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**ON THE VALUE-DISTRIBUTION OF THE DIFFERENCE
BETWEEN LOGARITHMS OF TWO SYMMETRIC POWER
 L -FUNCTIONS**

KOJI MATSUMOTO

*Graduate School of Mathematics, Nagoya University, Furocho, Chikusa-ku,
Nagoya 464-8602, Japan
kohjimat@math.nagoya-u.ac.jp*

YUMIKO UMEGAKI

*Department of Mathematical and Physical Sciences, Nara Women's University, Kitaoyoa
Nishimachi,
Nara 630-8506, Japan
ichihara@cc.nara-wu.ac.jp*

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We consider the value distribution of the difference between logarithms of two symmetric power L -functions at $s = \sigma > 1/2$. We prove that certain averages of those values can be written as integrals involving a density function which is constructed explicitly.

Keywords: Value-distribution; automorphic L -function; symmetric power L -function; density function.

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1. Introduction and the statement of main results

Let f be a primitive form of weight k and level N , which means that it is a normalized common Hecke eigen new form of weight k for $\Gamma_0(N)$. We denote by $S_k(N)$ the set of all cusp forms of weight k and level N . Any $f \in S_k(N)$ has a Fourier expansion at infinity of the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z}, \quad \lambda_f(1) = 1.$$

In the case f is a normalized common Hecke eigen form, the Fourier coefficients $\lambda_f(n)$ are real numbers. We consider the L -function

$$L(f, s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}$$

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associated with a primitive form f where $s = \sigma + i\tau \in \mathbb{C}$. This is absolutely convergent when $\sigma > 1$, but can be continued to the whole of \mathbb{C} as an entire function.

We denote by \mathbb{P} the set of all prime numbers. We know that $L(f, s)$ has the Euler product

$$\begin{aligned} L(f, s) &= \prod_{\substack{p \in \mathbb{P} \\ p|N}} (1 - \lambda_f(p)p^{-s})^{-1} \prod_{\substack{p \in \mathbb{P} \\ p \nmid N}} (1 - \lambda_f(p)p^{-s} + p^{-2s})^{-1} \\ &= \prod_{\substack{p \in \mathbb{P} \\ p|N}} (1 - \lambda_f(p)p^{-s})^{-1} \prod_{\substack{p \in \mathbb{P} \\ p \nmid N}} (1 - \alpha_f(p)p^{-s})^{-1} (1 - \beta_f(p)p^{-s})^{-1}, \end{aligned}$$

where $\beta_f(p)$ is the complex conjugate of $\alpha_f(p)$. Note that $\alpha_f(p)$ and $\beta_f(p)$ satisfy $\alpha_f(p) + \beta_f(p) = \lambda_f(p)$ and $|\alpha_f(p)| = |\beta_f(p)| = 1$. This Euler product is deduced from the relations

$$\lambda_f(p^\ell) = \begin{cases} \lambda_f^\ell(p) & p | N, \\ \sum_{h=0}^{\ell} \alpha_f^{\ell-h}(p) \beta_f^h(p) & p \nmid N. \end{cases} \quad (1.1)$$

From this relations, we see $|\lambda_f(n)| \leq d(n)$, where $d(n)$ is the number of divisors of n .

In the present paper, we consider the value of $\log L(\text{Sym}_f^\gamma, s) - \log L(\text{Sym}_f^\ell, s)$ at $s = \sigma > 1/2$, where the γ -th symmetric power L -function is defined by

$$L(\text{Sym}_f^\gamma, s) = \prod_{\substack{p \in \mathbb{P} \\ p|N}} (1 - \lambda_f(p^\gamma)p^{-s})^{-1} \prod_{\substack{p \in \mathbb{P} \\ p \nmid N}} \prod_{h=0}^{\gamma} (1 - \alpha_f^{\gamma-h}(p) \beta_f^h(p) p^{-s})^{-1}$$

for $\sigma > 1$. In the case $\gamma = 1$, clearly $L(\text{Sym}_f^1, s) = L(f, s)$. In general, it is believed that the symmetric power L -function could be continued to an entire function and would satisfy a functional equation. We suppose the analytic continuation and its holomorphy for $\sigma > 1/2$ in Assumption 1.1 below.

For $\sigma > 1$, we define

$$\log L(\text{Sym}_f^\gamma, s) = - \sum_{\substack{p \in \mathbb{P} \\ p|N}} \text{Log}(1 - \lambda_f(p^\gamma)p^{-s}) - \sum_{\substack{p \in \mathbb{P} \\ p \nmid N}} \sum_{h=0}^{\gamma} \text{Log}(1 - \alpha_f^{\gamma-h}(p) \beta_f^h(p) p^{-s}),$$

where Log means the principal branch. In the strip $1/2 < \sigma \leq 1$, we suppose it can be analytically continued to $\sigma > 1/2$ under Assumption 1.4 below, which claims that $L(\text{Sym}_f^\gamma, s)$ has no zero in the strip $1/2 < \sigma \leq 1$. In this paper we introduce the following two assumptions.

Assumption 1.1. Let f be a primitive form of even weight k where $2 \leq k < 12$ or $k = 14$. The level of f is q^m , where q is a prime number. For a fixed positive

integer γ , the symmetric power L -function $L(\text{Sym}_f^\gamma, s)$ is analytically continued to a holomorphic function in $\sigma > 1/2$. Moreover it satisfies the estimate

$$|L(\text{Sym}_f^\gamma, s)| \ll_\gamma q^m (|\tau| + 2) \quad (1.2)$$

for $1/2 < \sigma \leq 2$.

Remark 1.2. For the symmetric power L -function, if we obtain a suitable functional equation which is the same type as in Cogdell and Michel [2], we have the same estimate as (1.2) by using the Phlagmén-Lindelöf principle. As Cogdell and Michel mentioned in [2], this assumption is held in the case when f is a primitive form of weight 2 and of square-free level for the symmetric cube L -function, which is proved by Kim and Shahidi [13].

Remark 1.3. For a primitive form of weight k and level M , where k is an even positive integer and M is a positive integer, the automorphic L -function $L(f, s)$ is entire and it has a functional equation. The estimate of the form (1.2) holds for $L(f, s)$, that is

$$|L(\text{Sym}_f^1, s)| = |L(f, s)| \ll M (|\tau| + 2) \quad (1.3)$$

for $1/2 < \sigma \leq 2$.

Assumption 1.4. Let f be a primitive form of weight k which $2 \leq k < 12$ or $k = 14$. The level is q^m , where q is a prime number. For a fixed positive integer γ , the L -functions $L(\text{Sym}_f^\gamma, s)$ satisfies Generalized Riemann Hypothesis (GRH) which means that $L(\text{Sym}_f^\gamma, s)$ has no zero in the strip $1/2 < \sigma \leq 1$.

In this paper, we mainly consider two types of averages which are defined below. For the definitions of them, we first prepare the notations. Let q be a prime number. For any sequence $\{A_f\}$ over primitive forms $f \in S_k(q^m)$, where $2 \leq k < 12$ or $k = 14$, we use the symbol \sum' in the following sense:

$$\sum'_{f \in S_k(q^m)} A_f = \frac{1}{C_k(1 - C_q(m))} \sum_{\substack{f \in S_k(q^m) \\ f: \text{primitive form}}} \frac{A_f}{\langle f, f \rangle},$$

where C_k and $C_q(m)$ are constants defined by

$$C_k = \frac{(4\pi)^{k-1}}{\Gamma(k-1)}, \quad C_q(m) = \begin{cases} 0 & m = 1, \\ q(q^2 - 1)^{-1} & m = 2, \\ q^{-1} & m \geq 3. \end{cases}$$

These constants appeared in Lemma 3 in the second author [4] (see (2.1) below), which came from Petersson's formula.

We define the ‘‘partial’’ Euler product of the symmetric power L -function by

$$L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, s) = \prod_{p \in \mathbb{P}(q)} \prod_{h=0}^{\gamma} (1 - \alpha_f^{\gamma-h}(p) \beta_f^h(p) p^{-s})^{-1}$$

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for a primitive form f of level q^m , where q is a prime number and the subset $\mathbb{P}(q) \subset \mathbb{P}$ means the set of all prime numbers except for the fixed prime number q . Let $\mu > \nu \geq 1$ be integers with $\mu - \nu = 2$. By $Q(\mu)$ we denote the smallest prime number satisfying $2^\mu / \sqrt{Q(\mu)} < 1$. In this paper, we study two types of averages which are defined by

$$\begin{aligned} & \text{Avg}_{\text{prime}} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \lim_{\substack{q \rightarrow \infty \\ q: \text{prime} \\ m: \text{fixed}}} \sum'_{f \in S_k(q^m)} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \end{aligned} \quad (1.4)$$

and

$$\begin{aligned} & \text{Avg}_{\text{power}} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \lim_{\substack{m \rightarrow \infty \\ q: \text{fixed prime} \\ (\text{If } 1 \geq \sigma > 1/2, q \geq Q(\mu))}} \sum'_{f \in S_k(q^m)} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)), \end{aligned} \quad (1.5)$$

where Ψ is a \mathbb{C} -valued function defined on \mathbb{R} . On the above average $\text{Avg}_{\text{power}}$, we consider $q \geq Q(\mu)$ when $1 \geq \sigma > 1/2$. The reason is technical which will be mentioned in Section 5. The main theorem in the present paper is as follows.

Theorem 1.5. *Let $\mu > \nu \geq 1$ be integers with $\mu - \nu = 2$. Suppose Assumptions 1.1 and 1.4 when γ is μ and ν . Let k be an even integer which satisfies $2 \leq k < 12$ or $k = 14$. Then, for $\sigma > 1/2$, there exists a function $\mathcal{M}_\sigma : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ which can be explicitly constructed, and for which the formula*

$$\begin{aligned} & \text{Avg}_{\text{prime}} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \text{Avg}_{\text{power}} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Psi(u) \frac{du}{\sqrt{2\pi}} \end{aligned}$$

holds for any $\Psi : \mathbb{R} \rightarrow \mathbb{C}$ which is a bounded continuous function or a compactly supported characteristic function.

Perhaps it sounds artificial to consider the "difference" of the logarithms of two L -functions. However, actually, this is a natural analogue of the logarithm of single L -function. In fact, when $\gamma = 1$,

$$L_{\mathbb{P}(q)}(f, s) = L_{\mathbb{P}(q)}(\text{Sym}_f^1, s) = \prod_{p \in \mathbb{P}(q)} (1 - \alpha_f(p)p^{-s})^{-1} (1 - \beta_f(p)p^{-s})^{-1},$$

and so, its restricted version for a finite subset $\mathcal{P} \subset \mathbb{P}(q)$ is

$$L_{\mathcal{P}}(f, s) = \prod_{p \in \mathcal{P}} (1 - \alpha_f(p)p^{-s})^{-1} (1 - \beta_f(p)p^{-s})^{-1}.$$

The basic identity in the proof of Theorem 1.5 is (2.5) below. We see that the right-hand side of (2.5) with $\mu = 1$ is exactly equal to $\log L_{\mathcal{P}}(f, s)$. Therefore, if we

could handle the case $\mu = 1$, we would obtain the same type of result as the above theorem for $\log L_{\mathbb{P}(q)}(f, s)$.

Remark 1.6. Here we give some remarks on the assumptions required in the theorem.

- (1) On the assumptions on μ and ν . As discussed above, it is surely interesting to search for the density function, of the nature similar to the above, for the average of $\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, s)$ itself, especially in the case of $\mu = 1$. However so far we cannot obtain such a density function by our method. The reason is explained in Remark 4.2 below (in Section 4). Therefore we consider the difference between the logarithm of symmetric μ -th power L -function and that of symmetric ν -th power L -function where μ and ν are of the same parity. However we have another difficulty in the case $\mu - \nu > 2$. It is explained in Remark 6.1 below (in Section 6). Hence Theorem 1.5 is shown only in the case $\mu - \nu = 2$. In particular $\mu = \nu + 2 \geq 3$, so we cannot handle the case $\mu = 1$.
- (2) The restriction on the weight k is necessary to prove (2.1) below.
- (3) The limit theorems proved for the Riemann zeta-function or Dirichlet L -functions (such as Bohr and Jessen [1] and Ihara and the first author [7], mentioned below) are unconditional. This is because, instead of the Riemann hypothesis, some mean value formulas are available in those cases. However our Theorem 1.5 assumes the analogue of the GRH (Assumption 2), because it seems difficult to prove suitable mean value results for symmetric power L -functions.

We mention a corollary of the $\text{Avg}_{\text{prime}}$ part of the theorem. Consider the following different type of averages, involving summations with respect to levels:

$$\begin{aligned} & \text{Avg}_{\text{primesum}} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \lim_{X \rightarrow \infty} \frac{1}{\pi(X)} \sum_{\substack{q \leq X \\ q:\text{prime} \\ m:\text{fixed}}} \sum_{f \in S_k(q^m)}' \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)), \end{aligned}$$

where $\pi(X)$ denotes the number of prime numbers not larger than X , and

$$\begin{aligned} & \text{Avg}_{\text{primepowersum}} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \lim_{X \rightarrow \infty} \frac{1}{\pi^*(X)} \sum_{\substack{q^m \leq X \\ q:\text{prime} \\ m \geq 1}} \sum_{f \in S_k(q^m)}' \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)), \end{aligned}$$

where $\pi^*(X)$ denotes the number of all pairs (q, m) of a prime number q and a positive integer m with $q^m \leq X$.

Corollary 1.7. *Under the same assumptions of Theorem 1.5, we have*

$$\begin{aligned} & \text{Avg}_{\text{primesum}} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \text{Avg}_{\text{primepowersum}} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Psi(u) \frac{du}{\sqrt{2\pi}}. \end{aligned}$$

Remark 1.8. Theorem 1.5 can be generalized to the case of any average defined by some limit, which is different from those in (1.4) and (1.5), but satisfies the condition that $q^m \rightarrow \infty$. (For example, $q \rightarrow \infty$ with $m = m(q)$ moving arbitrarily.) In fact, from the proof we can see that the only necessary limit procedure is $q^m \rightarrow \infty$.

The first result of this type is due to Bohr and Jessen [1]. Let $\zeta(s)$ be the Riemann zeta-function. Bohr and Jessen proved that, when Ψ is a compactly supported characteristic function defined on \mathbb{C} , the formula

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \Psi(\log \zeta(s + i\tau)) d\tau = \int_{\mathbb{C}} \mathcal{M}_{\zeta, \sigma}(w) \Psi(w) \frac{dudv}{\sqrt{2\pi}} \quad (1.6)$$

holds for $\Re s > 1/2$ (where $w = u + iv$), with a certain density function $\mathcal{M}_{\zeta, \sigma}$. The analogue for the logarithmic derivative $\zeta'/\zeta(s)$ was first proved by Kershner and Wintner [12].

