Semibricks and Koenig–Yang correspondences in τ-tilting theory
（τ傾理論における半煉瓦とKoenig–Yang対応）

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Preface

In the representation theory of finite-dimensional algebras A over a field K, we study the category $\mathsf{mod}\,A$ of finite-dimensional A-modules. In particular, projective modules and simple modules are basic and important. As it is well-known, the algebra A itself is a typical example of *progenerator*, that is, A is projective and every module M in $\mathsf{mod}\,A$ admits a surjection $A^{\oplus s} \to M$ from a direct sum of copies of A. On the other hand, we can say that the simple A-modules are the minimal units of the module category $\mathsf{mod}\,A$, because every module M in $\mathsf{mod}\,A$ is filtered by simple A-modules. There have been many generalizations and studies of projective modules and simple modules. Among them, the recent studies by Koenig–Yang [KY] and Adachi–Iyama–Reiten [AIR] are significant.

First, in [KY] and its derivation [BY], they showed the existence of bijections between many sets of important notions defined in derived categories, such as the set $\mathsf{2\text{-}silt}\,A$ of *2-term silting objects* in the perfect derived category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$, and the set $\mathsf{2\text{-}smc}\,A$ of *2-term simple-minded collections* in the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$. Here, $\mathsf{2\text{-}silt}\,A$ and $\mathsf{2\text{-}smc}\,A$ are generalizations in derived categories of progenerators and the set of simple modules, respectively. The details of these notions and the bijections are explained in Section 2 of Part 1 in this thesis.

Second, τ-tilting theory introduced in [AIR] is a theory where progenerators are generalized as support τ-tilting modules in the module category $\mathsf{mod}\,A$. They constructed a bijection from the set $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ of support τ-tilting modules to the set $\mathsf{f\text{-}tors}\,A$ of functorially finite torsion classes in $\mathsf{mod}\,A$ given by $M \mapsto \mathsf{Fac}\,M$. They also showed that M is maximal among Ext-projective objects in $\mathsf{Fac}\,M$ with respect to direct sums (see Proposition 1.6 in Part 1). Thus, we can say M is a "progenerator" of $\mathsf{Fac}\,M$. They also constructed a bijection $\mathsf{2\text{-}silt}\,A \ni P \mapsto H^0(P) \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,A$, which is a bijection between two notions generalizing progenerators in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ and $\mathsf{mod}\,A$ (see Proposition 2.6 in Part 1).

Including these, there exist one-to-one correspondences between many notions defined in the derived categories and the module category. These correspondences are called *Koenig–Yang correspondences*, which were improved in recent studies by Ingalls–Thomas [IT] and Marks–Šťovíček [MS]. However, Koenig–Yang correspondences do not involve a generalized notion of the simple modules in the module category $\mathsf{mod}\,A$. In order to fill in this blank, we use a notion of *semibricks* in this thesis.

We state the definition and some properties of semibricks briefly. First, a module $S \in \mathsf{mod}\,A$ is called a *brick* if the endomorphism ring $\mathsf{End}_A(S)$ is a division ring. Second, a set $\mathcal{S}$ of bricks is called a *semibrick* if any two different elements $S, S' \in \mathcal{S}$ satisfy $\mathsf{Hom}_A(S, S') = 0$. These definitions are written in Definition 1.1 in Part 1 of this thesis. For example, a set of simple A-modules is a semibrick. We write $\mathsf{Filt}\,\mathcal{S}$ for the subcategory of modules in $\mathsf{mod}\,A$ filtered by objects in a semibrick $\mathcal{S}$. By a result by Ringel [Rin], $\mathsf{Filt}\,\mathcal{S}$ is an abelian subcategory of $\mathsf{mod}\,A$, and then, $\mathcal{S}$ is nothing but the set of simple objects in $\mathsf{Filt}\,\mathcal{S}$.

In general, there exist much more semibricks than other objects such as support τ-tilting modules. For example, if A is the path algebra of the Kronecker quiver $1 \rightrightarrows 2$, then there exists a semibrick $\mathcal{S}$ consisting of infinitely many bricks. By definition, any subset of a semibrick is also a semibrick, so A admits uncountably many semibricks. On the other hand, the cardinality of the set $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ of support τ-tilting modules is countable for any finite-dimensional algebra A, see [DIJ, Section 6].

To solve this problem, we define the left finiteness of semibricks in terms of *torsion classes* in Definition 1.2 in Part 1 of this thesis. More precisely, a semibrick $\mathcal{S}$ is said to be *left finite* if the smallest torsion class $\mathsf{T}(\mathcal{S})$ is functorially finite. We write $\mathsf{f_L\text{-}sbrick}\,A$ for the set of left finite semibricks. In this thesis, we construct bijections $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A \to \mathsf{f_L\text{-}sbrick}\,A$ and $\mathsf{2\text{-}smc}\,A \to$

$\mathsf{f_L}\text{-}\mathsf{sbrick}\,A$, and complete Koenig–Yang correspondences as follows:

$$\begin{tikzcd}
 & \text{progenerators} & \text{the simple modules} \\
\text{in the derived categories} & \mathsf{2\text{-}silt}\,A \arrow[r, "{[\text{KY, BY}]}"] \arrow[d, "{[\text{AIR}]}"'] & \mathsf{2\text{-}smc}\,A \arrow[d, "\textbf{Theorem 2}"'] \\
\text{in the module category} & \mathsf{s\tau\text{-}tilt}\,A \arrow[r, "\textbf{Theorem 1}"] & \mathsf{f_L\text{-}sbrick}\,A
\end{tikzcd}.$$

This thesis consists of two parts. Part 1 is based on [Asa1] and Part 2 is based on [Asa2]. In Part 1, we mainly deal with the properties of semibricks holding for any finite-dimensional algebra A. On the other hand, in Part 2, we investigate bricks and semibricks in the case that A is the preprojective algebra of Dynkin type $\mathbb{A}_n$ or $\mathbb{D}_n$, by using the general results obtained in Part 1 and the combinatorics of the corresponding Coxeter groups.

In Part 1, we unify Koenig–Yang correspondences by using left finite semibricks. First, we construct a canonical bijection $\mathsf{s\tau\text{-}tilt}\,A \to \mathsf{f_L\text{-}sbrick}\,A$, which is a bijection between two notions generalizing progenerators and the simple modules in the module category $\mathsf{mod}\,A$ in Section 1.

Theorem 1 (Theorem 1.3)**.** *There exists a bijection* $\mathsf{s\tau\text{-}tilt}\,A \to \mathsf{f_L\text{-}sbrick}\,A$ *defined as* $M \mapsto \mathsf{ind}(M/\mathsf{rad}_B\,M)$*, where* $B := \mathsf{End}_A(M)$*.*

It also follows from this bijection that the cardinality of every left finite semibrick is at most n_A, where n_A is the number of isomorphic classes of simple A-modules, see Corollary 1.10.

Next, we construct a bijection $\mathsf{2\text{-}smc}\,A \to \mathsf{f_L\text{-}sbrick}\,A$, which is a bijection between two notions generalizing the simple modules in the derived category $\mathsf{D^b}(\mathsf{mod}\,A)$ and the module category $\mathsf{mod}\,A$.

Theorem 2 (Theorem 2.3 (1))**.** *There exist a bijections*

$$? \cap \mathsf{mod}\,A\colon\ \mathsf{2\text{-}smc}\,A \to \mathsf{f_L\text{-}sbrick}\,A$$

given by $\mathcal{X} \mapsto \mathcal{X} \cap \mathsf{mod}\,A$*.*

Combining recent results by [AIR, BY, KY] and ours, we have the following commutative diagram of bijections. In this diagram, the arrows with labels in rectangles denote our new bijections obtained in Theorems 1 and 2. See also Section 0 and Theorem 2.3 (2) in Part 1 for the details.

$$\begin{tikzcd}
\mathsf{s\tau^{-1}\text{-}tilt}\,A \arrow[r, "\mathsf{Sub}"] \arrow[rr, bend left, "\boxed{M \mapsto \mathsf{ind}(\mathsf{soc}_B\,M)}"] & \mathsf{f\text{-}torf}\,A & \mathsf{f_R\text{-}sbrick}\,A \arrow[l, "\mathsf{F}"'] \\
\mathsf{2\text{-}cosilt}\,A \arrow[u, "H^{-1}"] \arrow[dr, "{I \mapsto ({}^{\perp}I[<0], {}^{\perp}I[>0])}"] & & \\
 & \mathsf{int\text{-}t\text{-}str}\,A \arrow[uu, "{(\text{heart})[-1] \cap \mathsf{mod}\,A}"'] \arrow[dd, "{(\text{heart}) \cap \mathsf{mod}\,A}"] \arrow[r, "\text{simples of heart}"] & \mathsf{2\text{-}smc}\,A \arrow[uu, "\boxed{?[-1] \cap \mathsf{mod}\,A}"'] \arrow[dd, "\boxed{? \cap \mathsf{mod}\,A}"'] \\
\mathsf{2\text{-}silt}\,A \arrow[uu, "\nu"] \arrow[ur, "{P \mapsto (P[<0]^{\perp}, P[>0]^{\perp})}"'] \arrow[d, "H^0"'] & & \\
\mathsf{s\tau\text{-}tilt}\,A \arrow[r, "\mathsf{Fac}"] \arrow[rr, bend right, "\boxed{M \mapsto \mathsf{ind}(M/\mathsf{rad}_B\,M)}"'] & \mathsf{f\text{-}tors}\,A & \mathsf{f_L\text{-}sbrick}\,A \arrow[l, "\mathsf{T}"']
\end{tikzcd}$$

We remark that if A is *τ-tilting finite* (that is, $\mathsf{s\tau\text{-}tilt}\,A$ is a finite set), then every torsion(-free) class is functorially finite [DIJ]. In this case, all the semibricks in $\mathsf{mod}\,A$ are left finite and right finite, so every semibrick is a finite set. Thus, in the case that A is τ-tilting finite, there is a bijection from the set $\mathsf{sbrick}\, A$ of semibricks to the set of modules whose endomorphism rings are products of division rings, given by $\mathcal{S} \mapsto \bigoplus_{S \in \mathcal{S}} S$. Under this isomorphism, we will identify $\mathcal{S}$ and $\bigoplus_{S \in \mathcal{S}} S$ in Part 2.

In Part 2, we study bricks and semibricks over the *preprojective algebras* $\mathit{\Pi}$ of Dynkin type $\mathit{\Delta} = \mathbb{A}_n, \mathbb{D}_n$. The relationship between $\mathit{\Pi}$ and the Coxeter group W of type $\mathit{\Delta}$ has been investigated by using the ideal $I(w)$ associated to $w \in W$ introduced in [BIRS, IR]. In particular, Mizuno [Miz1] showed that there exists a bijection $W \to \mathsf{s\tau\text{-}tilt}\, \mathit{\Pi}$ given by $w \mapsto I(w)$; hence, $\mathit{\Pi}$ is τ-tilting finite. Combining it with our results in Part 1, we obtain a bijection $S(?) \colon W \to \mathsf{sbrick}\, \mathit{\Pi}$. The semibrick $S(w)$ is expressed as $\mathsf{soc}_B\, \mathit{\Pi}(w)$ by using the quotient module $\mathit{\Pi}(w) := \mathit{\Pi}/I(w)$ seen as a module over its endomorphism ring $B := \mathsf{End}_{\mathit{\Pi}}(\mathit{\Pi}(w))$. Our goal in Part 2 is describing the semibrick $S(w)$ for each $w \in W$.

For this purpose, we use the property of the Coxeter group W as a *lattice*. The Coxeter group W has a partial order $\le$ called the *right weak order*, and then $(W, \le)$ is a lattice [BB]. From the point of view of lattice theory, it is important to consider the set $\mathsf{j\text{-}irr}\, W$ of join-irreducible elements in W and the canonical join representation $w = \bigvee_{i=1}^{m} w_i$ ($w_i \in \mathsf{j\text{-}irr}\, W$) of a given element $w \in W$, which was introduced by Reading [Rea]. They are strongly related to the representation theory of $\mathit{\Pi}$. First, the bijection $S(?)$ is restricted to a bijection $\mathsf{j\text{-}irr}\, W \to \mathsf{brick}\, \mathit{\Pi}$, where $\mathsf{brick}\, \mathit{\Pi}$ is the set of bricks in $\mathsf{mod}\, \mathit{\Pi}$, see Proposition 2.2. Moreover, for a given element $w \in W$, the canonical join representation of w gives the decomposition of the semibrick $S(w)$ into bricks as follows.

Theorem 3 (Corollary 2.3)**.** *Let $w \in W$ and take $w_1, w_2, \ldots, w_m \in \mathsf{j\text{-}irr}\, W$ such that $S(w) = \bigoplus_{i=1}^{m} S(w_i)$. Then, $w = \bigvee_{i=1}^{m} w_i$ holds, and it is the canonical join representation of w in W.*

Next, we construct an algorithm to obtain a quiver representation of the brick $S(w)$ associated to each $w \in \mathsf{j\text{-}irr}\, W$ from the expression of w as a signed permutation in Theorem 3.1 and Corollary 3.3 for type $\mathbb{A}_n$, and in Theorem 3.7 and Corollary 3.10 for type $\mathbb{D}_n$. Since the result for type $\mathbb{D}_n$ is complicated, we state the result for type $\mathbb{A}_n$ here.

Theorem 4 (Theorem 3.1, Corollary 3.3)**.** *Let $w \in \mathsf{j\text{-}irr}\, W(\mathbb{A}_n)$ with its unique descent l. Then, the brick $S(w)$ is given as follows.*

- *Set $R := w([l+1, n+1])$, $a := w(l)$, $b := w(l+1)$, and $V := [b, a-1]$.*
- *The brick $S(w)$ has a K-basis $(\langle i \rangle)_{i \in V}$, where $\langle i \rangle$ belongs to $e_i S(w)$.*
- *For each $i \in V$, place a symbol i denoting the K-vector subspace $K\langle i \rangle$.*
- *For each $i \in V \setminus \{\max V\}$, we write exactly one arrow between i and $i+1$, where the orientation is $i \to i+1$ if $i+1 \in R$ and $i \leftarrow i+1$ if $i+1 \notin R$.*

If $\mathit{\Delta} = \mathbb{A}_8$ and $w = (2, 5, 8, 1, 3, 4, 6, 7, 9) \in \mathsf{j\text{-}irr}\, W(\mathbb{A}_8)$, then Theorem 4 gives

$$S(w) = 1 \leftarrow 2 \rightarrow 3 \rightarrow 4 \leftarrow 5 \rightarrow 6 \rightarrow 7.$$

For a general element $w \in W$, we give an algorithm to determine the canonical join representation of w in Propositions 4.4 (type $\mathbb{A}_n$) and 4.8 (type $\mathbb{D}_n$). Consequently, we have the explicit form of the semibrick $S(w)$ as desired, see Theorems 4.6 (type $\mathbb{A}_n$) and 4.10 (type $\mathbb{D}_n$).

For example, if $\mathit{\Delta} = \mathbb{A}_8$ and $w = (4, 9, 3, 6, 2, 8, 5, 1, 7) \in W(\mathbb{A}_8)$, then the canonical join representation of w is $w_2 \vee w_4 \vee w_6 \vee w_7$, where

$$\begin{aligned} w_2 &:= (1, 2, 4, 9, 3, 5, 6, 7, 8), & w_4 &:= (1, 3, 4, 6, 2, 5, 7, 8, 9), \\ w_6 &:= (1, 2, 3, 4, 6, 8, 5, 7, 9), & w_7 &:= (2, 3, 4, 5, 1, 6, 7, 8, 9). \end{aligned}$$

Thus, the semibrick $S(w)$ is the direct sum of the following four bricks:

$$\begin{aligned}
S(w_2) &= \phantom{1 \leftarrow 2 \leftarrow{}} 3 \leftarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8,\\
S(w_4) &= \phantom{1 \leftarrow{}} 2 \leftarrow 3 \leftarrow 4 \rightarrow 5 \phantom{{} \rightarrow 6 \rightarrow 7 \rightarrow 8},\\
S(w_6) &= \phantom{1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \rightarrow{}} 5 \leftarrow 6 \rightarrow 7 \phantom{{} \rightarrow 8},\\
S(w_7) &= 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \phantom{{} \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8}.
\end{aligned}$$

Part 1. Semibricks

0. INTRODUCTION

In representation theory of a finite-dimensional algebra A over a field K, the notion of (semi)simple modules is fundamental. By Schur's lemma, they satisfy the following properties:

- the endomorphism ring of a simple module is a division algebra,
- there exists no nonzero homomorphism between two nonisomorphic simple modules.

A module M in $\mathsf{mod}\, A$ is called a *brick* if its endomorphism ring is a division algebra. This notion is a generalization of simple modules, and it has long been studied in representation theory [Rin, Gab1, Gab2]. Typical examples of bricks are given as preprojective modules and preinjective modules over a finite-dimensional hereditary algebra. Sometimes, it is useful to consider sets of isoclasses of pairwise Hom-orthogonal bricks. We simply call them *semibricks*, and define $\mathsf{sbrick}\, A$ as the set of semibricks. It follows from a classical result by Ringel [Rin] that the semibricks $\mathcal{S}$ correspond bijectively to the wide subcategories $\mathcal{W}$ of $\mathsf{mod}\, A$, that is, the subcategories which are closed under taking kernels, cokernels, and extensions. Under this correspondence, $\mathcal{S}$ is the set of the simple objects in $\mathcal{W}$, and $\mathcal{W}$ consists of all the A-modules filtered by bricks in $\mathcal{S}$. Moreover, bricks and wide subcategories have close relationship with ring epimorphisms and universal localizations [Ste, Sch, GL].

In this part, we assign a condition for semibricks. We say that a semibrick $\mathcal{S}$ is *left finite* if the smallest torsion class $\mathsf{T}(\mathcal{S}) \subset \mathsf{mod}\, A$ containing $\mathcal{S}$ is functorially finite. We write $\mathsf{f_L}\text{-}\mathsf{sbrick}\, A$ for the set of left finite semibricks. Right finite semibricks and the set $\mathsf{f_R}\text{-}\mathsf{sbrick}\, A$ are defined dually by using torsion-free classes. We define left-finiteness and right-finiteness of wide subcategories in the same way as semibricks. We write $\mathsf{f_L}\text{-}\mathsf{wide}\, A$ (resp. $\mathsf{f_R}\text{-}\mathsf{wide}\, A$) for the set of left (resp. right) finite wide subcategories of $\mathsf{mod}\, A$. Clearly, Ringel's bijection is restricted to bijections $\mathsf{f_L}\text{-}\mathsf{sbrick}\, A \to \mathsf{f_L}\text{-}\mathsf{wide}\, A$ and $\mathsf{f_R}\text{-}\mathsf{sbrick}\, A \to \mathsf{f_R}\text{-}\mathsf{wide}\, A$.

Recently, Adachi–Iyama–Reiten [AIR] obtained a bijection from the set $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ of basic *support τ-tilting modules* in $\mathsf{mod}\, A$ to the set $\mathsf{f}\text{-}\mathsf{tors}\, A$ of functorially finite torsion classes in $\mathsf{mod}\, A$, where M is sent to $\mathsf{Fac}\, M$. Support τ-tilting modules are a generalization of tilting modules and are defined by using the Auslander–Reiten translation τ. Adachi–Iyama–Reiten also proved that there are operations called *mutations* of support τ-tilting modules, that is, constructing a new support τ-tilting module by changing one indecomposable direct summand of a given one. They showed that such mutations are nothing but the adjacency relations with respect to the inclusions of the corresponding torsion classes.

The following first main result shows that the support τ-tilting modules also correspond bijectively to the left finite semibricks, where $\mathsf{ind}\, N$ denotes the set of isoclasses of indecomposable direct summands of N. This is an extension of a result by Demonet–Iyama–Jasso [DIJ].

Theorem 0.1 (Theorem 1.3)**.** *We have the following maps, where $B := \mathsf{End}_A(M)$ in each case.*

1. *There exists a surjection $\tau\text{-}\mathsf{rigid}\, A \to \mathsf{f_L}\text{-}\mathsf{sbrick}\, A$ defined as $M \mapsto \mathsf{ind}(M/\mathsf{rad}_B\, M)$. Under this map, the left finite semibricks for two τ-rigid modules $M, M' \in \tau\text{-}\mathsf{rigid}\, A$ coincide if and only if $\mathsf{Fac}\, M = \mathsf{Fac}\, M'$.*
2. *There exists a bijection $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A \to \mathsf{f_L}\text{-}\mathsf{sbrick}\, A$ defined as $M \mapsto \mathsf{ind}(M/\mathsf{rad}_B\, M)$.*

For example, this map sends the progenerator A to the set of simple A-modules. In the proof of Theorem 0.1, we obtain a canonical bijection $\mathsf{T}\colon \mathsf{f_L}\text{-}\mathsf{sbrick}\, A \to \mathsf{f}\text{-}\mathsf{tors}\, A$, see Proposition 1.9. Consequently, we recover Marks–Šťovíček [MS] bijection between $\mathsf{f_L}\text{-}\mathsf{wide}\, A$ and $\mathsf{f}\text{-}\mathsf{tors}\, A$. As an application of Theorem 0.1, we have the following result.

Corollary 0.2 (Corollary 1.10)**.** *If a semibrick $\mathcal{S}$ is either left finite or right finite, then $\#\mathcal{S} \leq n_A$ holds, where n_A is the number of isoclasses of simple A-modules.*

This corollary does not generally hold for arbitrary semibricks. For example, if A is the Kronecker algebra, then there exists a semibrick consisting of infinitely many bricks.

In Subsection 1.2, we use the map in Theorem 0.1 to label the exchange quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ with bricks, that is, to define a map from the set of arrows in the exchange quiver to the set of bricks, see Definition 1.14.

Subsection 1.3 is devoted to further study of wide subcategories. Our aim in this subsection is to describe each left finite wide subcategory $\mathcal{W}$ of $\mathsf{mod}\,A$ as the module category $\mathsf{mod}\,A'$ of some finite-dimensional algebra A'. We can take $M \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ corresponding to $\mathcal{W}$, then we have an equivalnce $\mathsf{Hom}_A(M,?)\colon \mathsf{Fac}\,M \to \mathsf{Sub}_B\,DM$ by a Brenner–Butler type theorem, where $B := \mathsf{End}_A(M)$. We restrict this equivalence to $\mathcal{W} \subset \mathsf{Fac}\,M$. The following result shows that we can regard $\mathcal{W}$ naturally as $\mathsf{mod}\,B/\langle e\rangle$ given by a certain explicit idempotent $e \in B$.

Theorem 0.3 (Theorem 1.27)**.** *The equivalence* $\mathsf{Hom}_A(M,?)\colon \mathsf{Fac}\,M \to \mathsf{Sub}_B\,DM$ *is restricted to an equivalence* $\mathsf{Hom}_A(M,?)\colon \mathcal{W} \to \mathsf{mod}\,B/\langle e\rangle$*.*

In Subsection 1.4, we consider semibricks for a factor algebra A/I with $I \subset A$ an ideal. We clearly have an inclusion $\mathsf{sbrick}\,A/I \subset \mathsf{sbrick}\,A$, and we give the following sufficient condition so that the equality $\mathsf{sbrick}\,A/I = \mathsf{sbrick}\,A$ holds: I is generated by some elements belonging to the intersection of the center $Z(A)$ and the radical $\mathsf{rad}\,A$. This condition on the ideal I was originally considered by Eisele–Janssens–Raedschelders [EJR]. In this situation, we also prove that $\mathsf{f_L}\text{-}\mathsf{sbrick}\,A/I = \mathsf{f_L}\text{-}\mathsf{sbrick}\,A$ and recover their canonical bijection $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A \to \mathsf{s}\tau\text{-}\mathsf{tilt}\,A/I$.

