BOSON-FERMION CORRESPONDENCE FROM FACTORIZATION SPACES

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Abstract. We give a proof of the boson-fermion correspondence, an isomorphism of lattice and fermion vertex algebras, in terms of a natural isomorphism of corresponding factorization spaces.

0. Introduction

0.1. The notion of factorization space is a non-linear analog of the factorization algebra, which was introduced by Beilinson and Drinfeld [BD04] in their theory of chiral algebras. The notion of chiral algebra was introduced to give a geometric framework of vertex algebras, and to study the geometric Langlands correspondence [BD].

A factorization space over a scheme X is a family of ind-schemes $\mathcal{G}_I$ over X^I for each finite set I with "factorization structure" given by isomorphisms between these $\mathcal{G}_I$'s. Given a factorization space on X, one can apply a linearization procedure, and if X is a smooth curve then one obtains a factorization algebra, an equivalent notion of chiral algebra.

Although the notion of factorization space looks highly complicated at first glance, it fits various kinds of moduli problems over algebraic curves very well. We may say that factorization spaces capture the intimate connection between two-dimensional conformal field theories and moduli problems on curves.

Along this line, one can ask a question: can one enhance various properties of vertex algebras to the level of factorization spaces? In this note, we consider this kind of problem for the boson-fermion correspondence.

Recall that the two-dimensional boson-fermion correspondence [F81] is stated as an isomorphism

$$V_{\mathbb{Z}} \simeq \bigwedge \tag{0.1}$$

between the lattice vertex algebra $V_{\mathbb{Z}}$ attached to the lattice $\mathbb{Z}$ and the free fermion vertex super algebra $\bigwedge$. Here we borrowed the notations in [FB04]. The essence of the correspondence is that the vertex operators

$$V_{\pm}(z) := :e^{\pm\varphi(z)}:, \quad \varphi(z) := q + a_0 \log(z) - \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{a_n}{n} z^{-n}$$

acting on $V_{\mathbb{Z}}$ obey the same commutation relation with the fermionic operators $\psi(z), \psi^*(z)$ acting on $\bigwedge$. Here a_n's and q denote the Heisenberg generators with the commutation relation $[a_m, a_n] = m\delta_{m+n,0}$ and $[q, a_n] = \delta_{n,0}$. Thus schematically we have

$$V_+(z) \longleftrightarrow \psi(z), \quad V_-(z) \longleftrightarrow \psi^*(z) \tag{0.2}$$

of vertex operators. See [FB04, §5.3] for the detail.

Our main statement in this note is Theorem 5.1 giving an isomorphism

$$\mathcal{G}(X,\mathbb{Z}) \simeq \mathcal{G}r(X, \mathrm{SC}(1)) \tag{0.3}$$

between factorization spaces over a smooth curve X. Here $\mathcal{G}(X,\mathbb{Z})$ is the factorization space arising from the Picard variety of X (moduli space of line bundles on X). $\mathcal{G}r(X, \mathrm{SC}(1))$ is the Beilinson-Drinfeld Grassmannian for the special Clifford group $\mathrm{SC}(1)$. See §4.1 for the notations of Clifford groups.

These factorization spaces are equipped with factorization super linear bundles respectively, and the twisted linearization procedure give the lattice and the Clifford chiral algebras. These chiral algebras correspond to the vertex algebras $V_{\mathbb{Z}}$ and $\bigwedge$ respectively. Thus the isomorphism (0.3) can be considered as an enhanced version of the boson-fermion correspondence (0.1).

0.2. **Organization.** Let us explain the organization of this article.

In §1 we recall the definitions of factorization spaces and factorization line bundles. We also recall the Beilinson-Drinfeld Grassmann $\mathcal{G}r(X, G)$ for a smooth algebraic curve X and a reductive group G. The space $\mathcal{G}r(X, G)$ is a standard example of factorization space,

§2 reviews the linearization procedure which produces a chiral algebra from factorization space.

In §3 we recall the factorization space $\mathcal{G}(X,\mathbb{Z})$ arises from the Picard functor. By the linearization procedure it gives the lattice vertex algebra $V_{\mathbb{Z}}$.

In §4 we consider the Beilinson-Drinfeld Grassmann $\mathcal{G}r(X, \mathrm{SC}(Q))$ for the Clifford group $\mathrm{SC}(Q)$ attached to the non-degenerate quadratic form Q. We explain that by the linearization procedure it yields the Clifford chiral algebra.

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The final §5 gives the proof of the main theorem. The proof is not difficult, and the key observation is that we have a natural identification of moduli stacks

$$\mathcal{M}(X, \mathrm{SC}(1)) \simeq \mathrm{Pic}(X) \times \mathrm{Pic}(X)$$

of the moduli space of SC(1)-bundles on X with two pieces of Picard varieties. It reflects the correspondence (0.2) of vertex operators $(V_+, V_-) \longleftrightarrow (\psi, \psi^*)$.

The ingredients in §§1–3 are basically known facts, and the main sources of our presentation are [BD04, FB04]. The content in §4 is also a variant of [FB04, Chap. 20].

0.3. **Notation.** We will work over the field $\mathbb{C}$ of complex numbers unless otherwise stated. The symbol $\otimes$ denotes the tensor product of $\mathbb{C}$-linear spaces.

For a scheme or an algebraic stack Z, we denote by $\mathcal{O}_Z$, Θ_Z, Ω_Z and $\mathcal{D}_Z$ the structure sheaf, the tangent sheaf, the sheaf of 1-forms and the sheaf of differential operators on Z respectively (if they are defined). By "an $\mathcal{O}$-module on Z" we mean a quasi-coherent sheaf on Z. By "a $\mathcal{D}$-module on Z" we mean a sheaf of *right* $\mathcal{D}_Z$-modules quasi-coherent as $\mathcal{O}_Z$-modules.