Ihara [5] discovered that the same type of results can be shown for certain mean values of $L'/L(s, \chi)$ with respect to characters, where $L(s, \chi)$ denotes the Dirichlet (or Hecke) L -function attached to the character χ , including also the function field case. Ihara's work was strengthened, and extended to the log L case, in several joint papers of Ihara and the first author [6], [7], [8], [9]. Recently, Mourtada and Murty [14] obtained an analogous result for the mean value of $L'/L(s, \chi)$ with respect to discriminants.

In those former results, the function Ψ is defined on \mathbb{C} , and the right-hand side of the formula is an integral over \mathbb{C} . However in our Theorem 1.5, the function Ψ is defined on \mathbb{R} , and the right-hand side is an integral over \mathbb{R} . This is one remarkable difference of our present work from the former researches.

The plan of this paper is as follows. Section 2 is the preparation, with the proof of Corollary 1.7. In Section 3 we construct the density function \mathcal{M}_σ , in Section 4 we state the key lemma (Lemma 4.3) and prove it in the case $\sigma > 1$, in Section 5 we prepare certain approximation of $L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, s)$ to prove the key lemma, in Section 6 we prove the key lemma for $1 \geq \sigma > 1/2$ and finally, in Section 7 we will complete the proof of Theorem 1.5. The basic structure of our argument is similar to the previous work by Ihara and the first author [7].

2. Preparations

Bohr and Jessen [1] used the Kronecker-Weyl theorem on uniform distribution of sequences as an essential tool in the proof of (1.6). In Ihara [5], the corresponding tool is the orthogonality relation of characters.

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In our present situation, the corresponding useful tool is Petersson's well-known formula (see, e.g., [10]). In the proof of our main Theorem 1.5, we will use the following formula ((2.1) below) for a prime number q , which was shown in Lemma 3 in the second author [4]. This formula embodies the essence of Petersson's formula, in the form suitable for our present aim.

When $2 \leq k < 12$ or $k = 14$, for the primitive form f of weight k and level q^m , we have

$$\sum'_{f \in S_k(q^m)} \lambda_f(n) = \delta_{1,n} + \begin{cases} O_k(n^{(k-1)/2} q^{-k+1/2}) & m = 1, \\ O_k(n^{(k-1)/2} q^{m(-k+1/2)} q^{k-3/2}) & m \geq 2, \end{cases} \quad (2.1)$$

where $\delta_{1,n} = 1$ if $n = 1$ and 0 otherwise. We denote the error term in (2.1) by $n^{(k-1)/2} E(q^m)$, that is

$$\sum'_{f \in S_k(q^m)} \lambda_f(n) - \delta_{1,n} = n^{(k-1)/2} E(q^m).$$

Then we have

$$E(q^m) \ll q^{-k+1/2} \quad (2.2)$$

for any m , and

$$E(q^m) \ll \begin{cases} q^{-3/2} & m = 1, \\ q^{-5/2} & m = 2, \\ q^{-1-m} & m \geq 3, \end{cases} \ll q^{-m-1/2} \quad (2.3)$$

for any $m \geq 1$. Also in the case $n = 1$, the formula (2.1) implies

$$\sum'_{f \in S_k(q^m)} \lambda_f(1) = \sum'_{f \in S_k(q^m)} 1 = 1 + E(q^m) \ll 1. \quad (2.4)$$

Let \mathcal{P} be a subset of \mathbb{P} and q a fixed prime number. For a primitive form f of weight k and level q^m , define

$$L_{\mathcal{P}}(\text{Sym}_f^\gamma, s) = \prod_{p \in \mathcal{P}} L_p(\text{Sym}_f^\gamma, s),$$

where

$$L_p(\text{Sym}_f^\gamma, s) = \begin{cases} (1 - \lambda_f(p^\gamma) p^{-s})^{-1} & p = q, \\ \prod_{h=0}^{\gamma} (1 - \alpha_f^{\gamma-h}(p) \beta_f^h(p) p^{-s})^{-1} & p \neq q. \end{cases}$$

Especially we write

$$L_{\mathcal{P}}(f, s) = L_{\mathcal{P}}(\text{Sym}_f^1, s).$$

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Further, for integers $\mu > \nu > 0$ with $\mu - \nu = 2$, we put

$$L_{\mathcal{P}}(\mathrm{Sym}_f^\mu, \mathrm{Sym}_f^\nu, s) = \frac{L_{\mathcal{P}}(\mathrm{Sym}_f^\mu, s)}{L_{\mathcal{P}}(\mathrm{Sym}_f^\nu, s)}.$$

This can be defined for $\sigma > 1/2$ under Assumption 1.4.

Now let \mathcal{P} be a finite subset of $\mathbb{P}(q)$. We define the topological group $\mathcal{T}_{\mathcal{P}}$ by

$$\mathcal{T}_{\mathcal{P}} = \prod_{p \in \mathcal{P}} \mathcal{T},$$

where $\mathcal{T} = \{t \in \mathbb{C} \mid |t| = 1\}$. For a fixed $\sigma > 1/2$, we consider the function

$$g_{\sigma, \mathcal{P}}(t_{\mathcal{P}}) = \sum_{p \in \mathcal{P}} g_{\sigma, p}(t_p)$$

on $t_{\mathcal{P}} = (t_p)_{p \in \mathcal{P}} \in \mathcal{T}_{\mathcal{P}}$, where

$$g_{\sigma, p}(t_p) = -\log(1 - t_p p^{-\sigma}).$$

For any $s = \sigma + i\tau$ ($\sigma > 1/2$) we have

$$\begin{aligned} & \log L_{\mathcal{P}}(\mathrm{Sym}_f^\mu, \mathrm{Sym}_f^\nu, s) \\ &= \log L_{\mathcal{P}}(\mathrm{Sym}_f^\mu, s) - \log L_{\mathcal{P}}(\mathrm{Sym}_f^\nu, s) \\ &= \sum_{p \in \mathcal{P}} \left(-\sum_{h=0}^{\mu} \log(1 - \alpha_f^{\mu-h}(p) \beta_f^h(p) p^{-s}) + \sum_{h=0}^{\nu} \log(1 - \alpha_f^{\nu-h}(p) \beta_f^h(p) p^{-s}) \right) \\ &= \sum_{p \in \mathcal{P}} \left(-\log(1 - \alpha_f^\mu(p) p^{-s}) - \log(1 - \beta_f^\mu(p) p^{-s}) \right) \\ &= \sum_{p \in \mathcal{P}} \left(g_{\sigma, p}(\alpha_f^\mu(p) p^{-i\tau}) + g_{\sigma, p}(\beta_f^\mu(p) p^{-i\tau}) \right) \\ &= g_{\sigma, \mathcal{P}}(\alpha_f^\mu(\mathcal{P}) \mathcal{P}^{-i\tau}) + g_{\sigma, \mathcal{P}}(\beta_f^\mu(\mathcal{P}) \mathcal{P}^{-i\tau}), \end{aligned} \tag{2.5}$$

where $\alpha_f^\mu(\mathcal{P}) = (\alpha_f^\mu(p))_{p \in \mathcal{P}}$, $\beta_f^\mu(\mathcal{P}) = (\beta_f^\mu(p))_{p \in \mathcal{P}}$ and $\mathcal{P}^{-i\tau} = (p^{-i\tau})_{p \in \mathcal{P}}$. In the above equation, we used the fact that $\beta_f(p)$ is the complex conjugate of $\alpha_f(p)$, and hence $\alpha_f^{\mu-h}(p) \beta_f^h(p) = \alpha_f^{\nu-(h-1)}(p) \beta_f^{h-1}(p)$ ($1 \leq h \leq \mu - 1$). Hereafter, we sometimes write $\alpha_f(p) = e^{i\theta_f(p)}$ and $\beta_f(p) = e^{-i\theta_f(p)}$.

In the case $\sigma > 1$, we deal with the value $L_{\mathbb{P}(q)}(\mathrm{Sym}_f^\gamma, \sigma + i\tau)$ as the limit of the value $L_{\mathcal{P}}(\mathrm{Sym}_f^\gamma, \sigma + i\tau)$ as \mathcal{P} tends to $\mathbb{P}(q)$. In fact, from (2.5) we have

$$\begin{aligned} & \log L_{\mathbb{P}(q)}(\mathrm{Sym}_f^\mu, \mathrm{Sym}_f^\nu, \sigma + i\tau) \\ &= \lim_{\substack{\mathcal{P} \rightarrow \mathbb{P}(q) \\ \mathcal{P} \subset \mathbb{P}(q)}} (g_{\sigma, \mathcal{P}}(\alpha_f^\mu(\mathcal{P}) \mathcal{P}^{-i\tau}) + g_{\sigma, \mathcal{P}}(\beta_f^\mu(\mathcal{P}) \mathcal{P}^{-i\tau})). \end{aligned} \tag{2.6}$$

In the case $1 \geq \sigma > 1/2$, we will prove the relation between $\log L_{\mathbb{P}(q)}(\mathrm{Sym}_f^\gamma, \sigma)$ and $\log L_{\mathcal{P}}(\mathrm{Sym}_f^\gamma, \sigma)$ with a suitable finite subset $\mathcal{P} \subset \mathbb{P}(q)$ depending on q^m and will consider the averages of them. This will be given in Lemma 5.1 in Section 5.

Now we conclude this section with the proof of Corollary 1.7.

Proof of Corollary 1.7. Write

$$A(q^m) = \sum'_{f \in \mathcal{S}_k(q^m)} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)).$$

The $\text{Avg}_{\text{prime}}$ part of the theorem implies that, for any $\varepsilon > 0$, there exists a $Q_0 = Q_0(\varepsilon)$ for each fixed m such that

$$\left| A(q^m) - \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Psi(u) \frac{du}{\sqrt{2\pi}} \right| < \varepsilon$$

for any prime $q > Q_0$. This clearly implies the first part of the corollary. As for the second part, first note that $\pi^*(X)$ in the denominator can be replaced by $\pi(X)$, because $\lim_{X \rightarrow \infty} \pi(X)/\pi^*(X) = 1$. We divide the sum as

$$\begin{aligned} & \frac{1}{\pi(X)} \sum_{\substack{q^m \leq X \\ q: \text{prime} \\ m \geq 1}} A(q^m) \\ &= \frac{1}{\pi(X)} \sum_{q \leq X} A(q) + \frac{1}{\pi(X)} \sum_{2 \leq m \leq \lfloor \log X / \log 2 \rfloor} \sum_{q \leq X^{1/m}} A(q^m). \end{aligned} \quad (2.7)$$

Using (2.4) and the fact that Ψ is bounded, we find that $A(q^m)$ is bounded. Hence the second term on the right-hand side of (2.7) is

$$\ll \frac{1}{\pi(X)} \sum_m \pi(X^{1/m}) \leq \frac{1}{\pi(X)} \sum_m \pi(X^{1/2}) \ll X^{-1/2} \log X,$$

which tends to 0 as $X \rightarrow \infty$. Lastly we apply the case $m = 1$ of the first part of the corollary to the first term on the right-hand side of (2.7) to find that it tends to the desired integral. \square

3. The density function \mathcal{M}_σ

Now we start the proof of our main theorem. In this section we first construct the density function $\mathcal{M}_{\sigma, \mathcal{P}}$ for a finite set $\mathcal{P} \subset \mathbb{P}(q)$. By $|\mathcal{P}|$ we denote the number of the elements of \mathcal{P} .

Proposition 3.1. *For any $\sigma > 0$, there exists a non-negative function $\mathcal{M}_{\sigma, \mathcal{P}}$ defined on \mathbb{R} which satisfies following two properties.*

- The support of $\mathcal{M}_{\sigma, \mathcal{P}}$ is compact.
- For any continuous function Ψ on \mathbb{R} , we have

$$\int_{\mathbb{R}} \mathcal{M}_{\sigma, \mathcal{P}}(u) \Psi(u) \frac{du}{\sqrt{2\pi}} = \int_{\mathcal{T}_{\mathcal{P}}} \Psi(2\Re(g_{\sigma, \mathcal{P}}(t_{\mathcal{P}}))) d^* t_{\mathcal{P}},$$

where $d^* t_{\mathcal{P}}$ is the normalized Haar measure of $\mathcal{T}_{\mathcal{P}}$. In particular, taking $\Psi \equiv 1$, we have $\int_{\mathbb{R}} \mathcal{M}_{\sigma, \mathcal{P}}(u) \frac{du}{\sqrt{2\pi}} = 1$.

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Proof. We construct the function $\mathcal{M}_{\sigma, \mathcal{P}}$ by using the method similar to that in Ihara and the first author [7].

In the case $|\mathcal{P}| = 1$ namely $\mathcal{P} = \{p\}$, we define a one-to-one correspondence from the open set $(-\pi, 0)$ to its image $A(\sigma, p) \subset \mathbb{R}$ by

$$u = u(\theta) = -2 \log |1 - e^{i\theta} p^{-\sigma}|$$

for $\theta \in (-\pi, 0)$. In fact, since

$$\frac{du}{d\theta} = -\frac{2p^{-\sigma} \sin(\theta)}{|1 - e^{i\theta} p^{-\sigma}|^2}, \quad (3.1)$$

we see that u is monotonically increasing with respect to θ , hence one to one. The definition of $\mathcal{M}_{\sigma, \mathcal{P}} = \mathcal{M}_{\sigma, p}$ is

$$\mathcal{M}_{\sigma, p}(u) = \begin{cases} \frac{|1 - e^{i\theta} p^{-\sigma}|^2}{-\sqrt{2\pi} \sin(\theta) p^{-\sigma}} & u \in A(\sigma, p), \\ 0 & \text{otherwise.} \end{cases}$$

This function satisfies the properties of Proposition 3.1. In fact, using (3.1) we obtain

$$\begin{aligned} & \int_{\mathbb{R}} \Psi(u) \mathcal{M}_{\sigma, p}(u) \frac{du}{\sqrt{2\pi}} = \int_{A(\sigma, p)} \Psi(u) \mathcal{M}_{\sigma, p}(u) \frac{du}{\sqrt{2\pi}} \\ &= \lim_{t_1, t_2 \rightarrow 0} \int_{-\pi+t_1}^{-t_2} \Psi(-2 \log |1 - e^{i\theta} p^{-\sigma}|) \\ & \quad \times \frac{|1 - e^{i\theta} p^{-\sigma}|^2}{(-\sqrt{2\pi} \sin \theta p^{-\sigma})} \cdot \frac{(-2 \sin(\theta) p^{-\sigma})}{|1 - e^{i\theta} p^{-\sigma}|^2} \cdot \frac{d\theta}{\sqrt{2\pi}} \\ &= \lim_{t_1, t_2 \rightarrow 0} \frac{1}{\pi} \int_{-\pi+t_1}^{-t_2} \Psi(-2 \log |1 - e^{i\theta} p^{-\sigma}|) d\theta \\ &= \frac{1}{\pi} \int_0^\pi \Psi(-2 \log |1 - e^{i\theta} p^{-\sigma}|) d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^\pi \Psi(-2 \log |1 - e^{i\theta} p^{-\sigma}|) d\theta \\ &= \int_{\mathcal{T}_p} \Psi(-2 \log |1 - t_p p^{-\sigma}|) d^* t_p \\ &= \int_{\mathcal{T}_p} \Psi(2\Re(g_{\sigma, p}(t_p))) d^* t_p. \end{aligned}$$

In the case $|\mathcal{P}| > 1$, we construct the function $\mathcal{M}_{\sigma, \mathcal{P}}$ by the convolution product of $\mathcal{M}_{\sigma, \mathcal{P}'}$ and $\mathcal{M}_{\sigma, p}$ for $\mathcal{P} = \mathcal{P}' \cup \{p\}$ inductively, that is

$$\mathcal{M}_{\sigma, \mathcal{P}}(u) = \int_{\mathbb{R}} \mathcal{M}_{\sigma, \mathcal{P}'}(u') \mathcal{M}_{\sigma, p}(u - u') \frac{du'}{\sqrt{2\pi}}.$$

It is easy to show that this function satisfies the statements of Proposition 3.1. \square

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Secondly, for the purpose of considering $\lim_{|\mathcal{P}| \rightarrow \infty} \mathcal{M}_{\sigma, \mathcal{P}}$, we define the Fourier transform of $\mathcal{M}_{\sigma, \mathcal{P}}$.