Furthermore, we apply semibricks to study the derived category $\mathsf{D^b}(\mathsf{mod}\,A)$ in Section 2. A *simple-minded collection* in $\mathsf{D^b}(\mathsf{mod}\,A)$ is a set of isoclasses of objects in $\mathsf{D^b}(\mathsf{mod}\,A)$ satisfying the conditions in Schur's Lemma and some additional conditions, see Definition 2.1 for details. For our purpose, it is useful to consider simple-minded collections which are *2-term*, that is, the ith cohomology $H^i(X)$ of every object X in a simple-minded collection vanishes if $i \neq -1, 0$. A simple-minded collection in $\mathsf{D^b}(\mathsf{mod}\,A)$ always consists of n_A objects, and if it is 2-term, each of the objects belongs to either $\mathsf{mod}\,A$ or $(\mathsf{mod}\,A)[1]$, see [KY, BY]. We write $\mathsf{2\text{-}smc}\,A$ for the set of 2-term simple-minded collections in $\mathsf{D^b}(\mathsf{mod}\,A)$. Our next main theorem gives bijections between the sets $\mathsf{2\text{-}smc}\,A$, $\mathsf{f_L}\text{-}\mathsf{sbrick}\,A$, and $\mathsf{f_R}\text{-}\mathsf{sbrick}\,A$.

Theorem 0.4 (Theorem 2.3 (1))**.** *There exist bijections*

$$? \cap \mathsf{mod}\,A\colon\ \mathsf{2\text{-}smc}\,A \to \mathsf{f_L}\text{-}\mathsf{sbrick}\,A \quad \text{and} \quad ?[-1] \cap \mathsf{mod}\,A\colon\ \mathsf{2\text{-}smc}\,A \to \mathsf{f_R}\text{-}\mathsf{sbrick}\,A$$

given by $\mathcal{X} \mapsto \mathcal{X} \cap \mathsf{mod}\,A$ *and* $\mathcal{X} \mapsto \mathcal{X}[-1] \cap \mathsf{mod}\,A$*.*

Consequently, a 2-term simple-minded collection in $\mathsf{D^b}(\mathsf{mod}\,A)$ is a union of a right finite semibrick shifted by [1] and a left finite semibrick.

In recent years, thanks to many authors such as [AIR, BY, IT, KY, MS], canonical bijections between the sets of important objects containing the following ones have been discovered. Here, $\mathsf{proj}\,A$ (resp. $\mathsf{inj}\,A$) denotes the full subcategory of $\mathsf{mod}\,A$ consisting of all the projective (resp. injective) modules.

(a) The set $\mathsf{f_L}\text{-}\mathsf{sbrick}\,A$ of left finite semibricks in $\mathsf{mod}\,A$.
(a′) The set $\mathsf{f_R}\text{-}\mathsf{sbrick}\,A$ of right finite semibricks in $\mathsf{mod}\,A$.
(b) The set $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ of isoclasses of basic support τ-tilting A-modules in $\mathsf{mod}\,A$.
(b′) The set $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A$ of isoclasses of basic support τ^{-1}-tilting A-modules in $\mathsf{mod}\,A$.
(c) The set $\mathsf{f\text{-}tors}\,A$ of functorially finite torsion classes in $\mathsf{mod}\,A$.
(c′) The set $\mathsf{f\text{-}torf}\,A$ of functorially finite torsion-free classes in $\mathsf{mod}\,A$.
(d) The set $\mathsf{f_L}\text{-}\mathsf{wide}\,A$ of left finite wide subcategories of $\mathsf{mod}\,A$.
(d′) The set $\mathsf{f_R}\text{-}\mathsf{wide}\,A$ of right finite wide subcategories of $\mathsf{mod}\,A$.
(e) The set $\mathsf{2\text{-}smc}\,A$ of 2-term simple-minded collections in $\mathsf{D^b}(\mathsf{mod}\,A)$.
(f) The set $\mathsf{int\text{-}t\text{-}str}\,A$ of intermediate t-structures with length heart in $\mathsf{D^b}(\mathsf{mod}\,A)$.
(g) The set $\mathsf{2\text{-}silt}\,A$ of 2-term silting objects in $\mathsf{K^b}(\mathsf{proj}\,A)$.
(g′) The set $\mathsf{2\text{-}cosilt}\,A$ of 2-term cosilting objects in $\mathsf{K^b}(\mathsf{inj}\,A)$.

If A is the path algebra of a Dynkin quiver Q, Ingalls–Thomas [IT] showed that there are also one-to-one correspondences between the above sets and the set of clusters in the cluster algebra for the quiver Q, which was introduced by Fomin–Zelevinsky [FZ].

Figure 1 shows some of the known bijections and our new bijections (arrows with labels in rectangles). We prove the following result.

Theorem 0.5 (Theorem 2.3 (2))**.** *The diagram in Figure 1 below is commutative and all the maps are bijective. In this diagram,* $\mathcal{T} \in \mathsf{f\text{-}tors}\,A$ *corresponds to* $\mathcal{F} \in \mathsf{f\text{-}torf}\,A$ *if and only if* $(\mathcal{T}, \mathcal{F})$ *is a torsion pair in* $\mathsf{mod}\,A$*.*

$$\begin{tikzcd}
\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A \arrow[r, "\mathsf{Sub}"] \arrow[rr, bend left, "\boxed{M \mapsto \mathsf{ind}(\mathsf{soc}_B M)}"] & \mathsf{f\text{-}torf}\,A & \mathsf{f_R\text{-}sbrick}\,A \arrow[l, "\mathsf{F}"'] \\
2\text{-}\mathsf{cosilt}\,A \arrow[u, "H^{-1}"] \arrow[dr, "I \mapsto ({}^{\perp}I[<0], {}^{\perp}I[>0])"] & & \\
 & \mathsf{int\text{-}t\text{-}str}\,A \arrow[uu, "(\text{heart})[-1]\cap\mathsf{mod}\,A"'] \arrow[dd, "(\text{heart})\cap\mathsf{mod}\,A"] \arrow[r, "\text{simples of heart}"] & 2\text{-}\mathsf{smc}\,A \arrow[uu, "\boxed{?[-1] \cap \mathsf{mod}\,A}"'] \arrow[dd, "\boxed{? \cap \mathsf{mod}\,A}"] \\
2\text{-}\mathsf{silt}\,A \arrow[uu, "\nu"] \arrow[ur, "P \mapsto (P[<0]^{\perp}, P[>0]^{\perp})"'] \arrow[d, "H^0"'] & & \\
\mathsf{s}\tau\text{-}\mathsf{tilt}\,A \arrow[r, "\mathsf{Fac}"] \arrow[rr, bend right, "\boxed{M \mapsto \mathsf{ind}(M/\mathsf{rad}_B M)}"'] & \mathsf{f\text{-}tors}\,A & \mathsf{f_L\text{-}sbrick}\,A \arrow[l, "\mathsf{T}"']
\end{tikzcd}$$

FIGURE 1. The commutative diagram

In Subsection 2.3, we study left finite wide subcategories $\mathcal{W}$ in a similar way to Subsection 1.3, but we use $P \in 2\text{-}\mathsf{silt}\,A$ and $C := \mathsf{End}_{\mathsf{D^b}(\mathsf{mod}\,A)}(P)$ corresponding to $\mathcal{W}$, instead of $M \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ and $B = \mathsf{End}_A(M)$. There is an equivalence $\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\,A)}(P, ?) \colon \mathcal{H} \to \mathsf{mod}\,C$ [IY], where $\mathcal{H}$ is the heart of $(P[<0]^{\perp}, P[>0]^{\perp}) \in \mathsf{int\text{-}t\text{-}str}\,A$. For an explicitly defined idempotent $f \in C$, we have the following equivalence, which is parallel to Theorem 0.3.

Theorem 0.6 (Theorem 2.15)**.** *The equivalence* $\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\,A)}(P, ?) \colon \mathcal{H} \to \mathsf{mod}\,C$ *is restricted to an equivalence* $\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\,A)}(P, ?) \colon \mathcal{W} \to \mathsf{mod}\,C/\langle f \rangle$*.*

Moreover, there is a canonical surjection $\varphi \colon C \to B$ of K-algebras, and so we have a fully faithful functor $\mathsf{mod}\,B \to \mathsf{mod}\,C$. The surjection φ induces a surjection $\varphi \colon C/\langle f \rangle \to B/\langle e \rangle$ and a fully faithful functor $\mathsf{mod}\,B/\langle e \rangle \to \mathsf{mod}\,C/\langle f \rangle$. This embedding is actually an equivalence; hence, $\varphi \colon C/\langle f \rangle \to B/\langle e \rangle$ is an isomorphism of algebras, see Theorem 2.16.

In Subsection 2.4, we investigate 2-term silting objects in $\mathsf{K^b}(\mathsf{proj}\,A)$ and 2-term simple-minded collections in $\mathsf{D^b}(\mathsf{mod}\,A)$ from the point of view of the Grothendieck groups.

Let $P \in 2\text{-}\mathsf{silt}\,A$. Then, P has exactly n_A nonisomorphic indecomposable direct summands [KY]. The set $\{P_1, P_2, \ldots, P_{n_A}\}$ of indecomposable direct summands of P gives a $\mathbb{Z}$-basis of the Grothendieck group $K_0(\mathsf{K^b}(\mathsf{proj}\,A))$ [AI]. On the other hand, the corresponding 2-term simple-minded collection $\mathcal{X} \in 2\text{-}\mathsf{smc}\,A$ has n_A distinct elements, which are denoted by $X_1, X_2, \ldots, X_{n_A}$. They induce a $\mathbb{Z}$-basis of the Grothendieck group $K_0(\mathsf{D^b}(\mathsf{mod}\,A))$ [KY].

There is a natural $\mathbb{Z}$-bilinear form $\langle ?, ? \rangle \colon K_0(\mathsf{K^b}(\mathsf{proj}\,A)) \times K_0(\mathsf{D^b}(\mathsf{mod}\,A)) \to \mathbb{Z}$ given as

$$\langle P, X \rangle := \sum_{j \in \mathbb{Z}} (-1)^j \dim_K \mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\,A)}(P, X[j]).$$

We prove that the $\mathbb{Z}$-basis $\{P_1, P_2, \dots, P_{n_A}\}$ of $K_0(\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A))$ and the $\mathbb{Z}$-basis $\{X_1, X_2, \dots, X_{n_A}\}$ of $K_0(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A))$ satisfy the following "duality" with respect to the bilinear form $\langle ?, ? \rangle$.

Theorem 0.7 (Theorem 2.17). *There exists a permutation* $\sigma\colon \{1, 2, \dots, n_A\} \to \{1, 2, \dots, n_A\}$ *satisfying*

$$\langle P_k, X_l \rangle = \begin{cases} \mathsf{dim}_K \, \mathsf{End}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)}(X_l) & (\sigma(k) = l) \\ 0 & (\sigma(k) \neq l) \end{cases}.$$

We give examples of semibricks and wide subcategories in Section 3.

In Subsection 3.1, we calculate the numbers $a_{n,l} := \# \, \mathsf{sbrick}\, A_{n,l}$ and $b_{n,l} := \# \, \mathsf{sbrick}\, B_{n,l}$ for the following two families $(A_{n,l})_{n,l \geq 1}$ and $(B_{n,l})_{n,l \geq 1}$ of Nakayama algebras:

$$\begin{tikzcd} A_{n,l}\colon & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n, & \text{all the paths of length } l \text{ are } 0, \end{tikzcd}$$

$$\begin{tikzcd} B_{n,l}\colon & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[lll, rounded corners, to path={ -- ([yshift=-3ex]\tikztostart.south) -| (\tikztotarget)}], & \text{all the paths of length } l \text{ are } 0. \end{tikzcd}$$

Our result is expressed with the Catalan numbers $(c_n)_{n \geq 0}$. In particular, a path algebra of type $\mathbb{A}_n$ has exactly c_{n+1} semibricks.

Theorem 0.8 (Theorem 3.1). *Let* $n, l \geq 1$ *be integers. The following equations hold:*

$$a_{n,l} = c_{n+1} \quad (n = 1, 2, \dots, l), \qquad a_{n,l} = 2a_{n-1,l} + \sum_{i=2}^{l} c_{i-1} a_{n-i,l} \quad (n \geq l+1),$$

$$b_{n,l} = (n+1)c_n \quad (n = 1, 2, \dots, l), \qquad b_{n,l} = 2b_{n-1,l} + \sum_{i=2}^{l} c_{i-1} b_{n-i,l} \quad (n \geq l+1).$$

Since Nakayama algebras are representation-finite, every semibrick is left finite. Thus, by Theorem 0.1, there is a bijection between the sets $\mathsf{sbrick}\, A$ and $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$. The latter set was also investigated by Adachi [Ada].

Subsection 3.2 deals with functorially finiteness of wide subcategories. We write $\mathsf{f}\text{-}\mathsf{wide}\, A$ for the set of functorially finite wide subcategories of $\mathsf{mod}\, A$. If A is hereditary, then $\mathsf{f}\text{-}\mathsf{wide}\, A$ coincides with $\mathsf{f_L}\text{-}\mathsf{wide}\, A$ [IT]. We show that there is an inclusion $\mathsf{f_L}\text{-}\mathsf{wide}\, A \subset \mathsf{f}\text{-}\mathsf{wide}\, A$ for an arbitrary algebra A, and we give an example of a tilted algebra A which does not satisfy the equality $\mathsf{f_L}\text{-}\mathsf{wide}\, A = \mathsf{f}\text{-}\mathsf{wide}\, A$ (Example 3.13). Moreover, we prove the following result in the case that K is an algebraically closed field.

Theorem 0.9 (Theorem 3.14). *Let* H *be a hereditary algebra,* $T \in \mathsf{mod}\, H$ *be a tilting module, and* $A := \mathsf{End}_H(T)$. *Then the following assertions hold.*

(1) *If* $\mathsf{Sub}_H\, \tau T$ *has only finitely many indecomposable* H*-modules, then* $\mathsf{f}\text{-}\mathsf{wide}\, A = \mathsf{f_L}\text{-}\mathsf{wide}\, A$. *If* $\mathsf{Fac}_H\, T$ *has only finitely many indecomposable* H*-modules, then* $\mathsf{f}\text{-}\mathsf{wide}\, A = \mathsf{f_R}\text{-}\mathsf{wide}\, A$.

(2) *If* T *is either preprojective or preinjective, then* $\mathsf{f}\text{-}\mathsf{wide}\, A = \mathsf{f_L}\text{-}\mathsf{wide}\, A = \mathsf{f_R}\text{-}\mathsf{wide}\, A$.

(3) *Assume that* H *is a hereditary algebra of extended Dynkin type. We decompose* T *as* $T_{\mathrm{pp}} \oplus T_{\mathrm{reg}} \oplus T_{\mathrm{pi}}$ *with a preprojective module* T_{pp}*, a regular module* T_{reg}*, and a preinjective module* T_{pi}*. If* $T_{\mathrm{reg}} \neq 0$ *and* $T_{\mathrm{pi}} = 0$*, then we have* $\mathsf{f}\text{-}\mathsf{wide}\, A \supsetneq \mathsf{f_R}\text{-}\mathsf{wide}\, A$*, and if* $T_{\mathrm{reg}} \neq 0$ *and* $T_{\mathrm{pp}} = 0$*, then we have* $\mathsf{f}\text{-}\mathsf{wide}\, A \supsetneq \mathsf{f_L}\text{-}\mathsf{wide}\, A$.

As a consequence of Theorem 0.9 and [SimS, XVII.3], we obtain the following criterion in the case that H is a hereditary algebra of extended Dynkin type.

Corollary 0.10 (Corollary 3.16). *We use the setting of Theorem 3.14 (3). Then, the tilting* H*-module* T *satisfies one of the conditions (1)–(5) in the following table, which shows whether* $\mathsf{f}\text{-}\mathsf{wide}\, A = \mathsf{f_L}\text{-}\mathsf{wide}\, A$ *or* $\mathsf{f}\text{-}\mathsf{wide}\, A = \mathsf{f_R}\text{-}\mathsf{wide}\, A$ *holds in each case.*

No.	$T_{\rm pp}$	$T_{\rm reg}$	$T_{\rm pi}$	$\mathsf{f\text{-}wide}\,A = \mathsf{f_L\text{-}wide}\,A$	$\mathsf{f\text{-}wide}\,A = \mathsf{f_R\text{-}wide}\,A$
(1)	$\neq 0$	$= 0$	$= 0$	Yes	Yes
(2)	$= 0$	$= 0$	$\neq 0$	Yes	Yes
(3)	$\neq 0$	$\neq 0$	$= 0$	Yes	No
(4)	$= 0$	$\neq 0$	$\neq 0$	No	Yes
(5)	$\neq 0$		$\neq 0$	Yes	Yes

0.1. **Notation.** We use the symbol "$\subset$" as the meaning of "$\subseteq$". The composition of maps or functors $f\colon X \to Y$ and $g\colon Y \to Z$ is denoted by gf.

Throughout of this part, K is a field and A is a finite-dimensional K-algebra. The category of finite-dimensional right A-modules is denoted by $\mathsf{mod}\,A$, and its full subcategory consisting of all the projective (resp. injective) A-modules is denoted by $\mathsf{proj}\,A$ (resp. $\mathsf{inj}\,A$). Unless otherwise stated, algebras and modules are finite-dimensional, and subcategories are full subcategories which are closed under taking isomorphic objects. The bounded derived category of $\mathsf{mod}\,A$ is denoted by $\mathsf{D}^{\rm b}(\mathsf{mod}\,A)$, and the homotopy category of the bounded complex category over $\mathsf{proj}\,A$ (resp. $\mathsf{inj}\,A$) is denoted by $\mathsf{K}^{\rm b}(\mathsf{proj}\,A)$ (resp. $\mathsf{K}^{\rm b}(\mathsf{inj}\,A)$).

For each full subcategory $\mathcal{C} \subset \mathsf{mod}\,A$, we define full subcategories of $\mathsf{mod}\,A$ as follows:

- $\mathsf{add}\,\mathcal{C}$ is the additive closure of $\mathcal{C}$,
- $\mathsf{Fac}\,\mathcal{C}$ consists of the factor modules of objects in $\mathsf{add}\,\mathcal{C}$,
- $\mathsf{Sub}\,\mathcal{C}$ consists of the submodules of objects in $\mathsf{add}\,\mathcal{C}$,
- $\mathsf{Filt}\,\mathcal{C}$ consists of the objects M such that there exists a sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ with $M_i/M_{i-1} \in \mathsf{add}\,\mathcal{C}$,
- $\mathcal{C}^\perp$ and ${}^\perp\mathcal{C}$ are defined as $\mathcal{C}^\perp := \{M \in \mathsf{mod}\,A \mid \mathsf{Hom}_A(\mathcal{C}, M) = 0\}$ and ${}^\perp\mathcal{C} := \{M \in \mathsf{mod}\,A \mid \mathsf{Hom}_A(M, \mathcal{C}) = 0\}$.

If the set of objects of $\mathcal{C}$ is a finite set $\{M_1, \ldots, M_n\}$, then we write $\mathsf{add}(M_1, \ldots, M_n)$ for $\mathsf{add}\,\mathcal{C}$, and other full subcategories are similarly denoted. In order to emphasize the algebra A, we sometimes use notations such as $\mathsf{add}_A\,\mathcal{C}$.

For $M \in \mathsf{mod}\,A$, the notation $\mathsf{ind}\,M$ denotes the set of isoclasses of indecomposable direct summands of M, and we set $|M| := \#(\mathsf{ind}\,M)$. If $M \cong \bigoplus_{i=1}^m M_i^{n_i}$ with M_i indecomposable, $M_i \not\cong M_j$ $(i \neq j)$, and $n_i \geq 1$, then $\mathsf{ind}\,M = \{M_1, \ldots, M_m\}$ and $|M| = m$ hold.

For a set $X \subset A$, the notation $\langle X \rangle$ denotes the two-sided ideal of A generated by X. We define a functor D as the K-dual $\mathsf{Hom}_K(?, K)$.

We often identify an object and its isoclass.

1. SEMIBRICKS IN MODULE CATEGORIES

In this section, we consider the sets of pairwise Hom-orthogonal bricks called *semibricks*, and give bijections between the semibricks and other concepts in module categories, especially *support τ-tilting modules*.

1.1. **Bijections I.**

We first recall the definition of bricks, and give the definition of semibricks.

Definition 1.1. We define the following notions.

(1) A module S in $\mathsf{mod}\,A$ is called a *brick* if $\mathsf{End}_A(S)$ is a division K-algebra. We write $\mathsf{brick}\,A$ for the set of isoclasses of bricks in $\mathsf{mod}\,A$.

(2) A subset $\mathcal{S} \subset \mathsf{brick}\,A$ is called a *semibrick* if $\mathsf{Hom}_A(S_1, S_2) = 0$ holds for any $S_1 \neq S_2 \in \mathcal{S}$. We write $\mathsf{sbrick}\,A$ for the set of semibricks in $\mathsf{mod}\,A$.

By Schur's lemma, a simple A-module is a brick, and a set of isoclasses of simple A-modules is a semibrick.

Note that we allow a semibrick to be an infinite set. This occurs, for example, in the case of the Kronecker quiver algebra. Any subset of a semibrick is also a semibrick, so if there is a semibrick consisting of infinitely many bricks in $\mathsf{mod}\,A$, then there are uncountably many semibricks in $\mathsf{mod}\, A$. In this part, we treat the semibricks satisfying some condition on torsion pairs. Each of such semibricks has $|A|$ bricks at most, see Corollary 1.10.

We recall some fundamental properties of torsion pairs.

A full subcategory $\mathcal{T} \subset \mathsf{mod}\, A$ is called a *torsion class* if $\mathcal{T}$ is closed under taking extensions and factor modules, and a full subcategory $\mathcal{F} \subset \mathsf{mod}\, A$ is called a *torsion-free class* if $\mathcal{F}$ is closed under taking extensions and submodules. We define $\mathsf{tors}\, A$ as the set of torsion classes in $\mathsf{mod}\, A$ and $\mathsf{torf}\, A$ as the set of torsion-free classes in $\mathsf{mod}\, A$. For a torsion class $\mathcal{T}$, the corresponding torsion pair is $(\mathcal{T}, \mathcal{T}^{\perp})$, and for a torsion-free class $\mathcal{F}$, the corresponding torsion pair is $({}^{\perp}\mathcal{F}, \mathcal{F})$. Note that ${}^{\perp}\mathcal{C} \in \mathsf{tors}\, A$ and $\mathcal{C}^{\perp} \in \mathsf{torf}\, A$ hold for any full subcategory $\mathcal{C} \subset \mathsf{mod}\, A$.

Let $\mathcal{C} \subset \mathsf{mod}\, A$ be a full subcategory. We write $\mathsf{T}(\mathcal{C})$ for the smallest torsion class containing $\mathcal{C}$, and $\mathsf{F}(\mathcal{C})$ for the smallest torsion-free class containing $\mathcal{C}$. It is well-known that $\mathsf{T}(\mathcal{C}) = \mathsf{Filt}(\mathsf{Fac}\, \mathcal{C})$ and $\mathsf{F}(\mathcal{C}) = \mathsf{Filt}(\mathsf{Sub}\, \mathcal{C})$ hold, see [MS, Lemma 3.1].

We mainly consider functorially finite torsion classes and torsion-free classes (see [MS, Section 2] for the definition of functorially finiteness) in this part. We write $\mathsf{f\text{-}tors}\, A$ for the set of functorially finite torsion classes in $\mathsf{mod}\, A$ and $\mathsf{f\text{-}torf}\, A$ for the set of functorially finite torsion-free classes in $\mathsf{mod}\, A$. By [Sma, Theorem], for a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathsf{mod}\, A$, $\mathcal{T} \in \mathsf{f\text{-}tors}\, A$ holds if and only if $\mathcal{F} \in \mathsf{f\text{-}torf}\, A$.

With these preparations on torsion pairs, we define left finiteness and right finiteness of semibricks as follows.

Definition 1.2. Let $\mathcal{S} \in \mathsf{sbrick}\, A$.

(1) The semibrick $\mathcal{S}$ is said to be *left finite* if $\mathsf{T}(\mathcal{S})$ is functorially finite. We write $\mathsf{f_L\text{-}sbrick}\, A$ for the set of left finite semibricks in $\mathsf{mod}\, A$.

