Anomaly and holographic local renormalization group

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Abstract

In order to scrutinize quantum field theories (QFTs), we discuss trace anomalies. In particular we compute them holographically with gauge fields in various dimensions including odd dimensions. As a result, we have generalized holographic c-theorem in four dimensions and shown that holographic trace anomalies do not exist in all odd dimensions.

Contents

1 Introduction

Quantum field theory (QFT) is one of the most successful theories in physics, which has been applied to particle physics, as well as to condensed matter physics, cosmological physics, and so on. However, a general (typically strongly coupled) QFT is difficult to study. For example, we cannot satisfactorily answer even fundamental questions such as what degrees of freedom are contained in a theory, what phase the theory is in (such as whether a symmetry is spontaneously broken or not). One of the most promising approaches to address these problems is to exploit symmetries.

One way to employ symmetries is to use anomalies. Anomaly was first discovered through a study of pion decay $\pi^0 \to \gamma\gamma$ amplitude [1]. Later, through endeavours to generalize the anomaly to higher dimensions [2], it was realized that anomalies are closely related to algebraic topology. Concretely speaking, it was found that anomalies originate from nontriviality of specific bundles [3].

In the Wilsonian renormalization group (RG) [4], one integrates out higher momentum (a.k.a. heavy) modes and deals just with lower momentum (a.k.a. light) modes. Thus it is intuitively expected that the 'number' of degrees of freedom decreases monotonically under RG flows. In fact, the intuition was proved by Zamolodchikov [5] in two dimensions. The generalizations of the theorem to higher spacetime dimensions are also known [6, 7, 8, 9, 10]. The theorems can be used to constrain QFTs [11], and open a way to (at least partially) answer the problems raised above. For example, suppose we are given a QFT in the ultraviolet (UV) with some global symmetries and consider deformations of the theory with relevant operators to trigger RG flows to some infrared (IR) theory. It is in general difficult to judge whether some of the global symmetries are spontaneously broken or not in the IR. However, since the putative SSB entails Nambu-Goldstone bosons, the symmetry breaking raises the 'number' of degrees of freedom. Thus we can conclude that too many SSB violates the monotonicity, hence the phase is ruled out.

Why the combination of RG flows and anomalies can constrain QFTs? A powerful tool called anomaly matching [12] is underlying the logic. Suppose a symmetry has an anomaly in a UV theory, and the symmetry is preserved all along RG flows. Since the anomaly is a defining property of the theory, the anomaly must match at every energy scale, especially in the IR. Sometimes both weakly coupled points and strongly coupled ones lie on the same RG flow. For example, we know asymptotically free description of QCD_4 in the UV, while the theory becomes strongly coupled in the IR. Therefore by computing anomalies in weakly coupled points, and using the anomaly matching, we can probe strongly coupled QFTs. Since the tool is the only method human beings have to address strongly coupled theories, it has been applied to various situations: check of dualities [13, 14, 15], study of phases of strongly coupled theories [16], and so on.

QFTs have intrinsic ambiguities caused by the degrees of freedom to add local counterterms. The ambiguity is inherited to anomalies. We review this point, which is needed to understand our results, the holographic c-theorem, in section 2.

In section 3, we report our results on holographic c-theorem, and holographic trace anomaly in the presence of gauge fields in various dimensions. In case of even dimensional QFTs, it was known that holographic c-function decreases monotonically under RG flows [29], however, the analysis was limited to QFTs without background gauge fields. We incorporated background gauge fields in the computation, and taking the ambiguity caused by local counterterms, we have generalized the theorem for QFTs with global symmetries. In case of odd dimensional QFTs, there were no calculations so far (to the author's knowledge) because it was believed that there were no odd-dimensional trace anomalies. However, recently it was pointed out [36] that there may appear trace anomalies even in odd dimensions if background scalar and gauge fields are incorporated. Therefore we have applied the holographic computation to three-dimensional QFTs with these background fields. As a result, we found trace anomalies do appear but their coefficients are proportional to beta-functions. Hence the putative anomalies vanish on (conformal) fixed points. We also clarify what is responsible for the vanishment. Since the reasoning continues to hold in all odd-dimensional QFTs, our result proves the absence of holographic trace anomalies in odd-dimensional CFTs. Since the holographic computations have close relation to a more general framework called local renormalization group (LRG), one will find exactly the same equations. In fact, LRG equations naturally appears as flow equations, which play the central role in the holographic computation of trace anomalies. We also comment on the relation in section 3. In the last subsection, we classify coupling constants, and introduce a notion of the conformal manifold.

Another way to exploit symmetries is to impose larger symmetries such as conformal symmetry or supersymmetry (SUSY). The larger symmetries are, the more constraints imposed on theories, and we can extract stronger claims. Since most of the general QFTs can be obtained by relevant deformations of conformal field theories (CFTs), CFTs are good places to start. Some deformations do not break conformal symmetries, resulting in another CFT. The operators which do not break conformal symmetry are called exactly marginal. The coefficients of the exactly marginal operators, called exactly marginal couplings, can thus be interpreted to label a continuous family of CFTs. The set is called a conformal manifold. Some of the manifolds are known to have interesting geometry. For example, it is known that conformal manifolds of two-dimensional $\mathcal{N} = (2, 2)$ SCFTs and four-dimensional $\mathcal{N} = 2$ SCFTs are Kähler [17, 18, 19]. We review these facts, and report partial results of our work in progress [20] in the appendix C. The project is about branched four-sphere partition functions.

In the calculation of the holographic trace anomaly, which is explained in the body of this thesis, the flow equation (a.k.a. Hamiltonian constraint) plays a central role. The author found the flow equation is similar to the antibracket, which is introduced in the Batalin-Vilkovisky (BV) formalism. The observation gives another simple and systematic derivation of first-class constraints including the flow equation. We report the derivation in another appendix B.

In the other appendices, we collect technical details, namely, notations (appendix A) and construction of SUSY on $\mathbb{S}^1_l \times \mathbb{H}^3_r$ (appendix D).

2 Anomaly

In this section, we review general aspects of anomalies.

Anomalies appear when it is impossible to respect all physical requirements simultaneously. For instance, in case of the chiral anomaly mentioned in the introduction, the system classically has a chiral symmetry and a gauge symmetry, however, we can respect at most one of them at the quantum level. Anomalies in gauge symmetries (in UV complete theories) lead to mathematical inconsistency because gauge symmetry is just a redundancy of description, and if it were violated physics began to depend on how we describe Nature, which is clearly inconsistent. Hence we usually respect the gauge symmetry, and the chiral current is no longer conserved at the quantum level.

Let us review the chiral anomaly in more detail. Consider a $(2n)$-dimensional system which consists of chiral fermion(s) coupled to background gauge field(s)[1]. The system is described by a fermion partition function $Z_\psi(A)$, or equivalently by a fermion effective action $\Gamma_\psi(A)$[2]

$$\exp\Big\{-\Gamma_\psi(A)\Big\} \equiv Z_\psi(A) := \int \mathcal{D}\psi \exp\Big\{-S[\psi, A]\Big\}. \tag{2.1}$$

We can easily compute the current coupled to the background gauge field(s) A via Noether's procedure;

$$J^\mu(x) = \frac{\delta \Gamma_\psi}{\delta A_\mu(x)}.$$

On the other hand, under an infinitesimal gauge transformation $A \mapsto A + D\alpha$, the fermion partition function changes as

$$\begin{aligned}
\Gamma_\psi(A) &\mapsto \Gamma_\psi(A + D\alpha) \\
&= \Gamma_\psi(A) + \mathrm{tr} \int d^{2n}x D_\mu \alpha \frac{\delta \Gamma_\psi}{\delta A_\mu} \\
&= \Gamma_\psi(A) - \mathrm{tr} \int d^{2n}x \alpha D_\mu \frac{\delta \Gamma_\psi}{\delta A_\mu} \\
&= \Gamma_\psi(A) - \mathrm{tr} \int d^{2n}x \alpha D_\mu J^\mu.
\end{aligned}$$

The integration by parts is allowed because α is implicitly assumed to have a compact support. Naively, one expects the fermion effective action be invariant under the transformation, or equivalently the current be covariantly conserved. However, this is not the case when there

[1] We follow the condensed matter convention, namely, background gauge fields are denoted by upper case letters, say A, and dynamical gauge fields are denoted by lower case letters, say a.

[2] It is common to put variables of functionals in square bracket, however, we save square brackets for equivalent classes $[A]$, and denote functionals with round brackets in this section.

is an anomaly. It is known that the current is not conserved [2]

$$D_\mu J^{a\mu} = \frac{i^n}{(2\pi)^n(n-1)!}\int_0^1 dt(1-t)\mathrm{Str}\Big[\lambda^a d(AF_t^{n-1})\Big], \tag{2.2}$$

where λ^a is a generator of the gauge group with $\alpha = \alpha^a\lambda^a$. Hence the fermion effective action is not gauge invariant $\Gamma_\psi(A) \neq \Gamma_\psi(A+D\alpha)$. The anomaly is called the non-Abelian anomaly. In four dimensions, it reduces to the familiar form

$$D_\mu J^{a\mu} = -\frac{1}{24\pi^2}\mathrm{tr}\Big\{\lambda^a d(AdA + \frac{1}{2}A^3)\Big\}.$$

Note that the formula (2.2) is an *operatorial* equation, meaning that the equality must be understood inside correlators. Since A is a background gauge field, the RHS is a c-number, and the violation is mild in the sense that the anomaly shows up just in contact terms[3]. The formula, however, continues to hold even when the gauge fields are dynamical

$$D_\mu J^{a\mu} = \frac{i^n}{(2\pi)^n(n-1)!}\int_0^1 dt(1-t)\mathrm{Str}\left[\lambda^a d(af_t^{n-1})\right]. \tag{2.3}$$

Now that the RHS is a q-number, rather than a c-number, and the symmetry is explicitly broken. Since gauge symmetries must be respected in mathematically consistent (UV complete) theories, the violation indicates that we are not allowed to gauge when there is an anomaly (2.2). We would like to emphasize that a theory with the anomaly (2.2) is completely consistent as long as the gauge fields are background fields. The anomaly is a sign of obstruction to gauge the symmetry. The obstruction (2.2) is called a 't Hooft anomaly.

We expressed the anomaly in terms of differential forms to emphasize the relation between anomaly and algebraic geometry. One would notice that the anomaly is completely fixed by geometrical content, and there is no room for Lagrangian dependence. Indeed, under a weak assumption, it is proved that anomalies just depend on representations [21]. We would not reproduce the proof because it is not essential for our discussion. Here we would like to see the relation between anomaly and representation, which is crucial to understand the crux of anomaly.

Consider a gauge group G and its representation ρ. We write the representation space $V_{\rho(G)}^{\dim\rho}$. Hence a fermion in the representation takes value in the vector space

$$\psi(x) \in V_{\rho(G)}^{\dim\rho}.$$

Since the fermion effective action is formally given by the sum of integrated connected k-point functions of current operators

$$\int d^{2n}x_1\cdots d^{2n}x_k A_{\mu_1}^{a_1}(x_1)\cdots A_{\mu_k}^{a_k}(x_k)\langle J^{a_1\mu_1}(x_1)\cdots J^{a_k\mu_k}(x_k)\rangle_{\text{con.}},$$

[3]For example, the four-dimensional anomaly can be 'detected' by taking functional derivatives with respect to the background gauge fields twice. Then the anomaly appears as a contact term in three-point functions of current operators.

it suffers from UV divergences, and we need to regularize them to get mathematically meaningful results.

Suppose ρ is a (pseudo)real representation. This means there are G-invariant two-tensors ω. If the two-tensor is antisymmetric, ρ is called pseudoreal, and if the tensor is symmetric, ρ is called real. In case of pseudoreal representations, we can write gauge invariant mass terms $\omega_{ab}\psi^a\psi^b$, so we can regularize the UV divergences gauge invariantly employing the Pauli-Villars (PV) prescription. Hence there is no ambiguity in the fermion effective action when the fermion belongs to pseudoreal representations. In case of real representations, naively we cannot write gauge invariant mass terms because of Fermi statistics $\omega_{ab}\psi^a\psi^b = 0$, however, if another copy of the theory is introduced, we can do write gauge invariant mass terms $\omega_{ab}\psi^a\tilde{\psi}^b$, where $\tilde{\psi}$ is a fermion of the copy. Thus we can regularize the doubled theory gauge invariantly. This means the partition function squared $Z_\psi(A)^2$ is completely gauge invariant. We conclude that anomalies of real fermions can only appear as sign ambiguities. A famous example of this ambiguity was discussed in [22][4]. See [23], for example, for an excellent review.

Next, let us consider a fermion in a complex representation ρ. In this case, G-invariant two-tensor does not exist. Thus gauge invariance forbids mass term of the fermion, and we cannot use the PV regularization. Since the fermion takes value in the vector space $V^{\dim\rho}_{\rho(G)}$ over the complex field $\mathbb{C}$, the fermion effective action is also complex valued (in general)

$$\Gamma_\psi(A) \in \mathbb{C}.$$

Now, consider a fermion in the complex conjugate representation $\bar{\rho}$. The corresponding fermion effective action is given by the complex conjugate of $\Gamma_{\psi\in\rho}$. Furthermore, since $\rho+\bar{\rho}$ is a real representation, we can regularize the corresponding fermion effective action gauge invariantly. Thus we have learned that there is no ambiguity in real parts of fermion effective actions

$$\Gamma_{\psi\in\rho} + \Gamma_{\psi\in\bar{\rho}} = 2\mathrm{Re}\Gamma_{\psi\in\rho}.$$

In other words, only imaginary parts of fermion effective actions suffer from gauge non-invariance. In terms of the fermion partition function

$$Z_\psi(A) = |Z_\psi(A)|e^{-i\mathrm{Im}\Gamma_\psi(A)}, \tag{2.4}$$

this is equivalent to say $Z_\psi(A)$ suffers from phase ambiguity[5]

$$e^{-i\mathrm{Im}\Gamma_\psi(A)} \neq e^{-i\mathrm{Im}\Gamma_\psi(A+D\alpha)}.$$

[4]In the paper, Witten considered a tensor product representation $\rho := \mathrm{spin}(O(5)) \otimes \square(SU(2))$. Since $\mathrm{spin}(O(5))$ is a pseudoreal representation, having an $O(5)$-invariant antisymmetric two-tensor $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$, and $\square(SU(2))$ is also a pseudoreal representation, having an $SU(2)$-invariant antisymmetric tensor $\omega_{ab} = -\omega_{ba}$, the tensor product representation ρ has a *symmetric* invariant two-tensor $\varepsilon\omega$, meaning ρ is real, rather than pseudoreal. Thus the fermion partition functions with an odd number of dynamical doublets suffer from sign ambiguity, leading to mathematical inconsistency known as $SU(2)$ anomaly.

[5]We can consider the phase as a map from the space of gauge configurations $\mathcal{A}$ to $U(1)$

$$e^{-i\mathrm{Im}\Gamma_\psi} : \mathcal{A} \to U(1).$$

The modulus is given by

$$|Z_\psi(A)| = e^{-\mathrm{Re}\Gamma_\psi(A)},$$

and it is completely gauge invariant

$$|Z_\psi(A)| = |Z_\psi(A + D\alpha)|.$$

As we mentioned, it is also possible to prove anomalies just depend on representations and not on specific Lagrangians under a weak assumption [21].

Let us see how exactly the phase ambiguity appears. From the above discussion, we have learned that

$$T^a(x) := -\left(D_\mu \frac{\delta}{\delta A_\mu(x)}\right)^a \equiv -\partial^x_\mu \frac{\delta}{\delta A^a_\mu(x)} - f^{abc} A^b_\mu(x) \frac{\delta}{\delta A^c_\mu(x)} \tag{2.5}$$

generates gauge transformations. One can easily compute the generators satisfy an algebra

$$[T^a(x), T^b(y)] = f^{abc}\delta^{(2n)}(x - y)T^c(y). \tag{2.6}$$

Acting both sides of the algebra on the fermion effective action, we obtain

$$T^a(x)\mathfrak{A}^b(y) - T^b(y)\mathfrak{A}^a(x) = f^{abc}\delta^{(2n)}(x - y)\mathfrak{A}^c(y), \tag{2.7}$$

where

$$\mathfrak{A}^a(x) := iT^a(x)\Gamma_\psi(A) \equiv -iD_\mu\langle J^{a\mu}(x)\rangle \tag{2.8}$$

is the anomaly (2.2). To be more precise, since we have just seen that real parts of fermion effective actions are gauge invariant $T^a(x)\mathrm{Re}\Gamma_\psi(A) = 0$, we can equivalently write

$$\mathfrak{A}^a(x) = -T^a(x)\mathrm{Im}\Gamma_\psi(A).$$

In terms of the fermion partition function, this is equivalent to

$$T^a(x)Z_\psi(A) = i\mathfrak{A}^a(x)Z_\psi(A). \tag{2.9}$$

The relation (2.7) is called the Wess-Zumino (WZ) consistency condition [24]. It is clear from this argument that anomalies have to satisfy the WZ condition to respect the algebra (2.6). Turning the logic around, the WZ condition can be used to fix forms of anomalies.

It is natural to identify two gauge configurations which are connected with a gauge transformation $g \in \mathcal{G}$. The identification results in a quotient space $\mathcal{A}/\mathcal{G}$ called gauge orbit space. The map induces another 'map' on the space of gauge orbits

$$\widetilde{e^{-i\mathrm{Im}\Gamma_\psi}} : \mathcal{A}/\mathcal{G} \to U(1),$$

however, because of the ambiguity, it is multivalued (in general). Therefore we should recognize the 'map' as a local section. We will elaborate on this point below.

One should notice that not all of (2.8) immediately leads to (genuine) anomalies. In QFTs, there is a freedom (or ambiguity) to add local counterterms (equivalently to change schemes). Since those terms which can be tuned away with suitable local counterterms do not exist in the scheme, we do not recognize those terms as true contributions to anomalies. Therefore anomalies are usually defined as 'cohomologically nontrivial,' which means the terms cannot be tuned away with local counterterms. On the other hand, terms which can be tuned away with local counterterms are called cohomologically trivial, and we customary do not think them as intrinsic anomalies. A typical example of the cohomologically trivial term is the $\Box R$ piece in the four-dimensional trace anomaly. In fact, the term can be tuned away with a local counterterm R^2. Due to this inevitable ambiguity in anomalies, there was a confusion soon after the anomaly was found because two computations did not match. As is evident now, the reason of the mismatch was that they did not match local counterterms. Because of this freedom (or ambiguity), there are in principle infinite numbers of forms of anomalies. Two typical forms are known as consistent anomaly which obey the WZ consistency condition, and covariant anomaly which has a form $\text{tr}\lambda^a F^n$. The former appears in $(n+1)$-point functions of $(V-A)$-currents, while the latter appears in correlations functions of one axial current and n vector currents. Since the latter is covariant, it is easier to compute. Because background gauge fields commute, we have to symmetrize the result[6] to get the (leading term of the) former from the latter, which obviously violates Bose symmetry. Once we have obtained leading terms, we can reproduce the other terms by imposing the WZ condition. For an explicit construction of the responsible local counterterms, see [25].

Although we have considered only infinitesimal gauge transformations, it is also instructive to consider finite gauge transformations

$$Z_\psi(A^g) = e^{-2\pi i\omega_1(A;g)}Z_\psi(A). \tag{2.10}$$

A function which depends on one group element, such as ω_1, is called a 1-cochain. Since we still insist on group composition law $g_2 \circ g_1 = g_{12}$, we must have

$$e^{-2\pi i[\omega_1(A;g_1)+\omega_1(A^{g_1};g_2)]}Z_\psi(A) = Z_\psi((A^{g_1})^{g_2}) = Z_\psi(A^{g_{12}}) = e^{-2\pi i\omega_1(A;g_{12})}Z_\psi(A),$$

or

$$\omega_1(A^{g_1};g_2) - \omega_1(A;g_{12}) + \omega_1(A;g_1) = 0 \mod \mathbb{Z}.$$

A function which satisfies this relation is called a 1-cocycle[7]. Thus we have seen in (2.10) that the non-Abelian anomaly appears as a (nontrivial) 1-cocycle.

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[6] Schematically, for the leading terms $\text{tr}\lambda(dA)^n$ we have

$$(\text{consistent}) = \frac{1}{n+1}(\text{covariant}).$$

[7] More generally, a function $\omega_k(A;g_1,\cdots,g_k)$ which depends on k group elements is called k-cochain. On

Usually, (2.9) is considered as a quantum breaking of classical symmetry, however, it is possible to give another interpretation [26]. Let us define a new generator by

$$U^a(x) := T^a(x) - i\mathfrak{A}^a(x). \tag{2.11}$$

By definition, it is immediate that

$$U^a(x)Z_\psi(A) = 0. \tag{2.12}$$

Furthermore, thanks to the WZ condition, one can easily show the new generators still satisfy the algebra (2.6):

$$[U^a(x), U^b(y)] = f^{abc}\delta^{(2n)}(x-y)U^c(y). \tag{2.13}$$

Thus it is possible to say that a presence of anomalies modifies representations of the gauge groups. In fact, the modification is exactly realized by additional Chern-Simons (CS) $(2n+1)$-form in one higher dimensional spacetime, meaning that the total $(2n+1)$-dimensional system with the original theory living on the $(2n)$-dimensional boundary is completely gauge invariant.

Yet another interpretation is known. In the presence of non-Abelian anomalies, the commutation relations of Gauss's law generators are modified. To be precise, the commutator admits a central extension as a Schwinger term, which is a 2-cocycle (see for example [27]). This fact that central extensions lead to anomalies is exactly the same as the fact that trace anomalies in two-dimensions manifest as a central extension of the Witt algebra $W = (\{L_n\}_{n\in\mathbb{Z}}, +, \overset{\mathbb{C}}{\cdot}, [,], *)$, called Virasoro algebra

$$\text{Vir} := W \oplus \mathbb{C}Z = (\{L_n, Z\}_{n\in\mathbb{Z}}, +, \overset{\mathbb{C}}{\cdot}, [,], *).$$

This vantage point has triggered recent developments starting from [16].

We have seen various interpretations of (non-Abelian) anomalies, however, we still do not understand a fundamental question; what is essential (or responsible) for the presence of anomalies? As we will see below, it will turn out that nontriviality of specific bundles gives rise to anomalies. Since the fact is also true for other anomalies, we obtain a transparent criterion when anomalies appear by studying relevant bundles.

It is convenient to work on $\mathbb{S}^{2n}$. Consider a group of transformations

$$\begin{array}{ccccc} \mathcal{G} & : & \mathbb{S}^{2n} & \to & G \\ \Downarrow & & \Downarrow & & \Downarrow \\ g & : & x & \mapsto & g(x) \end{array},$$

k-cochains, an operator called coboundary operator Δ acts by

$$\Delta\omega_k := \omega_k(A^{g_1}; g_2, \cdots, g_{k+1}) + \cdots + (-)^l \omega_k(A; g_1, \cdots, g_{l,l+1}, \cdots, g_{k+1}) + \cdots + (-)^{k+1}\omega_k(A; g_1, \cdots, g_k).$$

Since the coboundary operator satisfies $\Delta^2 = 0$, we can define Δ-cohomology. So k-cochain is called a k-cocycle if $\Delta\omega_k = 0$, and k-coboundary if there exists $(k-1)$-cochain such that $\omega_k = \Delta\omega_{k-1}$. The latter is also called trivial. An analogy with differential forms might be helpful.

where G is assumed to be simply connected, i.e., $\pi_1(G) = 0$. Pick a one-parameter family g_θ of these such that (i) $\forall x \in \mathbb{S}^{2n},\ g_0(x) = id_G = g_{2\pi}(x)$, and (ii) $\forall x \in \mathbb{S}^{2n},\ g_\theta(x) = g_{\theta+2\pi}(x)$. Such a family of maps draw a circle (or a loop) in $\mathcal{G}$. The loop is classified by the homotopy group $\pi_1(\mathcal{G})$. The family of maps is homotopically equivalent to a map from $\mathbb{S}^{2n+1}$ to G [8]. Such maps are classified by $\pi_{2n+1}(G)$. Since we are considering the same family of maps $g_\#$, we conclude

$$\pi_1(\mathcal{G}) = \pi_{2n+1}(G). \tag{2.14}$$

Now, pick a reference gauge configuration $A_0 \in \mathcal{A}$, and consider its gauge transformations by the one-parameter family $g_\#$

$$A_\theta := A_0^{g_\theta}.$$

The resulting orbit draws a one-dimensional circle $\mathbb{S}^1 \subset \mathcal{A}$. In addition, by introducing another parameter $t \in [0, 1]$, we interpolate A_θ inwards to form a two-dimensional disc $D^2 \subset \mathcal{A}$

$$A_{t,\theta} := tA_\theta,$$

which is bounded by A_θ.

It is natural to identify two background gauge configurations which are connected with a gauge transformation. The identification results in a quotient space

$$\mathcal{A}/\mathcal{G}.$$

By construction, the boundary A_θ of the disc is identified to a point in the quotient space, resulting in a two-sphere $\mathbb{S}^2 \subset \mathcal{A}/\mathcal{G}$. Whether the two-sphere is contractible or not is equivalent to whether the loop in $\mathcal{A}$ is trivial or not. The topology of the two-sphere is classified by the second homotopy group $\pi_2(\mathcal{A}/\mathcal{G})$. Using a well-known theorem in mathematics

$$\pi_2(\mathcal{A}/\mathcal{G}) = \ker\{\pi_1(\mathcal{G}) \to \pi_1(\mathcal{A})\}$$

and the topological triviality of $\mathcal{A}$, i.e., $\pi_1(\mathcal{A}) = 0$, we get

$$\pi_2(\mathcal{A}/\mathcal{G}) = \pi_1(\mathcal{G}).$$

Combinining the result with (2.14), we conclude

$$\pi_2(\mathcal{A}/\mathcal{G}) = \pi_{2n+1}(G). \tag{2.15}$$

In other words, whether the sphere is contractible or not is determined by $\pi_{2n+1}(G)$.

[8] One way to define $\mathbb{S}^d$ is as a d-dimensional hypercube $B^d := \left([0,1]^d\right)$ with its boundary ∂B^d identified. Thus the family $g_\# : \mathbb{S}^1 \times \mathbb{S}^{2n} \to G$ can be recognized as a map from $B^{2n+1} = [0, 2\pi] \times B^{2n}$ to G with 'boundary conditions' $g_0(x) = id_G = g_{2\pi}(x)$ and assigning a single element $g'(\theta)$ on ∂B^{2n} for each θ. Because of the assumption $\pi_1(G) = 0$, the loop $g'(\theta)$ is contractible, and gives id_G on ∂B^{2n} for all θ. To sum up, thanks to the two conditions, the map assigns id_G on $\partial B^{2n+1} = \partial\left([0, 2\pi] \times B^{2n}\right) = \mathbb{S}^{2n+1}$, and is homotopically equivalent to another map $\mathbb{S}^{2n+1} \to G$.

