

ISOTOPES OF HURWITZ ALGEBRAS

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ABSTRACT. We study the class of all algebras that are isotopic to a Hurwitz algebra. In the general case, isomorphism classes of such algebras are shown to correspond to orbits of a certain group action. Our main focus is the isotopes of Hamilton's quaternions, for which a more detailed, and geometrically intuitive description of the isomorphism classes is given. As an application, we also demonstrate how some results concerning the classification of finite-dimensional composition algebras can be deduced from our general results.

1. INTRODUCTION

Let k be a field, and V a vector space over k . We shall say that a quadratic form $q : V \rightarrow k$ is *non-degenerate* if the associated bilinear form $\langle x, y \rangle = q(x+y) - q(x) - q(y)$ is non-degenerate (i.e., $\langle x, V \rangle = 0$ only if $x = 0$). A *composition algebra* is a non-zero (not necessarily associative) algebra A over a field k , equipped with a non-degenerate quadratic form $n : A \rightarrow k$ such that $n(ab) = n(a)n(b)$ for all $a, b \in A$. The form n is usually called the *norm* of A . A composition algebra possessing an identity element is called a *Hurwitz algebra*. Every Hurwitz algebra has dimension one, two, four or eight (thus, in particular, it is finite dimensional), and can be constructed via an iterative method known as the *Cayley-Dickson process*.¹ The facts about Hurwitz algebras referred in this section are described in detail in [15], Chapter VIII.

Two algebras A and B over a field k are said to be *isotopic* if there exist invertible linear maps $\alpha, \beta, \gamma : A \rightarrow B$ such that $\gamma(xy) = \alpha(x)\beta(y)$ for all $x, y \in A$. Clearly, isotopy is an equivalence relation amongst k -algebras. If A and B are isotopic then there exist $\alpha, \beta \in \text{GL}(A)$ such that the algebra (A, \circ) , with multiplication $x \circ y = \alpha(x)\beta(y)$, is isomorphic to B . The algebra (A, \circ) is called the *principal isotope* of A determined by α and β , and is denoted by $A_{\alpha, \beta}$.

Several important classes of non-associative algebras can be constructed by isotopy from the Hurwitz algebras. Examples include:

- (1) All finite-dimensional composition algebras [14, p. 957]. This includes in particular all finite-dimensional absolute valued algebras, which are precisely the finite-dimensional composition algebras over \mathbb{R} whose norm is anisotropic (i.e., $n(x) = 0$ only if $x = 0$). However, there exist composition algebras without identity element that are infinite dimensional, and thus not isotopic to any Hurwitz algebra; see e.g. [4, 8].
- (2) All division algebras of dimension two over a field of characteristic different from two [17].
- (3) All finite-dimensional division algebras A that are not isotopic to an associative algebra, and satisfy the following property: for all non-zero $a \in A$ there exists an element $b \in A$ such that $b(ax) = x$ for all $x \in A$. [6]

The purpose of the present article is to describe, in a uniform way, all isotopes of Hurwitz algebras. Generalising ideas that have earlier been used in more specialised situations (for example in [3, 17]), we give a general description of all algebras isotopic to a Hurwitz algebra, encompassing also the case of the ground field k having characteristic two. Our principal result is a comprehensive geometric description of all isotopes of Hamilton's quaternion algebra \mathbb{H} – a class of real division algebras that has not been studied before.

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¹Some authors have used a weaker notion of non-degeneracy for the norm n , requiring that $n(x+y) = n(y)$ for all y implies $x = 0$. This definition gives rise to additional unital composition algebras over fields k of characteristic two, in form of purely inseparable field extensions of k [14]. If $\text{char } k \neq 2$, the two definitions are equivalent.

Isotopes of Hurwitz algebras of dimension two is a special case of Petersson's [17] description of all 2-dimensional algebras, which is also based on the concept of isotopy. There is a large number of articles on finite-dimensional composition algebras and absolute valued algebras (some examples are [2, 3, 8, 14, 18]), most of which use the concept of isotopy as a central tool.

This article is organised as follows. In Section 2, a general description is given of the category of isotopes of a Hurwitz algebra A . A more elaborate study of the case where (A, n) is a Euclidean space is given in Section 3, bringing about the announced description of isotopes of \mathbb{H} . Finally, Section 4 treats composition algebras, showing how a description of these can be deduced from the results in Section 2.

From here on, let k denote a field. All algebras are, unless otherwise stated, assumed to be finite dimensional over k . Every element a of an algebra A determines linear endomorphisms L_a and R_a of A , defined by $L_a(x) = ax$ and $R_a(x) = xa$ respectively. For a principal isotope $A_{\alpha, \beta} = (A, \circ)$ of A , we use the notation $L_a^\circ(x) = a \circ x$ and $R_a^\circ(x) = x \circ a$. An algebra A is said to be a *division algebra* if $A \neq 0$ and L_a and R_a are bijective for all non-zero $a \in A$. Moreover, A is *alternative* if the identities $x^2y = x(xy)$ and $xy^2 = (xy)y$ hold for all $x, y \in A$.

All Hurwitz algebras are alternative. Two-dimensional Hurwitz algebras are quadratic étale algebras, i.e., either separable field extensions of k or isomorphic to $k \times k$. The Hurwitz algebras of dimension four are all *quaternion algebras*, that is, all four-dimensional central simple associative algebras. Eight-dimensional Hurwitz algebras are precisely the central simple alternative algebras that are not associative [21] (these are called *octonion algebras*). A Hurwitz algebra A is commutative if and only if $\dim A \leq 2$, and associative if and only if $\dim A \leq 4$.

Any element x in a Hurwitz algebra $A = (A, n)$ satisfies $x^2 = \langle x, 1 \rangle x - n(x)1$. Hence, the norm n is uniquely determined by the algebra structure of A , and every algebra morphism of Hurwitz algebras that respects the identity element also preserves the norm. Every Hurwitz algebra of dimension at least 2 has a unique non-trivial involution² $\kappa : A \rightarrow A$, $x \mapsto \bar{x}$ satisfying $x + \bar{x} \in k1$ and $x\bar{x} = \bar{x}x = n(x)1$ for all $x \in A$. Moreover, two Hurwitz algebras are isomorphic if and only if their respective norms are equivalent (this was first proved in [12] in characteristic different from two). The quadratic forms occurring as norms of Hurwitz algebras are precisely the m -fold Pfister forms over k , for $m \in \{0, 1, 2, 3\}$. If A is a Hurwitz algebra and $a \in A$, then L_a and R_a are invertible if and only if $n(a) \neq 0$. This is also equivalent to the existence of an inverse a^{-1} of a in A : since $a\bar{a} = \bar{a}a = n(a)1$, we have $a^{-1} = n(a)^{-1}\bar{a}$ if $n(a) \neq 0$. Moreover, $L_a^{-1} = L_{a^{-1}}$ and $R_a^{-1} = R_{a^{-1}}$ in this case. The invertible elements of any alternative algebra A form a so-called *Moufang loop* under multiplication (the concept was introduced by Moufang in [16] under the name *quasi-group*), denoted by A^* .

For any algebra A , the *nucleus* is defined as $N(A) = \{a \in A \mid (xy)z = x(yz) \text{ for } a \in \{x, y, z\}\}$. The nucleus is an associative subalgebra of A . If A is a Hurwitz algebra, then $a \in N(A)$ if and only if $(xa)y = x(ay)$ for all $x, y \in A$. If $\dim A \leq 4$ then A is associative and thus $N(A) = A$; the nucleus of an eight-dimensional Hurwitz algebra is $k1$.

A *similitude* of a non-zero quadratic space $V = (V, q)$ is an invertible linear map $\varphi : V \rightarrow V$ such that $q(\varphi(x)) = \mu(\varphi)q(x)$ for all $x \in V$, where $\mu(\varphi) \in k$ is a scalar independent of x . The element $\mu(\varphi)$ is called the *multiplier* of φ . If $l = \dim V$ is even, then $\det(\varphi) = \pm\mu(\varphi)^{l/2}$ [15, 12A]. If $\text{char } k \neq 2$, a similitude φ satisfying $\det(\varphi) = \mu(\varphi)^{l/2}$ are said to be *proper*. In the characteristic two case, a similitude $\varphi : V \rightarrow V$ is defined to be proper if its Dickson invariant (see [15, 12.12]) is zero. The group of all similitudes of V is denoted by $\text{GO}(V, q)$, or $\text{GO}(V)$ for short. The proper similitudes form a normal subgroup $\text{GO}^+(V) \subset \text{GO}(V)$ of index two. Elements in $\text{GO}(V) \setminus \text{GO}^+(V)$ are called *improper* similitudes. The map $\mu : \text{GO}(V) \rightarrow k^*$, $\varphi \mapsto \mu(\varphi)$ is a group homomorphism, the kernel of which is the orthogonal group $\text{O}(V)$. We write $\text{O}^+(V) = \text{O}(V) \cap \text{GO}^+(V)$, or $\text{SO}(V) = \text{O}^+(V)$ in case $k = \mathbb{R}$ and (V, q) is a Euclidean space.

