$$

which satisfies $\omega_{t_0} = \omega_0$, $\omega_{t_{\mathrm{f}}} = \omega_{\mathrm{f}}$, and $\dot{\omega}_{t_0} = \dot{\omega}_{t_{\mathrm{f}}} = 0$. Here, we consider three cases with different final times $t_{\mathrm{f}} = 0.2, 0.5, 2.0$ and set $t_0 = 0$ and $(\omega_0, \omega_{\mathrm{f}}) = (2, 4)$ (see Fig. 3.1). In Fig. 3.2, we show some phase-space trajectories of the CPO, which are the solutions of Eqs. (3.10) and (3.11). For every final time t_{f}, we can find that the final points of the trajectories are always on the same targeted energy shells $\mathcal{E}_{t_{\mathrm{f}}}^{(\mu)} = \frac{\omega_{t_{\mathrm{f}}}}{2\omega_{t_0}} = \frac{\omega_{\mathrm{f}}}{2\omega_0}$ and $\mathcal{E}_{t_{\mathrm{f}}}^{(\nu)} = \frac{\omega_{t_0}\omega_{t_{\mathrm{f}}}}{2} = \frac{\omega_0\omega_{\mathrm{f}}}{2}$ with the aid of the counter-adiabatic term, implying the success of the TT algorithm. On the other hand, $E_{t_{\mathrm{f}}}^{(\mu)}$ and $E_{t_{\mathrm{f}}}^{(\nu)}$ at the final points of the trajectories given by Eqs. (2.121) and (2.122), respectively, vary depending on t_{f}, failing to achieve the same final state as that of the adiabatic processes unless a sufficiently large t_{f} is taken.

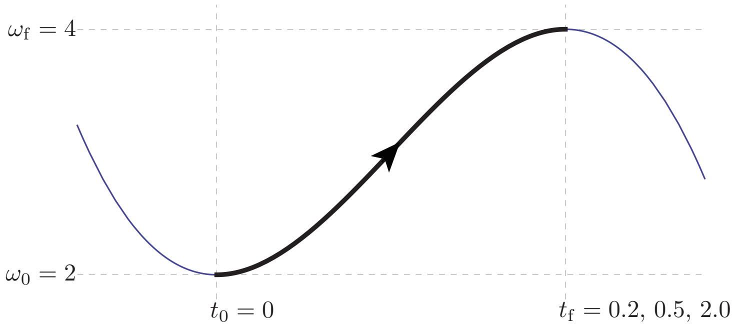

FIGURE 3.1: Time evolution of the angular frequency Eq. (3.43) given as the cubic function.

In Fig. 3.3, we show the time dependence of the parameters $Q_t^{\rm TT}$ and Q_t^* together with the quantities $\frac{\mathcal{E}_t^{(X)}}{\Omega_t}$ and $\frac{E_t^{(X)}}{\omega_t}$ ($X := \mu$ or ν) in the insets, for the trajectories in Fig. 3.2. We can find that the equality $\frac{\mathcal{E}_t^{(X)}}{\Omega_t} = J^{(X)}$ holds at the final points at $t = t_{\rm f}$ as expected. That is, $Q_t^{\rm TT}$ is unity at every final time $t_{\rm f}$ we chose, but it is not so for the intermediate times . Without the counter-diabatic Hamiltonian $\hat{H}_t^{\rm cd}$, the areas enclosed by the trajectories were well defined. In Fig. 3.3, $\frac{E_t^{(X)}}{\omega_t} = J^{(X)}$ holds for an arbitrary time t only if a sufficiently large $t_{\rm f}$ is taken.

In Fig. 3.4, we show two transition probabilities $P_{t,t_0}^{0,0}$ and $P_{t,t_0}^{1,1}$ obtained using the time evolution of $Q_t^{\rm TT}$ and Q_t^* in Fig. 3.3. By a selection rule [55], only transitions between even or odd quantum-number states are allowed (see Appendix A.7).

We note that, in the fastest case of $t_{\rm f} = 0.2$ in Fig. 3.3, Ω_t temporarily attains an imaginary value after $Q_t^{\rm TT}$ shows diverging behavior as $\Omega_t \to 0$. Because $\hat{H}_{\rm TT}$ in Eq. (3.3) has a continuous energy spectrum in this time interval, the transition probabilities in Eq. (3.7) using the discrete energy spectrum cannot be defined there (the gray-shaded regions in the insets of Fig. 3.4 represent these time intervals). As the time evolution approaches the final state, however, Ω_t becomes a real number again.

3.5 Measure of adiabaticity from dynamical-invariant perspectives

3.5.1 Canonical transformation of the CPO

In Sec. 3.2.2, we defined the transition probability Eq. (3.7) between the instantaneous eigenstates of the TT Hamiltonian and calculated its TPGF (3.15). Here we introduce the transition probability between the instantaneous eigenstates of the adiabatic Hamiltonian,

$$\bar{P}_{t,t_0}^{m,n} := \left| \iint_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}x_0 \langle m; \omega_t | x \rangle U_{t,t_0}^{\rm TT}(x|x_0) \langle x_0 | n; \omega_{t_0} \rangle \right|^2, \tag{3.44}$$

where $U_{t,t_0}^{\rm TT}(x|x_0)$ was given in Eq. (3.9). We can also write $\bar{P}_{t,t_0}^{m,n} = |\langle m; \omega_t | \hat{U}_{t,t_0}^{\rm TT} | n; \omega_{t_0} \rangle|^2$. We let X_t be a solution of the EoM

$$\ddot{X}_t + \tilde{\Omega}_t^2 X_t = 0, \qquad X_t = \mu_t \text{ or } \nu_t, \tag{3.45}$$

with different initial conditions of $(\mu_{t_0}, \dot{\mu}_{t_0}) = (0, 1)$ and $(\nu_{t_0}, \dot{\nu}_{t_0}) = (1, 0)$. We regard the variables X_t and $\dot{X}_t$ as a pair of *classical* canonical variables, that is, X_t and $P_t^{(X)} := \dot{X}_t$, respectively. The

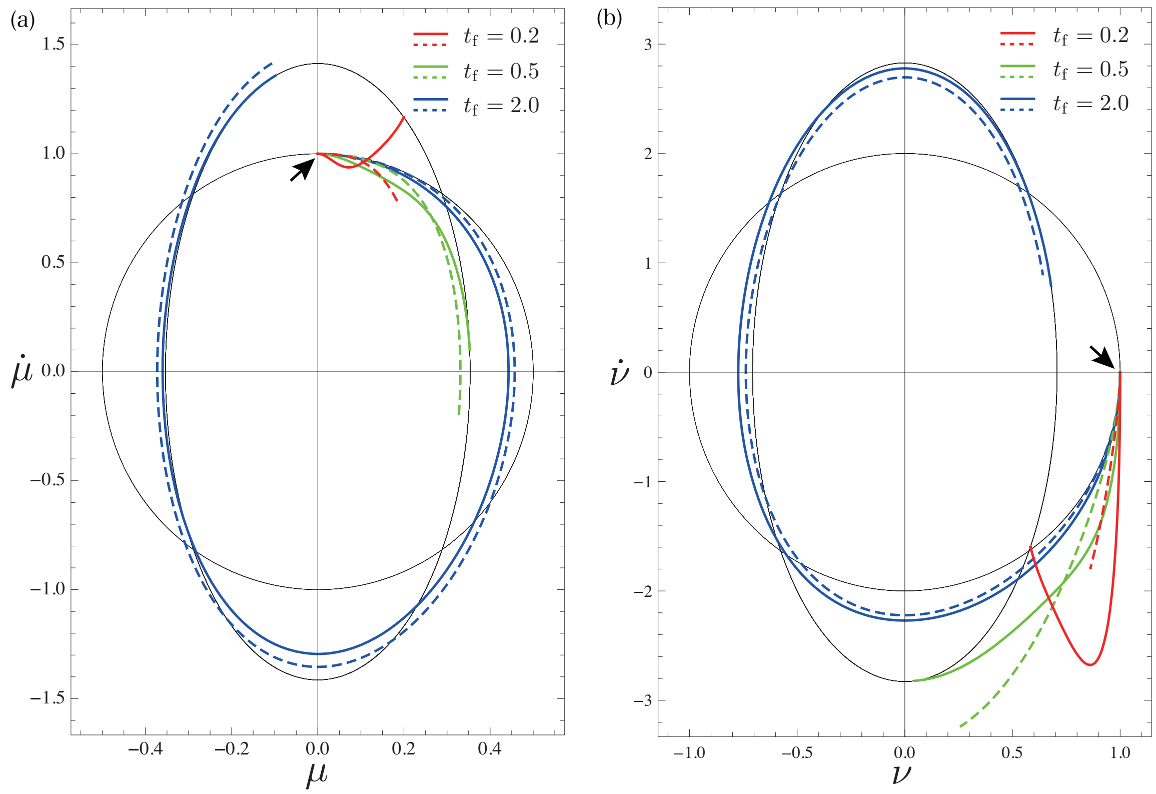

FIGURE 3.2: The trajectories of the classical parametric oscillator (CPO) in the phase space. (a) The colored solid curves are solutions μ_t of Eq. (3.10). The colored dashed curves are solutions μ_t of Eq. (2.121). The final time t_f is varied. The monochromatic ellipses are the energy shells at the initial and the final times. (b) The colored solid curves are solutions ν_t of Eq. (3.11). The colored dashed curves are solutions ν_t of Eq. (2.122). The arrows point the initial points $(0, 1)$ for (a) and $(1, 0)$ for (b). The initial conditions are chosen to have $\mathcal{E}^{(\mu)}_{t_0} = \frac{1}{2}$ for (a) and $\mathcal{E}^{(\nu)}_{t_0} = \frac{\omega_0^2}{2} = 2$ for (b). The final boundary conditions are chosen to have $\mathcal{E}^{(\mu)}_{t_f} = \frac{\omega_f}{2\omega_0} = 1$ for (a) and $\mathcal{E}^{(\nu)}_{t_f} = \frac{\omega_0\omega_f}{2} = 4$ for (b).

Hamiltonian for Eq. (3.45) is given as

$$H^{(X)}_t \equiv H^{(X)}(X_t, P^{(X)}_t) := \frac{1}{2}(P^{(X)2}_t + \tilde{\Omega}^2_t X^2_t). \tag{3.46}$$

We then introduce the generating function of a type II canonical transformation between $(X_t, P^{(X)}_t)$ and $(x^{(X)}_t, p^{(X)}_t)$ [23, 66] as

$$F(X_t, p^{(X)}_t, t) := p^{(X)}_t X_t - \frac{X_t^2}{4}\frac{\dot{\omega}_t}{\omega_t}. \tag{3.47}$$

By using Eq. (3.47), we find that these canonical variables are related as

$$\begin{cases} P^{(X)}_t = \dfrac{\partial F(X_t, p^{(X)}_t, t)}{\partial X_t} = p^{(X)}_t - \dfrac{X_t}{2}\dfrac{\dot{\omega}_t}{\omega_t}, & (3.48) \\ x^{(X)}_t = \dfrac{\partial F(X_t, p^{(X)}_t, t)}{\partial p^{(X)}_t} = X_t. & (3.49) \end{cases}$$

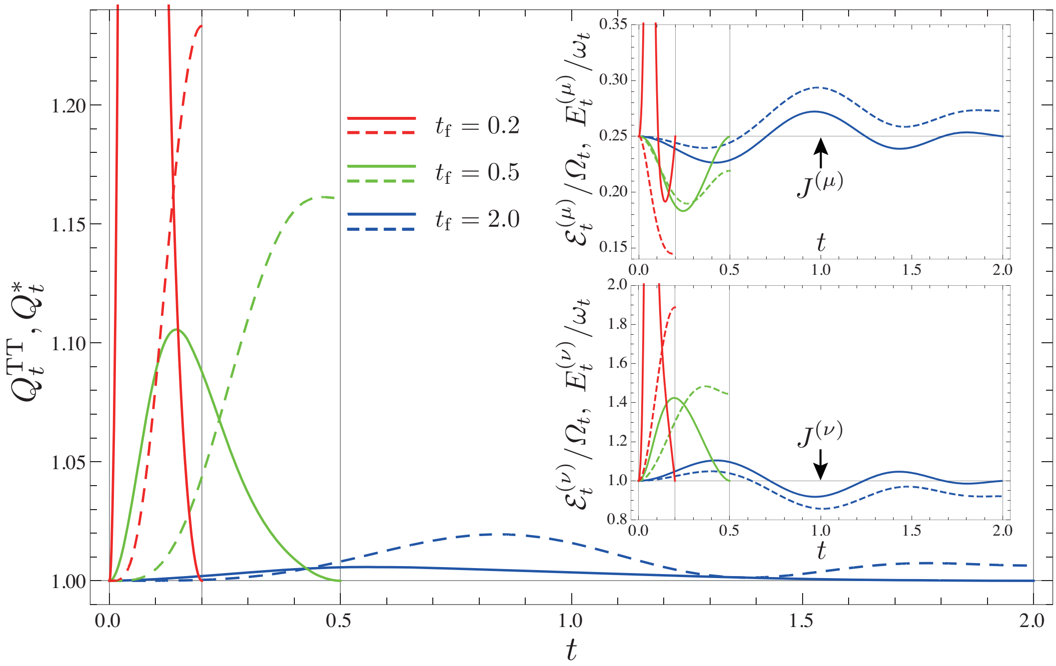

FIGURE 3.3: The adiabaticity measures as a function of the time t. If the value of these measures at the final time is unity, the complete transitionless control is attained. Q_t^* (dashed curve) is the Husimi's parameter for the quantum parametric oscillator without the counter-diabatic control. $Q_t^{\rm TT}$ (solid curve) is defined for the quantum parametric oscillator with the counter-diabatic control. The insets show comparison of the quantities $\frac{\mathcal{E}_t^{(X)}}{\Omega_t}$ and $\frac{E_t^{(X)}}{\omega_t}$ with the ideal invariants $J^{(X)}$.

By using Eqs. (3.46)–(3.49), we can define the Hamiltonian for the canonical variables $(x_t^{(X)}, p_t^{(X)})$ as

$$H^{\rm TT(X)}(x_t^{(X)}, p_t^{(X)}) := H^{(X)}(X_t, P_t^{(X)}) + \frac{\partial F(X_t, p_t^{(X)}, t)}{\partial t}, \tag{3.50}$$

which turns out to be

$$H_t^{\rm TT(X)} = H_t^{\rm ad(X)} + H_t^{\rm cd(X)}, \tag{3.51}$$

where

$$H_t^{\rm ad(X)} := \frac{1}{2}(p_t^{(X)2} + \omega_t^2 x_t^{(X)2}), \tag{3.52}$$

$$H_t^{\rm cd(X)} := -\frac{1}{2}\frac{\dot{\omega}_t}{\omega_t} x_t^{(X)} p_t^{(X)}. \tag{3.53}$$

The Hamiltonian $H_t^{\rm cd(X)}$ in Eq. (3.53) is the same as the counter-dissipative Hamiltonian in Eq. (2.81) [23]. The Hamilton equation of Eq. (3.51) is given as

$$\begin{cases} \dot{x}_t^{(X)} = \dfrac{\partial H_t^{\rm TT(X)}}{\partial p_t^{(X)}} = p_t^{(X)} - \dfrac{x_t^{(X)}}{2}\dfrac{\dot{\omega}_t}{\omega_t}, & (3.54) \\ \dot{p}_t^{(X)} = -\dfrac{\partial H_t^{\rm TT(X)}}{\partial x_t^{(X)}} = \dfrac{p_t^{(X)}}{2}\dfrac{\dot{\omega}_t}{\omega_t} - \omega_t^2 x_t^{(X)}. & (3.55) \end{cases}$$

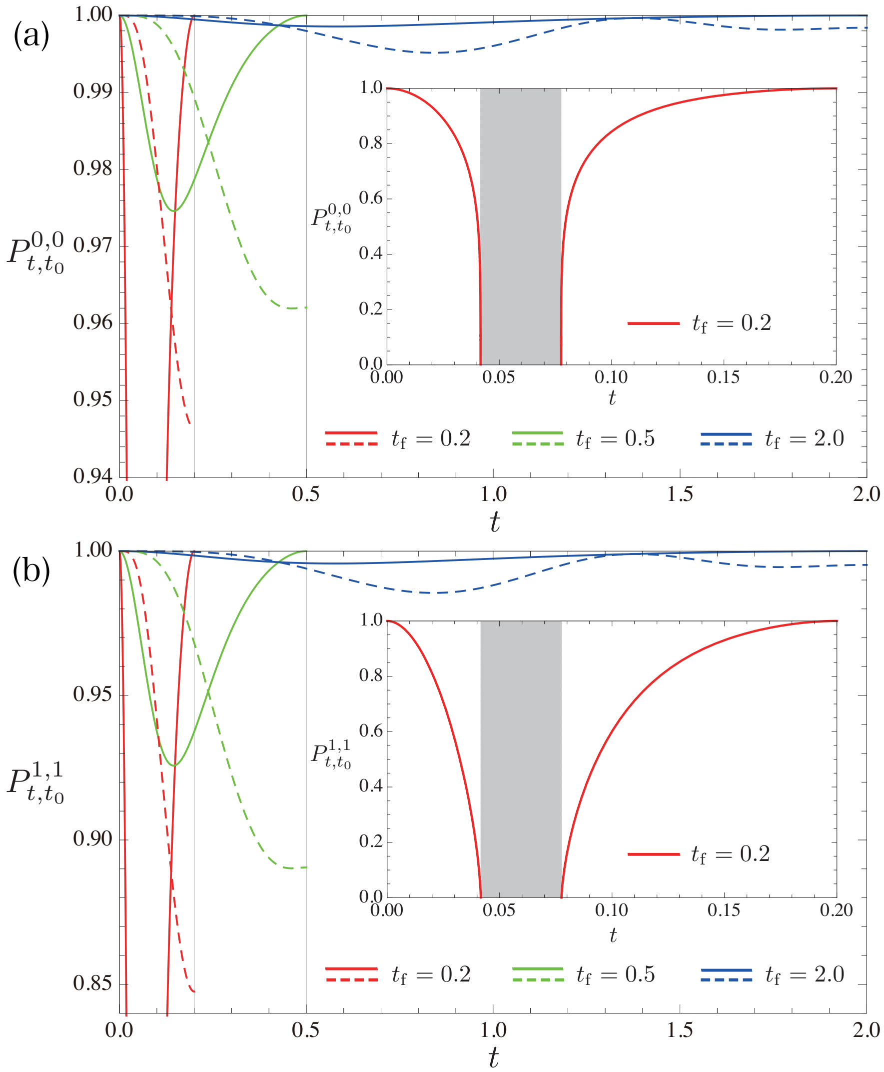

FIGURE 3.4: Transition probabilities between (a) even and (b) odd quantum-number states, $n, m = 0$ and $n, m = 1$, respectively, as functions of time t. The solid (dashed) curves represent the probabilities with (without) the counter-diabatic term. The gray-shaded regions in the insets represent the time intervals in which the Ω_t becomes imaginary, and hence the instantaneous energy spectrum becomes continuous for the case of $t_f = 0.2$.

From Eq. (3.54), the generalized momentum $p_t^{(X)}$ can be written by the generalized position $x_t^{(X)}$ as $p_t^{(X)} = \dot{x}_t^{(X)} + \frac{x_t^{(X)}}{2}\frac{\dot{\omega}_t}{\omega_t}$. By eliminating $p_t^{(X)}$ from Eq. (3.55), we have the same EoM as Eq. (3.45):

$$\ddot{x}_t^{(X)} + \tilde{\Omega}_t^2 x_t^{(X)} = 0. \tag{3.56}$$

The initial conditions of the canonical variables $(x_t^{(X)}, p_t^{(X)})$ are $(x_{t_0}^{(\mu)}, p_{t_0}^{(\mu)}) = (0, 1)$ and $(x_{t_0}^{(\nu)}, p_{t_0}^{(\nu)}) = \left(1, \frac{1}{2}\frac{\dot{\omega}_{t_0}}{\omega_{t_0}}\right)$.

3.5.2 Extended Husimi's measure of adiabaticity

By using the variables $(x_t^{(X)}, p_t^{(X)})$, we can rewrite the propagator [Eq. (3.9)] as

$$U_{t,t_0}^{\mathrm{TT}}(x|x_0) = \sqrt{\frac{M}{2\pi i\hbar x_t^{(\mu)}}}\exp\left[\frac{iM}{2\hbar x_t^{(\mu)}}\{p_t^{(\mu)}x^2 - 2xx_0 + (x_t^{(\nu)} - p_{t_0}^{(\nu)}x_t^{(\mu)})x_0^2\}\right]. \tag{3.57}$$

For the propagator Eq. (3.57), we calculate the TPGF as

$$\begin{aligned}\bar{\mathcal{P}}_{t,t_0}^{u,v} &:= \sum_{n,m=0}^{\infty} u^n v^m \bar{P}_{t,t_0}^{m,n} \\ &= \sqrt{\frac{2}{\bar{Q}_t^{\mathrm{TT}}(1-u^2)(1-v^2) + (1+u^2)(1+v^2) - 4uv}}.\end{aligned} \tag{3.58}$$

In this case, a measure of adiabaticity is given as [13]

$$\bar{Q}_t^{\mathrm{TT}} := \omega_{t_0}\frac{\bar{E}_t^{(\mu)}}{\omega_t} + \omega_{t_0}^{-1}\frac{\bar{E}_t^{(\nu)}}{\omega_t}; \qquad \dot{\omega}_{t_0} = 0, \tag{3.59}$$

which we call the extended Husimi's measure of adiabaticity. We have introduced the two classical energies as

$$\bar{E}_t^{(X)} := \frac{1}{2}(p_t^{(X)2} + \omega_t^2 x_t^{(X)2}). \tag{3.60}$$

Note that $\bar{E}_t^{(X)}$ is a function of the canonical variables $(x_t^{(X)}, p_t^{(X)})$, and is equal to $H_t^{\mathrm{ad}(X)}$ of Eq. (3.52). We can easily show $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\bar{E}_t^{(X)}}{\omega_t} = 0$ by using Eqs. (3.54) and (3.55). Therefore, whereas $\frac{\bar{E}_t^{(X)}}{\omega_t}$ is conserved as the adiabatic invariant for the usual QPO driven by slowly changing $H_t^{\mathrm{ad}(X)}$, it becomes the dynamical invariant that is precisely conserved for the QPO driven by $H_t^{\mathrm{TT}(X)}$ [23]. From the initial value of $x_t^{(X)}$ and $p_t^{(X)}$, we find $\bar{Q}_t^{\mathrm{TT}} = 1$ for arbitrary time $t \in [t_0, t_\mathrm{f}]$. This means $\bar{P}_{t,t_0}^{m,n} = \delta_{m,n}$, that is, a transitionless time evolution has been achieved at all time.

3.5.3 Expression of $\bar{Q}_t^{\mathrm{TT}}$ using the Ermakov-Lewis invariant

Here we introduce two types of the EL invariants in terms of the CPO, μ_t and ν_t, as

$$\bar{I}_t^{\mathrm{EL}} := \begin{cases} \dfrac{\omega_{t_0}}{2}\left[(\dot{\rho}_t\mu_t - \rho_t\dot{\mu}_t)^2 + W_t^2\left(\dfrac{\mu_t}{\rho_t}\right)^2\right] =: \bar{I}_t^{\mathrm{EL}(\mu)}, & (3.61) \\ \dfrac{1}{2\omega_{t_0}}\left[(\dot{\rho}_t\nu_t - \rho_t\dot{\nu}_t)^2 + W_t^2\left(\dfrac{\nu_t}{\rho_t}\right)^2\right] =: \bar{I}_t^{\mathrm{EL}(\nu)}. & (3.62) \end{cases}$$

In the above, the time-dependent variable ρ_t is defined as a solution of the Ermakov equation [61–63],

$$\ddot{\rho}_t + \tilde{\Omega}_t^2\rho_t = \frac{W_t^2}{\rho_t^3}, \quad \text{with} \quad \rho_t = \sqrt{\frac{\omega_{t_0}^2\mu_t^2 + \nu_t^2}{\omega_{t_0}}}. \tag{3.63}$$

[13] Without the condition of $\dot{\omega}_{t_0} = 0$, the parameter $\bar{Q}_t^{\mathrm{TT}}$ becomes

$$\bar{Q}_t^{\mathrm{TT}} = \omega_{t_0}\frac{\bar{E}_t^{(\mu)}}{\omega_t} + \omega_{t_0}^{-1}\frac{\bar{E}_t^{(\nu)}}{\omega_t} + p_{t_0}^{(\nu)}\frac{p_{t_0}^{(\nu)}\bar{E}_t^{(\mu)} - 2(p_t^{(\mu)}p_t^{(\nu)} + \omega_t^2 x_t^{(\mu)}x_t^{(\nu)})}{\omega_t\omega_{t_0}},$$

which is generally time-dependent.