(2) The semibrick $\mathcal{S}$ is said to be *right finite* if $\mathsf{F}(\mathcal{S})$ is functorially finite. We write $\mathsf{f_R\text{-}sbrick}\, A$ for the set of right finite semibricks in $\mathsf{mod}\, A$.

One may wonder whether a subset of a left finite semibrick is left finite again. It does not hold in general, see Example 3.13.

We recall the notion of support τ-tilting modules, which was introduced by Adachi–Iyama–Reiten [AIR]. Let $M \in \mathsf{mod}\, A$ and $P \in \mathsf{proj}\, A$, then the A-module M is called a *τ-rigid module* if M satisfies $\mathsf{Hom}_A(M, \tau M) = 0$ for the Auslander–Reiten translation τ, and the pair (M, P) is called a *τ-rigid pair* if M is τ-rigid and $\mathsf{Hom}_A(P, M) = 0$. We write $\tau\text{-}\mathsf{rigid}\, A$ for the set of isoclasses of basic τ-rigid modules in $\mathsf{mod}\, A$. Here, M is said to be *basic* if M can be decomposed into $\bigoplus_{i=1}^{m} M_i$ with M_i indecomposable and $M_i \not\cong M_j$ for $i \neq j$. We remark that any τ-rigid module M satisfies $\mathsf{Ext}_A^1(M, \mathsf{Fac}\, M) = 0$, see [AS2, Proposition 5.8].

Assume that (M, P) is a τ-rigid pair. We say that (M, P) is a *support τ-tilting pair* if $|M| + |P| = |A|$, and that (M, P) is an *almost support τ-tilting pair* if $|M| + |P| = |A| - 1$. An A-module M is called a *support τ-tilting module* if there exists $P \in \mathsf{proj}\, A$ such that (M, P) is a support τ-tilting pair. We write $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ for the set of isoclasses of basic support τ-tilting modules in $\mathsf{mod}\, A$, and we use both notations $M \in \mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ and $(M, P) \in \mathsf{s}\tau\text{-}\mathsf{tilt}\, A$. We remark that if (M, P) and (M, Q) are support τ-tilting pairs, then $\mathsf{add}\, P = \mathsf{add}\, Q$ holds [AIR, Proposition 2.3].

The notions of τ^{-1}-rigid modules and support τ^{-1}-tilting modules are also defined dually, see the paragraphs just before [AIR, Theorem 2.15]. We write $\tau^{-1}\text{-}\mathsf{rigid}\, A$ for the set of isoclasses of basic τ^{-1}-rigid modules in $\mathsf{mod}\, A$ and $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$ for the set of isoclasses of basic support τ^{-1}-tilting modules in $\mathsf{mod}\, A$.

Now, we are ready to state our first main theorem, which gives a large extension of a bijection given in [DIJ, Theorem 4.1]. Our theorem gives a bijection between the support τ-tilting modules and the left finite semibricks, and its dual bijection.

Theorem 1.3. *We have the following maps, where $B := \mathsf{End}_A(M)$ in each case.*

(1) *There exists a surjection $\tau\text{-}\mathsf{rigid}\, A \to \mathsf{f_L\text{-}sbrick}\, A$ defined as $M \mapsto \mathsf{ind}(M/\mathsf{rad}_B\, M)$. Under this map, the left finite semibricks for two τ-rigid modules $M, M' \in \tau\text{-}\mathsf{rigid}\, A$ coincide if and only if $\mathsf{Fac}\, M = \mathsf{Fac}\, M'$.*

(2) *There exists a bijection* $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A \to \mathsf{f_L}\text{-}\mathsf{sbrick}\,A$ *defined as* $M \mapsto \mathsf{ind}(M/\mathsf{rad}_B\,M)$.

(3) *There exists a surjection* $\tau^{-1}\text{-}\mathsf{rigid}\,A \to \mathsf{f_R}\text{-}\mathsf{sbrick}\,A$ *defined as* $M \mapsto \mathsf{ind}(\mathsf{soc}_B\,M)$. *Under this map, the right finite semibricks for two* τ^{-1}*-rigid modules* $M, M' \in \tau^{-1}\text{-}\mathsf{rigid}\,A$ *coincide if and only if* $\mathsf{Sub}\,M = \mathsf{Sub}\,M'$.

(4) *There exists a bijection* $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A \to \mathsf{f_R}\text{-}\mathsf{sbrick}\,A$ *defined as* $M \mapsto \mathsf{ind}(\mathsf{soc}_B\,M)$.

We remark that the two conditions $\#\,\mathsf{s}\tau\text{-}\mathsf{tilt}\,A < \infty$ and $\mathsf{f}\text{-}\mathsf{tors}\,A = \mathsf{tors}\,A$ are equivalent [DIJ, Theorem 3.8, Proposition 3.9]. In this case, A is said to be *τ-tilting finite*, and we have the equations $\mathsf{f_L}\text{-}\mathsf{sbrick}\,A = \mathsf{f_R}\text{-}\mathsf{sbrick}\,A = \mathsf{sbrick}\,A$ and the bijections $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A \to \mathsf{sbrick}\,A$ and $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A \to \mathsf{sbrick}\,A$.

In the rest of this subsection, we prove Theorem 1.3. To investigate the maps in this theorem in detail, we sometimes use the following notation.

Definition 1.4. Let $M \in \tau\text{-}\mathsf{rigid}\,A$ and $B := \mathsf{End}_A(M)$. We decompose M as $M = \bigoplus_{i=1}^m M_i$ with M_i indecomposable, and then define

$$L := \mathsf{rad}_B\,M, \quad N := M/L, \quad L_i := \sum_{f \in \mathsf{rad}_A(M,M_i)} \mathsf{Im}\,f \subset M_i, \quad N_i := M_i/L_i$$

for $i = 1, 2, \ldots, m$. Moreover, we set $I := \{i \in \{1, 2, \ldots, m\} \mid N_i \neq 0\}$.

Clearly, $L = \bigoplus_{i=1}^m L_i$ and $N = \bigoplus_{i=1}^m N_i$ hold. Using this notation, we first show the well-definedness of the maps in Theorem 1.3.

Lemma 1.5. *In the setting of Definition 1.4, the following assertions hold.*

(1) *We have* $\mathsf{Ext}^1_A(M, L) = 0$.
(2) *The module* N_i *is a brick or zero for each* i.
(3) *If* $i \neq j$, *we have* $\mathsf{Hom}_A(M_i, N_j) = 0$ *and* $\mathsf{Hom}_A(N_i, N_j) = 0$.
(4) *The* A*-module* N *is basic and* $\mathsf{ind}\,N = \{N_i \mid i \in I\} \in \mathsf{sbrick}\,A$.
(5) *The torsion class* $\mathsf{T}(N)$ *is equal to* $\mathsf{Fac}\,M$.
(6) *We have* $\mathsf{ind}\,N = \{N_i \mid i \in I\} \in \mathsf{f_L}\text{-}\mathsf{sbrick}\,A$.

Proof. (1) Because M is τ-rigid, it is enough to show that $L \in \mathsf{Fac}\,M$. Take a K-basis $f_1, \ldots, f_s \colon M \to M$ of $\mathsf{rad}\,B \subset B = \mathsf{End}_A(M)$, and set $f := \begin{bmatrix} f_1 & \cdots & f_s \end{bmatrix} : M^{\oplus s} \to M$. Then, $\mathsf{Im}\,f = \mathsf{rad}_B\,M$ holds, so $L \in \mathsf{Fac}\,M$. Thus, we have the assertion.

(2) It is sufficient to show that any nonzero endomorphism $f \colon N_i \to N_i$ is an isomorphism. Consider the exact sequence $0 \to L_i \xrightarrow{\mu_i} M_i \xrightarrow{\pi_i} N_i \to 0$. Since $\mathsf{Ext}^1_A(M_i, L_i) = 0$ follows from (1), we obtain that $\mathsf{Hom}_A(M_i, \pi_i)$ is surjective. Thus, there exists $g \colon M_i \to M_i$ such that $\pi_i g = f\pi_i$. Because $f \neq 0$, the map g is an isomorphism (otherwise, we have $\mathsf{Im}\,g \subset L_i$, which yields $f = 0$, a contradiction). Therefore, f is also an isomorphism.

(3) Let $i \neq j$ and $f \colon M_i \to N_j$. Consider the exact sequence $0 \to L_j \xrightarrow{\mu_j} M_j \xrightarrow{\pi_j} N_j \to 0$. Since $\mathsf{Ext}^1_A(M_i, L_j) = 0$ follows from (1), we get that $\mathsf{Hom}_A(M_i, \pi_j)$ is surjective. Thus, there exists $g \colon M_i \to M_j$ such that $\pi_j g = f$. Because M is basic, we get $M_i \not\cong M_j$. Therefore, we have $\mathsf{Im}\,g \subset L_j$ and $f = 0$. We get that $\mathsf{Hom}_A(M_i, N_j) = 0$ for all $i \neq j$. Because $N_i \in \mathsf{Fac}\,M_i$, we also obtain that $\mathsf{Hom}_A(N_i, N_j) = 0$.

(4) We obtain the assertions by (2) and (3).

(5) Since M is τ-rigid, $\mathsf{Fac}\,M$ is a torsion class [AS2, Theorem 5.10]. Thus, the inclusion $\mathsf{T}(N) \subset \mathsf{Fac}\,M$ holds. To prove that $\mathsf{Fac}\,M \subset \mathsf{T}(N)$, it suffices to show that $M \in \mathsf{T}(N)$. We define $f \colon M^{\oplus s} \to M$ as in (1), then $f((\mathsf{rad}^t_B\,M)^{\oplus s}) = \mathsf{rad}^{t+1}_B\,M$ holds for all $t \geq 0$. Therefore, $\mathsf{rad}^{t+1}_B\,M/\mathsf{rad}^{t+2}_B\,M \in \mathsf{Fac}(\mathsf{rad}^t_B\,M/\mathsf{rad}^{t+1}_B\,M)$. By induction, we have $\mathsf{rad}^t_B\,M/\mathsf{rad}^{t+1}_B\,M \in \mathsf{Fac}\,N$ for all $t \geq 0$. Moreover, there exists t_0 such that $\mathsf{rad}^{t_0}_B\,M = 0$. Then the sequence $M \supset \mathsf{rad}_B\,M \supset \cdots \supset \mathsf{rad}^{t_0}_B\,M = 0$ shows that $M \in \mathsf{Filt}(\mathsf{Fac}\,N) = \mathsf{T}(N)$. Thus, the assertion $\mathsf{T}(N) = \mathsf{Fac}\,M$ is proved.

(6) By (4), it remains to show $\mathsf{T}(N) \in \textsf{f-tors}\, A$. We have obtained $\mathsf{T}(N) = \mathsf{Fac}\, M \in \mathsf{tors}\, A$ in (5), and $\mathsf{Fac}\, M$ is functorially finite in $\mathsf{mod}\, A$ by [AS1, Proposition 4.6]. Thus, we have the assertion. ☐

We recall an important property of support τ-tilting modules from [AIR], which associates the support τ-tilting modules and the functorially finite torsion classes bijectively.

Proposition 1.6. [AIR, Theorem 2.7] *There exists a bijection* $\mathsf{Fac}\colon \mathsf{s}\tau\text{-}\mathsf{tilt}\, A \to \textsf{f-tors}\, A$ *defined as* $M \mapsto \mathsf{Fac}\, M$. *The inverse* $\textsf{f-tors}\, A \to \mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ *is given by sending each* $\mathcal{T} \in \textsf{f-tors}\, A$ *to the direct sum of indecomposable Ext-projective objects in* $\mathcal{T}$.

To show Theorem 1.3, we study the operation T taking the smallest torsion class in the following lemmas.

Lemma 1.7. *Let* $\mathcal{S} \in \mathsf{sbrick}\, A$, *then the following assertions hold.*

1. *Let* $S \in \mathcal{S}$. *Every nonzero homomorphism* $f\colon M \to S$ *with* $M \in \mathsf{T}(\mathcal{S})$ *is surjective. Moreover, we have* $\mathsf{Ker}\, f \in \mathsf{T}(\mathcal{S})$.
2. *Let* $L \in \mathsf{T}(\mathcal{S})$ *and assume* $L \neq 0$. *If every nonzero homomorphism* $f\colon M \to L$ *with* $M \in \mathsf{T}(\mathcal{S})$ *is surjective and satisfies* $\mathsf{Ker}\, f \in \mathsf{T}(\mathcal{S})$, *then we have* $L \in \mathcal{S}$.

Proof. (1) Since $M \in \mathsf{T}(\mathcal{S}) = \mathsf{Filt}(\mathsf{Fac}\, \mathcal{S})$, there exists a sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_{l-1} \subset M_l = M$ with $M_i/M_{i-1} \in \mathsf{Fac}\, \mathcal{S}$. Because $f \neq 0$, we can take $k := \min\{i \mid f(M_i) \neq 0\} \geq 1$. Then f induces a nonzero map $\overline{f}\colon M/M_{k-1} \to S$ with $\overline{f}(M_k/M_{k-1}) \neq 0$. Because $M_k/M_{k-1} \in \mathsf{Fac}\, \mathcal{S}$, there exists a homomorphism $g\colon N \to M/M_{k-1}$ with $N \in \mathsf{add}\, \mathcal{S}$ and $\mathsf{Im}\, g = M_k/M_{k-1}$, and we have a nonzero map $\overline{f}g \neq 0\colon N \to S$. Then there exists an indecomposable direct summand S_1 of N such that $(\overline{f}g)|_{S_1} \neq 0\colon S_1 \to S$. It is clear that S_1 is a brick in $\mathcal{S}$, and $(\overline{f}g)|_{S_1}\colon S_1 \to S$ is isomorphic because $\mathcal{S}$ is a semibrick. Thus, $\overline{f}$ is surjective, and so is f.

The isomorphism $(\overline{f}g)|_{S_1}\colon S_1 \to M/M_{k-1} \to S$ implies that the epimorphism $\overline{f}\colon M/M_{k-1} \to S$ splits. Thus, we have $S \oplus \mathsf{Ker}\, \overline{f} \cong M/M_{k-1} \in \mathsf{Filt}(\mathsf{Fac}\, \mathcal{S}) = \mathsf{T}(\mathcal{S})$. Therefore, $\mathsf{Ker}\, \overline{f}$ belongs to $\mathsf{T}(\mathcal{S})$. Clearly, there exists an exact sequence $0 \to M_{k-1} \to \mathsf{Ker}\, f \to \mathsf{Ker}\, \overline{f} \to 0$, and M_{k-1} also belongs to $\mathsf{Filt}(\mathsf{Fac}\, \mathcal{S}) = \mathsf{T}(\mathcal{S})$. Therefore, we have $\mathsf{Ker}\, f \in \mathsf{T}(\mathcal{S})$.

(2) Since $L \in \mathsf{T}(\mathcal{S}) = \mathsf{Filt}(\mathsf{Fac}\, \mathcal{S})$ and $L \neq 0$, there exists a nonzero map $f\colon S \to L$ with $S \in \mathcal{S}$. By assumption, f is surjective and $\mathsf{Ker}\, f \in \mathsf{T}(\mathcal{S})$. An inclusion $\mathsf{Ker}\, f \to S$ is zero or surjective by (1), and then $f \neq 0$ implies $\mathsf{Ker}\, f = 0$. Therefore, f is isomorphic; hence, we have $L \in \mathcal{S}$. ☐

Lemma 1.8. *The map* $\mathsf{T}\colon \mathsf{sbrick}\, A \to \mathsf{tors}\, A$ *is injective. Moreover, it is restricted to an injection* $\mathsf{T}\colon \mathsf{f_L}\text{-}\mathsf{sbrick}\, A \to \textsf{f-tors}\, A$.

Proof. Let semibricks $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfy $\mathsf{T}(\mathcal{S}_1) = \mathsf{T}(\mathcal{S}_2)$.

We claim that $\mathcal{S}_1 \subset \mathcal{S}_2$. Let $S_1 \in \mathcal{S}_1$. By $S_1 \in \mathsf{T}(\mathcal{S}_1) = \mathsf{T}(\mathcal{S}_2) = \mathsf{Filt}(\mathsf{Fac}\, \mathcal{S}_2)$, there exists a nonzero map $f\colon S_2 \to S_1$ with $S_2 \in \mathcal{S}_2$. Because $S_2 \in \mathsf{T}(\mathcal{S}_2) = \mathsf{T}(\mathcal{S}_1)$, Lemma 1.7 (1) implies that f is surjective. Similarly, we have a surjection $g\colon S_1' \to S_2$ with $S_1' \in \mathcal{S}_1$. Then $fg\colon S_1' \to S_1$ is surjective. Because $\mathcal{S}_1$ is a semibrick, fg is isomorphic. Since g is surjective, f is isomorphic. Thus, we have $S_1 \cong S_2 \in \mathcal{S}_2$. Now, we have proved that $\mathcal{S}_1 \subset \mathcal{S}_2$.

By symmetry, we also obtain that $\mathcal{S}_2 \subset \mathcal{S}_1$; hence, $\mathcal{S}_1 = \mathcal{S}_2$. This implies the injectivity.

The restriction $\mathsf{T}\colon \mathsf{f_L}\text{-}\mathsf{sbrick}\, A \to \textsf{f-tors}\, A$ is well-defined by definition, and it is injective. ☐

Now, we can show Theorem 1.3.

Proof of Theorem 1.3. We prove (1) and (2). By applying (1) and (2) to the opposite algebra A^{op} and taking the K-duals, we obtain (3) and (4).

The maps in (1) and (2) are well-defined by Lemma 1.5.

Lemma 1.5 also implies that the following diagram is commutative:

$$\begin{tikzcd}
\mathsf{s}\tau\text{-}\mathsf{tilt}\,A \arrow[d, "\text{incl.}"'] \arrow[rd, "\mathsf{Fac}"] & & \\
\tau\text{-}\mathsf{rigid}\,A \arrow[r, "\mathsf{Fac}"] \arrow[rr, rounded corners, to path={-- ([yshift=-3ex]\tikztostart.south) -| (\tikztotarget)}, "M\mapsto\mathsf{ind}(M/\mathsf{rad}_B\,M)"'] & \mathsf{f}\text{-}\mathsf{tors}\,A & \mathsf{f_L}\text{-}\mathsf{sbrick}\,A \arrow[l, "\mathsf{T}"'] .
\end{tikzcd}$$

Here, $\mathsf{Fac}\colon \tau\text{-}\mathsf{rigid}\,A \to \mathsf{f}\text{-}\mathsf{tors}\,A$ is surjective by Proposition 1.6, so $\mathsf{T}\colon \mathsf{f_L}\text{-}\mathsf{sbrick}\,A \to \mathsf{f}\text{-}\mathsf{tors}\,A$ is surjective. By Lemma 1.8, $\mathsf{T}\colon \mathsf{f_L}\text{-}\mathsf{sbrick}\,A \to \mathsf{f}\text{-}\mathsf{tors}\,A$ is also injective, so it is bijective. Thus, the map in (1) is surjective, and two τ-rigid modules $M, M' \in \tau\text{-}\mathsf{rigid}\,A$ are sent to the same left finite semibrick if and only if $\mathsf{Fac}\,M = \mathsf{Fac}\,M'$. We can prove the bijectivity of the map in (2) in a similar way, since $\mathsf{Fac}\colon \mathsf{s}\tau\text{-}\mathsf{tilt}\,A \to \mathsf{f}\text{-}\mathsf{tors}\,A$ is bijective by Proposition 1.6. ☐

In the proof of Theorem 1.3 above, we have already obtained the next result.

Proposition 1.9. *The following conditions hold, where* $B := \mathsf{End}_A(M)$ *in each case.*

(1) *The map* $\mathsf{T}\colon \mathsf{f_L}\text{-}\mathsf{sbrick}\,A \to \mathsf{f}\text{-}\mathsf{tors}\,A$ *is a bijection, and we have the following commutative diagram of bijections:*

$$\begin{tikzcd}
\mathsf{s}\tau\text{-}\mathsf{tilt}\,A \arrow[r, "\mathsf{Fac}"] \arrow[rr, rounded corners, to path={-- ([yshift=-3ex]\tikztostart.south) -| (\tikztotarget)}, "M\mapsto\mathsf{ind}(M/\mathsf{rad}_B\,M)"'] & \mathsf{f}\text{-}\mathsf{tors}\,A & \mathsf{f_L}\text{-}\mathsf{sbrick}\,A \arrow[l, "\mathsf{T}"'] .
\end{tikzcd}$$

(2) *The map* $\mathsf{F}\colon \mathsf{f_R}\text{-}\mathsf{sbrick}\,A \to \mathsf{f}\text{-}\mathsf{torf}\,A$ *is a bijection, and we have the following commutative diagram of bijections:*

$$\begin{tikzcd}
\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A \arrow[r, "\mathsf{Sub}"] \arrow[rr, rounded corners, to path={-- ([yshift=-3ex]\tikztostart.south) -| (\tikztotarget)}, "M\mapsto\mathsf{ind}(\mathsf{soc}_B\,M)"'] & \mathsf{f}\text{-}\mathsf{torf}\,A & \mathsf{f_R}\text{-}\mathsf{sbrick}\,A \arrow[l, "\mathsf{F}"'] .
\end{tikzcd}$$

Therefore, the functorially finite torsion classes in $\mathsf{mod}\,A$ correspond bijectively not only to the support τ-tilting modules, which are in the "projective" side (Proposition 1.6), but also to the left finite semibricks, which are in the "simple" side (Lemma 1.7). These are connected by the map in Theorem 1.3.

We have the next corollary on the cardinalities of semibricks.

Corollary 1.10. *If* $\mathcal{S} \in \mathsf{sbrick}\,A$ *is either left finite or right finite, then* $\#\mathcal{S} \le |A|$ *holds.*

Proof. We argue the left finite case. The other case is similarly proved.

Let $\mathcal{S} \in \mathsf{f_L}\text{-}\mathsf{sbrick}\,A$. Then, we can take $M \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ such that $\mathsf{ind}(M/\mathsf{rad}_B\,M) = \mathcal{S}$ by Theorem 1.3. Thus, we have $\#\mathcal{S} = |M/\mathsf{rad}_B\,M| \le |M|$ by Lemma 1.5 (4). By the definition of support τ-tilting modules, $|M| \le |A|$ holds. Therefore, we have $\#\mathcal{S} \le |A|$. ☐

1.2. **Labeling the exchange quiver with bricks I.** The map in Theorem 1.3 gives us a way of labeling the exchange quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ with bricks, that is, to label each arrow with a brick. Here, the exchange quiver is the quiver expressing the *mutations*.

We briefly recall the definition of mutations in $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ from [AIR, Definition 2.19]. Let $(M, P) \neq (N, Q) \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,A$. If there exists an almost support τ-tilting pair which is a common direct summand of (M, P) and (N, Q), then we say that (N, Q) is a *mutation* of (M, P). By [AIR, Theorem 2.18], for each direct indecomposable summand L of M or P, there uniquely exists a mutation of (M, P) at L. If N is a mutation of M, then exactly one of $\mathsf{Fac}\,M \supsetneq \mathsf{Fac}\,N$ or $\mathsf{Fac}\,M \subsetneq \mathsf{Fac}\,N$ holds. If $\mathsf{Fac}\,M \supsetneq \mathsf{Fac}\,N$, then N is called a *left mutation* of M, and otherwise N is called a *right mutation* of M. We remark that if N is the left mutation of M at M_1 with $M = M_1 \oplus M_2$, then we have $\mathsf{Fac}\,N = \mathsf{Fac}\,M_2$.

The exchange quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ is a quiver with its vertices the elements of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$, and there is an arrow from M to N if and only if N is a left mutation of M. For any vertex M, the number of arrows which are either from or to M is always $|A|$.

The following lemma is important, and will be used later.

Lemma 1.11. [DIJ, Example 3.5] *If N is a left mutation of M in $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$, then there exists no torsion class $\mathcal{T} \in \mathsf{tors}\,A$ satisfying $\mathsf{Fac}\,M \supsetneq \mathcal{T} \supsetneq \mathsf{Fac}\,N$.*

We analogously define mutations in $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A$. Assume that N is a mutation of M in $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A$. In this part, we call N a left mutation of M if $\mathsf{Sub}\,M \subsetneq \mathsf{Sub}\,N$ and a right mutation of M if $\mathsf{Sub}\,M \supsetneq \mathsf{Sub}\,N$.