For any Hurwitz algebra $A = (A, n)$, define $\text{G}(A)$ to be the set of all $\varphi \in \text{GO}(A)$ for which there exist $\varphi_1, \varphi_2 \in \text{GO}(A)$ such that $\varphi(xy) = \varphi_1(x)\varphi_2(x)$ for all $x, y \in A$. Now, by what is known as

²By an involution is meant a self-inverse isomorphism $A \rightarrow A^{\text{op}}$.

the principle of *triality* for Hurwitz algebras [19, 3.2],

$$G(A) = \begin{cases} \text{GO}^+(A) & \text{if } \dim A \geq 4, \\ \text{GO}(A) & \text{if } \dim A \leq 2. \end{cases}$$

We call φ_1 and φ_2 *triality components* of φ . It is not difficult to see that any other pair of triality components of φ is of the form $(R_w^{-1} \varphi_1, L_w \varphi_2)$ for some $w \in N(A)^*$, and that $\varphi_1 = R_{\varphi_2(1)}^{-1} \varphi$ and $\varphi_2 = L_{\varphi_1(1)}^{-1} \varphi$. If A is associative, then

$$\varphi(xy) = \varphi_1(x)\varphi_2(y) = (\varphi(x)\varphi_2(1)^{-1})(\varphi_1(1)^{-1}\varphi(y)) = \varphi(x)\varphi(1)^{-1}\varphi(y).$$

Given a Hurwitz algebra A , triality components φ_1, φ_2 of any $\varphi \in G(A)$ are again elements of $G(A)$. Moreover, $(\varphi_1)^{-1}$ and $(\varphi_2)^{-1}$ are a pair of triality components of φ^{-1} . We write $\text{PG}(A) = G(A)/(k^*\mathbb{I})$.

A *groupoid* is a category in which every morphism is an isomorphism. Every action of a group G on a set X gives rise to a groupoid with object class X , and morphisms $x \rightarrow y$ being the set of group elements $g \in G$ satisfying $g \cdot x = y$. We call this the *groupoid of the G -action on X* . Given a vector space V over k , and a quadratic form $q : V \rightarrow k$, set $\text{PGL}(V) = \text{GL}(V)/(k^*\mathbb{I})$, $\text{PGO}(V, q) = \text{GO}(V, q)/(k^*\mathbb{I})$ and $\text{PGO}^+(V, q) = \text{GO}^+(V, q)/(k^*\mathbb{I})$. The orbit/coset (of some group action/normal subgroup) represented by an element in a set/group will be denoted using square brackets around the element in question. For example, if $\alpha \in \text{GL}(A)$, the element in $\text{PGL}(A)$ represented by α is written as $[\alpha]$. If V is a Euclidean space then $\text{Pds}(V)$ denotes the set of positive definite symmetric endomorphisms of V , and $\text{SPds}(V) = \text{Pds}(V) \cap \text{SL}(A)$. The set of isomorphisms from an object A to an object B in a category \mathcal{C} is denoted by $\text{Iso}(A, B) = \text{Iso}_{\mathcal{C}}(A, B)$. Throughout, C_2 denotes the cyclic group of order two, generated by the canonical involution in a Hurwitz algebra of dimension at least two: $C_2 = \langle \kappa \rangle = \{\mathbb{I}, \kappa\}$.

2. GENERAL DESCRIPTION

The following lemma is essentially due to Albert [1, Theorem 7], for general (not necessarily associative) algebras. The formulation has been sharpened slightly and adapted to our situation.

Lemma 1. *Let A be a Hurwitz algebra, and $\alpha, \beta \in \text{GL}(A)$. The isotope $A_{\alpha, \beta}$ is unital if and only if $\alpha = R_a, \beta = L_b$ for some $a, b \in A^*$. The identity element in A_{R_a, L_b} is $(ab)^{-1}$.*

Proof. The isotope $A_{\alpha, \beta} = (A, \circ)$ is unital if and only if there exists an element $e \in A$ such that $L_e^\circ = R_e^\circ = \mathbb{I}_A$. Now $e \circ x = \alpha(e)\beta(x)$, that is, $L_e^\circ = L_{\alpha(e)}\beta$, so $L_e^\circ = \mathbb{I}_A$ if and only if $L_{\alpha(e)}$ is invertible $\beta = L_{\alpha(e)}^{-1}$. Now $L_{\alpha(e)}$ being invertible means that $\alpha(e) \in A^*$, and thus $\beta = L_{\alpha(e)}^{-1} = L_{\alpha(e)^{-1}}$. Similarly, $R_e^\circ = R_{\beta(e)}\alpha$ equals the identity map if and only if $\alpha = R_{\beta(e)^{-1}} = R_{\beta(e)^{-1}}$.

It readily verified that $(ab)^{-1} \circ x = x = x \circ (ab)$ in A_{R_a, L_b} . \square

Proposition 2. *Let A be a Hurwitz algebra. Any isotope B of A that has an identity element is again a Hurwitz algebra, isomorphic to A , and $n_B = n_A(1_B)^{-1}n_A$.*

Proof. By Lemma 1, any principal isotope $B = (A, \circ)$ of A that is unital has the form $B = A_{R_c, L_d}$. Defining $n_B(x) = n_A(cd)n_A(x)$ for all $x \in B$, we have

$$\begin{aligned} n_B(x \circ y) &= n_B((xc)(dy)) = n_A(cd)n_A((xc)(dy)) \\ &= n_A(cd)n_A(x)n_A(cd)n_A(y) = n_B(x)n_B(y) \end{aligned}$$

so B is a Hurwitz algebra. Moreover, $L_{cd} : (B, n_B) \rightarrow (A, n_A)$ is an isometry. Being isometric as quadratic spaces, A and B are isomorphic algebras. Since $1_B = (cd)^{-1}$, it is clear that $n_B = n_A(cd)n_A = n_A(1_B)^{-1}n_A$. \square

From Proposition 2 follows the simple but important observation that, if two isotopes $A_{\alpha, \beta}$ and $B_{\gamma, \delta}$ of Hurwitz algebras A and B are isomorphic to each other, then $A \simeq B$.

Corollary 3. *Let A be a Hurwitz algebra, and $\alpha, \beta, \gamma, \delta \in \text{GL}(A)$. Any isomorphism $\varphi : A_{\alpha, \beta} \rightarrow A_{\gamma, \delta}$ is a similitude of (A, n_A) with multiplier $n_A(\varphi(1_A))$.*

Proof. Let $\varphi : A_{\alpha,\beta} \rightarrow A_{\gamma,\delta}$ be an isomorphism. It is straightforward to verify that φ is also an isomorphism $A \rightarrow B$, where $B = A_{\gamma\varphi\alpha^{-1}\varphi,\delta\varphi\beta^{-1}\varphi}$. Thus, in particular, φ is an orthogonal map $(A, n_A) \rightarrow (B, n_B)$, and $\varphi(1_A) = 1_B$. By Proposition 2, $n_B = n_A(1_B)^{-1}n_A$, so $n_A(\varphi(x)) = n_A(1_B)n_B(\varphi(x)) = n_A(\varphi(1_A))n_A(x)$. \square

The following proposition describes all isomorphisms between isotopes of Hurwitz algebras. It generalises the description of eight-dimensional absolute-valued algebras given in [3, Theorem 4.3].

Proposition 4. *Let $A = (A, n)$ be a Hurwitz algebra and $\alpha, \beta, \gamma, \delta \in \text{GL}(A)$. A map $\varphi : A_{\alpha,\beta} \rightarrow A_{\gamma,\delta}$ is an isomorphism if and only if $\varphi \in \text{G}(A)$ and there exist triality components φ_1, φ_2 of φ such that*

$$\begin{cases} \gamma = \varphi_1\alpha\varphi^{-1}, \\ \delta = \varphi_2\beta\varphi^{-1}. \end{cases}$$

Remark 5. Note that, since any pair (φ_1, φ_2) of triality components of φ satisfy $\varphi_1 = \text{R}_{\varphi_2(1)}^{-1}$, $\varphi_2 = \text{L}_{\varphi_1(1)}^{-1}$, we could also have written

$$\begin{cases} \gamma = \varphi_1\alpha\varphi^{-1} = \text{R}_{\varphi_2(1_A)}^{-1}\varphi\alpha\varphi^{-1}, \\ \delta = \varphi_2\beta\varphi^{-1} = \text{L}_{\varphi_1(1_A)}^{-1}\varphi\beta\varphi^{-1} \end{cases}$$

in the proposition above.

Proof. A direct computation shows that if $\varphi \in \text{G}(A)$ and $\gamma = \varphi_1\alpha\varphi^{-1}$ and $\delta = \varphi_2\beta\varphi^{-1}$, then $\varphi \in \text{Iso}(A_{\alpha,\beta}, A_{\gamma,\delta})$.