As we have seen in Eq. (3.40), the EL invariants are equal to a half of the squared classical Wronskian as

$$\bar{I}_t^{\rm EL} = \bar{I}_t^{{\rm EL}(\mu)} = \bar{I}_t^{{\rm EL}(\nu)} = \frac{W_t^2}{2} = \frac{1}{2}. \tag{3.64}$$

By using Eqs. (3.61) and (3.62), Eqs. (3.27) and (3.28), and $\rho_t = \frac{1}{\sqrt{\omega_t}}$ as the solution of Eq. (3.63), we can rewrite $\frac{\bar{E}_t^{(\mu)}}{\omega_t}$ in Eq. (3.59) as

$$\frac{\bar{E}_t^{(\mu)}}{\omega_t} = \frac{\bar{I}_t^{{\rm EL}(\mu)}}{\omega_{t_0}}, \quad \frac{\bar{E}_t^{(\nu)}}{\omega_t} = \bar{I}_t^{{\rm EL}(\nu)}\omega_{t_0}; \qquad \dot{\omega}_{t_0} = 0. \tag{3.65}$$

We then obtain

$$\bar{Q}_t^{\rm TT} = \bar{I}_t^{{\rm EL}(\mu)} + \bar{I}_t^{{\rm EL}(\nu)} = 1; \qquad \dot{\omega}_{t_0} = 0. \tag{3.66}$$

Therefore, $\bar{Q}_t^{\rm TT}$ itself is also an dynamical invariant.

3.5.4 Expression of a quantum version of the EL invariant using quantum Wronakians

We introduce the following *quantum* Wronskians [67] as

$$\hat{G}_t^{(\mu)} := \sqrt{M\omega_{t_0}}\left(p_t^{(\mu)}\hat{x} - x_t^{(\mu)}\frac{\hat{p}}{M}\right), \tag{3.67}$$

$$\hat{G}_t^{(\nu)} := -\sqrt{\frac{M}{\omega_{t_0}}}\left(p_t^{(\nu)}\hat{x} - x_t^{(\nu)}\frac{\hat{p}}{M}\right). \tag{3.68}$$

By differentiating them with respect to time t and by using Eq. (3.56), we have [14]

$$\frac{{\rm d}\hat{G}_t^{(\mu)}}{{\rm d}t} = \sqrt{M\omega_{t_0}}(\ddot{x}_t^{(\mu)} + \tilde{\Omega}_t^2 x_t^{(\mu)})\hat{x} = 0, \tag{3.70}$$

$$\frac{{\rm d}\hat{G}_t^{(\nu)}}{{\rm d}t} = -\sqrt{\frac{M}{\omega_{t_0}}}(\ddot{x}_t^{(\nu)} + \tilde{\Omega}_t^2 x_t^{(\nu)})\hat{x} = 0. \tag{3.71}$$

The commutation relation between the quantum Wronskian $\hat{G}_t^{(\mu)}$ and $\hat{G}_t^{(\nu)}$ is related to the classical Wronskian as [67]

$$[\hat{G}_t^{(\mu)}, \hat{G}_t^{(\nu)}] = i\hbar W_t. \tag{3.72}$$

If the classical Wronskian is unity, the quantum Wronskians are canonical conjugate.

We introduce the following invariant as

$$\begin{aligned}\hat{I}_t^{\rm EL} &:= \frac{1}{2}(\hat{G}_t^{(\mu)2} + \hat{G}_t^{(\nu)2}) \\ &= \frac{M}{2}\left\{\left[\left(\dot{\rho}_t + \frac{\rho_t}{2}\frac{\dot{\omega}_t}{\omega_t}\right)\hat{x} - \rho_t\frac{\hat{p}}{M}\right]^2 + W_t^2\left(\frac{\hat{x}}{\rho_t}\right)^2\right\},\end{aligned} \tag{3.73}$$

[14]From Eqs. (3.67)–(3.71) and their forms at an initial values, $\hat{G}_t^{(\mu)}|_{t=t_0} = \sqrt{M\omega_{t_0}}\hat{x}$ and $\hat{G}_t^{(\nu)}|_{t=t_0} = \sqrt{\frac{M}{\omega_{t_0}}}\left(\frac{\hat{p}}{M} - \frac{\hat{x}}{2}\frac{\dot{\omega}_{t_0}}{\omega_{t_0}}\right)$, we find that these quantum Wronskians are constant operators of $\hat{x}$ and $\hat{p}$, that is,

$$\hat{G}_t^{(\mu)} = \sqrt{M\omega_{t_0}}\hat{x}, \quad \hat{G}_t^{(\nu)} = \sqrt{\frac{M}{\omega_{t_0}}}\left(\frac{\hat{p}}{M} - \frac{\hat{x}}{2}\frac{\dot{\omega}_{t_0}}{\omega_{t_0}}\right). \tag{3.69}$$

where we used Eqs. (3.67) and (3.68). Eq. (3.73) can be regarded as the *quantum* EL invariant [67] for the QPO with the counter-diabatic term. The quantum EL invariant can be expressed in terms of the quantum Wronskians as is similar to the classical case (see Eqs. (3.61), (3.62), and (3.64)). Since the quantum Wronskians, $\hat{G}_t^{(\mu)}$ and $\hat{G}_t^{(\nu)}$, are conserved quantities as verified at Eqs. (3.70) and (3.71), any polynomials of these operators with time-independent coefficients are also conserved quantities. Since $\rho_t = \frac{1}{\sqrt{\omega_t}}$ and $W_t = 1$, the invariant $\hat{I}_t^{\mathrm{EL}}$ is equal to the ratio of the adiabatic Hamiltonian to the angular frequency,

$$\hat{I}_t^{\mathrm{EL}} = \frac{\hat{H}_t^{\mathrm{ad}}}{\omega_t}; \qquad \dot{\omega}_{t_0} = 0. \tag{3.74}$$

Because of $\hat{I}_t^{\mathrm{EL}} = \omega_{t_0}^{-1} \hat{I}_t^{\mathrm{LR}}$ from Eq. (2.33), we find that $\hat{I}_t^{\mathrm{EL}}$ is the LR invariant [21].

We can express $\hat{I}_t^{\mathrm{EL}}$ by using a Bosonic operator as follows [21]. From Eqs. (3.72) and (3.74), we have

$$\hat{I}_t^{\mathrm{EL}} |n;\omega_t\rangle = \hbar\left(n + \frac{W_t}{2}\right)|n;\omega_t\rangle; \qquad \dot{\omega}_{t_0} = 0. \tag{3.75}$$

Therefore, we can infer from Eq. (3.75) that another form of the quantum EL invariant becomes

$$\hat{I}_t^{\mathrm{EL}} = \hbar\left(\hat{A}_t^\dagger \hat{A}_t + \frac{W_t}{2}\right) = \hbar\left(\hat{a}_t^{(\rho)\dagger}\hat{a}_t^{(\rho)} + \frac{W_t}{2}\right), \tag{3.76}$$

where

$$\hat{A}_t := \frac{1}{\sqrt{2\hbar}}(\hat{G}_t^{(\mu)} + i\hat{G}_t^{(\nu)}), \tag{3.77}$$

$$\hat{a}_t^{(\rho)} := \sqrt{\frac{M}{2\hbar}}\left\{W_t \frac{\hat{x}}{\rho_t} - i\left[\left(\dot{\rho}_t + \frac{\rho_t}{2}\frac{\dot{\omega}_t}{\omega_t}\right)\hat{x} - \rho_t \frac{\hat{p}}{M}\right]\right\}, \tag{3.78}$$

with relations $[\hat{A}_t, \hat{A}_t^\dagger] = [\hat{a}_t, \hat{a}_t^\dagger] = W_t$. Note that $\hat{A}_t$ is clearly a time-*independent* operator and $\hat{A}_t \neq \hat{a}_t$, i.e.,

$$\hat{a}_t^{(\rho)} = e^{-i\theta_t^{(\rho)}\hat{I}_t^{\mathrm{EL}}/\hbar}\hat{A}_t e^{i\theta_t^{(\rho)}\hat{I}_t^{\mathrm{EL}}/\hbar} = \hat{A}_t e^{i\theta_t^{(\rho)}}, \tag{3.79}$$

where $\theta_t^{(\rho)} := \int_{t_0}^t \frac{W_\tau}{\rho_\tau^2} d\tau$. Also, by defining a time-independent number operator as

$$\hat{n}_t^{(\rho)} := \hat{A}_t^\dagger \hat{A}_t = \hat{a}_t^{(\rho)\dagger}\hat{a}_t^{(\rho)}, \tag{3.80}$$

we can write $\hat{I}_t^{\mathrm{EL}} = \hbar\left(\hat{n}_t^{(\rho)} + \frac{W_t}{2}\right)$.

3.6 Concluding remarks for chapter 3

We give a few remarks on the main result of the TPGF of the QPO with the counter-diabatic term given by Eq. (3.15).

First, we note that the phase-amplitude method is formally applicable to the CPO in Eqs. (2.121) and (2.122) induced from the usual QPO without the counter-diabatic term; that is, it is formally applicable to the original case considered by Husimi [55]. In this case, the Ermakov equation given by Eq. (3.29) is replaced with

$$\ddot{\rho}_t + \omega_t^2 \rho_t = \frac{W_t^2}{\rho_t^3}. \tag{3.81}$$

With this replacement, the simple identification $\dot{\theta}_t = \omega_t$ in Eq. (3.35) is no longer applicable, and we cannot obtain a simple solution such as Eq. (3.37) for this case in general. However, in the adiabatic approximation of $\ddot{\rho}_t \simeq 0$, the Ermakov equation in Eq. (3.81) has the solution given by Eq. (3.37) as an adiabatic solution [1]:

$$\rho_t \simeq \frac{1}{\sqrt{\omega_t}}. \tag{3.82}$$

This implies that the Ermakov equation given by Eq. (3.29) for the QPO with the counter-diabatic term can have the adiabatic solution of the original Ermakov equation given by Eq. (3.81) for the usual QPO as the exact solution. The Schrödinger equation with the TT Hamiltonian can have the adiabatic solution of the original Schrödinger equation as the exact solution. This is an interpretation of the TT algorithm applied to the QPO based on the TPGF approach via Husimi's method.

Second, Beau et al. [1] recently introduced the ratio of the *diabatic* (non-adiabatic) mean energy to the adiabatic one as a measure of adiabaticity of a quantum heat engine, which is represented by a scale factor satisfying the Ermakov equation. This adiabaticity measure is equivalent to the original Husimi's measure of adiabaticity in the case of scale-invariant systems [68]. In contrast, our parameter is derived from the direct calculation of the probability generating function Eq. (3.15) and includes the adiabaticity measure as a special case. By substituting Eqs. (3.30) and (3.31) into Eq. (3.16) and using Eq. (3.37), we can express Q_t^{TT} as

$$Q_t^{\mathrm{TT}} = \frac{1}{2\Omega_t}\left(\dot{\rho}_t^2 + \Omega_t^2 \rho_t^2 + \frac{W_t^2}{\rho_t^2}\right) + \frac{\rho_t}{\Omega_t}\left(\dot{\rho}_t + \frac{\rho_t}{2}\frac{\dot{\omega}_t}{\omega_t}\right)\frac{\dot{\omega}_t}{\omega_t}. \tag{3.83}$$

This expression agrees with the *diabatic* factor in Ref. [1] if we replace Ω_t with ω_t and if we impose $\dot{\omega}_{t_0} = 0$.

In this chapter, we have studied the QPO with the counter-diabatic term with the TT algorithm based on the TPGF approach. By applying Husimi's method, we have obtained the propagator of the QPO with the counter-diabatic term with the two linearly independent solutions of the corresponding CPO. By calculating the TPGF from the propagator, we have found that it contains a simple time-dependent parameter that characterizes the success of the TT algorithm. The key to obtaining this simple parameter was the explicit solutions of the CPO derived based on the phase-amplitude method. We have illustrated this result by showing the trajectories of the CPO on the classical phase space and the time dependence of our parameter by using a specific form of the angular frequency. We have also introduced the extended Husimi's measure of adiabaticity to characterize the TT algorithm from the dynamical-invariant perspectives. Main points of this chapter are summarized in Tab. 3.1.

\	Without TT algorithm		With TT algorithm		
Hamiltonian of the QPO	$\hat{H}_t^{\mathrm{ad}} = \frac{\hat{p}^2}{2M} + \frac{M}{2}\omega_t^2\hat{x}^2$		$\hat{H}_t^{\mathrm{TT}} = \hat{H}_t^{\mathrm{ad}} + \hat{H}_t^{\mathrm{cd}}$, where $\hat{H}_t^{\mathrm{cd}} = -\frac{1}{2}\frac{\dot{\omega}_t}{\omega_t}\frac{\hat{x}\hat{p}+\hat{p}\hat{x}}{2}$		
Schrödinger eq.	$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\lvert\Psi(t)\rangle = \hat{H}_t^{\mathrm{ad}}\lvert\Psi(t)\rangle$		$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\lvert\Psi(t)\rangle = \hat{H}_t^{\mathrm{TT}}\lvert\Psi(t)\rangle$		
Instantaneous eigenrelation	$\hat{H}_t^{\mathrm{ad}}\lvert n;\omega_t\rangle = E_{n,t}^{\mathrm{ad}}\lvert n,\omega_t\rangle$, where $E_{n,t}^{\mathrm{ad}} = \hbar\omega_t\left(n+\frac{1}{2}\right)$		$\hat{H}_t^{\mathrm{TT}}\lvert n;\Omega_t\rangle = E_{n,t}^{\mathrm{TT}}\lvert n,\Omega_t\rangle$, where $E_{n,t}^{\mathrm{TT}} = \hbar\Omega_t\left(n+\frac{1}{2}\right)$ with $\Omega_t = \sqrt{\omega_t^2 - \left(\frac{1}{2}\frac{\dot{\omega}_t}{\omega_t}\right)^2}$		
Propagator	$U_{t,t_0}^{\mathrm{H}}(x\lvert x_0) = \sqrt{\frac{M}{2\pi i\hbar\mu_t}}\exp\left[\frac{iM}{2\hbar\mu_t}\left(\dot{\mu}_t x^2 - 2xx_0 + \nu_t x_0^2\right)\right]$		$U_{t,t_0}^{\mathrm{TT}}(x\lvert x_0) = \sqrt{\frac{M}{2\pi i\hbar\mu_t}}\exp\left[\frac{iM}{2\hbar}\left\{\left(\frac{\dot{\mu}_t}{\mu_t} + \frac{1}{2}\frac{\dot{\omega}_t}{\omega_t}\right)x^2 - \frac{2xx_0}{\mu_t} + \left(\frac{\nu_t}{\mu_t} - \frac{1}{2}\frac{\dot{\omega}_{t_0}}{\omega_{t_0}}\right)x_0^2\right\}\right]$		
Time-evolution unitary op.	$\hat{U}_{t,t_0}^{\mathrm{H}} = \iint_{\mathbb{R}^2}\mathrm{d}x\mathrm{d}x_0\lvert x\rangle U_{t,t_0}^{\mathrm{H}}(x\lvert x_0)\langle x_0\rvert$		$\hat{U}_{t,t_0}^{\mathrm{TT}} = \iint_{\mathbb{R}^2}\mathrm{d}x\mathrm{d}x_0\lvert x\rangle U_{t,t_0}^{\mathrm{TT}}(x\lvert x_0)\langle x_0\rvert$		
Transition Probability	$P_{t,t_0}^{(\mathrm{H})m,n} = \lvert\langle m;\omega_t\rvert\hat{U}_{t,t_0}^{\mathrm{H}}\lvert n;\omega_{t_0}\rangle\rvert^2$		$\bar{P}_{t,t_0}^{m,n} = \lvert\langle m;\omega_t\rvert\hat{U}_{t,t_0}^{\mathrm{TT}}\lvert n;\omega_{t_0}\rangle\rvert^2$	$P_{t,t_0}^{m,n} = \lvert\langle m;\Omega_t\rvert\hat{U}_{t,t_0}^{\mathrm{TT}}\lvert n;\Omega_{t_0}\rangle\rvert^2$	
EoMs of the CPO in Husimi's method	$\ddot{\mu}_t + \omega_t^2\mu_t = 0$, $(\mu_{t_0},\dot{\mu}_{t_0}) = (0,1)$ $\ddot{\nu}_t + \omega_t^2\nu_t = 0$, $(\nu_{t_0},\dot{\nu}_{t_0}) = (1,0)$		$\ddot{\mu}_t + \tilde{\Omega}_t^2\mu_t = 0$, $(\mu_{t_0},\dot{\mu}_{t_0}) = (0,1)$ $\ddot{\nu}_t + \tilde{\Omega}_t^2\nu_t = 0$, $(\nu_{t_0},\dot{\nu}_{t_0}) = (1,0)$ $\tilde{\Omega}_t = \sqrt{\Omega_t^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{\omega}_t}{\omega_t}} = \sqrt{\omega_t^2 - \frac{3}{4}\frac{\dot{\omega}_t^2}{\omega_t^2} + \frac{1}{2}\frac{\ddot{\omega}_t}{\omega_t}}$		
Ermakov eq.	$\ddot{\rho}_t + \omega_t^2\rho_t = \frac{W_t^2}{\rho_t^3}$, where $W_t = \dot{\mu}_t\nu_t - \mu_t\dot{\nu}_t = 1$		$\ddot{\rho}_t + \tilde{\Omega}_t^2\rho_t = \frac{W_t^2}{\rho_t^3}$, where $W_t = \dot{\mu}_t\nu_t - \mu_t\dot{\nu}_t = 1$		
Condition of the frequency	\	$\dot{\omega}_t \simeq 0,\ ^{\forall}t$	$\dot{\omega}_{t_0} = 0$		$\dot{\omega}_{t_0} = \dot{\omega}_{t_{\mathrm{f}}} = 0$
Sol. of Schrödinger eq.	$\lvert\Psi(t)\rangle = \hat{U}_{t,t_0}^{\mathrm{H}}\lvert\Psi(t_0)\rangle$	$\lvert\Psi(t)\rangle \simeq \sum_{n=0}^{\infty}C_n e^{i\xi_{n,t}}\lvert n;\omega_t\rangle$	$\lvert\Psi(t)\rangle = \sum_{n=0}^{\infty}C_n e^{i\xi_{n,t}}\lvert n;\omega_t\rangle$		
Sols. of the classical EoM	$\mu_t = \frac{\rho_t}{\sqrt{\omega_{t_0}}}\sin\left(\int_{t_0}^{t}\frac{W_\tau}{\rho_\tau^2}\mathrm{d}\tau\right)$ $\nu_t = \sqrt{\omega_{t_0}}\rho_t\cos\left(\int_{t_0}^{t}\frac{W_\tau}{\rho_\tau^2}\mathrm{d}\tau\right)$	$\mu_t \simeq \frac{1}{\sqrt{\omega_t\omega_{t_0}}}\sin\left(\int_{t_0}^{t}\omega_\tau\mathrm{d}\tau\right)$ $\nu_t \simeq \sqrt{\frac{\omega_{t_0}}{\omega_t}}\cos\left(\int_{t_0}^{t}\omega_\tau\mathrm{d}\tau\right)$	$\mu_t = \frac{1}{\sqrt{\omega_t\omega_{t_0}}}\sin\left(\int_{t_0}^{t}\omega_\tau\mathrm{d}\tau\right)$ $\nu_t = \sqrt{\frac{\omega_{t_0}}{\omega_t}}\cos\left(\int_{t_0}^{t}\omega_\tau\mathrm{d}\tau\right)$		
Sol. of Ermakov eq.	$\rho_t = \sqrt{\frac{\omega_{t_0}^2\mu_t^2+\nu_t^2}{\omega_{t_0}}}$	$\rho_t \simeq \frac{1}{\sqrt{\omega_t}}$ $\because\ \ddot{\rho}_t \simeq 0$	$\rho_t = \frac{1}{\sqrt{\omega_t}}$		
Measure of adiabaticity	$Q_t^* \geq 1,\ ^{\forall}t$	$Q_t^* \simeq 1,\ ^{\forall}t$	$\bar{Q}_t^{\mathrm{TT}} = 1,\ ^{\forall}t$	$Q_t^{\mathrm{TT}} = \frac{\omega_t}{\Omega_t},\ ^{\forall}t$	$Q_{t_0}^{\mathrm{TT}} = Q_{t_{\mathrm{f}}}^{\mathrm{TT}} = 1,\ ^{\forall}t_{\mathrm{f}}$

TABLE 3.1: Main points of chapter 3.

Chapter 4

Gaming the Quantum as a Discrete-time Control

Most of this chapter is based on my work [69].

In the previous chapter 2 and 3, we have considered the continuous-time evolution and the adiabatic control while in here we shall discuss a discrete-time unitary evolution in a quantum two-level system.

4.1 Introductory remarks for chapter 4

"Game" familiar to many people will be Go, Shogi, card games, etc. More than two "agents (or players [15])" follow the rules and aim for their victory. The issue of applied mathematics, which analyzes the influence of a set of player's behaviors (strategy) on each "victory/defeat," is called game theory. Game theory was originally introduced by von Neumann and Morgenstein [35]. Nash mathematically formulated the composition of cooperation and non-cooperation among players in the game and also introduced a basic concept, Nash equilibrium, which is applied in an analysis of almost all of non-cooperative games [70]. The range of applications of game theory is wide, for example, social science, biology, computer science, political science, and, more recently, physics [71]. In a usual setting of a game, each player can select one of a few numbers of possible operations. The results of the game are determined by the strategies of all players.

The game theory analyzes all "game situations." Game situations mean that there exist multiple decision-making entities depending on each other for their purposes. In game theory, we formalize such game situation using a mathematical model and analyze the cooperation and non-cooperation among players. This mathematical model is called a game.

The Prisoner's Dilemma [72] is a game which shows that evolution of cooperative behaviors among selfish agents is possible. The Hawk-Dove Game [73] is a model which indicates that competition between cooperative and aggressive agents can reach a Nash equilibrium. There recently exists interest in applying game-theoretic techniques in physics. The games are necessarily idealizations of social and physical situations.

On the other hand, quantum computers perform computations by exploiting the quantum mechanical principles of superposition, entanglement, non-locality, and interference [74]. The upsurge of interest in quantum computing has been accompanied with increasing attention to research on quantum information processing [75].

A game including quantum natures was independently brought by Meyer [37] and Eisert et al. [38] in 1999. This occurrence was a starting point of quantum game theory. Naively, this theory can be formulated by expressing a unitary operator as a quantum player's operation and by replacing the classical probability with a quantum amplitude. Since states of quantum games should be regarded as quantum states, possible quantum game flows could be described as quantum circuits. In order

[15] A player is an essential element of the game which is the subject of decision-making. Depending on the situation we analyzed, players are individuals such as consumers, investors, various organizations such as companies, organizations and political parties, and a wider variety of entities such as governments and nations.

to know the result of the quantum game, we need to do a projection measurement. Quantum games keep coherence until we have measured the state of the quantum game. Quantum-game-theoretic approaches could be applied to quantum communication [76] or quantum computing [77] protocols.

In a seminal paper [37], Meyer originated the "quantum penny flip game" in quantum game theory. In non-cooperative games, he attempted to apply game theory to quantum mechanics in order to make a thorough investigation of equilibrium behaviors of quantum algorithms. In the quantum penny flip game, two players manipulate one invisible coin and try to control the final state of the coin. One player is allowed to use quantum mechanical operations on the coin while another player is allowed to use only classical operations which is commutative unitary. Meyer found a typical strategy that always guarantees a secure victory of the quantum player against the classical one.