For a morphism $f : Z_1 \to Z_2$, the symbols $f^{\cdot}$ and $f_{\cdot}$ denote the inverse and direct image functors of $\mathcal{O}$-modules respectively.

Finally we will use the symbols $\mathcal{O} := k[[t]]$ for the algebra of formal series and $\mathcal{K} := k((t))$ for the field of formal Laurent series.

1. Factorization space

We follow [BD04, §3.10.16], [FB04, §20.4.1], [G99, Chap. 5] and [KV04].

1.1. **The category $\mathcal{S}$ of finite sets and surjections.** The following category $\mathcal{S}$ will be used repeatedly.

Definition 1.1. Let $\mathcal{S}$ be the category of non-empty finite sets and surjections. For $\pi : J \twoheadrightarrow I$ in $\mathcal{S}$ and $i \in I$ we set $J_i := \pi^{-1}(i) \subset J$.

Let X be a smooth complex curve as before. For $\pi : J \twoheadrightarrow I$ in $\mathcal{S}$, denote by

$$\Delta^{(\pi)} \equiv \Delta^{(J/I)} : X^I \hookrightarrow X^J$$

the natural embedding of X^I into X^J. The image of $\Delta^{(\pi)}$ consists of $(x_j)_{j \in J}$ such that $x_j = x_{j'}$ if $\pi(j) = \pi(j')$. Also set

$$U^{(\pi)} \equiv U^{(J/I)} := \left\{ (x_j)_{j \in J} \in X^J \mid x_j \neq x_{j'} \text{ if } \pi(j) \neq \pi(j') \right\}.$$

Denote the open embedding $U^{(\pi)} \subset X^J$ by

$$j^{(\pi)} \equiv j^{(J/I)} : U^{(\pi)} \hookrightarrow X^J.$$

1.2. **Recollection on ind-schemes.** Before introducing the factorization spaces, we need to recall the notion of ind-scheme since otherwise we have no good examples. We follow [KV04, §1.1] for the presentation.

For any category $\mathcal{C}$, an ind-object of $\mathcal{C}$ means a filtering inductive system over $\mathcal{C}$. Ind-objects form a category where a morphism is a collection of morphisms in $\mathcal{C}$ between the objects appearing in the inductive systems, satisfying some compatibility conditions. An ind-object will be represented by the symbol "$\varinjlim_i$"C_i with $C_i \in \mathcal{C}$.

Denote by $\mathcal{S}ch$ the category of separated schemes over $\mathbb{C}$. An *ind-scheme* is an ind-object of $\mathcal{S}ch$ represented by an inductive system of schemes. Formal schemes are ind-schemes, like

$$\mathrm{Spf}\, k[[t]] = \text{“}\varinjlim_n\text{”} \,\mathrm{Spec}\, k[t]/(t^{n+1}).$$

A *strict ind-scheme* is an ind-scheme whose associated inductive system given by closed embeddings of quasi-compact schemes.

Recall that a scheme S is equivalent to the functor of points $F_S : T \mapsto \mathrm{Hom}_{\mathcal{S}ch}(T, S)$. Considering $\mathcal{S}ch$ as the Zariski site, this functor is a sheaf of sets on the site $\mathcal{S}ch$. If one calls such a sheaf a $\mathbb{C}$-space, then an ind-scheme is a $\mathbb{C}$-space represented by an inductive system of schemes.

One can define an ind-scheme over a scheme Z similarly by replacing the category $\mathcal{S}ch$ by the category $\mathcal{S}ch_Z$ of schemes over Z. For a morphism $f : Z_1 \to Z_2$ of schemes, denote by f_* and f^* the push-forward and the pull-back of ind-schemes over Z_i's.

Also we have a notion of formal smoothness for ind-schemes as in case of the ordinary schemes. An ind-scheme X over a scheme Z is called formally smooth if any morphism $T \to X$ from an affine scheme T over Z lift to $T' \to X$, where T' is a first order thickenings of T.

1.3. **Definition of factorization space.** Now we turn to our main object, the factorization space. It is a non-linear analog of factorization algebra, and may be considered as a sheaf on the Ran space [R00] (see also [BD04, Chap. 4]). Recall once again the category $\mathcal{S}$ in Definition 1.1.

Definition. Let X be a scheme. A *factorization space* $\mathcal{G}$ on X consists of the following data.

- A formally smooth ind-scheme $\mathcal{G}_I$ over X^I for each $I \in \mathcal{S}$.
- An isomorphism
$$\nu^{(\pi)} \equiv \nu^{(J/I)} : \Delta^{(\pi)*}\mathcal{G}_J \xrightarrow{\sim} \mathcal{G}_I$$
of ind-schemes over X^I for each $\pi : J \twoheadrightarrow I$ in $\mathcal{S}$.
- An isomorphism
$$\kappa^{(\pi)} \equiv \kappa^{(J/I)} : j^{(\pi)*}\big(\prod_{i\in I} \mathcal{G}_{J_i}\big) \xrightarrow{\sim} j^{(\pi)*}\mathcal{G}_J$$
of ind-schemes over $U^{(\pi)}$ for each $\pi : J \twoheadrightarrow I$ in $\mathcal{S}$, called the *factorization isomorphism.*

These should satisfy the following compatibility conditions.