When $\mathcal{P} = \{p\}$, we define $\widetilde{\mathcal{M}}_{\sigma, p}$ by

$$\widetilde{\mathcal{M}}_{\sigma, p}(x) = \int_{\mathbb{R}} \mathcal{M}_{\sigma, p}(u) \psi_x(u) \frac{du}{\sqrt{2\pi}},$$

where $\psi_x(u) = e^{ixu}$. (The Fourier transform is sometimes defined by using e^{-ixu} instead of e^{ixu} , but here we follow the notation in [5] and [7].) As Ihara and the first author discussed in p.644 of [7], we can show

$$\widetilde{\mathcal{M}}_{\sigma, p}(x) = O\left((1 + |x|)^{-1/2}\right)$$

by using the Jessen-Wintner Theorem [11]. We define $\widetilde{\mathcal{M}}_{\sigma, \mathcal{P}}(x)$ by

$$\widetilde{\mathcal{M}}_{\sigma, \mathcal{P}}(x) = \prod_{p \in \mathcal{P}} \widetilde{\mathcal{M}}_{\sigma, p}(x).$$

Then we have

$$\widetilde{\mathcal{M}}_{\sigma, \mathcal{P}}(x) \ll \left((1 + |x|)^{-|\mathcal{P}|/2}\right) \quad (3.2)$$

from the above estimate of $\widetilde{\mathcal{M}}_{\sigma, p}$. On the other hand, we have the trivial bound

$$\left|\widetilde{\mathcal{M}}_{\sigma, \mathcal{P}}(x)\right| \leq 1. \quad (3.3)$$

We can show the following properties (a), (b) and (c). The proofs of them are also similar to [7], pp.645-646.

- (1) $\widetilde{\mathcal{M}}_{\sigma, \mathcal{P}}(x) \in L^t$ ($t \in [1, \infty]$).
- (2) For any subsets \mathcal{P}' and \mathcal{P} of $\mathbb{P}(q)$ with $\mathcal{P}' \subset \mathcal{P}$, from (3.3) we can see

$$\left|\widetilde{\mathcal{M}}_{\sigma, \mathcal{P}}(x)\right| \leq \left|\widetilde{\mathcal{M}}_{\sigma, \mathcal{P}'}(x)\right|.$$

- (3) Let $y \in \mathbb{N}$, and put $\mathcal{P}_y = \{p \in \mathbb{P}(q) \mid p \leq y\} \subset \mathbb{P}(q)$. We can show the existence of $\lim_{y \rightarrow \infty} \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}_y}(x)$ for $\sigma > 1/2$. We denote it by $\widetilde{\mathcal{M}}_{\sigma}(x)$. For any $a > 0$, this convergence is uniform on $|x| \leq a$.

These properties yield the next proposition which is the analogue of Proposition 3.4 in [7].

Proposition 3.2. *For $\varepsilon > 0$ and $\sigma \geq 1/2 + \varepsilon$, there exists*

$$\widetilde{\mathcal{M}}_{\sigma}(x) = \lim_{y \rightarrow \infty} \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}_y}(x),$$

whose convergence is uniform in $x \in \mathbb{R}$. For each $\sigma > 1/2$, the above convergence is L^t -convergence and the function $\widetilde{\mathcal{M}}_{\sigma}(x)$ belongs to L^t ($1 \leq t \leq \infty$).

By using (3.2) and (3.3), we have

$$\widetilde{\mathcal{M}}_{\sigma}(x) = O\left((1 + |x|)^{-n/2}\right) \quad (3.4)$$

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for any $n \in \mathbb{N}$. We also have

$$|\widetilde{\mathcal{M}}_\sigma(x)| \leq 1. \quad (3.5)$$

Finally, we define the function $\mathcal{M}_\sigma(u)$. For any finite set $\mathcal{P} \subset \mathbb{P}(q)$, we have

$$\int_{\mathbb{R}} \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}}(x) \psi_{-u}(x) \frac{dx}{\sqrt{2\pi}} = \mathcal{M}_{\sigma, \mathcal{P}}(u).$$

This is the Fourier inverse transform. We define

$$\mathcal{M}_\sigma(u) = \int_{\mathbb{R}} \widetilde{\mathcal{M}}_\sigma(x) \psi_{-u}(x) \frac{dx}{\sqrt{2\pi}},$$

where we can see that the right-hand side of this equation is convergent by using (3.4).

Proposition 3.3. *For $\sigma > 1/2$, the function \mathcal{M}_σ satisfies following five properties.*

- $\lim_{y \rightarrow \infty} \mathcal{M}_{\sigma, \mathcal{P}_y}(u) = \mathcal{M}_\sigma(u)$ and this convergence is uniform in u .
- The function $\mathcal{M}_\sigma(u)$ is continuous in u and non-negative.
- $\lim_{u \rightarrow \infty} \mathcal{M}_\sigma(u) = 0$.
- The functions $\mathcal{M}_\sigma(u)$ and $\widetilde{\mathcal{M}}_\sigma(x)$ are Fourier duals of each other.
- $\int_{\mathbb{R}} \mathcal{M}_\sigma(u) \frac{du}{\sqrt{2\pi}} = 1$.

This is the analogue of Proposition 3.5 in [7] and the proof is similar.

4. The key lemma

For a fixed $\sigma > 1/2$, $\tau \in \mathbb{R}$ and a finite set $\mathcal{P} \subset \mathbb{P}(q)$, we put

$$\Phi_{\sigma, \tau, \mathcal{P}}(t_{\mathcal{P}}, t'_{\mathcal{P}}) = \sum_{p \in \mathcal{P}} (g_{\sigma, p}(t_p p^{-i\tau}) + g_{\sigma, p}(t'_p p^{-i\tau})),$$

where $t_{\mathcal{P}} = (t_p)_{p \in \mathcal{P}}$, $t'_{\mathcal{P}} = (t'_p)_{p \in \mathcal{P}} \in \mathcal{T}_{\mathcal{P}}$. From (2.5) we see that

$$\psi_x \circ \Phi_{\sigma, \tau, \mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P})) = \psi_x(\log L_{\mathcal{P}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma + i\tau)),$$

where $\psi_x(u) = \exp(ixu)$. Therefore, to prove our Theorem 1.5, it is important to consider two averages

$$\text{avg}_{\text{prime}}(\psi_x \circ \Phi_{\sigma, \tau, \mathcal{P}}) = \lim_{\substack{q \rightarrow \infty \\ q: \text{prime} \\ m: \text{fix}}} \sum'_{f \in S_k(q^m)} \psi_x \circ \Phi_{\sigma, \tau, \mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P}))$$

and

$$\text{avg}_{\text{power}}(\psi_x \circ \Phi_{\sigma, \tau, \mathcal{P}}) = \lim_{\substack{m \rightarrow \infty \\ q: \text{fixed prime} \\ (\text{If } 1 \geq \sigma > 1/2, q \geq Q(\mu))}} \sum'_{f \in S_k(q^m)} \psi_x \circ \Phi_{\sigma, \tau, \mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P})).$$

Our first aim in this section is to show the following

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Lemma 4.1. *Let $\mu > \nu \geq 1$ be integers with $\mu - \nu = 2$ and \mathcal{P} be a finite subset of $\mathbb{P}(q)$. In the case $2 \leq k < 12$ or $k = 14$, we have*

$$\begin{aligned} \text{avg}_{\mathfrak{S}_{\text{prime}}}(\psi_x \circ \Phi_{\sigma,\tau,\mathcal{P}}) &= \text{avg}_{\mathfrak{S}_{\text{power}}}(\psi_x \circ \Phi_{\sigma,\tau,\mathcal{P}}) \\ &= \int_{\mathcal{T}_{\mathcal{P}}} \psi_x(\Phi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t_{\mathcal{P}}^{-1})) d^*t_{\mathcal{P}}. \end{aligned}$$

The above convergence is uniform in $|x| \leq R$ for any $R > 0$.

Proof. Let $1 > \varepsilon' > 0$. Considering the Taylor expansion we find that there exist an $M_p = M_p(\varepsilon', R) \in \mathbb{N}$ and $d_{m_p} \in \mathbb{C}$ ($0 \leq m_p \leq M_p$) such that $\psi_x \circ g_{\sigma,p}$ can be approximated by a polynomial as

$$\left| \psi_x \circ g_{\sigma,p}(t_p) - \sum_{m_p=0}^{M_p} d_{m_p} t_p^{m_p} \right| < \varepsilon', \quad (4.1)$$

uniformly on \mathcal{T} with respect to t_p and also on $|x| \leq R$ with respect to x . Replacing t_p by $t_p p^{-i\tau}$, we have

$$\left| \psi_x \circ g_{\sigma,p}(t_p p^{-i\tau}) - \sum_{m_p=0}^{M_p} c_{m_p} t_p^{m_p} \right| < \varepsilon',$$

where $c_{m_p} = d_{m_p} p^{-i\tau m_p}$. Write

$$\Psi_{\sigma,\tau,p}(t_p; M_p) = \sum_{m_p=0}^{M_p} c_{m_p} t_p^{m_p}$$

and define

$$\Psi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t'_{\mathcal{P}}; M_{\mathcal{P}}) = \prod_{p \in \mathcal{P}} \Psi_{\sigma,\tau,p}(t_p; M_p) \Psi_{\sigma,\tau,p}(t'_p; M_p),$$

where $M_{\mathcal{P}} = (M_p)_{p \in \mathcal{P}}$.

Let $\varepsilon'' > 0$. Choosing ε' (depending on $|\mathcal{P}|$ and ε'') sufficiently small, we obtain

$$|\psi_x \circ \Phi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t'_{\mathcal{P}}) - \Psi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t'_{\mathcal{P}}; M_{\mathcal{P}})| < \varepsilon'', \quad (4.2)$$

again uniformly on \mathcal{T} with respect to t_p and also on $|x| \leq R$ with respect to x . In fact, since

$$\begin{aligned} &\psi_x \circ \Phi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t'_{\mathcal{P}}) \\ &= \prod_{p \in \mathcal{P}} \psi_x(g_{\sigma,p}(t_p p^{-i\tau})) \psi_x(g_{\sigma,p}(t'_p p^{-i\tau})) \\ &= \prod_{p \in \mathcal{P}} (\Psi_{\sigma,\tau,p}(t_p; M_p) + O(\varepsilon')) (\Psi_{\sigma,\tau,p}(t'_p; M_p) + O(\varepsilon')) \\ &= \Psi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t'_{\mathcal{P}}; M_{\mathcal{P}}) + (\text{remainder terms}), \end{aligned}$$

we obtain (4.2).

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The first step of the proof of the lemma is to express the average of the value of $\psi_x \circ \Phi_{\sigma,\tau,\mathcal{P}}$ by using $\Psi_{\sigma,\tau,\mathcal{P}}$. Let $\varepsilon > 0$. From (4.2) with $t_{\mathcal{P}} = \alpha_f^\mu(\mathcal{P})$, $t'_{\mathcal{P}} = \beta_f^\mu(\mathcal{P})$, $\varepsilon'' = \varepsilon/2$ and (2.4), we have

$$\left| \sum'_{f \in S_k(q^m)} \psi_x \circ \Phi_{\sigma,\tau,\mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P})) - \sum'_{f \in S_k(q^m)} \Psi_{\sigma,\tau,\mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P}); M_{\mathcal{P}}) \right| < \sum'_{f \in S_k(q^m)} \varepsilon'' = \frac{\varepsilon}{2}(1 + O(E(q^m))). \quad (4.3)$$

Therefore, if q^m is sufficiently large, from (2.3) we see that

$$\left| \sum'_{f \in S(q^m)} \psi_x \circ \Phi_{\sigma,\tau,\mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P})) - \sum'_{f \in S(q^m)} \Psi_{\sigma,\tau,\mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P}); M_{\mathcal{P}}) \right| < \varepsilon. \quad (4.4)$$

As the second step, we calculate $\Psi_{\sigma,\tau,\mathcal{P}}$ as follows;

$$\begin{aligned} & \Psi_{\sigma,\tau,\mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P}); M_{\mathcal{P}}) \\ &= \prod_{p \in \mathcal{P}} \left(\sum_{m_p=0}^{M_p} c_{m_p} e^{\mu i m_p \theta_f(p)} \right) \left(\sum_{n_p=0}^{M_p} c_{n_p} e^{-\mu i n_p \theta_f(p)} \right) \\ &= \prod_{p \in \mathcal{P}} \left(\sum_{m_p=0}^{M_p} c_{m_p}^2 \right. \\ & \quad + \sum_{m_p=0}^{M_p} \sum_{\substack{n_p=0 \\ m_p < n_p}}^{M_p} c_{m_p} e^{\mu i m_p \theta_f(p)} c_{n_p} e^{-\mu i n_p \theta_f(p)} \\ & \quad \left. + \sum_{m_p=0}^{M_p} \sum_{\substack{n_p=0 \\ m_p > n_p}}^{M_p} c_{m_p} e^{\mu i m_p \theta_f(p)} c_{n_p} e^{-\mu i n_p \theta_f(p)} \right) \\ &= \prod_{p \in \mathcal{P}} \left(\sum_{m_p=0}^{M_p} c_{m_p}^2 \right. \\ & \quad \left. + \sum_{m_p=0}^{M_p} \sum_{\substack{n_p=0 \\ m_p < n_p}}^{M_p} c_{m_p} c_{n_p} (e^{\mu i (m_p - n_p) \theta_f(p)} + e^{\mu i (n_p - m_p) \theta_f(p)}) \right). \end{aligned} \quad (4.5)$$