It has been demonstrated that quantum players are more predominant than classical players [37]. We here define the meaning of "predominance" in a game as follows. As we will see at Sec. 4.2.2 concretely, if player Q can neutralize another player P's operations while player P cannot do so against player Q, player Q is more predominant than player P. This definition implies that more predominant player Q can freely decide the final state of a game if less predominant player P has just two operations. Regarding games with malicious rules [78] in which a classical player can win against a full quantum player [79], this classical player does not have any predominance because he may not be able to lose on purpose, i.e., he cannot freely decide the final state of the game. Being able to win does not constitute predominance. In this chapter, we consider that the meaning of "predominance" is more strict than that of "advantage" which is often used in other papers.

Although there have been numerous discussions about games with quantum vs. classical players and quantum vs. quantum players, there have been few discussions about games without any ancillary systems with quantum vs. restricted quantum players. Such games would be useful in identifying the precise quantum behavior that leads predominance. Namely, the following questions could be answered. What are the conditions for the existence of the predominance/advantage of a full quantum player under some restrictions of another quantum player? How much restriction allows predominance/advantage of the full quantum player?

Strategies in the penny flip game can be regarded as a kind of information processing. The quantum penny flip game was introduced to investigate the possible influence of quantum mechanics on information processing [37, 38, 80]. The purpose of this chapter is to investigate whether quantum operations can recover the state of a system disturbed by a classical agent. Our standpoint can be classified as "*gaming the quantum* [81]," which purports to be one of the natural approaches to exploring the quantum landscape for situations that are biasedly or unbiasedly restricted.

The remainder of this chapter is organized as follows. In Sec. 4.2, we review the classical/quantum penny flip game. In Sec. 4.3, we change the set of classical player P's commutative operations to various non-commutative ones (see Sec. 4.3.1–4.3.4), find an example of quantum player Q's winning strategy, and calculate the general solutions of player Q's winning strategies. In Sec. 4.4, we conclude the chapter 4, discuss other research, and mention future work.

4.2 Classical/Quantum penny flip game

Here, we introduce the simple "penny flip game," which is a main subject of this section.

4.2.1 Classical version

The classical penny flip game was introduced by Meyer [37]. This game has the following rules:

i) Players P and Q have a common penny coin.
ii) The initial state of the penny is heads; the penny is in a box, making it invisible to the players.
iii) Each player can choose whether to flip the penny.

iv) The players can see neither the current state of the penny nor the other player's previous operation.
v) The sequence of operations is Q → P → Q.
vi) If the final state is heads (i.e., the final state is equal to the initial state), Q wins; otherwise, P wins.

The payoff matrix of the game is given in Table 4.1, in which F, N, and NF represent the actions of flip, no flip, and no flip after flip, respectively. The numbers in the matrix are the payoffs for each player; the first index is for player P, and the second index is for player Q. For example, $(-1, 1)$ means that player P loses and player Q wins because the final state is heads. It is easily verified that the probability of each player winning is $\frac{1}{2}$, and that there exists no pure strategy under Nash equilibrium [70]. The probabilities of the choices of each player are denoted as $\vec{p} := (p_N, p_F)$ and $\vec{q} := (q_{NN}, q_{NF}, q_{FN}, q_{FF})$, respectively. The payoff functions are defined as the expectation of an individual player as $u_{\mathsf{P}}(\vec{p}, \vec{q}) = -u_{\mathsf{Q}}(\vec{p}, \vec{q}) = (1 - 2p_N)[1 - 2(q_{NF} + q_{FN})]$. The mixed-strategy Nash equilibria are given at $\vec{p}^* = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\vec{q}^* = \left(q^*_{NN}, q^*_{NF}, \frac{1}{2} - q^*_{NF}, \frac{1}{2} - q^*_{NN}\right)$, where q^*_{NF} and q^*_{NN} may take any value in the range $\left[0, \frac{1}{2}\right]$. Hence, player P's optimal strategy is to choose either F or N with equal probability, and player Q's optimal strategies are to choose either the same or different operations with equal probability. We find that the average payoffs of both players are zero at the Nash equilibrium. Altogether, the classical penny flip game is a symmetric, zero-sum, and fair game.

		Q			
	(P, Q)	NN	NF	FN	FF
P	N	$(-1, 1)$	$(1, -1)$	$(1, -1)$	$(-1, 1)$
	F	$(1, -1)$	$(-1, 1)$	$(-1, 1)$	$(1, -1)$

TABLE 4.1: Payoff matrix of the classical penny flip game.

4.2.2 Quantum version

In discussing unitary quantum operations, we use the following notation: $\hat{\vec{\sigma}} := (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$, where $\hat{\sigma}_1 \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\hat{\sigma}_2 \doteq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\hat{\sigma}_3 \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are Pauli matrices.

The quantum penny flip game was formulated by Meyer [37]. In the classical penny flip game, a penny coin takes one of two states: heads or tails. Meyer introduced a two-state quantum system through the spin of a "quantum coin." In this case, we have to account for quantum properties such as superposition and unitary transformation. In the quantum penny flip game, only player Q can employ a "quantum operation." Namely, the quantum player can apply arbitrary unitary transformations whereas the classical player can apply only Abelian unitary transformations. Moving forward, player P's and Q's quantum payoff functions are defined as $\$_{\mathsf{P}} = -\$_{\mathsf{Q}} = 1 - 2|\langle f|i\rangle|^2$, where $|i\rangle$ and $|f\rangle$ are the initial and final states of the coin, respectively. Meyer showed that player Q wins every time if he uses the Hadamard transformation:

$$|0\rangle \xrightarrow[\hat{H}]{\mathsf{Q}} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow[\hat{\sigma}_1 \text{ or } \hat{\mathbb{1}}]{\mathsf{P}} \left\{ \begin{array}{ll} \frac{|1\rangle + |0\rangle}{\sqrt{2}} & \text{if P applies } \hat{\sigma}_1 \\ \frac{|0\rangle + |1\rangle}{\sqrt{2}} & \text{if P applies } \hat{\mathbb{1}} \end{array} \right\} \xrightarrow[\hat{H}]{\mathsf{Q}} |0\rangle, \tag{4.1}$$

where $|0\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ denotes "heads" (i.e., spin up), $|1\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ denotes "tails" (i.e., spin down), $\hat{H} = \frac{\hat{\sigma}_1 + \hat{\sigma}_3}{\sqrt{2}} \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the Hadamard transformation, the Pauli matrix $\hat{\sigma}_1$ flips the penny coin, and the identity matrix $\hat{\mathbb{1}}$ leaves the penny coin unchanged. In the first step, player Q applies the Hadamard transformation, $\hat{H}$, which puts the coin into the equal-weight superposition state of heads and tails. In the second step, player P can choose whether to flip the coin, but the superposed state of the coin remains unchanged by either operation selected by player P. In the third step, player Q again applies the Hadamard transformation $\hat{H}$, which puts the coin back to the initial state because of $\hat{H}^2 = \hat{\mathbb{1}}$. Thus, player Q always wins when they open the box. Hence, in the penny flip game, the quantum strategy is perfectly advantageous against any classical strategy. It is worth noting that the intermediate state $|+x\rangle$ is a simultaneous eigenstate of player P's operations $\hat{\mathbb{1}}$ and $\hat{\sigma}_1$. This fact implies that the quantum player Q nullifies the operations of player P, $\hat{\sigma}_1$ or $\hat{\mathbb{1}}$ (see Fig. 4.1), that is, player Q is predominant.

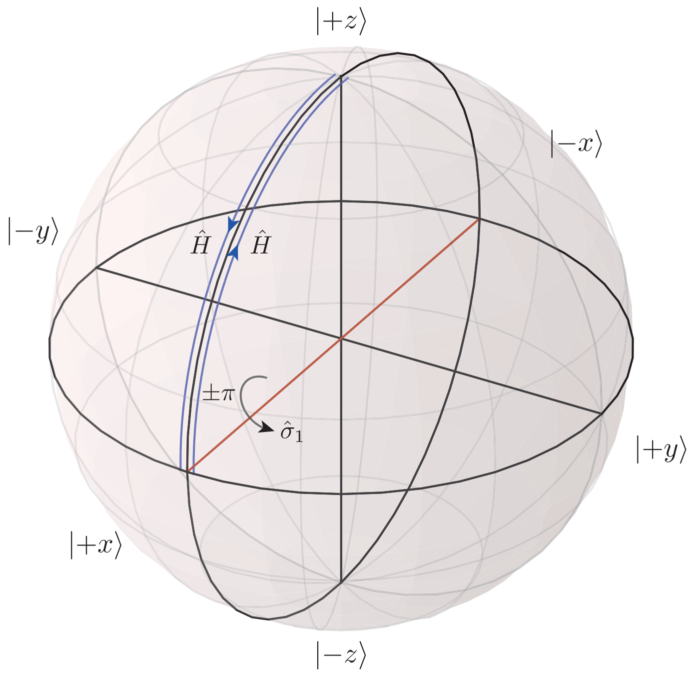

FIGURE 4.1: Winning quantum strategy of Meyer drawn on the Bloch sphere. Here, we set $|\pm x\rangle \doteq \frac{1}{\sqrt{2}}\binom{1}{\pm 1}$, $|\pm y\rangle \doteq \frac{1}{\sqrt{2}}\binom{1}{\pm i}$, $|+z\rangle = |0\rangle$ and $|-z\rangle = |1\rangle$. The Hadamard transformation, which is the operation of player Q, converts the $|+z\rangle$ state to the $|+x\rangle$ state. The operation of $\hat{\sigma}_1$, which is the coin-flip operation by player P, is a rotation around the x-axis by π radians. Player P cannot change the $|+x\rangle$ state by applying the coin-flip operation (see Eq. (4.1)).

However, the game proceeds differently if both players are allowed to play with quantum strategies. Meyer showed that the one-sided advantage is lost in this case (see Theorem 2 of Ref. [37]). Although the strategy provided by Meyer is only one of many winning strategies, his example demonstrates the predominance of quantum strategies. Chappell et al. [41] provided all of the unitary transformations that are winning strategies for player Q:

$$\hat{U}_{\mathsf{Q}}^{(1)}(\theta,\phi) = e^{i\delta_1}\exp\left[i\frac{\theta}{2}\left(a, b\cot\frac{\theta}{2}, ab\right)\cdot\hat{\vec{\sigma}}\right], \quad \hat{U}_{\mathsf{Q}}^{(2)}(\theta,\phi) = e^{i\delta_2}e^{i\phi\hat{\sigma}_3/2}\hat{U}_{\mathsf{Q}}^{(1)\dagger}, \tag{4.2}$$

where $\hat{U}_{\mathsf{Q}}^{(1)}$ and $\hat{U}_{\mathsf{Q}}^{(2)}$ are player Q's first and second operations, respectively, $a = \pm\sqrt{\frac{1}{2}\left(1-\cot^2\frac{\theta}{2}\right)}$, $b = \pm 1$, $|\theta| \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, and $\phi, \delta_1, \delta_2 \in [0, 2\pi)$. By selecting $(\theta, \phi, \delta_1, \delta_2) = \left(\pi, 0, -\frac{\pi}{2}, -\frac{\pi}{2}\right)$, the

Chappell transformation becomes $\hat{U}_{\mathsf{Q}}^{(1)} = \hat{U}_{\mathsf{Q}}^{(2)} = \frac{\hat{\sigma}_1+\hat{\sigma}_3}{\sqrt{2}} = \hat{H}$, which corresponds to Meyer's solution.

4.3 Modified quantum penny flip game

In the previous section, we saw that player Q can change the state of the coin into a simultaneous eigenstate of the possible operations of player P if the operations of P are mutually commutative. This is the winning strategy for player Q. As a direct extension of this observation, we propose a question: if player P is allowed to use a restricted class of non-commutative unitary operations, does player Q have a winning strategy? Meyer [37] showed that if player P is also allowed to use any unitary operation, player Q has no winning strategies. Thus, to interpret the question, we must define the class of the operations available to player P.

4.3.1 Non-Abelian strategy and winning counter-strategy

To begin with, we consider a simple modification of the strategy of player P by allowing him to use $\hat{\sigma}_3$ instead of $\hat{\mathbb{1}}$ as the non-flipping operation. Player P still uses $\hat{\sigma}_1$ as the flipping operation. These operators are non-commutative: $[\hat{\sigma}_3, \hat{\sigma}_1] = 2i\hat{\sigma}_2 \neq 0$; therefore, they generate a non-Abelian group. In this case, there is no longer a simultaneous eigenstate of player P's operations. Nevertheless, we found a winning strategy for player Q:

$$\hat{U}_{\mathsf{Q}}^{(1)} \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad \hat{U}_{\mathsf{Q}}^{(2)} \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} i & -1 \\ 1 & -i \end{pmatrix}. \tag{4.3}$$

The game proceeds as follows:

$$|0\rangle \xrightarrow[\hat{U}_{\mathsf{Q}}^{(1)}]{\mathsf{Q}} \frac{|0\rangle + i|1\rangle}{\sqrt{2}} \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \xrightarrow[\hat{\sigma}_1 \text{ or } \hat{\sigma}_3]{\mathsf{P}} \begin{Bmatrix} i\frac{|0\rangle - i|1\rangle}{\sqrt{2}} & \text{if P applies } \hat{\sigma}_1 \\ \frac{|0\rangle - i|1\rangle}{\sqrt{2}} & \text{if P applies } \hat{\sigma}_3 \end{Bmatrix} \xrightarrow[\hat{U}_{\mathsf{Q}}^{(2)}]{\mathsf{Q}} \begin{Bmatrix} -|0\rangle, \\ i|0\rangle. \end{Bmatrix} \tag{4.4}$$

Thus, the final state of the coin is always equivalent to the initial state, heads. This means that the operations given in Eq. (4.3) constitute a winning strategy for player Q.

This strategy utilizes two special states, $|\pm y\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$, for which both operations $\hat{\sigma}_1$ and $\hat{\sigma}_3$, i.e., those available to player P, have the same effect except for a phase change. Namely, player P must flip the coin through the operations, which implies player Q is predominant even if player P's operations are non-commutative. Thus, player Q can always know the state of the coin (see Fig. 4.2).

By using a method similar to Chappell et al. [41], we can obtain all winning strategies for player Q in the modified game in which player P uses $\hat{\sigma}_1$ and $\hat{\sigma}_3$. The complete set of the winning strategies are the unitary operators

$$\hat{U}_{\mathsf{Q}}^{(1)}(\theta, \phi) = e^{i\delta_1} \exp\left[i\frac{\theta}{2}\left(b \cot\frac{\theta}{2}, ab, a\right) \cdot \hat{\vec{\sigma}}\right], \quad \hat{U}_{\mathsf{Q}}^{(2)}(\theta, \phi) = e^{i\delta_2} e^{i\phi\hat{\sigma}_3/2} \hat{U}_{\mathsf{Q}}^{(1)\dagger} \hat{\sigma}_3, \tag{4.5}$$

which are parameterized by the same variables as Eq. (4.2). By selecting $(\theta, \phi, \delta_1, \delta_2) = \left(\frac{\pi}{2}, 0, 0, \frac{\pi}{2}\right)$, the general solution in Eq. (4.5) is reduced to Eq. (4.3).

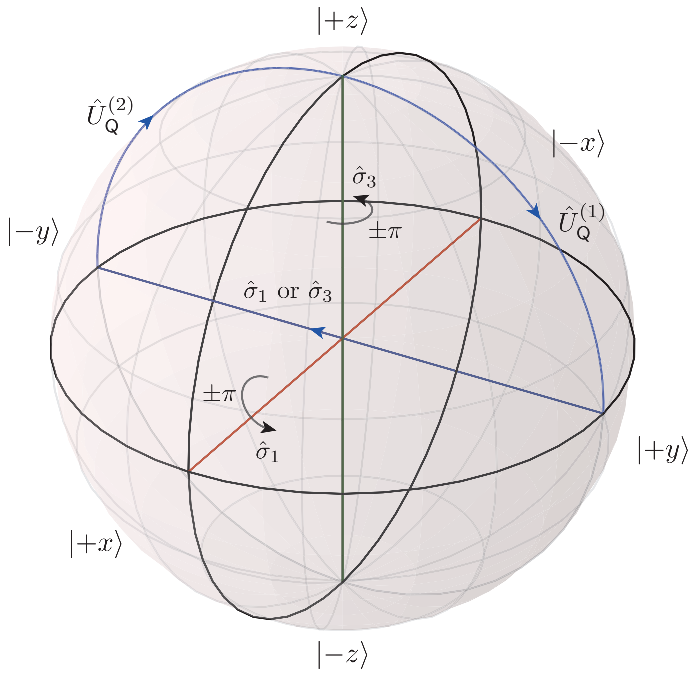

FIGURE 4.2: Winning strategy against operations $\hat{\sigma}_1$ and $\hat{\sigma}_3$ on the Bloch sphere. Player P always converts the state $|+y\rangle$ into $|-y\rangle$ (see Eq. (4.4)).

4.3.2 Non-Abelian strategy with phase variables and winning counter-strategy

In this game variant, we introduce a modified flipping operator $\hat{F}$ and a modified non-flipping operator $\hat{N}$ for player P:

$$\hat{F}(\alpha) := e^{i\alpha\hat{\sigma}_3/2}\hat{\sigma}_1 \doteq \begin{pmatrix} 0 & e^{i\alpha/2} \\ e^{-i\alpha/2} & 0 \end{pmatrix}, \quad \hat{N}(\beta) := e^{i\beta\hat{\sigma}_3/2} \doteq \begin{pmatrix} e^{i\beta/2} & 0 \\ 0 & e^{-i\beta/2} \end{pmatrix}, \tag{4.6}$$

where $\alpha, \beta \in \mathbb{R}$. In a classical sense, operator $\hat{F}$ flips the coin whereas operator $\hat{N}$ does not, but both introduce phase changes to the quantum state of the coin. In general, they are non-commutative: $[\hat{F}(\alpha), \hat{N}(\beta)] = 2e^{i\alpha\hat{\sigma}_3/2}\hat{\sigma}_2 \sin\frac{\beta}{2} \neq 0$ if $\beta \notin 2\pi\mathbb{Z}$. By using the group composition law of $SU(2)$ [82], the modified flipping operation can be rewritten as $i\hat{F}(\alpha) = \exp\left[\pm i\frac{\pi}{2}\left(\cos\frac{\alpha}{2}, -\sin\frac{\alpha}{2}, 0\right) \cdot \hat{\vec{\sigma}}\right]$ whose rotation (i.e., flipping) axes are in the same plane of the Bloch sphere. Even if we replace α with $\alpha + 2\pi\mathbb{Z}$, the rotation axis is unchanged. This is equivalent to the fact that the commutation relation, $[\hat{F}(\alpha), \hat{F}(\alpha')] = 2i\hat{\sigma}_3 \sin\frac{\alpha-\alpha'}{2}$, is zero. We call the operators in Eq. (4.6) a phase-variable strategy, and we call player P using this strategy a phase-variable player. Various operations can be derived from this general case provided that plus–minus signs are arbitrary:

- By selecting $\alpha, \beta \in 4\pi\mathbb{Z}$, player P's operations become $(\hat{F}, \hat{N}) = (\hat{\sigma}_1, \hat{\mathbb{1}})$, i.e., those in Meyer's setting.
- By selecting $\alpha, \beta \in 2(2\mathbb{Z}+1)\pi$, player P's operations become $(\hat{F}, \hat{N}) = -(\hat{\sigma}_1, \hat{\mathbb{1}})$, i.e., those in Meyer's setting, except for a sign change.
- By selecting $\alpha \in 4\pi\mathbb{Z}$, $\beta \in (4\mathbb{Z}+1)\pi$, player P's operations become $(\hat{F}, \hat{N}) = (\hat{\sigma}_1, i\hat{\sigma}_3)$, i.e., those defined in Sec. 4.3.1 except for a phase change.
- By selecting $\alpha \in 2(2\mathbb{Z}+1)\pi$, $\beta \in (4\mathbb{Z}-1)\pi$, player P's operations become $(\hat{F}, \hat{N}) = -(\hat{\sigma}_1, i\hat{\sigma}_3)$, i.e., those defined in Sec. 4.3.1, except for a phase change.

The difference of the phase is not important in these arguments.

We seek a winning strategy for player Q against the phase-variable player P. We use density matrix representation of the coin state to deal with classical and quantum operations on the same footing. By using density matrix representation, the game flow is illustrated as:

$$\hat{\rho}_0 \xrightarrow[\hat{U}_{\mathsf{Q}}^{(1)}]{\mathsf{Q}} \hat{\rho}_1 \xrightarrow[\hat{F}(\alpha)\text{ or }\hat{N}(\beta)]{\mathsf{P}} \hat{\rho}_2 \xrightarrow[\hat{U}_{\mathsf{Q}}^{(2)}]{\mathsf{Q}} \hat{\rho}_3. \tag{4.7}$$

The initial state of the coin is assumed to be heads, $\hat{\rho}_0 := |0\rangle\langle 0|$. Player Q applies a unitary transformation $\hat{U}_{\mathsf{Q}}^{(1)}$ on the coin, yielding $\hat{\rho}_1 := \hat{U}_{\mathsf{Q}}^{(1)}\hat{\rho}_0\hat{U}_{\mathsf{Q}}^{(1)\dagger}$. In the next step, player P applies the flipping operation $\hat{F}(\alpha)$ with probability p or the non-flipping operation $\hat{N}(\beta)$ with probability $1-p$. Thus, the density matrix is transformed to $\hat{\rho}_2 := p\hat{F}\hat{\rho}_1\hat{F}^\dagger + (1-p)\hat{N}\hat{\rho}_1\hat{N}^\dagger$. The phase parameters α and β can be adjusted to yield the strongest strategy for player P. In the final step, player Q applies another unitary transformation $\hat{U}_{\mathsf{Q}}^{(2)}$, which yields $\hat{\rho}_3 := \hat{U}_{\mathsf{Q}}^{(2)}\hat{\rho}_2\hat{U}_{\mathsf{Q}}^{(2)\dagger}$. Thus, the density matrix of the final state is

$$\hat{\rho}_3 = p\hat{U}_{\mathsf{Q}}^{(2)}\hat{F}\hat{U}_{\mathsf{Q}}^{(1)}\hat{\rho}_0\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{F}^\dagger\hat{U}_{\mathsf{Q}}^{(2)\dagger} + (1-p)\hat{U}_{\mathsf{Q}}^{(2)}\hat{N}\hat{U}_{\mathsf{Q}}^{(1)}\hat{\rho}_0\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{N}^\dagger\hat{U}_{\mathsf{Q}}^{(2)\dagger}. \tag{4.8}$$

A perfect strategy for player Q requires that $\hat{\rho}_3 = \hat{\rho}_0$ for arbitrary flip probability p; thus, the following equations must hold:

$$\left.\begin{aligned} &\hat{U}_{\mathsf{Q}}^{(2)}\hat{F}\hat{U}_{\mathsf{Q}}^{(1)}\hat{\rho}_0\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{F}^\dagger\hat{U}_{\mathsf{Q}}^{(2)\dagger} && (4.9)\\ &\hat{U}_{\mathsf{Q}}^{(2)}\hat{N}\hat{U}_{\mathsf{Q}}^{(1)}\hat{\rho}_0\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{N}^\dagger\hat{U}_{\mathsf{Q}}^{(2)\dagger} && (4.10)\end{aligned}\right\} = \hat{\rho}_0.$$

By using $\hat{\rho}_0 = \frac{\hat{\mathbb{1}}+\hat{\sigma}_3}{2}$, we can rewrite Eq. (4.10) as $[\hat{U}_{\mathsf{Q}}^{(2)}\hat{N}\hat{U}_{\mathsf{Q}}^{(1)}, \hat{\sigma}_3] = 0$. From this, we can derive a relation between $\hat{U}_{\mathsf{Q}}^{(1)}$ and $\hat{U}_{\mathsf{Q}}^{(2)}$, i.e., $\hat{U}_{\mathsf{Q}}^{(2)}\hat{N}\hat{U}_{\mathsf{Q}}^{(1)} = e^{i\delta_2}e^{i\phi\hat{\sigma}_3/2}$, which is equivalent to

$$\hat{U}_{\mathsf{Q}}^{(2)} = e^{i\delta_2}e^{i\phi\hat{\sigma}_3/2}\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{N}^\dagger(\beta) \tag{4.11}$$

where $\phi, \delta_2 \in [0, 2\pi)$. By substituting Eq. (4.11) into Eq. (4.9), we obtain

$$e^{i\phi\hat{\sigma}_3/2}\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{N}^\dagger\hat{F}\hat{U}_{\mathsf{Q}}^{(1)}\hat{\rho}_0\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{F}^\dagger\hat{N}\hat{U}_{\mathsf{Q}}^{(1)}e^{-i\phi\hat{\sigma}_3/2} = \hat{\rho}_0. \tag{4.12}$$