With these preparations, we can show that (non-Abelian) anomalies originate from non-triviality of specific bundles, called determinant bundles. We have explained that the fermion partition functions can be considered as maps

$$Z_\psi : \mathcal{A} \to \mathbb{C}.$$

When we mod out the configuration space by gauge transformations, the map induces another 'map'

$$\widetilde{Z_\psi} : \mathcal{A}/\mathcal{G} \to \mathbb{C}.$$

Since the gauge ambiguity manifests as a $U(1)$ phase transformation, we can form a bundle, called determinant bundle, over the gauge orbit space

$$DET = (DET, \pi, \mathcal{A}/\mathcal{G}, (\{\widetilde{Z_\psi}\} \subset)\mathbb{C}, U(1)). \tag{2.16}$$

We would like to show that (non-Abelian) anomalies appear when the bundle is nontrivial. Suppose there was no anomaly. Then Z_ψ assigns a unique value to all gauge equivalent configurations $A \in \mathcal{A}$ (possibly by choosing appropriate local counterterms). This is equivalent to say that $\widetilde{Z_\psi}$ is a well-defined map, assigning a single value for each gauge orbit $[A] \in \mathcal{A}/\mathcal{G}$. Thus the transition functions cannot have nontrivial winding numbers. Since the determinant bundle and its associated principal $U(1)$ bundle

$$P(DET) = (P(DET), \pi_P, \mathcal{A}/\mathcal{G}, U(1)) \tag{2.17}$$

share the same transition functions, triviality of the transition functions of the former implies the latter admits a global section, meaning that $P(DET)$ is trivial. This is equivalent to DET itself be trivial. We conclude

$$\text{no non-Abelian anomaly} \Rightarrow \text{trivial } DET.$$

This is equivalent to its contrapositive statement

$$\text{nontrivial } DET \Rightarrow \exists \text{non-Abelian anomaly}.$$

In the proof, we exploited topological (non)triviality of transition functions of DET, or equivalently those of $P(DET)$. Transition functions which take values in $U(1)$ are classified by $\pi_m(U(1))$, which is nonzero only when $m = 1$. Therefore we only have to consider subspaces $\mathbb{S}^{m+1} = \mathbb{S}^2 (\subset \mathcal{A}/\mathcal{G})$, in which $\mathbb{S}^m = \mathbb{S}^1$ is embedded as an overlap of patches where transition functions are defined. This is why we considered $\mathbb{S}^2$ above. Whether two-spheres in the gauge orbit space are contractible or not is controlled by $\pi_2(\mathcal{A}/\mathcal{G})$, which is equal to the winding number transition functions over $\mathbb{S}^1$ have, which is counted by $\pi_1(U(1))$. To sum up, $P(DET)$ is nontrivial when transition functions have nonvanishing winding numbers, which is counted by $\mathbb{Z} = \pi_1(U(1))$, which is equal to $\pi_2(\mathcal{A}/\mathcal{G}) = \pi_{2n+1}(G)$. We conclude

$$\pi_{2n+1}(G) = \mathbb{Z} \Rightarrow \exists \text{non-Abelian anomaly}. \tag{2.18}$$

An interpretation of the claim should be clear; since the homotopy group is nontrivial, there exists a nontrivial element of the group. Using the element, we can construct a nontrivial transition function. Thus the determinant bundle cannot be trivial, resulting in multivalued (or ambiguous) $\widetilde{Z_\psi}$, a phenomenon called (non-Abelian) anomaly. In other words, $\widetilde{Z_\psi}$ is a local section $\widetilde{Z_\psi} \in \Gamma(\mathcal{A}/\mathcal{G}, DET)$ of the determinant bundle. Moreover, a careful analysis shows that the winding number of transition functions of the bundle is equivalent to the Dirac index in $(2n+2)$ dimensions [3], reproducing the well-known fact. The extra two dimensions have a natural interpretation; one way to compute (non-Abelian) anomalies in $(2n)$-dimensional spacetime is to start from $(2n+2)$-dimensional Chern character

$$\frac{1}{(n+1)!}\mathrm{Str}\left(\frac{iF}{2\pi}\right)^{n+1}, \tag{2.19}$$

which is half the $(2n+2)$-dimensional chiral (a.k.a. Abelian) anomaly. It is well-known that the form can be written as a derivative of the CS $(2n+1)$-form

$$\omega_{2n+1} = \frac{2\pi}{(n+1)!}\left(\frac{i}{2\pi}\right)^{n+1} \times (n+1)\mathrm{Str}\int_0^1 dt A F_t^n,$$

where $F_t := tF + (t^2 - t)A^2$. We have multiplied by (2π) because (2.19) is an integer, and it becomes a well-defined Feynman amplitude e^{-S} with the factor. The result reduces to the familiar form

$$-\frac{1}{4\pi}\mathrm{Str}\left(AdA + \frac{2}{3}A^3\right)$$

in three dimensions, or $n = 1$. In the absence of boundary, the CS form is completely gauge invariant (also under 'large gauge transformations' if an overall coefficient called CS level is an integer), however, if there is a boundary, since gauge transformations $A \mapsto A + D\alpha$ yield a total derivative term, the infinitesimal transformation gives rise to a $(2n)$-dimensional surface term

$$\omega_{2n}^1 = \frac{2\pi}{(n+1)!}\left(\frac{i}{2\pi}\right)^{n+1} \times n(n+1)\int_0^1 dt(1-t)\mathrm{Str}\left[\alpha d(AF_t^{n-1})\right],$$

which can be interpreted as the non-Abelian anomaly (2.2). The procedure to compute $(2n)$-dimensional non-Abelian anomaly from $(2n+2)$-dimensional Chern character is known as anomaly descent. From our discussion above, the two extra directions can be recognized as the parameters (t, θ), which label the disc $D^2 \subset \mathcal{A}$. More physical interpretation is also known; suppose there is a conserved current coupled to G in $(2n+1)$ dimensions. In the presense of $(2n)$-dimensional boundary, the current induces nonzero divergence on the boundary, which is nothing but the $(2n)$-dimensional non-Abelian anomaly. Since the anomaly can be understood as a flow from the $(2n+1)$-dimensional bulk to the $(2n)$-dimensional boundary, the mechanism is known as anomaly inflow [26, 28].

(2.18) gives an answer to the question raised above, namely, (non-Abelian) anomalies originate from nontriviality of (determinant) bundles. Then some questions come up immediately: (i) How the nontriviality of the bundle appears? In other words, we know that field contents control whether anomalies appear or not. Then how the field contents affect the nontriviality of the bundle? (ii) If a bundle is nontrivial, there are nontrivial holonomies under parallel transformations along a loop. Then what are the holonomies of the (determinant) bundles? (iii) What is the relation of the BRST cohomology with ghost number one and the nontriviality of the bundle? If the BRST cohomology with ghost number one is trivial, then there would be no anomaly, so that the bundle would also be trivial. (iv) Why the nontriviality of the bundle appears at the quantum level? In other words, how the (determinant) bundles 'detect' whether the theories are classical or quantum?

At least, the author does not know satisfactory answers to these questions. We would like to address these problems in the future.

3 Holographic trace anomaly

One of the most fundamental questions in QFTs concerns the degrees of freedom; What is the 'number' of degrees of freedom? An anomaly called trace (a.k.a. conformal or Weyl) anomaly is expected to count the 'number' [9]. We would like to discuss holographic trace anomalies in this section.

3.1 Formalism

Beside a slight difference between (bulk) actions in even or odd dimensions, we perform exactly the same analysis in any spacetime dimensions [30]. Thus it would be instructive to explain the general formalism in advance.

Suppose one considers a bulk action in $(d+1)$-dimensional spacetime:

$$S_{\text{bulk}}[\hat{\mathcal{F}}] = \int_{M^{d+1}} d^{d+1}X\sqrt{\hat{\gamma}}\mathcal{L}_{\text{bulk}}(\hat{\mathcal{F}}, \partial_\tau \hat{\mathcal{F}}) - 2\int_{\Sigma^d} d^d x\sqrt{\hat{h}}\hat{K}, \tag{3.1.1}$$

where we have collectively denoted bulk (dynamical) fields $\mathcal{F}$, and quantities with hat are off-shell, i.e., not necessarily solutions of equations of motion. The second term is nothing but the Gibbons-Hawking (GH) term, which is needed to treat a gravitational system in the Hamiltonian formalism. The GH term is defined on a d-dimensional hypersurface $\Sigma^d := \{X \in M^{d+1} | \tau = \text{const.}\}$. We have already implicitly used the Arnowitt-Deser-Misner (ADM) decomposition

$$ds^2 = \hat{\gamma}_{MN}dX^M dX^N = \hat{N}^2(x,\tau)d\tau^2 + \hat{h}_{\mu\nu}(dx^\mu + \hat{\lambda}^\mu(x,\tau)d\tau)(dx^\nu + \hat{\lambda}^\nu(x,\tau)d\tau) \tag{3.1.2}$$

[9] This expectation is indeed proved in two and four dimensional general QFTs [5, 10] and in an arbitrary dimensional QFTs with holographic duals [29].

to introduce the GH term. That is, the extrinsic curvature $K_{\mu\nu}$ is defined by

$$K_{\mu\nu} := \frac{1}{2N}\Big(\partial_\tau h_{\mu\nu} - \nabla_\mu\lambda_\nu - \nabla_\nu\lambda_\mu\Big), \tag{3.1.3}$$

and $K := h^{\mu\nu}K_{\mu\nu}$. We recognize τ as the 'time' direction. N and λ are called Lapse function and shift function, respectively. Employing the ADM decomposition, we rewrite (3.1.1) as

$$S_{\rm bulk}[\hat{\mathcal{F}}] = \int d^dx d\tau \sqrt{\hat{h}}\mathcal{L}_{d+1} + (\text{GH}). \tag{3.1.4}$$

Since the volume integral suffers from divergence originating from the infinite volume of M^{d+1}, we regularize the integral by introducing a cut-off surface at $\tau = \tau_0$[10]. As in the traditional Hamiltonian formalism, we define conjugate momenta by

$$\hat{\pi} := \frac{\partial \mathcal{L}_{d+1}}{\partial(\partial_\tau \hat{\mathcal{F}})}. \tag{3.1.5}$$

Using the conjugate momenta, we further rewrite the bulk action in the first-order form

$$S_{\rm bulk}[\hat{\mathcal{F}}] = \int d^dx d\tau \sqrt{\hat{h}}\Big\{\hat{\pi}\partial_\tau\hat{\mathcal{F}} + \cdots\Big\}. \tag{3.1.6}$$

In general, the first-order form contains some auxiliary fields which do not have τ derivatives reflecting the symmetries of the bulk action. Thus they serve as Lagrange multipliers $\hat{\varphi}$, and their equations of motion produce constraints

$$\frac{1}{\sqrt{\hat{h}}}\frac{\delta S_{\rm bulk}}{\delta\hat{\varphi}} \approx 0. \tag{3.1.7}$$

Conversely, one can show the constraints ensure the symmetries.

Since the equations of motion one obtains from the bulk action are second order, one needs to specify two boundary conditions to fix a solution. Usually one imposes Dirichlet boundary conditions[11] on the cut-off surface $\tau = \tau_0$;

$$\bar{\mathcal{F}}(x, \tau = \tau_0) = \mathcal{F}(x), \tag{3.1.8}$$

where bar implies the field is on-shell, i.e., it is a solution of the equations of motion. Substituting the solution $\bar{\mathcal{F}}(x,\tau)$ into the bulk action, one arrives at an on-shell action as a functional of the Dirichlet boundary conditions (3.1.8);

$$S_{\rm OS}[\mathcal{F};\tau_0] := S_{\rm bulk}[\hat{\mathcal{F}} = \bar{\mathcal{F}}]. \tag{3.1.9}$$

[10]See for example [31].

[11]Another 'boundary' condition is implicit. Namely, one usually requires the solutions be regular in the interior of the bulk.

The Gubser-Klebanov-Polyakov-Witten (GKP-Witten) [32] prescription claims that the boundary 'values' $\mathcal{F}(x)$ are recognized as background fields coupled to operators $\mathcal{O}_{\mathcal{F}}$ in the dual QFT. Thus (connected) correlation functions of the dual QFT (in the presence of background fields $\mathcal{F}$) can be computed by taking functional derivatives of the on-shell action (which is identified with the generating functional $W[\mathcal{F}]$ of connected correlators [33, 32]) with respect to the background fields $\mathcal{F}$;

$$\langle \mathcal{O}_{\mathcal{F}}(x) \rangle_{\text{con.}} \sim \frac{\delta S_{\text{OS}}}{\delta \mathcal{F}(x)}. \tag{3.1.10}$$

The variation of the on-shell action typically has a form

$$\delta S_{\text{OS}}[\mathcal{F}; \tau_0] = -\int d^d x \sqrt{h} \bar{\pi}(x, \tau_0) \delta \mathcal{F}(x). \tag{3.1.11}$$

As a result one gets

$$\bar{\pi}(x, \tau_0) = -\frac{1}{\sqrt{h}} \frac{\delta S_{\text{OS}}}{\delta \mathcal{F}(x)}, \quad \frac{\partial S_{\text{OS}}}{\partial \tau_0} = 0. \tag{3.1.12}$$

These are called Hamilton-Jacobi (HJ) equations. Substituting the HJ equations in the constraints (3.1.7), one finally obtains the flow equations, which typically has a form

$$\{S_{\text{OS}}, S_{\text{OS}}\} = \mathcal{L}_d. \tag{3.1.13}$$

We will explain the definitions of the bracket or $\mathcal{L}_d$ in the explicit computations below.

So as to study the flow equation systematically, one usually assigns an additive number called weight to each element. Essentially, we count the numbers of derivatives by weights w. Hence, we assign $w = 1$ to derivatives ∂. So as to make covariant derivatives also have weights one, we also assign $w = 1$ to gauge fields A. In addition, since integrals are 'inverse' of derivatives, we assign $w = -d$ to volume elements $d^d x$. Since functional derivatives remove the volume element, it is natural to assign $w[\delta/\delta\mathcal{F}] = -w[d^d x \mathcal{F}] = d - [\mathcal{F}]$. These assignments are summarized in Table 1.

elements	weight
$d^d x$	$-d$
$\phi^I(x), h_{\mu\nu}(x), \Gamma[\phi, A, h]$	0
$\partial_\mu, A^a_\mu(x)$	1
$R, R_{\mu\nu}, R_{\mu\nu\rho\sigma}, \partial^2, \ldots$	2
$\delta/\delta A^a_\mu(x)$	$d-1$
$\delta/\delta h_{\mu\nu}(x), \delta/\delta\phi^I(x)$	d

Table 1: assignment of weights in QFT_d

As we have briefly explained, the on-shell action contains information of correlation functions. This means the on-shell action has non-local parts. The non-locality of an on-shell action can be understood as follows; We have solved equations of motion to get on-shell fields with a Dirichlet boundary condition. The boundary condition serves as a source. To interpolate the boundary 'value' $\mathcal{F}(y)$ to the inside of the bulk, say a point X, we sum up all the propagation from the sources to X. The propagation is given by Green's function $G(X,y)$. In short, a value of the on-shell field at X is schematically given by

$$\bar{\mathcal{F}}(X) \sim \int d^d y G(X,y)\mathcal{F}(y).$$

Since the Green's function is the 'inverse' of kinetic operators[12], it exhibits non-locality. The integral also leads to non-locality. Hence, an on-shell action, which is given as a functional of on-shell fields, also contains non-local parts. We separate the non-local parts from the local parts (modulo ambiguities to add finite local counterterms [34]);

$$\frac{1}{2\kappa_{d+1}^2} S_{\mathrm{OS}}[\mathcal{F};\tau_0] \equiv \frac{1}{2\kappa_{d+1}^2} S_{\mathrm{loc}}[\mathcal{F};\tau_0] - \Gamma[\mathcal{F};\tau_0], \tag{3.1.14}$$

where $2\kappa_{d+1}^2$ is the $(d+1)$-dimensional Newton constant. Since we assume the bulk (and also its dual) theories are some kind of effective theories, we employ the totalitarian's principle to parametrize the local action, that is, we write down all terms that are not forbidden by symmetries with general coefficient functionals;

$$S_{\mathrm{loc}}[\mathcal{F};\tau_0] = \int d^d x \sqrt{h}\mathcal{L}_{\mathrm{loc}} = \int d^d x \sqrt{h} \sum_{w=0,2,\cdots} [\mathcal{L}_{\mathrm{loc}}]_w \tag{3.1.15}$$

with for example

$$\begin{aligned}
[\mathcal{L}_{\mathrm{loc}}]_0 &= W(\phi), \\
[\mathcal{L}_{\mathrm{loc}}]_2 &= -\Phi(\phi)R_{(d)} + \frac{1}{2}M_{IJ}(\phi)\boldsymbol{\nabla}^\mu\phi^I\boldsymbol{\nabla}_\mu\phi^J, \\
&\vdots \,.
\end{aligned}$$

See the explicit calculations below for the concrete forms. The coefficient functionals are undetermined at this point, however, some of them are fixed by flow equations. In fact, since the weight is additive, terms with different weights do not mix. The situation is very much like the power counting in effective field theories (EFTs). Thanks to the decomposition, one can solve each part of the flow equation independently. One will find that part of the flow equations with lower weights serve to fix the coefficient functionals. See the explicit examples below.

[12] For example, in the case of free massive scalars, the Green's function can be schematically written as $1/(\Box + m^2)$.

The trace anomalies we were looking for appear in the $w = d$ part of the flow equation because vevs of the stress tensor are given by $\langle T^{\mu\nu}(x)\rangle \sim \delta\Gamma/\delta h_{\mu\nu}(x)$. Traces of the stress tensors $\langle T^{\mu}{}_{\mu}(x)\rangle$ in general enjoy contributions from one-point functions of the QFTs. One can identify their coefficients as beta-function(al)s [13]. Since the beta-function(al)s vanish in CFTs, the trace anomalies are defined as the 'values' $\langle T^{\mu}{}_{\mu}(x)\rangle$ at the zeros of the beta-function(al)s.

Before delving in concrete theories, we explain a close relation between the holographic computation described above and the framework called local renormalization group (LRG) [35].

In the usual Wilsonian RG, one performs global scale transformations with appropriate modifications of coupling constants, which are specified by beta-functions, so as not to disturb IR physics. On the other hand, in the LRG, one performs local scale transformations, e.g., Weyl transformations. In order not to disturb IR physics, we have to promote coupling constants to coupling functions $\lambda \mapsto \lambda(x)$, i.e., background fields. Our starting point in LRG is a generating functional (a.k.a. vacuum functional or Schwinger functional) $W[h,\phi]$:

$$e^{-W[h,\lambda]} := \int \mathcal{D}X \exp\left\{-S[X,h] - \int d^dx\sqrt{h}\lambda^I(x)O_I(x)\right\}. \tag{3.1.16}$$

Here, we collectively expressed coupling functions by λ. In actual problems, it may contain background scalar fields $\phi(x)$, background gauge fields $A(x)$, or more general background fields. Being background fields, $h(x)$ or $\lambda(x)$ serve as devices to compute correlation functions of the theory S:

$$\frac{2}{\sqrt{h}}\frac{\delta W[h,\lambda]}{\delta h^{\mu\nu}(x)} = \langle T_{\mu\nu}(x)\rangle, \quad \frac{1}{\sqrt{h}}\frac{\delta W[h,\lambda]}{\delta\lambda^I(x)} = \langle O_I(x)\rangle. \tag{3.1.17}$$

As stated, LRG transformations are local scale transformations realized by, say Weyl transformations $h_{\mu\nu}(x) \mapsto e^{2\sigma(x)}h_{\mu\nu}(x)$, which do not disturb IR physics. As in the usual RG, we have to modify background fields appropriately. The modifications are specified by beta-functionals. Hence, LRG transformations are generated by an operator called LRG operator

$$\Delta_\sigma := \int d^dx\sigma(x)\left(2h^{\mu\nu}(x)\frac{\delta}{\delta h^{\mu\nu}(x)} - \beta^I[\lambda(x)]\frac{\delta}{\delta\lambda^I(x)}\right) \tag{3.1.18}$$

[13] This fact can be understood as follows; As explained in [31], we can recognize the extra direction in the bulk as the renormalization scale. Hence we can study how theories change by moving in the direction, or by taking derivatives with the radial coordinate. The manipulation brings down the (integrated) trace of the stress tensor. The manipulation thus amounts to compute the one-point function $\langle T^{\mu}{}_{\mu}(x)\rangle$. On the other hand, we know that the change can be absorbed by redefining coupling constants. The change is nothing but beta-functions. In fact, if we take derivatives of $\exp\{-\int\lambda O\}$ with respect to the radial coordinate, we get (integrated) one-point functions multiplied by beta-functions. These arguments are concisely summarized in the operator

$$\Delta_\sigma := \int d^dx\sigma(x)\left(2h_{\mu\nu}\frac{\delta}{\delta h_{\mu\nu}} + \beta^I\frac{\delta}{\delta\phi^I} + \beta^a_\mu\frac{\delta}{\delta A^a_\mu} + \cdots\right).$$

called local renormalization group operator.

with an arbitrary (scalar) function $\sigma(x)$. The formal functional β is defined through an equation called LRG equation which essentially states the Schwinger functional be invariant under the LRG transformations. The equation is an analogue of the Callan-Symanzik equation. In other words, the equation guarantees invariance of IR physics. Since we have to define QFTs on curved spaces to exploit LRG, there are in general trace anomalies. Taking the anomalies into account, we should impose LRG transformation invariance on the Schwinger functional up to the anomaly

$$\Delta_\sigma W[h,\lambda] = \mathcal{A}_\sigma^{\text{anomaly}}[h,\lambda]. \tag{3.1.19}$$

The RHS is a functional of background fields only, which cannot be tuned away with local counterterms made of background fields, as explained in section 2.

Since Weyl transformations commute, we must have

$$\forall \sigma, \sigma', \quad [\Delta_\sigma, \Delta_{\sigma'}] = 0. \tag{3.1.20}$$

Acting (3.1.20) on the Schwinger functional, we obtain WZ consistency condition on $\mathcal{A}^{\text{anomaly}}$:

$$\forall \sigma, \sigma', \quad \Delta_\sigma \mathcal{A}_{\sigma'}^{\text{anomaly}} - \Delta_{\sigma'} \mathcal{A}_\sigma^{\text{anomaly}} = 0. \tag{3.1.21}$$

As we have emphasized in section 2, $\mathcal{A}^{\text{anomaly}}$ is not unique due to the intrinsic ambiguity of QFTs to add local counterterms.

To get a grip of the general formalism of LRG explained above, let us study a simple example of two-dimensional QFT with background scalar fields $\lambda^I(x)$. The LRG operator is given by

$$\Delta_\sigma := \int d^2x \sigma(x) \left(2h^{\mu\nu}(x) \frac{\delta}{\delta h^{\mu\nu}(x)} - \beta^I[\lambda(x)] \frac{\delta}{\delta \lambda^I(x)} \right).$$

The most general anomaly functional is given by

$$\begin{aligned}\mathcal{A}_\sigma^{\text{anomaly}}[h,\lambda] = &\int d^2x \sqrt{h} \sigma(x) \left(\frac{1}{2} C[\lambda] R(x) - \frac{1}{2} G_{IJ}[\lambda] \partial_\mu \lambda^I(x) \partial^\mu \lambda^J(x) \right) \\ &- \int d^2x \sqrt{h} \partial_\mu \sigma(x) D_I[\lambda] \partial^\mu \lambda^I(x),\end{aligned}$$

where R is the Ricci scalar. The WZ condition (3.1.20) gives

$$\partial_I C[\lambda] = G_{IJ}[\lambda] \beta^J[\lambda] - \mathfrak{L}_\beta D_I[\lambda],$$

where

$$\mathfrak{L}_v D_I := v^J \partial_J D_I + \partial_I v^J D_J$$

is the Lie derivative along a vector field v. A redefinition

$$\tilde{C} := C + D_I \beta^I$$

gives

$$\partial_I \tilde{C} = G_{IJ}\beta^J + (\partial_I D_J - \partial_J D_I)\beta^J$$

and

$$\beta^I \partial_I \tilde{C} = G_{IJ}\beta^I\beta^J .$$

The Schwinger functional suffers from an ambiguity to add local counterterms

$$\delta W[h,\lambda] = -\int d^2x\sqrt{h}\left(\frac{1}{2}c[\lambda]R(x) - \frac{1}{2}g_{IJ}[\lambda]\partial_\mu\lambda^I(x)\partial^\mu\lambda^J(x)\right).$$

The local counterterms cause ambiguities

$$\begin{aligned}
\delta C &= \beta^I\partial_I c,\\
\delta G_{IJ} &= \mathfrak{L}_\beta g_{IJ},\\
\delta D_I &= -\partial_I c + g_{IJ}\beta^J,\\
\delta\tilde{C} &= g_{IJ}\beta^I\beta^J.
\end{aligned}$$

One can easily check that the consistency condition is invariant under the ambiguity, confirming the expectation that putative 'anomalies' caused by local counterterms are cohomologically trivial. Comparing the WZ condition in terms of $\tilde{C}$ and the ambiguity δG_{IJ}, one realizes that if one can show an existence of g_{IJ} such that $G_{IJ} + \mathfrak{L}_\beta g_{IJ}$ is positive definite, $\tilde{C}$ decreases monotonically under RG flows, which is nothing but the c-theorem. In fact, it was shown that there exists such a g_{IJ} [35], proving the c-theorem.

In the holographic computations below, one would find a lot of similar expressions in LRG.

3.2 Even dimensions

Since we have elucidated the general formalism in detail in the previous section, we would be brief in this and the next sections.

Let us begin with the case of QFTs in $d = 2n$ spacetime dimensions. A bulk action

$$\begin{aligned}
S_{\text{bulk}}[\hat{\phi},\hat{A},\hat{\gamma}] = &\int_{M^{2n+1}} d^{2n+1}X\sqrt{\hat{\gamma}}\Big\{V(\hat{\phi}) - \hat{R}_{(2n+1)} + \frac{1}{2}L_{IJ}(\hat{\phi})\hat{\gamma}^{MN}\boldsymbol{\nabla}_M\hat{\phi}^I\boldsymbol{\nabla}_N\hat{\phi}^J + \frac{1}{4}B(\hat{\phi})\hat{F}^a_{MN}\hat{F}^{aMN}\Big\}\\
&- 2\int_{\Sigma^{2n}} d^{2n}x\sqrt{\hat{h}}\hat{K}.
\end{aligned} \tag{3.2.1}$$

reduces to the first-order form:

$$
\begin{aligned}
S_{\rm bulk} = \int d^{2n}x d\tau \sqrt{\hat{h}} \Bigg\{ & \hat{\pi}_I \partial_\tau \hat{\phi}^I + \hat{\pi}^{a\mu} \partial_\tau \hat{A}^a_\mu + \hat{\pi}^{\mu\nu} \partial_\tau \hat{h}_{\mu\nu} \\
& + \hat{N} \Big[\frac{1}{2n-1} (\hat{\pi}^\mu_\mu)^2 - \hat{\pi}^2_{\mu\nu} - \frac{1}{2} L^{IJ}(\hat{\phi}) \hat{\pi}_I \hat{\pi}_J - \frac{1}{2B(\hat{\phi})} \hat{h}^{\mu\nu} \hat{\pi}^a_\mu \hat{\pi}^a_\nu \\
& \qquad + V(\hat{\phi}) - \hat{R}_{(2n)} + \frac{1}{2} L_{IJ}(\hat{\phi}) \hat{h}^{\mu\nu} \hat{\boldsymbol{\nabla}}_\mu \hat{\phi}^I \hat{\boldsymbol{\nabla}}_\nu \hat{\phi}^J + \frac{1}{4} B(\hat{\phi}) \hat{F}^a_{\mu\nu} \hat{F}^{a\mu\nu} \Big] \\
& + \hat{\lambda}^\mu \Big[2 \hat{\nabla}^\nu \hat{\pi}_{\mu\nu} - \hat{\pi}_I \hat{\boldsymbol{\nabla}}_\mu \hat{\phi}^I - \hat{F}^a_{\mu\nu} \hat{\pi}^{a\nu} \Big] \\
& + \hat{A}^a_\tau \Big[\hat{\boldsymbol{\nabla}}^a_{b\nu} \hat{\pi}^{b\nu} - (iT^a \hat{\phi})^I \hat{\pi}_I \Big] \Bigg\}.
\end{aligned} \tag{3.2.2}
$$

The canonical momenta conjugate to $\hat{\phi}$, $\hat{A}$, and $\hat{h}$ are respectively given by

$$
\hat{\pi}_I := \frac{\partial \mathcal{L}_{2n+1}}{\partial(\partial_\tau \hat{\phi}^I)} = \frac{1}{\hat{N}} L_{IJ}(\hat{\phi}) (\hat{\boldsymbol{\nabla}}_\tau \hat{\phi}^J - \hat{\lambda}^\mu \hat{\boldsymbol{\nabla}}_\mu \hat{\phi}^J), \tag{3.2.3}
$$

$$
\hat{\pi}^{a\mu} := \frac{\partial \mathcal{L}_{2n+1}}{\partial(\partial_\tau \hat{A}^a_\mu)} = \frac{1}{\hat{N}^3} B(\hat{\phi}) \Big[\hat{N}^2 \hat{h}^{\mu\nu} \hat{F}^a_{\tau\nu} - \hat{\lambda}^\nu (\hat{N}^2 \hat{h}^{\rho\mu} + \hat{\lambda}^\rho \hat{\lambda}^\mu) \hat{F}^a_{\nu\rho} \Big], \tag{3.2.4}
$$

$$
\hat{\pi}^{\mu\nu} := \frac{\partial \mathcal{L}_{2n+1}}{\partial(\partial_\tau \hat{h}_{\mu\nu})} = \hat{K}^{\mu\nu} - \hat{h}^{\mu\nu} \hat{K}. \tag{3.2.5}
$$

As informed in the previous section, (3.2.2) does not have τ derivatives of N, λ, and A_τ, which reflects diffeomorphism invariance in τ- and x-directions, and gauge symmetries of the action, respectively. The three fields are auxiliary fields, and their equations of motion yield the first-class constraints called Hamiltonian constraint, momentum constraint, and Gauss's law, respectively;

$$
\begin{aligned}
\hat{H} := \frac{1}{\sqrt{\hat{h}}} \frac{\delta S}{\delta \hat{N}} = & \frac{1}{2n-1} (\hat{\pi}^\mu_\mu)^2 - \hat{\pi}^2_{\mu\nu} - \frac{1}{2} L^{IJ}(\hat{\phi}) \hat{\pi}_I \hat{\pi}_J - \frac{1}{2B(\hat{\phi})} \hat{h}^{\mu\nu} \hat{\pi}^a_\mu \hat{\pi}^a_\nu \\
& + V(\hat{\phi}) - \hat{R}_{(2n)} + \frac{1}{2} L_{IJ}(\hat{\phi}) \hat{h}^{\mu\nu} \hat{\boldsymbol{\nabla}}_\mu \hat{\phi}^I \hat{\boldsymbol{\nabla}}_\nu \hat{\phi}^J + \frac{1}{4} B(\hat{\phi}) \hat{F}^a_{\mu\nu} \hat{F}^{a\mu\nu} \approx 0,
\end{aligned} \tag{3.2.6}
$$

$$
\hat{P}_\mu := \frac{1}{\sqrt{\hat{h}}} \frac{\delta S}{\delta \hat{\lambda}^\mu} = 2 \hat{\nabla}^\nu \hat{\pi}_{\mu\nu} - \hat{\pi}_I \hat{\boldsymbol{\nabla}}_\mu \hat{\phi}^I - \hat{F}^a_{\mu\nu} \hat{\pi}^{a\nu} \approx 0, \tag{3.2.7}
$$

$$
\hat{G}^a := \frac{1}{\sqrt{\hat{h}}} \frac{\delta S}{\delta \hat{A}^a_\tau} = \hat{\boldsymbol{\nabla}}^a_{b\nu} \hat{\pi}^{b\nu} - (iT^a \hat{\phi})^I \hat{\pi}_I \approx 0. \tag{3.2.8}
$$