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We put $n_p - m_p = r_p$. Using (1.1), we see that

$$\begin{aligned}
 & \Psi_{\sigma, \tau, \mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P}); M_{\mathcal{P}}) \\
 &= \prod_{p \in \mathcal{P}} \left(\sum_{m_p=0}^{M_p} c_{m_p}^2 + \sum_{r_p=1}^{M_p} \sum_{n_p=r_p}^{M_p} c_{n_p-r_p} c_{n_p} (e^{\mu i r_p \theta_f(p)} + e^{-\mu i r_p \theta_f(p)}) \right) \\
 &= \prod_{p \in \mathcal{P}} \left(\sum_{m_p=0}^{M_p} c_{m_p}^2 + \sum_{r_p=1}^{M_p} \sum_{n_p=r_p}^{M_p} c_{n_p-r_p} c_{n_p} \left((e^{\mu i r_p \theta_f(p)} \right. \right. \\
 &\quad \left. \left. + \sum_{\ell=1}^{\mu r_p-1} e^{i(\mu r_p-2\ell)\theta_f(p)} + e^{-\mu i r_p \theta_f(p)} \right) - \sum_{\ell=1}^{\mu r_p-1} e^{i(\mu r_p-2\ell)\theta_f(p)} \right) \\
 &= \prod_{p \in \mathcal{P}} \left(\sum_{m_p=0}^{M_p} c_{m_p}^2 + \sum_{r_p=1}^{M_p} \sum_{n_p=r_p}^{M_p} c_{n_p-r_p} c_{n_p} (\lambda_f(p^{\mu r_p}) - \lambda_f(p^{\mu r_p-2})) \right).
 \end{aligned}$$

Since $\mu = \nu + 2 \geq 3$, by using (2.1), we obtain

$$\begin{aligned}
 & \sum'_{f \in S(q^m)} \Psi_{\sigma, \tau, \mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P}); M_{\mathcal{P}}) \\
 &= \sum'_{f \in S_k(q^m)} \prod_{p \in \mathcal{P}} \left(\sum_{m_p=0}^{M_p} c_{m_p}^2 + \sum_{r_p=1}^{M_p} \sum_{n_p=r_p}^{M_p} c_{n_p-r_p} c_{n_p} (\lambda_f(p^{\mu r_p}) - \lambda_f(p^{\mu r_p-2})) \right) \\
 &= \prod_{p \in \mathcal{P}} \sum_{m_p=0}^{M_p} c_{m_p}^2 + O(E(q^m)),
 \end{aligned}$$

where the implied constant of the error term depends on \mathcal{P} , μ and $M_{\mathcal{P}} = M_{\mathcal{P}}(\varepsilon', R)$ (hence depends on ε under the above choice of ε'). But still, this error term can be smaller than ε for sufficiently large q^m . Combining this with (4.4), we obtain

$$\left| \sum'_{f \in S(q^m)} \psi_x \circ \Phi_{\sigma, \tau, \mathcal{P}}(\alpha_f(\mathcal{P})^\mu, \beta_f(\mathcal{P})^\mu) - \prod_{p \in \mathcal{P}} \sum_{m_p=0}^{M_p} c_{m_p}^2 \right| < 2\varepsilon. \quad (4.6)$$

As the final step, we calculate the integral in the statement of Lemma 4.1. For

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any $\varepsilon > 0$, using (4.2), we have

$$\begin{aligned}
 & \int_{\mathcal{T}_{\mathcal{P}}} \psi_x(\Phi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t_{\mathcal{P}}^{-1})) d^* t_{\mathcal{P}} \\
 &= \int_{\mathcal{T}_{\mathcal{P}}} (\psi_x(\Phi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t_{\mathcal{P}}^{-1})) - \Psi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t_{\mathcal{P}}^{-1}; M_{\mathcal{P}}) + \Psi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t_{\mathcal{P}}^{-1}; M_{\mathcal{P}})) d^* t_{\mathcal{P}} \\
 &= \int_{\mathcal{T}_{\mathcal{P}}} \Psi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t_{\mathcal{P}}^{-1}; M_{\mathcal{P}}) d^* t_{\mathcal{P}} + O(\varepsilon) \\
 &= \int_{\mathcal{T}_{\mathcal{P}}} \prod_{p \in \mathcal{P}} \left(\sum_{m_p=0}^{M_p} c_{m_p} t_p^{m_p} \right) \left(\sum_{n_p=0}^{M_p} c_{n_p} t_p^{-n_p} \right) d^* t_{\mathcal{P}} + O(\varepsilon) \\
 &= \prod_{p \in \mathcal{P}} \sum_{m_p=0}^{M_p} c_{m_p}^2 + O(\varepsilon). \tag{4.7}
 \end{aligned}$$

From (4.6) and (4.7) we find that the identity in the statement of Lemma 4.1 holds with the error $O(\varepsilon)$, but this error can be arbitrarily small, so the assertion of Lemma 4.1 follows. \square

Remark 4.2. In the above proof, the function $\Psi_{\sigma,\tau,\mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P}); M_{\mathcal{P}})$ is expressed by

$$e^{\mu i m_p \theta_f(p)} e^{-\mu i n_p \theta_f(p)}, \quad (m_p, n_p \geq 0, \mu \geq 3)$$

(see (4.5)). When $m_p \neq n_p$ these are written by $\lambda_f(p^{\mu r_p})$ and $\lambda_f(p^{\mu r_p - 2})$ ($r_p \geq 1$) and, as shown above, they are included in the error terms by (2.1). If we try to study averages of $\log L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, s)$ itself (without considering the difference) by the same method as in this paper, we have to handle the terms of the form

$$e^{i m_p \theta_f(p)} e^{-i n_p \theta_f(p)} \quad (m_p - n_p = \pm 2).$$

However, these terms produce other “main” terms by (2.4), since $\alpha_f^2(p) + \beta_f^2(p) = \lambda_f(p^2) - 1$. This invalidates the above argument, so our method, as it is, cannot be applied to $\log L(\text{Sym}_f^\gamma, s)$.

When $\tau = 0$, Proposition 3.1 and Lemma 4.1 imply

$$\begin{aligned}
 & \text{avg}_{\text{prime}}(\psi_x \circ \Phi_{\sigma,0,\mathcal{P}}) = \text{avg}_{\text{power}}(\psi_x \circ \Phi_{\sigma,0,\mathcal{P}}) \\
 &= \int_{\mathcal{T}_{\mathcal{P}}} \psi_x(\Phi_{\sigma,0,\mathcal{P}}(t_{\mathcal{P}}, t_{\mathcal{P}}^{-1})) d^* t_{\mathcal{P}} \\
 &= \int_{\mathcal{T}_{\mathcal{P}}} \psi_x \left(\sum_{p \in \mathcal{P}} (g_{\sigma,p}(t_p) + g_{\sigma,p}(t_p^{-1})) \right) d^* t_{\mathcal{P}} \\
 &= \int_{\mathcal{T}_{\mathcal{P}}} \psi_x(g_{\sigma,\mathcal{P}}(t_{\mathcal{P}}) + \overline{g_{\sigma,\mathcal{P}}(t_{\mathcal{P}})}) d^* t_{\mathcal{P}} \\
 &= \int_{\mathcal{T}_{\mathcal{P}}} \psi_x(2\Re(g_{\sigma,\mathcal{P}}(t_{\mathcal{P}}))) d^* t_{\mathcal{P}} = \int_{\mathbb{R}} \mathcal{M}_{\sigma,\mathcal{P}}(u) \psi_x(u) \frac{du}{\sqrt{2\pi}}, \tag{4.8}
 \end{aligned}$$

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uniformly in $|x| \leq R$. This fact deduces the case $\sigma > 1$ of the following key lemma.

Lemma 4.3. *Let $\mu > \nu \geq 1$ be integers with $\mu - \nu = 2$. Suppose Assumption 1.1 and 1.4. In the case $2 \leq k < 12$ or $k = 14$, for $\sigma > 1/2$ and $\psi_x(u) = \exp(ixu)$, we have*

$$\begin{aligned} & \text{Avg}_{\text{prime}} \psi_x(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \text{Avg}_{\text{power}} \psi_x(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \psi_x(u) \frac{du}{\sqrt{2\pi}} \end{aligned}$$

The above convergence is uniform in $|x| \leq R$ for any $R > 0$.

We note that this lemma is actually a special case $\Psi = \psi_x$ in our main Theorem 1.5. To show this lemma is the main body of the proof of Theorem 1.5.

Proof in the case $\sigma > 1$. Since $\sigma > 1$, we find a sufficiently large finite subset $\mathcal{P} \subset \mathbb{P}(q)$ for which it holds that

$$|L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, s) - L_{\mathcal{P}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, s)| < \varepsilon$$

and $|\widetilde{\mathcal{M}}_{\sigma, \mathcal{P}}(x) - \widetilde{\mathcal{M}}_\sigma(x)| < \varepsilon$ for any $x \in \mathbb{R}$ and any $\varepsilon > 0$. The last inequality is provided by Proposition 3.2. We can choose the above \mathcal{P} which does not depend on q^m . Using this \mathcal{P} , we have

$$\begin{aligned} & \left| \sum'_{f \in S_k(q^m)} \psi_x(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)) - \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \psi_x(u) \frac{du}{\sqrt{2\pi}} \right| \\ & \leq \left| \sum'_{f \in S_k(q^m)} \left(\psi_x(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)) - \psi_x(\log L_{\mathcal{P}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)) \right) \right| \\ & \quad + \left| \sum'_{f \in S_k(q^m)} \psi_x(\log L_{\mathcal{P}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)) - \int_{\mathbb{R}} \mathcal{M}_{\sigma, \mathcal{P}}(u) \psi_x(u) \frac{du}{\sqrt{2\pi}} \right| \\ & \quad + \left| \int_{\mathbb{R}} \mathcal{M}_{\sigma, \mathcal{P}}(u) \psi_x(u) \frac{du}{\sqrt{2\pi}} - \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \psi_x(u) \frac{du}{\sqrt{2\pi}} \right| \\ & = S_1 + S_2 + S_3, \end{aligned}$$

say. We remind the relation

$$|\psi_x(u) - \psi_x(u')| \ll |x| \cdot |u - u'| \tag{4.9}$$

for $u \in \mathbb{R}$ (see Ihara [5] (6.5.19) or Ihara-Matsumoto [7]). We see that

$$S_1 \ll |x| \sum'_{f \in S_k(q^m)} \left(|\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma) - \log L_{\mathcal{P}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)| \right)$$

and

$$S_3 = \left| \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}}(x) - \widetilde{\mathcal{M}}_\sigma(x) \right|.$$

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Therefore $|S_1|$ and $|S_3|$ are $O(\varepsilon)$ for large $|\mathcal{P}|$, with the implied constant depending on R . As for the estimate on $|S_2|$, we use (4.8), whose convergence is uniform on $|x| \leq R$. This completes the proof. \square

In the next two sections we will give the proof of Lemma 4.3 when $1 \geq \sigma > 1/2$.

5. The approximation of $L_{\mathbb{P}(q)}$ under GRH

In this section, we suppose Assumptions 1.1 and 1.4. This section is the first step of the proof of Lemma 4.3 for $1 \geq \sigma > 1/2$.

In this section, we study the approximation of $L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, s)$ by $L_{\mathcal{P}}(\text{Sym}_f^\gamma, s)$ with suitable \mathcal{P} which depends on the level of the primitive form f (see Lemma 5.1 below). Recall that the level of f is q^m and q is a prime number.

Let the sets $\mathcal{P}_{\log q^m}$ and $\mathcal{P}_{\log q^m}^+$ be defined by

$$\mathcal{P}_{\log q^m} = \{p \in \mathbb{P}(q) \mid p \leq \log q^m\}$$

and

$$\mathcal{P}_{\log q^m}^+ = \mathcal{P}_{\log q^m} \cup \{q\}.$$

Then

$$\log L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, s) = \log L(\text{Sym}_f^\gamma, s) + \log(1 - \lambda_f(q^\gamma)q^{-s}),$$

$$\log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\gamma, s) = \log L_{\mathcal{P}_{\log q^m}^+}(\text{Sym}_f^\gamma, s) + \log(1 - \lambda_f(q^\gamma)q^{-s})$$

and

$$\begin{aligned} & \log L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, s) - \log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\gamma, s) \\ &= \log L(\text{Sym}_f^\gamma, s) - \log L_{\mathcal{P}_{\log q^m}^+}(\text{Sym}_f^\gamma, s). \end{aligned}$$

Define

$$F(\text{Sym}_f^\gamma, s) = \frac{L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, s)}{L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\gamma, s)}.$$

Since

$$\begin{aligned} & \log L(\text{Sym}_f^\gamma, s) \\ &= -\log(1 - \lambda_f(q^\gamma)q^{-s}) - \sum_{p \neq q} \sum_{h=0}^{\gamma} \log(1 - \alpha_f^{\gamma-h}(p)\beta^h(p)p^{-s}) \\ &= \sum_{\ell=1}^{\infty} \frac{\lambda_f^\ell(q^\gamma)}{\ell q^{\ell s}} + \sum_{p \neq q} \sum_{h=0}^{\gamma} \sum_{\ell=1}^{\infty} \frac{\alpha_f^{(\gamma-h)\ell}(p)\beta^{h\ell}(p)}{\ell p^{\ell s}} \end{aligned}$$

for $\sigma > 1$, we can write

$$\log F(\text{Sym}_f^\gamma, s) = \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \sum_{\ell=1}^{\infty} \frac{c_{f,\gamma,p}(\ell)}{\ell p^{\ell s}} \quad (5.1)$$

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and

$$\frac{F'}{F}(\text{Sym}_f^\gamma, s) = - \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \sum_{\ell=1}^{\infty} \frac{c_{f,\gamma,p}(\ell) \log p}{p^{\ell s}} \quad (5.2)$$

for $\sigma > 1$, where the coefficients $c_{f,\gamma,p}$ are defined by

$$c_{f,\gamma,p}(\ell) = \sum_{h=0}^{\gamma} \alpha_f^{(\gamma-h)\ell}(p) \beta_f^{h\ell}(p), \quad (5.3)$$

for $p \neq q$. By Assumptions 1.1 and 1.4, the functions $\log L(\text{Sym}_f^\gamma, s)$ are holomorphic for $\sigma > 1/2$. By the argument of the proof of Lemma 3 in Duke [3], we obtain

$$\left| \frac{L'(\text{Sym}_f^\gamma, s)}{L(\text{Sym}_f^\gamma, s)} \right| \ll_{\epsilon,\gamma} \log q^m + \log(1 + |t|) \quad (5.4)$$

for $1/2 + \epsilon \leq \sigma \leq 2$ ($0 < \epsilon \leq 1$) from (1.2) and (1.3).