There is a one-to-one correspondence between $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ and $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A$ given by the following bijections:

$$\mathsf{s}\tau\text{-}\mathsf{tilt}\,A \xrightarrow[\cong]{\mathsf{Fac}} \mathsf{f}\text{-}\mathsf{tors}\,A \xrightarrow[\cong]{?^{\perp}} \mathsf{f}\text{-}\mathsf{torf}\,A \xleftarrow[\cong]{\mathsf{Sub}} \mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A.$$

We have the next easy observations on this map.

Proposition 1.12. *The above bijection $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A \to \mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A$ satisfies the following properties.*

(1) [AIR, Proposition 2.16] *Let $(M, P) \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,A$, and decompose $M = M_{\mathrm{np}} \oplus M_{\mathrm{pr}}$ so that M_{np} has no nonzero projective direct summand and that M_{pr} is projective. Then $(M, P) \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ maps to $(\tau M_{\mathrm{np}} \oplus \nu P, \nu M_{\mathrm{pr}}) \in \mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A$.*

(2) *Assume that $M, N \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ map to $M', N' \in \mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A$ respectively. Then N is a left mutation of M in $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ if and only if N' is a left mutation of M' in $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A$.*

(3) *The exchange quivers of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ and $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A$ are isomorphic by this bijection.*

Proof. Part (2) is easy, because the above bijection preserves direct summands by (1).

Part (3) is immediately deduced from (2). ☐

We also need the following detailed description of Theorem 1.3.

Proposition 1.13. *In the setting of Definition 1.4, let $M \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,A$. Then the following conditions are equivalent for $i = 1, 2, \ldots, m$.*

(a) *The module N_i is a brick.*

(b) *The module N_i is nonzero.*

(c) *The module M_i is not in $\mathsf{Fac} \bigoplus_{j \neq i} M_j$.*

(d) *There exists a left mutation of M at M_i in $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$.*

In particular, the number of left mutations of M is equal to $|M/\mathsf{rad}_B\, M|$.

Proof. The conditions (a) and (b) are equivalent by Lemma 1.5 (2).

Next, we show the equivalence of (b) and (c). For each i, set $B_i := \mathsf{End}_A(M_i)$ and

$$L_i' := \sum_{f \in \mathsf{rad}_A(M_j, M_i),\ j \neq i} \mathsf{Im}\, f, \quad L_i'' := \sum_{f \in \mathsf{rad}_A(M_i, M_i)} \mathsf{Im}\, f.$$

They are B_i-A-subbimodules of M_i and satisfy $L_i = L_i' + L_i''$. We can see that (b) holds if and only if $L_i' + L_i'' \neq M_i$, and that (c) holds if and only if $L_i' \neq M_i$. Therefore, it is sufficient to prove that $L_i' + L_i'' = M_i$ holds if and only if $L_i' = M_i$. Clearly, $L_i' = M_i$ implies $L_i' + L_i'' = M_i$. On the other hand, assume $L_i' + L_i'' = M_i$. Because $L_i'' = \mathsf{rad}_{B_i} M_i$, we have $L_i' = M_i$ by applying Nakayama's Lemma as left B_i-modules. Thus, (b) and (c) are equivalent.

The equivalence of (c) and (d) is proved in [AIR, Definition-Proposition 2.28]. ☐

Now, we are able to define labels of the exchange quivers.

Definition 1.14. We label the exchange quivers of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ and $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A$ with bricks as follows.

(1) Let $M \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ and decompose M as $M = \bigoplus_{i=1}^{m} M_i$ with M_i indecomposable. Assume that $M \to N$ is an arrow in the exchange quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$, and that N is the left mutation of M at M_i. Then we label this arrow with a brick $M_i / \sum_{f \in \mathsf{rad}_A(M, M_i)} \mathsf{Im}\, f$.

(2) Let $N' \in \mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$ and decompose N' as $N' = \bigoplus_{i=1}^{m} N'_i$ with N'_i indecomposable. Assume that $M' \to N'$ is an arrow in the exchange quiver of $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$, and that M' is the right mutation of N' at N'_i. Then we label this arrow with a brick $\bigcap_{f \in \mathsf{rad}_A(N'_i, N')} \mathsf{Ker}\, f$.

We came up with this labeling from mutations of 2-term simple-minded collections in the derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$, which are discussed in Subsection 2.2. This labeling is extended to the Hasse quiver of $\mathsf{tors}\, A$ in [DIRRT, Section 3.2].

Now we recall that there is a bijection between $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ and $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$ (see Proposition 1.12). Actually, the two definitions of labeling coincide under this bijection.

Theorem 1.15. *The bijection between* $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ *and* $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$ *preserves the labels of the exchange quivers of* $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ *and* $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$.

The above theorem is proved by the following lemma and proposition characterizing the labels of the exchange quivers.

Lemma 1.16. *Assume that* $M \to N$ *is an arrow in the exchange quiver of* $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ *labeled with a brick* S*, and that* $M, N \in \mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ *correspond to* $M', N' \in \mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$ *respectively. We set* $(\mathcal{T}_1, \mathcal{F}_1) := (\mathsf{Fac}\, M, \mathsf{Sub}\, M')$, $(\mathcal{T}_2, \mathcal{F}_2) := (\mathsf{Fac}\, N, \mathsf{Sub}\, N')$, *and* $\mathcal{C} := \mathcal{T}_1 \cap \mathcal{F}_2$. *Then we have the following assertions.*

(1) *The brick* S *belongs to* $\mathcal{C}$.
(2) *If* $L \in \mathcal{C}$ *and* $L \neq 0$, *then* $\mathsf{T}(\mathcal{T}_2 \cup \{L\}) = \mathcal{T}_1$. *In particular, we have* $\mathsf{T}(\mathcal{T}_2 \cup \{S\}) = \mathcal{T}_1$.
(3) *If* $L_1, L_2 \in \mathcal{C}$ *and* $L_1, L_2 \neq 0$, *then we have* $\mathsf{Hom}_A(L_1, L_2) \neq 0$.

Proof. There exists an indecomposable direct summand M_1 of M such that N is the left mutation of M at M_1. We decompose M as $M_1 \oplus M_2$, then we obtain $\mathsf{Fac}\, M_2 = \mathcal{T}_2$.

(1) It is clear that $S \in \mathcal{T}_1$. To prove that $S \in \mathcal{F}_2$, it is sufficient to show that $\mathsf{Hom}_A(M_2, S) = 0$, which follows from Lemma 1.5 (3). Thus, S belongs to $\mathcal{F}_2$; hence, we have $S \in \mathcal{C}$.

(2) Because $L \in \mathcal{F}_2$ and $L \neq 0$, we get $L \notin \mathcal{T}_2$. Thus, we have $\mathsf{T}(\mathcal{T}_2 \cup \{L\}) \supsetneq \mathcal{T}_2$. By the assumption $L \in \mathcal{T}_1$, we also have $\mathcal{T}_1 \supset \mathsf{T}(\mathcal{T}_2 \cup \{L\}) \supsetneq \mathcal{T}_2$. By Lemma 1.11, $\mathsf{T}(\mathcal{T}_2 \cup \{L\})$ must coincide with $\mathcal{T}_1$. In particular, we obtain that $\mathsf{T}(\mathcal{T}_2 \cup \{S\}) = \mathcal{T}_1$ by (1).

(3) Because $L_2 \in \mathcal{C} \subset \mathcal{F}_2$, we have $\mathsf{Hom}_A(\mathcal{T}_2, L_2) = 0$. We assume that $\mathsf{Hom}_A(L_1, L_2) = 0$, and deduce a contradiction. In this case, (2) implies that $\mathsf{Hom}_A(\mathcal{T}_1, L_2) = 0$, which is false, because $L_2 \in \mathcal{T}_1$ and $L_2 \neq 0$. Thus, we have $\mathsf{Hom}_A(L_1, L_2) \neq 0$. ☐

Proposition 1.17. *In the setting above, the following assertions hold.*

(1) *There exists a unique brick in* $\mathcal{C}$, *and it is* S.
(2) *The arrow* $M' \to N'$ *in the exchange quiver of* $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$ *is also labeled with* S.
(3) *The subcategory* $\mathcal{C}$ *coincides with* $\mathsf{Filt}\, S$.

Proof. (1) Let L be a brick in $\mathcal{C}$. By Lemma 1.16 (1), S is a brick belonging to $\mathcal{C}$, so it suffices to show that $L \cong S$. By Lemma 1.16 (3), there exists nonzero maps $f \colon L \to S$ and $g \colon S \to L$. By Lemma 1.7 (1), f is surjective. Thus, the map $gf \colon L \to L$ is nonzero, and it is an isomorphism, because L is a brick. Since f is surjective, we have $L \cong S$.

(2) Let S' be the brick on the arrow $M' \to N'$ in the exchange quiver of $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$. The dual of Lemma 1.16 (1) yields $S' \in \mathcal{C}$. Then (1) implies that $S' \cong S$.

(3) Clearly, $\mathsf{Filt}\, S \subset \mathcal{C}$ holds.

It remains to show that $\mathcal{C} \subset \mathsf{Filt}\, S$. Let $L \in \mathcal{C}$. If $L = 0$, then $L \in \mathsf{Filt}\, S$. Thus, we may assume $L \neq 0$, and we use induction on $\mathsf{dim}_K L$. By Lemma 1.16 (1) and (3), there exists a nonzero map $f \colon L \to S$, and by Lemma 1.7 (1), f is surjective.

If $\mathsf{dim}_K L = \mathsf{dim}_K S$, then we have $L \cong S \in \mathsf{Filt}\, S$.

If $\mathsf{dim}_K L > \mathsf{dim}_K S$, there exists a short exact sequence $0 \to \mathsf{Ker}\, f \to L \to S \to 0$. By Lemma 1.7 (1), $\mathsf{Ker}\, f \in \mathcal{T}_1$ holds. We also have $\mathsf{Ker}\, f \in \mathcal{F}_2$, because $L \in \mathcal{F}_2$. Thus, we have $\mathsf{Ker}\, f \in \mathcal{C}$, and then the induction hypothesis implies that $\mathsf{Ker}\, f \in \mathsf{Filt}\, S$, since $\mathsf{dim}_K \mathsf{Ker}\, f < \mathsf{dim}_K L$. Therefore, $L \in \mathsf{Filt}\, S$ is obtained.

The induction process is now complete. ☐

Proof of Theorem 1.15. It is an immediate result of Proposition 1.17 (2). ☐

We remark that the subcategory $\mathcal{C} = \mathsf{Filt}\, S$ of $\mathsf{mod}\, A$ in Proposition 1.17 is a *wide subcategory* of $\mathsf{mod}\, A$, see Subsection 1.3.

Example 1.18. Let A be the path algebra of the quiver $1 \to 2 \to 3$. Figure 2 below is the exchange quivers of $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ and $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$ labeled with bricks. The bricks in $\mathsf{ind}(M/\mathsf{rad}_B\, M) \in \mathsf{f_L}\text{-}\mathsf{sbrick}\, A$ for each $M \in \mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ and the bricks in $\mathsf{ind}(\mathsf{soc}_B\, M) \in \mathsf{f_R}\text{-}\mathsf{sbrick}\, A$ for each $M \in \mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$ are denoted by red letters.

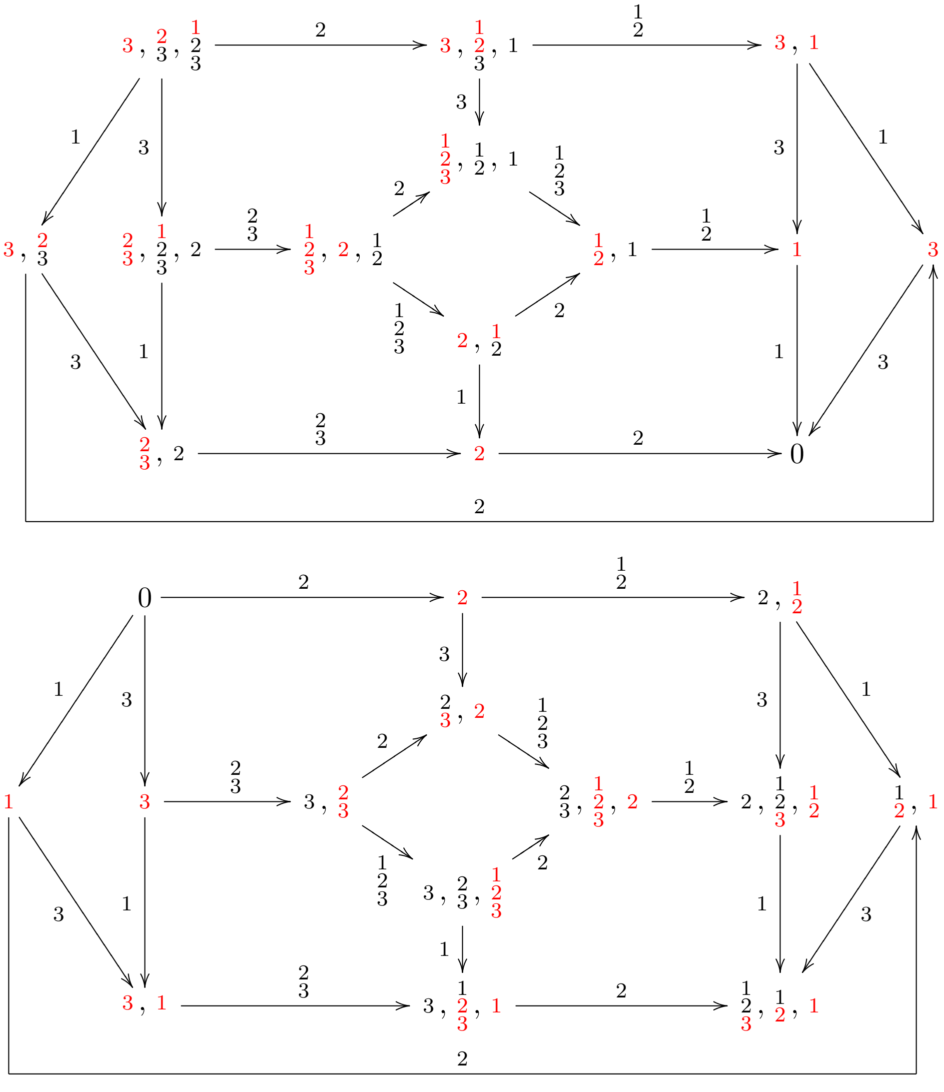


FIGURE 2. The exchange quivers of $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ and $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$

Labeling the exchange quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ with bricks is originally considered as *layer labeling* in the case that A is a preprojective algebra of Dynkin type Δ by Iyama–Reading–Reiten–Thomas [IRRT]. The definition of their layer labeling uses the bijection between $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ and the Coxeter group of Δ given by Mizuno [Miz1, Theorem 2.21], so it is rather different from the definition of our labeling. However, for the preprojective algebras of Dynkin type, layers are precisely bricks, and the layer on each arrow $M \to N$ belongs to $\mathcal{C}$ in Proposition 1.17, see [IRRT, Theorem 4.1]. Therefore, their layer labeling completely coincides with our labeling.

Example 1.19. Let A be the preprojective algebra of type $\mathbb{A}_3$. This is given by the following quiver and relations:

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \arrow[r, shift left, "\gamma"] & 3 \arrow[l, shift left, "\delta"] \end{tikzcd}, \quad \alpha\beta = 0, \quad \beta\alpha = \gamma\delta, \quad \delta\gamma = 0.$$

Figure 3 below is the exchange quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ labeled with bricks. The bricks in the semibrick $\mathsf{ind}(M/\mathsf{rad}_B\, M) \in \mathsf{f_L}\text{-}\mathsf{sbrick}\, A$ for each $M \in \mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ are denoted by red letters. Compare this labeling with the layer labeling of the exchange quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ [IRRT, Figure 2].

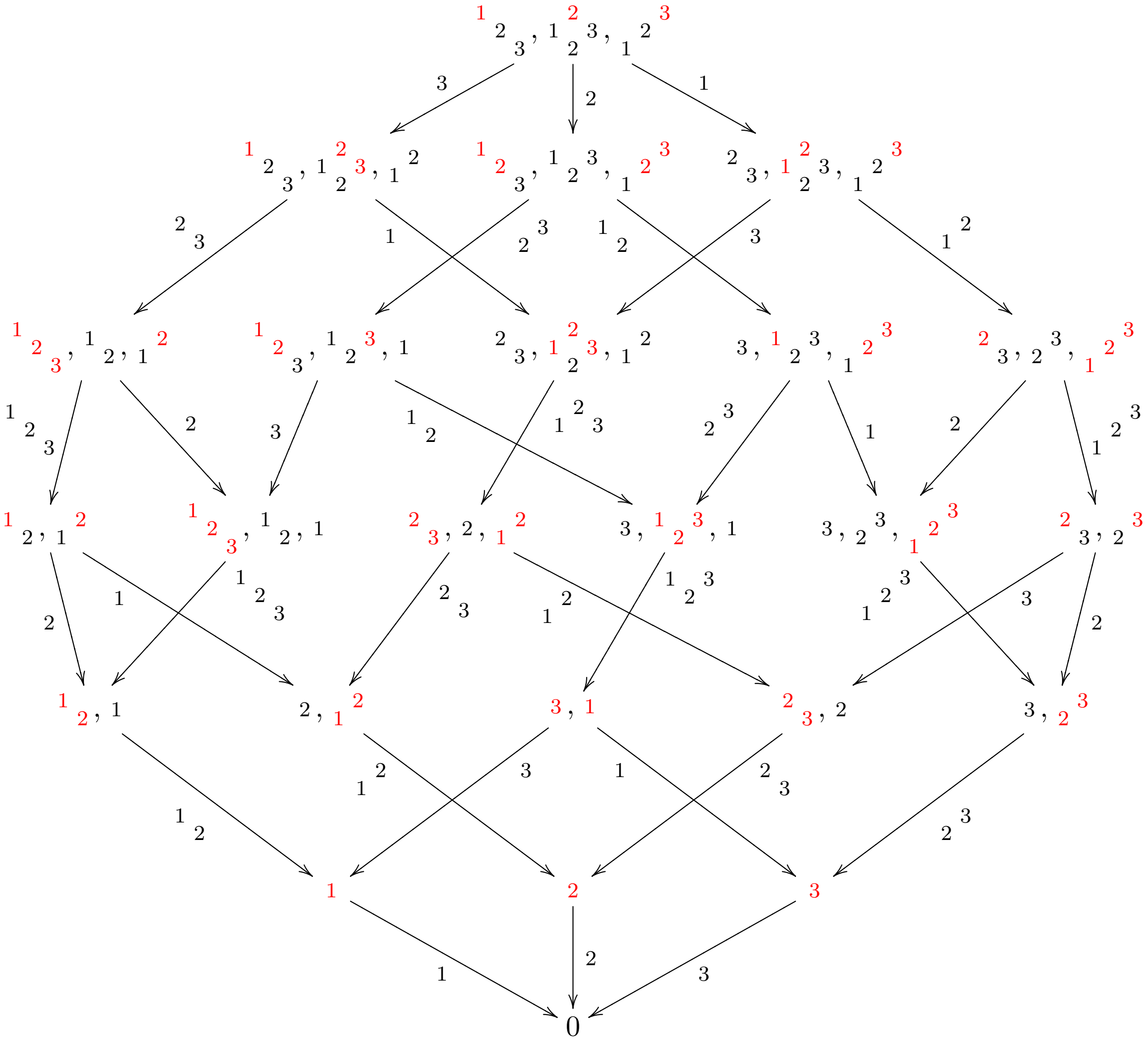


FIGURE 3. The exchange quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$

Now we recall a result of Jasso [Jas] on *reductions* of support τ-tilting modules.

Fix $U \in \tau\text{-}\mathsf{rigid}\, A$. For $M \in \mathsf{mod}\, A$, we have a canonical exact sequence $0 \to \mathsf{t}M \to M \to \mathsf{f}M \to 0$ with $\mathsf{t}M \in \mathsf{Fac}\, U$ and $\mathsf{f}M \in U^{\perp}$.

There uniquely exists the *Bongartz completion* T of the τ-rigid module U, that is, the module $T \in \mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ satisfying $U \in \mathsf{add}\, T$ and $\mathsf{Fac}\, T = {}^{\perp}(\tau U)$ [AIR, Theorem 2.10]. We define two subsets $\mathsf{s}\tau\text{-}\mathsf{tilt}_U\, A \subset \mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ and $\mathsf{f}\text{-}\mathsf{tors}_U\, A \subset \mathsf{f}\text{-}\mathsf{tors}\, A$ by

$$\mathsf{s}\tau\text{-}\mathsf{tilt}_U\, A := \{M \in \mathsf{s}\tau\text{-}\mathsf{tilt}\, A \mid U \in \mathsf{add}\, M\}, \quad \mathsf{f}\text{-}\mathsf{tors}_U\, A := \{\mathcal{T} \in \mathsf{f}\text{-}\mathsf{tors}\, A \mid \mathsf{Fac}\, U \subset \mathcal{T} \subset {}^{\perp}(\tau U)\}.$$

Let $M \in \mathsf{s\tau\text{-}tilt}\, A$, then $M \in \mathsf{s\tau\text{-}tilt}_U\, A$ holds if and only if $\mathsf{Fac}\, U \subset \mathsf{Fac}\, M \subset {}^{\perp}(\tau U)$ [AIR, Proposition 2.9]. Therefore, the bijection $\mathsf{Fac}\colon \mathsf{s\tau\text{-}tilt}\, A \to \mathsf{f\text{-}tors}\, A$ is restricted to a bijection $\mathsf{Fac}\colon \mathsf{s\tau\text{-}tilt}_U\, A \to \mathsf{f\text{-}tors}_U\, A$.

We consider the algebra $C := \mathsf{End}_A(T)/[U]$, where $[U]$ denotes the ideal of $\mathsf{End}_A(T)$ consisting of the morphisms factoring through some objects in $\mathsf{add}\, U$. Then, there is an equivalence $\varPhi := \mathsf{Hom}_A(T, ?)\colon U^{\perp} \cap {}^{\perp}(\tau U) \to \mathsf{mod}\, C$ [Jas, Theorem 3.8]. We also remark that $M \in \mathsf{s\tau\text{-}tilt}_U\, A$ implies $\mathsf{Fac}\, M \cap U^{\perp} \subset U^{\perp} \cap {}^{\perp}(\tau U)$.

Under this preparation, the following result holds.

Proposition 1.20. [Jas, Theorems 3.14, 3.16, Corollary 3.17] *There exist bijections* $\mathsf{f\text{-}tors}_U\, A \to \mathsf{f\text{-}tors}\, C$ *and* $\mathsf{s\tau\text{-}tilt}_U\, A \to \mathsf{s\tau\text{-}tilt}\, C$ *given by* $\mathcal{T} \mapsto \varPhi(\mathcal{T} \cap U^{\perp})$ *and* $M \mapsto \varPhi(\mathsf{f}M)$. *Moreover, they are compatible with mutations and satisfy the following commutative diagram:*

$$\begin{tikzcd}
\mathsf{s\tau\text{-}tilt}_U\, A \arrow[r, "\mathcal{T}\mapsto\varPhi(\mathcal{T}\cap U^{\perp})"] \arrow[d, "\mathsf{Fac}"] & \mathsf{s\tau\text{-}tilt}\, C \arrow[d, "\mathsf{Fac}"] \\
\mathsf{f\text{-}tors}_U\, A \arrow[r, "M\mapsto\varPhi(\mathsf{f}M)"] & \mathsf{f\text{-}tors}\, C
\end{tikzcd} .$$

This proposition says that the exchange quiver of $\mathsf{s\tau\text{-}tilt}\, C$ is isomorphic to the full subquiver of the exchange quiver of $\mathsf{s\tau\text{-}tilt}\, A$ consisting of the elements in $\mathsf{s\tau\text{-}tilt}_U\, A$. We have the following assertion on the labels of these exchange quivers.