For the other direction, assume that $\varphi \in \text{Iso}(A_{\alpha,\beta}, A_{\gamma,\delta})$. This means that $\varphi(\alpha(x)\beta(y)) = \gamma\varphi(x) \cdot \delta\varphi(y)$ for all $x, y \in A$, and setting $z = \alpha(x)$, $w = \beta(y)$ gives

$$\varphi(zw) = \gamma\varphi\alpha^{-1}(z) \cdot \delta\varphi\beta^{-1}(w).$$

Inserting $w = 1_A$ into this equation yields $\varphi(z) = \gamma\varphi\alpha^{-1}(z) \cdot \delta\varphi\beta^{-1}(1_A)$, so $\gamma\varphi\alpha^{-1} = \text{R}_{\delta\varphi\beta^{-1}(1_A)}^{-1}\varphi$. By Corollary 3, we know that φ is a similitude with multiplier $n_A(\varphi(1_A))$, so it follows that $\gamma\varphi\alpha^{-1}$ is a similitude with multiplier $\frac{n_A(\varphi(1_A))}{n_A(\delta\varphi\beta^{-1}(1_A))}$. Similarly, $\delta\varphi\beta^{-1}$ is a similitude with multiplier $\frac{n_A(\varphi(1_A))}{n_A(\gamma\varphi\alpha^{-1}(1_A))}$. Setting $\varphi_1 = \gamma\varphi\alpha^{-1}$ and $\varphi_2 = \delta\varphi\beta^{-1}$ gives $\varphi(zw) = \varphi_1(z)\varphi_2(w)$ with $\varphi_1, \varphi_2 \in \text{GO}(A)$, which means that $\varphi \in \text{G}(A)$. Moreover, $\gamma = \varphi_1\alpha\varphi^{-1}$ and $\delta = \varphi_2\beta\varphi^{-1}$, as required. \square

We record the following observations for future use. The first statement is a consequence of the fact, referred to in the introduction, that $x \in N(A)$ if and only if $(ax)b = a(xb)$ for all $a, b \in A$; the second follows from Proposition 4.

Lemma 6. *Let A be a Hurwitz algebra, $\alpha, \beta, \gamma, \delta \in \text{GL}(A)$ and $\rho \in k^*$.*

- (1) $A_{\alpha,\beta} = A_{\gamma,\delta}$ if and only if $\alpha = \text{R}_w^{-1}\gamma$, $\beta = \text{L}_w\delta$ for some $w \in N(A)^*$.
- (2) The homothety $h_\rho(x) = \rho x$ on A defines an isomorphism $A_{\rho\mathbb{I},\mathbb{I}} \rightarrow A$.

The set $T = \{([\varphi_0], [\varphi_1], [\varphi_2]) \in \text{PG}(A)^3 \mid \varphi_0(xy) = \varphi_1(x)\varphi_2(y), \forall x, y \in A\}$ is a group under component-wise multiplication. The kernel of the group epimorphism

$$\pi_0 : T \rightarrow \text{PG}(A), (\varphi_0, \varphi_1, \varphi_2) \mapsto \varphi_0$$

is

$$N = \{([\mathbb{I}_A], [\text{R}_w^{-1}], [\text{L}_w]) \mid w \in N(A)^*\} \simeq N(A)^*/(k^*1),$$

hence π_0 induces an isomorphism $T/N \xrightarrow{\sim} \text{PG}(A)$. Now T acts on $\text{PGL}(A)^2$ by

$$(1) \quad ([\varphi_0], [\varphi_1], [\varphi_2]) \cdot ([\alpha], [\beta]) = ([\varphi_1\alpha\varphi_0^{-1}], [\varphi_2\beta\varphi_0^{-1}]).$$

We denote by $X_A = \text{PGL}(A)^2/N$ the orbit set of the induced N -action on $\text{PGL}(A)^2$. Then (1) induces an action of $\text{PG}(A) \simeq T/N$ on X_A , given by

$$(2) \quad \varphi \cdot [\alpha, \beta] = [\varphi_1\alpha\varphi^{-1}, \varphi_2\beta\varphi^{-1}]$$

for $[\alpha, \beta] \in X_A$ and $\varphi \in \text{PG}(A)$. Here φ_1, φ_2 are any pair of triality components of φ , that is, $(\varphi, \varphi_1, \varphi_2) \in \pi_0^{-1}(\varphi) \subset T$.

We use the following notational convention: if $\gamma, \delta \in \text{GL}(A)$, then the element of X_A represented by $([\gamma], [\delta]) \in \text{PGL}(A)$ is denoted by $[\gamma, \delta]$ (rather than $[[\gamma], [\delta]]$).

Let $\mathcal{X}(A) = {}_{\text{PG}(A)}X_A$ be the groupoid of the action (2). For any Hurwitz algebra A , let $\mathcal{I}(A)$ denote the category of principal isotopes of A , and $\tilde{\mathcal{I}}(A)$ the groupoid obtained from $\mathcal{I}(A)$ by removing all non-isomorphisms between the objects. Note that if A is a division algebra then so are all its isotopes, and thus any non-zero morphism in $\mathcal{I}(A)$ is an isomorphism in this case.

The essence of our findings so far is summarised in the following theorem.

Theorem 7. *For any Hurwitz algebra A , the categories $\tilde{\mathcal{I}}(A)$ and $\mathcal{X}(A)$ are equivalent. An equivalence $\mathcal{F}_A : \tilde{\mathcal{I}}(A) \rightarrow \mathcal{X}(A)$ is given by $\mathcal{F}_A(A_{\alpha, \beta}) = [\alpha, \beta]$ and $\mathcal{F}_A(\varphi) = [\varphi]$.*

Proof. Let $(\alpha, \beta), (\gamma, \delta) \in \text{GL}(A)^2$. If $A_{\alpha, \beta} = A_{\gamma, \delta}$ then, by Lemma 6,

$$(\gamma, \delta) = (\mathbf{R}_w^{-1} \alpha, \mathbf{L}_w \beta) = (\mathbb{I}_A, \mathbf{R}_w^{-1}, \mathbf{L}_w) \cdot (\alpha, \beta)$$

for some $w \in N(A)^*$, and hence the elements $([\alpha], [\beta])$ and $([\gamma], [\delta])$ in $\text{PGL}(A)^2$ represent the same object in $X_A = \text{PGL}(A)^2/N$. Proposition 4 guarantees that $[\varphi] \cdot [\alpha, \beta] = [\gamma, \delta]$ whenever $\varphi \in \text{Iso}(A_{\alpha, \beta}, A_{\gamma, \delta})$. This shows that \mathcal{F}_A is well defined. Clearly, \mathcal{F}_A is surjective on objects, hence dense as a functor.

If $\varphi, \psi \in \text{Iso}(A_{\alpha, \beta}, A_{\gamma, \delta})$ and $\mathcal{F}_A(\varphi) = \mathcal{F}_A(\psi)$, then $\varphi = \rho\psi$ for some $\rho \in k^*$, so $\rho\mathbb{I}_A = \varphi\psi^{-1} \in \text{Aut}(A_{\alpha, \beta})$. This implies $\rho = 1$ and $\varphi = \psi$; hence \mathcal{F}_A is faithful. Fullness is clear from the construction. \square

Remark 8. (1) If k is a Euclidean field³ (e.g., $k = \mathbb{R}$), then the group $\text{PGO}(A, n)$ is canonically isomorphic to $\text{O}(A, n)/(\pm\mathbb{I})$, via composition of inclusion and quotient projection: $\text{O}(A, n)/(\pm\mathbb{I}) \hookrightarrow \text{GO}(A, n)/(\pm\mathbb{I}) \rightarrow \text{GO}(A, n)/(k^*\mathbb{I}) = \text{PGO}(A, n)$. This also induces an isomorphism $\text{O}^+(A, n)/(\pm\mathbb{I}) \rightarrow \text{PGO}^+(A, n)$.

(2) The set $X_A = \text{PGL}(A)^2/N$ may also be viewed as the orbit set $\text{PGL}(A)^2/N(A)^*$ of the $N(A)^*$ -action on $\text{PGL}(A)^2$ given by $w \cdot ([\alpha], [\beta]) = ([\mathbf{R}_w^{-1} \alpha], [\mathbf{L}_w \beta])$.

(3) Assume that A is associative. The maps $\pi_i : T \rightarrow \text{PG}(A)$, $(\varphi_0, \varphi_1, \varphi_2) \mapsto \varphi_i$, $i = 1, 2$, are group epimorphisms, and since N is normal in T , the images $\pi_1(N) = \{[\mathbf{R}_a] \mid a \in A^*\} \subset \text{PG}(A)$ and $\pi_2(N) = \{[\mathbf{L}_a] \mid a \in A^*\} \subset \text{PG}(A)$ of N under π_1 and π_2 are normal subgroups of $\text{PG}(A)$. It follows that $\mathbf{R}_{A^*} = \{\mathbf{R}_a \mid a \in A^*\}$ and $\mathbf{L}_{A^*} = \{\mathbf{L}_a \mid a \in A^*\}$ are normal subgroups of $\text{G}(A)$.