Furthermore, Eq. (4.12) can be rewritten as $[\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{N}^\dagger\hat{F}\hat{U}_{\mathsf{Q}}^{(1)}, \hat{\sigma}_3] = 0$, which implies that $\hat{U} := \hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{N}^\dagger\hat{F}\hat{U}_{\mathsf{Q}}^{(1)}$ is a linear combination of $\hat{\mathbb{1}}$ and $\hat{\sigma}_3$. Because $\hat{U}$ is an arbitrary unitary transformation, $\hat{U}$ satisfies $\hat{U}^\dagger\hat{U} = \hat{\mathbb{1}}$. Furthermore, we need to seek $\hat{U}$ satisfying $\hat{U} \neq \pm\hat{\mathbb{1}}$. We consider the case $\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{N}^\dagger\hat{F}\hat{U}_{\mathsf{Q}}^{(1)} = \pm\hat{\sigma}_3$, i.e.,

$$\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{N}^\dagger\hat{F} = \pm\hat{\sigma}_3\hat{U}_{\mathsf{Q}}^{(1)\dagger}. \tag{4.13}$$

The unitary operator $\hat{U}_{\mathsf{Q}}^{(1)}$ can be parameterized as

$$\hat{U}_{\mathsf{Q}}^{(1)} = e^{i\delta_1}e^{i\theta\vec{n}\cdot\hat{\vec{\sigma}}/2} = e^{i\delta_1}\left(\cos\frac{\theta}{2}\hat{\mathbb{1}} + i\sin\frac{\theta}{2}\vec{n}\cdot\hat{\vec{\sigma}}\right), \tag{4.14}$$

with the parameters $\theta \in \mathbb{R}$, $\vec{n} = (n_1, n_2, n_3) \in \mathbb{S}^2 \subset \mathbb{R}^3$, and $\delta_1 \in [0, 2\pi)$. By substituting Eq. (4.14) into Eq. (4.13), we obtain the relation

$$\left[\cos\frac{\theta}{2}\hat{\mathbb{1}} - i\sin\frac{\theta}{2}\Big(n_1\hat{\sigma}_1 + n_2\hat{\sigma}_2 + n_3\hat{\sigma}_3\Big)\right]\hat{N}^\dagger\hat{F} = b\hat{\sigma}_3\left[\cos\frac{\theta}{2}\hat{\mathbb{1}} - i\sin\frac{\theta}{2}\Big(n_1\hat{\sigma}_1 + n_2\hat{\sigma}_2 + n_3\hat{\sigma}_3\Big)\right], \tag{4.15}$$

where $b = \pm 1$. We want to find the parameters θ and $\hat{n}$ satisfying Eq. (4.15). From Eq. (4.6), we have $\hat{N}^\dagger \hat{F} = \cos\frac{\Delta}{2}\hat{\sigma}_1 - \sin\frac{\Delta}{2}\hat{\sigma}_2$, where $\Delta := \alpha - \beta$. By comparing both sides of Eq. (4.15), this equation is satisfied if

$$\begin{cases} \left(n_1 \cos\frac{\Delta}{2} - n_2 \sin\frac{\Delta}{2}\right)\sin\frac{\theta}{2} = b n_3 \sin\frac{\theta}{2}, & (4.16) \\ \left(n_1 \sin\frac{\Delta}{2} + n_2 \cos\frac{\Delta}{2}\right)\sin\frac{\theta}{2} = b\cos\frac{\theta}{2}, & (4.17) \end{cases}$$

which implies $\sin\frac{\theta}{2} \neq 0$. From Eqs. (4.16) and (4.17), we obtain

$$n_2 = b\cot\frac{\theta}{2}\sec\frac{\Delta}{2} - n_1\tan\frac{\Delta}{2}, \quad n_3 = bn_1\sec\frac{\Delta}{2} - \cot\frac{\theta}{2}\tan\frac{\Delta}{2}. \tag{4.18}$$

By substituting these into the constraint $n_1^2 + n_2^2 + n_3^2 = 1$, we obtain

$$n_1^2 - 2bn_1\cot\frac{\theta}{2}\sin\frac{\Delta}{2} + \frac{1}{2}\left[\cot^2\frac{\theta}{2}\left(1+\sin^2\frac{\Delta}{2}\right) - \cos^2\frac{\Delta}{2}\right] = 0, \tag{4.19}$$

and hence,

$$n_1 = b\cot\frac{\theta}{2}\sin\frac{\Delta}{2} + a\cos\frac{\Delta}{2}, \tag{4.20}$$

where $a := \pm\sqrt{\frac{1}{2}\left(1-\cot^2\frac{\theta}{2}\right)}$ and $\left|\cot\frac{\theta}{2}\right| \leq 1$ (i.e., $|\theta| \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$). By substituting Eq. (4.20) into Eq. (4.18), we obtain

$$n_2 = b\cot\frac{\theta}{2}\cos\frac{\Delta}{2} - a\sin\frac{\Delta}{2}, \quad n_3 = ab. \tag{4.21}$$

By combining these with Eqs. (4.11) and (4.14), we obtain the winning strategy for player Q:

$$\hat{U}_{\mathsf{Q}}^{(1)}(\theta,\phi;\alpha,\beta) = e^{i\delta_1}\exp\left[i\frac{\theta}{2}\left(b\cot\frac{\theta}{2}\sin\frac{\Delta}{2} + a\cos\frac{\Delta}{2},\ b\cot\frac{\theta}{2}\cos\frac{\Delta}{2} - a\sin\frac{\Delta}{2},\ ab\right)\cdot\hat{\vec{\sigma}}\right], \tag{4.22}$$

$$\hat{U}_{\mathsf{Q}}^{(2)}(\theta,\phi;\alpha,\beta) = e^{i\delta_2}\hat{N}(\phi)\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{N}^\dagger(\beta) = e^{i\delta_2}e^{i\phi\hat{\sigma}_3/2}\hat{U}_{\mathsf{Q}}^{(1)\dagger}e^{-i\beta\hat{\sigma}_3/2}. \tag{4.23}$$

Even if player P can change the phase, player Q always possesses winning strategies independent of the probability p, with the provision that player Q knows player P's values of α and β. Thus, player Q is always at least advantageous.

4.3.3 Unrestricted strategy and winning counter-strategy

In the above game variants, the operations of player P must be coin flipping or non-flipping operations in the classical sense. Here, we discard this restriction. Player P is allowed to use one of two arbitrary unitary operators, $\hat{U}_{\mathsf{P}}^{(1)}$ and $\hat{U}_{\mathsf{P}}^{(2)}$. They do not necessarily yield definitive heads or tails states when they act on a coin in the heads state. Instead, they can yield superposition states of heads and tails. In this sense, player P also becomes a quantum player. Player P applies $\hat{U}_{\mathsf{P}}^{(1)}$ to the coin with probability p or $\hat{U}_{\mathsf{P}}^{(2)}$ with probability $1-p$. In this section, we seek a winning strategy for player Q.

By using density matrices, the game flow is illustrated as

$$\hat{\rho}_0 \xrightarrow[\hat{U}_{\mathsf{Q}}^{(1)}]{\mathsf{Q}} \hat{\rho}_1 \xrightarrow[\{\hat{U}_{\mathsf{P}}^{(k)}\}_{k=1,2}]{\mathsf{P}} \hat{\rho}_2 \xrightarrow[\hat{U}_{\mathsf{Q}}^{(2)}]{\mathsf{Q}} \hat{\rho}_3. \tag{4.24}$$

The final state of the coin is

$$\hat{\rho}_3 := p\hat{U}_{\mathsf{Q}}^{(2)}\hat{U}_{\mathsf{P}}^{(1)}\hat{U}_{\mathsf{Q}}^{(1)}\hat{\rho}_0\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{U}_{\mathsf{P}}^{(1)\dagger}\hat{U}_{\mathsf{Q}}^{(2)\dagger} + (1-p)\hat{U}_{\mathsf{Q}}^{(2)}\hat{U}_{\mathsf{P}}^{(2)}\hat{U}_{\mathsf{Q}}^{(1)}\hat{\rho}_0\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{U}_{\mathsf{P}}^{(2)\dagger}\hat{U}_{\mathsf{Q}}^{(2)\dagger} \tag{4.25}$$

where $p \in [0,1]$. We would like to find $\hat{U}_{\mathsf{Q}}^{(1)}$ and $\hat{U}_{\mathsf{Q}}^{(2)}$ that yield $\hat{\rho}_3 = \hat{\rho}_0$ for arbitrary p. Via arguments similar to the previous section, we obtain the equation $[\hat{U}_{\mathsf{Q}}^{(2)}\hat{U}_{\mathsf{P}}^{(2)}\hat{U}_{\mathsf{Q}}^{(1)}, \hat{\sigma}_3] = 0$. From this, we have

$$\hat{U}_{\mathsf{Q}}^{(2)} = e^{i\delta_2}e^{i\theta_2\hat{\sigma}_3/2}\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{U}_{\mathsf{P}}^{(2)\dagger}, \tag{4.26}$$

where $\delta_2, \theta_2 \in [0, 2\pi)$. By substituting Eq. (4.26) into Eq. (4.25), we obtain

$$e^{i\theta_2\hat{\sigma}_3/2}\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{U}_{\mathsf{P}}^{(2)\dagger}\hat{U}_{\mathsf{P}}^{(1)}\hat{U}_{\mathsf{Q}}^{(1)}\hat{\rho}_0\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{U}_{\mathsf{P}}^{(1)\dagger}\hat{U}_{\mathsf{P}}^{(2)}\hat{U}_{\mathsf{Q}}^{(1)}e^{-i\theta_2\hat{\sigma}_3/2} = \hat{\rho}_0. \tag{4.27}$$

By using $\hat{\rho}_0 = \frac{\hat{\mathbb{1}}+\hat{\sigma}_3}{2}$, we can rewrite Eq. (4.27) as $[\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{U}_{\mathsf{P}}^{(2)\dagger}\hat{U}_{\mathsf{P}}^{(1)}\hat{U}_{\mathsf{Q}}^{(1)}, \hat{\sigma}_3] = 0$, which implies

$$\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{U}_{\mathsf{P}}^{(2)\dagger}\hat{U}_{\mathsf{P}}^{(1)} = e^{i\delta_3}e^{i\gamma\hat{\sigma}_3/2}\hat{U}_{\mathsf{Q}}^{(1)\dagger}, \tag{4.28}$$

with the parameters $\delta_3, \gamma \in \mathbb{R}$. The unitary operators $\hat{U}_{\mathsf{Q}}^{(1)}$ and $\hat{U}_{\mathsf{P}}^{(k)}$ can be parameterized as

$$\hat{U}_{\mathsf{Q}}^{(1)} = e^{i\delta_1}e^{i\theta_1\vec{n}\cdot\hat{\vec{\sigma}}/2} = e^{i\delta_1}\left(\cos\frac{\theta_1}{2}\hat{\mathbb{1}} + i\sin\frac{\theta_1}{2}\vec{n}\cdot\hat{\vec{\sigma}}\right), \tag{4.29}$$

$$\hat{U}_{\mathsf{P}}^{(k)} = e^{i\xi_k}e^{i\phi_k\vec{m}_k\cdot\hat{\vec{\sigma}}/2} = e^{i\xi_k}\left(\cos\frac{\phi_k}{2}\hat{\mathbb{1}} + i\sin\frac{\phi_k}{2}\vec{m}_k\cdot\hat{\vec{\sigma}}\right), \tag{4.30}$$

with the parameters $k\ (=1,2)$, $\delta_1, \xi_k, \theta_1, \phi_k \in \mathbb{R}$, $\vec{n} := (n_1, n_2, n_3) \in \mathbb{S}^2$, and $\vec{m}_k := (m_{k1}, m_{k2}, m_{k3}) \in \mathbb{S}^2$. Here, we need to obtain $\hat{U}_{\mathsf{Q}}^{(1)}$ satisfying Eq. (4.28). By using the law of spherical trigonometry [82], we can rewrite the left-hand side of Eq. (4.28) as

$$\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{U}_{\mathsf{P}}^{(2)\dagger}\hat{U}_{\mathsf{P}}^{(1)} = e^{i(\xi_1-\xi_2-\delta_1)}e^{i\Phi\vec{\mathfrak{M}}\cdot\hat{\vec{\sigma}}/2} = e^{i(\xi_1-\xi_2-\delta_1)}\left(\cos\frac{\Phi}{2}\hat{\mathbb{1}} + i\sin\frac{\Phi}{2}\vec{\mathfrak{M}}\cdot\hat{\vec{\sigma}}\right), \tag{4.31}$$

where

$$\cos\frac{\varphi}{2} := \cos\frac{\phi_1}{2}\cos\frac{\phi_2}{2} + \vec{m}_1\cdot\vec{m}_2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}, \tag{4.32}$$

$$\vec{M} := \frac{\vec{m}_1\sin\frac{\phi_1}{2}\cos\frac{\phi_2}{2} - \vec{m}_2\cos\frac{\phi_1}{2}\sin\frac{\phi_2}{2} - \vec{m}_1\times\vec{m}_2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}}{\sin\frac{\varphi}{2}} \in \mathbb{S}^2, \tag{4.33}$$

$$\cos\frac{\Phi}{2} := \cos\frac{\theta_1}{2}\cos\frac{\varphi}{2} + \vec{M}\cdot\vec{n}\sin\frac{\theta_1}{2}\sin\frac{\varphi}{2}, \tag{4.34}$$

$$\vec{\mathfrak{M}} := \frac{\vec{M}\sin\frac{\varphi}{2}\cos\frac{\theta_1}{2} - \vec{n}\cos\frac{\varphi}{2}\sin\frac{\theta_1}{2} - \vec{M}\times\vec{n}\sin\frac{\varphi}{2}\sin\frac{\theta_1}{2}}{\sin\frac{\Phi}{2}} \in \mathbb{S}^2. \tag{4.35}$$

Similarly, the right-hand side of Eq. (4.28) is rewritten as

$$e^{i\delta_3}e^{i\gamma\hat{\sigma}_3/2}\hat{U}_{\mathsf{Q}}^{(1)\dagger} = e^{i(\delta_3-\delta_1)}\left(\cos\frac{\Theta}{2}\hat{\mathbb{1}} + i\sin\frac{\Theta}{2}\vec{N}\cdot\hat{\vec{\sigma}}\right), \tag{4.36}$$

where

$$\cos\frac{\Theta}{2} := \cos\frac{\gamma}{2}\cos\frac{\theta_1}{2} + n_3\sin\frac{\gamma}{2}\sin\frac{\theta_1}{2}, \tag{4.37}$$

$$\vec{N} := -\frac{1}{\sin\frac{\Theta}{2}}\begin{pmatrix} \left(n_1\cos\frac{\gamma}{2}+n_2\sin\frac{\gamma}{2}\right)\sin\frac{\theta_1}{2} \\ \left(n_2\cos\frac{\gamma}{2}-n_1\sin\frac{\gamma}{2}\right)\sin\frac{\theta_1}{2} \\ n_3\sin\frac{\theta_1}{2}\cos\frac{\gamma}{2}-\cos\frac{\theta_1}{2}\sin\frac{\gamma}{2} \end{pmatrix} \in \mathbb{S}^2 \tag{4.38}$$

From Eqs. (4.31) and (4.36), we obtain the following relation:

$$\cos\frac{\Phi}{2}\hat{\mathbb{1}} + i\sin\frac{\Phi}{2}\vec{\mathfrak{M}}\cdot\hat{\vec{\sigma}} = e^{i(\delta_3+\xi_2-\xi_1)}\left(\cos\frac{\Theta}{2}\hat{\mathbb{1}} + i\sin\frac{\Theta}{2}\vec{N}\cdot\hat{\vec{\sigma}}\right). \tag{4.39}$$

Because $\cos\frac{\Phi}{2}, \sin\frac{\Phi}{2}, \cos\frac{\Theta}{2}, \sin\frac{\Theta}{2} \in \mathbb{R}$, it must be true that $e^{i(\delta_3+\xi_2-\xi_1)} \in \mathbb{R}$. We choose the value of δ_3 so as to satisfy $\delta_3+\xi_2-\xi_1 \in \pi\mathbb{Z}$. Namely, we find $e^{i(\delta_3+\xi_2-\xi_1)} =: c$, where $c = \pm 1$. The value of c is decided from the start of the game. We obtain a system of linear equations:

$$\begin{cases} \cos\frac{\Phi}{2} = c\cos\frac{\Theta}{2}, & (4.40) \\ \vec{\mathfrak{M}}\sin\frac{\Phi}{2} = c\vec{N}\sin\frac{\Theta}{2}. & (4.41) \end{cases}$$

Since Eqs. (4.40) and (4.41) are equivalent to a system of four linear equations with three unknowns n_1, n_2, and n_3, only three of the four equations are mutually independent. By selecting Eq. (4.41), we obtain the matrix equation:

$$\hat{V}\vec{n} = \cos\frac{\theta_1}{2}\begin{pmatrix} M_1\sin\frac{\varphi}{2} \\ M_2\sin\frac{\varphi}{2} \\ M_3\sin\frac{\varphi}{2} - c\sin\frac{\gamma}{2} \end{pmatrix}, \tag{4.42}$$

where

$$\hat{V} := \sin\frac{\theta_1}{2}\begin{pmatrix} \cos\frac{\varphi}{2}-c\cos\frac{\gamma}{2} & -\left(M_3\sin\frac{\varphi}{2}+c\sin\frac{\gamma}{2}\right) & M_2\sin\frac{\varphi}{2} \\ M_3\sin\frac{\varphi}{2}+c\sin\frac{\gamma}{2} & \cos\frac{\varphi}{2}-c\cos\frac{\gamma}{2} & -M_1\sin\frac{\varphi}{2} \\ -M_2\sin\frac{\varphi}{2} & M_1\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2}-c\cos\frac{\gamma}{2} \end{pmatrix}. \tag{4.43}$$

Notably, Eq. (4.42) can be solved if the inverse matrix $\hat{V}^{-1}$ exists, i.e., if the determinant of matrix V is non-zero. The determinant of matrix $\hat{V}$, is calculated as

$$\det\hat{V} = 2c\sin^3\frac{\theta_1}{2}\left(\cos\frac{\varphi}{2}-c\cos\frac{\gamma}{2}\right)\left(M_3\sin\frac{\varphi}{2}\sin\frac{\gamma}{2}-\cos\frac{\varphi}{2}\cos\frac{\gamma}{2}+c\right). \tag{4.44}$$

We find that winning strategies actually exist for player Q when player P is allowed to use two arbitrary $U(2)$ operations $\hat{U}_{\mathsf{P}}^{(k)}$,

$$\hat{U}_{\mathsf{Q}}^{(1)} = e^{i\delta_1}e^{i\theta_1\vec{n}\cdot\hat{\vec{\sigma}}/2}, \quad \hat{U}_{\mathsf{Q}}^{(2)} = e^{i\delta_2}e^{i\theta_2\hat{\sigma}_3/2}\hat{U}_{\mathsf{Q}}^{(1)\dagger}\hat{U}_{\mathsf{P}}^{(2)\dagger}, \tag{4.45}$$

where the Bloch vector for player Q's winning strategies is given by

$$\vec{n} = -\frac{\cot\frac{\theta_1}{2}}{M_3\sin\frac{\varphi}{2}\sin\frac{\gamma}{2}-\cos\frac{\varphi}{2}\cos\frac{\gamma}{2}+c}\begin{pmatrix} \left(M_1\cos\frac{\gamma}{2}-M_2\sin\frac{\gamma}{2}\right)\sin\frac{\varphi}{2} \\ \left(M_1\sin\frac{\gamma}{2}+M_2\cos\frac{\gamma}{2}\right)\sin\frac{\varphi}{2} \\ M_3\sin\frac{\varphi}{2}\cos\frac{\gamma}{2}+\cos\frac{\varphi}{2}\sin\frac{\gamma}{2} \end{pmatrix}. \tag{4.46}$$

Player Q should choose parameters θ_1, γ, and $c = \pm 1$ such that Eq. (4.44) is non-zero so as to make Eq. (4.46) converge, with the provision that player Q knows player P's values of φ [Eq. (4.32)] and $\vec{M}$ [Eq. (4.33)]. When player Q operates the strategies in Eq. (4.45), player Q always wins independent of the probability p. Thus, player Q is always at least advantageous.

4.3.4 Multiple strategy and winning counter-strategy

Finally, we propose an even more general game. We allow player P to choose one of ℓ elements $\{\hat{U}_{\mathrm{P}}^{(j)}\}_{j=1,\cdots,\ell}$ of the group $U(2)$ as his/her operation. We call this a multiple strategy. We seek to evaluate the existence of a winning strategy for player Q in this game. If all the operators given to player P are mutually commutative, a simultaneous eigenvector of these operators exists, and this vector is invariant under operations of player P. Hence, in this case, player Q always wins by transforming the initial state vector to the simultaneous eigenvector at the first step and transforming it back to the initial state at the final step.

We can also consider when player P has one of ℓ elements of the group $U(2)$, which are divided into two types of unitary operations. We allow player P to choose s modified flipping operations $\{\hat{F}(\alpha_{k_{\mathrm{F}}})\}_{k_{\mathrm{F}}=1,\cdots,s} := \{e^{i\alpha_{k_{\mathrm{F}}}\hat{\sigma}_3/2}\hat{\sigma}_1\}_{k_{\mathrm{F}}=1,\cdots,s}$ and $\ell - s$ modified non-flipping operations $\{\hat{N}(\beta_{k_{\mathrm{N}}})\}_{k_{\mathrm{N}}=s+1,\cdots,\ell} := \{e^{i\beta_{k_{\mathrm{N}}}\hat{\sigma}_3/2}\}_{k_{\mathrm{N}}=s+1,\cdots,\ell}$. Player P has at least one of each type of unitary operation, i.e., $1 \le s \le \ell - 1$.

If all of player P's modified flipping operations $\{\hat{F}(\alpha_{k_{\mathrm{F}}})\}_{k_{\mathrm{F}}}$ are mutually commutative and all of their modified non-flipping operations $\{\hat{N}(\beta_{k_{\mathrm{N}}})\}_{k_{\mathrm{N}}}$ are equal to identity $\hat{\mathbb{1}}$, i.e., $\beta_{k_{\mathrm{N}}} \in \pi\mathbb{Z}$ for all k_{N}, we can easily deduce that player Q always has a complete set of winning strategies because simultaneous eigenstates exist for player P, similar to Sec. 4.2.2.

If all $\{\hat{F}(\alpha_{k_{\mathrm{F}}})\}_{k_{\mathrm{F}}}$ are mutually commutative and all $\{\hat{N}(\beta_{k_{\mathrm{N}}})\}_{k_{\mathrm{N}}}$ are not equal to identity $\hat{\mathbb{1}}$, no winning strategies exist for player Q in general. However, only for $s = 1$ or $\ell - 1$, winning strategies do exist for player Q because examinations such as that in Sec. 4.3.2 are always available.

4.4 Concluding remarks of chapter 4

Meyer proposed a quantum version of the penny flip game in which player P is allowed to use only classical operations, i.e., flipping or non-flipping, on the coin whereas player Q is allowed to use any unitary transformation. Meyer showed that there is a winning strategy for player Q; player Q always wins by transforming the initial coin state to a superposition state that is the simultaneous eigenvector of the flipping operation $\hat{\sigma}_1$ and the non-flipping operation $\hat{\mathbb{1}}$, and is hence invariant under any operation of player P. Therefore, player Q is always predominant.

In this chapter, we proposed and analyzed four generalizations of the quantum penny flip game.

In the first generalization, we allow player P to use $\hat{\sigma}_1$ and $\hat{\sigma}_3$ as his/her operations; in contrast to Meyer's game, these operations are non-commutative and do not admit simultaneous eigenvectors. Even in this game, we found a simple example and a complete set of winning strategies for player Q. After the first winning operation of player Q, the two possible operations of player P yield equivalent states; therefore, player Q can restore the coin state into the same initial state through his second operation. This scheme is common among all the winning strategies. Then, player Q is always predominant even if player P's operations are non-commutative.