(1) $\nu^{(\pi)}$'s are compatible with compositions of surjections π.
(2) For any $\pi : J \twoheadrightarrow I$ and $\rho : K \twoheadrightarrow J$, we should have
$$\kappa^{(K/J)} = \kappa^{(K/I)}\big(\boxtimes_{i\in I}\kappa^{(K_i/J_i)}\big).$$
(3) For any $J \twoheadrightarrow I$ and $K \twoheadrightarrow J$, we should have
$$\nu^{(K/J)}\Delta^{(K/J)*}\big(\kappa^{(K/I)}\big) = \kappa^{(J/I)}\big(\boxtimes_{i\in I}\nu^{(K_i/J_i)}\big).$$

We can attach a factorization space $\mathcal{G}$ with the structure morphism $r^{(I)} : \mathcal{G}_I \to X^I$ of ind-schemes.
Let us also introduce the object corresponding to the unit of factorization algebra.

Definition. A *unit* of a factorization space $\mathcal{G}$ on a scheme X is the data of morphisms
$$u^{(I)} : X^I \longrightarrow \mathcal{G}_I$$
of ind-schemes for each $I \in \mathcal{S}$ such that for any morphism $f : U \to \mathcal{G}_{\{1\}}$ with open $U \subset X$ we have

- $u^{(\{1\})} \boxtimes f$, which can be seen as a morphism $U^2 \setminus \Delta \to \mathcal{G}_{\{1,2\}}$ by $\kappa^{(\pi:\{1,2\}\twoheadrightarrow\{1\})}$, extends to a morphism $U^2 \to \mathcal{G}_{\{1,2\}}$.
- $\Delta^{(\pi),*}(u^{(\{1\})} \boxtimes f) = f$.

Fact 1.2. A factorization space $\mathcal{G}$ on X together with a unit has a connection along X.

Let us explain precisely what a connection on $\mathcal{G}$ along X is. Assume that we are given

- an local Artin scheme I of length 1,
- a morphism $f : I \times S \to X$ of schemes with S a scheme,
- $g_0 : I_0 \times S \to \mathcal{G}$ of ind-schemes with $I_0 := I_{\mathrm{red}} \simeq \mathrm{Spec}(k)$ the reduced scheme of I

such that $r^{(\{1\})} \circ g_0 : I_0 \times S \to \mathcal{G} \to X$ coincides with the composition $I_0 \times S \to I \times S \to X$. A connection on $\mathcal{G}$ along X is equivalent to the property that for given (I, f, g_0) there is a map $g : I \times S \to \mathcal{G}$ such that $r \circ g = f$ extending g_0.

Definition. A factorization space with a unit will be called a *factorization monoid.*

We followed [KV04] for the terminology. In [BD04, §3.10.16] a factorization monoid is called a chiral monoid.
We also need a line bundle over factorization space.

Definition 1.3. Let $\mathcal{G}$ be a factorization space over X. A *factorization line bundle* $\mathcal{L}$ over $\mathcal{G}$ is a collection of line bundles $\mathcal{L}_I$ on $\mathcal{G}_I$ together with isomorphisms
$$j^{(J/I)\cdot}\mathcal{L}_I \xrightarrow{\sim} j^{(J/I)\cdot}\big(\otimes_{i\in I}\mathcal{L}_{J_i}\big)$$
over $U^{(J/I)}$ which should satisfy the factorization property.

Now we want to explain a standard example of factorization space, namely the Beilinson-Drinfeld Grassmannian $\mathcal{G}r(X, G)$. As a preparation we will recall some facts on the moduli space of G-bundles on an algebraic curve in the next subsection.

1.4. **Moduli space of G-bundles on curve.** Let G be a reductive algebraic group over $\mathbb{C}$. Recall that the *affine Grassmannian*

$$G(\mathcal{K})/G(\mathcal{O}) = G\big(k((z))\big)/G\big(k[[z]]\big)$$

can be considered as the moduli space of G-bundles on the disk $D := \operatorname{Spec}\mathcal{O}$ together with a trivialization on the punctured disk $D^\times := \operatorname{Spec}\mathcal{K}$. Strictly speaking, $G(\mathcal{K})$ is an ind-scheme, and $G(\mathcal{K})/G(\mathcal{O})$ is a formally smooth strict ind-scheme.

Denote by $\mathcal{M}(X,G)$ the category of G-torsors on X. By the result of [DS95], we have

Fact 1.4. Let X be a smooth algebraic curve and $x \in X$ be a point.

(1) A choice of local coordinate z at x gives an identification

$$\mathcal{G}r(X,G)_x := \{(\mathcal{P},\varphi) \mid \mathcal{P} \in \mathcal{M}(X,G),\ \varphi : \text{trivialization of } \mathcal{P}|_{X\setminus\{x\}}\} \xrightarrow{\sim} G(\mathcal{K})/G(\mathcal{O}). \tag{1.1}$$

(2) If G is semi-simple, then any G-bundle on $X \setminus \{x\}$ is trivial.

Let $\mathfrak{M}(X,G)$ be the moduli stack of G-torsors on a smooth projective curve X. We have a natural morphism $\mathcal{G}r(X,G)_x \to \mathfrak{M}(X,G)$ by forgetting the trivialization φ. If G is semisimple, then Fact 1.4 implies the following adelic description of $\mathfrak{M}(X,G)$.

$$\mathfrak{M}(X,G) \simeq G(\mathcal{K}_x)_{\text{out}}\backslash G(\mathcal{K}_x)/G(\mathcal{O}_x).$$

Here $G(\mathcal{K}_x)_{\text{out}}$ denotes the space of regular functions $X \setminus \{x\} \to G$, which is naturally a subgroup of $G(\mathcal{K}_x)$.

1.5. **Beilinson-Drinfeld Grassmannian.** Let X be a smooth algebraic curve as before. Denote by $\mathcal{S}ch$ the category of schemes over $\mathbb{C}$.