A Dirichlet boundary condition at $\tau = \tau_0$

$$
\bar{\phi}^I(x, \tau = \tau_0) = \phi^I(x), \quad \bar{A}_\mu(x, \tau = \tau_0) = A_\mu(x), \quad \bar{h}_{\mu\nu}(x, \tau = \tau_0) = h_{\mu\nu}(x)
$$

(and the regularity of the resulting solutions in the interior of the bulk) fixes a solution. Substitution of the classical solutions into (3.2.2) yields the on-shell action

$$S_{\rm OS}[\phi, A, h; \tau_0] := \int d^{2n}x \int_{\tau_0}^{\infty} d\tau \sqrt{\bar{h}} \Big\{ \bar{\pi}_I \partial_\tau \bar{\phi}^I + \bar{\pi}^{a\mu} \partial_\tau \bar{A}^a_\mu + \bar{\pi}^{\mu\nu} \partial_\tau \bar{h}_{\mu\nu} \Big\}. \tag{3.2.9}$$

Its variation

$$\delta S_{\rm OS} = -\int d^{2n}x \sqrt{h} \Big\{ \bar{\pi}^I(x,\tau_0) \delta\phi^I(x) + \bar{\pi}^{a\mu}(x,\tau_0) \delta A^a_\mu(x) + \bar{\pi}^{\mu\nu}(x,\tau_0) \delta h_{\mu\nu}(x) \Big\} \tag{3.2.10}$$

yields HJ equations

$$\bar{\pi}^I(x,\tau_0) = -\frac{1}{\sqrt{h}} \frac{\delta S_{\rm OS}}{\delta\phi^I(x)}, \quad \bar{\pi}^{a\mu}(x,\tau_0) = -\frac{1}{\sqrt{h}} \frac{\delta S_{\rm OS}}{\delta A^a_\mu(x)}, \quad \bar{\pi}^{\mu\nu}(x,\tau_0) = -\frac{1}{\sqrt{h}} \frac{\delta S_{\rm OS}}{\delta h_{\mu\nu}(x)}, \quad \frac{\partial S_{\rm OS}}{\partial \tau_0} = 0. \tag{3.2.11}$$

Inserting the HJ equations in the Hamiltonian constraint (3.2.6), we obtain the flow equation:

$$\{S_{\rm OS}, S_{\rm OS}\} = \mathcal{L}_{2n}, \tag{3.2.12}$$

where

$$\{S_{\rm OS}, S_{\rm OS}\} := \left(\frac{1}{\sqrt{h}}\right)^2 \left[-\frac{1}{2n-1} \left(h_{\mu\nu} \frac{\delta S_{\rm OS}}{\delta h_{\mu\nu}} \right)^2 + \left(\frac{\delta S_{\rm OS}}{\delta h_{\mu\nu}} \right)^2 + \frac{1}{2} L^{IJ}(\phi) \frac{\delta S_{\rm OS}}{\delta \phi^I} \frac{\delta S_{\rm OS}}{\delta \phi^J} + \frac{1}{2B(\phi)} h_{\mu\nu} \frac{\delta S_{\rm OS}}{\delta A^a_\mu} \frac{\delta S_{\rm OS}}{\delta A^a_\nu} \right] \tag{3.2.13}$$

and

$$\mathcal{L}_{2n} := V(\phi) - R_{(2n)} + \frac{1}{2} L_{IJ}(\phi) \boldsymbol{\nabla}^\mu \phi^I \boldsymbol{\nabla}_\mu \phi^J + \frac{1}{4} B(\phi) F^a_{\mu\nu} F^{a\mu\nu}. \tag{3.2.14}$$

We claimed the constraints originate from the symmetries of the on-shell action. One can indeed see the constraints guarantee these symmetries. Let us begin with the Gauss's law. Using (3.2.8), we have

$$\begin{aligned}
0 &= \int d^dx \sqrt{h} \alpha^a \left(-(iT^a\phi)^I \pi_I + \boldsymbol{\nabla}^a{}_{b\mu} \pi^{b\mu} \right) \\
&= \int d^dx \left\{ \alpha^a (iT^a\phi)^I \frac{\delta S_{\rm OS}}{\delta \phi^I} + \boldsymbol{\nabla}_\mu \alpha^a \frac{\delta S_{\rm OS}}{\delta A^a_\mu} \right\} \\
&= \int d^dx \left(\delta^{\rm gauge}_\alpha \phi^I \frac{\delta S_{\rm OS}}{\delta \phi^I} + \delta^{\rm gauge}_\alpha A^a_\mu \frac{\delta S_{\rm OS}}{\delta A^a_\mu} \right) = \delta^{\rm gauge}_\alpha S_{\rm OS},
\end{aligned}$$

where

$$\delta^{\rm gauge}_\alpha \phi^I := \alpha^a (iT^a\phi)^I, \quad \delta^{\rm gauge}_\alpha A^a_\mu := \boldsymbol{\nabla}_\mu \alpha^a \equiv \nabla_\mu \alpha^a + f^a{}_{bc} A^b_\mu \alpha^c$$

are nothing but the usual gauge transformations. We can see the diffeomorphism invariance in x-directions in the same way. Using (3.2.7) we have

$$
\begin{aligned}
0 &= \int d^dx\sqrt{h}\epsilon^\mu\left(-\pi_I \boldsymbol{\nabla}_\mu\phi^I - F^a_{\mu\nu}\pi^{a\nu} + 2\nabla^\nu\pi_{\mu\nu}\right)\\
&= \int d^dx\left\{\epsilon^\mu\boldsymbol{\nabla}_\mu\phi^I\frac{\delta S_{\rm OS}}{\delta\phi^I} + \epsilon^\mu F^a_{\mu\nu}\frac{\delta S_{\rm OS}}{\delta A^a_\nu} + (\nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu)\frac{\delta S_{\rm OS}}{\delta h_{\mu\nu}}\right\}\\
&= \delta_\epsilon S_{\rm OS} - \int d^dx\sqrt{h}\epsilon^\mu A^a_\mu\left\{-(iT^a\phi)^I\pi_I + \boldsymbol{\nabla}^a{}_{b\nu}\pi^{b\nu}\right\},
\end{aligned}
$$

where

$$
\delta_\epsilon\phi^I := \mathcal{L}_\epsilon\phi^I \equiv \epsilon^\mu\partial_\mu\phi^I, \quad \delta_\epsilon A^a_\mu := \mathcal{L}_\epsilon A^a_\mu \equiv \epsilon^\nu\partial_\nu A^a_\mu + \partial_\mu\epsilon^\nu A^a_\nu, \quad \delta_\epsilon h_{\mu\nu} := \mathcal{L}_\epsilon h_{\mu\nu} \equiv \nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu
$$

are nothing but the Lie derivatives along an arbitrary vector field ϵ.

Going back to the analysis of the flow equations, we divide the on-shell action as in (3.1.14). Employing the totalitarian's principle, we parametrize the local Lagrangians as

$$
\mathcal{L}_{\rm loc} \equiv \sum_{w=0,2,4,\cdots}[\mathcal{L}_{\rm loc}]_w, \tag{3.2.15}
$$

where

$$
[\mathcal{L}_{\rm loc}]_0 := W(\phi), \tag{3.2.16}
$$

$$
[\mathcal{L}_{\rm loc}]_2 := -\Phi(\phi)R_{(2n)} + \frac{1}{2}M_{IJ}(\phi)\boldsymbol{\nabla}^\mu\phi^I\boldsymbol{\nabla}_\mu\phi^J, \tag{3.2.17}
$$

and so on. We also define the corresponding actions by

$$
S_{{\rm loc};w-2n} := \int d^{2n}x\sqrt{h}[\mathcal{L}_{\rm loc}]_w. \tag{3.2.18}
$$

See the Table 1.

We are now ready to compute holographic trace anomalies. To compute the quantity, substitute (3.1.14) in (3.2.12). Those terms with $w < 2n$ fix some coefficient functionals; for example, $w = 0$ part yields

$$
V = -\frac{2n}{4(2n-1)}W^2 + \frac{1}{2}L^{IJ}\partial_I W\partial_J W. \tag{3.2.19}
$$

For $w = 2$

$$
-1 = \frac{2n-2}{2(2n-1)}W\Phi - L^{IJ}\partial_I W\partial_J\Phi, \tag{3.2.20}
$$

$$
\begin{aligned}
\frac{1}{2}L_{IJ} = &-\frac{2n-2}{4(2n-1)}WM_{IJ} - L^{KL}\partial_K W\Gamma_{L;IJ}\\
&- W\partial_I\partial_J\Phi - \frac{1}{2B}M_{IK}M_{JL}(T^a\phi)^K(T^a\phi)^L,
\end{aligned} \tag{3.2.21}
$$

$$
0 = W\partial_K\Phi + L^{IJ}\partial_I W M_{JK}. \tag{3.2.22}
$$

For $w = 4$,

$$\frac{1}{4} B F^a_{\mu\nu} F^{a\mu\nu} = [\{S_{\rm OS}, S_{\rm OS}\}]_4. \tag{3.2.23}$$

For $w = 2n \neq 4$,

$$\begin{aligned}[\mathcal{L}_{2n}]_{2n} = \frac{2\kappa^2_{2n+1} W}{2(2n-1)} \frac{2}{\sqrt{h}} h_{\mu\nu} \frac{\delta\Gamma}{\delta h_{\mu\nu}} - \frac{2\kappa^2_{2n+1}}{\sqrt{h}} L^{IJ} \partial_I W \frac{\delta\Gamma}{\delta\phi^J} \\ - \frac{2\kappa^2_{2n+1}}{hB} h_{\mu\nu} \frac{\delta[S_{\rm loc}]_2}{\delta A^a_\mu} \frac{\delta\Gamma}{\delta A^a_\nu} + [\{S_{\rm loc}, S_{\rm loc}\}]_{2n}.\end{aligned} \tag{3.2.24}$$

According to the GKP-Witten prescription, correlation functions of local operators $O_I(x)$, $J^{a\mu}(x)$, and $T^{\mu\nu}(x)$ in the dual QFTs can be computed by taking functional derivatives with their dual external fields $\phi^I(x)$, $A^a_\mu(x)$, and $h_{\mu\nu}(x)$, respectively;

$$\langle O^I(x)\rangle := \frac{1}{\sqrt{h}} \frac{\delta\Gamma}{\delta\phi^I(x)}, \quad \langle J^{a\mu}(x)\rangle := \frac{1}{\sqrt{h}} \frac{\delta\Gamma}{\delta A^a_\mu(x)}, \quad \langle T^{\mu\nu}(x)\rangle := \frac{2}{\sqrt{h}} \frac{\delta\Gamma}{\delta h_{\mu\nu}(x)}. \tag{3.2.25}$$

Substituting (3.1.10) into (3.2.24), and solving it in terms of the trace of the stress tensor, one obtains

$$\begin{aligned}\langle T^\mu{}_\mu\rangle = \frac{2(2n-1)}{W} L^{IJ} \frac{1}{\sqrt{h}} \frac{\delta S_{{\rm loc};0-2n}}{\delta\phi^I} \langle O^J\rangle + \frac{2(2n-1)}{BW} h_{\mu\nu} \frac{1}{\sqrt{h}} \frac{\delta S_{{\rm loc};2-2n}}{\delta A^a_\mu} \langle J^{a\nu}\rangle \\ + \frac{2(2n-1)}{2\kappa^2_{2n+1} W} \Big([\mathcal{L}_{2n}]_{2n} - [\{S_{\rm loc}, S_{\rm loc}\}]_{2n} \Big),\end{aligned} \tag{3.2.26}$$

The coefficients of the one-point functions in the RHS define beta-functions:

$$\beta^I(\phi) := -\frac{2(2n-1)}{W(\phi)} L^{IJ}(\phi) \frac{1}{\sqrt{h}} \frac{\delta S_{{\rm loc};0-2n}}{\delta\phi^I} = -\frac{2(2n-1)}{W(\phi)} L^{IJ}(\phi) \partial_J W(\phi), \tag{3.2.27}$$

$$\beta^a_\mu(\phi, A) := -\frac{2(2n-1)}{B(\phi)W(\phi)} h_{\mu\nu} \frac{1}{\sqrt{h}} \frac{\delta S_{{\rm loc};2-2n}}{\delta A^a_\nu} = \frac{2(2n-1)}{B(\phi)W(\phi)} M_{IJ}(\phi) (iT^a\phi)^I \boldsymbol{\nabla}_\mu \phi^J. \tag{3.2.28}$$

Note that if we substitute the expressions (3.2.27) and (3.2.28), (3.2.26) exactly has the form of the LRG operator (3.1.18). As in [36], we define $\beta^a_\mu(\phi, A) \equiv \rho^a_I(\phi) \boldsymbol{\nabla}_\mu \phi^I$, i.e.,

$$\rho^a_I(\phi) = \frac{2(2n-1)}{B(\phi)W(\phi)} M_{IJ}(\phi) (iT^a\phi)^J.$$

Then one can easily check that the suggested properties [37] of the vector beta-functions are indeed satisfied: (i) The gradient property $\beta_\mu \propto \delta S/\delta A$ can be obviously seen to be satisfied from the definition (3.2.28). (ii) The orthogonal relation

$$\rho^a_I \beta^I = \frac{4(2n-1)^2}{BW} (iT^a\phi)^I \partial_I \Phi = 0$$

follows where we have used (3.2.22) and the gauge invariance of Φ

$$\forall\alpha, \quad \delta_\alpha^{\text{gauge}}\Phi(\phi) = \alpha^a(iT^a\phi)^I\partial_I\Phi(\phi) = 0.$$

(iii) The Higgs-like relations can be shown as follows; define a local RG operator

$$\Delta_\sigma := \int d^{2n}x\sigma(x)\Big\{2h_{\mu\nu}\frac{\delta}{\delta h_{\mu\nu}(x)} + \beta^I\frac{\delta}{\delta\phi^I(x)} + \rho_I^a\boldsymbol{\nabla}_\mu\phi^I\frac{\delta}{\delta A_\mu^a(x)}\Big\},$$

and compare coefficients of n-point functions. Then one obtains anomalous dimensions

$$\gamma^I{}_J = -\partial_J\beta^I + \rho_J^a(iT^a\phi)^I, \quad \gamma^a{}_b = \rho_I^c\delta_{bc}(iT^a\phi)^I. \tag{3.2.29}$$

These expressions manifest the suggested Higgs-like relations. (iv) Finally, the equivalence

$$\beta_\mu = 0 \iff \boldsymbol{\nabla}_\mu J^{a\mu} = 0$$

can be shown to be satisfied. $\Leftarrow$ follows from the Gauss's law (3.2.8). Using the HJ equations, (3.2.8) can be rewritten as

$$\left(\boldsymbol{\nabla}_\mu\frac{\delta S_{\text{OS}}}{\delta A_\mu(x)}\right)^a - (iT^a\phi)^I\frac{\delta S_{\text{OS}}}{\delta\phi^I(x)} = 0.$$

Since local pieces of the on-shell action are gauge invariant by definition, only non-local piece yields nontrivial result

$$\boldsymbol{\nabla}_\mu J^{a\mu} = (iT^a\phi)^I O_I \tag{3.2.30}$$

as an operator equation. Since O_I is an operator (not redundant in general), it cannot always be zero. Hence $\boldsymbol{\nabla}_\mu J^{a\mu} = 0$ requires $(iT^a\phi)^I = 0$, and $\beta_\mu^a = 0$ follows immediately from the expression (3.2.28). On the other hand, $\Rightarrow$ can be shown by case analysis. Since we are assuming M_{IJ} to be invertible, there are two possibilities which realize $\beta_\mu = 0$, namely $(iT^a\phi)^I = 0$ or $\boldsymbol{\nabla}_\mu\phi^I = 0$. In the case of the former, current conservation follows immediately from (3.2.30). In the latter case, since $(iT^a\phi)^I \neq 0$ in general, ϕ does not belong to the singlet. Recalling the definition of the covariant derivative

$$\boldsymbol{\nabla}_\mu\phi^I \equiv \partial_\mu\phi^I - A_\mu^a(iT^a\phi)^I,$$

we need $\phi^I = \text{const.} = 0$ for $\boldsymbol{\nabla}_\mu\phi^I = 0$ to be satisfied for an arbitrary A. Thus we also get the current conservation.

It is also possible to slightly generalize the holographic a-theorem in the presence of the gauge fields. We just discuss four dimensions here. Generalizations to other dimensions may be tedious but are straightforward.

The most general form of the trace anomaly in four dimensions are given by [38]

$$\langle T^\mu{}_\mu\rangle\Big|_{\text{anomaly}} = -\frac{1}{4}AE_4 + CW_{\mu\nu\rho\sigma}^2 - E^{\mu\nu}G_{IJ}\boldsymbol{\nabla}_\mu\phi^I\boldsymbol{\nabla}_\nu\phi^J + 2\boldsymbol{\nabla}_\mu(W_IE^{\mu\nu}\boldsymbol{\nabla}_\nu\phi^I) + \cdots,$$

where $E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$ is the Euler density, $W^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2$ is the Weyl tensor squared, and $E^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}\gamma^{\mu\nu}R$ is the Einstein tensor. The other terms are not important for our discussion. Comparing this form with the explicit expression obtained from (3.2.26), we obtain

$$A = -\frac{12\Phi^2}{2\kappa_5^2 W}, \quad C = -\frac{3\Phi^2}{2\kappa_5^2 W}, \quad G_{IJ} = -\frac{12}{2\kappa_5^2 W^2}\Phi\partial_{(I}W\partial_{J)}\Phi + \frac{6}{2\kappa_5^2 W}\partial_I\Phi\partial_J\Phi - \frac{6}{2\kappa_5^2 W}\Phi M_{IJ},$$

where $\partial_{(I}W\partial_{J)}\Phi := \frac{1}{2}\partial_I W\partial_J\Phi + (I \leftrightarrow J)$. These expressions are obtained in a shceme $S_{\mathrm{loc};4-4} = 0$, however, we can 'modify' these by adding local counterterms [34]. Local counterterms, which are consistent with symmetries, have the same form as the trace anomaly

$$S_{\mathrm{loc};4-4} = \int d^4x\sqrt{h}\Big\{ - E^{\mu\nu} g_{IJ} \boldsymbol{\nabla}_\mu \phi^I \boldsymbol{\nabla}_\nu \phi^J + \cdots \Big\}.$$

The local counterterms cause shifts

$$\tilde{A} \mapsto \tilde{A}' := \tilde{A} + g_{IJ}\beta^I\beta^J, \quad G_{IJ} \mapsto G'_{IJ} := G_{IJ} + \mathfrak{L}_\beta g_{IJ},$$

where

$$\tilde{A} := A + W_I\beta^I.$$

As discussed in [29], holographic c-function in d-dimensions are defined as $c_h \sim 1/W^{d-1}$. Chosen the coefficient correctly,

$$c_h := -\frac{27}{2\kappa_5^2}\frac{1}{W^3} \tag{3.2.31}$$

matches C given above at fixed points. Then using flow equations with lower weights, one can show

$$\beta^I\partial_I c_h = \frac{1}{2}c_h L_{IJ}\beta^I\beta^J. \tag{3.2.32}$$

Using the definition (3.2.31), we can rewrite our scalar beta-function (3.2.27) as

$$\beta^I = \frac{2}{c_h}L^{IJ}\partial_J c_h. \tag{3.2.33}$$

With the positivity of c_h, which is needed to interpret c_h as a 'number' of degrees of freedom, and the positive definiteness of L_{IJ}, (3.2.33) manifests the gradient flow nature of holographic RG flows. (3.2.32) tells us that $c_h L_{IJ}$ be identified with the Zamolodchikov metric. Furthermore, in [39], we identified

$$\tilde{A}' = 4c_h$$

by considering the most general form of local counterterms.

3.3 Odd dimensions

It was believed that there is no odd-dimensional trace anomalies mainly because one cannot write suitable terms. However, Nakayama pointed out [36] that one can do write candidate terms if one incorporates background scalar and gauge fields. For concreteness, we concentrate on the case of three-dimensional QFTs. We will find that the putative three-dimensional holographic trace anomalies vanish on conformal fixed points. We will clarify what is responsible for the vanishing holographic trace anomalies. Since the reasoning also holds in other odd-dimensional CFTs, our analysis exhaustively rules out the possibility of nonzero odd-dimensional holograhic trace anomalies.

The analysis is exactly the same. Since we have to supply the Levi-Civita tensor to obtain the candidate terms, we introduce the θ term. This is the only difference. Then a bulk action

$$
\begin{aligned}
&S_{\text{bulk}}[\hat{\phi},\hat{A},\hat{\gamma}]\\
&=\int_{M^4} d^4X\sqrt{\hat{\gamma}}\left\{V(\hat{\phi})-\hat{R}_{(4)}+\frac{1}{2}L_{IJ}(\hat{\phi})\hat{\gamma}^{MN}\boldsymbol{\nabla}_M\hat{\phi}^I\boldsymbol{\nabla}_\nu\hat{\phi}^J+\frac{1}{4}B(\hat{\phi})\hat{F}^a_{MN}\hat{F}^{aMN}+\frac{1}{4}\Theta\epsilon^{MNPQ}\hat{F}^a_{MN}\hat{F}^a_{PQ}\right\}\\
&\quad-2\int_{\Sigma^3} d^3x\sqrt{\hat{h}}\hat{K}
\end{aligned}
\tag{3.3.1}
$$

reduces to the first-order action

$$
\begin{aligned}
S_{\text{bulk}}=\int d^3xd\tau\sqrt{\hat{h}}\Bigg\{&\hat{\pi}^I\partial_\tau\hat{\phi}^I+\hat{\pi}^{a\mu}\partial_\tau\hat{A}^a_\mu+\hat{\pi}^{\mu\nu}\partial_\tau\hat{h}_{\mu\nu}\\
&+\hat{N}\left[\frac{1}{2}\hat{\pi}^2-\hat{\pi}^2_{\mu\nu}-\frac{1}{2}L^{IJ}(\hat{\phi})\hat{\pi}^I\hat{\pi}^J-\frac{1}{2B(\hat{\phi})}\hat{h}^{\mu\nu}\hat{\pi}^a_\mu\hat{\pi}^a_\nu-\frac{\Theta}{B(\hat{\phi})}\epsilon^{\mu\nu\rho}_{(3)}\hat{\pi}^a_\mu\hat{F}^a_{\nu\rho}\right.\\
&\quad\left.+V(\hat{\phi})-\hat{R}_{(3)}+\frac{1}{2}L^{IJ}(\hat{\phi})\hat{h}^{\mu\nu}\boldsymbol{\nabla}_\mu\hat{\phi}^I\boldsymbol{\nabla}_\nu\hat{\phi}^J+\left(\frac{1}{4}B(\hat{\phi})+\frac{\Theta^2}{B(\hat{\phi})}\right)\hat{F}^a_{\mu\nu}\hat{F}^{a\mu\nu}\right]\\
&+\hat{\lambda}^\mu\left[2\hat{\nabla}^\nu\hat{\pi}_{\mu\nu}-\hat{\pi}^I\boldsymbol{\nabla}_\mu\hat{\phi}^I-\hat{F}^a_{\mu\nu}\hat{\pi}^{a\nu}\right]\\
&+\hat{A}^a_\tau\left[\boldsymbol{\nabla}_\mu\hat{\pi}^{a\mu}-(iT^a\hat{\phi})^I\hat{\pi}^I\right],
\end{aligned}
\tag{3.3.2}
$$

where

$$\hat{\pi}^I := \frac{\partial \mathcal{L}_4}{\partial(\partial_\tau \hat{\phi}^I)} = \frac{1}{\hat{N}} L^{IJ}(\hat{\phi}) \left(\hat{\boldsymbol{\nabla}}_\tau \hat{\phi}^J - \hat{\lambda}^\mu \hat{\boldsymbol{\nabla}}_\mu \hat{\phi}^J \right), \tag{3.3.3}$$

$$\begin{aligned}\hat{\pi}^{a\mu} :=& \frac{\partial \mathcal{L}_4}{\partial(\partial_\tau \hat{A}^a_\mu)} = \frac{1}{\hat{N}^3} B(\hat{\phi}) \left[\hat{N}^2 \hat{h}^{\mu\nu} \hat{F}^a_{\tau\nu} - \hat{\lambda}^\nu \left(\hat{N}^2 \hat{h}^{\rho\mu} + \hat{\lambda}^\rho \hat{\lambda}^\mu \right) \hat{F}^a_{\nu\rho} \right] - \hat{N} \Theta \epsilon^{\mu\nu\rho\tau}_{(4)} \hat{F}^a_{\nu\rho} \\ &= \frac{1}{\hat{N}} B(\hat{\phi}) \left[\hat{h}^{\mu\nu} \hat{F}^a_{\tau\nu} - \hat{\lambda}^\nu \hat{h}^{\rho\mu} \hat{F}^a_{\nu\rho} \right] - \Theta \epsilon^{\mu\nu\rho}_{(3)} \hat{F}^a_{\nu\rho},\end{aligned} \tag{3.3.4}$$

$$\hat{\pi}^{\mu\nu} := \frac{\partial \mathcal{L}_4}{\partial(\partial_\tau \hat{h}_{\mu\nu})} = \hat{K}^{\mu\nu} - \hat{h}^{\mu\nu} \hat{K}. \tag{3.3.5}$$

Equations of motion of the auxiliary fields N, λ, and A_τ produce first-class constraints

$$\begin{aligned}\hat{H} :=& \frac{1}{\sqrt{\hat{h}}} \frac{\delta \boldsymbol{S}}{\delta \hat{N}} \\ =& \frac{1}{2} \hat{\pi}^2 - \hat{\pi}^2_{\mu\nu} - \frac{1}{2} L^{IJ} \hat{\pi}^I \hat{\pi}^J - \frac{1}{2B} \hat{h}^{\mu\nu} \hat{\pi}^a_\mu \hat{\pi}^a_\nu - \frac{\Theta}{B} \epsilon^{\mu\nu\rho}_{(3)} \hat{\pi}^a_\mu \hat{F}^a_{\nu\rho} \\ &+ V - \hat{R}_{(3)} + \frac{1}{2} L^{IJ} \hat{h}^{\mu\nu} \hat{\boldsymbol{\nabla}}_\mu \hat{\phi}^I \hat{\boldsymbol{\nabla}}_\nu \hat{\phi}^J + \left(\frac{1}{4} B + \frac{\Theta^2}{B} \right) \hat{F}^a_{\mu\nu} \hat{F}^{a\mu\nu} \approx 0,\end{aligned} \tag{3.3.6}$$

$$\hat{P}_\mu := \frac{1}{\sqrt{\hat{h}}} \frac{\delta \boldsymbol{S}}{\delta \hat{\lambda}^\mu} = 2 \hat{\nabla}^\nu \hat{\pi}_{\mu\nu} - \hat{\pi}^I \hat{\boldsymbol{\nabla}}_\mu \hat{\phi}^I - \hat{F}^a_{\mu\nu} \hat{\pi}^{a\nu} \approx 0, \tag{3.3.7}$$

$$\hat{G}^a := \frac{1}{\sqrt{\hat{h}}} \frac{\delta \boldsymbol{S}}{\delta \hat{A}^a_\tau} = \hat{\boldsymbol{\nabla}}_\mu \hat{\pi}^{a\mu} - (i T^a \hat{\phi})^I \hat{\pi}^I \approx 0. \tag{3.3.8}$$

A solution of the EOMs with a Dirichlet boundary condition at $\tau = \tau_0$

$$\bar{\phi}^I(x, \tau = \tau_0) = \phi^I(x), \quad \bar{A}_\mu(x, \tau = \tau_0) = A_\mu(x), \quad \bar{h}_{\mu\nu}(x, \tau = \tau_0) = h_{\mu\nu}(x)$$

provides an on-shell action

$$\begin{aligned}S_{\mathrm{OS}}[\phi, A, h; \tau_0] :=& S_{\mathrm{bulk}}[\hat{\phi} = \bar{\phi}, \hat{A} = \bar{A}, \hat{h} = \bar{h}] \\ &= \int d^3x \int_{\tau_0}^\infty d\tau \sqrt{\bar{h}} \left\{ \bar{\pi}^I \partial_\tau \bar{\phi}^I + \bar{\pi}^{a\mu} \partial_\tau \bar{A}^a_\mu + \bar{\pi}^{\mu\nu} \partial_\tau \bar{h}_{\mu\nu} \right\}.\end{aligned} \tag{3.3.9}$$

Its variation

$$\delta S_{\mathrm{OS}}[\phi, A, h; \tau_0] = - \int d^3x \sqrt{h} \left\{ \bar{\pi}^I(x, \tau_0) \delta \phi^I(x) + \bar{\pi}^{a\mu}(x, \tau_0) \delta A^a_\mu(x) + \bar{\pi}^{\mu\nu}(x, \tau_0) \delta h_{\mu\nu}(x) \right\} \tag{3.3.10}$$

yields HJ equations

$$\bar{\pi}^I(x, \tau_0) = -\frac{1}{\sqrt{h}} \frac{\delta S_{\mathrm{OS}}}{\delta \phi^I(x)}, \quad \bar{\pi}^{a\mu}(x, \tau_0) = -\frac{1}{\sqrt{h}} \frac{\delta S_{\mathrm{OS}}}{\delta A^a_\mu(x)}, \quad \bar{\pi}^{\mu\nu}(x, \tau_0) = -\frac{1}{\sqrt{h}} \frac{\delta S_{\mathrm{OS}}}{\delta h_{\mu\nu}(x)}, \quad \frac{\partial S_{\mathrm{OS}}}{\partial \tau_0} = 0. \tag{3.3.11}$$

Substituting the HJ equations in the Hamiltonian constraint (3.3.6), one obtains the flow equation

$$\{S_{\mathrm{OS}}, S_{\mathrm{OS}}\} = \mathcal{L}_3 \tag{3.3.12}$$

where

$$\begin{aligned}\{S_{\mathrm{OS}}, S_{\mathrm{OS}}\} := \left(\frac{1}{\sqrt{h}}\right)^2 \Bigg[& -\frac{1}{2}\left(h_{\mu\nu}\frac{\delta S_{\mathrm{OS}}}{\delta h_{\mu\nu}}\right)^2 + \left(\frac{\delta S_{\mathrm{OS}}}{\delta h_{\mu\nu}}\right)^2 + \frac{1}{2}L^{IJ}(\phi)\frac{\delta S_{\mathrm{OS}}}{\delta \phi^I}\frac{\delta S_{\mathrm{OS}}}{\delta \phi^J} \\ & + \frac{1}{2B(\phi)}h_{\mu\nu}\frac{\delta S_{\mathrm{OS}}}{\delta A^a_\mu}\frac{\delta S_{\mathrm{OS}}}{\delta A^a_\nu} - \frac{\Theta}{B(\phi)}\sqrt{h}\epsilon^{\mu\nu\rho}_{(3)}\frac{\delta S_{\mathrm{OS}}}{\delta A^{a\mu}}F^a_{\nu\rho}\Bigg]\end{aligned} \tag{3.3.13}$$

and

$$\mathcal{L}_3 := V(\phi) - R_{(3)} + \frac{1}{2}L^{IJ}(\phi)\boldsymbol{\nabla}^\mu\phi^I\boldsymbol{\nabla}_\mu\phi^J + \left(\frac{1}{4}B(\phi) + \frac{\Theta^2}{B(\phi)}\right)F^a_{\mu\nu}F^{a\mu\nu}. \tag{3.3.14}$$

Invariance of the on-shell action under various symmetry transformations can be shown in the same way as in even dimensions.