In the second generalization, we allow player P to use phase-changing flipping and non-flipping operations. In this game, we also found a complete set of winning strategies for player Q, with the provision that player Q knows the values of the parameters α and β in player P's operations. Thus, player Q is always at least advantageous.

In the third generalization, we allow player P to use two arbitrary unitary operations. Even in this game, player Q has a set of winning strategies with a suitable choice of parameters. This fact implies that non-commutativity, phase, and the number of generators of unitary operations are completely unrelated to the existence of winning strategies. Thus, player Q is always at least advantageous.

In the fourth generalization, we allow player P to use $\ell \geq 3$ elements of phase-changing flipping and non-flipping operations. Even in this game, player Q has a set of winning strategies if some conditions are satisfied. Meyer's original game and our first, second, and third generalizations are special cases of this fourth generalized game. Consequently, we found that even if player P has non-Abelian mixed strategies, there were cases in which player Q has a set of winning strategies.

In these games, the purpose of player Q was to restore the initial state at the end whereas the purpose of player P was to change the coin from the initial state. In this context, a winning strategy for player Q is equivalent to restoration of the initial state against player P. Furthermore, the conditions for the existence of winning strategies were similar to the classification of interference such that the initial state can be always restored.

Chapter 5

Concluding Remarks

In this dissertation, we have discussed quantum control through two approaches, adiabatic processes and gaming flows, as continuous- and discrete-time quantum transports.

In chapter 3, by applying Husimi's method, we have derived the propagator of a quantum parametric oscillator (QPO) with the counter-diabatic Hamiltonian realizing an adiabatic evolution in an arbitrary short time [Eq. (3.9)]. This propagator is written with two linearly independent solutions of a corresponding classical parametric oscillator (CPO). By using this propagator, we defined two kinds of transition probabilities, $P^{m,n}_{t,t_0}$ [Eq. (3.7)] and $\bar{P}^{m,n}_{t,t_0}$ [Eq. (3.44)]. The former is a transition probability between the instantaneous eigenstates of the adiabatic Hamiltonian, and the latter is a transition probability between the instantaneous eigenstates of the TT Hamiltonian (the adiabatic Hamiltonian plus the counter-diabatic Hamiltonian). By introducing two measures of adiabaticity, Q^{TT}_t [Eq. (3.21)] and $\bar{Q}^{\mathrm{TT}}_t$ [Eq. (3.59)], we obtained concise expressions of the generating functions for these transition probabilities. From the analysis of Q^{TT}_t, we found one aspect of shortcuts to adiabaticity (STA): Whereas the state itself is given by the exact solution $|\Psi(t)\rangle = e^{i\xi_{n,t}}|n;\vec{\lambda}_t\rangle$ that preserves the quantum number, the transition probability $P^{m,n}_{t,t_0}$ becomes $\delta_{m,n}$ only at the endpoints of the time evolution because of $Q^{\mathrm{TT}}_t = 1$ only at these points. Namely, its intermediate state may be highly *diabatic* with respect to the instantaneous eigenstate of the TT Hamiltonian. From the analysis of $\bar{Q}^{\mathrm{TT}}_t$, which consists of the linear summation of the adiabatic invariant or the Ermakov-Lewis (EL) invariant of the CPO, we found that the transition probability $\bar{P}^{m,n}_{t,t_0}$ always becomes $\delta_{m,n}$ because of $\bar{Q}^{\mathrm{TT}}_t = 1$ all the time during the time evolution. We also introduced the quantum EL invariant [Eq. (3.73)] written with the quantum Wronskian [Eqs. (3.67) and (3.68)]. In the presence of the counter-diabatic Hamiltonian, we found that the quantum EL invariant is proportional to the Lewis-Riesenfeld (LR) invariant. This LR invariant may be interpreted as a *quantum version of* the adiabatic invariant [Eq. (3.74)].

The results in chapter 3 may have broader applications. In statistical mechanics, there exists a method possessing a similar motivation to STA, which is called shortcuts to isothermality (STI) [83]. The other methods such as engineered swift equilibration approach [84] and stochastic shortcuts using flow-fields [51] are also similar to STI. STA realizes an adiabatic process in an arbitrary short time, whereas STI realizes an isothermal process from an equilibrium state to another one in an arbitrary short time. In STI, the time-evolution equation is the Fokker-Planck equation which determines the probability distribution of a Brownian particle, instead of the Schödinger equation. No matter how fast the system evolves, STI enables the shape of the probability distribution of the Brownian particle to be preserved exactly. Suppose that system evolves with an original Hamiltonian including a set of time-dependent external parameters. STI gives us an auxiliary Hamiltonian that cancels out deviation from an instantaneous equilibrium distribution of the original Hamiltonian. We should impose the boundary condition such that the auxiliary Hamiltonian vanishes at two endpoints of the driving process. STI inherits the idea of STA for isolate quantum systems [13, 15, 16, 22–24, 85–87]. Since the Fokker-Planck equation is formally equivalent to the Schrödinger equation, we may solve it by applying Husimi's method. It would be interesting to find a measure of isothermality and several invariants characterizing STI.

In chapter 4, we have analyzed the quantum penny flip game as a problem of quantum error correction. In the original game, the malicious third party that schemes to disturb quantum state had two commutative operations. Since there existed a simultaneous eigenstate of his/her operations, we were able to invalidate one of his/her operations. This result could be trivial. We then gave the third party "power," i.e., non-commutativity. There exists no longer a simultaneous eigenstate. Even in this setting, we showed that the third party's operation could be neutralized. We have challenged how much we could do without using any ancillae, i.e., without relying on quantum entanglement at all. I hope that this research will be a key to achieve quantum error correction with minimal ancillae.

It is my expectation that this work provides a new perspective on other quantum games in various fields such as finance [88, 89]. In the quantum prisoner's dilemma [38], the quantum Hawk-Dove game [90], and the quantum stag hunt game [91], replacing a classical player's operations with a restricted set of quantum operations could change properties of the game. Especially in the quantum penny flip game [37] and our modified games, the goals of the two players are to either save the initial state or disturb it. To guarantee victory, player Q needs to set a suitable intermediate state. This situation can also be represented as quantum information processing, that is, player Q can be regarded as a sender/receiver of information and player P can be regarded as an eavesdropper.

Appendix A

Supplementary Materials for Chapter 3

We list some complicated calculus and also include essential techniques. This appendix is based on the contents from Ref. [57].

A.1 Derivation of energy eigenfunction in Eq. (3.6)

From the definition of the vacuum state $\hat{b}_t|0;\Omega_t\rangle \equiv 0$ and by using Eq. (3.5), we can easily obtain the normalized energy eigenfunction of the vacuum state for position x as

$$\langle x|0;\Omega_t\rangle = \left(\frac{M\Omega_t}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{\zeta_t M\Omega_t}{2\hbar}x^2\right). \tag{A.1}$$

For a function $f(x)$, by using the relation

$$\left(\sqrt{a}x - \frac{1}{\sqrt{a}}\frac{\partial}{\partial x}\right)^n f(x) = (-1)^n e^{ax^2/2}\frac{1}{a^{n/2}}\frac{\partial^n}{\partial x^n}(e^{-ax^2/2}f(x)), \tag{A.2}$$

we can also obtain the normalized energy eigenfunction of the n-th excited state in Eq. (3.6) as follows:

$$\begin{aligned}
\langle x|n;\Omega_t\rangle &= \frac{1}{\sqrt{n!}}\langle x|\hat{b}_t^{\dagger n}|0;\Omega_t\rangle \\
&= \frac{\zeta^{*n/2}}{\sqrt{2^n n!}}\left(\sqrt{\frac{\zeta_t^* M\Omega_t}{\hbar}}x - \sqrt{\frac{\hbar}{\zeta_t^* M\Omega_t}}\frac{\partial}{\partial x}\right)^n \langle x|0;\Omega_t\rangle \\
&= \frac{1}{\sqrt{2^n n!}}(-1)^n \exp\left(\frac{\zeta_t^* M\Omega_t}{2\hbar}x^2\right)\left(\frac{\hbar}{M\Omega_t}\right)^{n/2}\frac{\partial^n}{\partial x^n}\left[\exp\left(-\frac{\zeta_t^* M\Omega_t}{2\hbar}x^2\right)\langle x|0;\Omega_t\rangle\right] \\
&= \frac{1}{\sqrt{2^n n!}}\left(\frac{M\Omega_t}{\pi\hbar}\right)^{1/4}(-1)^n \exp\left(\frac{M\Omega_t}{\hbar}x^2\right)\left(\frac{\hbar}{M\Omega_t}\right)^{n/2} \\
&\quad \times \left[\frac{\partial^n}{\partial x^n}\exp\left(-\frac{M\Omega_t}{\hbar}x^2\right)\right]\exp\left(-\frac{\zeta_t M\Omega_t}{2\hbar}x^2\right) \\
&= \frac{1}{\sqrt{2^n n!}}\left(\frac{M\Omega_t}{\pi\hbar}\right)^{1/4}\mathrm{H}_n\left(\sqrt{\frac{M\Omega_t}{\hbar}}x\right)\exp\left(-\frac{\zeta_t M\Omega_t}{2\hbar}x^2\right),
\end{aligned} \tag{A.3}$$

where $\mathrm{H}_n(\cdot)$ are the n-th-degree Hermite polynomials [Eq. (2.110)].

A.2 Derivation of propagator in Eq. (3.9)

Based on Husimi's method [55], we derive Eq. (3.9). For the TT Hamiltonian of the QPO in Eq. (3.2), the x-representation of the wave function

$$\langle x|\Psi(t)\rangle = \int_{\mathbb{R}} \mathrm{d}x_0 U^{\mathrm{TT}}_{t,t_0}(x|x_0)\langle x_0|\Psi(t_0)\rangle, \qquad t\in[t_0,\infty), \tag{A.4}$$

satisfies the Schröginger equation

$$i\hbar\frac{\partial}{\partial t}\langle x|\Psi(t)\rangle = \langle x|\hat{H}^{\mathrm{TT}}_t|\Psi(t)\rangle, \tag{A.5}$$

where $U^{\mathrm{TT}}_{t,t_0}(x|x_0)$ is the propagator. Here, we assume the following Gaussian form of the propagator as the specific anstaz [55]:

$$U^{\mathrm{TT}}_{t,t_0}(x|x_0) = \sqrt{\frac{M}{2\pi i\hbar\mu_t}}e^{i(\alpha_t x^2+\beta_t x x_0+\gamma_t x_0^2)/\hbar}, \tag{A.6}$$

where the coefficients μ_t, α_t, β_t, and γ_t are time-dependent real-valued functions. By substituting Eq. (B.18) with Eq. (A.6) into the Schröginger equation given by Eq. (B.19), we find that four coupled ordinary differential equations (ODEs) for the coefficients μ_t, α_t, β_t, and γ_t follow:

$$\begin{cases} \alpha_t = \dfrac{M}{2}\left(\dfrac{\dot{\mu}_t}{\mu_t} + \dfrac{1}{2}\dfrac{\dot{\omega}_t}{\omega_t}\right), & \text{(A.7)}\\ \dot{\alpha}_t + \dfrac{2}{M}\alpha_t^2 - \dfrac{\dot{\omega}_t}{\omega_t}\alpha_t + \dfrac{M}{2}\omega_t^2 = 0, & \text{(A.8)}\\ \dot{\beta}_t + \left(\dfrac{2}{M}\alpha_t - \dfrac{1}{2}\dfrac{\dot{\omega}_t}{\omega_t}\right)\beta_t = 0, & \text{(A.9)}\\ \dot{\gamma}_t + \dfrac{\beta_t^2}{2M} = 0. & \text{(A.10)} \end{cases}$$

By substituting Eq. (A.7) into Eq. (A.8), we obtain

$$\ddot{\mu}_t + \tilde{\Omega}_t^2\mu_t = 0. \tag{A.11}$$

By substituting Eq. (A.7) into Eq. (A.9) and by solving Eq. (A.9) with respect to β_t, we have

$$\beta_t = \frac{C_1}{\mu_t}, \tag{A.12}$$

with the integral constant C_1. Next, by substituting Eq. (A.12) into Eq. (A.10) and by solving Eq. (A.10) with respect to γ_t, we also have

$$\gamma_t = -\frac{C_1^2}{2M}\int_{t_0}^{t}\frac{\mathrm{d}\tau}{\mu_\tau^2} + C_2, \tag{A.13}$$

with the integral constant C_2. From Eqs. (A.7), (A.12), and (A.13), the dynamics of α_t, β_t, and γ_t can be determined by using the solution of μ_t satisfying Eq. (A.11).

We now determine the initial condition of μ_t in Eq. (A.11) and the integral constants C_1 and C_2 according to the following argument: to represent the wave function $\langle x|\Psi(t)\rangle$ from an arbitrary initial wave function $\langle x_0|\Psi(t_0)\rangle$, the propagator $U^{\mathrm{TT}}_{t,t_0}(x|x_0)$ in Eq. (A.6) needs to satisfy $\lim_{t\to t_0+0} U^{\mathrm{TT}}_{t,t_0}(x|x_0) = \delta(x-x_0)$. Therefore, it is natural to assume the following asymptotic form of the propagator :

$$U^{\mathrm{TT}}_{t,t_0}(x|x_0)|_{t\simeq t_0} = \sqrt{\frac{M}{2\pi i\hbar(t-t_0)}}\exp\left[\frac{iM}{2\hbar}\frac{(x-x_0)^2}{t-t_0}\right]e^{iF(x,x_0)/\hbar}(1+O(t-t_0)), \tag{A.14}$$

where $F(x,x_0)$ is a function that satisfies $F(x_0,x_0) = 0$. Althrough a Taylor expansion of μ_t with respect to t around t_0 and from Eqs. (A.6) and (A.14), we have

$$\begin{aligned}\mu_t|_{t\simeq t_0} &= \mu_{t_0} + \dot{\mu}_{t_0}(t-t_0) + O((t-t_0)^2)\\ &= t - t_0 + O((t-t_0)^2),\end{aligned} \tag{A.15}$$

from which we can determine μ_{t_0} and $\dot{\mu}_{t_0}$ as

$$\mu_{t_0} = 0, \quad \dot{\mu}_{t_0} = 1, \tag{A.16}$$

as the initial condition of Eq. (A.11). By using $\ddot{\mu}_{t_0} = 0$ obtained from Eqs. (A.11) and (A.16), we modify Eq. (A.15) as

$$\mu_t|_{t\simeq t_0} = t - t_0 + O((t-t_0)^3). \tag{A.17}$$

From Eq. (A.7), we can find

$$\alpha_t|_{t\simeq t_0} = \frac{M}{2}\left(\frac{1}{t-t_0} + \frac{1}{2}\frac{\dot{\omega}_{t_0}}{\omega_{t_0}}\right) + O(t-t_0). \tag{A.18}$$

From Eqs. (A.12) and (A.17), we then find

$$\beta_t|_{t\simeq t_0} = \frac{C_1}{t-t_0} + O(t-t_0). \tag{A.19}$$

To determine the asymptotic form of γ_t in Eq. (A.13), we introduce a solution ν_t as a linearly independent solution of μ_t, which satisfies the same equation as Eq. (A.11) but with a different initial condition:

$$\nu_{t_0} = 1, \quad \dot{\nu}_{t_0} = 0, \tag{A.20}$$

where the Wronskian $W_t = \dot{\mu}_t\nu_t - \mu_t\dot{\nu}_t$ is unity as in Eq. (3.13). From this Wronskian, we obtain

$$\frac{\nu_t}{\mu_t} = -\int_{t_0}^{t}\frac{\mathrm{d}\tau}{\mu_\tau^2}. \tag{A.21}$$

From Eqs. (A.13) and (A.21), we have

$$\gamma_t = \frac{C_1^2}{2M}\frac{\nu_t}{\mu_t} + C_2. \tag{A.22}$$

From Eqs. (A.15), (A.16), and (A.20), we obtain the asymptotic form of γ_t as

$$\gamma_t|_{t\simeq t_0} = \frac{C_1^2}{2M}\frac{1}{t-t_0} + C_2 + O(t-t_0). \tag{A.23}$$

By substituting Eqs. (A.15), (A.18), (A.19), and (A.23) into Eq. (A.6), we have the following asymptotic form of the propagator:

$$\begin{aligned} U^{\mathrm{TT}}_{t,t_0}(x|x_0)|_{t\simeq t_0} &= \sqrt{\frac{M}{2\pi i\hbar(t-t_0)}}\exp\left[\frac{i}{\hbar}\left\{\frac{M}{2}\left(\frac{1}{t-t_0}+\frac{1}{2}\frac{\dot\omega_{t_0}}{\omega_{t_0}}\right)x^2\right.\right. \\ &\quad\left.\left.+\frac{C_1}{t-t_0}xx_0+\left(\frac{C_1^2}{2M}\frac{1}{t-t_0}+C_2\right)x_0^2\right\}+O(t-t_0)\right] \\ &= \sqrt{\frac{M}{2\pi i\hbar(t-t_0)}}\exp\left[\frac{iM}{2\hbar}\frac{(x-x_0)^2}{t-t_0}\right]e^{iF(x,x_0)/\hbar}(1+O(t-t_0)), \end{aligned} \tag{A.24}$$

where the function $F(x,x_0)$ is

$$F(x,x_0) = \frac{M}{4}\frac{\dot\omega_{t_0}}{\omega_{t_0}}x^2 + \frac{C_1+M}{t-t_0}xx_0 + \left(\frac{C_1^2-M^2}{2M}\frac{1}{t-t_0}+C_2\right)x_0^2. \tag{A.25}$$

Since $F(x_0,x_0) = 0$ is required in the limit of $t \to t_0 + 0$, we must set $C_1 = -M$ and $C_2 = -\frac{M}{4}\frac{\dot\omega_{t_0}}{\omega_{t_0}}$. We then obtain

$$\beta_t = -\frac{M}{\mu_t}, \tag{A.26}$$

$$\gamma_t = \frac{M}{2}\left(\frac{\nu_t}{\mu_t} - \frac{1}{2}\frac{\dot\omega_{t_0}}{\omega_{t_0}}\right). \tag{A.27}$$

By substituting Eqs. (A.7), (A.26) and (A.27) into Eq. (A.6), we finally obtain the propagator given by Eq. (3.9).

The Husimi's method can be applied to a generalized quantum parametric oscillator (GQPO) [Eq. (B.1)] as given in Appendix B.

A.3 Derivation of probability generating function in Eq. (3.15)

By using Mehler's formula and the energy eigenfunction given by Eq. (A.3) (Eq. (3.6)), we obtain the following relation:

$$\sum_{n=0}^{\infty} z^n\langle n;\Omega_t|x\rangle\langle y|n;\Omega_t\rangle = \sqrt{\frac{M\Omega_t}{\pi\hbar(1-z^2)}}\exp\left[-\frac{M\Omega_t}{2\hbar}\frac{(1+z^2)(x^2+y^2)-4zxy}{1-z^2} - \frac{iM}{4\hbar}\frac{\dot\omega_t}{\omega_t}(x^2-y^2)\right]. \tag{A.28}$$

By using Eq. (A.28), we can calculate the probability generating function as

$$\begin{aligned} \mathcal{P}^{u,v}_{t,t_0} &= \sum_{n,m=0}^{\infty} u^n v^m P^{m,n}_{t,t_0} \\ &= \sum_{n,m=0}^{\infty} u^n v^m \left|\iint_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}x_0 \langle m;\Omega_t|x\rangle U^{\mathrm{TT}}_{t,t_0}(x|x_0)\langle x_0|n;\Omega_{t_0}\rangle\right|^2 \\ &= \iiiint_{\mathbb{R}^4} \mathrm{d}x\mathrm{d}x_0\mathrm{d}x'\mathrm{d}x'_0 U^{\mathrm{TT}*}_{t,t_0}(x|x_0)U^{\mathrm{TT}}_{t,t_0}(x'|x'_0) \\ &\quad\times \sum_{m=0}^{\infty} v^m\langle m;\Omega_t|x\rangle\langle x'|m;\Omega_t\rangle \sum_{n=0}^{\infty} u^n\langle n;\Omega_{t_0}|x_0\rangle\langle x'_0|n;\Omega_{t_0}\rangle \end{aligned}$$

$$= \frac{2}{\mu_t}\left(\frac{M}{2\pi\hbar}\right)^2\sqrt{\frac{\Omega_t\Omega_{t_0}}{(1-u^2)(1-v^2)}}\int_{\mathbb{R}^4}\mathrm{d}\vec{x}\exp\left(-\frac{M}{2\hbar}\vec{x}\cdot A\vec{x}\right)$$
$$= \frac{2}{\mu_t}\sqrt{\frac{\Omega_t\Omega_{t_0}}{(1-u^2)(1-v^2)\det A}}, \tag{A.29}$$

where we defined

$$\vec{x} := \begin{pmatrix} x \\ x_0 \\ x' \\ x'_0 \end{pmatrix}, \quad A := \begin{pmatrix} \frac{1+v^2}{1-v^2}\Omega_t + i\left(\frac{\dot{\mu}_t}{\mu_t} + \frac{\dot{\omega}_t}{\omega_t}\right) & -\frac{i}{\mu_t} & -\frac{2v}{1-v^2}\Omega_t & 0 \\ -\frac{i}{\mu_t} & \frac{1+u^2}{1-u^2}\Omega_{t_0} + i\frac{\nu_t}{\mu_t} & 0 & -\frac{2u}{1-u^2}\Omega_{t_0} \\ -\frac{2v}{1-v^2}\Omega_t & 0 & \frac{1+v^2}{1-v^2}\Omega_t - i\left(\frac{\dot{\mu}_t}{\mu_t} + \frac{\dot{\omega}_t}{\omega_t}\right) & \frac{i}{\mu_t} \\ 0 & -\frac{2u}{1-u^2}\Omega_{t_0} & \frac{i}{\mu_t} & \frac{1+u^2}{1-u^2}\Omega_{t_0} - i\frac{\nu_t}{\mu_t} \end{pmatrix}. \tag{A.30}$$

and used the following formula of the Gaussian integral:

$$\int_{\mathbb{R}^n}\mathrm{d}\vec{x}e^{-a\vec{x}\cdot A\vec{x}} = \sqrt{\frac{(\pi/a)^n}{\det A}}, \tag{A.31}$$

provided $a > 0$, $\vec{x} \in \mathbb{R}^n$, and the n-by-n matrix A is symmetric. By using the Wronskian given by Eq. (3.13), we obtain

$$\det A = \frac{1}{\mu_t^2}\frac{2\Omega_t\Omega_{t_0}}{(1-u^2)(1-v^2)}\left[Q_t^{\mathrm{TT}}(1-u^2)(1-v^2) + (1+u^2)(1+v^2) - 4uv\right]. \tag{A.32}$$

Here, Q_t^{TT} is given as Eq. (3.16). Then, we finally obtain Eq. (3.15) by substituting Eq. (A.32) into Eq. (A.29).

A.4 Derivation of Ermakov equation in Eq. (3.29) from Wronskian

Here, we derive the Ermakov equation in Eq. (3.29) [61–63]. By differentiating Eq. (3.27) with respect to time t and using Eqs. (3.10) and (3.13), we can show the following relation:

$$0 = \frac{\mathrm{d}W_t^{(\mu)}}{\mathrm{d}t}$$
$$= -\frac{\rho_t\mu_t}{\sqrt{\Omega_{t_0}^{-1}\rho_t^2 - \mu_t^2}}\left[\frac{\ddot{\rho}_t\mu_t - \rho_t\ddot{\mu}_t}{\mu_t} - \frac{(\dot{\rho}_t\mu_t - \rho_t\dot{\mu}_t)^2}{\rho_t(\Omega_{t_0}^{-1}\rho_t^2 - \mu_t^2)}\right]$$
$$= \frac{\mu_t W_t^{(\mu)}}{\dot{\rho}_t\mu_t - \rho_t\dot{\mu}_t}\left(\ddot{\rho}_t + \tilde{\Omega}_t^2\rho_t - \frac{W_t^{(\mu)2}}{\rho_t^3}\right). \tag{A.33}$$

From the above, we have the Ermakov equation of ρ_t for μ_t:

$$\ddot{\rho}_t + \tilde{\Omega}_t^2\rho_t = \frac{W_t^{(\mu)2}}{\rho_t^3}. \tag{A.34}$$

Similarly, by differentiating Eq. (3.28) with respect to time t and using Eqs. (3.11) and (3.13), we can obtain the following relation:

$$0 = \frac{\mathrm{d}W_t^{(\nu)}}{\mathrm{d}t}$$

$$= \frac{\rho_t \nu_t}{\sqrt{\Omega_{t_0}\rho_t^2 - \nu_t^2}} \left[\frac{\ddot{\rho}_t \nu_t - \rho_t \ddot{\nu}_t}{\nu_t} - \frac{(\dot{\rho}_t \nu_t - \rho_t \dot{\nu}_t)^2}{\rho_t(\Omega_{t_0}\rho_t^2 - \nu_t^2)} \right]$$

$$= \frac{\nu_t W_t^{(\nu)}}{\dot{\rho}_t \nu_t - \rho_t \dot{\nu}_t} \left(\ddot{\rho}_t + \tilde{\Omega}_t^2 \rho_t - \frac{W_t^{(\nu)2}}{\rho_t^3} \right). \tag{A.35}$$

From the above, we have the Ermakov equation of ρ_t for ν_t:

$$\ddot{\rho}_t + \tilde{\Omega}_t^2 \rho_t = \frac{W_t^{(\nu)2}}{\rho_t^3}. \tag{A.36}$$

Because $W_t = W_t^{(\mu)} = W_t^{(\nu)} = 1$, we obtain Eq. (3.29) from Eqs. (A.34) and (A.36).