For $I \in \mathcal{S}$, consider the functor which maps $S \in \mathcal{S}ch$ to the data $(f^I, \mathcal{P}, \varphi)$ consisting of

- a morphism $f^I : S \to X^I$ of schemes
- a G-torsor $\mathcal{P}$ on $S \times X$
- a trivialization φ of $\mathcal{P}$ on $S \times X \setminus \big\{\Gamma(f_i^I)\big\}_{i\in I}$, where $f_i^I : S \to X$ is the composition of f^I with the i-th factor projection $X^I \to X$, and $\Gamma(s) \subset S \times X$ is the graph of $s : S \to X$.

By Fact 1.4, this functor can be represented by an ind-scheme $\mathcal{G}r(X,G)_I$. For $S = \operatorname{Spec} k$, the value $\mathcal{G}r(X,G)_I(S)$ of this functor is identified with

$$\Big\{\big(\mathcal{P}, (x_i)_{i\in I}, \varphi\big) \,\Big|\, \mathcal{P} \in \mathcal{M}_G(X),\ x_i \in X,\ \varphi : \text{trivialization of } \mathcal{P}|_{X\setminus\{x_i\}_{i\in I}}\Big\}.$$

We have a natural morphism

$$r^{(I)} : \mathcal{G}r(X,G)_I \longrightarrow X^I,$$

and the space $\mathcal{G}r(X,G)_x$ in the isomorphism (1.1) is the fiber of $r^{(I)}$ with $I = \{1\}$.

Fact 1.5. The collection

$$\mathcal{G}r(X,G) := \{\mathcal{G}r(X,G)_I\}_{I\in\mathcal{S}}$$

has a structure of factorization monoid on X, called the *Beilinson-Drinfeld Grassmannian.*

Here we explain the factorization isomorphism only for $|I| \le 2$. Denote by $\mathcal{G}r_{(x_i)_{i\in I}}$ the fiber of $r^{(I)}$ over $(x_i)_{i\in I} \in X^I$. By Fact 1.4, in the case $|I| = 1$, we have

$$\mathcal{G}r_{(x)} \simeq G(\mathcal{K})/G(\mathcal{O}).$$

If $x_1 \neq x_2$, then $(x_1,x_2) \in X^2 \setminus \Delta$, and we have a morphism $\mathcal{G}r_{(x_1,x_2)} \to \mathcal{G}r_{(x_1)} \times \mathcal{G}r_{(x_2)}$ by restricting the data to $X \setminus \{x_i\}$. We also have the other direction map $\mathcal{G}r_{(x_1)} \times \mathcal{G}r_{(x_2)} \to \mathcal{G}r_{(x_1,x_2)}$ by gluing the G-bundles over $X \setminus \{x_1,x_2\}$. If $x_1 = x_2$, then $(x_1,x_2) \in \Delta$, and we have an identification $\mathcal{G}r_{(x_1,x_2)} \xrightarrow{\sim} \mathcal{G}r_{(x_1)}$.

The unit on $\mathcal{G}r(X,G)$ is given by the trivial G-bundles. Namely, define $u^{(I)} : X^I \to \mathcal{G}r(X,G)_I$ by setting $u^{(I)}\big((x_i)_{i\in I}\big)$ to be the trivial G-bundle with the obvious trivialization away from x_i's.

$\mathcal{G}r(X,G)$ has a natural factorization line bundle. Assume the Lie algebra $\operatorname{Lie}(G)$ is equipped with an invariant inner product. It induces a central extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \widehat{G} \longrightarrow G(\mathcal{K}) \longrightarrow 1$$

of algebraic group ind-schemes, and hence a $\mathbb{G}_m$-torsor

$$\widehat{G}(\mathcal{K})/G(\mathcal{O}) \longrightarrow G(\mathcal{K})/G(\mathcal{O})$$

over the affine Grassmannian. It defines a factorization line bundle $\mathcal{L}(G)$ over $\mathcal{G}r(X,G)$.

2. Linearization

This section gives a recollection of linearization procedure established in [BD04, Chap. 3], following the description in [BD04] and [FB04, Chap. 20].

2.1. **Recollection on $\mathcal{D}$-modules.** Let us collect some notations on $\mathcal{D}$-modules.

Let Z be a smooth algebraic variety over $\mathbb{C}$. Recall that the canonical sheaf

$$\omega_Z := \wedge^{\dim Z}\Omega_Z$$

has a natural right $\mathcal{D}$-module structure determined by

$$\nu \cdot \tau := -\mathcal{L}ie_\tau(\nu), \quad \nu \in \omega_Z, \ \tau \in \Theta_Z \subset \mathcal{D}_Z$$

where $\mathcal{L}ie_\tau$ is the Lie derivative with respect to τ. Then the categories of left and right $\mathcal{D}$-modules are equivalent under the functor

$$\mathcal{L} \longmapsto \mathcal{L}^r := \omega_Z \otimes_{\mathcal{O}_Z} \mathcal{L},$$

where the structure of right $\mathcal{D}$-module on $\mathcal{L}^r$ is determined by

$$(\nu \otimes l) \cdot \tau := (\nu \cdot \tau) \otimes l - \nu \otimes (\tau \cdot l), \quad l \in \mathcal{L}.$$

The inverse functor is given by

$$\mathcal{M} \longmapsto \mathcal{M}^\ell := \omega_Z^{-1} \otimes_{\mathcal{O}_Z} \mathcal{M}. \tag{2.1}$$

Hereafter the term a "$\mathcal{D}$-module" means a right $\mathcal{D}$-module.