In order to study the flow equation systematically, we separate the on-shell action into local parts and the non-local part as (3.1.14). Employing the same weight assignment as in the table 1, we write

$$S_{\mathrm{loc}}[\phi, A, h] = \int d^3x\sqrt{h}\mathcal{L}_{\mathrm{loc}} = \int d^3x\sqrt{h}\sum_{w=0,2,3,\ldots}[\mathcal{L}_{\mathrm{loc}}]_w$$

with

$$[\mathcal{L}_{\mathrm{loc}}]_0 = W(\phi)\ , \tag{3.3.15}$$

$$[\mathcal{L}_{\mathrm{loc}}]_2 = -\Phi(\phi)R_{(3)} + \frac{1}{2}M^{IJ}(\phi)\boldsymbol{\nabla}^\mu\phi^I\boldsymbol{\nabla}_\mu\phi^J, \tag{3.3.16}$$

$$\begin{aligned}[\mathcal{L}_{\mathrm{loc}}]_3 = {} & \epsilon^{\mu\nu\rho}_{(3)}D^{IJK}(\phi)\boldsymbol{\nabla}_\mu\phi^I\boldsymbol{\nabla}_\nu\phi^J\boldsymbol{\nabla}_\rho\phi^K + \epsilon^{\mu\nu\rho}_{(3)}E^I(\phi)(F_{\mu\nu})^{IJ}\boldsymbol{\nabla}_\rho\phi^J \\ & + \epsilon^{\mu\nu\rho}_{(3)}\frac{k_{\mathrm{CS}}}{4\pi}\mathrm{tr}\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right).\end{aligned} \tag{3.3.17}$$

We would like to stress that there is no $w = 1$ term because one cannot construct a Lorentz singlet from just one derivative, or one gauge field. We also define

$$S_{{}_{\mathrm{loc};w-3}} := \int d^3x\sqrt{h}[\mathcal{L}_{\mathrm{loc}}]_w. \tag{3.3.18}$$

Thanks to the weight assignment, flow equation is decomposed in each weight, and those terms with lower weights fix some coefficient functionals: for example, $w = 0$:

$$V(\phi) = -\frac{3}{8}W^2 + \frac{1}{2}L^{IJ}\partial^I W\partial^J W, \tag{3.3.19}$$

$w = 2:$

$$-1 = \frac{1}{4}W - L^{IJ}\partial^I W \partial^J \Phi, \tag{3.3.20}$$

$$\begin{aligned}\frac{1}{2}L^{IJ} &= -\frac{1}{8}WM^{IJ} - L^{KL}\partial^K W \Gamma^{L;IJ} \\ &\quad - W\partial^I\partial^J\Phi - \frac{1}{2B}M^{IK}M^{JL}(T^a\phi)^K(T^a\phi)^L,\end{aligned} \tag{3.3.21}$$

$$0 = W\partial^K\Phi + L^{IJ}\partial^I W M^{JK}, \tag{3.3.22}$$

$w = 3:$

$$\begin{aligned}0 = \left(\frac{1}{\sqrt{h}}\right)^2 \Bigg\{ & 2\kappa_4^2 \left(h_{\rho\sigma}\frac{\delta S_{\mathrm{loc};0-3}}{\delta h_{\rho\sigma}}\right) h_{\mu\nu}\frac{\delta}{\delta h_{\mu\nu}}\left(\Gamma - \frac{1}{2\kappa_4^2}S_{\mathrm{loc};3-3}\right) - 4\kappa_4^2 \frac{\delta S_{\mathrm{loc};0-3}}{\delta h^{\mu\nu}}\frac{\delta}{\delta h_{\mu\nu}}\left(\Gamma - \frac{1}{2\kappa_4^2}S_{\mathrm{loc};3-3}\right) \\ & - 2\kappa_4^2 L^{IJ}\frac{\delta S_{\mathrm{loc};0-3}}{\delta\phi^I}\frac{\delta}{\delta\phi^J}\left(\Gamma - \frac{1}{2\kappa_4^2}S_{\mathrm{loc};3-3}\right) - \frac{2\kappa_4^2}{B}h_{\mu\nu}\frac{\delta S_{\mathrm{loc};2-3}}{\delta A^a_\mu}\frac{\delta}{\delta A^a_\nu}\left(\Gamma - \frac{1}{2\kappa_4^2}S_{\mathrm{loc};3-3}\right) \\ & - \frac{\Theta}{B}\sqrt{h}\epsilon_{(3)}^{\mu\nu\rho}\frac{\delta S_{\mathrm{loc};2-3}}{\delta A^{a\mu}}F^a_{\nu\rho}\Bigg\},\end{aligned} \tag{3.3.23}$$

$w = 4:$

$$\{S_{\mathrm{OS}}, S_{\mathrm{OS}}\}_{w=4} = \left(\frac{1}{4}B + \frac{\Theta^2}{B}\right)F^a_{\mu\nu}F^{a\mu\nu}. \tag{3.3.24}$$

We can solve the $w = 3$ part in terms of the trace of the stress tensor as before:

$$\begin{aligned}\langle T^\mu{}_\mu\rangle = & \frac{2}{2\kappa_4^2}h_{\mu\nu}\frac{1}{\sqrt{h}}\frac{\delta S_{\mathrm{loc};3-3}}{\delta h_{\mu\nu}} + \frac{4}{W}L^{IJ}\frac{1}{\sqrt{h}}\frac{\delta S_{\mathrm{loc};0-3}}{\delta\phi^I}\frac{1}{\sqrt{h}}\langle O'^J\rangle + \frac{4}{BW}h_{\mu\nu}\frac{1}{\sqrt{h}}\frac{\delta S_{\mathrm{loc};2-3}}{\delta A^a_\mu}\langle J'^{a\nu}\rangle \\ & + \frac{1}{2\kappa_4^2}\frac{4\Theta}{BW}\epsilon_{(3)}^{\mu\nu\rho}\frac{1}{\sqrt{h}}\frac{\delta S_{\mathrm{loc};2-3}}{\delta A^{a\mu}}F^a_{\nu\rho},\end{aligned} \tag{3.3.25}$$

where $\langle\mathcal{O}'\rangle$ is defined as a vev of an operator $\mathcal{O}$ with counterterms taken into account, e.g., $\langle O'^I\rangle := \frac{1}{\sqrt{h}}\frac{\delta}{\delta\phi^I}\left(\Gamma - \frac{1}{2\kappa_4^2}S_{\mathrm{loc};3-3}\right)$. This expression allows us to identify the coefficients of the vevs as beta-function(al)s:

$$\begin{aligned}\beta^I(\phi) :=& -\frac{4}{W}L^{IJ}\frac{1}{\sqrt{h}}\frac{\delta S_{\mathrm{loc};0-3}}{\delta\phi^J} \\ =& -\frac{4}{W}L^{IJ}\partial^J W,\end{aligned} \tag{3.3.26}$$

$$\begin{aligned}\beta^a_\mu(\phi, A) \equiv \rho^a_I(\phi)\boldsymbol{\nabla}_\mu\phi^I :=& -\frac{4}{BW}h_{\mu\nu}\frac{1}{\sqrt{h}}\frac{\delta S_{\mathrm{loc};2-3}}{\delta A^a_\nu} \\ =& \frac{4}{BW}M^{IJ}(iT^a\phi)^J\boldsymbol{\nabla}_\mu\phi^I.\end{aligned} \tag{3.3.27}$$

Furthermore, the first term on RHS of (3.3.25) is the only origin of the term so called 'Virial current'. However, since the three-dimensional theory is topological $\delta S_{\mathrm{loc};3-3}/\delta h_{\mu\nu}=0$, the term trivially vanishes, i.e., there is no Virial current in our theory.

One can check the suggested properties of the vector beta-function(al)s as in the case of even dimensions.

Now that we are ready to consider the three-dimensinal holographic trace anomaly. Explicit computation yields

$$\begin{aligned}
\langle T^{\mu}{}_{\mu}\rangle =& -\beta^I\langle O^I\rangle - \beta^a_\mu\langle J^{a\mu}\rangle - \frac{\Theta}{2\kappa_4^2}\epsilon^{\mu\nu\rho}_{(3)}\beta^a_\mu F^a_{\nu\rho} \\
&+ \frac{1}{2\kappa_4^2}\beta^I\frac{1}{\sqrt{h}}\frac{\delta S_{\mathrm{loc};3-3}}{\delta\phi^I} + \frac{1}{2\kappa_4^2}\beta^a_\mu\frac{1}{\sqrt{h}}\frac{\delta S_{\mathrm{loc};3-3}}{\delta A^a_\mu} \\
=& -\beta^I\langle O^I\rangle - \beta^a_\mu\langle J^{a\mu}\rangle \\
&+ \frac{1}{2\kappa_4^2}\epsilon^{\mu\nu\rho}_{(3)}\boldsymbol{\nabla}_\mu\phi^I\boldsymbol{\nabla}_\nu\phi^J\boldsymbol{\nabla}_\rho\phi^K\bigg\{ -2E\rho^{aI}(iT^a)^{JK} - 2\rho^{aI}\partial^J E(\phi iT^a)^K + \epsilon^{IJK}\beta^L\partial^L D \\
&\qquad - 3D\rho^{aI}\epsilon^{JKL}(iT^a\phi)^L - 3\partial^I D\epsilon^{JKL}\beta^L\bigg\} \\
&+ \frac{1}{2\kappa_4^2}\epsilon^{\mu\nu\rho}_{(3)}F^a_{\mu\nu}\boldsymbol{\nabla}_\rho\phi^I\bigg\{ -\Theta\rho^{aI} - \frac{k_{\mathrm{CS}}}{4\pi}C(r)\rho^{aI} + 2E(iT^a\beta)^I + \beta^K\partial^K E(iT^a\phi)^I \\
&\qquad - E\rho^{bI}(\phi\{T^a,T^b\}\phi) + \partial^I E(\phi iT^a\beta) - 3D\epsilon^{IJK}(iT^a\phi)^J\beta^K\bigg\}.
\end{aligned} \tag{3.3.28}$$

On the other hand, the most general form of the trace anomaly is given by

$$\langle T^{\mu}{}_{\mu}\rangle\Big|_{\mathrm{anomaly}} = \epsilon^{\mu\nu\rho}_{(3)}\left\{C_{IJK}\boldsymbol{\nabla}_\mu\phi^I\boldsymbol{\nabla}_\nu\phi^J\boldsymbol{\nabla}_\rho\phi^K + C^a_I F^a_{\mu\nu}\boldsymbol{\nabla}_\rho\phi^I\right\}. \tag{3.3.29}$$

Comparing the two expressions, we can identify the coefficients;

$$\begin{aligned}
C_{IJK} = \frac{1}{2\kappa_4^2}\bigg\{& -2E\rho^{aI}(iT^a)^{JK} - 2\rho^{aI}\partial^J E(\phi iT^a)^K + \epsilon^{IJK}\beta^L\partial^L D \\
& - 3D\rho^{aI}\epsilon^{JKL}(iT^a\phi)^L - 3\partial^I D\epsilon^{JKL}\beta^L\bigg\},
\end{aligned} \tag{3.3.30}$$

$$\begin{aligned}
C^a_I = \frac{1}{2\kappa_4^2}\bigg\{& -\Theta\rho^{aI} - \frac{k_{\mathrm{CS}}}{4\pi}C(r)\rho^{aI} + 2E(iT^a\beta)^I + \beta^K\partial^K E(iT^a\phi)^I \\
& - E\rho^{bI}(\phi\{T^a,T^b\}\phi) + \partial^I E(\phi iT^a\beta) - 3D\epsilon^{IJK}(iT^a\phi)^J\beta^K\bigg\}.
\end{aligned} \tag{3.3.31}$$

One would notice at once that all terms are proportional to beta-function(al)s[14]. Thus we see that the three-dimensional holographic trace anomaly vanishes on (conformal) fixed points $\beta = 0$.

Why the anomaly vanishes? Now that we have finished the explicit computation, we can easily see what is essential for the vanishing holographic trace anomalies in three (or more general odd) dimensions. We have explained at the end of section 3.1 that trace anomalies are included in the $w = d = 3$ part of the flow equations. Taking the form of the flow equation (3.3.13) into account, one notices that there are just a few terms which can contribute to the $w = 3$ part of the flow equation, thus to the trace anomaly. One should recall the weight assignment summarized in the Table 1. Let us begin with terms with the form

$$\frac{\delta S_{\mathrm{OS}}}{\delta \mathcal{F}} \frac{\delta S_{\mathrm{OS}}}{\delta \mathcal{F}}.$$

One would learn that the vevs $\langle O_I \rangle$ and $\langle J^{a\mu} \rangle$ have weights 3 and 2, respectively, remembering the weight assignments. As explained, coefficients of the vevs (3.1.10) are identified with beta-function(al)s. Thus they must have the forms $\beta^I \propto \delta S_{\mathrm{loc};0-3}/\delta\phi$ and $\beta^a_\mu \propto \delta S_{\mathrm{loc};2-3}/\delta A$. These are all contributions from non-local parts of S_{OS}. If one chooses S_{loc} for both S_{OS}, those terms can contribute to trace anomalies. The scalar field and metric parts are good places to start. Since functional derivatives with these fields completely cancel $w = -3$ from the volume elements of the local action, just pairs of local Lagrangians whose weight add up to three contribute to the trace anomaly. Because there is no local Lagrangian with $w = 1$, there is no pair with weights $3 = 1 + 2$, and all we have is $3 = 0 + 3$. The topological property $\delta S_{\mathrm{loc};3-3}/\delta h_{\mu\nu}$ kills potential contribution from the metric part of the pair. In addition, the scalar part is proportional to the (scalar) beta-function(al) $\delta S_{\mathrm{loc};0-3}/\delta\phi$. Thus all contributions to $\langle T^\mu{}_\mu \rangle$ from the scalar field and metric part is proportional to the (scalar) beta-function(al). Next, let us study the gauge field part. The functional derivatives with gauge fields do not completely cancel $w = -3$ from the volume elements. Hence just pairs of local Lagrangians whose weights sum up to five contribute to trace anomalies. Since there is no local Lagrangian with weight one, all we have is a pair $5 = 2 + 3$ [15], however, the contribution is again proportional to the (vector) beta-function(al). So far, all contributions to the trace anomaly from the terms $(\delta S_{\mathrm{OS}}/\delta\mathcal{F})^2$ are proportional to beta-function(al)s. Finally, let us look at the last term of (3.3.13). Since the fields strength F already has $w = 2$, for the term to contribute to the trace anomaly, the coefficient $\delta S_{\mathrm{OS}}/\delta A$ should have weight one. Because $\delta/\delta A$ already has $w = 2$, only the term $S_{\mathrm{loc};2-3} \in S_{\mathrm{OS}}$ can contribute, but as we have already seen, the term is nothing but the vector beta-function(al). To sum up, we have seen that all contributions to the holographic trace anomaly are proportional to beta-function(al)s as we have shown through explicit calculations. The absence of local Lagrangian with weight one was essential. From the above argument, we have learned that

[14] Since $\boldsymbol{\nabla}\phi$ in general does not vanish, we need $\rho^a_I = 0$ to achieve $\beta^a_\mu = 0$.

[15] Note that although we do have a pair $5 = 0 + 5$, $\delta S_{\mathrm{loc};0-3}/\delta A$ vanishes because $S_{\mathrm{loc};0-3}$ cannot contain gauge fields, and the pair cannot contribute to trace anomalies.

one needs a term with $w = 1$, which belongs to Lorentz and flavor singlets, in the local action to get nonzero holographic trace anomalies on fixed points. Repeating the same analysis, one can also show that one cannot get nonzero holographic trace anomalies in the general odd dimensions because there are no local Lagrangians with lower odd weights, i.e., $w < d$, if one just considers simple extensions of bulk actions. For a concrete example of bulk action extension, see the appendix E of [40]

3.4 Exactly marginal coupling

It would be instructive to make a comment on special classes of couplings which are called exactly marginal couplings. Coupling constants are classified into three classes according to their behavior under RG flows, that is, relevant, marginal, and irrelevant couplings. Relevant couplings λ_{rel} have positive mass dimensions $[\lambda_{\mathrm{rel}}] > 0$. Thus they become more and more important in the IR, hence the name relevant. On the other hand, irrelevant couplings λ_{irrel} has negative mass dimensions $[\lambda_{\mathrm{irrel}}] < 0$. Thus as the theory flows to the IR, effects of the coupling constants become less and less important, hence irrelevant. The last class, marginal couplings λ_{mar} have mass dimension zero $[\lambda_{\mathrm{mar}}] = 0$ at the classical level. Thus their lowest order beta-functions vanish. To see the behaviour of the marginal couplings under RG flows, we therefore have to scrutinize higher order terms of beta-functions. In other words, coupling constants enjoy quantum corrections to scaling dimensions in general, i.e., anomalous dimensions. If the marginal couplings receive quantum corrections to have positive scaling dimensions, they become more and more important in the IR, and they are called marginally relevant. On the other hand, if they result in negative scaling dimensions, they are called marginally irrelevant. Sometimes it happens that beta-functions remain zero to all orders. Then the couplings do not run under RG flows. These coupling constants are called exactly marginal. In general, it is difficult to judge whether a marginal coupling is exactly marginal or not, because one has to compute beta-functions to all orders[16]. Although it is difficult to get a condition which is equivalent to the exactly marginal character of a coupling constant, a necessary condition for a coupling constant be exactly marginal in QFTs with holographic duals can be obtained relatively easily. We have already encountered the condition in the analysis above. We have explicitly shown in (3.2.27) and (3.3.26) that the scalar beta-fucntion(al)s are proportional to $\partial_I W(\phi)$. For a 'coupling constant' ϕ^I to be exactly marginal, we therefore need $W(\phi) = \text{const.}$ as a necessary condition. If one recalls (3.2.19) or (3.3.19), the necessary condition requires $V(\phi) = \text{const.}$. In other words, bulk scalar fields, whose boundary 'values' are identified with coupling 'constants' according to the GKP-Witten prescription [32], can be exactly marginal if they only have cosmological constant $V(\phi) = \text{const.}$ as potential terms.

[16]In the presence of SUSY, a group theoretical criterion is known [41].

4 Conclusion

We explained various tools to study QFTs. There are many ways but we elucidated one of the most promising approaches, namely, the use of symmetries. Concretely speaking, we have discussed anomalies and larger symmetries such as conformal symmetry or SUSY.

In section 2, we reviewed various aspects of anomalies from the viewpoint of bundles. Especially, we have learned that anomalies originate from nontriviality of bundles over quotient spaces of background fields. This fact will be useful to exhaustively study when some still mysterious anomalies appear. In fact, the author is trying to apply the fact on three-dimensional trace anomaly and shortening anomaly.

The usual trace anomalies originate from nontriviality of bundles over conformal class, which is an equivalence class of background metric with two metrics identified if there exists a conformal map between them. The nontriviality has a root in characteristic classes, and we cannot construct characteristic classes of odd-forms just from metrics. Hence it was believed that there does not exist three-dimensional trace anomalies. Recently, however, it was pointed out that one can write candidate terms if one further incorporates background scalar and gauge fields [36]. In this situation, possible trace anomaly originates from nontriviality of another bundle over $\mathcal{S} \times \mathcal{A} \times \mathfrak{M}/\sim$, where $\phi \in \mathcal{S}$, $A \in \mathcal{A}$, and $\gamma \in \mathfrak{M}$ are configuration spaces of background scalar, gauge, and metric fields, respectively. Since we know little about this bundle, it is possible that the bundle is nontrivial, and we might find a nonzero three-dimensional trace anomaly. The author is trying to pursue this line.

Another anomaly the author is studying is the shortening anomaly [42]. This anomaly was found recently in two-dimensional SCFTs. The anomaly appears when one tries to promote exactly marginal coupling constants to chiral and twisted chiral background superfields at the same time in the presence of SUSY larger than $\mathcal{N} = (2, 2)$. Little is understood about the anomaly. For example, we still do not know whether there are shortening anomalies in other dimensions. We are trying to understand the anomaly by studying the relevant bundles [43]. The relevant bundles in this problem are those over background superfields, in which exactly marginal couplings reside, modded out by SUSY transformations.

In section 3, we reported our results on holographic trace anomalies in various dimensions taking gauge fields into account. In particular we generalized the proof of holographic c-theorem, and studied the three-dimensional trace anomaly mentioned above to find they are all proportional to β-functions, and vanish on (conformal) fixed points. We also clarified what is responsible for the vanishment. Since the reasoning also holds in other odd-dimensional CFTs, we have proved there does not exist odd-dimensional holographic trace anomalies.

In the holographic computations of trace anomalies, flow equations play pivotal roles. In the appendix B, we also explained another derivation of flow equations (a.k.a. Hamiltonian constraints) employing the BV formalism [44].

As an example of larger symmetries, we studied conformal symmetry with or without SUSY in the appendix C. Concretely speaking, we reviewed geometry of conformal manifolds, and in the last subsection, we reported partial results of our work in progress [20], which is on branched four-sphere partition functions.

Throughout our investigation to understand QFTs in this thesis, we encountered some unanswered questions (to the author's knowledge):

- What is responsible for nontriviality of bundles? In other words, how field contents affect nontriviality of the bundles?
- What are the holonomies of the bundles?
- What is the relation of BRST cohomology with ghost number one and the nontriviality of the bundle?
- Why the nontriviality of the bundle appears at the quantum level?
- If our conformal manifolds $\mathcal{M}^{\mathcal{N}=2}_{\mathbb{S}^1_{qr}\times\mathbb{H}^3_r}$ are Kähler, we can freely rescale the radius of $\mathbb{S}^1$ without destroying the Kählerity. In other words, the rescaling preserves $U(n)$ holonomy of the complex n-dimensional manifold $\mathcal{M}^{\mathcal{N}=2}_{\mathbb{S}^1_r\times\mathbb{H}^3_r}$. Then what is the most general deformations which is consistent with the $U(n)$ holonomy?
- In general, manipulations on spacetime manifolds break some SUSY. For example, $\mathbb{R}^n \to \mathbb{S}^n$ explicitly breaks shift symmetry of the Euclidean space, so does part of SUSY on $\mathbb{R}^n$. On the other hand, modifying global structures of spacetime manifolds is expected not to lift exactly marginal operators because beta-functions would continue to vanish regardless of global structures since beta-functions are determined by UV divergences, which are local in nature. Therefore we may find nontrivial conformal manifolds in the absence of SUSY by performing suitable manipulations to our initial spacetime manifolds. This possibility was recently addressed in [45].

We would like to return to these problems in the future.

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A Notations

In the holographic computations in section 3, we work in the Lorentzian signature. Bulk spacetime indices are denoted by $M, N, \cdots$, boundary spacetime indices by $\mu, \nu, \cdots$, and gauge indices by $a, b, \cdots$. In the section, fields with hat are off-shell.

In the other parts of the thesis, we will consistently work in the Euclidean signature. Local Lorentz indices are denoted by $a, b, \cdots$, and spacetimes indices by $\mu, \nu, \cdots$. 'Flavor' indices are always denoted by $I, J, \cdots$ throughout the thesis.