(4) If $\dim A = 8$, since $N(A) = k1$, we have $X_A = \text{PGL}(A)^2$.

If A is associative, then the group $\text{G}(A)$ has a particularly nice form.

Proposition 9. *If A is an associative Hurwitz algebra, then $\text{G}(A)$ can be written as a semi-direct product in the following ways:*

$$\text{G}(A) = \mathbf{L}_{A^*} \rtimes \text{Aut}(A) = \mathbf{R}_{A^*} \rtimes \text{Aut}(A).$$

In particular, \mathbf{L}_{A^} and \mathbf{R}_{A^*} are normal subgroups of $\text{G}(A)$.*

Proof. As already noted in Remark 8(2), \mathbf{L}_{A^*} and \mathbf{R}_{A^*} are normal subgroups of $\text{G}(A)$. Since any automorphism of A must fix the identity element 1_A in A , we have $\mathbf{L}_a \in \text{Aut}(A)$ for $a \in A^*$ if and only if $a = 1$. Thus $\mathbf{L}_{A^*} \cap \text{Aut}(A) = \{\mathbb{I}\}$. The inclusion $\text{Aut}(A) \subset \text{G}(A)$ is obvious. If $\varphi \in \text{G}(A)$, then

$$\mathbf{L}_{\varphi(1)}^{-1} \varphi(xy) = \mathbf{L}_{\varphi(1)}^{-1} (\varphi(x)\varphi(1)^{-1}\varphi(y)) = \mathbf{L}_{\varphi(1)}^{-1} \varphi(x) \cdot \mathbf{L}_{\varphi(1)^{-1}} \varphi(y) = \mathbf{L}_{\varphi(1)}^{-1} \varphi(x) \cdot \mathbf{L}_a^{-1} \varphi(y),$$

so $\mathbf{L}_{\varphi(1)}^{-1} \varphi \in \text{Aut}(A)$. This implies $\text{G}(A) = \mathbf{L}_{A^*} \text{Aut}(A)$, which shows that $\text{G}(A) = \mathbf{L}_{A^*} \rtimes \text{Aut}(A)$. The identity $\text{G}(A) = \mathbf{R}_{A^*} \rtimes \text{Aut}(A)$ is proved similarly. \square

Note that if $\dim A = 2$ then $\text{Aut}(A) = \{\mathbb{I}, \kappa\} = \text{C}_2$. If A is a quaternion algebra then it is central simple, and the Skolem-Noether Theorem [13, p. 222] gives $\text{Aut}(A) = \{\mathbf{L}_a \mathbf{R}_a^{-1} \mid a \in A^*\}$. Hence every $\varphi \in \text{GO}^+(A)$ can be written as $\varphi = \mathbf{L}_a \mathbf{L}_b \mathbf{R}_b^{-1} = \mathbf{L}_{ab} \mathbf{R}_{b^{-1}}$. This leads to the following result.

³A field k is called Euclidean if k^{*2} forms an ordering of k .

Corollary 10. *Every proper similitude of a quaternion algebra A has the form $L_a R_b$ for some $a, b \in A^*$. The kernel of the group epimorphism $A^* \times (A^*)^{\text{op}} \rightarrow \text{GO}^+(A)$, $(a, b) \mapsto L_a R_b$ is $\{(\rho, \rho^{-1}) \mid \rho \in k^*\} \subset A^* \times (A^*)^{\text{op}}$.*

Proof. From the argument in the paragraph preceding Corollary 10, it is clear that every $\varphi \in \text{GO}^+(A)$ can be written as $L_a R_b$ for some $a, b \in A^*$. For the second assertion, assume that $L_a R_b = \mathbb{1}_A$, i.e., that $axb = x$ for all $x \in A$. Setting $x = a^{-1}$ gives $b = a^{-1}$, and now $L_a R_{a^{-1}} = \mathbb{1}_A$ if and only if $a \in k^*1$. \square

If A is an associative Hurwitz algebra, then the map $L_a \psi \in L_{A^*} \rtimes \text{Aut}(A) = \text{G}(A)$ has a pair of triality components $(L_a \psi, \psi)$. Thus the action (2) of $[L_a \psi] \in \text{PG}(A)$ on $[\alpha, \beta] \in X_A$ can be written as

$$(3) \quad [L_a \psi] \cdot [\alpha, \beta] = [L_a \psi \alpha \psi^{-1} L_a^{-1}, \psi \beta \psi^{-1} L_a^{-1}] .$$

For $\dim A = 2$, this description of the groupoid $\mathcal{X}(A)$ is equivalent to the isomorphism criterion 1.12 in [17], applied to isotopes of quadratic étale algebras.

We conclude this section by introducing a numerical, easily computed isomorphism invariant. For l a positive integer, let $k^{*l} = \{\rho^l \in k^* \mid \rho \in k^*\} \subset k^*$.

Proposition 11. *Let A be a Hurwitz algebra of dimension $2l \geq 4$, and $\alpha, \beta \in \text{GL}(A)$. Then the pair $([\det(\alpha)], [\det(\beta)]) \in (k^*/k^{*l})^2$ is an isomorphism invariant for $A_{\alpha, \beta}$.*

Proof. Assuming the existence of an isomorphism $\varphi : A_{\alpha, \beta} \rightarrow A_{\gamma, \delta}$, we need to show that $\det(\alpha) \det(\gamma)^{-1} \in k^{*l}$ and $\det(\beta) \det(\delta)^{-1} \in k^{*l}$. For any $x \in A$, we have $\varphi L_{\alpha(x)} \beta = L_{\gamma\varphi(x)} \delta \varphi$ and hence $\varphi L_{\alpha(x)} \beta \varphi^{-1} = L_{\gamma\varphi(x)} \delta$, so

$$(4) \quad \det(L_{\alpha(x)}) \det(\beta) = \det(\varphi L_{\alpha(x)} \beta \varphi^{-1}) = \det(L_{\gamma\varphi(x)} \delta) = \det(L_{\gamma\varphi(x)}) \det(\delta).$$

Since the maps L_a and R_a are proper similitudes with multiplier $n_A(a)$ for any $a \in A^*$, the equation (4) implies that $n_A(\alpha(x))^l \det(\beta) = n_A(\gamma\varphi(x))^l \det(\delta)$ whenever $\alpha(x) \in A^*$. Hence, $\det(\beta) \det(\delta)^{-1} \in k^{*l}$. Similarly, the identity $\varphi R_{\beta(y)} \alpha \varphi^{-1} = R_{\delta\varphi(y)} \gamma$ implies that $\det(\alpha) \det(\gamma)^{-1} \in k^{*l}$. \square

Let A and l be as in Proposition 11. For $i, j \in k^*/k^{*l}$, setting

$$\mathcal{X}(A)_{i,j} = \left\{ [\alpha, \beta] \in \mathcal{X}(A) \mid ([\det(\alpha)], [\det(\beta)]) = (i, j) \in (k^*/k^{*l})^2 \right\} \subset \mathcal{X}(A),$$

the groupoid $\mathcal{X}(A)$ can be written as a coproduct

$$(5) \quad \mathcal{X}(A) = \coprod_{i,j \in k^*/k^{*l}} \mathcal{X}(A)_{i,j} .$$

Hence, any subcategory $\mathcal{A} \subset \mathcal{X}(A)$ can be classified by classifying each of the subcategories $\mathcal{A}_{i,j} = \mathcal{A} \cap \mathcal{X}(A)_{i,j} \subset \mathcal{X}(A)_{i,j}$.

As for the real ground field, $[\mathbb{R}^* : \mathbb{R}^{*l}] = 2$ for any even integer $l \geq 2$, the two cosets being represented by 1 and -1 , and the quotient projection $\mathbb{R}^* \rightarrow \mathbb{R}^*/\mathbb{R}^{*l}$ is given by $\rho \mapsto [\text{sign}(\rho)]$. If A is either \mathbb{H} or \mathbb{O} , $\alpha, \beta \in \text{GL}(A)$ and $A_{\alpha, \beta} = (A, \circ)$, then $\det(L_x^\circ) = \det(L_{\alpha(x)}) \det(\beta)$. Since $\det(L_{\alpha(x)}) = n(\alpha(x))^{(\dim A)/2} > 0$ for any $x \neq 0$, it follows that $\text{sign}(\det(L_x^\circ)) = \text{sign}(\det(\beta))$, and similarly $\text{sign}(\det(R_x^\circ)) = \text{sign}(\det(\alpha))$. This means that the decomposition $\mathcal{X}(A) = \coprod_{i,j \in \{-1,1\}} \mathcal{X}(A)_{i,j}$ here coincides with the “double sign” decomposition for real division algebras, introduced in [5].