2.2. **Linearization of factorization spaces.** Let $\mathcal{G}$ be a factorization monoid over a smooth variety X. Recall the morphisms $r^{(I)} : \mathcal{G}_I \to X^I$ and $u^{(I)} : X^I \to \mathcal{G}_I$. Now consider the $\mathcal{O}$-module

$$\mathcal{A}_{\mathcal{G},I} := r_*^{(I)} u_!^{(I)} \omega_{X^I}.$$

Here $r_*^{(I)}$ is the direct image functor of $\mathcal{O}$-modules by the morphism $r^{(I)}$, and $u_!^{(I)}$ is the direct image functor of $\mathcal{D}$-modules by $u^{(I)}$. Actually one must be careful of $u_!^{(I)}$ since it involves $\mathcal{D}$-modules on ind-schemes. See [KV04, §3] for the detail. This sheaf $\mathcal{A}_{\mathcal{G},I}$ can be considered as the space of delta functions on $\mathcal{G}_I$ along the section $u^{(I)}(X^I)$.

The connection on $\mathcal{G}$ along X given in Fact 1.2 defines a right $\mathcal{D}$-module structure on $\mathcal{A}_{\mathcal{G},I}$ and the section $u^{(I)}$ defines an embedding $\omega_{X^I} \hookrightarrow \mathcal{A}_{\mathcal{G},I}$. Then the axiom of factorization space implies

Fact 2.1. If X is a smooth curve, then the collection $\{\mathcal{A}_{\mathcal{G},I}^\ell\}_{I \in \mathcal{S}}$ has a structure of factorization algebra on X, and hence $\mathcal{A}_{\mathcal{G},\{1\}}$ has a structure of chiral algebra on X.

Here we used the notation (2.1). We also omit the definition of chiral and factorization algebras. See [BD04, §§3.1–3.4] or [FB04, Chap. 19,20]. We call the obtained chiral algebra $\mathcal{A}_{\mathcal{G},\{1\}}$ the *chiral algebra associated to* $\mathcal{G}$.

Let us explain the twisted version of this construction. Assume that a factorization space $\mathcal{G}$ over a curve X is equipped with a factorization line bundle $\mathcal{L} = \{\mathcal{L}_I\}$. Then consider the collection of $\mathcal{D}$-modules

$$\mathcal{A}_{\mathcal{G},I}^{\mathcal{L}} := r_*^{(I)}\big(\mathcal{L}_I \otimes_{\mathcal{G}_I} u_!^{(I)}(\omega_{X^I})\big).$$

The construction in Fact 2.1 can be applied to this family of $\mathcal{L}_I$-twisted sheaves. The result is

Fact 2.2. Assume that $\mathcal{G}$ is a factorization monoid over a smooth curve X, and that $\mathcal{G}$ is equipped with a factorization linear bundle $\mathcal{L}$. Then $\mathcal{A}_{\mathcal{G},\{1\}}^{\mathcal{L}}$ has a structure of chiral algebra on X.

We call the resulting chiral algebra the $\mathcal{L}$-*twisted chiral algebra associated to* $\mathcal{G}$.

3. Lattice chiral algebra

Now we recall the factorization monoid arising from the Picard functor whose associated chiral algebra is the lattice chiral algebra [BD04, §3.10], [G99, Chap. 6], [FB04, §§20.4.4–9].

Following the symbols in [FB04, Chap. 5], let $V_{\mathbb{Z}} = \oplus_{n\in\mathbb{Z}}\pi_n$ be the lattice vertex algebra associated to the lattice $\mathbb{Z}$. Each component π_n is the Fock representation of the Heisenberg algebra generated by $\{a_n \mid n \in \mathbb{Z}\}$ and 1 with the defining relation $[a_m, a_n] = m\delta_{m+n,0}$. Recall that $V_{\mathbb{Z}}$ is actually a vertex *super* algebra,

There is a universal chiral algebra $\mathcal{A}_{\mathbb{Z}}$ in the meaning of [BD04, §3.3.14] corresponding to the lattice vertex algebra $V_{\mathbb{Z}}$. We call it the *lattice chiral algebra*.

3.1. **Some super language.** Here we collect notations for super objects which will be used repeatedly.

For a commutative ring R, a super R-module means a $\mathbb{Z}/2\mathbb{Z}$-graded R-module. Denote by p the parity of homogeneous elements of M. Consider the tensor product of super R-modules with the commutativity constraint $a \otimes b = (-1)^{p(a)p(b)} b \otimes a$. A super R-line is an invertible object of the monoidal category formed by super R-modules with the tensor product described as above. Super R-lines give rise to a Picard groupoid.

One can consider a similar setting for any $\mathbb{C}$-linear category $\mathcal{M}$. Namely $\mathbb{Z}/2\mathbb{Z}$-graded objects form a $\mathbb{C}$-category $\mathcal{M}^s$, and if $\mathcal{M}$ is a monoidal category then $\mathcal{M}^s$ is also a super monoidal category with Koszul rule of sings as described above.

Below we will use super line bundles on a scheme (or stack) X. It is nothing but a super object in the monoidal category $\mathrm{Pic}(X)$ of line bundles on X. In particular, replacing line bundles by super line bundles in Definition 1.3, we have the notion of *factorization super line bundle* on a factorization space.

3.2. **Abelian Beilinson-Drinfeld Grassmannian and lattice chiral algebras.** Consider the Beilinson-Drinfeld Grassmannian $\mathcal{G}r(X, G)$ with $G = \mathrm{GL}(1)$. Since GL(1)-torsor is nothing but a line bundle, we have $\mathfrak{M}(X, \mathrm{GL}(1)) = \mathrm{Pic}(X)$. For $I \in \mathcal{S}$, we also have

$$\mathcal{G}r(X, \mathrm{GL}(1))_I = \left\{(\mathcal{L}, S, \varphi) \mid \mathcal{L} \in \mathrm{Pic}(X),\ S \in X^I,\ \varphi : \text{trivialization of } \mathcal{L}|_{X\setminus S}\right\}.$$

One can naturally construct a factorization *super* line bundle on $\mathcal{G}r(X, \mathrm{GL}(1))$ by the theta line bundle Θ on $\mathrm{Pic}(X)$. Choosing a theta characteristic $\Omega_X^{1/2}$, we can describe the fiber of Θ on $\mathcal{L} \in \mathrm{Pic}(X)$ as

$$\Theta|_{\mathcal{L}} \simeq \det H^*(\mathcal{L} \otimes_{\mathcal{O}_X} \Omega_X^{1/2}). \tag{3.1}$$

Considering Θ as an odd line bundle on $\mathrm{Pic}(X)$, we have a factorization super line bundle on $\mathcal{G}r(X, \mathrm{GL}(1))$ induced by Θ. Denote the resulting factorization super line bundle by the same letter $\Theta = \{\Theta_I\}$.