Now we restrict ourselves to the case $\gamma = \mu, \nu$. When $q \geq Q(\mu)$, we have

$$\begin{aligned} & \log L_{\mathcal{P}_{\log q^m}^+}(\text{Sym}_f^\gamma, s) + \log(1 - \lambda_f(q^\gamma)q^{-s}) \\ &= - \sum_{\substack{p \leq \log q^m \\ p \neq q}} \sum_{h=0}^{\gamma} \log(1 - \alpha_f^{\gamma-h}(p) \beta_f^h(p) p^{-s}) \end{aligned} \quad (5.5)$$

for $\sigma > 1/2$ and $\gamma = \mu, \nu$. Here, on the second term of the left-hand side of the above equation, since we know $\lambda_f(q) < d(q) = 2$ and $|\lambda_f(q^\gamma)q^{-s}| < 2^\gamma q^{-\sigma} < 2^\mu / \sqrt{q}$ from (1.1), we have $|\lambda_f(q^\gamma)q^{-s}| < 2^\mu / \sqrt{Q(\mu)} < 1$ and the above logarithm is well-defined. Differentiating the both sides of (5.5), we obtain

$$\begin{aligned} & \frac{L'_{\mathcal{P}_{\log q^m}^+}(\text{Sym}_f^\gamma, s)}{L_{\mathcal{P}_{\log q^m}^+}(\text{Sym}_f^\gamma, s)} \\ &= - \frac{\lambda_f(q^\gamma)q^{-s} \log q}{1 - \lambda_f(q^\gamma)q^{-s}} - \sum_{\substack{p \leq \log q^m \\ p \neq q}} \sum_{h=0}^{\gamma} \frac{\alpha_f^{\gamma-h}(p) \beta_f^h(p) p^{-s} \log p}{1 - \alpha_f^{\gamma-h}(p) \beta_f^h(p) p^{-s}}. \end{aligned}$$

The denominator $(1 - \lambda_f(q^\gamma)q^{-s})$ has a lower bound $1 - 2^\mu / \sqrt{Q(\mu)} > 0$ when $q \geq Q(\mu)$. Therefore the first term of the right-hand side of the above equation can be estimated by $q^{-1/2} \log q$. Hence, when $q \geq Q(\mu)$ and $\sigma > 1/2$, we have

$$\begin{aligned} \left| \frac{L'_{\mathcal{P}_{\log q^m}^+}(\text{Sym}_f^\gamma, s)}{L_{\mathcal{P}_{\log q^m}^+}(\text{Sym}_f^\gamma, s)} \right| & \ll q^{-1/2} \log q + \sum_{p \leq \log q^m} \frac{p^{-\sigma} \log p}{1 - p^{-\sigma}} \\ & \ll q^{-1/2} \log q + \sum_{p \leq \log q^m} \frac{\log p}{p^{1/2}} \\ & \ll q^{-1/2} \log q + (\log q^m)^{1/2} \\ & \ll (\log q^m)^{1/2} \end{aligned} \quad (5.6)$$

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by partial summation and the prime number theorem. From (5.4) and (5.6), we obtain

$$\frac{F'}{F}(\text{Sym}_f^\gamma, s) \ll \log q^m + \log(1 + |t|) \quad \left(\frac{1}{2} + \varepsilon \leq \sigma \leq 2\right), \quad (5.7)$$

under assumptions.

Now we assume $1/2 < \sigma \leq 1$, and put $\sigma = 1/2 + \delta$, $0 < \delta \leq 1/2$. The following argument is similar to the proof of Proposition 5 in Duke [3]. By Mellin's formula, we know

$$e^{-y} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} y^{-z} \Gamma(z) dz.$$

Therefore by (5.2), for $y > 0$, we have

$$\begin{aligned} & - \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \log p \sum_{\ell=1}^{\infty} \frac{c_{f,\gamma,p}(\ell)}{p^{\ell u}} e^{-p^\ell/x} \\ &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{F'}{F}(\text{Sym}_f^\gamma, u+z) x^z \Gamma(z) dz, \end{aligned}$$

where $u > 0$ and $x > 1$. Now assume $(1+\delta)/2 < u \leq 3/2$. By shifting the path of integration to $\Re z = (1+\delta)/2 - u$ and using (5.7), we have

$$\begin{aligned} & - \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \log p \sum_{\ell=1}^{\infty} \frac{c_{f,\gamma,p}(\ell)}{p^{\ell u}} e^{-p^\ell/x} \\ &= \frac{F'}{F}(\text{Sym}_f^\gamma, u) + O(x^{(1+\delta)/2-u} \log q^m). \end{aligned}$$

Integrating the above equation with respect to u from $\sigma = 1/2 + \delta$ to $3/2$ we obtain

$$\begin{aligned} & \log F(\text{Sym}_f^\gamma, 3/2) - \log F(\text{Sym}_f^\gamma, \sigma) = \int_{\sigma}^{3/2} \frac{F'}{F}(\text{Sym}_f^\gamma, u) du \\ &= - \int_{\sigma}^{3/2} \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \log p \sum_{\ell=1}^{\infty} \frac{c_{f,\gamma,p}(\ell)}{p^{\ell u}} e^{-p^\ell/x} du \\ & \quad + O\left(\int_{\sigma}^{3/2} x^{(1+\delta)/2-u} du \cdot \log q^m\right). \end{aligned}$$

Separating the term corresponding to $\ell = 1$ on the right-hand side, we see that

$$\begin{aligned} & \log F(\text{Sym}_f^\gamma, \sigma) - \log F(\text{Sym}_f^\gamma, 3/2) \\ &= - \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \frac{c_{f,\gamma,p}(1)}{p^{3/2}} e^{-p/x} + \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \frac{c_{f,\gamma,p}(1)}{p^\sigma} e^{-p/x} \\ & \quad + O_\delta \left(\sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \sum_{\ell=2}^{\infty} \frac{1}{\ell p^{\ell\sigma}} e^{-p^\ell/x} + \frac{x^{1/2-\sigma+\delta/2} \log q^m}{\log x} \right), \quad (5.8) \end{aligned}$$

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because from (5.3) we see that $c_{f,\gamma,p}(\ell) = O(\gamma)$.

On the right-hand side of (5.8), we have

$$\begin{aligned} & \sum_{p \geq \log q^m} \sum_{\ell=2}^{\infty} \frac{1}{\ell p^{\ell\sigma}} e^{-p^\ell/x} \\ & \ll \sum_{\ell=2}^{\infty} \frac{1}{\ell} \sum_{p \geq \log q^m} \frac{1}{p^{\ell/2+\ell\delta}} < \sum_{p \geq \log q^m} \frac{1}{p^{1+2\delta}} + \sum_{\ell=3}^{\infty} \frac{1}{\ell} \sum_{p \geq \log q^m} \frac{1}{p^{\ell/2+\ell\delta}} \\ & \ll \frac{1}{(\log q^m)^\delta} + \sum_{\ell=3}^{\infty} \frac{1}{\ell^2} \sum_{p \geq \log q^m} \frac{1}{p^{\ell/3+\ell\delta}} \ll \frac{1}{(\log q^m)^\delta} \end{aligned} \quad (5.9)$$

and

$$\sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \frac{c_{f,\gamma,p}(1)}{p^{3/2}} e^{-p/x} \ll \sum_{p > \log q^m} \frac{1}{p^{3/2}} \ll \sum_{n > \log q^m} \frac{1}{n^{3/2}} \ll (\log q^m)^{-1/2}.$$

Moreover, using (5.1) we have

$$\begin{aligned} |\log F(\text{Sym}_f^\gamma, 3/2)| & \ll \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \sum_{\ell=1}^{\infty} \frac{1}{\ell p^{3\ell/2}} \\ & \ll \sum_{\ell=1}^{\infty} \frac{1}{\ell^2} \sum_{p \geq \log q^m} \frac{1}{p^{3\ell/2-\ell/4}} < \sum_{\log q^m < n} \frac{1}{n^{5/4}} \ll (\log q^m)^{-1/4}. \end{aligned} \quad (5.10)$$

Letting $x = q^{m/4(k-1)\gamma}$ we obtain

$$\begin{aligned} & \log F(\text{Sym}_f^\gamma, \sigma) - \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \frac{c_{f,\gamma,p}(1)}{p^\sigma} e^{-p/q^{m/4(k-1)\gamma}} \\ & = O_{\delta,k,\gamma} \left((\log q^m)^{-\delta} + (\log q^m)^{-1/4} + (q^{m/4(k-1)\gamma})^{-\delta/2} \right). \end{aligned}$$

Since it is easy to see that $c_{f,\gamma,p}(1) = \lambda_f(p^\gamma)$ from (1.1) and (5.3), we now obtain the following lemma.

Lemma 5.1. *Suppose Assumptions 1.1 and 1.4. Let $Q(\mu)$ be the smallest prime number satisfying $2^\mu/\sqrt{Q(\mu)} < 1$ and f be a primitive form in $S_k(q^m)$, where $q > Q(\mu)$ is a prime. For fixed γ and $\sigma = 1/2 + \delta$ ($0 < \delta \leq 1/2$), we have*

$$\begin{aligned} & \log L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, \sigma) - \log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\gamma, \sigma) - \mathcal{S}_\gamma \\ & \ll (\log q^m)^{-\delta} + (\log q^m)^{-1/4} + (q^{m/4(k-1)\gamma})^{-\delta/2}, \end{aligned} \quad (5.11)$$

where

$$\mathcal{S}_\gamma = \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \frac{\lambda_f(p^\gamma)}{p^\sigma} e^{-p/q^{m/(k-1)\gamma}}.$$

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Remark 5.2. We can obtain a better estimate than (5.11). Instead of (5.9), we use

$$\begin{aligned} \sum_{p \geq \log q^m} \frac{1}{p^{1+2\delta}} &< \sum_{n \geq \log q^m} \frac{1}{n^{1+2\delta}} \ll \frac{1}{2\delta} (\log q^m)^{-2\delta}, \\ \sum_{\ell=3}^{\infty} \frac{1}{\ell} \sum_{p \geq \log q^m} \frac{1}{p^{\ell/2+\ell\delta}} &< \sum_{\ell=3}^{\infty} \frac{1}{\ell} \sum_{p \geq \log q^m} \frac{1}{p^{3/2} p^{\ell\delta}} \ll \sum_{\ell=3}^{\infty} \frac{1}{\ell} \sum_{p \geq \log q^m} \frac{1}{p^{3/2} \ell \delta \log p} \\ &\ll \sum_{p \geq \log q^m} \frac{1}{p^{3/2} \delta \log p} \ll \frac{1}{\delta \log \log q^m} \sum_{n \geq \log q^m} \frac{1}{n^{3/2}} \\ &\ll \frac{1}{\delta \log \log q^m \sqrt{\log q^m}}. \end{aligned}$$

Instead of (5.10), we use

$$\begin{aligned} \sum_{p \in \mathbb{P}(q)} \sum_{\mathcal{P}_{\log q^m}} \sum_{\ell=1}^{\infty} \frac{1}{\ell p^{3\ell/2}} &\ll \sum_{p \geq \log q^m} \frac{1}{p^{3/2}} + \sum_{p \geq \log q^m} \sum_{\ell \geq 2} \frac{1}{p^{3\ell/2}} \\ &\ll \sum_{n \geq \log q^m} \frac{1}{n^{3/2}} + \sum_{p \geq \log q^m} \frac{1}{p^{3/2}(p^{3/2}-1)} \\ &\ll \frac{1}{\sqrt{\log q^m}} + \sum_{p \geq \log q^m} \frac{1}{p^3} \\ &\ll \frac{1}{\sqrt{\log q^m}} + \frac{1}{(\log q^m)^2} \ll \frac{1}{\sqrt{\log q^m}}. \end{aligned}$$

Then the first and the second terms on the right-hand side of (5.11) can be replaced by

$$\ll_{\delta, \gamma} (\log q^m)^{-2\delta} + (\log q^m)^{-1/2}. \quad (5.12)$$

Further, if we change the choice of x as $x = (\log q^m)^{4+2/\delta}$ we find that the third term on the right-hand side of (5.11) is absorbed into (5.12) (though the definition of S_γ is also changed).

6. Proof of Lemma 4.3 for $1 \geq \sigma > 1/2$

We already proved Lemma 4.3 for $\sigma > 1$ in Section 4. In this section, we prove Lemma 4.3 for $1 \geq \sigma > 1/2$ by using (5.11) proved in the previous section, under Assumptions 1.1 and 1.4. We remind the relation

$$\int_{\mathbb{R}} \mathcal{M}_\sigma(u) \psi_x(u) \frac{du}{\sqrt{2\pi}} = \widetilde{\mathcal{M}}_\sigma(x).$$

Our aim is to prove that

$$\left| \sum'_{f \in S_k(q^m)} \psi_\xi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) - \widetilde{\mathcal{M}}_\sigma(x) \right| \quad (6.1)$$

tends to 0 as $q^m \rightarrow \infty$ for fixed $q \geq Q(\mu)$, when $1 \geq \sigma > 1/2$.