Let us consider a permutation $(\mu_1 \cdots \mu_n)$ of n spacetime indices. We denote its sign $[\mu_1 \cdots \mu_n]$, where we define

$$[12 \cdots n] = +1, \quad [01 \cdots (n-1)] = +1.$$

Then Levi-Civita tensors are defined by

$$\epsilon_{(E)}^{\mu_1 \cdots \mu_n} := -\frac{1}{\sqrt{\gamma}}[\mu_1 \cdots \mu_n], \quad \epsilon_{(L)}^{\mu_1 \cdots \mu_n} := -\frac{1}{\sqrt{\gamma}}[\mu_1 \cdots \mu_n],$$

where $\gamma := \pm \det \gamma_{\mu\nu}$ with plus sign in case of Euclidean signature and minus sign in case of Lorentzian one. The Levi-Civita connection is defined as usual

$$\Gamma^\rho_{\mu\nu} := \frac{1}{2}\gamma^{\rho\sigma}(\partial_\mu \gamma_{\sigma\nu} + \partial_\nu \gamma_{\mu\sigma} - \partial_\sigma \gamma_{\mu\nu}).$$

The spin connections are given by

$$\omega^{\mathrm{rigid}}{}_{\mu}^{ab} := -e^a_\nu \nabla_\mu e^{b\nu} = e^b_\nu \nabla_\mu e^{a\nu},$$

where e^a_μ is a tetrad

$$ds^2 = \gamma_{\mu\nu} dx^\mu dx^\nu = \delta_{ab} e^a_\mu e^b_\nu dx^\mu dx^\nu.$$

Using these connections, covariant derivatives in section 3 are defined by

$$\boldsymbol{\nabla}_M \phi^I := \nabla_M \phi^I - A^a_M (iT^a \phi)^I.$$

Spinors on n-dimensional (curved) manifolds with Euclidean signature are defined as usual. That is, spacetime gamma matrices are defined to obey the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\gamma^{\mu\nu},$$

while local Lorentz gamma matrices are defined by

$$\{\Gamma^a, \Gamma^b\} = 2\delta^{ab}.$$

These two are related by

$$\gamma_\mu = e^a_\mu \Gamma^a.$$

In section C.3, we consider Euclidean four-manifolds. The tangent space group which acts on the manifold is $SO(4) \cong SU(2)_+ \otimes SU(2)_-$. A left-handed spinor ζ_α belongs to the spin representaion of $SU(2)_+$, while a right-handed spinor $\tilde{\zeta}^{\dot{\alpha}}$ belongs to that of $SU(2)_-$. They do not mix under charge conjugation, and are independent spinors. Rather, charge conjugation raises or lowers spinor indices

$$(\zeta_\alpha)^* \equiv (\zeta^\dagger)^\alpha, \quad (\tilde{\zeta}^{\dot{\alpha}})^* \equiv (\tilde{\zeta}^\dagger)_{\dot{\alpha}}.$$

Four-dimensional tangent space Pauli matrices are given by

$$\sigma^a_{\alpha\dot{\alpha}} = (\boldsymbol{\sigma}, -i1_2), \quad \tilde{\sigma}^{a\dot{\alpha}\alpha} = (-\boldsymbol{\sigma}, -i1_2),$$

where $a = 1, \ldots, 4$ labels tangent space, and $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the usual Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Spacetime Pauli matrices are given by

$$\sigma_\mu = e^a_\mu \sigma^a.$$

As evident from the expression, the four-dimensional Pauli matrices are no longer complex conjugates of each other, however, we still have[17]

$$\tilde{\sigma}^{a\dot{\alpha}\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\sigma^a_{\beta\dot{\beta}},$$

as one can easily show by explicit computation. These satisfy

$$\sigma_a\tilde{\sigma}_b + \sigma_b\tilde{\sigma}_a = -2\delta_{ab}, \quad \tilde{\sigma}_a\sigma_b + \tilde{\sigma}_b\sigma_a = -2\delta_{ab}.$$

The antisymmetric matrices are defined by

$$(\sigma_{ab})_\alpha{}^\beta := \frac{1}{4}(\sigma_{a\alpha\dot{\alpha}}\tilde{\sigma}^{\dot{\alpha}\beta}_b - \sigma_{b\alpha\dot{\alpha}}\tilde{\sigma}^{\dot{\alpha}\beta}_a), \quad (\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}} := \frac{1}{4}(\tilde{\sigma}^{\dot{\alpha}\alpha}_a\sigma_{b\alpha\dot{\beta}} - \tilde{\sigma}^{\dot{\alpha}\alpha}_b\sigma_{a\alpha\dot{\beta}}).$$

The antisymmetric matrices equip us with covariant derivatives of the two-component spinors:

$$(\nabla_\mu\zeta)_\alpha := \left(\delta^\beta_\alpha\partial_\mu + \frac{1}{2}\omega^{\text{rigid}}_{\mu ab}(\sigma^{ab})_\alpha{}^\beta\right)\zeta_\beta, \quad (\nabla_\mu\tilde{\zeta})^{\dot{\alpha}} := \left(\delta^{\dot{\alpha}}_{\dot{\beta}}\partial_\mu + \frac{1}{2}\omega^{\text{rigid}}_{\mu ab}(\tilde{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}}\right)\tilde{\zeta}^{\dot{\beta}}.$$

Using the four-dimensional Pauli matrices, we define gamma matrices by[18]

$$\Gamma^a := \begin{pmatrix} 0 & i\sigma^a \\ i\tilde{\sigma}^a & 0 \end{pmatrix}.$$

[17]We use the convention

$$\varepsilon^{12} = +1, \quad \varepsilon^{21} = -1.$$

[18]The signs before Pauli matrices are just a matter of convention. All we need is the coefficients of Pauli matrices multiply to (-1), so that they satisfy the Clifford algebra. One merit of this convention is that it makes the gamma matrix Hermitian $(\Gamma^a)^\dagger = \Gamma^a$.

Using the gamma matrices, we further define

$$\Gamma^{ab} := \frac{1}{2}[\Gamma^a, \Gamma^b] = -2 \begin{pmatrix} \sigma^{ab} & 0 \\ 0 & \tilde{\sigma}^{ab} \end{pmatrix}.$$

We can combine two two-component spinors into a four-component spinor

$$\epsilon := \begin{pmatrix} \zeta_\alpha \\ \tilde{\zeta}^{\dot{\alpha}} \end{pmatrix}.$$

Van Proeyen defined covariant derivatives on the four-component spinor by

$$\nabla_\mu \epsilon := \left(\partial_\mu + \frac{1}{4} \omega^{\mathrm{VP}}_{\mu ab} \Gamma^{ab} \right) \epsilon.$$

Substituting the explicit form of Γ^{ab}, we obtain

$$\nabla_\mu \epsilon = \partial_\mu \begin{pmatrix} \zeta \\ \tilde{\zeta} \end{pmatrix} - \frac{1}{2} \omega^{\mathrm{VP}}_{\mu ab} \begin{pmatrix} \sigma^{ab} & 0 \\ 0 & \tilde{\sigma}^{ab} \end{pmatrix} \begin{pmatrix} \zeta \\ \tilde{\zeta} \end{pmatrix} = \begin{pmatrix} \left(\partial_\mu - \frac{1}{2} \omega^{\mathrm{VP}}_{\mu ab} \sigma^{ab}\right) \zeta \\ \left(\partial_\mu - \frac{1}{2} \omega^{\mathrm{VP}}_{\mu ab} \tilde{\sigma}^{ab}\right) \tilde{\zeta} \end{pmatrix}.$$

Therefore we must have

$$\omega^{\mathrm{rigid}}_{\mu ab} = -\omega^{\mathrm{VP}}_{\mu ab}$$

to make the covariant derivatives on two-spinors consistent.

A four-spinor with bar is defined by

$$\bar{\epsilon} := \epsilon^T \mathcal{C}$$

where $\mathcal{C}$ is the charge conjugation matrix, which is unitary.

B First-class constraints from Batalin-Vilkovisky (BV) formalism

B.1 BV formalism revisited

In the BV formalism [46, 47, 48], one assembles all fields (including BRST ghost fields introduced when one fixes a gauge) into a collection, and simply call it fields, e.g. $\mathcal{F}^n = (\phi, A, \gamma, c, \bar{c})$. For each component of the field, one introduces an external field collectively called an antifield, e.g. $K_n = (K_\phi, K_A, K_\gamma, K_c, K_{\bar{c}})$. Let us denote a space of (anti)field configurations $\mathcal{C} := \{\mathcal{F}, K\}$, and a space of functionals which take values in some field $\mathbb{K}$ as $\mathfrak{F}$, i.e., $\mathfrak{F} := \{F : \mathcal{C} \to \mathbb{K}\}$. Since $\mathfrak{F}$ can be equipped with three natural operations, namely, scalar multiplication over $\mathbb{K}$ $\overset{\mathbb{K}}{\cdot}$, addition $+$, and product $\cdot$, $(\mathfrak{F}, \overset{\mathbb{K}}{\cdot}, +, \cdot)$ becomes an algebra over

$\mathbb{K}$ with a unit element $1_{\mathfrak{F}}$. Next, one defines a bracket $(\cdot,\cdot) : \mathfrak{F}\times\mathfrak{F}\to\mathbb{K}$ called antibracket by

$$\forall F, G \in \mathfrak{F}, \quad (F,G) := \int d^D X \left\{ \frac{\delta^R F}{\delta \mathcal{F}^n(X)} \frac{\delta^L G}{\delta K_n(X)} - \frac{\delta^R F}{\delta K_n(X)} \frac{\delta^L G}{\delta \mathcal{F}^n(X)} \right\}, \tag{B.1}$$

where the superscripts R and L denote right and left derivatives, respectively, and the sum over n is understood. Note that the definition immediately yields

$$(\mathcal{F}^n(X), K_m(Y)) = \delta^n_m \delta^{(D)}(X-Y),$$

and antifields can be recognized as conjugate momenta to fields in terms of the antibracket. On $\mathfrak{F}$, one can define a $\mathbb{Z}_2$ valued function $\epsilon : \mathfrak{F}\to\mathbb{Z}_2$ called statistics. Using the function, $\mathfrak{F}$ can be divided into two; a part $\mathfrak{F}_0$ called even with $\epsilon = 0$ and another $\mathfrak{F}_1$ called odd with $\epsilon = 1$, i.e., $\mathfrak{F} = \mathfrak{F}_0 \oplus \mathfrak{F}_1$. We can also assign statistics to $\mathcal{C}$. On $\mathcal{C}$, one can further define a $\mathbb{Z}$ valued function called ghost number. Thus each component of $\mathfrak{F}$ can be further decomposed into smaller components with fixed ghost number n, i.e.,

$$\mathfrak{F} = \mathfrak{F}_0 \oplus \mathfrak{F}_1 = \left(\bigoplus_{n\in\mathbb{Z}} \mathfrak{F}_0^{(n)} \right) \oplus \left(\bigoplus_{n\in\mathbb{Z}} \mathfrak{F}_1^{(n)} \right).$$

A natural assignment with representations of a gauge group G is summarized in Table 2. The assignment is determined by requiring the action be even, have ghost number zero, and belong to gauge singlets. $\mathbb{Z}_2$ and $\mathbb{Z}$ are equipped with natural additions and products. Using the statistics, one can show the fundamental properties of the antibracket; $\forall F, G, H \in \mathfrak{F}$,

$$(F,G) = (-)^{(\epsilon(F)+1)(\epsilon(G)+1)}(G,F), \quad (-)^{\epsilon(F)\epsilon(H)+\epsilon(G)}(F,(G,H)) + (\text{cyclic terms}) = 0. \tag{B.2}$$

Then an extended action $S[\mathcal{F}, K]$ is defined as a solution of the (classical) master equation

$$(S,S) = 0. \tag{B.3}$$

Since the equation is a second order differential equation, one has to specify two boundary conditions to fix a solution. We choose

$$S[\mathcal{F}, K=0] = S_c[f], \quad -\frac{\delta^R S[\mathcal{F},K]}{\delta K_n(X)}\bigg|_{K_n=0} = R^n[f,C](X), \tag{B.4}$$

where we collectively write original fields in the classical action S_c as f, and 'ghost fields' introduced in the BV formalism as C. A general solution (we will discuss further generalization in short) which obey the boundary condition (B.4) is given by

$$S[\mathcal{F}, K] \equiv S_c[f] + S_K[\mathcal{F}, K], \tag{B.5}$$

where one can show that the source term S_K is linear in antifields if the algebras close off-shell [49]:

$$S_K[\mathcal{F}, K] := -\int d^D X R^n[\mathcal{F}] K_n, \tag{B.6}$$

(anti)field	$\epsilon[\cdot]$ mod2	ghost #	G
ϕ	0	0	r
A	0	0	Adj
γ	0	0	1
ω	1	1	1
$\bar{\omega}$	1	-1	1
B	0	0	1
c	1	1	Adj
$\bar{c}$	1	-1	Adj
b	0	0	Adj
K_ϕ	1	-1	$\bar{r}$
K_A	1	-1	Adj
K_γ	1	-1	1
K_ω	0	-2	1
$K_{\bar{\omega}}$	0	0	1
K_B	1	-1	1
K_c	0	-2	Adj
$K_{\bar{c}}$	0	0	Adj
K_b	1	-1	Adj

Table 2: Assignment of (quantum) numbers

where summation over n is understood. One notices at once that R^n is nothing but variations of fields $\mathcal{F}^n$; this fact can be seen by computing an antibracket $(S, \mathcal{F}^n)$

$$\begin{aligned}(S, \mathcal{F}^n(X)) &= -\int d^D Y \frac{\delta^R S}{\delta K_m(Y)} \frac{\delta^L \mathcal{F}^n(X)}{\delta \mathcal{F}^m(Y)} \\ &= R^n[\mathcal{F}(X)].\end{aligned}$$

Using the properties (B.2), one can easily show

$$\forall F \in \mathfrak{F}, \quad (S, (S, F)) = 0. \tag{B.7}$$

Thus a map defined by choosing the first entry of the antibracket as the extended action

$$(S, \cdot) : \mathfrak{F} \to \mathfrak{F}, \tag{B.8}$$

is nilpotent. This is why the BV formalism is said to generalize the BRST prescription. Being nilpotent, one can define the BV cohomology algebra on $\mathfrak{F}$;

$$\begin{aligned}\mathfrak{C} :=& \{F \in \mathfrak{F} | (S, F) = 0\}, \\ \mathfrak{E} :=& \{F \in \mathfrak{F} | \exists G \in \mathfrak{F} \ s.t. \ F = (S, G)\},\end{aligned} \tag{B.9}$$

where elements of these spaces are called closed and exact, respectively, as usual. We define an equivalence relation by

$$F \sim F' \overset{\text{def}}{\iff} F - F' \in \mathfrak{E}.$$

Then the BV cohomology algebra is defined by

$$\mathfrak{H} := \mathfrak{C}/\mathfrak{E}. \tag{B.10}$$

Note that since $\mathfrak{F}$ was equipped with three operations, $\mathfrak{H}$ is also equipped with them naturally induced in it, i.e., we have an algebra $(\mathfrak{H}, \overset{\mathbb{K}}{\cdot}, +, \cdot)$. One can easily show that the product depends just on equivalent classes, and thus is well-defined.

When one tries to quantize a classical theory S_c with gauge symmetries, one has to fix a gauge, however, armed with all these machinery, we can fix a gauge with ease; all we have to do is to add a cohomologically exact term (S, Ψ), where $\Psi \in \mathfrak{F}$ with some conditions explained in short, to the extended action

$$S[\mathcal{F}, K] := S_c[\mathcal{F}] + S_K[\mathcal{F}, K] + (S, \Psi) \equiv \int d^D X \mathcal{L}(\mathcal{F}, K). \tag{B.11}$$

Thanks to the nilpotency (B.7), the gauge fixed action also satisfies the (classical) master equation (B.3). Note that for the term (S, Ψ) to enter the modified action, Ψ has to be Grassmann odd. This is why Ψ is called a gauge fermion. Consideration on representation further requires Ψ be a Lorentz singlet. One would convince oneself by studying a gravitational system, i.e., the case in which $\mathcal{F}$ contains a metric γ. It is natural to require the BV map preserve representations of the Lorentz group because we expect a field $\mathcal{F}^n$ and its variation R^n to belong to the same representation. Then for the term (S, Ψ) to enter a gauge fixed action, which is a Lorentz singlet, Ψ itself must belong to a Lorentz singlet. The source term S_K of the extended action also has to belong to a Lorentz singlet. This observation and the definition (B.6) lead us to a conclusion that antifields be tensor densities when one considers a gravitational system. The result is analogous to the fact that canonical momenta in the Hamiltonian formulation of the general relativity are tensor densities [50]. This result also supports our observation to recognize antifields as canonical momenta conjugate to fields. Since we have required the variations R^n belong to the same representations, we can simply borrow the familiar expressions of the BRST transformations for R^n; $R^n[\mathcal{F}] = \boldsymbol{\delta}_{\mathrm{B}} \mathcal{F}^n$. For example we have

$$
\begin{aligned}
(S, \phi^I(X)) &= \boldsymbol{\delta}_{\mathrm{B}}\phi^I(X) = gc^a(X)\left(iT^a\phi(X)\right)^I, \\
(S, A^a_M(X)) &= \boldsymbol{\delta}_{\mathrm{B}}A^a_M(X) = \boldsymbol{\nabla}_M c^a(X), \\
(S, \gamma_{MN}(X)) &= \boldsymbol{\delta}_{\mathrm{B}}\gamma_{MN}(X) = -\gamma_{LN}(X)\nabla_M\omega^L(X) - \gamma_{ML}(X)\nabla_N\omega^L(X), \\
(S, c^a(X)) &= \boldsymbol{\delta}_{\mathrm{B}}c^a(X) = -\frac{1}{2}gf^a{}_{bc}c^b(X)c^c(X), \\
(S, \bar{c}^a(X)) &= \boldsymbol{\delta}_{\mathrm{B}}\bar{c}^a(X) = ib^a(X), \\
(S, b^a(X)) &= \boldsymbol{\delta}_{\mathrm{B}}b^a(X) = 0, \\
(S, \omega^M(X)) &= \boldsymbol{\delta}_{\mathrm{B}}\omega^M(X) = -\omega^N(X)\nabla_N\omega^M(X), \\
(S, \bar{\omega}_M(X)) &= \boldsymbol{\delta}_{\mathrm{B}}\bar{\omega}_M(X) = iB_M(X), \\
(S, B_M(X)) &= \boldsymbol{\delta}_{\mathrm{B}}B_M(X) = 0.
\end{aligned}
$$

With these observations, the author suggested a vantage point to interpret the antifields as first-class constraints. This suggestion may start to seem plausible if one notices the following points; (i) Since we have introduced antifields as independent external fields with resepect to the antibracket, i.e., $\forall K_m, K_n \in \mathcal{C}$, $(K_m(X), K_n(Y)) = 0$, they commute with respect to the antibracket. (ii) A linear combination of antifleds, that is the source term S_K, generates gauge transformations, i.e., $\forall \mathcal{F}^n \in \mathcal{C}$, $(S_K[\mathcal{F}, K], \mathcal{F}^n(X)) = R^n[\mathcal{F}(X)]$. This is reminiscent to the fact that a linear combination of first-class constraints in the canonical treatment of constrained systems à la Dirac serves as a generator of gauge transformations. (iii) The extended action was defined by adding a linear combination of antifields, namely the source term S_K, to the classical action, which is analogous to the fact that one adds a linear combination of first-class constraints to an original Hamiltonian to define an extended (or total) Hamiltonian.

With this new interpretation of antifields, the author further suggested the following identifications, which may lead us to a 'Lagrangian treatment' of constrained systems;

$$
\begin{aligned}
q &\leftrightarrow \mathcal{F}; \text{generalized coordinates}, \\
p &\leftrightarrow K; \text{canonical momenta}, \\
\{\cdot,\cdot\}_P &\leftrightarrow (\cdot,\cdot); \text{bracket}, \\
H &\leftrightarrow S; \text{time evolution generator}, \\
\phi &\leftrightarrow K; \text{first-class constraints}.
\end{aligned}
$$

B.2 Derivation of first-class constraints

In the traditional analysis in section 3.1, we started from a bulk action. We then computed its first-order form, on-shell action, and finally obtained first-class constraints including flow equations. The author has found a simple and systematic derivation of the first-class constraints starting from the same[19] 'bulk' action via the BV formalism explained above. We now move on to the derivation.

As explained in section B.1, a map $(S,\cdot):\mathfrak{F}\to\mathfrak{F}$ is nilpotent, and we can define BV cohomology algebra $\mathfrak{H}$. Thus an exact term (S,K_n) should be identified with $0\in\mathfrak{H}$. Furthermore, since the antifields can be recognized as first-class constraints, they are weakly zero, namely we should set them to zero after the computation[20]. Therefore we arrive at an equation

$$(S,K_n)\Big|_{K=0}\sim 0\in\mathfrak{H}. \tag{B.12}$$

This simple equation reproduces all (first-class) constraints. One does not have to compute canonical momenta, nor perform Legendre transformations to get Hamiltonians in this prescription. All one has to do is to put antifields one by one into the equation (B.12). Let us see whether the prescription correctly reproduce the first-class constraints including the flow equations we have obtained in the body of the thesis[21].

We will just consider even-dimensional case. A 'bulk' or classical action is given by (3.2.1);

$$\begin{aligned}&S_c[\phi,A,\gamma]\\&=\int_{M^{2n+1}}d^{2n+1}X\sqrt{\gamma}\left\{V(\phi)-R_{(2n+1)}+\frac{1}{2}L_{IJ}(\phi)\gamma^{MN}\boldsymbol{\nabla}_M\phi^I\boldsymbol{\nabla}_N\phi^J+\frac{1}{4}B(\phi)F^a_{MN}F^{aMN}\right\}.\end{aligned} \tag{B.13}$$

The source term S_K is given by

$$\begin{aligned}S_K[\mathcal{F},K]:=&-\int d^{2n+1}X\Big\{R^I_\phi K_{\phi I}+R_{A}{}^a_M K^{aM}_A+R_{\gamma MN}K^{MN}_\gamma+R^M_\omega K_{\omega M}\\&\qquad+R_{\bar\omega M}K^M_{\bar\omega}+R_{BM}K^M_B+R^a_cK^a_c+R^a_{\bar c}K^a_{\bar c}+R^a_bK^a_b\Big\} \qquad\qquad (\mathrm{B.14})\\=&-\int d^{2n+1}X\Big\{(\boldsymbol{\delta}_\mathrm{B}\phi)^IK_{\phi I}+(\boldsymbol{\delta}_\mathrm{B}A)^a_MK^{aM}_A+(\boldsymbol{\delta}_\mathrm{B}\gamma)_{MN}K^{MN}_\gamma+(\boldsymbol{\delta}_\mathrm{B}\omega)^MK_{\omega M}\\&\qquad+(\boldsymbol{\delta}_\mathrm{B}\bar\omega)_MK^M_{\bar\omega}+(\boldsymbol{\delta}_\mathrm{B}B)_MK^M_B+(\boldsymbol{\delta}_\mathrm{B}c)^aK^a_c+(\boldsymbol{\delta}_\mathrm{B}\bar c)^aK^a_{\bar c}+(\boldsymbol{\delta}_\mathrm{B}b)^aK^a_b\Big\}.\end{aligned}$$

We know the system has three first-class constraints, i.e., Hamiltonian constraint, momentum constraint, and Gauss's law. One can also pretend not to know the constraints and put antifields one after another. Then one will learn that candidate antibrackets are given by (S,K_N), (S,K^μ_λ), and $(S,K^{a\tau}_A)$, respectively. Here, the antifields are given by

$$K_N:=2NK^{\tau\tau}_\gamma,\quad K^\mu_\lambda:=2\lambda^\mu K^{\tau\tau}_\gamma+2K^{\mu\tau}_\gamma,\quad K^{\mu\nu}_h:=K^{\mu\nu}_\gamma-\lambda^\mu\lambda^\nu K^{\tau\tau}_\gamma, \tag{B.15}$$

[19] Although little attention was paid on possible surface terms, as mentioned in the paper.

[20] Even if one is not convinced with the identifications suggested in [44], the manipulation to set $K=0$ after the computation is always allowed because one introduces the antifields K as external fields.

[21] As remarked in [44], we pay little attention on possible surface terms. In addition, since we are just interested in classical (bulk) theory, we do not fix a gauge.

which can be obtained by computing BRST transformations of (3.1.2);

$$
\begin{aligned}
S_K &\ni -\int d^D X (\boldsymbol{\delta}_{\mathrm{B}}\gamma_{MN}) K_\gamma^{MN} \\
&= -\int d^D X \Bigg\{ (\boldsymbol{\delta}_{\mathrm{B}} N) 2N K_\gamma^{\tau\tau} + (\boldsymbol{\delta}_{\mathrm{B}}\lambda_\mu)\left[2\lambda^\mu K_\gamma^{\tau\tau} + 2K_\gamma^{\mu\tau}\right] + (\boldsymbol{\delta}_{\mathrm{B}} h_{\mu\nu})\left[K_\gamma^{\mu\nu} - \lambda^\mu\lambda^\nu K_\gamma^{\tau\tau}\right] \Bigg\}.
\end{aligned}
$$

Straightforward computation gives

$$
\begin{aligned}
(S, K_N) &\equiv (S, 2NK_\gamma^{\tau\tau}) \\
&= \sqrt{h}\Bigg\{ V(\phi) - R_{(2n)} + \frac{1}{2}L_{IJ}(\phi) h^{\mu\nu}\boldsymbol{\nabla}_\mu\phi^I\boldsymbol{\nabla}_\nu\phi^J + \frac{1}{4}B(\phi)h^{\mu\rho}h^{\nu\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \\
&\qquad + K^2 - K^{\mu\nu}K_{\mu\nu} - \frac{1}{2N^2}L_{IJ}(\phi)(\boldsymbol{\nabla}_\tau\phi^I - \lambda^\mu\boldsymbol{\nabla}_\mu\phi^I)(\boldsymbol{\nabla}_\tau\phi^J - \lambda^\nu\boldsymbol{\nabla}_\nu\phi^J) \\
&\qquad - \frac{B(\phi)}{2N^2}h^{\mu\nu}\Big(F^a_{\tau\mu} - \lambda^\rho F^a_{\rho\mu}\Big)\Big(F^a_{\tau\nu} - \lambda^\sigma F^a_{\sigma\nu}\Big)\Bigg\} + (K \text{ terms}) \sim 0,
\end{aligned} \tag{B.16}
$$

$$
\begin{aligned}
(S, K_\lambda^\mu) &\equiv (S, 2\lambda^\mu K_\gamma^{\tau\tau} + 2K_\gamma^{\mu\tau}) \\
&= \sqrt{h}\Bigg\{ \frac{2}{N}h^{\mu\nu}(R_{\tau\nu} - \lambda^\rho R_{\rho\nu}) - \frac{1}{N}L_{IJ}(\phi)h^{\mu\nu}(\boldsymbol{\nabla}_\tau\phi^I - \lambda^\rho\boldsymbol{\nabla}_\rho\phi^I)\boldsymbol{\nabla}_\nu\phi^J \\
&\qquad - \frac{B(\phi)}{N}h^{\mu\nu}h^{\rho\sigma}F^a_{\nu\sigma}(F^a_{\tau\rho} - \lambda^\alpha F^a_{\alpha\rho})\Bigg\} + (K \text{ terms}) \sim 0,
\end{aligned} \tag{B.17}
$$

$$
(S, K_A^{a\tau}) = \boldsymbol{\nabla}_\mu\Big[N\sqrt{h}B(\phi)F^{a\tau\mu}\Big] - \frac{\sqrt{h}}{N}L_{IJ}(\phi)\Big(\boldsymbol{\nabla}_\tau\phi^I - \lambda^\mu\boldsymbol{\nabla}_\mu\phi^I\Big)(iT^a\phi)^J - f^{abc}c^bK_A^{c\tau} \sim 0. \tag{B.18}
$$

We would like to emphasis that these are our final results. Note that we have not computed canonical momenta, nor Hamiltonian. In other words, all we used are Lagrangian variables. We have obtained the results at one step circumventing all tedious computations of, for example, first order action, on-shell action, or its variation.

One would notice that the expressions do not completely match our previous results. One can convince oneself that they do match completely if one rewrites these results in Hamiltonian variables, although we would like to emphasis again that this is not mandatory for our prescription (B.12). Canonical momenta are given by

$$\pi_I := \frac{\partial \mathcal{L}}{\partial(\partial_\tau \phi^I)} = \frac{\sqrt{h}}{N} L_{IJ}\Big(\boldsymbol{\nabla}_\tau \phi^J - \lambda^\mu \boldsymbol{\nabla}_\mu \phi^J\Big) + \omega^\tau K_{\phi I}, \tag{B.19}$$

$$\pi^{a\mu} := \frac{\partial \mathcal{L}}{\partial(\partial_\tau A^a_\mu)} = N\sqrt{h} B F^{a\tau\mu} = \frac{\sqrt{h}}{N} B\Big(h^{\mu\nu} F^a_{\tau\nu} - \lambda^\nu h^{\rho\mu} F^a_{\nu\rho}\Big), \tag{B.20}$$

$$\pi^{\mu\nu} := \frac{\partial \mathcal{L}}{\partial(\partial_\tau h_{\mu\nu})} = \sqrt{h}\Big(K^{\mu\nu} - h^{\mu\nu} K\Big). \tag{B.21}$$

Note that we have included $\sqrt{h}$ in $\mathcal{L}$ to make the canonical momenta tensor densities in accord with the standard canonical formulation of gravitational systems [50]. As a consequence, these are $\sqrt{h}$ times those (3.2.3), (3.2.4), and (3.2.5) (modulo terms proportional to antifields). In terms of these canonical variables, our results reduce to

$$\begin{aligned}
(S, K_N) = \sqrt{h}\Bigg\{ & V(\phi) - R_{(2n)} + \frac{1}{2} L_{IJ}(\phi) h^{\mu\nu} \boldsymbol{\nabla}_\mu \phi^I \boldsymbol{\nabla}_\nu \phi^J + \frac{1}{4} B(\phi) h^{\mu\rho} h^{\nu\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} \\
& + \frac{1}{h}\left(\frac{1}{2n-1}\pi^2 - \pi^{\mu\nu}\pi_{\mu\nu}\right) - \frac{1}{2h}(L^{-1}(\phi))^{IJ}\pi_I \pi_J - \frac{1}{2hB(\phi)} h_{\mu\nu}\pi^{a\mu}\pi^{a\nu}\Bigg\} \\
& + (K \text{ terms}) \sim 0, \\
(S, K^\mu_\lambda) = {} & 2\nabla_\nu \pi^{\mu\nu} - h^{\mu\nu}\pi_I \boldsymbol{\nabla}_\nu \phi^I - h^{\mu\nu} F^a_{\nu\rho}\pi^{a\rho} + (K \text{ terms}) \sim 0, \\
(S, K^{a\tau}_A) = {} & \boldsymbol{\nabla}_\mu \pi^{a\mu} - \pi_I (iT^a \phi)^I + (K \text{ terms}) \sim 0,
\end{aligned}$$

which perfectly agree with our previous expressions (3.2.6), (3.2.7), and (3.2.8) taking the slight difference of canonical momenta mentioned above.

C Conformal manifold

Most of the QFTs can be achieved by deforming CFTs. Hence a deformation problem of CFTs is an important problem, which is what we would like to discuss in this section. We review preliminary facts in the first two subsections, and report partial results on our project [20] in the last subsection.