3. ISOTOPES OF HAMILTON’S QUATERNIONS

In this section, we give a more detailed account for isotopes of real Hurwitz algebras whose underlying quadratic space is Euclidean. Such a Hurwitz algebra is isomorphic to either \mathbb{R} , \mathbb{C} , \mathbb{H} , or \mathbb{O} , and the isotopes are precisely the real division algebras that are isotopic to a Hurwitz algebra. The isotopes of \mathbb{C} are all the two-dimensional real division algebras, and their classification has been described in [7, 11]. While our goal is to study the isotopes of \mathbb{H} , some arguments hold also in the 8-dimensional case. Thus, let A be either \mathbb{H} or \mathbb{O} . Our main tool will be polar decomposition of linear maps.

Any $\alpha \in \text{GL}(A)$ can be written as $\alpha = \alpha' \lambda$, where $\det(\alpha') = |\det(\alpha)| > 0$ and $\lambda \in \text{C}_2 = \{\mathbb{I}, \kappa\}$. Polar decomposition now yields $\alpha' = \zeta \delta$, with $\zeta \in \text{SO}(A)$ and $\delta \in \text{Pds}(A)$. Hence $\alpha = \zeta \delta \lambda$, and this decomposition is unique [10, §14].

Passing to the projective setting, the above implies that every $\alpha \in \text{PGL}(A)$ uniquely determines $\zeta \in \text{SO}(A)/(\pm \mathbb{I})$, $\delta \in \text{SPds}(A) = \text{Pds}(A) \cap \text{SL}(A)$ and $\lambda \in \text{C}_2$ such that $\alpha = [\zeta \delta \lambda]$. As noted in Remark 8, $\text{SO}(A)/(\pm \mathbb{I}) \simeq \text{PGO}^+(A)$, and ζ can indeed be viewed as an element in $\text{PGO}^+(A)$ instead of $\text{SO}(A)/(\pm \mathbb{I})$. The cyclic group C_2 acts on $\text{PGO}^+(A)$ by $[\varphi]^\lambda = [\lambda \varphi \lambda]$ ($\lambda \in \text{C}_2$, $\varphi \in \text{GO}^+(A)$). Note that $[\text{L}_a]^\kappa = [\text{R}_a] = [\text{R}_a^{-1}]$ and $[\text{R}_a]^\kappa = [\text{L}_a] = [\text{L}_a^{-1}]$ in $\text{PGO}^+(A)$. We write $[\varphi]^{-\lambda}$ for $([\varphi]^{-1})^\lambda = ([\varphi]^\lambda)^{-1}$.

Let $\alpha, \beta \in \text{PGL}(A)$, $\alpha = [\zeta \delta \lambda]$ and $\beta = [\eta \epsilon \mu]$, where $\zeta, \eta \in \text{PGO}^+(A)$, $\delta, \epsilon \in \text{SPds}(A)$ and $\lambda, \mu \in \text{C}_2$. The action (2) of $[\varphi] \in \text{PGO}^+(A)$ on $[\alpha, \beta] \in X_A$ is given by

$$[\varphi] \cdot [\alpha, \beta] = [\varphi_1 \zeta \delta \lambda \varphi^{-1}, \varphi_2 \eta \epsilon \mu \varphi^{-1}] = [\varphi_1 \zeta \varphi^{-\lambda} (\varphi^\lambda \delta \varphi^{-\lambda}) \lambda, \varphi_2 \eta \varphi^{-\mu} (\varphi^\mu \epsilon \varphi^{-\mu}) \mu],$$

and $\varphi_1 \zeta \varphi^{-\lambda}, \varphi_2 \eta \varphi^{-\mu} \in \text{PGO}^+(A)$, $\varphi^\lambda \delta \varphi^{-\lambda}, \varphi^\mu \epsilon \varphi^{-\mu} \in \text{SPds}(A)$.

The group $N(A)^*$ acts on $\text{PGO}^+(A)^2$ by $w \cdot (\zeta, \eta) = ([\text{R}_w]^{-1} \zeta, [\text{L}_w] \eta)$, and we denote the orbit set of this action by $\text{PGO}^+(A)^2 / N(A)^*$. The argument in the preceding paragraph shows that the $\text{PGO}^+(A)$ -action on $\text{PGO}^+(A)^2 / N(A)^* \times \text{SPds}(A)^2 \times \text{C}_2^2$ by

$$(6) \quad [\varphi] \cdot ([\zeta, \eta], (\delta, \epsilon), (\lambda, \mu)) = ([\varphi_1 \zeta \varphi^{-\lambda}, \varphi_2 \eta \varphi^{-\mu}], (\varphi^\lambda \delta \varphi^{-\lambda}, \varphi^\mu \epsilon \varphi^{-\mu}), (\lambda, \mu))$$

is equivalent to the $\text{PGO}^+(A)$ -action (2) on X_A , via the map

$$m : \text{PGO}^+(A)^2 / N(A)^* \times \text{SPds}(A)^2 \times \text{C}_2^2 \rightarrow X_A, ([\zeta, \eta], (\delta, \epsilon), (\lambda, \mu)) \mapsto [\zeta \delta \lambda, \eta \epsilon \mu]$$

(i.e., $\varphi m = m \varphi$ for all $\varphi \in \text{PGO}^+(A)$). Hence, in particular, the groupoid

$$\mathcal{Y}(A) =_{\text{PGO}^+(A)} (\text{PGO}^+(A)^2 / N(A)^* \times \text{SPds}(A)^2 \times \text{C}_2^2)$$

of the action (6) is isomorphic to $\mathcal{X}(A)$:

Proposition 12. *The functor $\mathcal{G}_A : \mathcal{Y}(A) \rightarrow \mathcal{X}(A)$ defined by $\mathcal{G}_A(y) = m(y)$ for $y \in \mathcal{Y}(A)$, and $\mathcal{G}_A(\varphi) = \varphi$ for morphisms $\varphi \in \text{PGO}^+(A)$, is an isomorphism of categories.*

From here on we focus exclusively on the four-dimensional case, in which $A \simeq \mathbb{H}$. Corollary 10 implies that every element $\zeta \in \text{PGO}^+(\mathbb{H})$ has the form $\zeta = [\text{L}_a \text{R}_b]$ for some $a, b \in \mathbb{H}^*$. Since \mathbb{H} is associative, it follows that if $\varphi = [\text{L}_a \text{R}_b]$ then $\varphi_1 = [\text{L}_a]$, $\varphi_2 = [\text{R}_b]$ are triality components of φ . Recall that $N(\mathbb{H})^* = \mathbb{H}^*$ acts on $\text{PGO}^+(\mathbb{H})^2$ by $c \cdot (\zeta, \eta) = ([\text{R}_c^{-1}] \zeta, [\text{L}_c] \eta)$. Since $\text{L}_a \text{R}_b = \text{R}_b \text{L}_a$ for all $a, b \in \mathbb{H}$, the following identities hold in $\text{PGO}^+(\mathbb{H})^2 / N(\mathbb{H})^* = \text{PGO}^+(\mathbb{H})^2 / \mathbb{H}^*$:

$$[\text{L}_a \text{R}_b, \text{L}_c \text{R}_d] = c^{-1} \cdot [\text{L}_a \text{R}_b, \text{L}_c \text{R}_d] = [\text{R}_c \text{L}_a \text{R}_b, \text{R}_d] = [\text{L}_a \text{R}_{bc}, \text{R}_d].$$

Hence every element $[\zeta, \eta] \in \text{PGO}^+(\mathbb{H})^2 / \mathbb{H}^*$ can be written on the form $[\zeta, \eta] = [\text{L}_a \text{R}_b, \text{R}_c]$, with $a, b, c \in \mathbb{H}^*$. Now the action of $\varphi = [\text{L}_s \text{R}_t] \in \text{PGO}^+(\mathbb{H})$ ($s, t \in \mathbb{H}^*$) on $([\zeta, \eta], (\delta, \epsilon), (\lambda, \mu)) \in \mathcal{Y}(\mathbb{H})$ is given by

$$(7) \quad \varphi \cdot ([\zeta, \eta], (\delta, \epsilon), (\lambda, \mu)) = \left([\text{L}_s \text{L}_a \text{R}_b \text{L}_s^{-\lambda} \text{R}_t^{-\lambda}, \text{R}_t \text{R}_c \text{L}_s^{-\mu} \text{R}_t^{-\mu}], (\varphi^\lambda \delta \varphi^{-\lambda}, \varphi^\mu \epsilon \varphi^{-\mu}), (\lambda, \mu) \right).$$

For $(i, j) \in \{-1, 1\}^2$, set $\mathcal{Y}(\mathbb{H})_{i,j} = \mathcal{G}_A^{-1}(\mathcal{X}(\mathbb{H})_{i,j}) \subset \mathcal{Y}(\mathbb{H})$. Note that $([\zeta, \eta], (\delta, \epsilon), (\kappa^i, \kappa^j)) \in \mathcal{Y}(\mathbb{H})_{(-1)^i, (-1)^j}$. This gives the decomposition

$$\mathcal{Y}(\mathbb{H}) = \coprod_{i,j \in \{-1, 1\}} \mathcal{Y}(\mathbb{H})_{i,j},$$

and the action of $\varphi = \text{L}_s \text{R}_t$ on each of the cofactors $\mathcal{Y}(\mathbb{H})_{i,j}$ can be studied separately.