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First, using (4.9), we can see the following inequality:

$$\begin{aligned}
 & \left| \sum'_{f \in S_k(q^m)} \psi_x(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)) - \widetilde{\mathcal{M}}_\sigma(x) \right| \\
 & \leq \left| \sum'_{f \in S_k(q^m)} (\psi_x(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)) \right. \\
 & \quad \left. - \psi_x(\log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma))) \right| \\
 & \quad + \left| \sum'_{f \in S_k(q^m)} \psi_x(\log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)) - \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}_{\log q^m}}(x) \right| \\
 & \quad + \left| \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}_{\log q^m}}(x) - \widetilde{\mathcal{M}}_\sigma(x) \right| \\
 & \ll \sum'_{f \in S_k(q^m)} |x| \left(\left| \log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\mu, \sigma) - \mathcal{S}_\mu \right| + |\mathcal{S}_\mu| \right. \\
 & \quad \left. + \left| \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma) - \log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\nu, \sigma) - \mathcal{S}_\nu \right| + |\mathcal{S}_\nu| \right) \\
 & \quad + \left| \sum'_{f \in S_k(q^m)} \psi_x(\log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)) - \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}_{\log q^m}}(x) \right| \\
 & \quad + \left| \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}_{\log q^m}}(x) - \widetilde{\mathcal{M}}_\sigma(x) \right| \\
 & = \mathcal{X}_{\log q^m} + \mathcal{Y}_{\log q^m} + \mathcal{Z}_{\log q^m}, \tag{6.2}
 \end{aligned}$$

say. From Proposition 3.2, for any $\varepsilon > 0$, there exists a number $N_0 = N_0(\varepsilon)$ for which

$$\left| \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}_{\log q^m}}(x) - \widetilde{\mathcal{M}}_\sigma(x) \right| < \varepsilon$$

holds for any $q^m > N_0$, uniformly in $x \in \mathbb{R}$. Therefore

$$\lim_{q^m \rightarrow \infty} \mathcal{Z}_{\log q^m} = 0. \tag{6.3}$$

On the estimate of $\mathcal{X}_{\log q^m}$, by using (2.4) and (5.11), we find that

$$\begin{aligned}
 & \sum'_{f \in S_k(q^m)} |x| \left(\left| \log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\mu, \sigma) - \mathcal{S}_\mu \right| \right. \\
 & \quad \left. + \left| \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma) - \log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\nu, \sigma) - \mathcal{S}_\nu \right| \right) \rightarrow 0
 \end{aligned}$$

as q^m tends to ∞ , uniformly in $|x| \leq R$. Next, by the Cauchy-Schwarz inequality

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we have

$$\begin{aligned} & \sum'_{f \in S_k(q^m)} |\mathcal{S}_\gamma| \\ & \leq \left(\sum'_{f \in S_k(q^m)} 1^2 \right)^{1/2} \left(\sum'_{f \in S_k(q^m)} \left(\sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \frac{\lambda_f(p^\gamma)}{p^\sigma} e^{-p/q^{m/4(k-1)\gamma}} \right)^2 \right)^{1/2}. \end{aligned}$$

Here, the first factor is $O(1)$ by (2.4), while the second factor is

$$\begin{aligned} & \ll \left(\sum'_{f \in S_k(q^m)} \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \frac{e^{-2p/q^{m/4(k-1)\gamma}}}{p^{2\sigma}} \right. \\ & \quad \left. + \sum'_{f \in S_k(q^m)} \sum_{\substack{p, p' \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m} \\ p > p'}} \frac{\lambda_f(p^\gamma p'^\gamma)}{(pp')^\sigma} e^{-(p+p')/q^{m/4(k-1)\gamma}} \right)^{1/2} \\ & \ll \left(\sum'_{f \in S_k(q^m)} \frac{1}{(\log q^m)^\delta} \right. \\ & \quad \left. + \sum_{\substack{p, p' \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m} \\ p < p'}} \frac{(pp')^{(k-1)\gamma/2} E(q^m)}{(pp')^\sigma} e^{-(p+p')/q^{m/4(k-1)\gamma}} \right)^{1/2}. \end{aligned}$$

Noting

$$e^{-p/q^{m/4(k-1)\gamma}} \ll \left(\frac{q^{m/4(k-1)\gamma}}{p} \right)^{(k-1)\gamma/2+1/2}$$

we obtain

$$\begin{aligned} \sum'_{f \in S_k(q^m)} |\mathcal{S}_\gamma| & \ll \left(\sum'_{f \in S_k(q^m)} \frac{1}{(\log q^m)^\delta} \right. \\ & \quad \left. + E(q^m) \left(\sum_{p \in \mathbb{P}} \frac{p^{(k-1)\gamma/2}}{p^\sigma} \left(\frac{q^{m/4(k-1)\gamma}}{p} \right)^{(k-1)\gamma/2+1/2} \right)^2 \right)^{1/2} \\ & \ll \left(\frac{1}{(\log q^m)^\delta} + E(q^m) \left(\sum_{n=1}^{\infty} \frac{q^{(m/8+m/8(k-1)\gamma)}}{n^{\sigma+1/2}} \right)^2 \right)^{1/2} \\ & \ll \left(\frac{1}{(\log q^m)^\delta} + E(q^m) q^{m/2} \right)^{1/2} \\ & \ll (\log q^m)^{-\delta/2} \end{aligned} \tag{6.4}$$

by (2.3). Hence we see that

$$\lim_{q^m \rightarrow \infty} \mathcal{X}_{\log q^m} = 0 \tag{6.5}$$

uniformly in $|x| \leq R$.

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The remaining part of this section is devoted to the estimate of $\mathcal{Y}_{\log q^m}$.
We begin with the Taylor expansion

$$\psi_x(g_{\sigma,p}(t_p)) = \exp(ixg_{\sigma,p}(t_p)) = 1 + \sum_{n=1}^{\infty} \frac{(ix)^n}{n!} g_{\sigma,p}^n(t_p),$$

where

$$\begin{aligned} g_{\sigma,p}^n(t_p) &= (-\log(1 - t_p p^{-\sigma}))^n \\ &= \left(\sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{t_p}{p^\sigma} \right)^j \right)^n \\ &= \sum_{a=1}^{\infty} \left(\sum_{\substack{a=j_1+\dots+j_n \\ j_\ell \geq 1}} \frac{1}{j_1 j_2 \cdots j_n} \right) \left(\frac{t_p}{p^\sigma} \right)^a. \end{aligned}$$

Hence

$$\begin{aligned} \psi_x(g_{\sigma,p}(t_p)) &= 1 + \sum_{n=1}^{\infty} \frac{(ix)^n}{n!} \sum_{a=1}^{\infty} \left(\sum_{\substack{a=j_1+\dots+j_n \\ j_\ell \geq 1}} \frac{1}{j_1 j_2 \cdots j_n} \right) \left(\frac{t_p}{p^\sigma} \right)^a \\ &= 1 + \sum_{a=1}^{\infty} \sum_{n=1}^a \frac{(ix)^n}{n!} \left(\sum_{\substack{a=j_1+\dots+j_n \\ j_\ell \geq 1}} \frac{1}{j_1 j_2 \cdots j_n} \right) \left(\frac{t_p}{p^\sigma} \right)^a, \end{aligned}$$

which we can write as

$$\psi_x(g_{\sigma,p}(t_p)) = \sum_{a=0}^{\infty} G_a(p, x) t_p^a \tag{6.6}$$

with

$$G_a(p, x) = \begin{cases} 1 & a = 0, \\ \frac{1}{p^{a\sigma}} \sum_{n=1}^a \frac{(ix)^n}{n!} \left(\sum_{\substack{a=j_1+\dots+j_n \\ j_\ell \geq 1}} \frac{1}{j_1 j_2 \cdots j_n} \right) & a \geq 1. \end{cases}$$

Define

$$G_a(x) = \begin{cases} 1 & a = 0, \\ \sum_{n=1}^a \frac{x^n}{n!} \binom{a-1}{n-1} & a \geq 1. \end{cases}$$

This symbol is the same as (63) in Ihara and the first author [7]. Using this symbol we obtain

$$|G_a(p, x)| \leq \frac{1}{p^{a\sigma}} G_a(|x|) \tag{6.7}$$

(see (65) in [7]). We use (75), (78) and (79) in Ihara and the first author [7] below.

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From (2.5) and (6.6) we find that

$$\begin{aligned}
& \sum'_{f \in S_k(q^m)} \psi_x(\log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)) \\
&= \sum'_{f \in S_k(q^m)} \prod_{p \in \mathcal{P}_{\log q^m}} \psi_x(g_{\sigma,p}(\alpha_f^\mu(p)) + g_{\sigma,p}(\beta_f^\nu(p))) \\
&= \sum'_{f \in S_k(q^m)} \prod_{p \in \mathcal{P}_{\log q^m}} \left(\sum_{a_p=0}^{\infty} G_{a_p}(p, x) \alpha_f^{\mu a_p}(p) \right) \left(\sum_{b_p=0}^{\infty} G_{b_p}(p, x) \beta_f^{\nu b_p}(p) \right) \\
&= \sum'_{f \in S_k(q^m)} \prod_{p \in \mathcal{P}_{\log q^m}} \left(\sum_{a_p=0}^{\infty} G_{a_p}^2(p, x) \right. \\
&\quad \left. + \sum_{\substack{0 \leq a_p, b_p \\ a_p \neq b_p}} G_{a_p}(p, x) G_{b_p}(p, x) \alpha_f^{\mu a_p}(p) \beta_f^{\nu b_p}(p) \right) \tag{6.8}
\end{aligned}$$

and also, from Proposition 3.1 and (6.6),

$$\begin{aligned}
\widetilde{\mathcal{M}}_{\sigma, \mathcal{P}_{\log q^m}}(x) &= \int_{\mathbb{R}} \psi_x(u) \mathcal{M}_{\sigma, \mathcal{P}_{\log q^m}}(u) \frac{du}{\sqrt{2\pi}} \\
&= \int_{\mathcal{T}_{\mathcal{P}_{\log q^m}}} \psi_x(2\Re(g_{\sigma, \mathcal{P}_{\log q^m}}(t_{\mathcal{P}_{\log q^m}}))) d^* t_{\mathcal{P}_{\log q^m}} \\
&= \prod_{p \in \mathcal{P}_{\log q^m}} \int_{\mathcal{T}_p} \psi_x(2\Re(g_{\sigma,p}(t_p))) d^* t_p \\
&= \prod_{p \in \mathcal{P}_{\log q^m}} \int_{\mathcal{T}_p} \psi_x(g_{\sigma,p}(t_p)) \psi_x(g_{\sigma,p}(\bar{t}_p)) d^* t_p \\
&= \prod_{p \in \mathcal{P}_{\log q^m}} \int_{\mathcal{T}_p} \left(\sum_{a_p=0}^{\infty} G_{a_p}(p, x) t_p^{a_p} \right) \left(\sum_{b_p=0}^{\infty} G_{b_p}(p, x) t_p^{-b_p} \right) d^* t_p \\
&= \prod_{p \in \mathcal{P}_{\log q^m}} \left(\sum_{a_p=0}^{\infty} G_{a_p}^2(p, x) \right). \tag{6.9}
\end{aligned}$$

Write $\mathcal{P}_{\log q^m} = \{p_1, p_2, \dots, p_L\}$, where p_l means the l -th prime number. (So $L = \pi(\log q^m)$.) Substituting (6.8) and (6.9) into the definition of $\mathcal{Y}_{\log q^m}$, and

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noting (2.4), we obtain

$$\begin{aligned} \mathcal{Y}_{\mathcal{P}_{\log q^m}} &= \left| E(q^m) \prod_{p \in \mathcal{P}_{\log q^m}} \sum_{a_p=0}^{\infty} G_{a_p}^2(p, x) \right. \\ &\quad + \sum'_{f \in S_k(q^m)} \sum_{\substack{(j_1, \dots, j_L) \neq (0, \dots, 0) \\ j_\ell \in \{0, 1\}}} \prod_{\ell=1}^L \left(\sum_{a_{p_\ell}=0}^{\infty} G_{a_{p_\ell}}^2(p_\ell, x) \right)^{1-j_\ell} \\ &\quad \times \left. \left(\sum_{\substack{0 \leq a_{p_\ell}, b_{p_\ell} \\ a_{p_\ell} \neq b_{p_\ell}}} G_{a_{p_\ell}}(p_\ell, x) G_{b_{p_\ell}}(p_\ell, x) \alpha_f^{\mu a_{p_\ell}}(p_\ell) \beta_f^{\mu b_{p_\ell}}(p_\ell) \right)^{j_\ell} \right|. \end{aligned}$$

If we see that

$$\begin{aligned} \mathcal{Y}'_{\mathcal{P}_{\log q^m}} &= \left| \sum'_{f \in S_k(q^m)} \sum_{\substack{(j_1, \dots, j_L) \neq (0, \dots, 0) \\ j_\ell \in \{0, 1\}}} \prod_{\ell=1}^L \left(\sum_{a_{p_\ell}=0}^{\infty} G_{a_{p_\ell}}^2(p_\ell, x) \right)^{1-j_\ell} \right. \\ &\quad \times \left. \left(\sum_{\substack{0 \leq a_{p_\ell}, b_{p_\ell} \\ a_{p_\ell} \neq b_{p_\ell}}} G_{a_{p_\ell}}(p_\ell, x) G_{b_{p_\ell}}(p_\ell, x) \alpha_f^{\mu a_{p_\ell}}(p_\ell) \beta_f^{\mu b_{p_\ell}}(p_\ell) \right)^{j_\ell} \right| \quad (6.10) \end{aligned}$$

and

$$\mathcal{Y}''_{\mathcal{P}_{\log q^m}} = E(q^m) \prod_{p \in \mathcal{P}_{\log q^m}} \sum_{a_p=0}^{\infty} G_{a_p}^2(p, x) \quad (6.11)$$

tend to 0, then we obtain that $\mathcal{Y}_{\mathcal{P}_{\log q^m}}$ tends to 0 as q^m tends to ∞ .