C.1 Definition

Consider a CFT defined on a d-dimensional manifold (M, γ). A deformation of the CFT is defined by

$$S[X, \gamma, \lambda] := S_{\mathrm{CFT}}[X, \gamma] + \lambda^I \int_M d^d x \sqrt{\gamma}\, O_I(x). \tag{C.1}$$

In terms of the partition function, this can be written

$$Z[\gamma,\lambda]=\int\mathcal{D}X\exp\Big\{-S_{\text{CFT}}[X,\gamma]-\lambda^I\int_M d^dx\sqrt{\gamma}O_I(x)\Big\},\tag{C.2}$$

where we have collectively denoted dynamical fields X. Usually, the deformations break conformal symmetry of the original theory S_{CFT}. In fact, if one adds mass terms, they explicitly break conformal symmetry, and trigger an RG flow. However, sometimes it happens that the conformal symmetry is preserved under the deformations. In this case, each λ gives another CFT S, and the couplings λ can be recognized as labels of a continuous family of CFTs, which are connected to the reference CFT S_{CFT}, or $\lambda=0$. The family $\mathcal{M}:=\{\lambda\}$ is called conformal manifold. We will denote a conformal manifold of (S)CFTs on spacetime manifold M with $\mathcal{N}=\mathcal{N}_0$ SUSY as $\mathcal{M}_M^{\mathcal{N}=\mathcal{N}_0}$ in order to circumvent lengthy expressions such as 'conformal manifold of $\mathcal{N}=\mathcal{N}_0$ (S)CFTs on spacetime manifold M.' Furthermore, the manifolds are equipped with a natural metric called Zamolodchikov metric [5], which is Riemannian if the CFT is unitary. The metric is defined as coefficients of two-point functions of the deformation operators:

$$\langle O_I(x)O_J(y)\rangle_{\lambda,\text{con.}}\equiv\frac{g_{IJ}(\lambda)}{[d^2(x,y)]^d},\tag{C.3}$$

where we added the subscript λ to emphasize the correlation functions are evaluated in the presence of the deformations (C.1), and con. to emphasize it is a connected correlation function. $d(,)$ is the distance invariant under the isometries of M. We usually work in Euclidean signature, so the unitarity is equivalent to the reflection positivity [51]. Thus the Zamolodchikov metric is positive definite, and is a Riemannian metric in unitary CFTs. The fact especially guarantees an existence of the inverse g^{-1}.The conformal manifold $\mathcal{M}$ is equipped with the (Zamolodchikov) metric, and $(\mathcal{M},g)$ is promoted to a metric space (to be more precise a Riemannian manifold in unitary CFTs). The space is known to have fascinating properties. We will study some of them below.

It would be obvious from (C.1) that a motion on $\mathcal{M}$ is generated by exactly marginal operators. In other words, exactly marginal operators at $p\in\mathcal{M}$ span the tangent space $T_p\mathcal{M}$ of the conformal manifold. Collecting the tangent spaces, we can form the tangent bundle

$$T\mathcal{M}:=\bigcup_{p\in\mathcal{M}}T_p\mathcal{M}.$$

The presence of nontrivial (Zamolodchikov) metric results in a nontrivial holonomy in the tangent bundle. This observation led to interesting developments these days. See for example [52].

Before we start the investigation of the geometry of conformal manifolds, we need a small preparation. We would like to see some consequences led from the fact that O_Is must preserve conformal symmetries. For a deformation to preserve conformal symmetry, the deformation operator O_I must be exactly marginal. Otherwise, the deformation induces nonzero beta-functions, and the theory leaves the conformal fixed point.

Let us consider OPEs of the exactly marginal operators. From dimensional analysis and symmetries, the most general form of the OPE is given by

$$O_I(x)O_J(y) \sim \sum_{\Delta} C_{IJ\Delta} \frac{1}{[d^2(x,y)]^{d-\Delta/2}} O_\Delta(y) + C_{IJ}{}^K \frac{1}{\sqrt{\gamma}} \delta^{(d)}(x-y) O_{K'}(y), \tag{C.4}$$

where O_Δ is a scalar primary operator with scaling dimension Δ and $O_{K'}$ is a marginal operator not restricted to exactly marginal operators at this point. The factor $1/\sqrt{\gamma}$ is inserted to match transformation laws of both sides under diffeomorphism[22]. Note that the definition of the Zamolodchikov metric (C.3) is not restricted to flat spacetimes. On curved manifolds, one-point functions do not necessarily vanish. To understand the fact, let us briefly recall why one-point functions of primary operators with non-zero scaling dimensions vanish on flat spacetimes. By assumption of the translation invariance of the vacuum $P|vac\rangle = 0$, one-point functions must be independent of spacetime points:

$$\langle O_\Delta(x) \rangle = \langle e^{-iP\cdot x} O_\Delta(0) e^{iP\cdot x} \rangle = \langle O_\Delta(0) \rangle \equiv c_\Delta.$$

On the other hand, covariance under a dilation $x \mapsto x' = \alpha x$ implies

$$\langle O_\Delta(\alpha x) \rangle = \alpha^{-\Delta} \langle O_\Delta(x) \rangle.$$

[22] The Dirac's delta is not invariant under diffeomorphisms. Consider a transformation $x \mapsto \tilde{x}(x)$ with $\tilde{\gamma}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = \gamma_{\mu\nu} dx^\mu dx^\nu$, or $\det(\partial\tilde{x}/\partial x) = \sqrt{\gamma/\tilde{\gamma}}$. We would like to study how $\delta^{(d)}(\tilde{x}(x))$ behaves. Since the Dirac's delta has a support just at the vicinity of $\tilde{x} = 0$, we have to study zero(es) of the function $\tilde{x}$. Let us write a zero of the function as x_0. Then we can Taylor expand the function around $x = x_0$;

$$\begin{aligned} \tilde{x}(x) &= \tilde{x}(x_0 + (x - x_0)) \\ &= 0 + (x - x_0) \cdot \frac{\partial \tilde{x}}{\partial x}(x_0) + O\Big((x - x_0)^2\Big). \end{aligned}$$

Then the well-known properties of Dirac's delta yield

$$\begin{aligned} \delta^{(d)}(\tilde{x}(x)) &= \frac{1}{\left|\frac{\partial \tilde{x}}{\partial x}(x_0)\right|} \delta^{(d)}(x - x_0) \\ &= \sqrt{\frac{\tilde{\gamma}}{\gamma}} \delta^{(d)}(x - x_0). \end{aligned}$$

Thus the Dirac's delta is not diffeomorphism invariant, while a combination $\delta^{(d)}(x)/\sqrt{\gamma}$ is. This is of course consistent with the usual normalization of the Dirac's delta

$$1 \equiv \int d^d x \delta^{(d)}(x)$$

in flat space. The author would like to thank Taishi Ikeda for discussion on this point.

Combining the two relations, we obtain

$$c_\Delta \equiv \langle O_\Delta(\alpha x)\rangle = \alpha^{-\Delta} c_\Delta,$$

or

$$0 = c_\Delta(1-\alpha^{-\Delta}).$$

This implies $c_\Delta = 0$ unless $\Delta = 0$. In this analysis, translation invariance plays a central role. The invariance does not hold on curved manifolds in general. Furthermore, curved manifolds M have some scales r (such as a radius of a sphere). Hence in principle we can have

$$\langle O_\Delta(x)\rangle_M \sim r^{-\Delta}.$$

How can we determine the one-point functions? One should recall the gerenal rule of QFTs; an action is not sufficient to fix a theory, rather one should specify RG schemes (or equivalently local counterterms) to define a theory. This rule is also the case in our problem at hand, namely we should specify possible local counterterms.

On a curved manifold M we have Riemann tensors collectively denoted R, and we can use them to compensate for mass dimensions. In fact, we can write local counterterms [53]

$$\alpha_k \int_M d^dx\sqrt{\gamma}R^k\lambda_\Delta O_{\Delta-2k},$$

where λ_Δ is a spurion which couples to a primary operator with scaling dimension Δ. Since we can calculate the one-point function of the operator O_Δ by taking functional derivative of $\ln Z$ with respect to λ_Δ, the derivative brings down the local counterterm once from the action, and we obtain a contribution

$$\langle O_\Delta(x)\rangle_M \ni \#\alpha_k R^k \langle O_{\Delta-2k}(x)\rangle_M$$

or

$$\langle O_\Delta(x)\rangle_M \ni \#\alpha_k r^{-2k} \langle O_{\Delta-2k}(x)\rangle_M.$$

The local conterterm can also be interpreted as a mixing of O_Δ and $O_{\Delta-2k}$, and it manifests in a contact term

$$\langle T_{\mu\nu}(x)O_\Delta(y)\rangle_M \ni \#\alpha_k R^k \delta^{(d)}(x-y)\langle O_{\Delta-2k}(x)\rangle_M.$$

In particular, a primary operator with scaling dimension $\Delta \in 2\mathbb{N}$ can mix with the identity operator

$$\alpha_{\Delta/2}\int_M d^dx\sqrt{\gamma}R^{\Delta/2}\lambda_\Delta \mathbb{1}.$$

The presence of the local counterterms indicate that one-point functions on curved manifolds are intrinsically ambiguous (equivalently scheme dependent). We need to specify the local counterterms to define CFTs on curved manifolds. Local counterterms are not completely arbitrary. Some prefered symmetries force us to take specific schemes (or local counterterms). One can understand the fact easily recalling the experience in diagrammatic computation of chiral anomalies. In the example, gauge symmetry and axial symmetry compete with each other. Since we should respect gauge symmetry, we usually choose a scheme which makes the gauge symmetry manifest. This is also the case here. If we prefer to respect some symmetries, such as SUSY, we are forced to choose a specific scheme (or local counterterms) which respects the symmetries. As a result one-point functions can be physical. For example, we will see

$$Z[\gamma_{\mathbb{S}^4_r}, \lambda] = (\lambda\text{-indep. factor})e^{K(\lambda,\bar{\lambda})/12}$$

in the next subsection. This result implies

$$\frac{1}{\pi^2}\int_{\mathbb{S}^4_r} d^4x\sqrt{\gamma}\langle C_I(x)\rangle_{\mathbb{S}^4_r} = 32r^2\langle A_I(N)\rangle_{\mathbb{S}^4_r} = \frac{1}{12}\partial_I K(\lambda,\bar{\lambda}),$$

or

$$\langle A_I(N)\rangle_{\mathbb{S}^4_r} = \frac{1}{32\cdot 12r^2}\partial_I K(\lambda,\bar{\lambda}).$$

This one-point function is physical as a consequence of a scheme which respects SUSY. See [54] for another interesting example with non-vanishing one-point functions.

If there were (scalar) primary operators with non-zero scaling dimensions in the first term of the OPE (C.4), their one-point functions would not necessarily vanish, and they would disturb the definition (C.3). Thus (scalar) primaries with non-zero scaling dimensions are not allowed to appear in the non-local piece of the OPEs of exactly marginal operators. The OPE coefficient of the remaining term, which is proportional to the identity operator, is identified with the Zamolodchikov metric

$$C_{IJ0} = g_{IJ}(\lambda).$$

Next, let us study the contact term of the OPE. Suppose there were marginal operators in the term. Then we can show it leads to non-vanishing beta-functions. This can be easily seen by studying a connected two-point function of marginal operators which are not exactly marginal at order $\mathcal{O}(\lambda^2)$. This quantity can be computed by taking derivatives of $\langle O_{I'}(x)O_{J'}(y)\rangle_{\lambda,\text{con.}}$ with respect to λ twice. Recalling the partition function (C.2), we can compute the quantity by bringing down two integrated exactly marginal operators:

$$\partial_K\partial_L\langle O_{I'}(x)O_{J'}(y)\rangle_{\lambda,\text{con.}} = \int_M d^dz\sqrt{\gamma}\int_M d^dw\sqrt{\gamma}\langle O_{I'}(x)O_{J'}(y)O_K(z)O_L(w)\rangle_{\lambda,\text{con.}}.$$

Using the contact term of the OPE that we are studying for the pair of exactly marginal operators O_KO_L, one obtains a term of the form

$$C_{KL}{}^{M'}\int_M d^dz\sqrt{\gamma}\langle O_{I'}(x)O_{J'}(y)O_{M'}(z)\rangle_{\lambda,\text{con.}}.$$

Since the operators are marginal and not restricted to exactly marginal, the three-point function has non-local piece. The piece results in a contribution to $\langle O_{I'}(x)O_{J'}(y)\rangle_{\lambda,\text{con.}}|_{\lambda^2}$:

$$C_{KL}{}^{M'}\int_M d^dz\sqrt{\gamma}\frac{C_{I'J'M'}}{[d^2(x,y)]^{d/2}[d^2(y,z)]^{d/2}[d^2(z,x)]^{d/2}}.$$

The integral suffers from logarithmic UV divergences from the regions $z\to x,y$. For example, consider a region $z\to x$. $d^dz/[d^2(z,x)]^{d/2}$ obviously suffers from a logarithmic UV divergence. We have to regularize the divergence by necessarily introducing a renormalization scale μ. The factor $\ln\mu^2$ leads to non-zero beta-functions proportional to $C_{IJ}{}^{K'}$ in contradiction with the fact that the deformations are realized by exactly marginal operators. Therefore we conclude

$$C_{IJ}{}^{K'}=0,$$

namely, only exactly marginal operators can appear in the contact term.

To sum up the above analysis, we have found that OPEs of exactly marginal operators must have the following form:

$$O_I(x)O_J(y)\sim\frac{g_{IJ}(\lambda)}{[d^2(x,y)]^d}+C_{IJ}{}^K\frac{1}{\sqrt{\gamma}}\delta^{(d)}(x-y)O_K(y). \tag{C.5}$$

We can do more. The OPE coefficient of the contact term can be identified with the Christoffel symbol constructed from the Zamolodchikov metric [55]. To see this, take a derivative of (C.3) with respect to λ^K:

$$\frac{\partial_K g_{IJ}(\lambda)}{[d^2(x,y)]^d}=-\int_M d^dz\sqrt{\gamma}\langle O_I(x)O_J(y)O_K(z)\rangle_{\lambda,\text{con.}}.$$

The three-point function does not have non-local piece. To show this, suppose there was a non-local piece $1/[d^2(x,y)]^{d/2}[d^2(y,z)]^{d/2}[d^2(z,x)]^{d/2}$, where the form is fixed by conformal symmetry. Then as in our discussion above, the integral suffers from UV divergences from the region $z\to x,y$, which have to be regularized by introducing a factor $\ln\mu^2$. The factor contradicts the conformal symmetry, hence is forbidden. We conclude all we have is contact terms, say $O_I(x)O_K(z)\sim C_{IK}{}^L\delta^{(d)}(x-z)O_L(x)$.

Employing the OPE (C.5) and the two-point function (C.3), we can easily compute the RHS:

$$\begin{aligned}(\text{RHS})&=-\int_M d^dz\sqrt{\gamma}\left\{C_{IK}{}^L\frac{1}{\sqrt{\gamma}}\delta^{(d)}(x-z)\frac{g_{JL}(\lambda)}{[d^2(x,y)]^d}+C_{JK}{}^L\frac{1}{\sqrt{\gamma}}\delta^{(d)}(y-z)\frac{g_{IL}(\lambda)}{[d^2(x,y)]^d}\right\}\\&=-\frac{1}{[d^2(x,y)]^d}\left\{C_{IK}{}^Lg_{JL}(\lambda)+C_{JK}{}^Lg_{IL}(\lambda)\right\},\end{aligned}$$

or

$$\partial_Kg_{IJ}=-C_{IK}{}^Lg_{JL}-C_{JK}{}^Lg_{IL}. \tag{C.6}$$

In order to cancel some terms on the RHS of the expression, take a linear combination $(\partial_K g_{IJ} - \partial_I g_{JK} - \partial_K g_{IJ})$ to get

$$C_{IJ}{}^{L} g_{LK} = -\frac{1}{2}(\partial_I g_{JK} - \partial_I g_{JK} - \partial_K g_{IJ}),$$

or by multiplying both hand sides by g^{MK} we arrive at

$$C_{IJ}{}^{M}(\lambda) = -\frac{1}{2} g^{MK}(\lambda)(\partial_I g_{JK}(\lambda) - \partial_I g_{JK}(\lambda) - \partial_K g_{IJ}(\lambda)) = -\Gamma_{IJ}{}^{M}(\lambda). \tag{C.7}$$

This equation manifests the role of the OPE coefficient as the Christoffel symbol.

C.2 Geometry of conformal manifolds

It is known that conformal manifolds of CFTs in various dimensions with various numbers of SUSY are Kähler. There are various (equivalent) definitions of Kähler manifolds. One of them, which is suitable for our purpose, is given by the integrability

$$\partial_I g_{J\bar{K}} = \partial_J g_{I\bar{K}}, \tag{C.8}$$

and the condition

$$g_{IJ} = 0 = g_{\bar{I}\bar{J}}, \tag{C.9}$$

where I (resp. $\bar{I}$) is a holomorphic (resp. antiholomorphic) index. We will sketch out the proof of the fact that conformal manifolds of four-dimensional SCFTs be Kähler by Asnin [56] since we would like to discuss four-dimensional SCFTs in this thesis. For the case of other dimensions, see for example [17, 18].

Consider a four-dimensional SCFT with $\mathcal{N} = 1$. We consider superpotential deformations $W = \lambda^I \mathcal{O}_I$ of the SCFT. The exactly marginal operators are embedded in the top components of chiral superfields $O_I = \int d^2\theta \mathcal{O}_I = \mathcal{O}_I|_{\text{top}}$. Since the deformation is realized by adding the superpotential, there exists holomorphic coordinate system [57] on the conformal manifold.

In order to study whether the Zamolodchikov metric in our problem is Kähler, let us begin with (C.9). g_{IJ} can be obtained by performing superspace integrals $d^2\theta d^2\theta'$ of $\langle \mathcal{O}_I(x,\theta)\mathcal{O}_J(y,\theta')\rangle_{\lambda,\text{con.}}$. We can write some non-trivial function of (x, y, θ, θ') which is consistent with superconformal symmetry

$$\langle \mathcal{O}_I(x,\theta)\mathcal{O}_J(y,\theta')\rangle_{\lambda,\text{con.}} = f(x, y, \theta, \theta').$$

On the other hand, since the chiral superfield $\mathcal{O}_I$ is a superconformal primary, it is annihilated by superconformal generators $S^{\dot{\alpha}}$, i.e., $\{S^{\dot{\alpha}}, \mathcal{O}_I] = 0$. Performing $S^{\dot{\alpha}}$ on both hand sides of the two-point function, the LHS trivially vanishes, while the RHS does not. This implies that the function f itself mush vanish. Therefore we conclude $g_{IJ}(\lambda) = 0$. Similarly, one can show $g_{\bar{I}\bar{J}}(\lambda) = 0$.

Next, let us study whether the mixed components $g_{I\bar{J}}$ and $g_{\bar{I}J}$ satisfy the integrability (C.8). Take a derivative of both sides of

$$g_{J\bar{K}}(\lambda) = [(x-y)^2]^4 \langle O_J(x) \bar{O}_{\bar{K}}(y) \rangle_{\lambda,\text{con.}}$$

with respect to λ^I:

$$\begin{aligned}
\partial_I g_{J\bar{K}}(\lambda) &= [(x-y)^2]^4 \left\{ \langle \partial_I O_J(x) \bar{O}_{\bar{K}}(y) \rangle_{\lambda,\text{con.}} + \langle O_J(x) \partial_I \bar{O}_{\bar{K}}(y) \rangle_{\lambda,\text{con.}} - \int d^4z \langle O_J(x) \bar{O}_{\bar{K}}(y) O_I(z) \rangle_{\lambda,\text{con.}} \right\} \\
&= [(x-y)^2]^4 \left\{ \langle \partial_J O_I(x) \bar{O}_{\bar{K}}(y) \rangle_{\lambda,\text{con.}} - \int d^4z \langle O_J(x) \bar{O}_{\bar{K}}(y) O_I(z) \rangle_{\lambda,\text{con.}} \right\} \\
&= \partial_J g_{I\bar{K}}(\lambda),
\end{aligned}$$

where we have used the commutativity

$$\partial_I O_J = \partial_I \int d^2\theta \frac{\partial W}{\partial \lambda^J} = \partial_J \int d^2\theta \frac{\partial W}{\partial \lambda^I} = \partial_J O_I$$

for the first term, holomorphy

$$\partial_I \bar{O}_{\bar{K}} = \partial_I \int d^2\bar{\theta} \frac{\partial \bar{W}}{\partial \bar{\lambda}^K} = 0$$

for the second term, and the following fact for the last term; we have seen above that the three-point function can only have contact terms. In the presence of SUSY, the contanct terms have to be appropriately supersymmetrized. The consideration tells us that only two operators with the same chirality can admit contact terms

$$\mathcal{O}_I(x,\theta)\mathcal{O}_J(y,\theta') \sim C_{IJ}{}^K \delta^{(4)}(x-y)\delta^{(2)}(\theta-\theta')\mathcal{O}_K(y,\theta').$$

Note that the coefficient $C_{IJ}{}^K$ is symmetric in the lower indices (IJ).

Thus we have shown that conformal manifolds of four-dimensional SCFTs with $\mathcal{N} \geq 1$ are Kähler. For the technical details, see the original paper [56].

Once the Kählerity of the conformal manifold of a CFT defined on some spacetime manifold (M,γ) is shown, one can conclude at once that all conformal manifolds of CFTs defined on spacetime manifolds (N,γ') which is connected to (M,γ) via conformal maps are also Kähler without any computation. We now move on to a proof of this claim.

Suppose we have two d-dimensional metric spaces (M,γ) and (N,γ') connected with a conformal map $\varphi : M \to N$, i.e.,

$$\varphi^*\gamma' = \Omega^2 \cdot \gamma$$

for some smooth nonvanishing function $\Omega \in C^\infty(M)$. Under the conformal transformation, primary operators with spin J and scaling dimension Δ are transformed as

$$\mathcal{O}'^J_\Delta(x'(x)) = \Omega^{-(\Delta+J)}(x) \left(\frac{\partial x'}{\partial x}\right)^J \mathcal{O}^J_\Delta(x), \tag{C.10}$$

where tensor indices are implicit. Furthermore, suppose the conformal manifold of a CFT defined on (M, γ) is Kähler, namely $g_{IJ}(\lambda) = 0 = g_{\bar{I}\bar{J}}(\lambda)$ and $\partial_I g_{J\bar{K}}(\lambda) = \partial_J g_{I\bar{K}}(\lambda)$. Then we would like to address whether the Kählerity conditions (C.8) and (C.9) are satisfied by the Zamolodchikov metric g' of a CFT defined on (N, γ')[23]. Employing the transformation law (C.10) and the definition of the Zamolodchikov metric (C.3), we notice at once that (C.9) follows immediately:

$$\begin{aligned}
g'_{IJ}(\lambda) &\equiv [d^2_{\gamma'}(x', y')]^d \langle O'_I(x') O'_J(y')\rangle_{\gamma'} \\
&= [d^2_{\gamma'}(x', y')]^d \Omega^{-d}(x)\Omega^{-d}(y)\langle O_I(x) O_J(y)\rangle_{\gamma} \\
&= \left(\frac{d^2_{\gamma'}(x', y')}{d^2_{\gamma}(x, y)}\right)^d \Omega^{-d}(x)\Omega^{-d}(y) g_{IJ}(\lambda) = 0.
\end{aligned}$$

Here, d_γ denotes a distance measured with the metric γ. Since this equality must hold regardless of spacetime points[24], we conclude $g'_{IJ}(\lambda) = 0$. Similarly, one can show $g'_{\bar{I}\bar{J}}(\lambda) = 0$.

Next, let us study whether the integrability (C.8) is satisfied. This can be shown with ease by taking a derivative of $g'_{J\bar{K}}(\lambda)$ with respect to λ^I:

$$\begin{aligned}
\partial_I g'_{J\bar{K}}(\lambda) &= \partial_I \left\{[d^2_{\gamma'}(x', y')]^d \langle O'_J(x') \bar{O}'_{\bar{K}}(y')\rangle_{\gamma'}\right\} \\
&= \partial_I \left\{[d^2_{\gamma'}(x', y')]^d \Omega^{-d}(x)\Omega^{-d}(y)\langle O_J(x) \bar{O}_{\bar{K}}(y)\rangle_{\gamma}\right\} \\
&= \partial_I \left\{\left(\frac{d^2_{\gamma'}(x', y')}{d^2_{\gamma}(x, y)}\right)^d \Omega^{-d}(x)\Omega^{-d}(y) g_{J\bar{K}}(\lambda)\right\} \\
&= \left(\frac{d^2_{\gamma'}(x', y')}{d^2_{\gamma}(x, y)}\right)^d \Omega^{-d}(x)\Omega^{-d}(y) \partial_I g_{J\bar{K}}(\lambda) \\
&= \left(\frac{d^2_{\gamma'}(x', y')}{d^2_{\gamma}(x, y)}\right)^d \Omega^{-d}(x)\Omega^{-d}(y) \partial_J g_{I\bar{K}}(\lambda) \\
&= \partial_J g'_{I\bar{K}}(\lambda).
\end{aligned}$$

In the fourth line, we used the fact that the scaling dimension $\Delta = d$ of exactly marginal operators are independent of λ. Hence the derivative ∂_I just acts on the Zamolodchikov metric g.

To conclude, we have shown that if the conformal manifold of a CFT defined on a d-dimensional spacetime (M, γ) is Kähler, all the manifolds of CFTs defined on spacetime manifolds (N, γ') connected to (M, γ) via conformal maps are also Kähler as long as complex

[23] Since conformal maps do not preserve SUSY in general, we have to assume an existence of complex coordinate on $\mathcal{M}_N$. The author would like to thank Prof. Hiroaki Kanno for pointing out this point.

[24] In order to probe the Zamolodchikov metric, we are implicitly studying the correlator at separated points $x \neq y$. Thus the prefactor is well-defined. Furthermore, possible ambiguities caused by operator mixing discussed above do not enter our discussion here.

coordinates exist on $\mathcal{M}_N$. In other words, conformal maps do not disturb exactly marginal coupling dependence of two-point functions of exactly marginal operators[25]. Although the fact has been used implicitly in the literature, explicit proof is given for the first time to the best of author's knowledge.

Then compact spacetime manifolds are good places to work on because we can avoid IR divergences originating from infinite volumes of non-compact manifolds. One manifold with such a property is the n-sphere $\mathbb{S}^n$. Sphere partition functions are free of IR divergences, and are natural observables. This is the reason why the object has been studied extensively. See for example [17, 18, 19].

From now on we will focus on Euclidean spacetimes, which are conformally connected to d-spheres $\mathbb{S}^d$. Let us consider the (log of a) sphere partition function $\ln Z[\gamma_{\mathbb{S}^d_r}]$. The subscript r denotes the radius of the sphere. Since a generating functional (a.k.a Schwinger functional) $W[\gamma_{\mathbb{S}^d_r}]$ is defined by

$$\exp\Big\{ - W[\gamma_{\mathbb{S}^d_r}]\Big\} \equiv Z[\gamma_{\mathbb{S}^d_r}], \tag{C.11}$$

$\ln Z$ is nothing but W (up to minus sign). By definition, the generating functional W is a sum of all connected diagrams (or correlation functions). Thus in general, W suffers from UV divergences (and possibly IR ones if spacetime manifolds are non-compact, that is not the case because we are considering theories on spheres which are compact). The strongest divergence goes like ${\Lambda_{\text{UV}}}^d$. The dimensionful parameter should be compensated with the radius r of the sphere to make a dimensionless quantity. Λ_{UV} can enter W just in the combination $(\Lambda_{\text{UV}} r)$ because W is obviously dimensionless as it is in the exponent. The structure of the lower divergence depends on whether d is even or odd. The general expression is given by

$$W[\gamma_{\mathbb{S}^{2n}_r}] = A_{2n}(\Lambda_{\text{UV}} r)^{2n} + \cdots + A_2(\Lambda_{\text{UV}} r)^2 + A \ln(\Lambda_{\text{UV}} r) + A_0, \tag{C.12}$$

$$W[\gamma_{\mathbb{S}^{2n-1}_r}] = B_{2n-1}(\Lambda_{\text{UV}} r)^{2n-1} + \cdots + B_1(\Lambda_{\text{UV}} r) + F, \tag{C.13}$$

where the powers descend by two because of Euclidean symmetry. Now, we would like to ask how much of (C.12) and (C.13) are physical. That is, which parts of the generating functional cannot be tuned with local counterterms. Those terms which can be tuned with (finite) local counterterms are unphysical because these terms are scheme-dependent.

Let us begin with the case of general CFTs where an existence of SUSY is not assumed. Since our spacetime manifolds, d-spheres, are curved, local counterterms should respect diffeomorphism. Thus in particular a shift of the form $\Lambda_{\text{UV}} \mapsto \Lambda_{\text{UV}} + 1/r$ is not allowed, because we should recognize r as a component of the metric, and metrics themselves are not diffeomorphism invariant.