Theorem 1.21. *Let* $M \to N$ *be an arrow labeled with a brick* S *in the exchange quiver of* $\mathsf{s\tau\text{-}tilt}\, A$. *Assume* $M, N \in \mathsf{s\tau\text{-}tilt}_U\, A$ *and set* $Y := \varPhi(\mathsf{f}M)$ *and* $Z := \varPhi(\mathsf{f}N)$. *Then the corresponding arrow* $Y \to Z$ *in the exchange quiver of* $\mathsf{s\tau\text{-}tilt}\, C$ *is labeled with a brick* $\varPhi(S)$.

Proof. By Proposition 1.17 (1), the assertion holds if the brick $\varPhi(S)$ belongs to $\mathsf{Fac}\, Y \cap (\mathsf{Fac}\, Z)^{\perp}$. Proposition 1.17 (1) also implies that S is a brick in $\mathcal{T}_1 \cap (\mathcal{T}_2)^{\perp}$, where $\mathcal{T}_1 := \mathsf{Fac}\, M$ and $\mathcal{T}_2 := \mathsf{Fac}\, N$. By Proposition 1.20, we have $\mathsf{Fac}\, Y = \varPhi(\mathcal{T}_1 \cap U^{\perp})$ and $\mathsf{Fac}\, Z = \varPhi(\mathcal{T}_2 \cap U^{\perp})$, so it suffices to show $\varPhi(S) \in \varPhi(\mathcal{T}_1 \cap U^{\perp}) \cap \varPhi(\mathcal{T}_2 \cap U^{\perp})^{\perp}$.

We first prove $\varPhi(S) \in \varPhi(\mathcal{T}_1 \cap U^{\perp})$. It is enough to show $S \in \mathcal{T}_1 \cap U^{\perp}$. Since $U \in \mathcal{T}_2$, we get $(\mathcal{T}_2)^{\perp} \subset U^{\perp}$. We have seen $S \in \mathcal{T}_1 \cap (\mathcal{T}_2)^{\perp}$, so we get $S \in \mathcal{T}_1 \cap U^{\perp}$. Therefore, $\varPhi(S) \in \varPhi(\mathcal{T}_1 \cap U^{\perp})$.

Next, we show $\varPhi(S) \in \varPhi(\mathcal{T}_2 \cap U^{\perp})^{\perp}$, which is equivalent to $\mathsf{Hom}_C(\varPhi(\mathcal{T}_2 \cap U^{\perp}), \varPhi(S)) = 0$ by definition. By using the equivalence $\varPhi = \mathsf{Hom}_A(T, ?)\colon U^{\perp} \cap {}^{\perp}(\tau U) \to \mathsf{mod}\, C$, it suffices to get $\mathsf{Hom}_A(\mathcal{T}_2 \cap U^{\perp}, S) = 0$, which follows from $S \in (\mathcal{T}_2)^{\perp}$. Therefore, $\varPhi(S) \in \varPhi(\mathcal{T}_2 \cap U^{\perp})^{\perp}$.

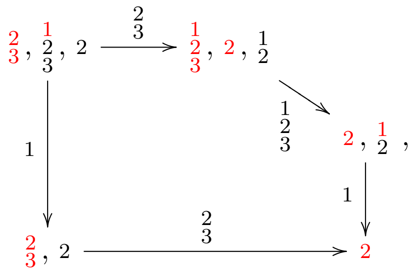
Now, we have proved $\varPhi(S) \in \varPhi(\mathcal{T}_1 \cap U^{\perp}) \cap \varPhi(\mathcal{T}_2 \cap U^{\perp})^{\perp}$. Then, the argument in the beginning gives the assertion. $\square$

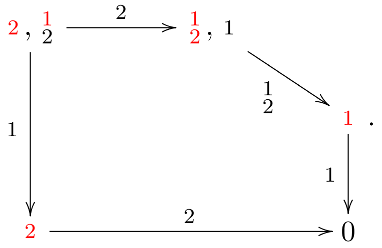
We give an example of Theorem 1.21.

Example 1.22. We use the setting of Example 1.18. Define $U, T_1, T_2 \in \mathsf{mod}\, A$ as follows:

$$U := \left(2 \right), \quad T_1 := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad T_2 := \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \right).$$

Then it is easy to see that $U \in \tau\text{-}\mathsf{rigid}\, A$ and that the Bongartz completion T of U is $T_2 \oplus T_1 \oplus U \in \mathsf{s\tau\text{-}tilt}\, A$. The algebra $C = \mathsf{End}_A(T)/[U]$ is isomorphic to the path algebra of the quiver $1 \to 2$, where T_1, T_2 correspond to the vertices $1, 2$. In the following two quivers, the left one is the full subquiver of the exchange quiver of $\mathsf{s\tau\text{-}tilt}\, A$ consisting of the elements in $\mathsf{s\tau\text{-}tilt}_U\, A$, and the right one is the exchange quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\, C$, and both of them are labeled with bricks:





We can easily check that the brick on each arrow of the left quiver is sent to the brick on the corresponding arrow of the right quiver by the functor $\Phi = \mathsf{Hom}_A(T, ?)$.

1.3. **Realizing wide subcategories as module categories I.** In this subsection, we study wide subcategories of the module category $\mathsf{mod}\, A$ by using our results on semibricks. Our goal here is describing a left finite wide subcategory of $\mathsf{mod}\, A$ as the module category $\mathsf{mod}\, E$ of some explicitly given algebra E.

First, we give the definition of wide subcategories.

Definition 1.23. A full subcategory $\mathcal{W} \subset \mathsf{mod}\, A$ is called a *wide subcategory* if $\mathcal{W}$ is a subcategory and closed under kernels, cokernels, and extensions of $\mathsf{mod}\, A$. We write $\mathsf{wide}\, A$ for the set of wide subcategories of $\mathsf{mod}\, A$.

Clearly, a wide subcategory of $\mathsf{mod}\, A$ is precisely an abelian subcategory of $\mathsf{mod}\, A$ closed under extensions. We recall the following important result deduced from [Rin, 1.2].

Proposition 1.24. *The map* $\mathsf{Filt}\colon \mathsf{sbrick}\, A \to \mathsf{wide}\, A$ *is a bijection, and its inverse* $\mathsf{wide}\, A \to \mathsf{sbrick}\, A$ *sends* $\mathcal{W} \in \mathsf{wide}\, A$ *to the set of isoclasses of simple objects in* $\mathcal{W}$*.*

It is easy to see that, if $\mathcal{S}$ is a semibrick and $\mathcal{W} = \mathsf{Filt}\, \mathcal{S}$ is the corresponding wide subcategory, then the torsion classes $\mathsf{T}(\mathcal{S})$ and $\mathsf{T}(\mathcal{W})$ coincide. Thus, we can define left finiteness and right finiteness for wide subcategories as in the case of semibricks. We write $\mathsf{f_L}\text{-}\mathsf{wide}\, A$ for the set of left finite wide subcategories, and $\mathsf{f_R}\text{-}\mathsf{wide}\, A$ for the set of right finite wide subcategories. Clearly, the bijection $\mathsf{Filt}\colon \mathsf{sbrick}\, A \to \mathsf{wide}\, A$ in Proposition 1.24 is restricted to bijections $\mathsf{f_L}\text{-}\mathsf{sbrick}\, A \to \mathsf{f_L}\text{-}\mathsf{wide}\, A$ and $\mathsf{f_R}\text{-}\mathsf{sbrick}\, A \to \mathsf{f_R}\text{-}\mathsf{wide}\, A$.

Now, we define $\mathsf{W_L}(\mathcal{T}) := \{M \in \mathcal{T} \mid \text{for any } f\colon L \to M \text{ in } \mathcal{T},\ \mathsf{Ker}\, f \in \mathcal{T}\}$ for $\mathcal{T} \in \mathsf{tors}\, A$ and $\mathsf{W_R}(\mathcal{F}) := \{M \in \mathcal{F} \mid \text{for any } f\colon M \to N \text{ in } \mathcal{F},\ \mathsf{Coker}\, f \in \mathcal{F}\}$ for $\mathcal{F} \in \mathsf{torf}\, A$. Each of them is a wide subcategory of $\mathsf{mod}\, A$, see [IT, Proposition 2.12] for the proof. We have well-defined maps $\mathsf{W_L}\colon \mathsf{tors}\, A \to \mathsf{wide}\, A$ and $\mathsf{W_R}\colon \mathsf{torf}\, A \to \mathsf{wide}\, A$. We recall the next result by Marks–Šťovíček.

Proposition 1.25. *The operations* T *and* $\mathsf{W_L}$ *satisfy the following properties.*

(1) [MS, Proposition 3.3] *A composition* $\mathsf{W_L} \circ \mathsf{T}\colon \mathsf{wide}\, A \to \mathsf{wide}\, A$ *is the identity. In particular, the map* $\mathsf{T}\colon \mathsf{wide}\, A \to \mathsf{tors}\, A$ *is injective.*
(2) [MS, Theorem 3.10] *The map* $\mathsf{T}\colon \mathsf{f_L}\text{-}\mathsf{wide}\, A \to \mathsf{f}\text{-}\mathsf{tors}\, A$ *is bijective. The inverse is given by* $\mathsf{W_L}\colon \mathsf{f}\text{-}\mathsf{tors}\, A \to \mathsf{f_L}\text{-}\mathsf{wide}\, A$*.*

We note that our Lemmas 1.7, 1.8 and Proposition 1.9 are an analogue of Proposition 1.25. We have $\mathsf{f}\text{-}\mathsf{tors}\, A \subset \mathsf{T}(\mathsf{wide}\, A) = \mathsf{T}(\mathsf{sbrick}\, A)$.

Summing up the result of Adachi–Iyama–Reiten, Marks–Šťovíček and ours, we have the next assertions.

Proposition 1.26. *We have the following commutative diagrams of bijections:*

$$\begin{tikzcd}
 & & \mathsf{f}_{\mathsf{L}}\text{-}\mathsf{wide}\,A \arrow[dl, shift left, "\mathsf{T}"] \arrow[d, shift left, "\text{simples}"] \\
\mathsf{s}\tau\text{-}\mathsf{tilt}\,A \arrow[r, shift left, "\mathsf{Fac}"] \arrow[rr, bend right=40, "M\mapsto\mathsf{ind}(M/\mathsf{rad}_B\,M)"'] & \mathsf{f}\text{-}\mathsf{tors}\,A \arrow[l, shift left, "\text{Ext-proj's}"] \arrow[ur, shift left, "\mathsf{W}_{\mathsf{L}}"] & \mathsf{f}_{\mathsf{L}}\text{-}\mathsf{sbrick}\,A \arrow[l, "\mathsf{T}"] \arrow[u, shift left, "\mathsf{Filt}"]
\end{tikzcd},$$

$$\begin{tikzcd}
 & & \mathsf{f}_{\mathsf{R}}\text{-}\mathsf{wide}\,A \arrow[dl, shift left, "\mathsf{F}"] \arrow[d, shift left, "\text{simples}"] \\
\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A \arrow[r, shift left, "\mathsf{Sub}"] \arrow[rr, bend right=40, "M\mapsto\mathsf{ind}(\mathsf{soc}_B\,M)"'] & \mathsf{f}\text{-}\mathsf{torf}\,A \arrow[l, shift left, "\text{Ext-inj's}"] \arrow[ur, shift left, "\mathsf{W}_{\mathsf{R}}"] & \mathsf{f}_{\mathsf{R}}\text{-}\mathsf{sbrick}\,A \arrow[l, "\mathsf{F}"] \arrow[u, shift left, "\mathsf{Filt}"]
\end{tikzcd}.$$

Proof. Propositions 1.6, 1.9, and 1.25 imply the assertion. □

Next we recall that a support τ-tilting A-module $M \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ is a tilting $A/\mathsf{ann}\,M$-module [AIR, Lemma 2.1, Proposition 2.2]. By Brenner–Butler's theorem, $\mathsf{Hom}_A(M, ?)\colon \mathsf{Fac}\,M \to \mathsf{Sub}_B\,DM$ is an equivalence, where $B := \mathsf{End}_A(M)$. It is also an equivalence of exact categories [DIJ, Proposition 3.2] and allows us to regard the wide subcategory $\mathcal{W}$ as a Serre subcategory of $\mathsf{mod}\,B$ as follows. We use the notation in Definition 1.4 here. Recall that $I = \{i \in \{1, 2, \dots, m\} \mid N_i \neq 0\}$.

Theorem 1.27. *In the setting of Definition 1.4, let* $M \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,A$. *Define* $e_i \in B$ *as the idempotent endomorphism* $M_i \to M \to M_i$ *for each* $i = 1, 2, \dots, m$ *and set* $e := 1 - \sum_{i\in I} e_i$. *Then the equivalence* $\mathsf{Hom}_A(M, ?)\colon \mathsf{Fac}\,M \to \mathsf{Sub}_B\,DM$ *is restricted to an equivalence*

$$\mathcal{W} \cong \mathsf{mod}\,B/\langle e\rangle.$$

Proof. Since $\mathsf{Hom}_A(M, ?)\colon \mathsf{Fac}\,M \to \mathsf{Sub}_B\,DM$ is an equivalence of exact categories [DIJ, Proposition 3.2], it is restricted to an equivalence $\mathcal{W} = \mathsf{Filt}\,\mathcal{S} \to \mathsf{Filt}_B\,\mathsf{Hom}_A(M, \mathcal{S})$. Thus, it suffices to show that $\mathsf{Hom}_A(M, \mathcal{S})$ coincides with the set of simple $B/\langle e\rangle$-modules. The cardinalities of these two sets coincide, because the number of isoclasses of simple $\mathsf{mod}\,B/\langle e\rangle$-modules is $\#I$, and it is equal to $\#\mathcal{S}$ by Lemma 1.5 (4). Therefore, we show that each element of $\mathsf{Hom}_A(M, \mathcal{S})$ is a simple $B/\langle e\rangle$-module; then, we obtain the desired equivalence. Let $S \in \mathcal{S}$.

We first show that $\mathsf{Hom}_A(M, S)$ is a simple B-module. Because $\mathsf{Hom}_A(M, S)$ is a nonzero B-module in $\mathsf{Sub}_B\,DM$, we can take a simple submodule Z of $\mathsf{Hom}_A(M, S)$. It is enough to obtain $\mathsf{Hom}_A(M, S) \cong Z$.

Since $M \in \mathsf{Sub}_B\,DM$, the submodule Z is also in $\mathsf{Sub}_B\,DM$. We define $f\colon Z \to \mathsf{Hom}_A(M, S)$ as the canonical inclusion. A quasi-inverse of $\mathsf{Hom}_A(M, ?)\colon \mathsf{Fac}\,M \to \mathsf{Sub}_B\,DM$ is given by $? \otimes_B M\colon \mathsf{Sub}_B\,DM \to \mathsf{Fac}\,M$, which sends f to a nonzero homomorphism $f \otimes_B M\colon Z \otimes_B M \to \mathsf{Hom}_A(M, S) \otimes_B M$ in $\mathsf{Fac}\,M$. Since $\mathsf{Hom}_A(M, S) \otimes_B M \cong S \in \mathcal{S}$ and $Z \otimes_B M \in \mathsf{Fac}\,M = \mathsf{T}(\mathcal{S})$, the nonzero homomorphism $f \otimes_B M$ is surjective and $\mathsf{Ker}\,f \in \mathsf{T}(\mathcal{S}) = \mathsf{Fac}\,M$ holds by Lemma 1.7 (1).

Thus, there exists a short exact sequence $0 \to \mathsf{Ker}\,f \to Z \otimes_B M \to S \to 0$ in $\mathsf{Fac}\,M$. By using the equivalence $\mathsf{Hom}_A(M, ?)\colon \mathsf{Fac}\,M \to \mathsf{Sub}_B\,DM$ of exact categories, we have a short exact sequence $0 \to \mathsf{Hom}_A(M, \mathsf{Ker}\,f) \to Z \to \mathsf{Hom}_A(M, S) \to 0$ in $\mathsf{Sub}_B\,DM$. Since Z is a simple B-module and $\mathsf{Hom}_A(M, S) \neq 0$, we obtain that $\mathsf{Hom}_A(M, S) \cong Z$ and that $\mathsf{Hom}_A(M, S)$ is a simple B-module.

Next, we show $\mathsf{Hom}_A(M, S) \in \mathsf{mod}\,B/\langle e\rangle$. By Lemma 1.5 (4), we can take $j \in I$ such that $S = N_j$. Clearly, $\mathsf{Hom}_A(M, N_j) \in \mathsf{mod}\,B/\langle e\rangle$ is equivalent to $\mathsf{Hom}_A(M_i, N_j) = 0$ for each

$i \in \{1, 2, \ldots, m\} \setminus I$. By Lemma 1.5 (3), the latter condition holds. We have $\mathsf{Hom}_A(M, S) \in \mathsf{mod}\, B/\langle e\rangle$.

These two properties yield that each element of $\mathsf{Hom}_A(M, \mathcal{S})$ is a simple $B/\langle e\rangle$-module. By the argument in the first paragraph, we get $\mathcal{W} \cong \mathsf{mod}\, B/\langle e\rangle$. ☐

We remark that Theorem 1.27 follows also from Jasso's reductions of τ-rigid modules. In the setting of Theorem 1.27, set $U := \bigoplus_{i \notin I} M_i$. Then, we can check that M is the Bongartz completion of U. From [Jas, Theorem 3.8], we have an equivalence $\mathsf{Hom}_A(M, ?)\colon U^{\perp} \cap {}^{\perp}(\tau U) \to \mathsf{mod}\, C$ with $C := \mathsf{End}_A(M)/[U]$. The algebra C is clearly $B/\langle e\rangle$. It follows from Theorem 1.21 that the simple objects of the category $U^{\perp} \cap {}^{\perp}(\tau U)$ are the elements in $\mathcal{S}$. Therefore, $U^{\perp} \cap {}^{\perp}(\tau U)$ coincides with $\mathcal{W}$, and we obtain the equivalence of Theorem 1.27.

As an application, we prove the next assertion. We write $\mathsf{f}\text{-}\mathsf{wide}\, A$ for the set of functorially finite wide subcategories of $\mathsf{mod}\, A$. This is also deduced from [MS, Proposition 3.3, Lemma 3.8].

Proposition 1.28. *We have an inclusion* $\mathsf{f_L}\text{-}\mathsf{wide}\, A \subset \mathsf{f}\text{-}\mathsf{wide}\, A$.

Proof. Let $\mathcal{W} \in \mathsf{f_L}\text{-}\mathsf{wide}\, A$ and take $M \in \mathsf{s\tau}\text{-}\mathsf{tilt}\, A$ corresponding to $\mathcal{W}$ in Proposition 1.26. Because $\mathsf{Fac}\, M \in \mathsf{f}\text{-}\mathsf{tors}\, A$, it suffices to show that $\mathcal{W}$ is functorially finite in $\mathsf{Fac}\, M$. By Theorem 1.27, this condition is equivalent to that $\mathsf{mod}\, B/\langle e\rangle$ is functorially finite in $\mathsf{Sub}_B\, DM$, which holds true because $\mathsf{mod}\, B/\langle e\rangle$ is functorially finite in $\mathsf{mod}\, B \supset \mathsf{Sub}_B\, DM$. ☐

1.4. **Semibricks for factor algebras.** Let $I \subset A$ be an ideal of a finite dimensional algebra A. Then we obviously have $\mathsf{sbrick}\, A/I \subseteq \mathsf{sbrick}\, A$, but the equality does not hold in general.

In this subsection, we first prove that $\mathsf{sbrick}\, A/I = \mathsf{sbrick}\, A$ holds in the following condition given by Eisele–Janssens–Raedschelders [EJR]:

the ideal $I \subset A$ is generated by a set $X \subset Z(A) \cap \mathsf{rad}\, A$.

Here, $Z(A)$ denotes the center of A. Moreover, by using semibricks, we give another proof of their theorem, which gives a canonical bijection $\mathsf{s\tau}\text{-}\mathsf{tilt}\, A \to \mathsf{s\tau}\text{-}\mathsf{tilt}\, A/I$.

For $M \in \mathsf{mod}\, A$ and every $a \in Z(A)$, we have an endomorphism $(\cdot a)\colon M \to M$. We can take a K-basis $a_1, a_2, \ldots, a_s$ of the vector subspace $KX \subset Z(A)$ generated by X, then the image of the homomorphism $\begin{bmatrix}(\cdot a_1) & \cdots & (\cdot a_s)\end{bmatrix}$ coincides with MI. Thus, the submodule $MI \subset M$ satisfies $MI \in \mathsf{Fac}\, M$.

From now on, we define $\overline{A} := A/I$ and $\overline{M} := M/MI$ for $M \in \mathsf{mod}\, A$. The operation $\overline{?}$ gives a right exact functor $\overline{?}\colon \mathsf{mod}\, A \to \mathsf{mod}\, \overline{A}$.

First, we observe that $\mathsf{sbrick}\, \overline{A} = \mathsf{sbrick}\, A$.

Proposition 1.29. *We have* $\mathsf{sbrick}\, A = \mathsf{sbrick}\, \overline{A}$.

Proof. It is sufficient to show that $S \in \mathsf{brick}\, A$ implies $S \in \mathsf{brick}\, \overline{A}$. By assumption, I is generated by the set $X \subset Z(A) \cap \mathsf{rad}\, A$. For each $x \in X$, the A-endomorphism $(\cdot x)\colon S \to S$ is nilpotent because $x \in \mathsf{rad}\, A$, and this map must be zero since $S \in \mathsf{brick}\, A$, so we have $Sx = 0$. Thus, $SI = 0$ holds; hence, we have $S \in \mathsf{brick}\, \overline{A}$. We have the assertion. ☐

In the rest of this subsection, we prove the following.

Theorem 1.30. *We have the following assertions.*

1. *We have* $\mathsf{f_L}\text{-}\mathsf{sbrick}\, A = \mathsf{f_L}\text{-}\mathsf{sbrick}\, \overline{A}$.
2. [EJR, Theorem 11] *There exists a bijection* $\mathsf{s\tau}\text{-}\mathsf{tilt}\, A \to \mathsf{s\tau}\text{-}\mathsf{tilt}\, \overline{A}$ *given as* $M \mapsto \overline{M}$*, which satisfies the following commutative diagram:*

$$\begin{tikzcd}
\mathsf{s\tau}\text{-}\mathsf{tilt}\, A \arrow[r, "\overline{?}"] \arrow[d, "\cong"', "M \mapsto \mathsf{ind}(M/\mathsf{rad}_B\, M)"] & \mathsf{s\tau}\text{-}\mathsf{tilt}\, \overline{A} \arrow[d, "\cong"', "M' \mapsto \mathsf{ind}(M'/\mathsf{rad}_{B'}\, M')"] \\
\mathsf{f_L}\text{-}\mathsf{sbrick}\, A \arrow[r, equal] & \mathsf{f_L}\text{-}\mathsf{sbrick}\, \overline{A}
\end{tikzcd}
\quad \begin{pmatrix} B := \mathsf{End}_A(M), \\ B' := \mathsf{End}_{\overline{A}}(M') \end{pmatrix}.$$

Moreover, this bijection preserves the mutations, and hence the exchange quivers of $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ *and* $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\overline{A}$ *are isomorphic.*

(3) *The labels of the exchange quivers of* $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ *and* $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\overline{A}$ *coincide.*

Eisele–Janssens–Raedschelders obtained the above result in a geometric way, but we show this result in terms of semibricks. There is another approach to this problem in [DIRRT, Section 5], which deals with lattice congruences on torsion classes.

The following lemmas are crucial.