A.5 Derivation of Ermakov equation (3.33) by phase-amplitude method

According to the phase-amplitude method [59], we rewrite μ_t and ν_t by using ρ_t defined in Eq. (3.26), from which a phase function can naturally be defined. Since $\dot{\rho}_t \mu_t - \rho_t \dot{\mu}_t = -\rho_t^2 \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mu_t}{\rho_t}$, the Wronskian given by Eq. (3.27) can be rewritten as

$$W_t^{(\mu)} = \frac{\rho_t^2}{\sqrt{\Omega_{t_0}^{-1} - \left(\frac{\mu_t}{\rho_t}\right)^2}} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mu_t}{\rho_t}. \tag{A.37}$$

This can easily be integrated to obtain

$$\mu_t = \frac{\rho_t}{\sqrt{\Omega_{t_0}}} \sin \int_{t_0}^{t} \frac{W_\tau^{(\mu)}}{\rho_\tau^2} \mathrm{d}\tau. \tag{A.38}$$

From Eqs. (3.13), (3.26), and (A.38), we also obtain

$$\nu_t = \sqrt{\Omega_{t_0}} \rho_t \cos \int_{t_0}^{t} \frac{W_\tau^{(\nu)}}{\rho_\tau^2} \mathrm{d}\tau. \tag{A.39}$$

We now introduce the phase function θ_t defined as

$$\theta_t := \int_{t_0}^{t} \frac{W_\tau^{(\mu)}}{\rho_\tau^2} \mathrm{d}\tau = \int_{t_0}^{t} \frac{W_\tau^{(\nu)}}{\rho_\tau^2} \mathrm{d}\tau. \tag{A.40}$$

By differentiating Eq. (A.40) with respect to time t, we can represent the Wronskian with ρ_t and θ_t as

$$W_t = \rho_t^2 \dot{\theta}_t. \tag{A.41}$$

By differentiating ρ_t in Eq. (A.41) with respect to time t twice, we have

$$\ddot{\rho}_t + \left(-\frac{3}{4}\frac{\ddot{\theta}_t^2}{\dot{\theta}_t^2} + \frac{1}{2}\frac{\dddot{\theta}_t}{\dot{\theta}_t} \right) \rho_t = 0. \tag{A.42}$$

Adding $\dot{\theta}_t^2 \rho_t$ to both sides of the above equation and defining

$$f_t := \sqrt{\dot{\theta}_t^2 - \frac{3}{4}\frac{\ddot{\theta}_t^2}{\dot{\theta}_t^2} + \frac{1}{2}\frac{\dddot{\theta}_t}{\dot{\theta}_t}}, \tag{A.43}$$

we obtain the Ermakov equation:

$$\ddot{\rho}_t + f_t^2 \rho_t = \dot{\theta}_t^2 \rho_t = \left(\frac{W_t}{\rho_t^2}\right)^2 \rho_t = \frac{W_t^2}{\rho_t^3}, \tag{A.44}$$

where we used Eq. (A.41).

A.6 Derivation of Wronskian in Eqs. (3.41) and (3.42)

By using Eqs. (3.37), (3.38), and (3.40), we obtain Eq. (3.41) as follows [65]:

$$\begin{aligned} W_t^{(\mu)} &= \frac{2I_t^{\mathrm{EL}(\mu)}}{W_t^{(\mu)}} \\ &= \frac{\omega_{t_0}}{W_t^{(\mu)}} \left[(\dot{\rho}_t \mu_t - \rho_t \dot{\mu}_t)^2 + W_t^{(\mu)2} \left(\frac{\mu_t}{\rho_t}\right)^2 \right] \\ &= \omega_{t_0} \left[\left(\dot{\mu}_t - \frac{\dot{\rho}_t}{\rho_t}\mu_t\right)^2 \frac{\rho_t^2}{W_t^{(\mu)}} + \mu_t^2 \frac{W_t^{(\mu)}}{\rho_t^2} \right] \\ &= \omega_{t_0} \left[\left(\dot{\mu}_t + \frac{\mu_t}{2}\frac{\dot{\omega}_t}{\omega_t}\right)^2 \frac{1}{\omega_t} + \mu_t^2 \omega_t \right] \\ &= \frac{2\omega_{t_0}}{\omega_t} \left[\mathcal{E}_t^{(\mu)} + \left(\dot{\mu}_t + \frac{\mu_t}{2}\frac{\dot{\omega}_t}{\omega_t}\right) \frac{\mu_t}{2}\frac{\dot{\omega}_t}{\omega_t} \right]. \end{aligned} \tag{A.45}$$

Similarly, by using Eqs. (3.37), (3.39), and (3.40), we obtain Eq. (3.42) as follows:

$$\begin{aligned} W_t^{(\nu)} &= \frac{2I_t^{\mathrm{EL}(\nu)}}{W_t^{(\nu)}} \\ &= \frac{1}{\omega_{t_0} W_t^{(\nu)}} \left[(\dot{\rho}_t \nu_t - \rho_t \dot{\nu}_t)^2 + W_t^{(\nu)2} \left(\frac{\nu_t}{\rho_t}\right)^2 \right] \\ &= \frac{1}{\omega_{t_0}} \left[\left(\dot{\nu}_t - \frac{\dot{\rho}_t}{\rho_t}\nu_t\right)^2 \frac{\rho_t^2}{W_t^{(\nu)}} + \nu_t^2 \frac{W_t^{(\nu)}}{\rho_t^2} \right] \\ &= \frac{1}{\omega_{t_0}} \left[\left(\dot{\nu}_t + \frac{\nu_t}{2}\frac{\dot{\omega}_t}{\omega_t}\right)^2 \frac{1}{\omega_t} + \nu_t^2 \omega_t \right] \\ &= \frac{2}{\omega_t \omega_{t_0}} \left[\mathcal{E}_t^{(\nu)} + \left(\dot{\nu}_t + \frac{\nu_t}{2}\frac{\dot{\omega}_t}{\omega_t}\right) \frac{\nu_t}{2}\frac{\dot{\omega}_t}{\omega_t} \right]. \end{aligned} \tag{A.46}$$

A.7 Exact explicit form of transition probabilities in Eq. (3.7)

Here, we derive an exact explicit form of the transition probabilities $P_{t,t_0}^{m,n}$ in Eq. (3.7) as a function of time t through the parameter Q_t^{TT} according to [55, 92]. Since the probability generating function $\mathcal{P}_{t,t_0}^{u,v}$ cannot be expanded in powers of u and v in an explicit series, we introduce the following transition amplitude $U_{t,t_0}^{m,n}$ [55]:

$$U_{t,t_0}^{m,n} := \iint_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}x_0 \langle m; \Omega_t | x \rangle U_{t,t_0}^{\mathrm{TT}}(x|x_0) \langle x_0 | n; \Omega_{t_0} \rangle, \tag{A.47}$$

where $P^{m,n}_{t,t_0} = |U^{m,n}_{t,t_0}|^2$. By using the generating function of the n-th-degree Hermite polynomials

$$e^{2xz-z^2} = \sum_{n=0}^{\infty} \frac{\mathrm{H}_n(x)}{n!} z^n, \tag{A.48}$$

we have

$$\sum_{n=0}^{\infty} \sqrt{\frac{2^n}{n!}} z^n \langle x|n;\Omega_t\rangle = \left(\frac{M\Omega_t}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{\zeta_t M\Omega_t}{2\hbar}x^2\right) \exp\left(2\sqrt{\frac{M\Omega_t}{\hbar}}xz - z^2\right). \tag{A.49}$$

We calculate the generating function of the transition amplitude $U^{m,n}_{t,t_0}$ as follows:

$$\begin{aligned}
\mathcal{U}^{u,v}_{t,t_0} &:= \sum_{m,n=0}^{\infty} \sqrt{\frac{2^{m+n-1}}{m!n!}} u^n v^m U^{m,n}_{t,t_0} \\
&= \frac{1}{\sqrt{2}} \iint_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}x_0 U^{\mathrm{TT}}_{t,t_0}(x|x_0) \sum_{m=0}^{\infty} \sqrt{\frac{2^m}{m!}} v^m \langle m;\Omega_t|x\rangle \sum_{n=0}^{\infty} \sqrt{\frac{2^n}{n!}} u^n \langle x_0|n;\Omega_{t_0}\rangle \\
&= \frac{M}{2\pi\hbar} \frac{(\Omega_t\Omega_{t_0})^{1/4}}{\sqrt{i\mu_t}} e^{-(u^2+v^2)} \int_{\mathbb{R}^2} \mathrm{d}\vec{x} \exp\Big[-\frac{M}{2\hbar} \underbrace{(\vec{x}\cdot B\vec{x} - 2\vec{b}\cdot\vec{x})}_{(\vec{x}-B^{-1}\vec{b})\cdot B(\vec{x}-B^{-1}\vec{b})-\vec{b}\cdot B^{-1}\vec{b}}\Big] \\
&= \frac{(\Omega_t\Omega_{t_0})^{1/4}}{\sqrt{i\mu_t \det B}} e^{-(u^2+v^2)} \exp\left(\frac{M}{2\hbar}\vec{b}\cdot B^{-1}\vec{b}\right) \\
&= \frac{(\Omega_t\Omega_{t_0})^{1/4}}{\sqrt{i\chi^{(-)}_t}} \exp\left(\frac{\chi^{(+)}_t u^2 - 4i\sqrt{\Omega_t\Omega_{t_0}}uv + \chi^{(+)*}_t v^2}{\chi^{(-)}_t}\right).
\end{aligned} \tag{A.50}$$

In the above, we have defined the following quantities:

$$\vec{x} := \begin{pmatrix} x \\ x_0 \end{pmatrix}, \quad \vec{b} := 2\sqrt{\frac{\hbar}{M}} \begin{pmatrix} \sqrt{\Omega_t}v \\ \sqrt{\Omega_{t_0}}u \end{pmatrix}, \tag{A.51}$$

$$B := \begin{pmatrix} \Omega_t - i\frac{\dot{\mu}_t}{\mu_t} & \frac{i}{\mu_t} \\ \frac{i}{\mu_t} & \Omega_{t_0} - i\frac{\nu_t}{\mu_t} \end{pmatrix}, \tag{A.52}$$

$$\chi^{(\pm)}_t := \Omega_{t_0}(\Omega_t\mu_t - i\dot{\mu}_t) \pm i(\Omega_t\nu_t - i\dot{\nu}_t). \tag{A.53}$$

Note that $\det B = \frac{\chi^{(-)}_t}{\mu_t}$ and $|\chi^{(\pm)}_t|^2 = 2\Omega_t\Omega_{t_0}(Q_t \mp 1)$, where

$$Q_t := \Omega_{t_0}\frac{\mathcal{E}^{(\mu)}_t}{\Omega_t} + \Omega^{-1}_{t_0}\frac{\mathcal{E}^{(\nu)}_t}{\Omega_t}, \tag{A.54}$$

which agrees with Q^{TT}_t in Eq. (3.23) if $\dot{\omega}_{t_0} = 0$ is imposed. For a usual QPO in the absence of $\hat{H}^{\mathrm{cd}}_t$, Q_t is identified as Husimi's measure of adiabaticity Q^*_t. From the symmetric property of $\mathcal{P}^{-u,-v}_{t,t_0} = \mathcal{P}^{u,v}_{t,t_0}$, we find $P^{m,n}_{t,t_0} = |U^{m,n}_{t,t_0}|^2 = 0$ if m and n are of different parity. Then, by expanding Eq. (A.50) explicitly in powers of u and v, we can obtain the matrix elements of $U^{m,n}_{t,t_0}$ as

$$U^{m,n}_{t,t_0} = (2\Omega_t\Omega_{t_0})^{1/4} \sqrt{\frac{m!n!}{2^{m+n-1}}} \sqrt{\frac{\chi^{(+)*m}_t \chi^{(+)n}_t}{i\chi^{(-)m+n+1}_t}} \sum_{s\geq 0}^{\min(m,n)} \frac{2^s\left(\frac{2}{1-Q_t}\right)^{s/2}}{s!\left(\frac{m-s}{2}\right)!\left(\frac{n-s}{2}\right)!}. \tag{A.55}$$

By applying the selection rule $m-n \in 2\mathbb{Z}$, the number s satisfies $s \in 2\mathbb{N}_0$ for $m,n \in 2\mathbb{N}_0$, and

$s \in 2\mathbb{N}_0 + 1$ for $m, n \in 2\mathbb{N}_0 + 1$. The explicit expression for the matrix elements of $U_{t,t_0}^{m,n}$ reads for even elements and odd elements [92], respectively, as

$$U_{t,t_0}^{2k,2l} = \frac{(2\Omega_t\Omega_{t_0})^{1/4}}{k!l!}\sqrt{\frac{(2k)!(2l)!}{2^{2(k+l)-1}}}\frac{\chi_t^{(+)*k}\chi_t^{(+)l}}{\chi_t^{(-)k+l}\sqrt{i\chi_t^{(-)}}}{}_2F_1\left(-k,-l;\frac{1}{2};\frac{2}{1-Q_t}\right), \tag{A.56}$$

$$U_{t,t_0}^{2k+1,2l+1} = -\frac{(2\Omega_t\Omega_{t_0})^{1/4}}{k!l!}\sqrt{\frac{(2k+1)!(2l+1)!}{2^{2(k+l)+1}}}\frac{\chi_t^{(+)*k}\chi_t^{(+)l}}{\chi_t^{(-)k+l}\sqrt{i\chi_t^{(-)}}}\frac{|\chi_t^{(+)}|}{\chi_t^{(-)}}\sqrt{\frac{2}{1-Q_t}}{}_2F_1\left(-k,-l;\frac{3}{2};\frac{2}{1-Q_t}\right), \tag{A.57}$$

where $k, l \in \mathbb{N}_0$ and ${}_2F_1$ is Gauss's hypergeometric function. We finally obtain the explicit closed form of transition probabilities as functions of the parameter Q_t, which reads for even elements and odd elements, respectively, as

$$P_{t,t_0}^{2k,2l} = \frac{(2k-1)!!(2l-1)!!}{(2k)!!(2l)!!}\sqrt{\frac{2}{Q_t+1}}\left(\frac{Q_t-1}{Q_t+1}\right)^{k+l}{}_2F_1{}^2\left(-k,-l;\frac{1}{2};\frac{2}{1-Q_t}\right), \tag{A.58}$$

$$P_{t,t_0}^{2k+1,2l+1} = \frac{(2k+1)!!(2l+1)!!}{(2k)!!(2l)!!}\left(\frac{2}{Q_t+1}\right)^{3/2}\left(\frac{Q_t-1}{Q_t+1}\right)^{k+l}{}_2F_1{}^2\left(-k,-l;\frac{3}{2};\frac{2}{1-Q_t}\right). \tag{A.59}$$

Appendix B

Generalized Quantum Parametric Oscillator

We consider the Hamiltonian of a generalized quantum parametric oscillator (GQPO)

$$\hat{H}_t = a_t\hat{p}^2 + b_t\hat{x}^2 + c_t\frac{\hat{x}\hat{p} + \hat{p}\hat{x}}{2} + f_t\hat{x} + g_t\hat{p} + h_t, \tag{B.1}$$

with a_t, b_t, c_t, f_t, g_t, and h_t being arbitrary real functions of time t. Since we wish to consider a case that the kinetic term $a_t\hat{p}^2$ in Eq. (B.1) always exists, we set $a_t > 0$ for arbitrary time t. The Schrödinger equation is given by

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = \hat{H}_t|\Psi(t)\rangle. \tag{B.2}$$

B.1 Generalized Husimi's method

B.1.1 Boson operator and eigenstate

By applying a technique in Ref. [93], we can rewrite the Hamiltonian Eq. (B.1) as

$$\hat{H}_t = \hbar\Omega_t\left(\hat{A}_t^\dagger\hat{A}_t + \frac{1}{2}\right) + f_t\hat{x} + g_t\hat{p} + h_t \tag{B.3}$$

$$= \hbar\Omega_t\left(\hat{B}_t^\dagger\hat{B}_t + |\Lambda_t|^2 + \frac{1}{2}\right), \tag{B.4}$$

where we defined

$$\Omega_t := \sqrt{4a_tb_t - c_t^2}, \tag{B.5}$$

$$\hat{A}_t := \frac{1}{2}\sqrt{\frac{\Omega_t}{\hbar a_t}}\left(\hat{x} + i\frac{2a_t\hat{p} + c_t\hat{x}}{\Omega_t}\right), \tag{B.6}$$

$$\hat{B}_t := \hat{A}_t + \Lambda_t, \tag{B.7}$$

$$\Lambda_t := \frac{1}{\sqrt{\hbar\Omega_t a_t}}\left(\frac{a_tf_t - c_tg_t}{\Omega_t} + i\frac{g_t}{2}\right) \in \mathbb{C}, \tag{B.8}$$

with $h_t := \hbar\Omega_t|\Lambda_t|^2$. The Bosonic operators $\hat{A}_t$ and $\hat{B}_t$ satisfy the Boson commutation relations $[\hat{A}_t, \hat{A}_t^\dagger] = [\hat{B}_t, \hat{B}_t^\dagger] = 1$. We here define a basis created by $\hat{B}_t^\dagger$ as $\sqrt{n!}|n;\Omega_t\rangle := \hat{B}_t^{\dagger n}|0;\Omega_t\rangle$ and do a vacuum as $\hat{B}_t|0;\Omega_t\rangle = 0$. The basis is an instantaneous eigenstate $|n;\Omega_t\rangle$,

$$\hat{H}_t|n;\Omega_t\rangle = E_{n,\Lambda,t}|n;\Omega_t\rangle, \tag{B.9}$$

where we defined an instantaneous eigenenergy as

$$E_{n,\Lambda,t} := \hbar\Omega_t\left(n + |\Lambda_t|^2 + \frac{1}{2}\right). \tag{B.10}$$

From the definition of the vacuum, we have the following condition as

$$0 = \langle x|\hat{B}_t|0;\Omega_t\rangle = \sqrt{\frac{\hbar a_t}{\Omega_t}}\left[\frac{\partial}{\partial x} + \left(1 + i\frac{c_t}{\Omega_t}\right)\frac{\Omega_t}{2\hbar a_t}x + \sqrt{\frac{\Omega_t}{\hbar a_t}}\Lambda_t\right]\langle x|0;\Omega_t\rangle. \tag{B.11}$$

We then obtain

$$\begin{aligned}\langle x|0;\Omega_t\rangle &= \mathcal{N}_{0,t}\exp\left[-\left\{\left(1 + i\frac{c_t}{\Omega_t}\right)\frac{\Omega_t}{4\hbar a_t}x^2 + \sqrt{\frac{\Omega_t}{\hbar a_t}}\Lambda_t x\right\}\right] \\ &= \mathcal{N}_{0,t}e^{\mathrm{Re}^2\{\Lambda_t\}}\exp\left[-\frac{\Omega_t}{4\hbar a_t}\left(x + 2\sqrt{\frac{\hbar a_t}{\Omega_t}}\mathrm{Re}\{\Lambda_t\}\right)^2\right. \\ &\quad \left. + i\left\{\frac{\Omega_t}{c_t}\mathrm{Im}^2\{\Lambda_t\} - \frac{c_t}{4\hbar a_t}\left(x + 2\frac{\sqrt{\hbar\Omega_t}a_t}{c_t}\mathrm{Im}\{\Lambda_t\}\right)^2\right\}\right].\end{aligned} \tag{B.12}$$

Since the normalized eigenfunction of the ground state for position x satisfies

$$1 = \|\langle x|0;\Omega_t\rangle\|^2 = |\mathcal{N}_{0,t}|^2 e^{2\mathrm{Re}^2\{\Lambda_t\}}\sqrt{\frac{2\pi\hbar a_t}{\Omega_t}}, \tag{B.13}$$

we have

$$\mathcal{N}_{0,t} = \left(\frac{\Omega_t}{2\pi\hbar a_t}\right)^{1/4} e^{-\mathrm{Re}^2\{\Lambda_t\}}. \tag{B.14}$$

Hence, the eigenfunction of the ground state is given by

$$\begin{aligned}\langle x|0;\Omega_t\rangle &= \left(\frac{\Omega_t}{2\pi\hbar a_t}\right)^{1/4} e^{-\mathrm{Re}^2\{\Lambda_t\}}\exp\left[-\left\{\left(1 + i\frac{c_t}{\Omega_t}\right)\frac{\Omega_t}{4\hbar a_t}x^2 + \sqrt{\frac{\Omega_t}{\hbar a_t}}\Lambda_t x\right\}\right] \\ &= \left(\frac{\Omega_t}{2\pi\hbar a_t}\right)^{1/4}\exp\left[-\frac{\Omega_t}{4\hbar a_t}\left(x + 2\sqrt{\frac{\hbar a_t}{\Omega_t}}\mathrm{Re}\{\Lambda_t\}\right)^2\right. \\ &\quad \left. + i\left\{\frac{\Omega_t}{c_t}\mathrm{Im}^2\{\Lambda_t\} - \frac{c_t}{4\hbar a_t}\left(x + 2\frac{\sqrt{\hbar\Omega_t}a_t}{c_t}\mathrm{Im}\{\Lambda_t\}\right)^2\right\}\right].\end{aligned} \tag{B.15}$$