For $([\zeta, \eta], (\delta, \epsilon), (\mathbb{I}, \mathbb{I})) \in \mathcal{X}(\mathbb{H})_{1,1}$, we have

$$\begin{aligned} \varphi \cdot ([\zeta, \eta], (\delta, \epsilon), (\mathbb{I}, \mathbb{I})) &= ([\text{L}_s \text{L}_a \text{R}_b \text{L}_s^{-1} \text{R}_t^{-1}, \text{R}_t \text{R}_c \text{L}_s^{-1} \text{R}_t^{-1}], (\varphi \delta \varphi^{-1}, \varphi \epsilon \varphi^{-1}), (\mathbb{I}, \mathbb{I})) \\ &= ([\text{R}_s^{-1} \text{R}_b \text{R}_t^{-1} \text{L}_s \text{L}_a \text{L}_s^{-1}, \text{R}_t \text{R}_c \text{R}_t^{-1}], (\varphi \delta \varphi^{-1}, \varphi \epsilon \varphi^{-1}), (\mathbb{I}, \mathbb{I})). \end{aligned}$$

In particular, if $s = 1$, $t = b$, so that $\varphi = \text{R}_b$, then

$$\varphi \cdot ([\zeta, \eta], (\delta, \epsilon), (\mathbb{I}, \mathbb{I})) = ([\text{L}_a, \text{R}_{bc b^{-1}}], (\text{R}_b \delta \text{R}_b^{-1}, \text{R}_b \epsilon \text{R}_b^{-1}), (\mathbb{I}, \mathbb{I})).$$

Hence every orbit in $\mathcal{Y}(\mathbb{H})_{1,1}$ contains an element of the form $([L_a, R_b], (\delta, \epsilon), (\mathbb{I}, \mathbb{I}))$. Moreover, the action of $\varphi = [L_s R_t]$ on such an element is given by

$$(8) \quad \begin{aligned} \varphi \cdot ([L_a, R_b], (\delta, \epsilon), (\mathbb{I}, \mathbb{I})) &= ([L_s L_a L_s^{-1} R_t^{-1}, R_t R_b L_s^{-1} R_t^{-1}], (\varphi \delta \varphi^{-1}, \varphi \epsilon \varphi^{-1}), (\mathbb{I}, \mathbb{I})) \\ &= ([R_{(st)^{-1}} L_{sas^{-1}}, R_{t^{-1}bt}], (\varphi \delta \varphi^{-1}, \varphi \epsilon \varphi^{-1}), (\mathbb{I}, \mathbb{I})) \end{aligned}$$

which has the form $([L_c, R_d], (\delta, \epsilon), (\mathbb{I}, \mathbb{I}))$ if and only if $t = s^{-1}$, that is, $\varphi = [L_s R_s^{-1}]$.

Denoting by $\mathcal{Z} = \mathbb{H}^*/\mathbb{R}^* \left((\mathbb{H}^*/\mathbb{R}^*)^2 \times \text{SPds}(\mathbb{H})^2 \right)$ the groupoid of the action

$$s \cdot ((a, b), (\delta, \epsilon)) = ((sas^{-1}, sbs^{-1}), (c_s \delta c_s^{-1}, c_s \epsilon c_s^{-1})),$$

the above implies that the functor $\mathcal{H}_{1,1} : \mathcal{Z} \rightarrow \mathcal{Y}(\mathbb{H})_{1,1}$ defined by

$$\mathcal{H}_{1,1}((a, b), (\delta, \epsilon)) = ([L_a, R_b], (\delta, \epsilon), (\mathbb{I}, \mathbb{I}))$$

on objects, and

$$\mathcal{H}_{i,j}(s) = [L_s R_s^{-1}]$$

on morphisms, is an equivalence of categories.

Next, consider $([\zeta, \eta], (\delta, \epsilon), (\kappa, \mathbb{I})) \in \mathcal{Y}(\mathbb{H})_{-1,1}$. In this case,

$$\begin{aligned} \varphi \cdot ([\zeta, \eta], (\delta, \epsilon), (\kappa, \mathbb{I})) &= ([L_s L_a R_b L_s^{-\kappa} R_t^{-\kappa}, R_t R_c L_s^{-1} R_t^{-1}], (\varphi^\kappa \delta \varphi^{-\kappa}, \varphi \epsilon \varphi^{-1}), (\kappa, \mathbb{I})) \\ &= ([L_s L_a L_t R_s^{-1} R_b R_s, R_t R_c R_t^{-1}], (\varphi^\kappa \delta \varphi^{-\kappa}, \varphi \epsilon \varphi^{-1}), (\kappa, \mathbb{I})) \end{aligned}$$

Setting $s = 1$ and $t = a^{-1}$ gives $\varphi = R_a^{-1}$ and

$$[R_a]^{-1} \cdot ([\zeta, \eta], (\delta, \epsilon), (\kappa, \mathbb{I})) = ([R_b, R_{aca^{-1}}], (L_a \delta L_a^{-1}, R_a^{-1} \epsilon R_a), (\kappa, \mathbb{I}))$$

so the orbit of $([\zeta, \eta], (\delta, \epsilon), (\kappa, \mathbb{I}))$ contains an element of the form $([R_a, R_b], (\delta', \epsilon'), (\kappa, \mathbb{I}))$. Similarly to the previous case with $\mathcal{Y}(\mathbb{H})_{1,1}$, the group elements $\varphi \in \text{PGO}^+(\mathbb{H})$ stabilising this form, so that $\varphi \cdot ([R_a, R_b], (\delta, \epsilon), (\kappa, \mathbb{I})) = ([R_c, R_d], (\delta', \epsilon'), (\kappa, \mathbb{I}))$ for some $c, d \in \mathbb{H}$, are precisely those of the form $\varphi = [L_s R_s^{-1}]$, $s \in \mathbb{H}^*$. Note that if $\varphi = [L_s R_s^{-1}]$ then $\varphi^\kappa = \varphi$, hence

$$\varphi \cdot ([R_a, R_b], (\delta, \epsilon), (\kappa, \mathbb{I})) = ([R_{sas^{-1}}, R_{sbs^{-1}}], (\varphi \delta \varphi^{-1}, \varphi \epsilon \varphi^{-1}), (\kappa, \mathbb{I})).$$

Again, this shows that there is an equivalence of categories $\mathcal{H}_{-1,1} : \mathcal{Z} \rightarrow \mathcal{Y}(\mathbb{H})_{-1,1}$ is given by $\mathcal{H}_{-1,1}(s) = [L_s R_s^{-1}]$ on morphisms and $\mathcal{H}_{-1,1}((a, b), (\delta, \epsilon)) = ([R_a, R_b], (\delta, \epsilon), (\kappa, \mathbb{I}))$ on objects.

Similar computations can be made for $\mathcal{Y}(\mathbb{H})_{1,-1}$ and $\mathcal{Y}(\mathbb{H})_{-1,-1}$, in each case resulting in an equivalence with the category \mathcal{Z} . The results are summarised in Theorem 13 below.

For $a \in \mathbb{H}^*$, set $c_a = [L_a R_a^{-1}] \in \text{PGO}^+(\mathbb{H})$. Note that the kernel of the morphism $\mathbb{H}^* \rightarrow \text{PGO}^+(\mathbb{H})$, $a \mapsto c_a$ is \mathbb{R}^* .

Theorem 13. *Let $(i, j) \in \{-1, 1\}^2$. Each of the subcategories $\mathcal{Y}(\mathbb{H})_{i,j}$ of $\mathcal{Y}(\mathbb{H})$ is equivalent to the groupoid $\mathcal{Z} = \mathbb{H}^*/\mathbb{R}^* \left((\mathbb{H}^*/\mathbb{R}^*)^2 \times \text{SPds}(\mathbb{H})^2 \right)$ of the action*

$$s \cdot ((a, b), (\delta, \epsilon)) = ((sas^{-1}, sbs^{-1}), (c_s \delta c_s^{-1}, c_s \epsilon c_s^{-1})).$$

An equivalence $\mathcal{H}_{i,j} : \mathcal{Z} \rightarrow \mathcal{Y}(\mathbb{H})_{i,j}$ is given by $\mathcal{H}_{i,j}(s) = c_s$ for morphisms $s \in \mathbb{H}^*/\mathbb{R}^*$ and

$$\begin{aligned} \mathcal{H}_{1,1}((a, b), (\delta, \epsilon)) &= ([L_a, R_b], (\delta, \epsilon), (\mathbb{I}, \mathbb{I})), & \mathcal{H}_{-1,1}((a, b), (\delta, \epsilon)) &= ([R_a, R_b], (\delta, \epsilon), (\kappa, \mathbb{I})), \\ \mathcal{H}_{1,-1}((a, b), (\delta, \epsilon)) &= ([L_a, L_b], (\delta, \epsilon), (\mathbb{I}, \kappa)), & \mathcal{H}_{-1,-1}((a, b), (\delta, \epsilon)) &= ([L_a, R_b], (\delta, \epsilon), (\kappa, \kappa)). \end{aligned}$$

Remark 14. Theorem 13 shows, in particular, that $\mathcal{X}(\mathbb{H})_{i,j} \simeq \mathcal{X}(\mathbb{H})_{i',j'}$ for all $(i, j), (i', j') \in \{-1, 1\}^2$.