Recall that the chiral algebra associated to a factorization monoid is constructed along the unit $u^{(\{1\})} : X \to \mathcal{G}_{\{1\}}$. In the present situation $\mathcal{G} = \mathcal{G}r(X, \mathrm{GL}(1))$, the unit is given by $x \mapsto (\mathcal{O}_X, \varphi)$ with φ the obvious trivialization away from x. Now we modify this construction and consider

$$u(n)_x : X \longrightarrow \mathcal{G}r(X, \mathrm{GL}(1))_x, \quad x \longmapsto (\mathcal{O}_X(nx), \varphi)$$

for $n \in \mathbb{Z}$. It induces $u(n)^{(\{1\})} : X \longrightarrow \mathcal{G}r(X, \mathrm{GL}(1))_{\{1\}}$ and $u(n)^{(I)} : X \longrightarrow \mathcal{G}r(X, \mathrm{GL}(1))_I$ for any $I \in \mathcal{S}$. Now define the $\mathcal{O}$-module

$$\mathcal{A}_{\mathbb{Z},I} := \bigoplus_{n\in\mathbb{Z}} r_*^{(I)} \left(\Theta_I \otimes u(n)_!^{(I)}(\omega_{X^I})\right).$$

The factorization structure makes $\mathcal{A}_{\mathbb{Z},\{1\}}$ into a chiral algebra. Denote it by $\mathcal{A}_{\mathbb{Z}}$.

Fact 3.1. The chiral algebra $\mathcal{A}_{\mathbb{Z}}$ is isomorphic to the lattice chiral algebra.

For later use let us restate this result. Set

$$\mathcal{G}(X, \mathbb{Z})_I := \{(\mathcal{L} \otimes \mathcal{O}_X(*S), S, \varphi) \mid \mathcal{L} \in \mathrm{Pic}(X), S \in X^I, \varphi : \text{trivialization of } \mathcal{L}|_{X\setminus S}\}$$

Here $\mathcal{O}_X(*S)$ means $\mathcal{O}_X(nS)$ for some $n \in \mathbb{Z}$. Then $\mathcal{G}(X, \mathbb{Z})$ is a factorization monoid with the unit $u : X \to \mathcal{G}(X, \mathbb{Z})_I$ given by the trivial line bundle. Θ is a factorization super line bundle on $\mathcal{G}(X, \mathbb{Z})$. Now $\mathcal{A}_{\mathbb{Z}}$ is isomorphic to the Θ-twisted chiral algebra associated to $\mathcal{G}(X, \mathbb{Z})$. Thus Fact 3.1 can be restated as

Proposition 3.2. The Θ-twisted chiral algebra associated to $\mathcal{G}(X, \mathbb{Z})$ is isomorphic to the lattice chiral algebra.

Remark. The Θ-twisted chiral algebra associated to $\mathcal{G}r(X, \mathrm{GL}(1))$ is nothing but the Heisenberg chiral algebra π_0^{ch}, and the corresponding vertex algebra π_0 is the 0-th part of the lattice vertex algebra $V_{\mathbb{Z}} = \oplus_{n\in\mathbb{Z}}\pi_n$.

4. Clifford chiral algebras

4.1. **Clifford bundles.** Here we collect some notations on the Clifford algebra. See [Bo, §IX.9] for the detail.

Let us denote by $\mathcal{C}l(Q)$ the *Clifford algebra* associated to a non-degenerate quadratic form Q on a linear space V of finite dimension. The definition is

$$\mathcal{C}l(Q) := T(V)/\langle v \otimes v - Q(v)1\rangle.$$

Denote by $\mathcal{C}l^+(Q)$ the even part and by $\mathcal{C}l^-(Q)$ the odd part of $\mathcal{C}l(Q)$. The involution $u \mapsto \pm u$ for $u \in \mathcal{C}l^{\pm}(Q)$ is denoted by α. The *Clifford group* $\mathrm{C}(Q)$ and the *special Clifford group* $\mathrm{SC}(Q)$ are defined to be

$$\mathrm{C}(Q) := \{u \in \mathcal{C}l^{\times}(Q) \mid \alpha(u)Vu^{-1} \subset V\}, \quad \mathrm{SC}(Q) := \mathrm{C}(Q) \cap \mathcal{C}l^+(Q),$$

where $\mathcal{C}l^{\times}(Q)$ is the group of invertibles in $\mathcal{C}l(Q)$. $\mathrm{SC}(Q)$ is a connected reductive group. The *spinor norm* is denoted by $N : \mathrm{SC}(Q) \to \mathbb{C}^{\times}$.

Using the notations in §1.4, we have a moduli stack $\mathfrak{M}(\mathrm{SC}(Q), X)$ of $\mathrm{SC}(Q)$-torsors on a smooth algebraic curve X. Applying Fact 1.5 to $G = \mathrm{SC}(Q)$. we have a factorization monoid $\mathcal{G}r\,(X, \mathrm{SC}(Q))$. The spinor norm N on $\mathrm{SC}(Q)$ induces a factorization super line bundle $\mathcal{L}(\mathrm{SC}(Q))$ by a similar argument as in the latter part of §1.5.