We first consider the second inner sum on the right-hand side of (6.10). Letting $a_p - b_p = r_p$ for the part of $a_p > b_p$ and letting $b_p - a_p = r_p$ for the part of $b_p > a_p$, we obtain

$$\begin{aligned} &\sum_{\substack{0 \leq a_p, b_p \\ a_p \neq b_p}} G_{a_p}(p, x) G_{b_p}(p, x) \alpha_f^{\mu a_p}(p) \beta_f^{\mu b_p}(p) \\ &= \sum_{0 \leq a_p < b_p} + \sum_{a_p > b_p \geq 0} \\ &= \sum_{r_p \geq 1} \sum_{b_p \geq r_p} G_{b_p - r_p}(p, x) G_{b_p}(p, x) \alpha_f^{\mu(b_p - r_p)}(p) \beta_f^{\mu b_p}(p) \\ &\quad + \sum_{r_p \geq 1} \sum_{a_p \geq r_p} G_{a_p}(p, x) G_{a_p - r_p}(p, x) \alpha_f^{\mu a_p}(p) \beta_f^{\mu(a_p - r_p)}(p) \\ &= \sum_{r_p \geq 1} \sum_{a_p \geq r_p} G_{a_p}(p, x) G_{a_p - r_p}(p, x) \left(\beta_f^{\mu r_p}(p) + \alpha_f^{\mu r_p}(p) \right) \\ &= \sum_{r_p \geq 1} \sum_{a_p \geq r_p} G_{a_p}(p, x) G_{a_p - r_p}(p, x) \left(\lambda_f(p^{\mu r_p}) - \lambda_f(p^{\mu r_p - 2}) \right), \end{aligned}$$

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where the last equation is deduced by

$$\begin{aligned} \lambda_f(p^{\mu r_p}) &= \sum_{h=0}^{\mu r_p} \alpha_f^{\mu r_p - h}(p) \beta_f^h(p) \\ &= \alpha_f^{\mu r_p} + \alpha_f^{\mu r_p - 2} + \alpha_f^{\mu r_p - 4} + \dots + \beta_f^{\mu r_p - 4} + \beta_f^{\mu r_p - 2} + \beta_f^{\mu r_p} \end{aligned}$$

which is from (1.1). Letting

$$G_{p,x}(r) = \sum_{a_p \geq r} G_{a_p}(p, x) G_{a_p - r}(p, x),$$

from (6.10) we obtain

$$\begin{aligned} \mathcal{Y}'_{\mathcal{P}_{\log q^m}} &= \left| \sum'_{f \in S_k(q^m)} \sum_{\substack{(j_1, \dots, j_L) \neq (0, \dots, 0) \\ j_\ell \in \{0, 1\}}} \prod_{\ell=1}^L \left(\sum_{a_{p_\ell}=0}^{\infty} G_{a_{p_\ell}}^2(p_\ell, x) \right)^{1-j_\ell} \right. \\ &\quad \times \left. \left(\sum_{r_{p_\ell} \geq 1} G_{a_{p_\ell}, x}(r_{p_\ell}) (\lambda_f(p_\ell^{\mu r_{p_\ell}}) - \lambda_f(p_\ell^{\mu r_{p_\ell} - 2})) \right)^{j_\ell} \right| \\ &= \left| \sum_{\substack{(j_1, \dots, j_L) \neq (0, \dots, 0) \\ j_\ell \in \{0, 1\}}} \prod_{\ell=1}^L \left(\sum_{a_{p_\ell}=0}^{\infty} G_{a_{p_\ell}}^2(p_\ell, x) \right)^{1-j_\ell} \right. \\ &\quad \times \left. \sum'_{f \in S_k(q^m)} \prod_{\ell=1}^L \left(\sum_{r_{p_\ell} \geq 1} G_{a_{p_\ell}, x}(r_{p_\ell}) (\lambda_f(p_\ell^{\mu r_{p_\ell}}) - \lambda_f(p_\ell^{\mu r_{p_\ell} - 2})) \right)^{j_\ell} \right|. \quad (6.12) \end{aligned}$$

Remark 6.1. Here we remark why we only consider the case $\mu - \nu = 2$ in the present paper. If we consider the case that ν has the same parity with μ but $\mu - \nu = 2h > 2$, and discuss analogously as above, then the factor of the form

$$\prod_{h=0}^{(\mu-\nu)/2-1} \prod_{\ell=1}^L \left(\sum_{r_{p_\ell} \geq 1} G_{a_{p_\ell}, x}(r_{p_\ell}) (\lambda_f(p_\ell^{(\mu-2h)r_{p_\ell}}) - \lambda_f(p_\ell^{(\mu-2h)r_{p_\ell} - 2})) \right)^{j_\ell}$$

appears. The summation of this factor over primitive forms cannot be included in the error term, because of (2.1).

Let us continue the argument. From (6.12) we obtain

$$\mathcal{Y}'_{\mathcal{P}_{\log q^m}} \leq \sum_{\substack{(j_1, \dots, j_L) \neq (0, \dots, 0) \\ j_\ell \in \{0, 1\}}} \prod_{\ell=1}^L |G_{p_\ell, x}(0)|^{1-j_\ell} \left| \sum'_{f \in S_k(q^m)} \sum_{1 < n} \mathcal{G}_x(n) \lambda_f(n) \right|,$$

where

$$\mathcal{G}_x(n) = \begin{cases} \prod_{\substack{1 \leq \ell \leq L \\ j_\ell = 1}} (-1)^{r''(p_\ell)} G_{p_\ell, x}(r_{p_\ell}) & n = \prod_{\substack{1 \leq \ell \leq L \\ j_\ell = 1}} p_\ell^{r'_{p_\ell}}, \\ 0 & \text{otherwise,} \end{cases}$$

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$r'_{p_\ell} = \mu r_{p_\ell}$ or $\mu r_{p_\ell} - 2$, and

$$r''(p_\ell) = \begin{cases} 0 & r'_{p_\ell} = \mu r_{p_\ell} \\ 1 & r'_{p_\ell} = \mu r_{p_\ell} - 2. \end{cases}$$

We divide the summation of $\mathcal{G}_x(n)\lambda_f(n)$ above into two parts according to the conditions $n \leq M$ and $n > M$, where M is a suitable constant depending on k , $\log q^m$ and p_ℓ ($1 \leq \ell \leq L$) defined below. We apply the formula (2.1) with $n \neq 1$ to the summation of $n \leq M$. And we use $\lambda_f(n) \ll n^\eta$ (where η is an arbitrarily small positive number which will be specified later) by the Ramanujan-Petersson estimate for the estimation of summation of $n > M$. We obtain

$$\begin{aligned} \mathcal{Y}'_{\mathcal{P}_{\log q^m}} &\leq \sum_{\substack{(j_1, \dots, j_L) \neq (0, \dots, 0) \\ j_\ell \in \{0, 1\}}} \prod_{\ell=1}^L |G_{p_\ell, x}(0)|^{1-j_\ell} \\ &\times \left(E(q^m) \sum_{1 < n \leq M} |\mathcal{G}_x(n)| n^{(k-1)/2} + \sum_{n > M} |\mathcal{G}_x(n)| n^\eta \right). \end{aligned} \quad (6.13)$$

From (75), (78) and (79) in [7] and (6.7) in this paper, we see that

$$\begin{aligned} |G_{p_\ell, x}(0)| &\leq \sum_{a_{p_\ell}=0}^{\infty} |G_{a_{p_\ell}}(p_\ell, x)|^2 \leq \sum_{a_{p_\ell}=0}^{\infty} \frac{1}{2a_{p_\ell}\sigma} G_{a_{p_\ell}}^2(|x|) \\ &\leq \left(\sum_{a_{p_\ell}=0}^{\infty} \frac{1}{a_{p_\ell}\sigma} G_{a_{p_\ell}}(|x|) \right)^2 = \left(\exp\left(\frac{|x|}{p_\ell^\sigma - 1}\right) \right)^2 \end{aligned} \quad (6.14)$$

and

$$\begin{aligned} |G_{p_\ell, x}(r_{p_\ell})| &\leq \sum_{a_{p_\ell} \geq r_{p_\ell}} |G_{a_{p_\ell}}(p_\ell, x) G_{a_{p_\ell} - r_{p_\ell}}(p_\ell, x)| \\ &\leq \sum_{a_{p_\ell} \geq r_{p_\ell}} \frac{1}{p_\ell^{a_{p_\ell}\sigma} p_\ell^{(a_{p_\ell} - r_{p_\ell})\sigma}} G_{a_{p_\ell}}(|x|) G_{a_{p_\ell} - r_{p_\ell}}(|x|) \\ &\leq \sum_{a'_{p_\ell}=0}^{\infty} \frac{1}{p_\ell^{(a'_{p_\ell} + r_{p_\ell})\sigma} p_\ell^{a'_{p_\ell}\sigma}} G_{a'_{p_\ell} + r_{p_\ell}}(|x|) G_{a'_{p_\ell}}(|x|) \\ &\leq \frac{1}{p^{r_{p_\ell}\sigma}} \sum_{a'_{p_\ell}=0}^{\infty} \frac{1}{p_\ell^{a'_{p_\ell}\sigma} p_\ell^{a'_{p_\ell}\sigma}} G_{a'_{p_\ell}}(|x|) G_{a'_{p_\ell}}(|x|) L_{r_{p_\ell}}(|x|) \\ &\leq \frac{L_{r_{p_\ell}}(|x|)}{p_\ell^{r_{p_\ell}\sigma}} \left(\exp\left(\frac{|x|}{p_\ell^\sigma - 1}\right) \right)^2 \\ &\leq \frac{1}{p^{r_{p_\ell}\sigma/2}} \left(\exp\left(\frac{|x|}{p_\ell^\sigma - 1}\right) \right)^2 \left(\exp\left(\frac{|x|}{p_\ell^{\sigma/2} - 1}\right) \right), \end{aligned} \quad (6.15)$$

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where

$$L_r(x) = \sum_{m=0}^r G_m(x)$$

(the same as [74] in [7]). Therefore, when

$$n = \prod_{\substack{1 \leq \ell \leq L \\ j_\ell = 1}} p_\ell^{r'_{p_\ell}}$$

with $r'_{p_\ell} = \mu r_{p_\ell}$ or $\mu r_{p_\ell} - 1$ for $r_{p_\ell} \geq 1$, from (6.15) we have

$$\begin{aligned} \mathcal{G}_x(n) &\leq \prod_{\substack{1 \leq \ell \leq L \\ j_\ell = 1}} |G_{p_\ell, x}(r_{p_\ell})| \\ &\leq \prod_{\substack{1 \leq \ell \leq L \\ j_\ell = 1}} \frac{1}{p_\ell^{r_{p_\ell} \sigma / 2}} \left(\exp \left(\frac{|x|}{p_\ell^\sigma - 1} \right) \right)^2 \left(\exp \left(\frac{|x|}{p_\ell^{\sigma/2} - 1} \right) \right) \\ &\leq \frac{1}{n^{\sigma/2\mu}} \prod_{\substack{1 \leq \ell \leq L \\ j_\ell = 1}} \left(\exp \left(\frac{|x|}{p_\ell^\sigma - 1} \right) \right)^2 \left(\exp \left(\frac{|x|}{p_\ell^{\sigma/2} - 1} \right) \right). \end{aligned} \quad (6.16)$$

From (6.13), (6.14) and (6.16), we obtain

$$\begin{aligned} \mathcal{Y}'_{P_{\log q^m}} &\ll \sum_{\substack{(j_1, \dots, j_L) \neq (0, \dots, 0) \\ j_\mu \in \{0, 1\}}} \prod_{\ell=1}^L \left(\exp \left(\frac{|x|}{p_\ell^\sigma - 1} \right) \right)^2 \left(\exp \left(\frac{|x|}{p_\ell^{\sigma/2} - 1} \right) \right)^{j_\ell} \\ &\times \left(E(q^m) \sum_{\substack{n \leq M \\ p \nmid n \text{ for } p > \log q^m}} \frac{n^{(k-1)/2}}{n^{\sigma/2\mu}} + \sum_{\substack{n > M \\ p \nmid n \text{ for } p > \log q^m}} \frac{n^\eta}{n^{\sigma/2\mu}} \right). \end{aligned}$$

Here we choose $\eta = \sigma/4\mu$. Then the second inner sum is

$$\begin{aligned} \sum_{\substack{n > M \\ p \nmid n \text{ for } p > \log q^m}} \frac{1}{n^{\sigma/4\mu}} &< \frac{1}{M^{\sigma/8\mu}} \sum_{\substack{n > M \\ p \nmid n \text{ for } p > \log q^m}} \frac{1}{n^{\sigma/8\mu}} \\ &< \frac{1}{M^{\sigma/8\mu}} \prod_{l=1}^L \frac{1}{1 - p_l^{-\sigma/8\mu}} = \frac{1}{M^{\sigma/8\mu}} \prod_{l=1}^L \frac{p_l^{\sigma/8\mu}}{p_l^{\sigma/8\mu} - 1} \\ &\ll \frac{1}{M^{\sigma/8\mu}} \prod_{l=1}^L \frac{p_l^{\sigma/8\mu}}{p_l^{\sigma/8\mu} - p_l^{\sigma/8\mu}/2} \ll \frac{2^L}{M^{\sigma/8\mu}}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \mathcal{Y}'_{\mathcal{P}_{\log q^m}} &\leq \sum_{\substack{(j_1, \dots, j_L) \neq (0, \dots, 0) \\ j_\ell \in \{0, 1\}}} \prod_{\ell=1}^L \left(\exp \left(\frac{|x|}{p_\ell^\sigma - 1} \right) \right)^2 \left(\exp \left(\frac{|x|}{p_\ell^{\sigma/2} - 1} \right) \right)^{j_\ell} \\ &\quad \times \left(E(q^m) M^{(k+1)/2 - \sigma/2\mu} + \frac{2^L}{M^{\sigma/8\mu}} \right) \\ &\leq 2^L \prod_{\ell=1}^L \left(\exp \left(\frac{|x|}{p_\ell^{1/4} - 1} \right) \right)^3 \\ &\quad \times \left(E(q^m) M^{(k+1)/2 - 1/4\mu} + \frac{2^L}{M^{1/16\mu}} \right), \end{aligned}$$

since $\sigma > 1/2$. For large ℓ , we know

$$\exp \left(\frac{|x|}{p_\ell^{1/4} - 1} \right) \leq \exp \left(\frac{R}{p_\ell^{1/4} - 1} \right) < 2 \quad (6.17)$$

for $|x| \leq R$, hence

$$\prod_{\ell=1}^L \left(\exp \left(\frac{|x|}{p_\ell^{1/4} - 1} \right) \right)^3 \ll_R 2^{3L}.$$

Therefore we obtain

$$\mathcal{Y}'_{\mathcal{P}_{\log q^m}} \leq 2^{4L} \left(E(q^m) M^{c(k, \mu)} + \frac{2^L}{M^{1/16\mu}} \right), \quad (6.18)$$

where $c(k, \mu) = (k+1)/2 - 1/4\mu$. Noting the fact that the number of the prime numbers less than $2^5 = 32$ is 11, we choose

$$M = \left(\frac{p_1 \cdots p_L}{(2^5)^{L-11}} \right)^{1/c(k, \mu)}.$$

Since $c(k, \mu) \leq 15/2 < 8$, we see that

$$\begin{aligned} M &= (p_1 \cdots p_{11})^{1/c(k, \mu)} \left(\frac{p_{12}}{2^5} \cdot \frac{p_{13}}{2^5} \cdot \frac{p_L}{2^5} \right)^{1/c(k, \mu)} \\ &> (p_1 \cdots p_{11})^{2/15} > (200560490130)^{1/8} > 25. \end{aligned}$$

For large m or q , by the prime number theorem we have

$$\begin{aligned} p_1 \cdots p_L &= \exp(\log p_1 + \cdots + \log p_L) \\ &= \exp(\log q^m + O((\log q^m) \exp(-c_1 \sqrt{\log \log q^m}))) \\ &\leq q^{m(1+c_2 \exp(-c_1 \sqrt{\log \log q^m}))} \end{aligned}$$

where c_1, c_2 are positive constants. By using (2.3), we have

$$\begin{aligned} 2^{4L} E(q^m) M^{c(k, \mu)} &\leq 2^{4L} E(q^m) \cdot \frac{p_1 \cdots p_L}{2^{5L-55}} \\ &\ll \frac{p_1 \cdots p_L}{2^L} \cdot E(q^m) \ll \frac{q^{m(1+c_2 \exp(-c_1 \sqrt{\log \log q^m}))}}{2^L} \cdot q^{-m}. \end{aligned} \quad (6.19)$$

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Again by the prime number theorem, we see that

$$2^L = 2^{\log q^m (1+o(1)) / \log \log q^m} = (q^m)^{\log 2(1+o(1)) / \log \log q^m},$$

so we find that the right-hand side of (6.19) is

$$\ll (q^m)^{c_2 \exp(-c_1 \sqrt{\log \log q^m}) - \log 2(1+o(1)) / \log \log q^m},$$

whose exponent is negative for large q^m . Therefore this tends to 0 as q^m tends to ∞ .