The image of the monomorphism $\mathbb{H}^*/\mathbb{R}^* \rightarrow \text{PGO}^+(\mathbb{H})$, $s \mapsto c_s$ is $\{\varphi \in \text{PGO}^+(\mathbb{H}) \mid \varphi(\mathbb{R}^*1) = \mathbb{R}^*1\}$, which can be identified with $\text{SO}_1(\mathbb{H}) = \{\varphi \in \text{SO}(\mathbb{H}) \mid \varphi(1) = 1\} \simeq \text{SO}(1, \frac{1}{\mathbb{H}})$. Thus the map $f : \mathbb{H}^*/\mathbb{R}^* \rightarrow \text{SO}_1(\mathbb{H})$, $s \mapsto c_s$ is an isomorphism, inducing an action of $\text{SO}_1(\mathbb{H})$ on the object set $(\mathbb{H}^*/\mathbb{R}^*)^2 \times \text{SPds}(\mathbb{H})^2$ of \mathcal{Z} , given by $\varphi \cdot ((a, b), (\delta, \epsilon)) = f^{-1}(\varphi) \cdot ((a, b), (\delta, \epsilon)) = ((\varphi(a), \varphi(b)), (\varphi \delta \varphi^{-1}, \varphi \epsilon \varphi^{-1}))$.

This allows for the following geometric interpretation of the category \mathcal{Z} . Elements in $\mathbb{H}^*/\mathbb{R}^*$ are viewed as lines through the origin in \mathbb{H} (that is, elements in the real projective space $\mathbb{P}(\mathbb{H})$), and $\delta \in \text{SPds}(A)$ is identified with the three-dimensional hyper-ellipsoid $E_\delta = \{x \in \mathbb{H} \mid \langle x, \delta(x) \rangle =$

$1\} \subset \mathbb{H}$. The set $\mathcal{E} = \{E_\delta \mid \delta \in \text{SPds}(\mathbb{H})\}$ consists of all hyper-ellipsoids centered in the origin and satisfying $\lambda_1 \lambda_2 \lambda_3 \lambda_4 = 1$, where $\lambda_i \in \mathbb{R}_{>0}$, $i = 1, 2, 3, 4$ are the lengths of the principal axes (which lie along the eigenspaces of δ). Moreover, \mathbb{H} is identified with \mathbb{R}^4 in the natural way, and $1_{\mathbb{H}} \subset \mathbb{H}$ with $V = \text{span}\{e_2, e_3, e_4\} \subset \mathbb{R}^4$. Now objects in \mathcal{Z} can be viewed as configurations in \mathbb{R}^4 consisting of two lines $a, b \in \mathbb{P}(\mathbb{R}^4)$, and two hyper-ellipsoids $E_\delta, E_\epsilon \in \mathcal{E}$. A morphism $(a, b, E_\delta, E_\epsilon) \rightarrow (a', b', E_{\delta'}, E_{\epsilon'})$ between two such configurations is an element $\varphi \in \text{SO}(V) \subset \text{SO}(\mathbb{R}^4)$ transforming one configuration to the other: $(\varphi(a), \varphi(b), \varphi(E_\delta), \varphi(E_\epsilon)) = (a', b', E_{\delta'}, E_{\epsilon'})$.

The above interpretation gives a picture of the category of isotopes of \mathbb{H} in elementary geometric terms, making it easy to read of whether or not two different isotopes of \mathbb{H} are isomorphic. To extract from this description a complete and irredundant list of representatives for the set of isomorphism classes of isotopes of \mathbb{H} , one would need to find a normal form for the described configurations of lines and hyper-ellipsoids under the natural action of $\text{SO}(V)$. While this task is in principle not difficult, it is by its nature quite technical, and we shall not pursue it further here. We note, however, that the object set of the groupoid \mathcal{Z} is a 24-dimensional manifold, with isomorphism classes being orbits under a continuous action of a 3-dimensional real Lie group.

4. COMPOSITION ALGEBRAS

In this section, as an example, we show how the theory of composition algebras fits into the general setting of Section 2. This leads to a short and easy proof of results by Stampfli-Rollier [20] on the structure four-dimensional composition algebras. As a bonus, our approach works also in characteristic two.

Lemma 15. *Let $A = (A, n_A)$ be a Hurwitz algebra, and $\alpha, \beta \in \text{GL}(A)$. The isotope $A_{\alpha, \beta}$ is a composition algebra if and only if $\alpha, \beta \in \text{GO}(A, n_A)$.*

Proof. If α, β are similitudes, then $A_{\alpha, \beta}$ is a composition algebra with respect to the norm $n = \mu(\alpha)\mu(\beta)n_A$. Conversely, suppose that $A_{\alpha, \beta} = (A, \circ)$ is a composition algebra with norm n . Let $a \in A$ be an element such that $n(a) \neq 0$, then $b = n(a)^{-1}(a \circ a)$ has norm 1, and thus the linear endomorphisms L_b° and R_b° of $A_{\alpha, \beta}$ are orthogonal with respect to n . Now the isotope $B = (A_{\alpha, \beta})_{(R_b^\circ)^{-1}, (L_b^\circ)^{-1}}$ is a Hurwitz algebra with norm n and identity element $b \circ b$.⁴ It follows that B is isotopic to A and thus, by Proposition 2, $n = \rho n_A$ for some $\rho \in k^*$. Hence

$$n_A(\alpha(x))n_A(\beta(y)) = n_A(\alpha(x)\beta(y)) = n_A(x \circ y) = \rho^{-1}n(x \circ y) = \rho^{-1}n(x)n(y) = \rho n_A(x)n_A(y)$$

for all $x, y \in A$. Inserting $y = 1_A$ into this equation gives $n_A(\alpha(x)) = \rho n_A(\beta(1))^{-1}n_A(x)$, so α is a similitude with multiplier $\rho n_A(\beta(1))^{-1}$ with respect to n_A . Similarly, β is a similitude with multiplier $\rho n_A(\alpha(1))^{-1}$. \square

Given a Hurwitz algebra A , let $\mathcal{X}^c(A) \subset \mathcal{X}(A)$ be the full subcategory formed by all images $[\alpha, \beta] = \mathcal{F}_A(A_{\alpha, \beta})$ where $A_{\alpha, \beta}$ is a composition algebra. Lemma 15 implies that $\mathcal{X}^c(A) = \text{PGO}(A, n_A)^2/N \subset \mathcal{X}(A)$.

For $\dim A \geq 2$, the property of being a proper respectively improper similitude is retained by the factors α, β of $[\alpha, \beta] \in \mathcal{X}^c(A)$ under the action of $\text{PG}(A)$. This means that for each $(i, j) \in \{-1, 1\}^2$, the subset

$$\mathcal{X}^{c(i, j)}(A) = (\text{PGO}^i(A) \times \text{PGO}^j(A)) / N(A)^* \subset \mathcal{X}^c(A)$$

(where we use the notational convention $\text{PGO}^{\pm 1}(A) = \text{PGO}^\pm(A)$) is invariant under this action, and the category $\mathcal{X}^c(A)$ decomposes as

$$\mathcal{X}^c(A) = \coprod_{i, j \in \{-1, 1\}} \mathcal{X}^{c(i, j)}(A).$$

This refines the decomposition (5) for $\dim A \geq 4$: $\mathcal{X}^{c(i, j)}(A) \subset \mathcal{X}^c(A)_{i, j}$ for all $i, j \in \{-1, 1\}$. If $-1 \notin k^l$, $l = (\dim A)/2$, then $\mathcal{X}^{c(i, j)}(A) = \mathcal{X}^c(A)_{i, j}$, but whereas the four sets $\mathcal{X}^{c(i, j)}(A)$ are always distinct, $\mathcal{X}^c(A)_{i, j} = \mathcal{X}^c(A)$ for all $i, j \in \{-1, 1\}$ in case $-1 \in k^l$.

⁴This argument, by which any finite-dimensional composition algebra is shown to be isotopic to a Hurwitz algebra with the same norm, is sometimes referred to as *Kaplansky's trick*, after [14, p. 957].