Finally let us set $\mathrm{SC}(n) := \mathrm{SC}(Q)$ with Q the standard quadratic form $z_1^2 + \cdots + z_n^2$ on $V = \mathbb{C}^n$. The involution α on $\mathbb{C}^n$ is nothing but the reflection $v \mapsto -v$ with respect to the origin.

4.2. **Clifford chiral algebras.** Let us introduce the Clifford chiral algebra following [BD04, §3.8.6] with minor modifications.

Let $\mathcal{V}, \mathcal{V}'$ be vector $\mathcal{D}_X$-bundles (locally projective right $\mathcal{D}_X$-module of finite rank) and assume that we have a skew-symmetric pairing $\langle\, , \rangle \in P_2^*(\{\mathcal{V}, \mathcal{V}'\}, \omega_X)$ in the $*$-pseudo-tensor category [BD04, Chap. 2] such that it vanishes on $\mathcal{V}$ and $\mathcal{V}'$ respectively. Then $\mathcal{V} \oplus \mathcal{V}' \oplus \omega_X$ is a Lie* algebra with the commutator $\langle\, , \rangle$. The resulting chiral algebra is denoted by $\mathcal{W}^{\mathrm{ch}}(\mathcal{V}, \mathcal{V}', \langle\, , \rangle)$.

For a positive integer n, let $\mathcal{V}_o := \mathcal{O}_X^{\oplus n}$ considered as $\mathcal{O}$-module, and $\mathcal{V} := \mathcal{V}_o \otimes_{\mathcal{O}_X} \mathcal{D}_X$ considered as $\mathcal{D}$-module. Set $\mathcal{V}^{\circ} := \mathcal{H}om^*(\mathcal{V}, \mathcal{O}_X)$ where $*$ denotes the linear dual. As in [BD04, §2.2.16], the linear pairing induces a non-degenerate pairing $\langle\, , \rangle \in P_2^*(\{\mathcal{V}[1], \mathcal{V}^{\circ}[-1]\}, \omega_X)$ with $[\pm 1]$ the shift as complexes. Now we set

$$\mathcal{C}l^{\mathrm{ch}}(n) := \mathcal{W}^{\mathrm{ch}}(\mathcal{V}[1], \mathcal{V}^{\circ}[-1], \langle\, , \rangle)$$

and call it the *fermionic chiral algebra*. Thus, strictly speaking, it is a chiral *super* algebra and generators behave as odd elements. It is a universal chiral algebra associated to the free fermionic vertex algebra $\bigwedge(n)$ of rank n.

4.3. **Relation to Beilinson-Drinfeld Grassmannian.** Recall that we set $\mathrm{SC}(n) := \mathrm{SC}(Q)$ with $Q = z_1^2 + \cdots + z_n^2$ on $V = \mathbb{C}^n$. Then we have the factorization monoid $\mathcal{G}r(X, \mathrm{SC}(n))$ with the factorization super line bundle $\mathcal{L}(\mathrm{SC}(n))$. Let us denote by $\mathcal{A}(\mathrm{SC}(n))$ the $\mathcal{L}(\mathrm{SC}(n))$-twisted chiral algebra associated to $\mathcal{G}r(X, \mathrm{SC}(n))$.

Now we have

Proposition 4.1. $\mathcal{A}(\mathrm{SC}(n))$ is isomorphic to $\mathcal{C}l^{\mathrm{ch}}(n)$ for any $n \in \mathbb{Z}_{\geq 1}$.

The proof is a variant of the argument in [FB04, Chap. 20]. It says that for a semi-simple G, the $\mathcal{L}(G)^{\otimes k}$-twisted chiral algebra associated to the Beilinson-Drinfeld Grassmannian $\mathcal{G}r(X, G)$ is the affine chiral algebra $\mathcal{V}_k^{\mathrm{ch}}(\mathfrak{g})$ of level $\mathbb{C}$. Here $\mathcal{V}_k^{\mathrm{ch}}(\mathfrak{g})$ is the chiral algebra counterpart of the affine vertex algebra $V_k(\mathfrak{g})$ of level k with $\mathfrak{g} := \mathrm{Lie}(G)$, and $\mathcal{L}(G)$ is the factorization line bundle induced by the normalized invariant inner product (see the latter part of §1.5).

Proof. We only discuss the case $n = 1$. The general case is quite similar. Denote by $\bigwedge = \bigwedge(1)$ the free fermionic vertex super algebra of rank 1, following the symbol in [FB04, §5.3]. Thus as a linear space $\bigwedge$ is the fermionic Fock space of the infinite-dimensional Clifford algebra $\mathcal{C}l_\infty$. The algebra $\mathcal{C}l_\infty$ is generated by $\{\psi_n, \psi_n^* \mid n \in \mathbb{Z}\}$ with the relation

$$[\psi_m, \psi_n]_+ = [\psi_m^*, \psi_n^*]_+ = 0, \quad [\psi_m, \psi_n^*]_+ = \delta_{m+n,0}.$$

The $\mathcal{C}l_\infty$-module $\bigwedge$ is generated by the element $|0\rangle$ such that

$$\psi_n |0\rangle = 0 \ (n \geq 0), \quad \psi_n^* |0\rangle = 0 \ (n > 0).$$

Note that for $Q = z^2$ on $V = \mathbb{C}$ we have $\mathcal{C}l(Q) \simeq \mathbb{C} \oplus \mathbb{C}$. By the description of linearization procedure, we have an identification $\mathcal{A}(\mathrm{SC}(1))_x \simeq \mathcal{C}l_\infty \otimes_{\mathcal{C}l_\infty^+} \mathbb{C}$ for any $x \in X$. Here $\mathbb{C}$ is the one-dimensional representation of $\mathcal{C}l_\infty^+$ corresponding to the fiber of $\mathcal{L}(\mathrm{SC}(1))_{\{1\}}$. Thus $\mathcal{A}(\mathrm{SC}(1))_x$ is identified with the free fermion vertex algebra $\bigwedge$ as linear space.