Lemma 1.31. *Let* $\mathcal{S} \in \mathsf{sbrick}\,\overline{A}$*, then we have the following assertions.*

(1) *We have* $\mathsf{T}_A(\mathcal{S}) \cap \mathsf{mod}\,\overline{A} = \mathsf{T}_{\overline{A}}(\mathcal{S})$*.*

(2) *Take an integer* $n > 0$ *such that* $I^n = 0$*. Then* $\mathsf{T}_A(\mathcal{S}) = (\mathsf{T}_{\overline{A}}(\mathcal{S}))^{*n} \subset \mathsf{mod}\, A$ *holds, where* $(\mathsf{T}_{\overline{A}}(\mathcal{S}))^{*n}$ *consists of the* A*-modules* L *having a sequence* $0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$ *with* $L_i/L_{i-1} \in \mathsf{T}_{\overline{A}}(\mathcal{S})$*.*

(3) *If* $\mathcal{S} \in \mathsf{f_L}\text{-}\mathsf{sbrick}\,\overline{A}$*, then* $\mathcal{S} \in \mathsf{f_L}\text{-}\mathsf{sbrick}\, A$*.*

(4) *We have* $\mathsf{f_L}\text{-}\mathsf{sbrick}\,\overline{A} \subset \mathsf{f_L}\text{-}\mathsf{sbrick}\, A$*.*

Proof. (1) This follows as $\mathsf{T}_A(\mathcal{S}) \cap \mathsf{mod}\,\overline{A} = \mathsf{Filt}_A(\mathsf{Fac}\,\mathcal{S}) \cap \mathsf{mod}\,\overline{A} = \mathsf{Filt}_{\overline{A}}(\mathsf{Fac}\,\mathcal{S}) = \mathsf{T}_{\overline{A}}(\mathcal{S})$.

(2) Clearly, we get that $\mathsf{T}_A(\mathcal{S}) \supset (\mathsf{T}_{\overline{A}}(\mathcal{S}))^{*n}$. We show that $\mathsf{T}_A(\mathcal{S}) \subset (\mathsf{T}_{\overline{A}}(\mathcal{S}))^{*n}$. Let $L \in \mathsf{T}_A(\mathcal{S})$. In the sequence $0 = LI^n \subset \cdots \subset LI \subset L$, we have $LI^t/LI^{t+1} \in \mathsf{T}_A(\mathcal{S}) \cap \mathsf{mod}\,\overline{A}$ for all $t \geq 0$. From (1), we get $LI^t/LI^{t+1} \in \mathsf{T}_{\overline{A}}(\mathcal{S})$. Thus, L belongs to $(\mathsf{T}_{\overline{A}}(\mathcal{S}))^{*n}$. Now, the inclusion $\mathsf{T}_A(\mathcal{S}) \subset (\mathsf{T}_{\overline{A}}(\mathcal{S}))^{*n}$ is proved; hence, $\mathsf{T}_A(\mathcal{S}) = (\mathsf{T}_{\overline{A}}(\mathcal{S}))^{*n}$.

(3) By assumption, $\mathsf{T}_{\overline{A}}(\mathcal{S})$ is functorially finite in $\mathsf{mod}\,\overline{A}$. Because $\mathsf{mod}\,\overline{A}$ is functorially finite in $\mathsf{mod}\, A$, the subcategory $\mathsf{T}_{\overline{A}}(\mathcal{S})$ is functorially finite in $\mathsf{mod}\, A$. By (2) and [SikS, Theorem 2.6], $\mathsf{T}_A(\mathcal{S}) = (\mathsf{T}_{\overline{A}}(\mathcal{S}))^{*n}$ is functorially finite in $\mathsf{mod}\, A$.

(4) It immediately follows from (3). □

Lemma 1.32. *Let* $M \in \mathsf{s}\tau\text{-}\mathsf{tilt}\, A$*, and set* $B := \mathsf{End}_A(M)$ *and* $B' := \mathsf{End}_{\overline{A}}(\overline{M})$*. Then we have the following properties.*

(1) *The functor* $\overline{?}$ *induces a* K*-algebra epimorphism* $\overline{?}_{M,M} \colon B \to B'$*, and it is restricted to a surjection* $\mathsf{rad}\, B \to \mathsf{rad}\, B'$*.*

(2) *As* A*-modules,* $M/\mathsf{rad}_B\, M \cong \overline{M}/\mathsf{rad}_{B'}\,\overline{M}$*.*

(3) *The* $\overline{A}$*-modules* $\overline{M}$ *belongs to* $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\overline{A}$*.*

(4) *We have* $\mathsf{f_L}\text{-}\mathsf{sbrick}\, A \subset \mathsf{f_L}\text{-}\mathsf{sbrick}\,\overline{A}$*.*

Proof. (1) The map $\overline{?}_{M,M} \colon B \to B'$ is clearly a K-algebra homomorphism. We prove its surjectivity. Let $\alpha \in B'$. Consider the exact sequence $0 \to MI \to M \xrightarrow{p} \overline{M} \to 0$. Since $MI \in \mathsf{Fac}\, M$ and $M \in \mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ imply $\mathsf{Ext}^1_A(M, MI) = 0$, we get that $\mathsf{Hom}_A(M, M) \to \mathsf{Hom}_A(M, \overline{M})$ is surjective. Thus, there exists $f \in \mathsf{Hom}_A(M, M) = B$ such that $pf = \alpha p$, and we obtain that $\overline{f} = \alpha$. Thus, we get a K-algebra epimorphism $\overline{?}_{M,M} \colon B \to B'$.

Next, let $f \in B$. For the remaining statement, it suffices to show that $\overline{f} \in \mathsf{rad}\, B'$ holds if and only if $f \in \mathsf{rad}\, B$. If $f \in \mathsf{rad}\, B$, then f is nilpotent, and so is $\overline{f}$. Thus, we have $\overline{f} \in \mathsf{rad}\, B'$. On the other hand, if $f \notin \mathsf{rad}\, B$, then there exists an indecomposable direct summand M_1 of M such that the component $f_{11} \colon M_1 \to M_1$ of f is isomorphic. Clearly, $\overline{f_{11}} \colon \overline{M_1} \to \overline{M_1}$ is an isomorphism. Here, $\overline{M_1} \neq 0$ holds, because $I \subset \mathsf{rad}\, A$. Thus, we have $\overline{f_{11}} \notin \mathsf{rad}\,\mathsf{End}_{\overline{A}}(\overline{M_1})$, and hence $\overline{f} \notin \mathsf{rad}\,\mathsf{End}_{\overline{A}}(\overline{M}) = \mathsf{rad}\, B'$. Therefore, $\overline{f} \in \mathsf{rad}\, B'$ holds if and only if $f \in \mathsf{rad}\, B$.

(2) By (1), the canonical epimorphism $M \to \overline{M}$ is restricted to an epimorphism

$$\mathsf{rad}_B\, M = \sum_{f \in \mathsf{rad}\, B} \mathsf{Im}\, f \to \sum_{g \in \mathsf{rad}\, B'} \mathsf{Im}\, g = \mathsf{rad}_{B'}\,\overline{M}$$

of A-modules, so we get $M/\mathsf{rad}_B\, M = \overline{M}/\mathsf{rad}_{B'}\,\overline{M}$ as A-modules.

(3) [DIRRT, Proposition 5.6 (a)] implies that $\overline{M}$ is a τ-rigid $\overline{A}$-module. On the other hand, we obtain an isomorphism $B/\mathsf{rad}\, B \cong B'/\mathsf{rad}\, B'$ of semisimple K-algebras from (1). By assumption,

M is basic, so $B/\mathsf{rad}\,B$ is a direct sum of $|M|$ division K-algebras. Thus, $B'/\mathsf{rad}\,B'$ is also a direct sum of $|M|$ division K-algebras; hence, $\overline{M}$ is basic and $|\overline{M}| = |M|$. These observations implies that $\overline{M}$ is a basic support τ-tilting $\overline{A}$-module.

(4) Let $\mathcal{S} \in \mathsf{f_L}\text{-}\mathsf{sbrick}\,A$, then Theorem 1.3 implies that there exists $M \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ such that $\mathcal{S} = \mathsf{ind}(M/\mathsf{rad}_B\,M)$. By (2), we obtain $\mathcal{S} = \mathsf{ind}(\overline{M}/\mathsf{rad}_{B'}\,\overline{M})$, and by (3), we have $\overline{M} \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,\overline{A}$. Applying Theorem 1.3 again, $\mathcal{S} = \mathsf{ind}(\overline{M}/\mathsf{rad}_{B'}\,\overline{M})$ belongs to $\mathsf{f_L}\text{-}\mathsf{sbrick}\,\overline{A}$. This implies the assertion. ☐

Now, we can finish the proof of the theorem.

Proof of Theorem 1.30. (1) It follows from Lemmas 1.31 (4) and 1.32 (4).

(2) The map $\overline{?}\colon \mathsf{s}\tau\text{-}\mathsf{tilt}\,A \to \mathsf{s}\tau\text{-}\mathsf{tilt}\,\overline{A}$ is well-defined by Lemma 1.32 (3), and the diagram is commutative by Lemma 1.32 (2). The vertical two maps in the diagram are bijective by Theorem 1.3, so the map $\overline{?}\colon \mathsf{s}\tau\text{-}\mathsf{tilt}\,A \to \mathsf{s}\tau\text{-}\mathsf{tilt}\,\overline{A}$ is also bijective. It is easy to see that this bijection preserves the mutations. Thus, the exchange quivers of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ and $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\overline{A}$ are isomorphic.

(3) Let $M \to N$ be an arrow in the exchange quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ labeled with a brick S. Lemma 1.32 (2) implies that $S \in \mathsf{ind}(\overline{M}/\mathsf{rad}_B\,\overline{M})$. By the definition of labels, S is on the unique arrow $\overline{M} \to \overline{L}$ with $S \notin \mathsf{Fac}\,\overline{L}$ in the exchange quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\overline{A}$. Thus, it suffices to show that $S \notin \mathsf{Fac}\,\overline{N}$. By the definition of labels again, we have $S \notin \mathsf{Fac}\,N$. Then $S \notin \mathsf{Fac}\,\overline{N}$ holds. Thus, the arrow $\overline{M} \to \overline{N}$ is labeled with S. ☐

2. SEMIBRICKS IN DERIVED CATEGORIES

In this section, we consider 2-term simple-minded collections in $\mathsf{D^b}(\mathsf{mod}\,A)$ to understand more properties of left finite or right finite semibricks. We call a triangulated subcategory $\mathcal{C}'$ of a triangulated category $\mathcal{C}$ a *thick subcategory* if $\mathcal{C}'$ is closed under direct summands in $\mathcal{C}$.

2.1. **Bijections II.** First, we give the definition of 2-term simple-minded collections in the derived category $\mathsf{D^b}(\mathsf{mod}\,A)$.

Definition 2.1. A set $\mathcal{X}$ of isoclasses of objects in $\mathsf{D^b}(\mathsf{mod}\,A)$ is called a *simple-minded collection* in $\mathsf{D^b}(\mathsf{mod}\,A)$ if it satisfies the following conditions:

- for any $X \in \mathcal{X}$, the endomorphism ring $\mathsf{End}_{\mathsf{D^b}(\mathsf{mod}\,A)}(X)$ is a division K-algebra,
- for any $X_1 \neq X_2 \in \mathcal{X}$, we have $\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\,A)}(X_1, X_2) = 0$,
- for any $X_1, X_2 \in \mathcal{X}$ and $n < 0$, we have $\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\,A)}(X_1, X_2[n]) = 0$,
- the smallest thick subcategory of $\mathsf{D^b}(\mathsf{mod}\,A)$ containing $\mathcal{X}$ is $\mathsf{D^b}(\mathsf{mod}\,A)$ itself.

A simple-minded collection $\mathcal{X}$ in $\mathsf{D^b}(\mathsf{mod}\,A)$ is said to be *2-term* if the ith cohomology $H^i(X)$ is 0 for any $i \neq -1, 0$ and any $X \in \mathcal{X}$. We write $\mathsf{2\text{-}smc}\,A$ for the set of 2-term simple-minded collections in $\mathsf{D^b}(\mathsf{mod}\,A)$.

Remark 2.2. We note the following basic properties holding for $\mathcal{X} \in \mathsf{2\text{-}smc}\,A$.

(1) [KY, Corollary 5.5] The cardinality $\#\mathcal{X}$ is equal to $|A|$.
(2) [BY, Remark 4.11] Every $X \in \mathcal{X}$ belongs to either $\mathsf{mod}\,A$ or $(\mathsf{mod}\,A)[1]$ up to isomorphisms in $\mathsf{D^b}(\mathsf{mod}\,A)$.

The following result is the main theorem of this section. It means that every 2-term simple-minded collection is divided into a left finite semibrick and a right finite semibrick, and there are bijections between these three notions.

Theorem 2.3. *We have the following assertions.*

(1) *There are bijections*

$$? \cap \mathsf{mod}\,A\colon\ \mathsf{2\text{-}smc}\,A \to \mathsf{f_L}\text{-}\mathsf{sbrick}\,A \quad \text{and} \quad ?[-1] \cap \mathsf{mod}\,A\colon\ \mathsf{2\text{-}smc}\,A \to \mathsf{f_R}\text{-}\mathsf{sbrick}\,A$$

given by $\mathcal{X} \mapsto \mathcal{X} \cap \mathsf{mod}\,A$ *and* $\mathcal{X} \mapsto \mathcal{X}[-1] \cap \mathsf{mod}\,A$*, respectively.*

(2) *The diagram in Figure 4 below is commutative and all the maps are bijective. In this diagram,* $\mathcal{T} \in \mathsf{f\text{-}tors}\, A$ *corresponds to* $\mathcal{F} \in \mathsf{f\text{-}torf}\, A$ *if and only if* $(\mathcal{T}, \mathcal{F})$ *is a torsion pair in* $\mathsf{mod}\, A$.

(3) *If* $\mathcal{X} \in \mathsf{2\text{-}smc}\, A$ *corresponds to* $\mathcal{S} \in \mathsf{f_L\text{-}sbrick}\, A$ *and* $\mathcal{S}' \in \mathsf{f_R\text{-}sbrick}\, A$, *then we have* $\mathcal{X} = \mathcal{S} \cup \mathcal{S}'[1]$ *and a torsion pair* $(\mathsf{T}(\mathcal{S}), \mathsf{F}(\mathcal{S}'))$ *in* $\mathsf{mod}\, A$.

$$\begin{tikzcd}
\mathsf{s\tau^{-1}\text{-}tilt}\, A \arrow[r, "\mathsf{Sub}"] \arrow[rr, bend left, "M \mapsto \mathsf{ind}(\mathsf{soc}_B\, M)"] & \mathsf{f\text{-}torf}\, A & \mathsf{f_R\text{-}sbrick}\, A \arrow[l, swap, "\mathsf{F}"] \\
\mathsf{2\text{-}cosilt}\, A \arrow[u, "H^{-1}"] \arrow[dr, "I \mapsto ({}^{\perp}I[<0], {}^{\perp}I[>0])"] & & \\
& \mathsf{int\text{-}t\text{-}str}\, A \arrow[uu, swap, "(\text{heart})[-1] \cap \mathsf{mod}\, A"] \arrow[dd, swap, "(\text{heart}) \cap \mathsf{mod}\, A"] \arrow[r, "\text{simples of heart}"] & \mathsf{2\text{-}smc}\, A \arrow[uu, swap, "?[-1] \cap \mathsf{mod}\, A"] \arrow[dd, swap, "? \cap \mathsf{mod}\, A"] \\
\mathsf{2\text{-}silt}\, A \arrow[uu, "\nu"] \arrow[d, "H^0"] \arrow[ur, swap, "P \mapsto (P[<0]^{\perp}, P[>0]^{\perp})"] & & \\
\mathsf{s\tau\text{-}tilt}\, A \arrow[r, "\mathsf{Fac}"] \arrow[rr, bend right, swap, "M \mapsto \mathsf{ind}(M/\mathsf{rad}_B\, M)"] & \mathsf{f\text{-}tors}\, A & \mathsf{f_L\text{-}sbrick}\, A \arrow[l, swap, "\mathsf{T}"]
\end{tikzcd}$$

($B := \mathsf{End}_A(M)$ *in each case*)

FIGURE 4. The commutative diagram

It follows that the correspondence between $\mathsf{s\tau\text{-}tilt}\, A$ and $\mathsf{s\tau^{-1}\text{-}tilt}\, A$ in Theorem 2.3 is the same one given before Proposition 1.12.

To prove this theorem, we use the notions and fundamental properties of 2-term silting objects in the homotopy category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ and intermediate t-structures in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$. We refer to [BY, KY] here.

An object $P \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ is called a *silting object* in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ if the following conditions are satisfied:

- for any $n > 0$, we have $\mathsf{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)}(P, P[n]) = 0$,
- the smallest thick subcategory of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ containing P is $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ itself.

A silting object P in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ is said to be *2-term* if it is isomorphic to some complex $(P^{-1} \to P^0)$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ with its components zero except for -1st and 0th ones. We write $\mathsf{2\text{-}silt}\, A$ for the set of isoclasses of basic 2-term silting objects in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$.

Dually, an object $I \in \mathsf{K}^{\mathrm{b}}(\mathsf{inj}\, A)$ is called a *2-term cosilting object* in $\mathsf{K}^{\mathrm{b}}(\mathsf{inj}\, A)$ if the following conditions are satisfied:

- for any $n > 0$, we have $\mathsf{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{inj}\, A)}(I, I[n]) = 0$,
- the smallest thick subcategory of $\mathsf{K}^{\mathrm{b}}(\mathsf{inj}\, A)$ containing I is $\mathsf{K}^{\mathrm{b}}(\mathsf{inj}\, A)$ itself,
- I is isomorphic to some complex $(I^{-1} \to I^0)$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{inj}\, A)$.

We write $\mathsf{2\text{-}cosilt}\, A$ for the set of isoclasses of basic ones. The Nakayama functor $\nu \colon \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A) \to \mathsf{K}^{\mathrm{b}}(\mathsf{inj}\, A)$ induces a bijection from $\mathsf{2\text{-}silt}\, A$ to $\mathsf{2\text{-}cosilt}\, A$. There is a formula called the Nakayama duality: $D\,\mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)}(P, X) \cong \mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)}(X, \nu P)$ for $P \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ and $X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$.

We next recall the notion of intermediate t-structures. The concept of t-structures was introduced by Beĭlinson–Bernstein–Deligne [BBD]. First, let $\mathcal{D}$ be a triangulated category and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a pair of additive full subcategories of $\mathcal{D}$. For simplicity, we define $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n]$ for $n \in \mathbb{Z}$. The pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is called a *t-structure* in $\mathcal{D}$ if it satisfies the following conditions:

- we have $\mathcal{D}^{\leq -1} \subset \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$, and $\mathsf{Hom}_{\mathcal{D}}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$,

- for every $X \in \mathcal{D}$, there exists a triangle $Y \to X \to Z \to Y[1]$ in $\mathcal{D}$ with $Y \in \mathcal{D}^{\leq 0}$ and $Z \in \mathcal{D}^{\geq 1}$.

In this case, it is easy to see that each of $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ determines the other. For a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ in $\mathcal{D}$, the intersection $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called the *heart* of the t-structure. It is well-known that the heart is an abelian category [BBD, Théorème 1.3.6].

If $\mathcal{D} = \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$, there is a canonical t-structure $(\mathcal{D}^{\leq 0}_{\mathrm{std}}, \mathcal{D}^{\geq 0}_{\mathrm{std}})$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$ defined with cohomologies, namely,

$$\mathcal{D}^{\leq 0}_{\mathrm{std}} := \{X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A) \mid H^i(X) = 0 \ (i > 0)\},$$
$$\mathcal{D}^{\geq 0}_{\mathrm{std}} := \{X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A) \mid H^i(X) = 0 \ (i < 0)\},$$

and it is called the *standard t-structure* in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$. We say that a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$ is *intermediate* with respect to the standard t-structure (or simply *intermediate*) if $\mathcal{D}^{\leq -1}_{\mathrm{std}} \subset \mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 0}_{\mathrm{std}}$, or equivalently, $\mathcal{D}^{\geq -1}_{\mathrm{std}} \supset \mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 0}_{\mathrm{std}}$ holds. We can see that every intermediate t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$ is *bounded*, that is,

$$\bigcup_{n \in \mathbb{Z}} \mathcal{D}^{\leq n} = \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A) = \bigcup_{n \in \mathbb{Z}} \mathcal{D}^{\geq n}$$

hold. The next lemma follows from the definition of intermediate t-structures.

Lemma 2.4. *Each intermediate t-structure* $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ *in* $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$ *with its heart* $\mathcal{H}$ *gives a torsion pair* $(\mathcal{H} \cap \mathsf{mod}\,A, \mathcal{H}[-1] \cap \mathsf{mod}\,A)$ *in* $\mathsf{mod}\,A$.

In this part, we only consider intermediate t-structures in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$ such that its heart is an abelian category with length. The set of such intermediate t-structures is denoted by $\mathsf{int\text{-}t\text{-}str}\,A$.

The following bijections are an important result by Brüstle–Yang [BY] on these concepts, and they are a "2-term" restrictions of Koenig–Yang bijections [KY, Theorem 6.1].

Proposition 2.5. [BY, Corollary 4.3] *We have the following bijections.*

(1) *There is a bijection* $\mathsf{2\text{-}silt}\,A \to \mathsf{int\text{-}t\text{-}str}\,A$ *given by* $P \mapsto (P[<\!0]^{\perp}, P[>\!0]^{\perp})$*, where*

$$\begin{aligned} P[<\!0]^{\perp} &= \{X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A) \mid \mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(P[n], X) = 0 \ (n < 0)\} \\ &= \{X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A) \mid \mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(X, \nu P[n]) = 0 \ (n < 0)\} = {}^{\perp}\nu P[<\!0], \\ P[>\!0]^{\perp} &= \{X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A) \mid \mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(P[n], X) = 0 \ (n > 0)\} \\ &= \{X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A) \mid \mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(X, \nu P[n]) = 0 \ (n > 0)\} = {}^{\perp}\nu P[>\!0]. \end{aligned}$$

(2) *There is a bijection* $\mathsf{int\text{-}t\text{-}str}\,A \to \mathsf{2\text{-}smc}\,A$ *defined as follows: each* $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}) \in \mathsf{int\text{-}t\text{-}str}\,A$ *is sent to the set of isoclasses of simple objects in the heart* $\mathcal{H} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$.

Note that the heart of the corresponding intermediate t-structure $(P[<\!0]^{\perp}, P[>\!0]^{\perp})$ for $P \in \mathsf{2\text{-}silt}\,A$ is

$$P[\neq\!0]^{\perp} = \{X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A) \mid \mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(P[n], X) = 0 \ (n \neq 0)\}.$$

It will play an important role in the proof of Theorem 2.3.

Proposition 2.5 gives bijections on notions in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ and $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$, but we also would like to know their relationship with notions in the module category. Here, we cite [AIR, BY].

Proposition 2.6. [AIR, Theorem 3.2] *The map* $\mathsf{2\text{-}silt}\,A \ni P \mapsto H^0(P) \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ *is bijective. The inverse is given as follows: each* $(M, P) \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ *with a minimal projective resolution* $P^{-1} \to P^0 \to M \to 0$ *of* M *is sent to* $(P^{-1} \oplus P \to P^0) \in \mathsf{2\text{-}silt}\,A$.

Proposition 2.7. [BY, Theorem 4.9] *We have a bijection* (heart)$\cap \mathsf{mod}\,A$: $\mathsf{int\text{-}t\text{-}str}\,A \to \mathsf{f\text{-}tors}\,A$ *defined as* $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}) \mapsto \mathcal{H} \cap \mathsf{mod}\,A$*, where* $\mathcal{H} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ *is the heart. Moreover, this bijection* *joins the following commutative diagram of bijections:*

$$\begin{tikzcd}
\mathsf{2\text{-}silt}\,A \arrow[r, "P\mapsto(P[<0]^{\perp},P[>0]^{\perp})"] \arrow[d, "H^0"] & \mathsf{int\text{-}t\text{-}str}\,A \arrow[d, "(\text{heart})\cap\mathsf{mod}\,A"] \\
\mathsf{s}\tau\mathsf{\text{-}tilt}\,A \arrow[r, "\mathsf{Fac}"] & \mathsf{f\text{-}tors}\,A
\end{tikzcd}.$$

In Proposition 2.7, we can easily see $\mathcal{H} \cap \mathsf{mod}\,A = \mathcal{D}^{\leq 0} \cap \mathsf{mod}\,A$ from the definition of intermediate t-structures, and the original statement in [BY] uses the latter.