If (A, n_A) is a Euclidean space, the groups $\text{PGO}(A)$ and $\text{PGO}^+(A)$ are canonically identified with $\text{O}(A)/(\pm\mathbb{I}_A)$ and $\text{SO}(A)/(\pm\mathbb{I}_A)$ respectively. This, together with Theorem 7, gives a description of all finite-dimensional absolute valued algebras equivalent to Theorem 4.3 in [3].

We proceed to describe all composition algebras isotopic to a fixed quaternion algebra $A = (A, n)$ over k . Every $[\alpha, \beta] \in \mathcal{X}^c(A)$ can be written as $[\alpha, \beta] = [\zeta\lambda, \eta\mu]$ with $\zeta, \eta \in \text{GO}^+(A)$ and $\lambda, \mu \in \text{C}_2$. Corollary 10 now gives $\zeta = [\text{L}_a \text{R}_b]$, $\eta = [\text{L}_c \text{R}_d]$ for some $a, b, c, d \in A^*$, and since $N(A) = A$, we have $[\alpha, \beta] = [\text{L}_a \text{R}_b \lambda, \text{L}_c \text{R}_d \mu] = [\text{L}_a \text{R}_{bc} \lambda, \text{R}_d]$ in X_A . Thus, analogously with the Euclidean case treated in Section 3, $[\alpha, \beta]$ can be written as $[\alpha, \beta] = [\text{L}_a \text{R}_b \lambda, \text{R}_c \mu]$. Moreover, one reads off that $[\alpha, \beta] \in \mathcal{X}^{c(\det(\lambda), \det(\mu))}(A)$.

Let $[\alpha, \beta] = [\text{L}_a \text{R}_b, \text{R}_c] \in \mathcal{X}^{c(1,1)}(A)$. Now $[\text{R}_b] \cdot [\alpha, \beta] = [\text{L}_a, \text{R}_{b^{-1}cb}]$, so every $\text{PGO}^+(A)$ -orbit in $\mathcal{X}^{c(1,1)}(A)$ contains an element of the form $[\text{L}_b, \text{R}_b]$. Similarly, every orbit in $\mathcal{X}^{c(-1,1)}(A)$ contains an element of the form $[\text{R}_a \kappa, \text{R}_b]$, every orbit in $\mathcal{X}^{c(1,-1)}(A)$ contains some $[\text{L}_a, \text{L}_b \kappa]$, and every orbit in $\mathcal{X}^{c(-1,-1)}(A)$ contains an element of the form $[\text{L}_a \kappa, \text{R}_b \kappa]$. By computations similar to (8), one proves that $\varphi \in \text{PGO}^+(A)$ stabilises each of these forms if and only if $\varphi = c_s$ for some $s \in A^*$, and that

$$\begin{aligned} c_s \cdot [\text{L}_a, \text{R}_b] &= [\text{L}_s \text{L}_a \text{L}_s^{-1}, \text{R}_s^{-1} \text{R}_b \text{R}_s] = [\text{L}_{sas^{-1}}, \text{R}_{sbs^{-1}}], \\ c_s \cdot [\text{R}_a \kappa, \text{R}_b] &= [\text{R}_s^{-1} \text{R}_a \text{R}_s \kappa, \text{R}_s^{-1} \text{R}_b \text{R}_s] = [\text{R}_{sas^{-1}} \kappa, \text{R}_{sbs^{-1}}], \\ c_s \cdot [\text{L}_a, \text{L}_b \kappa] &= [\text{L}_s \text{L}_a \text{L}_s^{-1}, \text{L}_s \text{L}_b \text{L}_s^{-1} \kappa] = [\text{L}_{sas^{-1}}, \text{L}_{sbs^{-1}} \kappa], \\ c_s \cdot [\text{L}_a \kappa, \text{R}_b \kappa] &= [\text{L}_s \text{L}_a \text{L}_s^{-1} \kappa, \text{R}_s^{-1} \text{R}_b \text{R}_s \kappa] = [\text{L}_{sas^{-1}} \kappa, \text{R}_{sbs^{-1}} \kappa]. \end{aligned}$$

This proves the following result, which gives an isomorphism criterion for all composition algebras isotopic to A .

Proposition 16. *Let A be a quaternion algebra. Each of the categories $\mathcal{X}^{c(i,j)}(A)$ is equivalent to the groupoid $\mathcal{Z}^c(A) = {}_{A^*/k^*}(A^*/k^*)^2$ of the action $s \cdot (a, b) = (sas^{-1}, sbs^{-1})$. Equivalences $\mathcal{H}_{i,j}^c : \mathcal{Z}^c(A) \rightarrow \mathcal{X}^c(A)_{i,j}$ are given by $\mathcal{H}_{i,j}^c(s) = c_s$ for morphisms, and*

$$\begin{aligned} \mathcal{H}_{1,1}^c(a, b) &= [\text{L}_a, \text{R}_b], & \mathcal{H}_{-1,1}^c(a, b) &= [\text{R}_a, \text{R}_b], \\ \mathcal{H}_{1,-1}^c(a, b) &= [\text{L}_a, \text{L}_b], & \mathcal{H}_{-1,-1}^c(a, b) &= [\text{L}_a, \text{R}_b]. \end{aligned}$$

A characterisation of all four-dimensional composition algebras over k for $\text{char } k \neq 2$, similar to Proposition 16, is given by Stampfli-Rollier in Section 3–4 of [20]. Her exposition also contains explicit isomorphism criteria in terms of the parameters a, b . The Euclidean case, comprising all composition algebras isotopic to \mathbb{H} , that is, all four-dimensional absolute valued algebras, has also been described by Ramírez Álvarez [18]. Forsberg, in his master thesis [9], refined that description to give an explicit cross-section for the isomorphism classes of these algebras.

REFERENCES

- [1] A. A. Albert. Non-associative algebras. I. Fundamental concepts and isotopy. *Ann. of Math. (2)*, 43:685–707, 1942.
- [2] Seidon Alsaody. An approach to finite-dimensional real division composition algebras through reflections. *Bull. Sci. Math.*, 139(4):357–399, 2015.
- [3] A. Calderón, A. Kaidi, C. Martín, A. Morales, M. Ramírez, and A. Rochdi. Finite-dimensional absolute-valued algebras. *Israel J. Math.*, 184:193–220, 2011.
- [4] J. A. Cuenca Mira. On one-sided division infinite-dimensional normed real algebras. *Publ. Mat.*, 36(2A):485–488 (1993), 1992.
- [5] E. Darpö and E. Dieterich. The double sign of a real division algebra of finite dimension greater than one. *Math. Nachr.*, 285(13):1635–1642, 2012.
- [6] Erik Darpö and José María Pérez Izquierdo. Autotopies and quasigroup identities: New aspects of non-associative division algebras. *Forum Math.*, 27(5):2691–2745, 2015.
- [7] E. Dieterich. Classification, automorphism groups and categorical structure of the two-dimensional real division algebras. *J. Algebra Appl.*, 4(5):517–538, 2005.
- [8] A. Elduque and H. C. Myung. On flexible composition algebras. *Comm. Algebra*, 21(7):2481–2505, 1993.
- [9] L. Forsberg. Four-dimensional absolute valued algebras. Master’s thesis, Uppsala University, 2009. U.U.D.M. Project Report 2009:9.
- [10] F. R. Gantmacher. *The theory of matrices. Vol. 1*. Translated by K. A. Hirsch. Chelsea Publishing Co., New York, 1959.

- [11] M. Hübner and H. P. Petersson. Two-dimensional real division algebras revisited. *Beiträge Algebra Geom.*, 45(1):29–36, 2004.
- [12] N. Jacobson. Composition algebras and their automorphisms. *Rend. Circ. Mat. Palermo*, 7(2):55–80, 1958.
- [13] N. Jacobson. *Basic Algebra II*. W.H. Freeman and Company, second edition, 1985.
- [14] I. Kaplansky. Infinite-dimensional quadratic forms admitting composition. *Proc. Amer. Math. Soc.*, 4:956–960, 1953.
- [15] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol. *The book of involutions*, volume 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998.
- [16] R. Moufang. Zur Struktur von Alternativkörpern. *Math. Ann.*, 110(1):416–430, 1935.
- [17] H. P. Petersson. The classification of two-dimensional nonassociative algebras. *Results Math.*, 37(1-2):120–154, 2000.
- [18] M. I. Ramírez Álvarez. On four-dimensional absolute-valued algebras. In *Proceedings of the International Conference on Jordan Structures (Málaga, 1997)*, pages 169–173, Málaga, 1999. Univ. Málaga.
- [19] T. A. Springer and F. D. Veldkamp. *Octonions, Jordan algebras and exceptional groups*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000.
- [20] C. Stampfli-Rollier. 4-dimensionale Quasikompositionsalgebren. *Arch. Math. (Basel)*, 40(6):516–525, 1983.
- [21] M. Zorn. Theorie der alternativen Ringe. *Abh. Math. Sem. Hamburg*, 8:123–147, 1931.

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