The factorization structure induces a connection on $\mathcal{A}^\ell(\mathrm{SC}(1))$ giving a left $\mathcal{D}$-module structure. Denote by $p(z)z^{1/2}dz$ and $p^*(z)z^{-1/2}dz$ the generating local sections of $\mathcal{A}^\ell(\mathrm{SC}(1))$. Here $z^{\pm 1/2}$ corresponds to the theta characteristic appearing in (3.1). Then we can calculate the commutation relation of $p(z)$ and $p^*(z)$ as in the affine chiral algebra case, we have $(z-w)[p(z)z^{1/2}, p^*(w)w^{-1/2}]_+ = 0$, which is nothing but the Clifford chiral super algebra relation $(z-w)[\psi(z)z^{1/2}, \psi^*(w)w^{-1/2}]_+ = 0$ with $\psi(z) := \sum_n \psi_n z^{-n}$ and $\psi^*(z) := \sum_n \psi_n^* z^{-n}$. $\square$

Remark. Let us mention a few topics related to our fermionic factorization spaces. These are suggested by anonymous referees, and we appreciate their comments.

(1) Recall that the even part of the free fermion vertex algebra $\bigwedge$ is isomorphic to the Virasoro vertex algebra Vir_c of central charge $c = 1/2$ (see [FRW] for example). It is natural to ask what the corresponding factorization space is. At present we have no good answer, but let us comment that for $c = -2(6\mu^2 - 6\mu + 1)$ with $\mu \in \mathbb{Z}$ one can construct a factorization space using decorated curves [FB04, §17.3, §20.4.13] which gives rise to Vir_c by linearization.

(2) We only discussed charged fermions $\{\psi_n, \psi_n^*\}_{n \in \mathbb{Z}}$ since we are motivated by the standard boson-fermion correspondence (0.1). For the non-charged fermions, namely the vertex super algebra generated by $\{\psi_n\}_{n \in \mathbb{Z}}$, it is enough to consider the Beilinson-Drinfeld Grassmannian of the group $\mathrm{Spin}^{\mathbb{C}}$. The linearization procedure then gives the vertex algebra attached to the Neveu-Schwarz free fermions.

(3) Another interesting question is how to construct a factorization space whose linearization gives Ramond fermions. This can be done by replacing the theta (super) line bundle Θ (see (3.1)) by the untwisted determinant line bundle $\det H^*(\mathcal{L})$. In other words, forget the theta characteristic $\Omega_X^{1/2}$.

5. Boson-Fermion correspondence via factorization spaces

Theorem 5.1. We have an isomorphism of factorization monoids

$$\mathcal{G}(X, \mathbb{Z}) \simeq \mathcal{G}r(X, \mathrm{SC}(1)).$$

Under this isomorphism we have an identification of factorization super line bundles

$$\Theta \simeq \mathcal{L}(\mathrm{SC}(1)).$$

Then Propositions 3.2 and 4.1 yield $\mathcal{A}_{\mathbb{Z}} \simeq \mathcal{C}l^{\mathrm{ch}}(1)$ as chiral algebras. In terms of vertex algebra, it implies $V_{\mathbb{Z}} \simeq \bigwedge$, which is nothing but the boson-fermion correspondence (0.1).

Proof of Theorem 5.1. Recall the notations of G-bundles in §1.4. Since we have $\mathrm{SC}(1) \simeq \mathrm{GL}(1)^{\times 2}$, it holds that

$$\mathfrak{M}(\mathrm{SC}(1), X) \simeq \mathfrak{M}(\mathrm{GL}(1), X)^{\times 2} \simeq \mathrm{Pic}(X)^{\times 2}.$$

Then by Fact 1.4 on the trivialization we have

$$\mathcal{G}r(X, \mathrm{SC}(1))_x \simeq \big(\mathcal{G}r(X, \mathrm{GL}(1))_x\big)^{\times 2} \simeq \mathcal{G}(X, \mathbb{Z})_x$$

for each $x \in X$. It yields the isomorphism $\mathcal{G}r(X, \mathrm{SC}(1))_I \simeq \mathcal{G}(X, \mathbb{Z})_I$ of ind-schemes for any $I \in \mathcal{S}$.

The factorization structures on $\mathcal{G}r(X, \mathrm{SC}(1))$ and $\mathcal{G}(X, \mathbb{Z})$ are the same essentially because they are both Beilinson-Drinfeld Grassmannians. The units coincide because they are given by trivial bundles. Thus we have $\mathcal{G}(X, \mathbb{Z}) \simeq \mathcal{G}r(X, \mathrm{SC}(1))$.

Recall that the factorization super line bundle $\mathcal{L}(\mathrm{SC}(1))$ comes from the spinor norm N on $\mathrm{SC}(1)$. Under the isomorphism $\mathrm{SC}(1) \simeq \mathrm{GL}(1)^{\times 2}$, N is given by $N(x, y) = xy$, and it can be identified with the linear pairing. Now the isomorphism $\Theta \simeq \mathcal{L}(\mathrm{SC}(1))$ is an obvious one. □

Remark. For $n \geq 2$ we have $\mathcal{G}(X, \mathbb{Z}^n) \not\simeq \mathcal{G}r(X, \mathrm{SC}(n))$ since $\mathrm{SC}(n) \not\simeq \mathrm{GL}(1)^{\times 2n}$.

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