Now, we start the proof of Theorem 2.3.

Lemma 2.8. *Let $P \in \mathsf{2\text{-}silt}\,A$, $\mathcal{H} := P[\neq 0]$ be the heart of the corresponding t-structure, $\mathcal{X}$ be the set of isoclasses of simple objects in $\mathcal{H}$, and $\mathcal{S} := \mathcal{X} \cap \mathsf{mod}\,A$ as in Theorem 2.3. Then we have $\mathcal{H} \cap \mathsf{mod}\,A = \mathsf{T}(\mathcal{S})$. Moreover, $\mathcal{S}$ belongs to $\mathsf{f_L\text{-}sbrick}\,A$.*

Proof. We first prove that $\mathsf{T}(\mathcal{S}) \subset \mathcal{H} \cap \mathsf{mod}\,A$. Because $\mathcal{H} \cap \mathsf{mod}\,A$ is a torsion class in $\mathsf{mod}\,A$ by Lemma 2.4, it is sufficient to see that $\mathcal{S} \subset \mathcal{H} \cap \mathsf{mod}\,A$. This is clear.

Next, we show the converse $\mathcal{H} \cap \mathsf{mod}\,A \subset \mathsf{T}(\mathcal{S})$. Since $H^0(\mathcal{H}) = \mathcal{H} \cap \mathsf{mod}\,A$ follows from [BY, Remark 4.11], it is enough to show $H^0(\mathcal{H}) \subset \mathsf{T}(\mathcal{S})$, that is, $H^0(X) \in \mathsf{T}(\mathcal{S})$ for all $X \in \mathcal{H}$. We use induction on the length l of X in the abelian category $\mathcal{H}$ with length.

If $l = 0$, then $X \cong 0$, so $H^0(X) = 0 \in \mathsf{T}(\mathcal{S})$.

If $l \geq 1$, there exists a short exact sequence $0 \to Y \to X \to Z \to 0$ in $\mathcal{H}$ with Y simple and Z of length $l-1$. By Remark 2.2 (2), we may assume that Y belongs to $\mathsf{mod}\,A$ or $(\mathsf{mod}\,A)[1]$. By the induction hypothesis, we can see $H^0(Z) \in \mathsf{T}(\mathcal{S})$. The short exact sequence is lifted to a triangle $Y \to X \to Z \to Y[1]$.

If $Y \in \mathsf{mod}\,A$, we have $Y \in \mathcal{S}$ and an exact sequence $Y \to H^0(X) \to H^0(Z) \to 0$ in $\mathsf{mod}\,A$. Set $M := \mathsf{Ker}(H^0(X) \to H^0(Z))$, then we get $M \in \mathsf{Fac}\,Y \subset \mathsf{T}(\mathcal{S})$. Moreover, there is a short exact sequence $0 \to M \to H^0(X) \to H^0(Z) \to 0$ in $\mathsf{mod}\,A$ with $H^0(Z) \in \mathsf{T}(\mathcal{S})$. Thus, the module $H^0(X)$ is also in $\mathsf{T}(\mathcal{S})$.

If $Y \in (\mathsf{mod}\,A)[1]$, we have an exact sequence $0 \to H^0(X) \to H^0(Z) \to 0$ in $\mathsf{mod}\,A$. Thus, we have $H^0(X) \cong H^0(Z)$ and $H^0(X) \in \mathsf{T}(\mathcal{S})$.

The induction is complete, and we have $\mathcal{H} \cap \mathsf{mod}\,A = H^0(\mathcal{H}) = \mathsf{T}(\mathcal{S})$.

Obviously, $\mathcal{S}$ is a semibrick, and $\mathsf{T}(\mathcal{S}) = \mathcal{H} \cap \mathsf{mod}\,A$ is functorially finite in $\mathsf{mod}\,A$ by Proposition 2.7. These mean $\mathcal{S} \in \mathsf{f_L\text{-}sbrick}\,A$. ☐

Proof of Theorem 2.3. (1)(2) From Proposition 2.7 and Lemma 2.8, it follows that the map $? \cap \mathsf{mod}\,A \colon \mathsf{2\text{-}smc}\,A \to \mathsf{f_L\text{-}sbrick}\,A$ is well-defined, and the following diagram is commutative:

$$\begin{tikzcd}
\mathsf{2\text{-}silt}\,A \arrow[r, "P\mapsto(P[<0]^{\perp},P[>0]^{\perp})"] \arrow[d, "H^0"] & \mathsf{int\text{-}t\text{-}str}\,A \arrow[r, "\text{simples of heart}"] \arrow[d, "(\text{heart})\cap\mathsf{mod}\,A"] & \mathsf{2\text{-}smc}\,A \arrow[d, "?\cap\mathsf{mod}\,A"] \\
\mathsf{s}\tau\mathsf{\text{-}tilt}\,A \arrow[r, "\mathsf{Fac}"] & \mathsf{f\text{-}tors}\,A & \mathsf{f_L\text{-}sbrick}\,A \arrow[l, "\mathsf{T}"']
\end{tikzcd}.$$

Because the other maps in this diagram are bijective by Propositions 1.9, 2.5 and 2.7, the map $? \cap \mathsf{mod}\,A \colon \mathsf{2\text{-}smc}\,A \to \mathsf{f_L\text{-}sbrick}\,A$ is bijective.

We also have the dual commutative diagram by considering the opposite algebra A^{op} and appropriate shifts of complexes, and thus, the map $?[-1] \cap \mathsf{mod}\,A \colon \mathsf{2\text{-}smc}\,A \to \mathsf{f_R\text{-}sbrick}\,A$ is bijective. Part (1) is proved.

The remaining part of the commutative diagram in (2) is obtained from the Nakayama duality and Theorem 1.3.

Moreover, Proposition 2.7 and Lemma 2.4 imply that $\mathcal{T} \in \mathsf{f\text{-}tors}\,A$ corresponds to $\mathcal{F} \in \mathsf{f\text{-}torf}\,A$ if and only if $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathsf{mod}\,A$.

(3) It is immediately deduced from (2) and Remark 2.2. ☐

As in Lemma 2.8, the map (heart) $\cap \operatorname{mod} A$: $\text{int-t-str}\, A \to \text{f-tors}\, A$ is equal to H^0(heart), and dually, the map (heart)$[-1] \cap \operatorname{mod} A$: $\text{int-t-str}\, A \to \text{f-torf}\, A$ is equal to H^{-1}(heart). Therefore, in the commutative diagram in Theorem 2.3 (2), we can regard the bottom row as the 0th cohomologies of the middle(-lower) row, and the top row as the -1st cohomologies of the middle(-upper) row.

2.2. **Labeling the exchange quiver with bricks II.** In this section, we label the exchange quiver of $\text{2-smc}\, A$ with bricks in a similar way to Subsection 1.2. First, we recall the fundamental properties of mutations in $\text{2-smc}\, A$ from [BY].

Proposition 2.9. [BY, Subsection 3.7] *Let $\mathcal{X} \in \text{2-smc}\, A$ and $\mathcal{S} := \mathcal{X} \cap \operatorname{mod} A$.*

1. *Assume $X \in \mathcal{X}$. Then there exists a left mutation of $\mathcal{X}$ at X in $\text{2-smc}\, A$ if and only if $X \in \mathcal{S}$.*
2. *For $S_0 \in \mathcal{S}$, a left mutation of $\mathcal{X}$ at S_0 uniquely exists, and it is given as follows:*
   - *define a full subcategory $\mathcal{E} := \operatorname{Filt} S_0$ of $\operatorname{mod} A$,*
   - *set $Y_{S_0} := S_0[1]$,*
   - *for any $S \in \mathcal{S} \setminus \{S_0\}$, take a left minimal $\mathcal{E}$-approximation $f_S \colon S[-1] \to E_S$ in $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$ and define Y_S as the mapping cone of f_S (we can see $Y_S \in \operatorname{mod} A$ and have an exact sequence $0 \to E_S \to Y_S \to S \to 0$ in $\operatorname{mod} A$),*
   - *for any $S'[1] \in \mathcal{X} \setminus \mathcal{S}$, take a left minimal $\mathcal{E}$-approximation $f_{S'[1]} \colon S' \to E_{S'[1]}$ in $\operatorname{mod} A$. If $f_{S'[1]}$ is injective, then set $Y_{S'[1]} := \operatorname{Coker} f_{S'[1]}$, and otherwise (in this case $f_{S'[1]}$ is surjective), set $Y_{S'[1]} := (\operatorname{Ker} f_{S'[1]})[1]$,*
   - *now $\mathcal{Y} := \{Y_X \mid X \in \mathcal{X}\}$ is the left mutation of $\mathcal{X}$ at S_0.*

We define labels of the exchange quiver of $\text{2-smc}\, A$ with bricks as follows.

Definition 2.10. In the setting of Proposition 2.9, we label the arrow $\mathcal{X} \to \mathcal{Y}$ in the exchange quiver of $\text{2-smc}\, A$ with a brick S_0.

The right mutation in $\text{2-smc}\, A$ at each object in $\mathcal{S}' := \mathcal{X}[-1] \cap \operatorname{mod} A$ is similarly defined. By definition, the labels on the arrows from $\mathcal{X}$ are the elements of $\mathcal{S}$, and the labels on the arrows to $\mathcal{X}$ are the elements of $\mathcal{S}'$.

We give an example, compare it with Example 1.18.

Example 2.11. Let A be the path algebra of the quiver $1 \to 2 \to 3$. Figure 5 below is the exchange quiver of $\text{2-smc}\, A$ labeled with bricks.

The three exchange quivers of $\mathsf{s}\tau\text{-tilt}\, A$, $\mathsf{s}\tau^{-1}\text{-tilt}\, A$, and $\text{2-smc}\, A$ in Examples 1.18 and 2.11 are naturally isomorphic by the maps in Theorem 2.3. This holds for an arbitrary finite dimensional K-algebra [BY, Corollary 4.3, Theorem 4.9]. Now, we show that the maps in Theorem 2.3 also preserve the labels of the exchange quivers.

Theorem 2.12. *The maps in Theorem 2.3 preserve the labels of the exchange quivers of $\mathsf{s}\tau\text{-tilt}\, A$, $\mathsf{s}\tau^{-1}\text{-tilt}\, A$, and $\text{2-smc}\, A$.*

Proof. Let $\mathcal{X} \to \mathcal{Y}$ be an arrow in the exchange quiver of $\text{2-smc}\, A$ labeled with a brick S. We have $S \in \mathcal{X} \cap \operatorname{mod} A$ and $S \in \mathcal{Y}[-1] \cap \operatorname{mod} A$ by definition. By Theorem 2.3, we can take $M \in \mathsf{s}\tau\text{-tilt}\, A$ corresponding to $\mathcal{X} \in \text{2-smc}\, A$ and $N' \in \mathsf{s}\tau^{-1}\text{-tilt}\, A$ corresponding to $\mathcal{Y} \in \text{2-smc}\, A$, and then $\mathsf{T}(\mathcal{X} \cap \operatorname{mod} A) = \operatorname{Fac} M$ and $\mathsf{F}(\mathcal{Y}[-1] \cap \operatorname{mod} A) = \operatorname{Sub} N'$ hold. Therefore, S is a brick belonging to $\operatorname{Fac} M \cap \operatorname{Sub} N'$. Now, Proposition 1.17 implies the assertion. ☐

2.3. **Realizing wide subcategories as module categories II.** We have investigated each left finite subcategory $\mathcal{W} \in \mathsf{f_L}\text{-wide}\, A$ in terms of the endomorphism algebra $B := \operatorname{End}_A(M)$ of the corresponding support τ-tilting module $M \in \mathsf{s}\tau\text{-tilt}\, A$ in Subsection 1.3. Now, instead of B, we use the endomorphism algebra $C := \operatorname{End}_{\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)}(P)$ of the corresponding 2-term silting objects $P \in \text{2-silt}\, A$ to study $\mathcal{W}$.

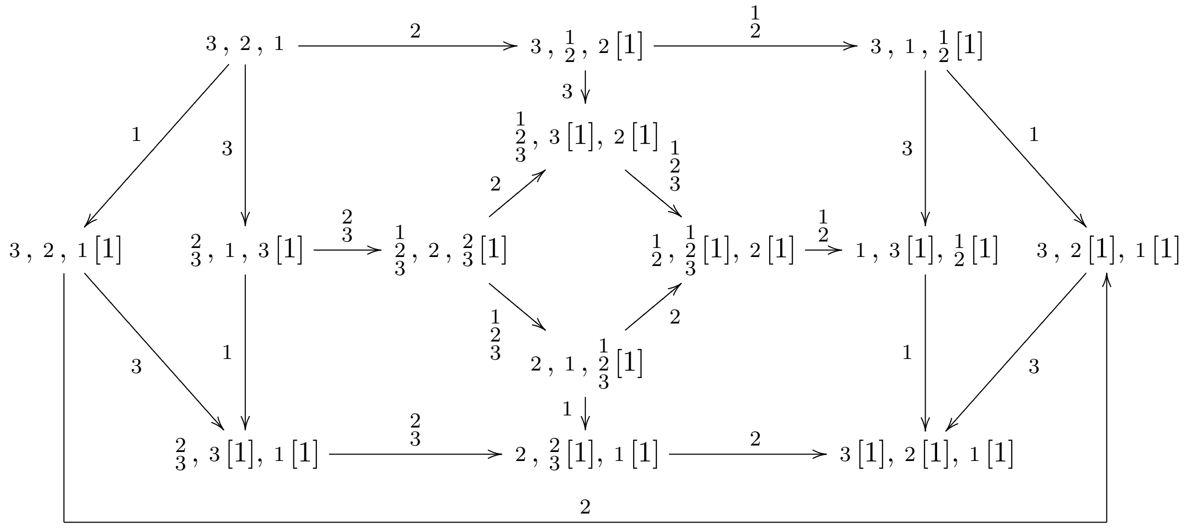


FIGURE 5. The exchange quiver of 2-smc A

From now on, we fix $P \in \mathsf{2\text{-}silt}\, A$ and set $C := \mathsf{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)}(P)$. The K-algebra C is isomorphic to $C' := \mathsf{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{inj}\, A)}(\nu P)$ by the Nakayama functor $\nu\colon \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A) \to \mathsf{K}^{\mathrm{b}}(\mathsf{inj}\, A)$. As in [IY, Proposition 4.8], the heart $\mathcal{H} = P[{\neq}0]^{\perp}$ of the corresponding t-structure $(P[{<}0]^{\perp}, P[{>}0]^{\perp})$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ is equivalent to $\mathsf{mod}\, C$ by the functor

$$\mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)}(P, ?)\colon \mathcal{H} \xrightarrow{\cong} \mathsf{mod}\, C.$$

By the Nakayama duality, we also have an equivalence $D\,\mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)}(?, \nu P)\colon \mathcal{H} \to \mathsf{mod}\, C'$. We also define

$$\begin{aligned} M' &:= H^{-1}(\nu P) \in \mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A, & M &:= H^{0}(P) \in \mathsf{s}\tau\text{-}\mathsf{tilt}\, A,\\ B' &:= \mathsf{End}_A(M'), & B &:= \mathsf{End}_A(M),\\ \mathcal{S}' &:= \mathsf{ind}(\mathsf{soc}_{B'} M') \in \mathsf{f_R}\text{-}\mathsf{sbrick}\, A, & \mathcal{S} &:= \mathsf{ind}(M/\mathsf{rad}_B M) \in \mathsf{f_L}\text{-}\mathsf{sbrick}\, A, \end{aligned}$$

and then Theorem 2.3 implies that $\mathcal{X} := \mathcal{S} \cup \mathcal{S}'[1] \in \mathsf{2\text{-}smc}\, A$.

Let $P_1, P_2, \ldots, P_n$ be the indecomposable direct summands of P, where $n = |A|$ by [KY, Corollary 5.1]. For $i = 1, 2, \ldots, n$, we set

$$\begin{aligned} M'_i &:= H^{-1}(\nu P_i), & M_i &:= H^{0}(P_i),\\ N'_i &:= \bigcap_{f \in \mathsf{rad}_A(M'_i, M')} \mathsf{Ker}\, f, & N_i &:= M_i / \sum_{f \in \mathsf{rad}_A(M, M_i)} \mathsf{Im}\, f. \end{aligned}$$

We obtain that N_i is a brick or zero from Lemma 1.5 (2). The module N'_i is also a brick or zero. Clearly, $\mathcal{S}' = \{N'_i \mid N'_i \neq 0\}$ and $\mathcal{S} = \{N_i \mid N_i \neq 0\}$ hold.

Lemma 2.13. *In the setting above, the following assertions hold for $i = 1, 2, \ldots, n$.*

1. *The module M_i is projective if and only if $M'_i = 0$.*
2. *We have $N_i \neq 0$ if and only if $N'_i = 0$.*
3. *We have $M_i \neq 0$ if and only if M'_i is not injective.*
4. *If M_i is projective then $N_i \neq 0$, and if $N_i \neq 0$ then $M_i \neq 0$.*

Proof. Parts (1), (3) and (4) are obvious.

(2) Because $\#\mathcal{S} + \#\mathcal{S}' = n$ by Remark 2.2, it is sufficient to show that the condition $N_i \neq 0$ implies $N'_i = 0$. We assume $N_i \neq 0$ and $N'_i \neq 0$ and deduce a contradiction.

Because N_i and $N'_i[1]$ are in $\mathcal{X} = \mathcal{S} \cup \mathcal{S}'[1] \in \mathsf{2\text{-}smc}\, A$, they are simple in $\mathcal{H}$ by Theorem 2.3. On the other hand, $N_i \in \mathsf{Fac}\, M_i$ implies $\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(P_i, N_i) \neq 0$, and $N'_i \in \mathsf{Sub}\, M'_i$ implies $\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(N'_i[1], \nu P_i) \neq 0$. By the Nakayama duality, we have $\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(P_i, N'_i[1]) \neq 0$.

Considering the equivalence $\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(P, ?) \colon \mathcal{H} \to \mathsf{mod}\, C$, the two simple objects N_i and $N'_i[1]$ in $\mathcal{H}$ must be isomorphic. It is a contradiction. □

We have a one-to-one correspondence between the indecomposable direct summands of P and the elements of $\mathcal{X}$ as follows. See also [KY, Lemma 5.3].

Lemma 2.14. *For $i \in \{1, 2, \ldots, n\}$, let $X_i := N_i$ if $N_i \neq 0$ and $X_i := N'_i[1]$ if $N'_i \neq 0$. Then we have $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$. Moreover, let $j \in \{1, 2, \ldots, n\}$ and define a division ring D_j as $\mathsf{End}_{\mathsf{D^b}(\mathsf{mod}\, A)}(X_j)$. Then,*

$$\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(P_i, X_j) \cong \begin{cases} D_j & (i = j) \\ 0 & (i \neq j) \end{cases}$$

as left D_j-modules.

Proof. Because $\mathcal{X} = \mathcal{S} \cup \mathcal{S}'[1]$, we have $\mathcal{X} = \{X_1, \ldots, X_n\}$.

Next, let $j \in \{1, 2, \ldots, n\}$. Then, X_j is a simple object in $\mathcal{H}$. Thus, by the equivalence $\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(P, ?) \colon \mathcal{H} \to \mathsf{mod}\, C$, the C-module $\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(P, X_j)$ is simple. This module is a simple module over its endomorphism algebra $\mathsf{End}_C(\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(P, X_j))$. By the functoriality of $\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(P, ?) \colon \mathcal{H} \to \mathsf{mod}\, C$, the module $\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(P, X_j)$ is a simple D_j-module. Thus, it is isomorphic to D_j as left D_j-modules.

Now, $D_j \cong \mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(P, X_j) \cong \bigoplus_{i=1}^{n} \mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(P_i, X_j)$ hold as left D_j-modules. Therefore, there uniquely exists i such that $\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(P_i, X_j) \neq 0$. As in the proof of Lemma 2.13, we have $\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(P_j, X_j) \neq 0$. Therefore, $\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(P_j, X_j)$ is isomorphic to D_j as left D_j-modules, and $\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(P_i, X_j)$ must be zero if $i \neq j$. □

By Lemma 2.13, we may assume that $P_1, P_2, \ldots, P_n$ and $k \leq l \leq m$ satisfy the following:

- M_i is projective if and only if $i = 1, \ldots, k$,
- $N_i \neq 0$ holds if and only if $i = 1, \ldots, l$,
- $M_i \neq 0$ holds if and only if $i = 1, \ldots, m$.

In this case, we have

$$\mathcal{S} = \{N_1, \ldots, N_l\}, \quad \mathcal{S}' = \{N'_{l+1}, \ldots, N'_n\}, \quad \mathcal{X} = \{N_1, \ldots, N_l, N'_{l+1}[1], \ldots, N'_n[1]\}.$$

The corresponding left finite wide subcategory is $\mathcal{W} = \mathsf{Filt}\, \mathcal{S}$, and the corresponding right finite wide subcategory is $\mathcal{W}' = \mathsf{Filt}\, \mathcal{S}'$.

We can see the two categories $\mathcal{W}, \mathcal{W}'[1]$ as Serre subcategories of $\mathcal{H}$ by the next theorem, which is the main result of this subsection.

We define $f_i \in C$ as the idempotent endomorphism $P \to P_i \to P$ in $\mathsf{K^b}(\mathsf{proj}\, A)$ and set $f := f_{l+1} + \cdots + f_n$ for $P \in \mathsf{2\text{-}silt}\, A$. We also define $f'_i \in C'$ as the idempotent endomorphism $\nu P \to \nu P_i \to \nu P$ in $\mathsf{K^b}(\mathsf{inj}\, A)$ and set $f' := f'_1 + \cdots + f'_l$ for $\nu P \in \mathsf{2\text{-}cosilt}\, A$.

Theorem 2.15. *In the setting above, we have the following equivalences.*

(1) *The equivalence $\mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(P, ?) \colon \mathcal{H} \to \mathsf{mod}\, C$ is restricted to an equivalence*

$$\mathcal{W} \cong \mathsf{mod}\, C/\langle f \rangle.$$

(2) *The equivalence $D\, \mathsf{Hom}_{\mathsf{D^b}(\mathsf{mod}\, A)}(?, \nu P) \colon \mathcal{H} \to \mathsf{mod}\, C'$ is restricted to an equivalence*

$$\mathcal{W}'[1] \cong \mathsf{mod}\, C'/\langle f' \rangle.$$

Proof. We only prove (1). Part (2) is shown similarly. Let $X \in \mathcal{H}$.

A simple object $X_i \in \mathcal{H}$ does not appear in the composition factors of X in $\mathcal{H}$ if and only if $\mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(P_i, X) = 0$, which is deduced from an exact functor $\mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(P_i, ?)\colon \mathcal{H} \to \mathsf{mod}\,K$ and Lemma 2.14. This is also equivalent to $\mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(P, X) \circ f_i = 0$.

The condition $X \in \mathcal{W}$ exactly means that none of $X_{l+1}, \dots, X_n$ appear in the composition factors of X in $\mathcal{H}$. By the observation above, this holds if and only if $\mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(P, X) \circ f = 0$. Therefore, the equivalence $\mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(P, ?)\colon \mathcal{H} \to \mathsf{mod}\,C$ is restricted to an equivalence $\mathcal{W} \cong \mathsf{mod}\,C/\langle f \rangle$. ☐

Now, we relate Theorem 2.15 to Theorem 1.27. The idempotent $e \in B$ defined in Theorem 1.27 is $e = \sum_{i=l+1}^{m} e_i$ in the current setting, where each $e_i \in B$ is the idempotent endomorphism $M_i \to M \to M_i$. There is an epimorphism $\varphi\colon C \to B$ of K-algebras defined as $g \mapsto H^0(g)$, which induces an epimorphism $\varphi\colon C/\langle f \rangle \to B/\langle e \rangle$. We dually define an idempotent $e' \in B'$ as $e' := \sum_{i=k+1}^{l} e'_i$, with each $e'_i \in B$ the idempotent endomorphism $M'_i \to M' \to M'_i$. Then we have a K-algebra epimorphism $\varphi'\colon C' \to B'$ defined as $g \mapsto H^{-1}(g)$, which induces an epimorphism $\varphi'\colon C'/\langle f' \rangle \to B'/\langle e' \rangle$.