[25]One would notice that this fact can be generalized to n-point functions of BPS operators with non-zero spins because spin J and scaling dimensions Δ in the exponent of conformal factors Ω are independent of exactly marginal couplings. The former, spin, is discrete, and it cannot be changed under the continuous deformations. The latter, scaling dimension of BPS operators, are protected, and are also independent of λ. Thus we can repeat the same analysis to conclude λ-dependence of n-point functions of BPS operators are not disturbed under conformal maps (at least when insertion points do not coincide).

In case of $d = 2n$, the strongest UV divergence can be tuned away with a local counterterm so called cosmological constant:

$$\#{\Lambda_{\rm UV}}^{2n} \int d^{2n}x \sqrt{\gamma_{\mathbb{S}^{2n}_r}}.$$

In fact, the volume integral gives some number multiplied by r^{2n}. Therefore by tuning # appropriately, we can tune away the A_{2n} term. Similarly, all A_{2n-2k} terms can be tuned away with local counterterms

$$\#{\Lambda_{\rm UV}}^{2n-2k} \int d^{2n}x \sqrt{\gamma_{\mathbb{S}^{2n}_r}} R^k,$$

where R collectively denotes Riemann tensor $R_{\mu\nu\rho\sigma}$, Ricci tensor $R_{\mu\nu}$, or Ricci scalar R. Therefore all terms except the coefficient A of the logarithmically divergent term are unphysical. The A term cannot be shifted with any local counterterms, and it is scheme-independent. Hence the coefficient A has a physical meaning[26].

In case of odd dimensions $d = 2n-1$, we can repeat the same analysis. All power divergent terms $B_{2n-1-2k}$ can be tuned away with a local counterterm

$$\#{\Lambda_{\rm UV}}^{2n-1-2k} \int d^{2n-1}x \sqrt{\gamma_{\mathbb{S}^{2n-1}_r}} R^k.$$

Indeed, one would convince oneself by dimensional analysis that the integral produces some number multiplied by $r^{2n-1-2k}$. The structure agrees with that of the term $B_{2n-1-2k}$, and if one chooses the coefficients # carefully, all power divergent terms $B_{2n-1-2k}$ can be tuned away. The constant piece F remains. It cannot be shifted with any local counterterms, and is scheme-independent. Thus the term F has a physical meaning[27].

When a CFT admit exactly marginal operators, the coefficients in (C.12),(C.13) become functions of exactly marginal couplings in general. Power divergences in this situation can also be tuned away with suitable coefficient functions of exactly marginal couplings. In order to interpret A or F as the 'number' of degrees of freedom, they must be independent of exactly marginal couplings. Happily, the coefficient functions are indeed independent of the exactly marginal couplings. Since A can be identified with the coefficients of trace anomalies, the fact $A(\lambda) = A =$ const. follows from the Wess-Zumino condition. The fact $F(\lambda) = F =$ const. can be shown using the fact that one-point functions of exactly marginal operators always vanish in odd-dimensions because marginal operators cannot mix with the identity operator.

[26]Indeed, it agrees with 'central charge' a, which is a coefficient of the Euler density E_{2n} in trace anomaly, and is believed that the number A can be recognized as the 'number' of degrees of freedom. As a matter of fact, this expectation is proved in two and four spacetime dimensions, although there are some subtleties coming from signatures of spacetimes. To be precise, two-dimensional version, called c-theorem [5], was proved in Euclidean signature to employ positivity of two-point functions, while four-dimensional version, called a-theorem [10], was proved in Lorentzian signature to employ the optical theorem.

[27]Similar to the even dimensional case, it is believed that F can be recognized as the 'number' of degrees of freedom [6, 7, 8].

So far, we have not assumed a presence of SUSY. If we assume its presence, local counterterms have to respect not only diffeomorphism but also local supersymmetry, that is, local counterterms have to respect symmetries of supergravity (SUGRA). Since local counterterms are more restricted, more terms in W remain physical. As a matter fact, the constant piece A_0 becomes physical [19].

The constant pieces are known to compute Kähler potentials in two-dimensional $\mathcal{N} = (2,2)$ SCFTs and four-dimensional $\mathcal{N} = 2$ SCFTs. To the author's knowledge, the fact in two dimension was first revealed through studies in string theory [58], and later the Kähler potentials were computed exactly via localization [18] and others. The fact was also given by another transparent proof via superconformal WT relation [19], in which the fact in four dimension was also shown. The latter proof was made complete in [59] by explicit construction of local counterterms which is responsible for the Kähler ambiguity of Kähler potentials. We will briefly review [59].

We consider a deformation problem[28] of four-dimensional $\mathcal{N} = 2$ SCFTs on $\mathbb{S}^4_r$

$$S[X, \gamma_{\mathbb{S}^4_r}, \lambda] := S_{\text{SCFT}}[X, \gamma_{\mathbb{S}^4_r}] + \frac{\lambda^I}{\pi^2}\int_{\mathbb{S}^4_r} \sqrt{\gamma} d^4\theta \mathcal{O}_I(x,\theta) + c.c..$$

The second derivative of the (log of the) partition function is formally given by integrated two-point function

$$\partial_I \partial_{\bar{J}} \ln Z[\gamma_{\mathbb{S}^4_r}, \lambda] = \frac{1}{\pi^4}\left\langle \int_{\mathbb{S}^4_r} d^d x \sqrt{\gamma} O_I(x) \int_{\mathbb{S}^4_r} d^d y \sqrt{\gamma} \bar{O}_{\bar{J}}(y) \right\rangle_{\text{con.}}.$$

Since the expression suffers from UV divergences from the region $y \to x$, we need to regularize it. In [59], massive regularization was employed. That is, they introduced a massive hypermultiplet. As a result, only a subgroup $SU(1|1)$ of the full superconformal symmetry $OSp(2|4)$ remains unbroken. The fermionic charges are constructed from conformal Killing spinors

$$\chi^j = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}\left(1 + \frac{i}{2r} x_m \Gamma^m\right)\chi_0^j,$$

with constraints

$$P_L \chi_0^j = 0, \quad \chi_0^i = \sigma_1^{ij} \varepsilon_{jk} \Gamma_1 \Gamma_2 \chi_0^k,$$

where P_L is the projection matrix to left-handed spinors. The fact that the conformal Killing spinors vanish on poles $N; x = 0$ and $S; x = \infty$ is important:

$$\chi_L(N) := P_L \chi(N) = 0 = \chi_R(S) := P_R \chi(S). \tag{C.14}$$

[28] In this argument, we depart from the normalization (C.1) of exactly marginal couplings λ and use the one of [59] which is multiplied by $1/\pi^2$ to make the final result (C.17) simpler.

Using some constraints and some algebras, one finds that the top component C of an $\mathcal{N}=2$ chiral multiplet $\mathcal{O}$ can be written almost exact:

$$C = \delta_{SU(1|1)} \left(\frac{\#}{||\chi_L||^2} \right) + (\text{total derivative}). \tag{C.15}$$

Note that this expression is valid only locally, namely on patches that do not include the North pole N.

Now consider a correlator of the integrated top component C and any $\delta_{SU(1|1)}$-closed operator O, i.e., $\delta_{SU(1|1)}O = 0$:

$$\left\langle \int_{\mathbb{S}^4_r} \sqrt{\gamma} C(x) O \right\rangle_{\text{con.}} .$$

By dividing the integration region $\mathbb{S}^4$ into a four-dimensional ball with radius ε which contains the North pole and the others, $\mathbb{S}^4_r = B^4_\varepsilon \cup (\mathbb{S}^4_r \backslash B^4_\varepsilon)$, one can see the contributions of the integrated top component 'localize' at the 'boundary'. In fact, in the limit $\varepsilon \to 0$, contributions from the ball would vanish

$$\lim_{\varepsilon \to 0} \left\langle \int_{B^4_\varepsilon} d^4x \sqrt{\gamma} C(x) O \right\rangle_{\text{con.}} = 0$$

because the integration measure $\sim \varepsilon^4$ cannot be compensated by the integrand to be finite[29]. Then we are left with

$$\lim_{\varepsilon \to 0} \left\langle \int_{\mathbb{S}^4_r \backslash B^4_\varepsilon} d^4x \sqrt{\gamma} C(x) O \right\rangle_{\text{con.}} .$$

Since the integration region no longer contain the North pole, we can safely exploit the expression (C.15) in the correlator. Since O is $\delta_{SU(1|1)}$-closed by definition, the $SU(1|1)$-exact term of C does not contribute:

$$\begin{aligned} \left\langle \int_{\mathbb{S}^4_r \backslash B^4_\varepsilon} d^4x \sqrt{\gamma} \left(\delta_{SU(1|1)} \# \right) O \right\rangle_{\text{con.}} &= \left\langle \delta_{SU(1|1)} \left(\int_{\mathbb{S}^4_r \backslash B^4_\varepsilon} d^4x \sqrt{\gamma} \# O \right) \right\rangle_{\text{con.}} \\ &\propto \left\langle [Q_{SU(1|1)}, \int_{\mathbb{S}^4_r \backslash B^4_\varepsilon} d^4x \sqrt{\gamma} \# O \} \right\rangle_{\text{con.}} = 0 \end{aligned}$$

because $SU(1|1)$ is assumed not to be spontaneously broken $Q_{SU(1|1)}|vacuum\rangle = 0$. Thus all we have is the total derivative term. Using the Stokes' theorem, we learn that all contributions come from the boundary $\partial(\mathbb{S}^4_r \backslash B^4_\varepsilon) = \mathbb{S}^3_\varepsilon$, which reduces to a point, the North pole, in the limit $\varepsilon \to 0$. Then one finds

$$\left\langle \int_{\mathbb{S}^4_r} d^4x \sqrt{\gamma} C(x) O \right\rangle_{\text{con.}} = 32\pi^2 r^2 \langle A(N) O \rangle_{\text{con.}}$$

[29]Here, it would be implicitly assumed that insertion point(s) of O (if O is a (product of) local operator(s)) are not contained in B^4_ε. The situation is realized by taking ε small enough unless some of the insertion points are placed exactly on N. Then singularities of the integrand are not enough to compensate for the vanishing volume element $\varepsilon^4 \to 0$.

where A is the bottom component of an $\mathcal{N}=2$ chiral multiplet. Similarly one obtains

$$\left\langle \int_{\mathbb{S}^4_r} d^4x \sqrt{\gamma}\bar{C}(x) O \right\rangle_{\text{con.}} = 32\pi^2 r^2 \langle \bar{A}(S) O\rangle_{\text{con.}}.$$

Here we have

$$\delta C = (\text{total derivative}) + (w-2)\#,$$

where w is the R-charge. Hence the integrated top component of an $\mathcal{N}=2$ chiral multiplet with $w=2$ is superconformal invariant on spacetime manifolds without boundary, such as $\mathbb{S}^4$, and it can be used as O in the above discussion. Therefore we arrive at

$$\begin{aligned} \partial_I \partial_{\bar{J}} \ln Z[\gamma_{\mathbb{S}^4_r}, \lambda] &= \frac{1}{\pi^4} \left\langle \int_{\mathbb{S}^4_r} d^dx \sqrt{\gamma} O_I(x) \int_{\mathbb{S}^4_r} d^dy \sqrt{\gamma} \bar{O}_{\bar{J}}(y) \right\rangle_{\text{con.}} \\ &= (32r^2)^2 \langle A_I(N) \bar{A}_{\bar{J}}(S)\rangle_{\text{con.}}. \end{aligned} \tag{C.16}$$

With the help of the superconformal WT relation $\langle A_I(N)\bar{A}_{\bar{J}}(S)\rangle_{\text{con.}} = \frac{r^4}{48}\langle C_I(N)\bar{C}_{\bar{J}}(S)\rangle_{\text{con.}}$, and the definition of the Zamolodchikov metric $\langle C_I(N)\bar{C}_{\bar{J}}(S)\rangle_{\text{con.}} \equiv \frac{1}{(2r)^8} g_{I\bar{J}}(\lambda)$, we arrive at

$$\partial_I \partial_{\bar{J}} \ln Z[\gamma_{\mathbb{S}^4_r}, \lambda] = \frac{1}{12} g_{I\bar{J}}(\lambda) = \frac{1}{12} \partial_I \partial_{\bar{J}} K(\lambda, \bar{\lambda}), \tag{C.17}$$

where in the second equality the fact that the conformal manifold $\mathcal{M}^{\mathcal{N}=2}_{\mathbb{R}^4}$ is Kähler, which we discussed above, is used. In other words, we have shown

$$Z[\gamma_{\mathbb{S}^4_r}, \lambda] = (\lambda\text{-indep. factor}) e^{K(\lambda,\bar{\lambda})/12}. \tag{C.18}$$

C.3 Branched sphere partition function

An interesting object called supersymmetric Rényi entropy (SRE) [60] was defined recently using an object which is similar to what we have just studied. The object is a branched sphere partition function

$$Z[\gamma_{\mathbb{S}^d_q}]. \tag{C.19}$$

A branched sphere $\mathbb{S}^d_q$ is a one-parameter generalization of the ordinary sphere $\mathbb{S}^d$. It has a metric

$$\begin{aligned} ds^2_{\mathbb{S}^d_q}/r^2 &= d\theta^2 + q^2 \sin^2\theta d\tau^2 + \cos^2\theta ds^2_{\mathbb{S}^{d-2}_1}, \\ \theta &\in [0,\pi/2), \quad \tau \in [0, 2\pi). \end{aligned} \tag{C.20}$$

where r is its radius. The parameter q is called a branching parameter. The branched sphere reduces to the ordinary sphere $\mathbb{S}^d$ in the limit $q \to 1$ as evident from the metric (C.20).

Since we have learned sphere partition functions compute Kähler potentials (in various spacetime dimensions with various numbers of SUSY) and the branched spheres reduce to the round spheres in the limit $q \to 1$, it is natural to ask whether the branched sphere partition functions (C.19) also compute Kähler potentials. We would like to address this problem in the rest of this section. These are partial results of our ongoing project [20].

Firstly, we notice that there is one problem with the branched sphere (C.20). The space suffers from a singularity. Consider a vicinity of the 'North pole' $\theta = 0$. In the vicinity, the τ component of the metric is approximated by

$$ds^2_{\mathbb{S}^d_q}/r^2 = \cdots + q^2\theta^2 d\tau^2 + \cdots .$$

This implies a period of τ is not 2π, rather $2\pi q$. In other words, a point $p \in \mathbb{S}^d_q$ does not come back to its original point p even if one shifts τ by 2π, rather one has to shift by $2\pi q$. Therefore there is a deficit angle $2\pi(q-1)$ around the 'North pole', and it leads to a singularity. In fact, Ricci scalar has Dirac's delta-like behaviour $R \sim (q-1)\delta(\theta)$. The singularity is known as a conical singularity.

It is a difficult problem to define QFTs on spaces with singularities. To avoid the problem, we perform a conformal map to another space which is free from singularities. Fortunately, such a map is known [61],[62]. Namely, there exists a conformal map from the branched sphere to a product space of a circle with radius qr times $(d-1)$-dimensional hyperbolic space with radius r

$$\varphi : \mathbb{S}^d_q \to \mathbb{S}^1_{qr} \times \mathbb{H}^{d-1}_r .$$

More concretely, the map is given by

$$\sinh\eta = -\cot\theta. \tag{C.21}$$

All effects of the branching is reflected in the rescaling of the $\mathbb{S}^1$ radius, and there is no singularity in the image. The fact can be understood explicitly from its metric

$$\begin{gathered} ds^2_{\mathbb{S}^1_{qr}\times\mathbb{H}^{d-1}_r} = (qr)^2 d\tau^2 + r^2\Big\{d\eta^2 + \sinh^2\eta\, ds^2_{\mathbb{S}^{d-2}_1}\Big\}, \\ \tau \in [0, 2\pi), \quad \eta \in [0, +\infty). \end{gathered} \tag{C.22}$$

A simple computation shows the map is indeed conformal

$$ds^2_{\mathbb{S}^d_q}/r^2 = \frac{1}{\cosh^2\eta} ds^2_{\mathbb{S}^1_{qr}\times\mathbb{H}^{d-1}_r}/r^2 .$$

To summarize so far, we launched a study of conformal manifolds $\mathcal{M}^{\mathcal{N}}_{\mathbb{S}^d_q}$. However, since the space suffers from (conical) singularity, we perform a conformal map, and work on $\mathbb{S}^1 \times \mathbb{H}^{d-1}$. The conformal map does not ruin our problem because conformal maps preserve Kählerity of conformal manifolds as we have shown. Thus if we can show Kählerity of conformal manifolds $\mathcal{M}^{\mathcal{N}}_{\mathbb{S}^1_{qr}\times\mathbb{H}^{d-1}_r}$, it immediately implies those on the branched spheres $\mathcal{M}^{\mathcal{N}}_{\mathbb{S}^d_q}$ are also Kähler, if we can make the 'manifold' mathematically meaningful.

Then our first task is to construct SUSY on $\mathbb{S}^1_l \times \mathbb{H}^{d-1}_r$. For definiteness, we concentrate on the case $d = 4$ from now on. The objective can be accomplished with ease employing the method called rigid SUSY initiated in [63]. To avoid a mess, we explain the construction in another section, Appendix D [30]. As a result one finds out that there exists conformal Killing spinors labeled by four complex constants ($c_1, c_2, \tilde{c}_1$, and $\tilde{c}_2$ in (D.27) and (D.34)). Furthermore, as explained in the same appendix, the conformal Killing spinors do not admit zeroes, and we cannot exploit the 'localization' method explained in the previous subsection. Therefore we have to perform a straightforward computation.

Since we have shown there exists SUSY on $\mathbb{S}^1_l \times \mathbb{H}^3_r$ with general l, we can safely set $l = r$ or $l = qr$, and define $\mathcal{N} = 2$ SCFTs on the manifolds.

We have two approaches in mind to address the Kählerity of $\mathcal{M}^{\mathcal{N}=2}_{\mathbb{S}^1_{qr}\times\mathbb{H}^3_r}$; one is to consider the case $|q-1| \ll 1$, and treat the rescaling of the $\mathbb{S}^1$ perturbatively, and another is to solve an ODE the partition function satisfies to get a result valid for general q, i.e., not necessarily close to 1. Let us explain these two approaches in this order.

We study SCFTs on $\mathbb{S}^1_{qr} \times \mathbb{H}^3_r$ with $|q-1| \ll 1$. In this case, we can treat the rescaling as a perturbation in $(q-1)$:

$$\gamma_{\mathbb{S}^1_{qr}\times\mathbb{H}^3_r} = \gamma_{\mathbb{S}^1_r\times\mathbb{H}^3_r} + \delta\gamma_{\tau\tau}$$

with $\delta\gamma_{\tau\tau} = (q^2-1)r^2$. Let us denote our $\mathcal{N} = 2$ SCFT on conformally flat $\mathbb{S}^1_r \times \mathbb{H}^3_r$

$$S_{\text{SCFT}}[X, \gamma_{\mathbb{S}^1_r\times\mathbb{H}^3_r}].$$

Exactly marginal deformations give new SCFTs

$$S[X, \gamma_{\mathbb{S}^1_r\times\mathbb{H}^3_r}, \lambda] := S_{\text{SCFT}}[X, \gamma_{\mathbb{S}^1_r\times\mathbb{H}^3_r}] + \lambda^K \int_{\mathbb{S}^1_r\times\mathbb{H}^3_r} d^4z\sqrt{\gamma}C_K(z) + \bar{\lambda}^{\bar{K}} \int_{\mathbb{S}^1_r\times\mathbb{H}^3_r} d^4z\sqrt{\gamma}\bar{C}_{\bar{K}}(z). \tag{C.23}$$

Since they are superpotential deformations, there exists holomorphic coordinate system [57] on the conformal manifold $\mathcal{M}^{\mathcal{N}=2}_{\mathbb{S}^1_r\times\mathbb{H}^3_r}$. The corresponding stress tensor is given by

$$T^{\mu\nu}(z) := \frac{2}{\sqrt{\gamma(z)}}\frac{\delta S[X, \gamma_{\mathbb{S}^1_r\times\mathbb{H}^3_r}, \lambda]}{\delta\gamma_{\mu\nu}(z)} = T^{\mu\nu}_{\text{SCFT}}(z) + \gamma^{\mu\nu}(z)\Big[\lambda^K C_K(z) + \bar{\lambda}^{\bar{K}}\bar{C}_K(z)\Big]. \tag{C.24}$$

Deformation of the background metric is realized by inserting the stress tensors:

$$S'[X, \gamma_{\mathbb{S}^1_{qr}\times\mathbb{H}^3_r}, \lambda] \equiv S[X, \gamma_{\mathbb{S}^1_r\times\mathbb{H}^3_r}, \lambda] + \delta\gamma_{\tau\tau}\int_{\mathbb{S}^1_r\times\mathbb{H}^3_r} d^4z\sqrt{\gamma}T^{\tau\tau}(z). \tag{C.25}$$

The corresponding partition function is given by

$$Z[\gamma_{\mathbb{S}^1_{qr}\times\mathbb{H}^3_r}, \lambda] := \int \mathcal{D}X \exp\Big\{-S[X, \gamma_{\mathbb{S}^1_r\times\mathbb{H}^3_r}, \lambda] - \delta\gamma_{\tau\tau}\int_{\mathbb{S}^1_r\times\mathbb{H}^3_r} d^4z\sqrt{\gamma}T^{\tau\tau}(z)\Big\}. \tag{C.26}$$

[30]Since we would like to discuss a rescaling of the $\mathbb{S}^1$ radius, we construct SUSY on $\mathbb{S}^1$ with general radius l rather than qr.

We would like to study the Zamolodchikov metric

$$\frac{g_{ij}(\lambda)}{[d_q^2(x,y)]^4} := \langle O_i(x)O_j(y)\rangle_{q,\text{con.}}$$
$$\equiv Z^{-1}[\gamma_{\mathbb{S}^1_{qr}\times\mathbb{H}^3_r},\lambda]\int \mathcal{D}X O_i(x)O_j(y)\exp\Big\{-S'[X,\gamma_{\mathbb{S}^1_{qr}\times\mathbb{H}^3_r},\lambda]\Big\} - \langle O_i(x)\rangle_q\langle O_j(y)\rangle_q, \tag{C.27}$$

where O_i are exactly marginal chiral (resp. anti-chiral) operator C_I (resp. $\bar{C}_{\bar{I}}$), and subscripts q denotes they are evaluated on the rescaled spacetime manifold $\mathbb{S}^1_{qr}\times\mathbb{H}^3_r$.

Let us study whether our Zamolodchikov metric is Kähler or not. All we have to do is to show two properties (C.8) and (C.9), as we reviewed in the previous subsection. After some straightforward computations, one finds

$$\frac{g_{IJ}(\lambda)}{[d_q^2(x,y)]^4} = \langle C_I(x)C_J(y)\rangle_{\text{con.}} - \delta\gamma_{\tau\tau}\int_{\mathbb{S}^1_r\times\mathbb{H}^3_r} d^4z\sqrt{\gamma}\langle C_I(x)C_J(y)T^{\tau\tau}(z)\rangle_{\text{con.}} + \mathcal{O}((q-1)^2), \tag{C.28}$$

$$\begin{aligned}\frac{\partial_K g_{I\bar{J}}(\lambda)}{[d_q^2(x,y)]^4} =& -[1+(q^2-1)]\int_{\mathbb{S}^1_r\times\mathbb{H}^3_r} d^4z\sqrt{\gamma}\langle C_I(x)\bar{C}_{\bar{J}}(y)C_K(z)\rangle_{\text{con.}}\\ &+\left\langle \frac{\partial C_I(x)}{\partial\lambda^K}\bar{C}_{\bar{J}}(y)\right\rangle_{\text{con.}} + \left\langle C_I(x)\frac{\partial \bar{C}_{\bar{J}}(y)}{\partial\lambda^K}\right\rangle_{\text{con.}}\\ &-\delta\gamma_{\tau\tau}\int_{\mathbb{S}^1_r\times\mathbb{H}^3_r} d^4z\sqrt{\gamma}\left\{\left\langle \frac{\partial C_I(x)}{\partial\lambda^K}\bar{C}_{\bar{J}}(y)T^{\tau\tau}(z)\right\rangle_{\text{con.}} + \left\langle C_I(x)\frac{\partial \bar{C}_{\bar{J}}(y)}{\partial\lambda^K}T^{\tau\tau}(z)\right\rangle_{\text{con.}}\right\}\\ &+\delta\gamma_{\tau\tau}\int_{\mathbb{S}^1_r\times\mathbb{H}^3_r} d^4z\sqrt{\gamma}d^4w\sqrt{\gamma}\langle C_I(x)\bar{C}_{\bar{J}}(y)C_K(z)T^{\tau\tau}(w)\rangle_{\text{con.}} + \mathcal{O}((q-1)^2).\end{aligned} \tag{C.29}$$

As we have emphasized in previous subsections, integrated correlation functions suffer from UV divergences. To extract mathematically meaningful results, we have to regularize them correctly. Without entering the technical parts, let us see what we can say at this point.

The first term of (C.28) is nothing but the chiral-chiral component of the Zamolodchikov metric on $\mathbb{S}^1_r\times\mathbb{H}^3_r$, which is conformal to $\mathbb{R}^4$, and we can use the knowledge that the component vanishes as proved by Asnin. In [64], it was shown that conformal symmetry requires that (non-local pieces of) the three-point function $\langle C_I(x)C_J(y)T^{\mu\nu}(z)\rangle_{\text{con.}}$ is proportional to $\langle C_I(x)C_J(y)\rangle_{\text{con.}}$, which vanishes as we have just seen.

The first term of (C.29) is evaluated on $\mathbb{S}^1_r\times\mathbb{H}^3_r$, which is conformal to $\mathbb{R}^4$, and we can use the knowledge that only operators with the same chirality admit contact terms. The contact term from the region $z\to x$ yields the Christoffel symbol $\Gamma^L_{IK}C_L$, which is symmetric in (IK). Since the exactly marginal operators can also be defined as

$$C_I(x) = \frac{\partial\mathcal{L}(x)}{\partial\lambda^I},$$

the second term is symmetric in (IK). The third term vanishes by holomorphy. The same reasoning leads us to a conclusion that the third line is also symmetric in (IK). Thus all terms we have seen is symmetric in (IK), and we expect that the other terms are also symmetirc in (IK), resulting in the integrability (C.8).

To accomplish the analysis, we have to consider integration of the four-point function carefully, which would be difficult. To circumvent the difficulty, we are trying another way; consider an ODE the (log of the) partition function $Z[\gamma_{\mathbb{S}^1_{qr}\times\mathbb{H}^3_r},\lambda]$ satisfies[31]

$$\frac{d}{dq}\ln Z[\gamma_{\mathbb{S}^1_{qr}\times\mathbb{H}^3_r},\lambda]=-\#q\int_{\mathbb{S}^1_{qr}\times\mathbb{H}^3_r}d^4x\sqrt{\gamma}\langle T^{\tau\tau}_{\mathrm{SCFT}}\rangle_{\mathbb{S}^1_{qr}\times\mathbb{H}^3_r}-\frac{1}{q}\lambda^I\int_{\mathbb{S}^1_{qr}\times\mathbb{H}^3_r}d^4x\sqrt{\gamma}\langle C_I(x)\rangle_{\mathbb{S}^1_{qr}\times\mathbb{H}^3_r}-c.c..$$

Since this is an ODE, there exists a unique solution once we specify an initial condition at $q=1$, where the spacetime manifold is conformal to four-sphere, and we know the partition function [65]

$$Z[\gamma_{\mathbb{S}^4_r},\lambda]=\left(\frac{r}{r_0}\right)^{-4a}e^{K(\lambda,\bar{\lambda})/12}.$$

To pursue this line, we have to determine the one-point functions which appear in the RHS of the ODE to fix the q-dependence of the ODE. The author is working on this point now.

Notice that we used the fact that the dimension of our manifold is four just to exploit Kählerity of $\mathcal{M}^{\mathcal{N}\geq 1}_{\mathbb{R}^4}$ and to ensure an existence of SUSY by explicitly computing conformal Killing spinors. Therefore we can repeat the same analysis in other spacetime dimensions as long as $\mathcal{M}_{\mathbb{R}^d}$ is Kähler and there exists SUSY. It would be interesting to study whether $\mathcal{M}^{\mathcal{N}=\mathcal{N}_0}_{\mathbb{S}^{2n}_q}$ (or $\mathcal{M}^{\mathcal{N}=\mathcal{N}_0}_{\mathbb{S}^1_{qr}\times\mathbb{H}^{2n-1}_r}$) is Kähler or not.