By using Eqs. (A.2) and (B.15), binomial expansion, and the following formula

$$\sum_{n=0}^{m}\binom{m}{n}(2x)^{m-n}\mathrm{H}_n(y) = \mathrm{H}_m(x+y), \tag{B.16}$$

we obtain the eigenfunction of the n-th excited state of the GQPO [Eq. (B.1)] as

$$\begin{aligned}\langle x|n;\Omega(t)\rangle &= \frac{1}{\sqrt{n!}}\langle x|\hat{B}_t^{\dagger n}|0;\Omega(t)\rangle = \frac{1}{\sqrt{n!}}\langle x|(\hat{A}_t^\dagger + \Lambda_t^*)^n|0;\Omega(t)\rangle \\ &= \frac{1}{\sqrt{n!}}\left[\frac{1}{2}\left(1 - i\frac{c_t}{\Omega_t}\right)\sqrt{\frac{\Omega_t}{\hbar a_t}}x - \sqrt{\frac{\hbar a_t}{\Omega_t}}\frac{\partial}{\partial x} + \Lambda_t^*\right]^n\langle x|0;\Omega_t\rangle \\ &= \frac{1}{\sqrt{n!}}\sum_{l=0}^{n}\binom{n}{l}\Lambda_t^{*n-l}\left[\frac{1}{2}\left(1 - i\frac{c_t}{\Omega_t}\right)\sqrt{\frac{\Omega_t}{\hbar a_t}}x - \sqrt{\frac{\hbar a_t}{\Omega_t}}\frac{\partial}{\partial x}\right]^l\langle x|0;\Omega_t\rangle\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2^n n!}} \sum_{l=0}^{n} \binom{n}{l} (\sqrt{2}\Lambda_t^*)^{n-l} \left(1 - i\frac{c_t}{\Omega_t}\right)^{l/2} \left[\sqrt{\left(1 - i\frac{c_t}{\Omega_t}\right)\frac{\Omega_t}{2\hbar a_t}}\, x - \frac{1}{\sqrt{\left(1 - i\frac{c_t}{\Omega_t}\right)\frac{\Omega_t}{2\hbar a_t}}}\frac{\partial}{\partial x}\right]^l \langle x|0;\Omega_t\rangle \\
&= \frac{1}{\sqrt{2^n n!}} \exp\left[\left(1 - i\frac{c_t}{\Omega_t}\right)\frac{\Omega_t}{4\hbar a_t}x^2\right] \sum_{l=0}^{n} \binom{n}{l} (\sqrt{2}\Lambda_t^*)^{n-l} \\
&\quad \times (-1)^l \frac{1}{\left(\frac{\Omega_t}{2\hbar a_t}\right)^{l/2}} \frac{\partial^l}{\partial x^l} \left\{ \exp\left[-\left(1 - i\frac{c_t}{\Omega_t}\right)\frac{\Omega_t}{4\hbar a_t}x^2\right] \langle x|0;\Omega_t\rangle \right\} \\
&= \frac{1}{\sqrt{2^n n!}} \left(\frac{\Omega_t}{2\pi\hbar a_t}\right)^{1/4} e^{-\mathrm{Re}^2\{\Lambda_t\} + \Lambda_t^2/2} \exp\left[\left(1 - i\frac{c_t}{\Omega_t}\right)\frac{\Omega_t}{4\hbar a_t}x^2\right] \sum_{l=0}^{n} \binom{n}{l} (\sqrt{2}\Lambda_t^*)^{n-l} \\
&\quad \times (-1)^l \frac{1}{\left(\frac{\Omega_t}{2\hbar a_t}\right)^{l/2}} \frac{\partial^l}{\partial x^l} \exp\left[-\frac{\Omega_t}{2\hbar a_t}\left(x + \sqrt{\frac{\hbar a_t}{\Omega_t}}\Lambda_t\right)^2\right] \\
&= \frac{1}{\sqrt{2^n n!}} \left(\frac{\Omega_t}{2\pi\hbar a_t}\right)^{1/4} \sum_{l=0}^{n} \binom{n}{l} (\sqrt{2}\Lambda_t^*)^{n-l} \mathrm{H}_l\left(\sqrt{\frac{\Omega_t}{2\hbar a_t}}\left(x + \sqrt{\frac{\hbar a_t}{\Omega_t}}\Lambda_t\right)\right) \\
&\quad \times e^{-\mathrm{Re}^2\{\Lambda_t\}} \exp\left[-\left\{\left(1 + i\frac{c_t}{\Omega_t}\right)\frac{\Omega_t}{4\hbar a_t}x^2 + \sqrt{\frac{\hbar a_t}{\Omega_t}}\Lambda_t x\right\}\right] \\
&= \frac{1}{\sqrt{2^n n!}} \left(\frac{\Omega_t}{2\pi\hbar a_t}\right)^{1/4} \mathrm{H}_n\left(\sqrt{\frac{\Omega_t}{2\hbar a_t}}\left(x + 2\sqrt{\frac{\hbar a_t}{\Omega_t}}\mathrm{Re}\{\Lambda_t\}\right)\right) \\
&\quad \times e^{-\mathrm{Re}^2\{\Lambda_t\}} \exp\left[-\left\{\left(1 + i\frac{c_t}{\Omega_t}\right)\frac{\Omega_t}{4\hbar a_t}x^2 + \sqrt{\frac{\hbar a_t}{\Omega_t}}\Lambda_t x\right\}\right] \\
&= \frac{1}{\sqrt{2^n n!}} \left(\frac{\Omega_t}{2\pi\hbar a_t}\right)^{1/4} \mathrm{H}_n\left(\sqrt{\frac{\Omega_t}{2\hbar a_t}}\left(x + 2\sqrt{\frac{\hbar a_t}{\Omega_t}}\mathrm{Re}\{\Lambda_t\}\right)\right) \\
&\quad \times \exp\left[-\frac{\Omega_t}{4\hbar a_t}\left(x + 2\sqrt{\frac{\hbar a_t}{\Omega_t}}\mathrm{Re}\{\Lambda_t\}\right)^2 + i\left\{\frac{\Omega_t}{c_t}\mathrm{Im}^2\{\Lambda_t\} - \frac{c_t}{4\hbar a_t}\left(x + 2\frac{\sqrt{\hbar\Omega_t a_t}}{c_t}\mathrm{Im}\{\Lambda_t\}\right)^2\right\}\right].
\end{aligned} \tag{B.17}
$$

B.1.2 Propagator of generalized quantum parametric oscillator

Let us consider the propagator $U_{t,t_0}(x|x_0)$ defined as

$$
\langle x|\Psi(t)\rangle = \int_{\mathbb{R}} \mathrm{d}x_0 U_{t,t_0}(x|x_0)\langle x_0|\Psi(t_0)\rangle, \qquad t \in [t_0, \infty), \tag{B.18}
$$

which satisfies the Schröginger equation

$$
i\hbar\frac{\partial}{\partial t}\langle x|\Psi(t)\rangle = \langle x|\hat{H}_t|\Psi(t)\rangle. \tag{B.19}
$$

This is a non-autonomous and inhomogeneous diffusion-type equation for the GQPOs. If $t < t_0$, the propagator $U_{t,t_0}(x|x_0)$ is zero. We assume the specific ansatz of the propagator:

$$
U_{t,t_0}(x|x_0) = \frac{1}{\sqrt{4\pi i\hbar a_t \mu_t}} e^{i(\alpha_t x^2 + \beta_t x x_0 + \gamma_t x_0^2 + \zeta_t x + \eta_t x_0 + \kappa_t)/\hbar}, \tag{B.20}
$$

where we introduced the time-dependent coefficients α_t, β_t, γ_t, ζ_t, η_t, κ_t, and μ_t. By substituting Eq. (B.20) into Eq. (B.19), we have

$$
0 = \Big\{ [\dot{\alpha}_t + 2\alpha_t(2\alpha_t a_t + c_t) + b_t]x^2 + [\dot{\beta}_t + \beta_t(4\alpha_t a_t + c_t)]x x_0 + (\dot{\gamma}_t + \beta_t^2 a_t)x_0^2
$$

$$+ [\dot{\zeta}_t + \zeta_t(4\alpha_t a_t + c_t) + 2\alpha_t g_t + f_t]x + [\dot{\eta}_t + \beta_t(2\zeta_t a_t + g_t)]x_0$$
$$+ \dot{\kappa}_t + \zeta_t(\zeta_t a_t + g_t) + h_t - \frac{i\hbar}{2}\left(4\alpha_t a_t + c_t - \frac{\dot{a}_t}{a_t} - \frac{\dot{\mu}_t}{\mu_t}\right)\Bigg\} U_{t,t_0}(x|x_0). \tag{B.21}$$

Eq. (B.21) can be reduced to the following seven coupled ordinary differential equations (ODEs):

$$\begin{cases} \dot{\alpha}_t + 2\alpha_t(2\alpha_t a_t + c_t) + b_t = 0, & \text{(B.22)} \\ \dot{\beta}_t + (4\alpha_t a_t + c_t)\beta_t = 0, & \text{(B.23)} \\ \dot{\gamma}_t + a_t\beta_t^2 = 0, & \text{(B.24)} \\ \dot{\zeta}_t + (4\alpha_t a_t + c_t)\zeta_t + f_t + 2\alpha_t g_t = 0, & \text{(B.25)} \\ \dot{\eta}_t + (g_t + 2\zeta_t a_t)\beta_t = 0, & \text{(B.26)} \\ \dot{\kappa}_t + (g_t + \zeta_t a_t)\zeta_t + h_t = 0, & \text{(B.27)} \\ 4\alpha_t a_t + c_t - \dfrac{\dot{a}_t}{a_t} - \dfrac{\dot{\mu}_t}{\mu_t} = 0. & \text{(B.28)} \end{cases}$$

From Eq. (B.28), we have

$$\alpha_t = \frac{1}{4a_t}\left(\frac{\dot{a}_t}{a_t} + \frac{\dot{\mu}_t}{\mu_t} - c_t\right), \tag{B.29}$$

$$\dot{\alpha}_t = \frac{1}{4a_t}\left[\frac{\ddot{a}_t}{a_t} - \left(\frac{\dot{a}_t}{a_t}\right)^2 + \frac{\ddot{\mu}_t}{\mu_t} - \left(\frac{\dot{\mu}_t}{\mu_t}\right)^2 - \frac{\dot{a}_t}{a_t}\left(\frac{\dot{a}_t}{a_t} + \frac{\dot{\mu}_t}{\mu_t} - c_t\right) - \dot{c}_t\right]. \tag{B.30}$$

By substituting Eq. (B.29) into Eq. (B.22), we have an EoM of a damped CPO as

$$\ddot{\mu}_t + \frac{\dot{a}_t}{a_t}\dot{\mu}_t + \tilde{\Omega}_t^2\mu_t = 0, \tag{B.31}$$

where we defined the new angular frequency as

$$\tilde{\Omega}_t := \sqrt{\Omega_t^2 - \dot{c}_t + \left(c_t - \frac{\dot{a}_t}{a_t}\right)\frac{\dot{a}_t}{a_t} + \frac{\ddot{a}_t}{a_t}}, \tag{B.32}$$

with the angular frequency Ω_t [Eq. (B.5)]. By substituting Eq. (B.29) into Eqs. (B.23) and (B.25), we can reduce Eqs. (B.23)–(B.27) to the following equations, which can be solved explicitly:

$$\dot{\beta}_t + \left(\frac{\dot{a}_t}{a_t} + \frac{\dot{\mu}_t}{\mu_t}\right)\beta_t = 0, \qquad \therefore\ \beta_t = \frac{A_1}{a_t\mu_t}, \tag{B.33}$$

$$\dot{\gamma}_t + \frac{A_1^2}{a_t\mu_t^2} = 0, \qquad \therefore\ \gamma_t = A_2 - A_1^2\int_{t_0}^{t}\frac{d\tau}{a_\tau\mu_\tau^2}, \tag{B.34}$$

$$\dot{\zeta}_t + \left(\frac{\dot{a}_t}{a_t} + \frac{\dot{\mu}_t}{\mu_t}\right)\zeta_t + f_t + 2\alpha_t g_t = 0, \qquad \therefore\ \zeta_t = \frac{1}{a_t\mu_t}\left[A_3 - \int_{t_0}^{t} d\tau\, a_\tau\mu_\tau(f_\tau + 2g_\tau\alpha_\tau)\right], \tag{B.35}$$

$$\dot{\eta}_t + A_1\frac{g_t + 2\zeta_t a_t}{a_t\mu_t} = 0, \qquad \therefore\ \eta_t = A_4 - A_1\int_{t_0}^{t} d\tau\frac{g_\tau + 2\zeta_\tau a_\tau}{a_\tau\mu_\tau}, \tag{B.36}$$

$$\dot{\kappa}_t + (g_t + \zeta_t a_t)\zeta_t + h_t = 0, \qquad \therefore\ \kappa_t = A_5 - \int_{t_0}^{t} d\tau[(g_\tau + \zeta_\tau a_\tau)\zeta_\tau + h_\tau], \tag{B.37}$$

provided that $\{A_i\}_{i=1,\cdots,5}$ are integral constants.

For the propagator with the Hamiltonian in Eq. (B.1), the asymptotic form of the solution $U_{t,t_0}(x|x_0)$ has been shown as [94]

$$\begin{aligned} U_{t,t_0}(x|x_0)|_{t\simeq t_0} &= \frac{1}{\sqrt{4\pi i\hbar a_{t_0}(t-t_0)}}\exp\Bigg[\frac{i}{\hbar}\frac{1}{4a_{t_0}}\Bigg\{\left(\frac{1}{t-t_0}-\frac{1}{2}\frac{\dot{a}_{t_0}}{a_{t_0}}-c_{t_0}\right)x^2 \\ &\quad -2\left(\frac{1}{t-t_0}-\frac{1}{2}\frac{\dot{a}_{t_0}}{a_{t_0}}\right)xx_0+\left(\frac{1}{t-t_0}-\frac{1}{2}\frac{\dot{a}_{t_0}}{a_{t_0}}+c_{t_0}\right)x_0^2-2g_{t_0}(x-x_0)\Bigg\}+O(t-t_0)\Bigg] \\ &= \frac{1}{\sqrt{4\pi i\hbar a_{t_0}(t-t_0)}}\exp\left[\frac{i}{\hbar}\frac{(x-x_0)^2}{4a_{t_0}}\right] \\ &\quad \times\exp\left[-\frac{i}{\hbar}\frac{1}{4a_{t_0}}\left\{\frac{1}{2}\frac{\dot{a}_{t_0}}{a_{t_0}}(x-x_0)^2+c_{t_0}(x^2-x_0^2)+2g_{t_0}(x-x_0)\right\}+O(t-t_0)\right]. \end{aligned} \tag{B.38}$$

We can confirm the consistency of the propagator at the initial time:

$$\begin{aligned} U_{t,t_0}(x|x_0)|_{t\to t_0+0} &= \delta(x-x_0)\exp\left[-\frac{i}{\hbar}\frac{1}{4a_{t_0}}\left\{\frac{1}{2}\frac{\dot{a}_{t_0}}{a_{t_0}}(x-x_0)^2+c_{t_0}(x^2-x_0^2)+2g_{t_0}(x-x_0)\right\}\right] \\ &= \delta(x-x_0). \end{aligned} \tag{B.39}$$

Thus, we find the asymptotic relations of the time-dependent coefficients $\alpha_t, \beta_t, \gamma_t, \zeta_t, \eta_t, \kappa_t$, and μ_t:

$$\begin{cases} \alpha_t|_{t\simeq t_0} = \dfrac{1}{4a_{t_0}}\left(\dfrac{1}{t-t_0}-\dfrac{1}{2}\dfrac{\dot{a}_{t_0}}{a_{t_0}}-c_{t_0}\right)+O(t-t_0), & \text{(B.40)} \\ \beta_t|_{t\simeq t_0} = -\dfrac{1}{2a_{t_0}}\left(\dfrac{1}{t-t_0}-\dfrac{1}{2}\dfrac{\dot{a}_{t_0}}{a_{t_0}}\right)+O(t-t_0), & \text{(B.41)} \\ \gamma_t|_{t\simeq t_0} = \dfrac{1}{4a_{t_0}}\left(\dfrac{1}{t-t_0}-\dfrac{1}{2}\dfrac{\dot{a}_{t_0}}{a_{t_0}}+c_{t_0}\right)+O(t-t_0), & \text{(B.42)} \\ \zeta_t|_{t\simeq t_0} = -2g_{t_0}+O(t-t_0), & \text{(B.43)} \\ \eta_t|_{t\simeq t_0} = 2g_{t_0}+O(t-t_0), & \text{(B.44)} \\ \kappa_t|_{t\simeq t_0} = O(t-t_0), & \text{(B.45)} \\ \mu_t|_{t\simeq t_0} = t-t_0+O^2(t-t_0). & \text{(B.46)} \end{cases}$$

From Eq. (B.46), we find the initial conditions of μ_t as

$$\mu_t|_{t\simeq t_0} \xrightarrow{t\to t_0+0} \mu_{t_0} = 0, \tag{B.47}$$

$$\dot{\mu}_t|_{t\simeq t_0} = 1+O(t-t_0) \xrightarrow{t\to t_0+0} \dot{\mu}_{t_0} = 1. \tag{B.48}$$

Hence, by solving Eq. (B.31) with respect to μ_t under the initial condition (B.47) and (B.48), we obtain the solutions of Eqs. (B.33)–(B.37) in succession.

Let ν_t be an another solution of Eq. (B.31) under the following initial condition

$$\nu_{t_0} = 1, \quad \dot{\nu}_{t_0} = 0. \tag{B.49}$$

Note that μ_t has the dimension of time, whereas ν_t is dimensionless. Let us consider the Wronskian $W_t := \dot{\mu}_t\nu_t - \mu_t\dot{\nu}_t$. Since $\frac{\mathrm{d}}{\mathrm{d}t}W_t = -\frac{\dot{a}_t}{a_t}W_t$, $W_{t_0} = 1$, and $a_t > 0$, we have

$$W_t = \frac{a_{t_0}}{a_t} > 0. \tag{B.50}$$

By using Eq. (B.50), we have

$$\frac{\nu_t}{\mu_t} = -a_{t_0}\int_{t_0}^{t}\frac{d\tau}{a_\tau\mu_\tau^2}. \tag{B.51}$$

By comparing Eq. (B.33) with Eq. (B.41), we have to set the integral constant as $A_1 = \frac{1}{2}\left(\frac{\mu_{t_0}}{2}\frac{\dot{a}_{t_0}}{a_{t_0}} - 1\right)$, then we get

$$\beta_t = \frac{1}{2a_t\mu_t}\left(\frac{\mu_{t_0}}{2}\frac{\dot{a}_{t_0}}{a_{t_0}} - 1\right). \tag{B.52}$$

From Eqs. (B.34) and (B.51), we have

$$\gamma_t = A_2 + \frac{1}{4a_{t_0}}\left(\frac{\mu_{t_0}}{2}\frac{\dot{a}_{t_0}}{a_{t_0}} - 1\right)^2\frac{\nu_t}{\mu_t}. \tag{B.53}$$

By comparing Eq. (B.42) with Eq. (B.53), we have to set the integral constant as $A_2 = \frac{1}{4a_{t_0}}\left(c_{t_0} - \frac{1}{2}\frac{\dot{a}_{t_0}}{a_{t_0}}\right)$, then we get

$$\gamma_t = \frac{1}{4a_{t_0}}\left[\left(\frac{\mu_{t_0}}{2}\frac{\dot{a}_{t_0}}{a_{t_0}} - 1\right)^2\frac{\nu_t}{\mu_t} - \frac{1}{2}\frac{\dot{a}_{t_0}}{a_{t_0}} + c_{t_0}\right]. \tag{B.54}$$

We then have to set $A_3 = -2a_{t_0}g_{t_0}\mu_{t_0}$, $A_4 = 2g_{t_0}$ and $A_5 = 0$. Hence, we obtain

$$\zeta_t = -\frac{1}{a_t\mu_t}\left\{2a_{t_0}g_{t_0}\mu_{t_0} + \int_{t_0}^{t}d\tau\,\mu_\tau\left[a_\tau f_\tau + \frac{g_\tau}{2}\left(\frac{\dot{a}_\tau}{a_\tau} + \frac{\dot{\mu}_\tau}{\mu_\tau} - c_\tau\right)\right]\right\}, \tag{B.55}$$

$$\eta_t = 2g_{t_0} + \frac{1}{2}\left(1 - \frac{\mu_{t_0}}{2}\frac{\dot{a}_{t_0}}{a_{t_0}}\right)\int_{t_0}^{t}d\tau\,\frac{g_\tau + 2a_\tau\zeta_\tau}{a_\tau\mu_\tau}, \tag{B.56}$$

$$\kappa_t = -\int_{t_0}^{t}d\tau[(g_\tau + a_\tau\zeta_\tau)\zeta_\tau + h_\tau]. \tag{B.57}$$

The time-dependent coefficients ζ_t, η_t, and κ_t of Eqs. (B.55)–(B.57) in the ansatz Eq. (B.20) can be determined when we have the solution μ_t in Eq. (B.31) under the initial conditions Eqs. (B.47) and (B.48).

Summarizing the above results, we obtain the propagator $U_{t,t_0}(x|x_0)$ of the GQPO [Eq. (B.1)]:

$$\begin{aligned} U_{t,t_0}(x|x_0) = \frac{1}{\sqrt{4\pi i\hbar a_t\mu_t}}\exp\Bigg[\frac{i}{\hbar}\frac{1}{4a_t}\Bigg\{&\left(\frac{\dot{a}_t}{a_t} + \frac{\dot{\mu}_t}{\mu_t} - c_t\right)x^2 + 2\left(\frac{\mu_{t_0}}{2}\frac{\dot{a}_{t_0}}{a_{t_0}} - 1\right)\frac{xx_0}{\mu_t} \\ &+ \frac{a_t}{a_{t_0}}\left[\left(\frac{\mu_{t_0}}{2}\frac{\dot{a}_{t_0}}{a_{t_0}} - 1\right)^2\frac{\nu_t}{\mu_t} - \frac{1}{2}\frac{\dot{a}_{t_0}}{a_{t_0}} + c_{t_0}\right]x_0^2\Bigg\}\Bigg]e^{i(\zeta_t x + \eta_t x_0 + \kappa_t)/\hbar}, \end{aligned} \tag{B.58}$$

where μ_t and ν_t are the solutions of the damped CPO with the angular frequency $\tilde{\Omega}_t$ [Eq. (B.32)] and the different initial conditions:

$$\ddot{\mu}_t + \frac{\dot{a}_t}{a_t}\dot{\mu}_t + \tilde{\Omega}_t^2\mu_t = 0, \qquad (\mu_{t_0}, \dot{\mu}_{t_0}) = (0, 1), \tag{B.59}$$

$$\ddot{\nu}_t + \frac{\dot{a}_t}{a_t}\dot{\nu}_t + \tilde{\Omega}_t^2\nu_t = 0, \qquad (\nu_{t_0}, \dot{\nu}_{t_0}) = (1, 0). \tag{B.60}$$

Since the Wronskian is always non-zero (i.e., $W_t = \frac{a_{t_0}}{a_t} > 0$) [Eq. (B.50)], the solutions μ_t and ν_t of Eqs. (B.59) and (B.60) are linearly independent, respectively.

Bibliography

[1] Mathieu Beau, Juan Diego Jaramillo, and Adolfo del Campo. Scaling-up quantum heat engines efficiently via shortcut to adiabaticity. *Entropy, 18(5), 16*, 2016(18(5)):168, April 2016. doi: 10.3390/e18050168. URL https://doi.org/10.3390/e18050168.

[2] Obinna Abah and Eric Lutz. Performance of shortcut-to-adiabaticity quantum engines, 2017. URL https://arxiv.org/abs/1707.09963.

[3] Tadashi Kadowaki and Hidetoshi Nishimori. Quantum annealing in the transverse Ising model. *Physical Review E*, 58(5355), November 1998. doi: 10.1103/PhysRevE.58.5355. URL https://doi.org/10.1103/PhysRevE.58.5355.

[4] Giuseppe E. Santoro and Erio Tosatti. Optimization using quantum mechanics: quantum annealing through adiabatic evolution. *Journal of Physics A: Mathematical and General*, 39 (R393), August 2006. doi: 10.1088/0305-4470/39/36/R01. URL https://doi.org/10.1088/0305-4470/39/36/R01.

[5] Xue-Ke Song, Hao Zhang, Qing Ai, Jing Qiu, and Fu-Guo Deng. Shortcuts to adiabatic holonomic quantum computation in decoherence-free subspace with transitionless quantum driving algorithm. *New Journal of Physics*, 18(023001), February 2016. doi: 10.1088/1367-2630/18/2/023001. URL https://doi.org/10.1088/1367-2630/18/2/023001.

[6] Kazutaka Takahashi. Shortcuts to adiabaticity for quantum annealing. *Physical Review A*, 95 (012309), January 2017. doi: 10.1103/PhysRevA.95.012309. URL https://doi.org/10.1103/PhysRevA.95.012309.

[7] Tova Feldmann and Ronnie Kosloff. Quantum four-stroke heat engine: Thermodynamic observables in a model with intrinsic friction. *Physical Review E*, 68(016101), July 2003. doi: 10.1103/PhysRevE.68.016101. URL https://doi.org/10.1103/PhysRevE.68.016101.

[8] Yair Rezek and Ronnie Kosloff. Irreversible performance of a quantum harmonic heat engine. *New Journal of Physics*, 8(83), May 2006. doi: 10.1088/1367-2630/8/5/083. URL https://doi.org/10.1088/1367-2630/8/5/083.

[9] Dionisis Stefanatos, Justin Ruths, and Jr-Shin Li. Frictionless atom cooling in harmonic traps: A time-optimal approach. *Physical Review A*, 82(063422), December 2010. doi: 10.1103/PhysRevA.82.063422. URL https://doi.org/10.1103/PhysRevA.82.063422.

[10] Max Born and Vladimir Aleksandrovich Fock. Beweis des Adiabatensatzes. *Zeitschrift für Physik*, 51(3–4):165–180, August 1928. doi: 10.1007/BF01343193. URL https://doi.org/10.1007/BF01343193.