Next, we have

$$\begin{aligned} \frac{2^{4L} 2^L}{M^{1/16\mu}} &= 2^{5L} \left(\frac{p_1 \cdots p_L}{2^{5L-55}} \right)^{-1/16\mu c(k,\mu)} \\ &\ll \frac{(2^L)^{5+5/16\mu c(k,\mu)}}{(p_1 \cdots p_L)^{1/16\mu c(k,\mu)}} \\ &\ll \left(\frac{(2^{80\mu c(k,\mu)+5})^L}{p_1 \cdots p_L} \right)^{1/16\mu c(k,\mu)}, \end{aligned}$$

so, putting $d(k, \mu) = 2^{80\mu c(k,\mu)+5}$, the above is

$$\begin{aligned} &= \left(\frac{d(k, \mu)}{p_1} \cdots \frac{d(k, \mu)}{p_{\pi(d(k,\mu))}} \frac{(d(k, \mu))^{L-\pi(d(k,\mu))}}{p_{\pi(d(k,\mu))+1} \cdots p_L} \right)^{1/16\mu c(k,\mu)} \\ &\ll_k \left(\frac{d(k, \mu)}{p_{\pi(d(k,\mu))+1}} \right)^{(L-\pi(d(k,\mu)))/16\mu c(k,\mu)}. \end{aligned} \quad (6.20)$$

Since the quantity in the parentheses is smaller than 1, we find that this also tends to 0 as q^m tends to ∞ . Therefore from (6.18) we conclude that $\mathcal{Y}'_{\mathcal{P}_{\log q^m}}$ tends to 0 as q^m tends to ∞ .

The idea of evaluating $\mathcal{Y}''_{\mathcal{P}_{\log q^m}}$, defined by (6.11), is essentially similar, but much simpler. First, using (6.14), we have

$$\begin{aligned} \mathcal{Y}''_{\mathcal{P}_{\log q^m}} &= E(q^m) \prod_{p \in \mathcal{P}_{\log q^m}} \sum_{a_p=0}^{\infty} G_{a_p}^2(p, x) \leq E(q^m) \prod_{p \in \mathcal{P}_{\log q^m}} |G_{p,x}(0)| \\ &\leq E(q^m) \prod_{p \in \mathcal{P}_{\log q^m}} \left(\exp \left(\frac{|x|}{p^\sigma - 1} \right) \right)^2. \end{aligned}$$

Then, by an argument similar to (6.17), the above is

$$\ll_R E(q^m) 2^L \ll E(q^m) e^{\log q^m / \log \log q^m} = E(q^m) (q^m)^{(1+o(1)) / \log \log q^m},$$

which tends to 0 as q^m tends to ∞ . Therefore we now arrive at the assertion

$$\lim_{q^m \rightarrow \infty} \mathcal{Y}_{\mathcal{P}_{\log q^m}} = 0. \quad (6.21)$$

Finally we see that Lemma 4.3 is established, by substituting (6.3), (6.5) and (6.21) into (6.2).

7. Completion of the proof of Theorem 1.5

The only remaining task now is to deduce the general statement of our Theorem 1.5 from Lemma 4.3. This can be done by using the general principle on the weak convergence of probability measures (as indicated in Remark 3.2 of [9]), but here we follow a more self-contained treatment given in Ihara and the first author [7]. In this section, we just explain the outline of the proof of Theorem 1.5, because the argument is the same as that in [7].

For any $\varepsilon > 0$, the aim of this section is to prove that

$$\sum'_{f \in S(q^m)} \Psi \circ \log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma) \quad (7.1)$$

tends to

$$\int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Psi(u) \frac{du}{\sqrt{2\pi}}$$

as q^m tends to ∞ .

We define the set Λ of the function Φ on \mathbb{C} by

$$\Lambda = \{\Phi \in L^1 \cap L^\infty \mid \Phi^\vee \in L^1 \cap L^\infty, (\Phi^\vee)^\wedge = \Phi\},$$

where Φ^\wedge means the Fourier transform of Φ and Φ^\vee means the Fourier inverse transform of Φ . We know

$$\Phi(u) = \int_{\mathbb{R}} \Phi^\wedge(x) \psi_{-u}(x) \frac{dx}{\sqrt{2\pi}} = \int_{\mathbb{R}} \Phi^\wedge(x) \psi_{-x}(u) \frac{dx}{\sqrt{2\pi}}.$$

Since we also know $\widetilde{\mathcal{M}}_\sigma \in \Lambda$ from Proposition 3.2 and Proposition 3.3, we have $\mathcal{M}_\sigma \in \Lambda$. Therefore we have

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Phi(u) \frac{du}{\sqrt{2\pi}} &= \int_{\mathbb{R}} \overline{\mathcal{M}_\sigma}^\wedge(x) \Phi^\wedge(x) \frac{dx}{\sqrt{2\pi}} \\ &= \int_{\mathbb{R}} \widetilde{\mathcal{M}}_\sigma(-x) \Phi^\wedge(x) \frac{dx}{\sqrt{2\pi}}. \end{aligned}$$

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In the case of $\Psi = \Phi \in \Lambda$, we can see

$$\begin{aligned}
 & \left| \sum'_{f \in S(q^m)} \Phi \circ \log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma) - \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Phi(u) \frac{du}{\sqrt{2\pi}} \right| \\
 &= \left| \sum'_{f \in S_k(q^m)} \int_{\mathbb{R}} (\Phi^\wedge(x) \psi_{-x}(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma))) \frac{dx}{\sqrt{2\pi}} \right. \\
 & \quad \left. - \int_{\mathbb{R}} \widetilde{\mathcal{M}}_\sigma(-x) \Phi^\wedge(x) \frac{dx}{\sqrt{2\pi}} \right| \\
 &= \left| \sum'_{f \in S_k(q^m)} \int_{\mathbb{R}} (\Phi^\wedge(-x) \psi_x(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma))) \frac{dx}{\sqrt{2\pi}} \right. \\
 & \quad \left. - \int_{\mathbb{R}} \widetilde{\mathcal{M}}_\sigma(x) \Phi^\wedge(-x) \frac{dx}{\sqrt{2\pi}} \right| \\
 &\leq \int_{\mathbb{R}} |\Phi^\wedge(-x)| \\
 & \quad \times \left| \sum'_{f \in S_k(q^m)} (\psi_x(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma))) - \widetilde{\mathcal{M}}_\sigma(x) \right| \frac{dx}{\sqrt{2\pi}}.
 \end{aligned}$$

We divide the above integral into two parts, $|x| \leq R$ and $|x| > R$ by sufficiently large R . Since ψ_x and $\widetilde{\mathcal{M}}_\sigma(x)$ are bounded (see (3.5)) and $\Phi^\wedge \in L^1$, the integral on $|x| > R$ is small for sufficiently large R . The other integral on $|x| \leq R$ is then also small by Lemma 4.3, for large q or m . Therefore the desired assertion holds for $\Psi \in \Lambda$.

In the case that Ψ is a compactly supported function on C^∞ , this is a element in the Schwartz space. Therefore $\Psi \in \Lambda$.

In the case that Ψ is a compactly supported continuous function, this is approximated by compactly supported functions in C^∞ . Therefore, in this case Theorem 1.5 is established. Especially, in the case that Ψ is a characteristic function on a compact subset, Ψ is approximated by compactly supported continuous function. Therefore, in this case the proof is complete.

Finally, we consider the case that Ψ is a bounded continuous function. For any $R > 0$, there exists a compactly supported continuous function Ψ_R such that $\Psi_R(x) = \Psi(x)$ for $|x| \leq R$ and $|\Psi_R(x)| \leq |\Psi(x)|$ for $|x| > R$. We already know that

$$\lim_{\substack{q \rightarrow \infty \\ \text{or } m \rightarrow \infty}} \sum'_{f \in S(q^m)} \Psi_R \circ \log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma) = \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Psi_R(u) \frac{du}{\sqrt{2\pi}},$$

where the above equation is proved for $q \geq Q(\mu)$ when $1 \geq \sigma > 1/2$ in the case of

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$m \rightarrow \infty$. For the right-hand side of this equation, we have

$$\begin{aligned} & \int_{\mathbb{R}} \mathcal{M}_{\sigma}(u) \Psi_R(u) \frac{du}{\sqrt{2\pi}} \\ &= \int_{|x|>R} \mathcal{M}_{\sigma}(u) \Psi_R(u) \frac{du}{\sqrt{2\pi}} + \int_{|x|\leq R} \mathcal{M}_{\sigma}(u) \Psi_R(u) \frac{du}{\sqrt{2\pi}} \\ &= \int_{|x|>R} \mathcal{M}_{\sigma}(u) \Psi_R(u) \frac{du}{\sqrt{2\pi}} + \int_{|x|\leq R} \mathcal{M}_{\sigma}(u) \Psi(u) \frac{du}{\sqrt{2\pi}} \\ &= \int_{|x|>R} \mathcal{M}_{\sigma}(u) (\Psi_R(u) - \Psi(u)) \frac{du}{\sqrt{2\pi}} + \int_{\mathbb{R}} \mathcal{M}_{\sigma}(u) \Psi(u) \frac{du}{\sqrt{2\pi}}. \end{aligned}$$

As for the former integral, we remind that \mathcal{M}_{σ} is non-negative to find

$$\int_{|x|>R} \mathcal{M}_{\sigma}(u) (\Psi_R(u) - \Psi(u)) \frac{du}{\sqrt{2\pi}} \ll \int_{|x|>R} \mathcal{M}_{\sigma}(u) \frac{du}{\sqrt{2\pi}}$$

which tends to 0 as R tends to ∞ , since we know

$$\int_{\mathbb{R}} \mathcal{M}_{\sigma}(u) \frac{du}{\sqrt{2\pi}} = 1$$

by Proposition 3.3. Hence we have

$$\begin{aligned} & \lim_{R \rightarrow \infty} \lim_{\substack{q \rightarrow \infty \\ \text{or } m \rightarrow \infty}} \sum'_{f \in S(q^m)} \Psi_R \circ \log L_{\mathbb{P}(q)}(\text{Sym}_f^{\mu}, \text{Sym}_f^{\nu}, \sigma) \\ &= \lim_{R \rightarrow \infty} \int_{\mathbb{R}} \mathcal{M}_{\sigma}(u) \Psi_R(u) \frac{du}{\sqrt{2\pi}} = \int_{\mathbb{R}} \mathcal{M}_{\sigma}(u) \Psi(u) \frac{du}{\sqrt{2\pi}}. \end{aligned} \quad (7.2)$$

Finally, by using the same argument as in p.675 in Ihara and the first author [7], from (7.2) we obtain

$$\lim_{\substack{q \rightarrow \infty \\ \text{or } m \rightarrow \infty}} \sum'_{f \in S(q^m)} \Psi \circ \log L_{\mathbb{P}(q)}(\text{Sym}_f^{\mu}, \text{Sym}_f^{\nu}, \sigma) = \int_{\mathbb{R}} \mathcal{M}_{\sigma}(u) \Psi(u) \frac{du}{\sqrt{2\pi}},$$

which is the conclusion of our Theorem 1.5.

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References

- [1] H. Bohr and B. Jessen, Über die Werteverteilung der Riemannschen Zetafunktion I, II, *Acta Math.* **54** (1930), 1-35 and *ibid.* **58** (1932), 1-55.
- [2] J. Cogdell and P. Michel, On the complex moments of symmetric power L -functions at $s = 1$, *Intern. Math. Res. Notices* **31** (2004), 1561-1617.
- [3] W. Duke, Extreme values of Artin L -functions and class numbers, *Compositio Math.* **136** (2003) 103-115.
- [4] Y. Ichihara, The first moment of L -functions of primitive forms on $\Gamma_0(p^\alpha)$ and a basis of old forms, *J. Number Theory* **131** (2011) 343-362.
- [5] Y. Ihara, On “ M -Functions” closely related to the distribution of L'/L -values, *Publ. RIMS, Kyoto Univ.* **44** (2008) 893-954.
- [6] Y. Ihara and K. Matsumoto, On L -functions over function fields: Power means of error-terms and distribution of L'/L -values, *RIMS Kôkyûroku Bessatsu* **B19** (2010), 221-247.
- [7] Y. Ihara and K. Matsumoto, On certain mean values and the value-distribution of logarithms of Dirichlet L -functions, *Quart. J. Math.* **62** (2011) 637-677.
- [8] Y. Ihara and K. Matsumoto, On $\log L$ and L'/L for L -functions and the associated “ M -functions”: connections in optimal cases, *Moscow Math. J.* **11** (2011), 73-111.
- [9] Y. Ihara and K. Matsumoto, On the value-distribution of logarithmic derivatives of Dirichlet L -functions, in *Analytic Number Theory, Approximation Theory, and Special Functions*, G. V. Milovanović and M. Th. Rassias (eds.), Springer, 2014, pp.79-91.
- [10] H. Iwaniec, *Topics in Classical Automorphic Forms*, Grad. Stud. Math. vol. 17, Amer. Math. Soc., Providence, 1997.
- [11] B. Jessen and A. Wintner, Distribution functions and the Riemann zeta function, *Trans. Amer. Math. Soc.* **38** (1935), 48-88.
- [12] R. Kershner and A. Wintner, On the asymptotic distribution of $\zeta'/\zeta(s)$ in the critical strip, *Amer. J. Math.* **59** (1937), 673-678.
- [13] H. Kim and F. Shahidi, Symmetric cube L -functions for GL_2 are entire, *Ann. of Math.* **150** (1999) 645-662.
- [14] M. Mourtada and V. K. Murty, Distribution of values of $L'/L(\sigma, \chi_D)$, *Moscow Math. J.* **15** (2015), 497-509.