D SUSY on $\mathbb{S}^1_l\times\mathbb{H}^3_r$

In this appendix, we compute conformal Killing spinors and superconformal variations on our manifold $\mathbb{S}^1_l\times\mathbb{H}^3_r$. This goal can be achieved employing a systematic method called rigid SUSY initiated by Festuccia and Seiberg [63]. In this method, one firstly embed a flat space Lagrangian into an off-shell SUGRA, where plenty of extra fields are present. In order to kill gravitational dynamical degrees of freedom while preserving field theoretical dynamical degrees of freedom, we take an appropriate scaling limit while sending the Planck scale to infinity. Symmetries of the underlying SUGRA, e.g., Kähler symmetry, require to turn on some background fields. Conformal Killing spinor equations are then obtained as SUSY conditions, which essentially says SUSY variations of fermionic variables should vanish. Otherwise, SUSY would be spontaneously broken. The conformal Killing spinors are their solutions. On the other hand, there is another way to obtain the conformal Killing spinor equations. Namely, one can embed the flat space theory to conformal SUGRA. Then

[31]We have not fixed the q-independent coefficient # of the first term, which depends on how we define the stress tensor.

conformal Killing spinor equations are again obtained as SUSY conditions. However, in this case, there are more background fields. Since the conformal Killing spinor equations should match, one can fix these background fields by comparing the resulting two equations. Substituting the background fields in the known superconformal transformations of conformal SUGRA, one can also obtain superconformal variations on a fixed background manifold, $\mathbb{S}^1_l \times \mathbb{H}^3_r$ in our problem.

D.1 Conformal Killing spinors

Since the manifold we want to consider

$$(\mathbb{S}^1_l \times \mathbb{H}^3_r, \gamma)$$

has the Ricci scalar

$$R[\mathbb{S}^1_l \times \mathbb{H}^3_r] = +\frac{6}{r^2} \tag{D.1}$$

in the convention of [63], we must turn on some background fields. The Kähler symmetry of the underlying SUGRA requires

$$R = +6\gamma_{\mu\nu}V^\mu V^\nu, \tag{D.2}$$

where V is a conserved auxiliary field in the 'new minimal' gravity multiplet, and some other conditions. Using the metric of $\mathbb{S}^1_l \times \mathbb{H}^3_r$ (in the polar coordinate)

$$ds^2_{\mathbb{S}^1_l \times \mathbb{H}^3_r} = l^2 d\tau^2 + r^2(d\eta^2 + \sinh^2\eta d\theta^2 + \sinh^2\eta \sin^2\theta d\phi^2), \tag{D.3}$$

we rewrite (D.2):

$$\begin{aligned}\frac{6}{r^2} &= R = 6\gamma_{\mu\nu}V^\mu V^\nu \\ &= 6\Big\{l^2 V^\tau V^\tau + r^2(V^\eta V^\eta + \sinh^2\eta V^\theta V^\theta + \sinh^2\eta \sin^2\theta V^\phi V^\phi)\Big\}.\end{aligned} \tag{D.4}$$

Solutions with nonzero background V fields in the $\mathbb{H}^3$ directions are allowed just in order to solve (D.4), however, they would lead to nonzero (constant) background gauge fields in these directions, thus to singularities at $\eta = 0$ because $\mathbb{S}^2 = \{x \in \mathbb{H}^3_r | \eta = \text{const.}\}$ shrinks at the point $\eta = 0$ and we cannot define a nonzero vector there. Therefore nonsingular solutions must have $V_{\mathbb{H}^3} = 0$ (similarly for A, which is also an auxiliary field in the 'new minimal' gravity multiplet). Then the solution of (D.4) is given by

$$V^\tau = \pm\frac{1}{rl} \quad \text{or} \quad V_\tau = \gamma_{\tau\tau}V^\tau = \pm\frac{l}{r}, \quad \text{the others} = 0. \tag{D.5}$$

Substituting (D.5) into the SUSY variation of the gravitino $\psi_{\mu\alpha}$, we obtain conformal Killing spinor equations as SUSY conditions $\delta\psi_\mu \stackrel{!}{=} 0$:

$$\begin{aligned}\Big((\nabla_\mu - iA_\mu)\zeta\Big)_\alpha &= \mp\frac{il}{r}\delta_{\mu,\tau}\zeta_\alpha \mp \frac{i}{rl}(\sigma_{\mu\tau})_\alpha{}^\beta\zeta_\beta,\\ \Big((\nabla_\mu + iA_\mu)\tilde{\zeta}\Big)^{\dot\alpha} &= \pm\frac{il}{r}\delta_{\mu,\tau}\tilde{\zeta}^{\dot\alpha} \pm \frac{i}{rl}(\tilde{\sigma}_{\mu\tau})^{\dot\alpha}{}_{\dot\beta}\tilde{\zeta}^{\dot\beta},\end{aligned} \tag{D.6}$$

where A is a background $U(1)_R$ gauge field.

Here, the metric (D.3) gives Levi-Civita connections

$$\begin{gathered}\Gamma^\nu_{\tau\rho} = 0, \quad \Gamma^\theta_{\eta\theta} = \coth\eta = \Gamma^\phi_{\eta\phi},\\ \Gamma^\eta_{\theta\theta} = -\sinh\eta\cosh\eta, \quad \Gamma^\phi_{\theta\phi} = \cot\theta,\\ \Gamma^\eta_{\phi\phi} = -\sinh\eta\cosh\eta\sin^2\theta, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta,\\ \text{and their counterparts}, \quad \text{the others} = 0.\end{gathered} \tag{D.7}$$

A natural choice of tetrads

$$e^1 = ld\tau, \quad e^2 = rd\eta, \quad e^3 = r\sinh\eta d\theta, \quad e^4 = r\sinh\eta\sin\theta d\phi \tag{D.8}$$

hence yields spin connections[32]

$$\begin{gathered}\omega^{ab}_\tau = 0 = \omega^{ab}_\eta, \quad \omega^{ab}_\theta = \cosh\eta(\delta^{a2}\delta^{b3} - \delta^{a3}\delta^{b2}),\\ \omega^{ab}_\phi = \cosh\eta\sin\theta(\delta^{a2}\delta^{b4} - \delta^{a4}\delta^{b2}) + \cos\theta(\delta^{a3}\delta^{b4} - \delta^{a4}\delta^{b3}).\end{gathered} \tag{D.9}$$

In addition, anti-symmetric spacetime matrices are given by

$$\begin{aligned}\sigma_{\eta\tau} &= \frac{irl}{2}\sigma^3 = \tilde{\sigma}_{\eta\tau},\\ \sigma_{\theta\tau} &= -\frac{irl\sinh\eta}{2}\sigma^2 = \tilde{\sigma}_{\theta\tau},\\ \sigma_{\phi\tau} &= \frac{irl\sinh\eta\sin\theta}{2}\sigma^1 = -\tilde{\sigma}_{\phi\tau}.\end{aligned} \tag{D.10}$$

Using the spin connections (D.9) and anti-symmetric matrices (D.10), we decompose (D.6)

[32] In order to simplify notations, we omit the superscript 'rigid' from now on. Thus spin connections without any label should be understood as those of [66].

in each direction:

$$\partial_\tau \zeta = i\left(A_\tau \mp \frac{l}{r}\right)\zeta, \tag{D.11}$$

$$0 = \partial_\eta \zeta \mp \frac{1}{2}\sigma^3 \zeta, \tag{D.12}$$

$$0 = \partial_\theta \zeta - \frac{i}{2}\begin{pmatrix} 0 & e^{\pm\eta} \\ e^{\mp\eta} & 0 \end{pmatrix}\zeta, \tag{D.13}$$

$$0 = \partial_\phi \zeta + \frac{1}{2}\begin{pmatrix} -i\cos\theta & -e^{\pm\eta}\sin\theta \\ e^{\mp\eta}\sin\theta & i\cos\theta \end{pmatrix}\zeta, \tag{D.14}$$

$$\partial_\tau \tilde{\zeta} = -i\left(A_\tau \mp \frac{l}{r}\right)\tilde{\zeta}, \tag{D.15}$$

$$0 = \partial_\eta \tilde{\zeta} \pm \frac{1}{2}\sigma^3 \tilde{\zeta}, \tag{D.16}$$

$$0 = \partial_\theta \tilde{\zeta} - \frac{i}{2}\begin{pmatrix} 0 & e^{\mp\eta} \\ e^{\pm\eta} & 0 \end{pmatrix}\tilde{\zeta}, \tag{D.17}$$

$$0 = \partial_\phi \tilde{\zeta} + \frac{1}{2}\begin{pmatrix} i\cos\theta & e^{\mp\eta}\sin\theta \\ -e^{\pm\eta}\sin\theta & -i\cos\theta \end{pmatrix}\tilde{\zeta},, \tag{D.18}$$

where we simply denote

$$\zeta := \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}, \quad \tilde{\zeta} := \begin{pmatrix} \tilde{\zeta}^{\dot{1}} \\ \tilde{\zeta}^{\dot{2}} \end{pmatrix}.$$

Suppose A_τ be a constant. Then the τ-directions can be solved at once:

$$\zeta(x) = \exp\left\{ i\left(A_\tau \mp \frac{l}{r}\right)\tau \right\}\zeta(\boldsymbol{x}), \quad \tilde{\zeta}(x) = \exp\left\{ -i\left(A_\tau \mp \frac{l}{r}\right)\tau \right\}\tilde{\zeta}(\boldsymbol{x}), \tag{D.19}$$

where for the notational economy, we simply denote $\mathbb{H}^3$-dependent part of the Killing spinors as $\zeta(\boldsymbol{x})$ or $\tilde{\zeta}(\boldsymbol{x})$. It may seem that there is a freedom to choose A_τ at our will, which thus affects the resulting SUSY algebras as explained in [63] and [66]. However conformal Killing spinors must satisfy (anti-)periodic boundary condition in the $\mathbb{S}^1_l$ direction

$$\zeta(\tau + 2\pi, \boldsymbol{x}) \stackrel{!}{=} \pm\zeta(\tau, \boldsymbol{x}), \quad \tilde{\zeta}(\tau + 2\pi, \boldsymbol{x}) \stackrel{!}{=} \pm\tilde{\zeta}(\tau, \boldsymbol{x}). \tag{D.20}$$

The boundary condition fixes the remaining degrees of freedom A_τ (mod 2π):

$$A_\tau = \pm\frac{l}{r} + 2\pi\nu. \quad (\nu \in \mathbb{Z} \text{ if } \zeta; \text{periodic}, \quad \nu \in \mathbb{Z} + \frac{1}{2} \text{ if } \zeta; \text{anti-periodic}) \tag{D.21}$$

Next, let us consider (D.14). The nontrivial matrix of the second term can be diagonalized:

$$M = SJS^{-1},$$

where

$$M := \begin{pmatrix} -i\cos\theta & -e^{\pm\eta}\sin\theta \\ e^{\mp\eta}\sin\theta & i\cos\theta \end{pmatrix}, \tag{D.22}$$

$$S := \begin{pmatrix} -ie^{\pm\eta}\cot\frac{\theta}{2} & ie^{\pm\eta}\tan\frac{\theta}{2} \\ 1 & 1 \end{pmatrix}, \quad J := \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} \frac{i}{2}e^{\mp\eta}\sin\theta & \sin^2\frac{\theta}{2} \\ -\frac{i}{2}e^{\mp\eta}\sin\theta & \cos^2\frac{\theta}{2} \end{pmatrix}. \tag{D.23}$$

If we define

$$\chi := S^{-1}\zeta, \tag{D.24}$$

(D.14) reduces to

$$0 = \partial_\phi\chi + \frac{1}{2}J\chi$$

because S does not depend on ϕ. Since the matrix is diagonal, we can solve the PDE at once:

$$\chi(\eta,\theta,\phi) = e^{\frac{i\phi}{2}\sigma^3}\chi(\eta,\theta). \tag{D.25}$$

One notices that (D.23) also diagonalizes (D.12) and (D.13). In fact they reduce to

$$\begin{aligned} 0 &= \partial_\eta\chi \pm \frac{1}{2}\chi, \\ 0 &= \partial_\theta\chi + \frac{1}{2}\begin{pmatrix} -\cot\frac{\theta}{2} & 0 \\ 0 & \tan\frac{\theta}{2} \end{pmatrix}\chi. \end{aligned}$$

These PDEs can also be solved with ease:

$$\chi(\eta,\theta) = e^{\mp\frac{\eta}{2}}e^{\frac{1}{2}\ln(\sin\theta) - \frac{1}{2}\ln\left(\cot\frac{\theta}{2}\right)\sigma^3}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{D.26}$$

where c_1, c_2; const.

Combining (D.19), (D.25), and (D.26), we arrive at

$$\chi(\tau,\eta,\theta,\phi) = \exp\left\{2\pi i\nu\tau \mp \frac{\eta}{2} + \frac{1}{2}\ln(\sin\theta) - \frac{1}{2}\ln\left(\cot\frac{\theta}{2}\right)\sigma^3 + \frac{i\phi}{2}\sigma^3\right\}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{D.27}$$

or by solving (D.24) in terms of ζ we obtain

$$\zeta(\tau,\eta,\theta,\phi) = \begin{pmatrix} -ie^{\pm\eta}\cot\frac{\theta}{2} & ie^{\pm\eta}\tan\frac{\theta}{2} \\ 1 & 1 \end{pmatrix}\chi(\tau,\eta,\theta,\phi). \tag{D.28}$$

In the same manner, let us solve $\tilde{\zeta}$. (D.18) can be diagonalized:

$$M = SJS^{-1}$$

where

$$M := \begin{pmatrix} i\cos\theta & e^{\mp\eta}\sin\theta \\ -e^{\pm\eta}\sin\theta & -i\cos\theta \end{pmatrix}, \tag{D.29}$$

$$S := \begin{pmatrix} ie^{\mp\eta}\tan\frac{\theta}{2} & -ie^{\mp\eta}\cot\frac{\theta}{2} \\ 1 & 1 \end{pmatrix}, \quad J := \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} -\frac{i}{2}e^{\pm\eta}\sin\theta & \cos^2\frac{\theta}{2} \\ \frac{i}{2}e^{\pm\eta}\sin\theta & \sin^2\frac{\theta}{2} \end{pmatrix}. \tag{D.30}$$

Thus a new basis

$$\tilde{\chi} := S^{-1}\tilde{\zeta} \tag{D.31}$$

diagonalizes (D.18):

$$0 = \partial_\phi \tilde{\chi} + \frac{1}{2}J\tilde{\chi},$$

and it can be solved at once:

$$\tilde{\chi}(\eta,\theta,\phi) = e^{\frac{i\phi}{2}\sigma^3}\tilde{\chi}(\eta,\theta). \tag{D.32}$$

(D.30) also diagonalizes the other components:

$$0 = \partial_\eta \tilde{\chi} \mp \frac{1}{2}\tilde{\chi},$$

$$0 = \partial_\theta \tilde{\chi} + \frac{1}{2}\begin{pmatrix} \tan\frac{\theta}{2} & 0 \\ 0 & -\cot\frac{\theta}{2} \end{pmatrix}\tilde{\chi}.$$

These PDEs are easily solved:

$$\tilde{\chi}(\eta,\theta) = e^{\pm\frac{\eta}{2}} e^{\frac{1}{2}\ln(\sin\theta)+\frac{1}{2}\ln\left(\cot\frac{\theta}{2}\right)\sigma^3}\begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix}, \tag{D.33}$$

where $\tilde{c}_1, \tilde{c}_2$; const.

Combining (D.19), (D.32), and (D.33) we arrive at

$$\tilde{\chi}(\tau,\eta,\theta,\phi) = \exp\left\{ -2\pi i\nu\tau \pm \frac{\eta}{2} + \frac{1}{2}\ln(\sin\theta) + \frac{1}{2}\ln\left(\cot\frac{\theta}{2}\right)\sigma^3 + \frac{i\phi}{2}\sigma^3 \right\}\begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix}, \tag{D.34}$$

or by solving (D.31) in terms of $\tilde{\zeta}$ we obtain

$$\tilde{\zeta}(\tau,\eta,\theta,\phi) = \begin{pmatrix} ie^{\mp\eta}\tan\frac{\theta}{2} & -ie^{\mp\eta}\cot\frac{\theta}{2} \\ 1 & 1 \end{pmatrix}\tilde{\chi}(\tau,\eta,\theta,\phi). \tag{D.35}$$

To sum up, we have computed conformal Killing spinors and found that there are four independent fermionic charges which are controlled by $c_1, c_2, \tilde{c}_1$, and $\tilde{c}_2$.

Four-component conformal Killing spinors which generate superconformal transformations on $\mathbb{S}^1_l \times \mathbb{H}^3_r$ are given by

$$\epsilon^j = \begin{pmatrix} u^j_\alpha \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\zeta + (-)^j i\sigma^2\zeta^*)_\alpha \\ 0 \end{pmatrix}, \quad \epsilon_j = (-)^j i\begin{pmatrix} 0 \\ \tilde{u}^{j\dot{\alpha}} \end{pmatrix} = -i\begin{pmatrix} 0 \\ \frac{1}{2}(\tilde{\zeta} + (-)^j i\sigma^2\tilde{\zeta}^*)^{\dot{\alpha}} \end{pmatrix}. \tag{D.36}$$

Let us study whether the conformal Killing spinors admit zeroes. If they have zeroes, then we can consult on the 'localization' computation explained in subsection C.2.

The problem we would like to consider is given as follows; Are there zeroes $x_0 = (\tau_0, \eta_0, \theta_0, \phi_0)$ which make $\epsilon^j(x_0) = 0$ or $\epsilon_j(x_0) = 0$ for arbitrary $c_1, c_2, \tilde{c}_1$, and $\tilde{c}_2$?

Using (D.36), we can write the first condition as

$$\zeta(x_0) + (-)^j i\sigma^2 \zeta^*(x_0) = 0.$$

Since ζ has two-components, this gives two (complex) conditions. To satisfy the condition, both real and imaginary parts have to vanish, and these are equivalent to four real conditions. We define

$$c_1 = a + ib, \quad c_2 = c + id$$

with $a, b, c, d \in \mathbb{R}$. By assumption, these four real numbers are independent, and their coefficients have to vanish by themselves. Then we get the four conditions

$$\begin{aligned}
0 &= -ie^{2\pi i\nu\tau_0} e^{\pm\eta_0/2} e^{i\phi_0/2} \cos\frac{\theta_0}{2} + (-)^j e^{-2\pi i\nu\tau_0} e^{\mp\eta_0/2} e^{-i\phi_0/2} \sin\frac{\theta_0}{2}, \\
0 &= e^{2\pi i\nu\tau_0} e^{\pm\eta_0/2} e^{i\phi_0/2} \cos\frac{\theta_0}{2} - (-)^j i e^{-2\pi i\nu\tau_0} e^{\mp\eta_0/2} e^{-i\phi_0/2} \sin\frac{\theta_0}{2}, \\
0 &= ie^{2\pi i\nu\tau_0} e^{\pm\eta_0/2} e^{-i\phi_0/2} \sin\frac{\theta_0}{2} + (-)^j e^{-2\pi i\nu\tau_0} e^{\mp\eta_0/2} e^{i\phi_0/2} \cos\frac{\theta_0}{2}, \\
0 &= -e^{2\pi i\nu\tau_0} e^{\pm\eta_0/2} e^{-i\phi_0/2} \sin\frac{\theta_0}{2} - (-)^j i e^{-2\pi i\nu\tau_0} e^{\mp\eta_0/2} e^{i\phi_0/2} \cos\frac{\theta_0}{2}.
\end{aligned}$$

Taking some linear combinations, one would learn these are equivalent to

$$\begin{aligned}
e^{2\pi i\nu\tau_0} e^{\pm\eta_0/2} e^{i\phi_0/2} \cos\frac{\theta_0}{2} &= 0 = e^{-2\pi i\nu\tau_0} e^{\mp\eta_0/2} e^{-i\phi_0/2} \sin\frac{\theta_0}{2}, \\
e^{2\pi i\nu\tau_0} e^{\pm\eta_0/2} e^{-i\phi_0/2} \sin\frac{\theta_0}{2} &= 0 = e^{-2\pi i\nu\tau_0} e^{\mp\eta_0/2} e^{i\phi_0/2} \cos\frac{\theta_0}{2}.
\end{aligned}$$

Interestingly, these are independent of j. Since the phases do not matter, these reduce to

$$\begin{aligned}
e^{\pm\eta_0/2} \cos\frac{\theta_0}{2} &= 0 = e^{\mp\eta_0/2} \sin\frac{\theta_0}{2}, \\
e^{\pm\eta_0/2} \sin\frac{\theta_0}{2} &= 0 = e^{\mp\eta_0/2} \cos\frac{\theta_0}{2}.
\end{aligned}$$

One would convince oneself that these conditions do not have solutions. That is ϵ^j does not admit zeroes. Exactly the same analysis shows that ϵ_j does not admit zeroes, neither[33]. Thus we cannot employ the clever 'localization' method, and we have to perform straightforward computations.

[33] There is a possibility that the result is just a matter of artifact due to our choice of coordinate system. Namely, it is logically possible that genuine zeroes of conformal Killing spinors are 'hidden' if we choose 'bad' coordinate systems. However, this would not be the case because conformal Killing spinors obtained in another coordinate system [66] do not have zeroes, neither.

D.2 Superconformal transformations

Next, let us move on to the computation of superconformal variations. Solving (D.36) for ζ or $\tilde{\zeta}$ in terms of u or $\tilde{u}$, we obtain

$$\begin{aligned} \zeta_\alpha = u^1_\alpha + u^2_\alpha, \quad \tilde{\zeta}^{\dot\alpha} = \tilde{u}^{1\dot\alpha} - \tilde{u}^{2\dot\alpha}, \\ (\zeta_\alpha)^* \equiv (\zeta^\dagger)^\alpha = (i\sigma^2)^{\alpha\beta}(u^1_\beta - u^2_\beta), \quad (\tilde{\zeta}^{\dot\alpha})^* \equiv (\tilde{\zeta}^\dagger)_{\dot\alpha} = (i\sigma^2)_{\dot\alpha\dot\beta}(\tilde{u}^{1\dot\beta} + \tilde{u}^{2\dot\beta}). \end{aligned} \tag{D.37}$$

Using these expressions, we rewrite the conformal Killing spinor equations:

$$\begin{aligned} 0 &= \partial_\tau u^j - 2\pi i\nu u^{j+1}, \quad 0 = \partial_\tau \tilde{u}^j - 2\pi i\nu \tilde{u}^{j+1}, \\ 0 &= \partial_\eta u^j \mp \frac{1}{2}\sigma^3 u^{j+1}, \quad 0 = \partial_\eta \tilde{u}^j \mp \frac{1}{2}\sigma^3 \tilde{u}^{j+1}, \\ 0 &= \nabla_\theta u^j \pm \frac{\sinh\eta}{2}\sigma^2 u^{j+1}, \quad 0 = \nabla_\theta \tilde{u}^j \pm \frac{\sinh\eta}{2}\sigma^2 \tilde{u}^{j+1}, \\ 0 &= \nabla_\phi u^j \mp \frac{\sinh\eta\sin\theta}{2}\sigma^1 u^{j+1}, \quad 0 = \nabla_\phi \tilde{u}^j \pm \frac{\sinh\eta\sin\theta}{2}\sigma^1 \tilde{u}^{j+1}, \end{aligned} \tag{D.38}$$

where $j = 1, 2$ and we use an abbreviation $j+1$ for $j+1 \mod 2$.

On the other hand, SUSY conditions $\delta\psi^j_\mu \overset{!}{=} 0 \overset{!}{=} \delta\psi_{\mu j}$ in the conformal SUGRA yield

$$\begin{aligned} 0 &= \nabla_\mu u^j + \frac{1}{2}(b_\mu + iA^{\mathrm{VP}}_\mu)u^j + V_{\mu}{}^{j}{}_{k} u^k - (-)^k \frac{1}{8} T^-_{ab}\varepsilon^{jk}\sigma^{ab}\sigma_\mu \tilde{u}^k - i\sigma_\mu \tilde{\xi}^j, \\ 0 &= -i\tilde{\sigma}_\mu \xi^j, \\ 0 &= -i\frac{1}{(-)^j i}\sigma_\mu \tilde{\xi}'_j, \\ 0 &= \nabla_\mu \tilde{u}^j + \frac{1}{2}(b_\mu - iA^{\mathrm{VP}}_\mu)\tilde{u}^j + V_{\mu j}{}^{k}\tilde{u}^k + \frac{i}{8}\frac{1}{(-)^j i} T^-_{ab}\varepsilon_{jk}\tilde{\sigma}^{ab}\tilde{\sigma}_\mu u^k - i\frac{1}{(-)^j i}\tilde{\sigma}_\mu \xi'_j, \end{aligned} \tag{D.39}$$

where $b, A^{\mathrm{VP}}, V, T^-, \eta$ are dilation, $U(1)$, $SU(2)$ background gauge fields, anti-symetric tensor, and four-spinor parameters of superconformal transformations. We use expressions in [67].

By comparing the two expressions of the conformal Killing spinor equations (D.38) and (D.39), one can fix some background fields in the conformal SUGRA:

$$\begin{aligned} \xi^j = \tilde{\xi}'_j = T^-_{ab} = b_\tau = A^{\mathrm{VP}}_\tau = V_{\tau}{}^{j}{}_{j} = V_{\tau j}{}^{j} = 0, \\ \tilde{\xi}^j = \pm\frac{1}{2r}\sigma^1 u^{j+1}, \quad \xi'_j = \mp(-)^j \frac{i}{2r}\sigma^1 \tilde{u}^{j+1}, \\ V_{\tau}{}^{j}{}_{j+1} = \pm\frac{il}{2r} - 2\pi i\nu = V_{\tau j}{}^{j+1}, \\ 0 = \frac{1}{2}(b_m + iA^{\mathrm{VP}}_m)u^j + V^j_{mk}u^k, \\ 0 = \frac{1}{2}(b_m - iA^{\mathrm{VP}}_m)\tilde{u}^j + V_{mj}{}^{k}\tilde{u}^k, \end{aligned} \tag{D.40}$$

where

$$\eta^j = \begin{pmatrix} \xi^j_\alpha = 0 \\ \tilde{\xi}^{j\dot{\alpha}} \end{pmatrix}, \quad \eta_j = \begin{pmatrix} \xi'_{j\alpha} \\ \tilde{\xi}'^{\dot{\alpha}}_j = 0 \end{pmatrix}. \tag{D.41}$$

Since non-zero constant background gauge fields in $\mathbb{H}^3$-directions would lead to singularities, we have

$$b_m = A^{\rm VP}_m = V^j_{mk} = V_{mj}{}^k = 0,$$

and we also get

$$D = 0$$

as a consequence of a SUSY condition $\delta\chi^i \overset{!}{=} 0$.

Since nonzero values of fermionic fields spontaneously break Poincaré symmetry, they must vanish:

$$\psi_\mu = 0 = \chi^i. \tag{D.42}$$

Substituting the background fields, we arrive at SUSY transformations on our manifold $\mathbb{S}^1_l \times \mathbb{H}^3_r$:[34]

$$\begin{aligned}
\delta A &= \frac{1}{2}\bar{\epsilon}^i \Psi_i, \\
\delta \Psi_i &= (\not{D}A)\epsilon_i + \frac{1}{2}B_{ij}\epsilon^j + \frac{1}{4}\Gamma \cdot G^- \varepsilon_{ij}\epsilon^j + 2wA\eta_i, \\
\delta B_{ij} &= \bar{\epsilon}_{(i}\not{D}\Psi_{j)} - \bar{\epsilon}^k \Lambda_{(i}\varepsilon_{j)k} + 2(1-w)\bar{\eta}_{(i}\Psi_{j)}, \\
\delta G^-_{ab} &= \frac{1}{4}\varepsilon^{ij}\bar{\epsilon}_i \not{D}\Gamma_{ab}\Psi_j + \frac{1}{4}\bar{\epsilon}^i\Gamma_{ab}\Lambda_i - \frac{1}{2}\varepsilon^{ij}(1+w)\bar{\eta}_i\Gamma_{ab}\Psi_j, \\
\delta \Lambda_i &= -\frac{1}{4}\Gamma \cdot G^- \overleftarrow{\not{D}}\epsilon_i - \frac{1}{2}\not{D}B_{ij}\epsilon_k\varepsilon^{jk} + \frac{1}{2}C\epsilon^j\varepsilon_{ij} - (1+w)B_{ij}\varepsilon^{jk}\eta_k + \frac{1}{2}(1-w)\Gamma \cdot G^-\eta_i, \\
\delta C &= -\varepsilon^{ij}\bar{\epsilon}_i \not{D}\Lambda_j + 2w\varepsilon^{ij}\bar{\eta}_i\Lambda_j,
\end{aligned} \tag{D.43}$$

where the superconformal covariant derivative would be defined by $D_\mu X := (\nabla_\mu - wb_\mu - iRA^{\rm VP}_\mu)X + \delta_{\mu,\tau}V_\tau X$ on a field X with Weyl weight w, $U(1)_R$ charge R, and in some nontrivial representation of the $SU(2)$. ∇ includes Levi-Civita connections or spin connections (in the convention of Van Proeyen, i.e., $\omega^{\rm VP}$) corresponding to the representations of the Poincaré group X belongs to. The dot '$\cdot$' implies contraction of tangent space indices; for example, $\Gamma \cdot G^- \equiv \Gamma^{ab}G^-_{ab}$.

[34] As explained in the appendix A, the definitions of the spin connections in [66] and [67] have the opposite sign. (D.43) is written in terms of $\omega^{\rm VP}$. Thus we have to be careful when we employ the explicit form of the spin connections (D.9), which are computed in the convention of [66], i.e., $\omega^{\rm rigid}$.

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