[11] Tosio Kato. On the Adiabatic Theorem of Quantum Mechanics. *Journal of the Physical Society of Japan*, 5(6):435–439, March 1950. doi: 10.1143/JPSJ.5.435. URL https://doi.org/10.1143/JPSJ.5.435.

[12] Erik Torrontegui, Sara Ibáñez, Sofia Martínez-Garaot, Michele Modugno, Adolfo del Campo, David Guéry-Odelin, Andreas Ruschhaupt, Xi Chen, and Juan Gonzalo Muga. Shortcuts

to Adiabaticity. *Advances In Atomic, Molecular, and Optical Physics 62, 117*, 62:117–169, July 2013. doi: 10.1016/B978-0-12-408090-4.00002-5. URL https://doi.org/10.1016/B978-0-12-408090-4.00002-5.

[13] Mustafa Demirplak and Stuart Alan Rice. Adiabatic Population Transfer with Control Fields. *The Journal of Physical Chemistry A*, 107(46):9937–9945, 2003. doi: 10.1021/jp030708a. URL http://dx.doi.org/10.1021/jp030708a.

[14] Mustafa Demirplak and Stuart Alan Rice. Assisted Adiabatic Passage Revisited. *The Journal of Physical Chemistry B*, 109(14):6838–6844, February 2005. doi: 10.1021/jp040647w. URL http://dx.doi.org/10.1021/jp040647w.

[15] Michael Victor Berry. Transitionless quantum driving. *Journal of Physics A: Mathematical and Theoretical*, 42(365303), August 2009. doi: 10.1088/1751-8113/42/36/365303. URL https://doi.org/10.1088/1751-8113/42/36/365303.

[16] Shumpei Masuda and Katsuhiro Nakamura. Fast-forward problem in quantum mechanics. *Physical Review A*, 78(062108), December 2008. doi: 10.1103/PhysRevA.78.062108. URL https://doi.org/10.1103/PhysRevA.78.062108.

[17] Shumpei Masuda and Katsuhiro Nakamura. Fast-forward of adiabatic dynamics in quantum mechanics. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 466(2116):1135–1154, November 2010. doi: 10.1098/rspa.2009.0446. URL https://doi.org/10.1098/rspa.2009.0446.

[18] Erik Torrontegui, Sofia Martínez-Garaot, Andreas Ruschhaupt, and Juan Gonzalo Muga. Shortcuts to adiabaticity: Fast-forward approach. *Physical Review A*, 86(013601), July 2012. doi: 10.1103/PhysRevA.86.013601. URL https://doi.org/10.1103/PhysRevA.86.013601.

[19] Shumpei Masuda and Stuart Alan Rice. Rapid coherent control of population transfer in lattice systems. *Physical Review A*, 89(033621), March 2014. doi: 10.1103/PhysRevA.89.033621. URL https://doi.org/10.1103/PhysRevA.89.033621.

[20] Christopher Jarzynski, Sebastian Deffner, Ayoti Patra, and Yiğit Subaşı. Fast forward to the classical adiabatic invariant. *Physical Review E*, 95(032122), March 2017. doi: 10.1103/PhysRevE.95.032122. URL https://doi.org/10.1103/PhysRevE.95.032122.

[21] H. R. Lewis and W. B. Riesenfeld. An Exact Quantum Theory of the Time-Dependent Harmonic Oscillator and of a Charged Particle in a Time-Dependent Electromagnetic Field. *Journal of Mathematical Physics*, 10(8):1458, August 1969. doi: 10.1063/1.1664991. URL https://doi.org/10.1063/1.1664991.

[22] Xi Chen, A. Ruschhaupt, Sebastian Schmidt, Adolfo del Campo, David Guéry-Odelin, and Juan Gonzalo Muga. Fast Optimal Frictionless Atom Cooling in Harmonic Traps: Shortcut to Adiabaticity. *Physical Review Letters*, 104(063002), February 2010. doi: 10.1103/PhysRevLett.104.063002. URL https://doi.org/10.1103/PhysRevLett.104.063002.

[23] Sebastian Deffner, Christopher Jarzynski, and Adolfo del Campo. Classical and Quantum Shortcuts to Adiabaticity for Scale-Invariant Driving. *Physical Review X*, 4(021013), April 2014. doi: 10.1103/PhysRevX.4.021013. URL https://doi.org/10.1103/PhysRevX.4.021013.

[24] Christopher Jarzynski. Generating shortcuts to adiabaticity in quantum and classical dynamics. *Physical Review A*, 88(040101(R)), October 2013. doi: 10.1103/PhysRevA.88.040101. URL https://doi.org/10.1103/PhysRevA.88.040101.

[25] Alberto Carlini, Akio Hosoya, Tatsuhiko Koike, and Yosuke Okudaira. Time-Optimal Quantum Evolution. *Physical Review Letters*, 96(060503), February 2006. doi: 10.1103/PhysRevLett.96.060503. URL https://doi.org/10.1103/PhysRevLett.96.060503.

[26] Alberto Carlini, Akio Hosoya, Tatsuhiko Koike, and Yosuke Okudaira. Time-optimal unitary operations. *Physical Review A*, 75(042308), April 2007. doi: 10.1103/PhysRevA.75.042308. URL https://doi.org/10.1103/PhysRevA.75.042308.

[27] Kazutaka Takahashi. How fast and robust is the quantum adiabatic passage? *Journal of Physics A: Mathematical and Theoretical*, 46(315304), July 2013. doi: 10.1088/1751-8113/46/31/315304. URL https://doi.org/10.1088/1751-8113/46/31/315304.

[28] Xi Chen, Erik Torrontegui, and Juan Gonzalo Muga. Lewis-Riesenfeld invariants and transitionless quantum driving. *Physical Review A*, 83(062116), June 2011. doi: 10.1103/PhysRevA.83.062116. URL https://doi.org/10.1103/PhysRevA.83.062116.

[29] Konrad Zuse. *Rechnender Raum*, volume 1. Braunschweig, 1969. URL http://www.worldscientific.com/worldscibooks/10.1142/8306. [English translation: Hector Zenil. A Computable Universe: Understanding and Exploring Nature as Computation. World Scientific Publishing Co., Inc. December 2012. ISBN: 978-981-4447-78-2.].

[30] Paola A. Zizzi. *Spacetime at the Planck Scale: The Quantum Computer View, The Foundations of Quantum Mechanics*, pages 345–358. World Scientific, 2012. doi: 10.1142/9789812773258_0030. URL https://doi.org/10.1142/9789812773258_0030.

[31] Jiangfeng Du, Hui Li, Xiaodong Xu, Mingjun Shi, Jihui Wu, Xianyi Zhou, and Rongdian Han. Experimental Realization of Quantum Games on a Quantum Computer. *Physical Review Letters*, 88(137902), March 2002. doi: 10.1103/PhysRevLett.88.137902. URL https://doi.org/10.1103/PhysRevLett.88.137902.

[32] Rafael Díaz. Quantum Boolean algebras. *Algebra and Discrete Mathematics*, 24(1):106–143, 2017. URL http://admjournal.luguniv.edu.ua/index.php/adm/article/view/82.

[33] Garrett Birkhoff and John von Neumann. The Logic of Quantum Mechanics. *Annals of Mathematics*, 37(4):823–843, October 1936. doi: 10.2307/1968621. URL https://doi.org/10.2307/1968621.

[34] George Whitelaw Mackey. *Mathematical Foundations of Quantum Mechanics*. Dover Publications, 2004. ISBN: 0-486-43517-2 (pbk.) (ISBN13: 9780486435176), [Originally published: New York: W. A. Benjamin, 1963, in series: Mathematical physics monograph series].

[35] John von Neumann and Oskar Morgenstern. *Theory of Games and Economic Behavior 60th Anniversary Commemorative Edition*, volume The Theory of Games and Economic Behaviour. Princeton University Press, 2007. ISBN: 978-0691130613.

[36] Martin A. Nowak and Karl Sigmund. Phage-lift for game theory. *Nature*, 398:367–368, 1999. doi: 10.1038/18761. URL http://dx.doi.org/10.1038/18761.

[37] David A. Meyer. Quantum Strategies. *Physical Review Letters*, 82(1052), February 1999. doi: 10.1103/PhysRevLett.82.1052. URL https://doi.org/10.1103/PhysRevLett.82.1052.

[38] Jens Eisert, Martin Wilkens, and Maciej Lewenstein. Quantum Games and Quantum Strategies. *Physical Review Letters*, 83(3077), October 1999. doi: 10.1103/PhysRevLett.83.3077. URL https://doi.org/10.1103/PhysRevLett.83.3077.

[39] Peter Williston Shor. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. *SIAM Journal on Computing*, 26(5):1484–1509, 1997. doi: 10.1137/S0097539795293172. URL https://doi.org/10.1137/S0097539795293172.

[40] Charles H. Bennett and David P. DiVincenzo. Quantum information and computation. *Nature*, 404:247–255, March 2000. doi: doi:10.1038/35005001. URL http://dx.doi.org/10.1038/35005001.

[41] James M. Chappell, Azhar Iqbal, Max A. Lohe, and Lorenz von Smekal. An Analysis of the Quantum Penny Flip Game using Geometric Algebra. *Journal of the Physical Society of Japan*, 78(5):054801, May 2009. doi: 10.1143/JPSJ.78.054801. URL https://doi.org/10.1143/JPSJ.78.054801.

[42] Kôdi Husimi and Hiroshi Ezawa. *Following a oscillation of a pendulum (Kôdi Husimi Collection II)*. Nihon Hyoron Sha co., Ltd., July 2013. ISBN:978-4-535-60347-9. [In Japanese: 伏見 康治・江沢 洋，ふりこの振動を追って（伏見康治コレクション第２巻），日本評論社].

[43] Dariusz Chruściński and Andrzej Jamiołkowski. *Geometric Phases in Classical and Quantum Mechanics*. Number v. 36 in Progress in mathematical physics. Birkhäuser, 2006. ISBN: 978-0-8176-8176-0.

[44] Yoshitaka Yamamoto and Koichi Nakamura. *Analytical Mechanics II*. Asakura Publishing Co., Ltd., 1998. ISBN: 978-4-254-13672-2. [In Japanese: 山本 義隆・中村 孔一，解析力学 II（朝倉物理学大系２），朝倉書店].

[45] Jun John Sakurai. *Modern Quantum Mechanics*. Addison Wesley, revised edition, 1993. ISBN: 978-0201539295.

[46] Shigefumi Naka. *Schrödinger equation – for a better understanding of quantum mechanics*, volume P-3 of *SGC Books*. Saiensu-Sha Co., Ltd. Publishers, new edition, 2007. ISBN: 978-4-7819-1175-5. [In Japanese: 仲 滋文，新版シュレーディンガー方程式　量子力学のよりよい理解のために，サイエンス社，SGC Books P-3].

[47] Hari Krovi, Maris Ozols, and Járémie Roland. Adiabatic condition and the quantum hitting time of Markov chains. *Physical Review A*, 82(022333), 2010. doi: 10.1103/PhysRevA.82.022333. URL https://doi.org/10.1103/PhysRevA.82.022333.

[48] Michael Victor Berry. Quantal phase factors accompanying adiabatic changes. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 392(1802): 45–57, March 1984. doi: 10.1098/rspa.1984.0023. URL https://doi.org/10.1098/rspa.1984.0023.

[49] Ayoti Patra and Christopher Jarzynski. Classical and Quantum Shortcuts to Adiabaticity in a Tilted Piston. *The Journal of Physical Chemistry B*, 121(15):3403–3411, October 2016. doi: 10.1021/acs.jpcb.6b08769. URL https://doi.org/10.1021/acs.jpcb.6b08769.

[50] Juan Gonzalo Muga, Xi Chen, Sara Ibáñez, Ion Lizuain, and Andreas Ruschhaupt. Transitionless quantum drivings for the harmonic oscillator. *Journal of Physics B: Atomic, Molecular and Optical Physics*, 43(085509), April 2010. doi: 10.1088/0953-4075/43/8/085509. URL https://doi.org/10.1088/0953-4075/43/8/085509.

[51] Ayoti Patra and Christopher Jarzynski. Shortcuts to adiabaticity using flow fields. *New Journal of Physics*, 19(125009), December 2017. doi: 10.1088/1367-2630/aa924c. URL https://doi.org/10.1088/1367-2630/aa924c.

[52] Wilhelm Magnu. On the exponential solution of differential equations for a linear operator. *Communications on Pure and Applied Mathematics*, 7(4):649–673, November 1954. doi: 10.1002/cpa.3160070404. URL `http://onlinelibrary.wiley.com/doi/10.1002/cpa.3160070404/references`.

[53] Jiawen Deng, Qing hai Wang, Zhihao Liu, Peter Hänggi, and Jiangbin Gong. Boosting work characteristics and overall heat-engine performance via shortcuts to adiabaticity: Quantum and classical systems. *Physical Review E*, 88(062122), December 2013. doi: 10.1103/PhysRevE.88.062122. URL `https://doi.org/10.1103/PhysRevE.88.062122`.

[54] Christopher Jarzynski. Geometric Phases and Anholonomy for a Class of Chaotic Classical Systems. *Physical Review Letters*, 74(1732), March 1995. doi: 10.1103/PhysRevLett.74.1732. URL `https://doi.org/10.1103/PhysRevLett.74.1732`.

[55] Kôdi Husimi. Miscellanea in Elementary Quantum Mechanics, II. *Progress of Theoretical Physics*, 9(4):381–402, April 1953. doi: 10.1143/ptp/9.4.381. URL `https://doi.org/10.1143/ptp/9.4.381`.

[56] Sebastian Deffner and Eric Lutz. Nonequilibrium work distribution of a quantum harmonic oscillator. *Physical Review E*, 77(021128), February 2008. doi: 10.1103/PhysRevE.77.021128. URL `https://doi.org/10.1103/PhysRevE.77.021128`.

[57] Hiroaki Mishima and Yuki Izumida. Transition probability generating function of a transitionless quantum parametric oscillator. *Physical Review E*, 96(012133), July 2017. doi: 10.1103/PhysRevE.96.012133. URL `https://doi.org/10.1103/PhysRevE.96.012133`.

[58] Hiroaki Mishima and Yuki Izumida. In preparation.

[59] Nils Andersson. Accurate regge pole positions and residues determined by phase-amplitude formulae. *Journal of Physics A: Mathematical and General*, 26(19):5085–5097, March 1993. doi: 10.1088/0305-4470/26/19/036. URL `https://doi.org/10.1088/0305-4470/26/19/036`.

[60] Hyeong-Chan Kim, Min-Ho Lee, Jeong-Young Ji, and Jae Kwan Kim. Heisenberg-picture approach to the exact quantum motion of a time-dependent forced harmonic oscillator. *Physical Review A*, 53(3767), June 1996. doi: 10.1103/PhysRevA.53.3767. URL `https://doi.org/10.1103/PhysRevA.53.3767`.

[61] Vasilij Petrovich Ermakov. Équations différentielles du deuxième ordre. Conditions d'intégrabilité sous forme finale. *Universita Izvestia Kiev*, 20(9):1–25, 1880. doi: 10.2298/AADM0802123E. URL `https://doi.org/10.2298/AADM0802123E`. [English translation: A. O. Harin and Peter. G. L. Leach. Second-order differential equations: Conditions of complete integrability. *Applicable Analysis and Discrete Mathematics*, **2**, 123–145, 2008].

[62] Edmund Pinney. The nonlinear differential equation $y'' + p(x)y + cy^{-3} = 0$. *Proceedings of the American Mathematical Society*, 1:681, 1950. doi: 10.1090/S0002-9939-1950-0037979-4. URL `https://doi.org/10.1090/S0002-9939-1950-0037979-4`.

[63] Peter G. L. Leach and K. Andriopoulos. The Ermakov equation: A commentary. *Applicable Analysis and Discrete Mathematics*, 2(2):146–157, 2008. doi: 10.2298/AADM0802146L. URL `https://doi.org/10.2298/AADM0802146L`.

[64] H. R. Lewis. Classical and Quantum Systems with Time-Dependent Harmonic-Oscillator-Type Hamiltonians. *physical Review Letters*, 18(13):510–512, March 1967. doi: 10.1103/PhysRevLett.18.510. URL `https://doi.org/10.1103/PhysRevLett.18.510`.

[65] Manuel Fernández-Guasti. The Wronskian and the Ermakov-Lewis Invariant. *International Mathematical Forum*, 4(13–16):795–804, 2009. doi: 10.12988/imf. URL http://www.m-hikari.com/imf-password2009/13-16-2009/fernandezIMF13-16-2009.pdf.

[66] Herbert Goldstein, Charles P. Poole Jr., and John L. Safko. *Classical Mechanics*, volume Classical Mechanics. Addison-Wesley, Reading, Massachusetts, 3rd edition, 2001. ISBN: 978-0201657029.

[67] Manuel Fernández-Guasti. *Nonlinear Phenomena Research Perspectives (chapter 6, "The Nonlinear Amplitude Equation in Harmonic Phenomena," pp. 177–223)*. Nova Science Publisher, Inc., 2007. URL http://investigacion.izt.uam.mx/mfg/arti/06-10/nonlinear-phen-amp7pc.pdf.

[68] Juan Diego Jaramillo, Mathieu Beau, and Adolfo del Campo. Quantum supremacy of many-particle thermal machines. *New Journal of Physics*, 18(075019), April 2016. doi: 10.1088/1367-2630/18/7/075019. URL https://doi.org/10.1088/1367-2630/18/7/075019.

[69] Hiroaki Mishima. Non-Abelian strategies in quantum penny flip game. *Progress of Theoretical and Experimental Physics*, 2018(1):013A04, January 2018. doi: 10.1093/ptep/ptx182. URL https://doi.org/10.1093/ptep/ptx182.

[70] John Forbes Nash. Equilibrium points in n-person games. volume 36, pages 48–49. National Academy of Sciences, January 1950. URL http://www.jstor.org/stable/88031.

[71] Derek Abbott, Paul. C. W. Davies, and Cosma R. Shalizi. Order from disorder: the role of noise in creative processes. A special issue on game theory and evolutionary processes – overview. *Fluctuation and Noise Letters*, 2(4):C1–C12, December 2002. doi: 10.1142/S0219477502001020. URL https://doi.org/10.1142/S0219477502001020.

[72] William Poundstone. *Prisoner's dilemma*. Oxford University Press, 1st anchor books edition, 1993. ISBN: 978-0385415804.

[73] John Maynard Smith. *Evolution and the Theory of Games*, volume Evolution and the Theory of Games. Cambridge University Press, 1982. doi: 10.1017/CBO9780511806292. URL https://doi.org/10.1017/CBO9780511806292.

[74] Jens Eisert and Michael M. Wolf. *Quantum Computing. In: Zomaya A.Y. (eds) Handbook of Nature-Inspired and Innovative Computing*. Springer, Boston, MA, 2006. doi: 10.1007/0-387-27705-6_8. URL https://doi.org/10.1007/0-387-27705-6_8.

[75] Michael Aaron Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 10th anniversary edition, 2010. ISBN: 978-1-107-00217-3.

[76] Howard Edward Brandt. Qubit devices and the issue of quantum decoherence. *Progress in Quantum Electronics*, 22(5):257–370, November 1999. doi: 10.1016/S0079-6727(99)00003-8. URL https://doi.org/10.1016/S0079-6727(99)00003-8.

[77] Chiu Fan Lee and Neil Fraser Johnson. Exploiting randomness in quantum information processing. *Physics Letters A*, 301(5–6):343–349, September 2002. doi: 10.1016/S0375-9601(02)01088-5. URL https://doi.org/10.1016/S0375-9601(02)01088-5.

[78] Namit Anand and Colin Benjamin. Do quantum strategies always win? *Quantum Information Processing*, 14(11):4027–4038, November 2015. doi: 10.1007/s11128-015-1105-y. URL https://doi.org/10.1007/s11128-015-1105-y.

[79] Jeongho Bang, Junghee Ryu, Marcin Pawłowski, Byoung Seung Ham, and Jinhyoung Lee. Quantum-mechanical machinery for rational decision-making in classical guessing game. *Scientific Reports*, 6(21424), February 2016. doi: 10.1038/srep21424. URL https://doi.org/10.1038/srep21424.

[80] Faisal Shah Khan and Simon James Daniel Phoenix. Mini-maximizing two qubit quantum computations. *Quantum Information Processing*, 12(12):3807–3819, December 2013. doi: 10.1007/s11128-013-0640-7. URL https://doi.org/10.1007/s11128-013-0640-7.

[81] Faisal Shah Khan and Simon James Daniel Phoenix. Gaming the Quantum. *Quantum Information & Computation*, 3(3&4):0231–0244, March 2013. URL http://www.rintonpress.com/xxqic13/qic-13-34/0231-0244.pdf.

[82] Walter Geller. *The VNR Concise Encyclopedia of Mathematics 2nd edition, Chap. 12*. Van Nostrand Reinhold, 2nd edition edition, 1989. doi: 10.1007/978-1-4684-8237-9. URL https://doi.org/10.1007/978-1-4684-8237-9.

[83] Geng Li, Haitao (H. T.) Quan, and Zhanchun (Z. C.) Tu. Shortcuts to isothermality and nonequilibrium work relations. *Physical Review E*, 96(012144), July 2017. doi: 10.1103/PhysRevE.96.012144. URL https://doi.org/10.1103/PhysRevE.96.012144.

[84] Ignacio A. Martínez, Artyom Petrosyan, David Guéry-Odelin, Emmanuel Trizac, and Sergio Ciliberto. Engineered swift equilibration of a Brownian particle. *Nature Physics*, 12:843–846, May 2016. doi: 10.1038/nphys3758. URL https://doi.org/10.1038/nphys3758.

[85] Agapi Emmanouilidou, Xingguang (X. G.) Zhao, Ping Ao, and Qian Niu. Steering an Eigenstate to a Destination. *Phys. Rev. Lett. 85, 1626*, 85(1626), August 2000. doi: 10.1103/PhysRevLett.85.1626. URL https://doi.org/10.1103/PhysRevLett.85.1626.

[86] Adolfo del Campo. Shortcuts to Adiabaticity by Counterdiabatic Driving. *Physical Review Letters*, 111(100502), September 2013. doi: 10.1103/PhysRevLett.111.100502. URL https://doi.org/10.1103/PhysRevLett.111.100502.

[87] Zhanchun (Z. C.) Tu. Stochastic heat engine with the consideration of inertial effects and shortcuts to adiabaticity. *Physical Review E*, 89(052148), May 2014. doi: 10.1103/PhysRevE.89.052148. URL https://doi.org/10.1103/PhysRevE.89.052148.

[88] Matthias Hanauske, Jennifer Kunz, Steffen Bernius, and Wolfgang König. Doves and hawks in economics revisited: An evolutionary quantum game theory based analysis of financial crises. *Physica A: Statistical Mechanics and its Applications*, 389(21):5084–5102, November 2010. doi: 10.1016/j.physa.2010.06.007. URL https://doi.org/10.1016/j.physa.2010.06.007.

[89] Anjan V. Thakor. Game Theory in Finance. *Financial Management*, 20(1):71–94, 1991. URL http://www.jstor.org/stable/3666098.

[90] Ahmad Nawaz and Abdul Hameed Toor. Evolutionarily Stable Strategies in Quantum Hawk–Dove Game. *Chinese Physics Letters*, 27(5):050303, July 2010. doi: 10.1088/0256-307X/27/5/050303. URL https://doi.org/10.1088/0256-307X/27/5/050303.

[91] Norihito Toyota. Quantization of the stag hunt game and the Nash equilibrilum, 2003. URL https://arxiv.org/abs/quant-ph/0307029.

[92] Sebastian Deffner, Obinna Abah, and Eric Lutz. Quantum work statistics of linear and nonlinear parametric oscillators. *Chemical Physics*, 375(2):200–208, October 2010. doi: 10.1016/j.chemphys.2010.04.042. URL https://doi.org/10.1016/j.chemphys.2010.04.042.

[93] Min-Ho Lee, Hyeong-Chan Kim, and Jeong-Young Ji. Exact wave functions and geometric phases of a generalized driven oscillator. *Journal of the Korean Physical Society*, 31:560, 1997.

[94] Erwin Suazo, Sergei K. Suslov, and José M. Vega-Guzmán. The Riccati System and a Diffusion-Type Equation. *Mathematics*, 2(2):96–118, May 2014. doi: 10.3390/math2020096. URL https://doi.org/10.3390/math2020096.