Theorem 2.16. *The following diagrams are commutative up to isomorphisms of functors:*

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "\mathsf{Hom}_A(M,?)", "\cong"'] \arrow[d, equal] & \mathsf{mod}\,B/\langle e \rangle \arrow[r, "\mathrm{nat}"] \arrow[d, "\varphi", "\cong"'] & \mathsf{mod}\,B \arrow[d, "\varphi"] \\
\mathcal{W} \arrow[r, "\mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(P,?)", "\cong"'] & \mathsf{mod}\,C/\langle f \rangle \arrow[r, "\mathrm{nat}"] & \mathsf{mod}\,C
\end{tikzcd},$$

$$\begin{tikzcd}
\mathcal{W}' \arrow[r, "\mathsf{Hom}_A(?,M')", "\cong"'] \arrow[d, "{[1]}", "\cong"'] & \mathsf{mod}\,B'/\langle e' \rangle \arrow[r, "\mathrm{nat}"] \arrow[d, "\varphi'", "\cong"'] & \mathsf{mod}\,B' \arrow[d, "\varphi'"] \\
\mathcal{W}'[1] \arrow[r, "\mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(?,\nu P)", "\cong"'] & \mathsf{mod}\,C'/\langle f' \rangle \arrow[r, "\mathrm{nat}"] & \mathsf{mod}\,C'
\end{tikzcd}.$$

In particular, $\varphi\colon C/\langle f \rangle \to B/\langle e \rangle$ and $\varphi'\colon C'/\langle f' \rangle \to B'/\langle e' \rangle$ are isomorphisms.

Proof. We show the commutativity of the first diagram. The second one is similarly proved.

It is enough to show that the functor $\mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(P, ?)\colon \mathcal{W} \to \mathsf{mod}\,C/\langle f \rangle$ is isomorphic to the following functor: each $X \in \mathcal{W}$ is sent to $\mathsf{Hom}_A(M, X)$ regarded as an object in $\mathsf{mod}\,C$, where C acts on $\mathsf{Hom}_A(M, X)$ by $g \cdot h = g \circ H^0(h)$ for $g \in \mathsf{Hom}_A(M, X)$ and $h \in C$.

We consider a natural morphism $\pi\colon P \to M$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$. By straightforward calculation, a map $\mathsf{Hom}_A(\pi, X)\colon \mathsf{Hom}_A(M, X) \to \mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(P, X)$ for $X \in \mathcal{W}$ is an isomorphism of K-vector spaces functorial in X. If it is a functorial isomorphism of C-modules with the C-action on $\mathsf{Hom}_A(M, X)$ defined as above, then the diagram is commutative. This is easily deduced from the formula $g \circ H^0(h) \circ \pi = g \circ \pi \circ h$ for $g \in \mathsf{Hom}_A(M, X)$ and $h \in C$.

Now, we have proved that φ induces an equivalence between $\mathsf{mod}\,B/\langle e \rangle$ and $\mathsf{mod}\,C/\langle f \rangle$. Because $C/\langle f \rangle$ and $B/\langle e \rangle$ are basic algebras, $\varphi\colon C/\langle f \rangle \to B/\langle e \rangle$ is an isomorphism. ☐

2.4. **Grothendieck groups and semibricks.**

In this subsection, we observe semibricks and 2-term simple-minded collections from the point of view of Grothendieck groups.

We first briefly recall the definition of the Grothendieck groups of triangulated categories. Let $\mathcal{D}$ be an essentially small triangulated category. Then the Grothendieck group $K_0(\mathcal{D})$ of $\mathcal{D}$ is the abelian group generated by all isomorphic classes $[X]$ in $\mathcal{D}$ and bounded by the relation $[X] - [Y] + [Z] = 0$ for each triangle $X \to Y \to Z \to X[1]$. In this part, we consider the case $\mathcal{D} = \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$ and the case $\mathcal{D} = \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$. It is clear that $[X[1]] = -[X]$.

For simplicity, we assume that A is a basic finite-dimensional K-algebra. Let $n := |A|$ and $e_1, e_2, \dots, e_n$ be the primitive idempotents in A.

It is well-known that the Grothendieck group $K_0(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A))$ has the family $([e_i(A/\mathsf{rad}\,A)])_{i=1}^n$ of isoclasses of simple modules as a $\mathbb{Z}$-basis [Hap, III.1.2]. If $e_i(A/\mathsf{rad}\,A)$ appears c_i times in the composition factors of a module $M \in \mathsf{mod}\,A$, then in $K_0(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A))$, the element $[M]$ is equal to $\sum_{i=1}^n c_i[e_i(A/\mathsf{rad}\,A)]$, and for each complex $X = (X^a \to X^{a+1} \to \cdots \to X^b) \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$, the element $[X]$ is equal to $\sum_{j=a}^b (-1)^j [X^j]$.

Similarly, the Grothendieck group $K_0(\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A))$ has the family $([e_iA])_{i=1}^n$ of isoclasses of indecomposable projective modules as a $\mathbb{Z}$-basis. If a projective module $P \in \mathsf{proj}\,A$ is decomposed as $P \cong \bigoplus_{i=1}^n (e_iA)^{\oplus g_i}$, then $[P] = \sum_{i=1}^n g_i[e_iA]$ holds, and for each complex $P = (P^a \to P^{a+1} \to \cdots \to P^b) \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$, the element $[P]$ is equal to $\sum_{j=a}^b (-1)^j [P^j]$.

For these two Grothendieck groups, there is a natural bilinear form

$$\langle ?, ? \rangle \colon K_0(\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)) \times K_0(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)) \to \mathbb{Z}$$

defined as

$$\langle P, X \rangle := \sum_{j \in \mathbb{Z}} (-1)^j \,\mathsf{dim}_K \,\mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(P, X[j])$$

for any $P \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ and $X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$. It is easy to see that

$$\langle e_iA, e_j(A/\mathsf{rad}\,A) \rangle = \begin{cases} \mathsf{dim}_K \,\mathsf{End}_A(e_i(A/\mathsf{rad}\,A)) & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

The ring $\mathsf{End}_A(e_i(A/\mathsf{rad}\,A))$ is a division K-algebra, since $e_i(A/\mathsf{rad}\,A)$ is a simple A-module. We define an $n \times n$ diagonal matrix $\boldsymbol{D}$ so that its (i,i) entry is $\mathsf{dim}_K \,\mathsf{End}_A(e_i(A/\mathsf{rad}\,A))$.

Under this preparation, we fix $P \in \mathsf{2\text{-}silt}\,A$. Decompose $P \cong \bigoplus_{k=1}^n P_k$, and define a vertical vector $\boldsymbol{g}_k = \begin{bmatrix} (g_k)_1 & (g_k)_2 & \cdots & (g_k)_n \end{bmatrix}^{\mathrm{T}} \in \mathbb{Z}^n$ so that $[P_k] = \sum_{i=1}^n (g_k)_i [e_iA]$ holds in $K_0(\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A))$ for each $k \in \{1, 2, \ldots, n\}$. Here, $?^{\mathrm{T}}$ means the transpose. The vector $\boldsymbol{g}_k$ is called the *g-vector* of $[P_k]$. Since $P \in \mathsf{2\text{-}silt}\,A$, the g-vectors $\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n$ generate $\mathbb{Z}^n$ [AI, Theorem 2.27].

We take $\mathcal{X} \in \mathsf{2\text{-}smc}\,A$ corresponding to $P \in \mathsf{2\text{-}silt}\,A$ in Proposition 2.5. By Lemma 2.14, we may assume that $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$ and that $\mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(P_k, S_l) = 0$ holds for $k \neq l$. We define a vertical vector $\boldsymbol{c}_k = \begin{bmatrix} (c_k)_1 & (c_k)_2 & \cdots & (c_k)_n \end{bmatrix}^{\mathrm{T}} \in \mathbb{Z}^n$ so that $[X_k] = \sum_{i=1}^n (c_k)_i [e_i(A/\mathsf{rad}\,A)]$ holds in $K_0(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A))$ for each $k \in \{1, 2, \ldots, n\}$. We here call $\boldsymbol{c}_k$ the *c-vector* of $[X_k]$. Since $\mathcal{X} \in \mathsf{2\text{-}smc}\,A$, the c-vectors $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n$ generate $\mathbb{Z}^n$ [KY, Lemma 3.3].

It is clear that $\langle P_k, X_l \rangle = (\boldsymbol{g}_k)^{\mathrm{T}} \cdot \boldsymbol{D} \cdot \boldsymbol{c}_l$. Actually, this value is given as follows.

Theorem 2.17. *Let $k, l \in \{1, 2, \ldots, n\}$. Then,*

$$\langle P_k, X_l \rangle = \begin{cases} \mathsf{dim}_K \,\mathsf{End}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(X_l) & (k = l) \\ 0 & (k \neq l) \end{cases}$$

Proof. Let $k, l \in \{1, 2, \ldots, n\}$. Proposition 2.5 implies $\mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(P_k, X_l[j]) = 0$ for any $j \neq 0$. Thus, $\langle P_k, X_l \rangle$ coincides with $\mathsf{dim}_K \,\mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(P_k, X_l)$. Now, the assertion follows from Lemma 2.14. □

We define $n \times n$ matrices $\boldsymbol{G}$ and $\boldsymbol{C}$ as $\boldsymbol{G} := \begin{bmatrix} \boldsymbol{g}_1 & \boldsymbol{g}_2 & \cdots & \boldsymbol{g}_n \end{bmatrix}$ and $\boldsymbol{C} := \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \cdots & \boldsymbol{c}_n \end{bmatrix}$. Then, the above theorem means $\boldsymbol{G}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{C} = \boldsymbol{D}'$, where $\boldsymbol{D}'$ is the $n \times n$ diagonal matrix whose (i,i) entry is $\mathsf{dim}_K \,\mathsf{End}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(X_i)$. Since the g-vectors generate $\mathbb{Z}^n$ and so do the c-vectors, $\boldsymbol{G}$ and $\boldsymbol{C}$ are invertible matrices on $\mathbb{Z}$. Therefore, the Smith normal forms of $\boldsymbol{D}$ and $\boldsymbol{D}'$ are the same. In particular, if $\boldsymbol{D}$ is the identity matrix, then so is $\boldsymbol{D}'$. We do not know whether the diagonal matrices $\boldsymbol{D}$ and $\boldsymbol{D}'$ always coincide up to reordering of their entries, but if the 2-term silting object $P \in \mathsf{2\text{-}silt}\,A$ can be obtained from $A \in \mathsf{2\text{-}silt}\,A$ or $A[1] \in \mathsf{2\text{-}silt}\,A$ by repeating mutations, then we have $\boldsymbol{D} = \boldsymbol{D}'$ up to reordering of their entries, see [KY, Remark 7.7].

3. EXAMPLES

3.1. **Semibricks for Nakayama algebras.** In this subsection, we calculate the numbers of semibricks for the following Nakayama algebras. For integers $n, l \geq 1$, let $A_{n,l}$ be the algebra given by the following quiver with relations:

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n\ , \quad \text{all the paths of length } l \text{ are } 0,$$

and $B_{n,l}$ be the algebra given by the following quiver with relations:

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[lll, bend left] \end{tikzcd}\ , \quad \text{all the paths of length } l \text{ are } 0.$$

In this subsection, we calculate $a_{n,l} := \#\,\mathsf{sbrick}\,A_{n,l}$ and $b_{n,l} := \#\,\mathsf{sbrick}\,B_{n,l}$. For $n = 0$, let $A_{0,l} := 0$; hence $a_{0,l} = 1$, since $\#\,\mathsf{sbrick}\,A_{0,l} = \{\emptyset\}$. For convenience, we also set $a_{n,l} = 0$ for $n < 0$.

To state our result, we recall the nth Catalan numbers:

$$c_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)! \cdot n!} \quad (n \geq 0).$$

They satisfy the following equation:

$$(*) \qquad c_{n+1} = \sum_{i=0}^{n} c_i c_{n-i} \quad (n \geq 0).$$

The next equations are the main result of this subsection.

Theorem 3.1. *Let $n, l \geq 1$ be integers.*

(1) *The following equations hold:*

$$a_{n,l} = c_{n+1} \quad (n = 1, 2, \ldots, l), \qquad a_{n,l} = 2a_{n-1,l} + \sum_{i=2}^{l} c_{i-1} a_{n-i,l} \quad (n \geq 1).$$

(2) *The following equations hold:*

$$b_{n,l} = (n+1)c_n \quad (n = 1, 2, \ldots, l), \qquad b_{n,l} = 2b_{n-1,l} + \sum_{i=2}^{l} c_{i-1} b_{n-i,l} \quad (n \geq l+1).$$

(3) *Let $\xi_1, \xi_2, \ldots, \xi_l \in \mathbb{C}$ be the roots of the polynomial $F_l(X) := X^l - 2X^{l-1} - \sum_{i=2}^{l} c_{i-1} X^{l-i}$ with multiplicities. Then we have*

$$a_{n,l} = \sum_{\substack{t_1, t_2 \ldots, t_l \in \mathbb{Z}_{\geq 0}, \\ t_1 + t_2 + \cdots + t_l = n}} \xi_1^{t_1} \xi_2^{t_2} \cdots \xi_l^{t_l} \quad \text{and} \quad b_{n,l} = \xi_1^n + \xi_2^n + \cdots + \xi_l^n \quad (n \geq 1).$$

We remark that $F_l(X)$ is the characteristic polynomial of the common recurrence relation of the sequences $(a_{n,l})_{n=1}^{\infty}$ and $(b_{n,l})_{n=1}^{\infty}$ stated in parts (1) and (2) of the above theorem.

From Theorem 3.1, we obtain Tables 1 and 2 below.

		n						
		1	2	3	4	5	6	7
	1	2	4	8	16	32	64	128
	2	2	5	12	29	70	169	408
	3	2	5	14	37	98	261	694
l	4	2	5	14	42	118	331	934
	5	2	5	14	42	132	387	1130
	6	2	5	14	42	132	429	1298
	7	2	5	14	42	132	429	1430

TABLE 1. The table of $a_{n,l}$

		n						
		1	2	3	4	5	6	7
	1	2	4	8	16	32	64	128
	2	2	6	14	34	82	198	478
	3	2	6	20	50	132	354	940
l	4	2	6	20	70	182	504	1430
	5	2	6	20	70	252	672	1920
	6	2	6	20	70	252	924	2508
	7	2	6	20	70	252	924	3432

TABLE 2. The table of $b_{n,l}$

The algebras $A_{n,l}$ and $B_{n,l}$ are representation-finite, so every semibrick is left finite. For each of these algebras, Theorem 1.3 implies that the number of semibricks coincides with the number of support τ-tilting modules. Support τ-tilting modules over Nakayama algebras were studied in the paper [Ada], which contains the tables of the numbers of support τ-tilting modules for the algebras $A_{n,l}$ and $B_{n,l}$ [Ada, Tables 3 and 5]. Compare them with our Tables 1 and 2 above.

For $u \in \{1, 2, \dots, n\}$, we write e_u for the idempotent of $A_{n,l}$ and $B_{n,l}$ for the vertex u, and set $e_U := \sum_{u\in U} e_u$ for each subset $U \subset \{1, 2, \dots, n\}$. The notation $\mathsf{supp}\, M := \{u \in \{1, 2, \dots, n\} \mid Me_i \neq 0\}$ denotes the support of a module M over $A_{n,l}$ or $B_{n,l}$. We define (cyclic) intervals $[u, v], [[u, v]] \subset \{1, 2, \dots, n\}$ for $u, v \in \{1, 2, \dots, n\}$ as

$$[u, v] = \begin{cases} \{u, u + 1, \dots, v\} & (u \leq v) \\ \emptyset & (u > v) \end{cases}, \quad [[u, v]] = \begin{cases} \{u, u + 1, \dots, v\} & (u \leq v) \\ \{u, u + 1, \dots, n, 1, 2, \dots, v\} & (u > v) \end{cases}.$$

Since $A_{n,l}$ and $B_{n,l}$ are Nakayama algebras, for $u, v \in \{1, 2, \dots, n\}$, there exists at most one brick X satisfying $\mathsf{top}\, X = S_u$ and $\mathsf{soc}\, X = S_v$, where S_u and S_v are the simple modules corresponding to the vertices u and v, respectively. If such an X exists, then we set $S_{u,v} := X$.

First, we remark the next property, which follows from $A_{n,l} = A_{n,n}$ for $n \leq l$.

Lemma 3.2. *Let $n, l \geq 1$ be integers with $n \leq l$. Then we have $a_{n,l} = a_{n,n}$.*

The next proposition is a keypoint to determine the values of $a_{n,l}$.

Proposition 3.3. *For any integers $n, l \geq 1$, the following equation holds:*

$$a_{n,l} = 2a_{n-1,l} + \sum_{i=2}^{l} a_{i-2,l}a_{n-i,l}.$$

Proof. Let $m := \mathsf{min}\{n, l\}$. A brick $S_{n-i+1,n}$ is well-defined for $i = 1, \dots, m$.

Let $\mathcal{S} \in \mathsf{sbrick}\, A_{n,l}$. Because $\mathcal{S}$ is a semibrick, $\mathcal{S}$ satisfies exactly one of the following conditions $(0), (1), \dots, (m)$:

(0) no brick $S \in \mathcal{S}$ satisfies $n \in \mathsf{supp}\, S$,

(i) ($i \in \{1, 2, \dots, m\}$) the brick $S_{n-i+1,n}$ belongs to $\mathcal{S}$.

We define $\mathsf{sb}(i)$ as the subset of $\mathsf{sbrick}\, A_{n,l}$ consisting of the semibricks satisfying the condition (i) for each $i \in \{0, 1, \dots, m\}$. Then $\# \mathsf{sbrick}\, A_{n,l} = \sum_{i=0}^{m} \# \mathsf{sb}(i)$ holds.

First, we clearly have $\mathsf{sb}(0) = \mathsf{sbrick}\, A_{n,l}/\langle e_n\rangle$ and $A_{n,l}/\langle e_n\rangle \cong A_{n-1,l}$. Thus, we get that $\# \mathsf{sb}(0) = a_{n-1,l}$.

Second, we have a bijection $\mathsf{sb}(1) \to \mathsf{sbrick}\, A_{n,l}/\langle e_n\rangle$ defined as $\mathcal{S} \mapsto \mathcal{S} \setminus \{S_{n,n}\}$. Because $A_{n,l}/\langle e_n\rangle \cong A_{n-1,l}$, we get that $\# \mathsf{sb}(1) = a_{n-1,l}$.

Next, for each $i \in \{2, 3, \dots, m\}$, there exists a bijection

$$\begin{aligned} &\mathsf{sb}(i) \to \mathsf{sbrick}\, A_{n,l}/\langle e_{[n-i+1,n]}\rangle \times \mathsf{sbrick}\, A_{n,l}/\langle 1 - e_{[n-i+2,n-1]}\rangle \\ &\quad \mathcal{S} \mapsto (\{S \in \mathcal{S} \mid \mathsf{supp}\, S \cap [n - i + 1, n] = \emptyset\}, \{S \in \mathcal{S} \mid \mathsf{supp}\, S \subset [n - i + 2, n - 1]\}). \end{aligned}$$

The inverse map is given by $(\mathcal{S}_1, \mathcal{S}_2) \mapsto \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{S_{n-i+1,n}\}$. Since there are isomorphisms of algebras $A_{n,l}/\langle e_{[n-i+1,n]}\rangle \cong A_{n-i,l}$ and $A_{n,l}/\langle 1 - e_{[n-i+2,n-1]}\rangle \cong A_{i-2,l}$ (including $i = 2$), we have $\# \mathsf{sb}(i) = a_{i-2,l}a_{n-i,l}$.

Now, we obtain the equations

$$a_{n,l} = \# \mathsf{sbrick}\, A_{n,l} = \sum_{i=0}^{m} \# \mathsf{sb}(i) = 2a_{n-1,l} + \sum_{i=2}^{m} a_{i-2,l}a_{n-i,l}.$$

This equation implies the assertion, because $a_{n,l} = 0$ for $n < 0$. $\square$

We can regard the equation in Proposition 3.3 as a recurrence relation on n with the coefficients $a_{n,l}$. We determine the values of these coefficients.

Lemma 3.4. *We have $a_{n,l} = c_{n+1}$ if $0 \leq n \leq l$ and $1 \leq l$.*

Proof. We use induction on n. Note that we have defined $a_{n,l} = 0$ for $n < 0$.

For $n = 0, 1$, we have $a_{0,l} = 1 = c_1$ and $a_{1,l} = 2 = c_2$ as desired.

If $n \geq 2$, by Proposition 3.3 and the induction hypothesis, we have

$$a_{n,l} = 2a_{n-1,l} + \sum_{i=2}^{n} a_{i-2,l}a_{n-i,l} = 2c_n + \sum_{i=2}^{n} c_{i-1}c_{n-i+1} = \sum_{i=0}^{n} c_i c_{n-i} \overset{(*)}{=} c_{n+1},$$

and the assertion holds. The induction process is now complete. □

Now, we have a recurrence relation of the sequence $(a_{n,l})_{n=1}^{\infty}$ for each $l \geq 1$.

Proof of Theorem 3.1 (1). Proposition 3.3 and Lemma 3.4 imply the assertion. □

Next, we consider the algebra $B_{n,l}$.

Lemma 3.5. *We have* $\mathsf{sbrick}\, B_{n,n} = \mathsf{sbrick}\, B_{n,l}$ *and* $b_{n,l} = b_{n,n}$ *if* $1 \leq n \leq l$.

Proof. It is easy to see that $B_{n,n}$ is a quotient algebra of $B_{n,l}$, and that any brick in $\mathsf{mod}\, B_{n,l}$ belongs to $\mathsf{mod}\, B_{n,n}$. Thus the assertions follow. □

The value $b_{n,l}$ can be calculated from the sequence $(a_{n,l})_{n=1}^{\infty}$.

Proposition 3.6. *For any integers* $n, l \geq 1$, *the following equation holds:*

$$b_{n,l} = 2a_{n-1,l} + \sum_{i=2}^{l} i c_{i-1} a_{n-i,l}.$$

Proof. We set $m := \mathsf{min}\{n, l\}$.

Let $\mathcal{S} \in \mathsf{sbrick}\, B_{n,l}$. Because $\mathcal{S}$ is a semibrick, $\mathcal{S}$ satisfies exactly one of the following conditions $(0), (1), \ldots, (m)$:

- (0) no brick $S \in \mathcal{S}$ satisfies $n \in \mathsf{supp}\, S$,
- (i) $(i \in \{1, 2, \ldots, m\})$ there exists some brick $S \in \mathcal{S}$ satisfying $n \in \mathsf{supp}\, S$, and $\mathsf{max}\{\mathsf{dim}_K S \mid S \in \mathcal{S},\ n \in \mathsf{supp}\, S\} = i$ holds.

We define $\mathsf{sb}(i)$ as the subset of $\mathsf{sbrick}\, B_{n,l}$ consisting of the semibricks satisfying the condition (i) for each $i \in \{0, 1, \ldots, m\}$. Then $\#\,\mathsf{sbrick}\, B_{n,l} = \sum_{i=0}^{m} \#\,\mathsf{sb}(i)$ holds.

First, we clearly have $\mathsf{sb}(0) = \mathsf{sbrick}\, B_{n,l}/\langle e_n \rangle$ and $B_{n,l}/\langle e_n \rangle \cong A_{n-1,l}$. Thus, we get that $\#\,\mathsf{sb}(0) = a_{n-1,l}$.

Second, we have a bijection $\mathsf{sb}(1) \to \mathsf{sbrick}\, B_{n,l}/\langle e_n \rangle$ defined as $\mathcal{S} \mapsto \mathcal{S} \setminus \{S_{n,n}\}$. Because $B_{n,l}/\langle e_n \rangle \cong A_{n-1,l}$, we get that $\#\,\mathsf{sb}(1) = a_{n-1,l}$.

Next, let $i \in \{2, 3, \ldots, m\}$. For each pair (u, v) satisfying $n \in [[u, v]]$ and $\#[[u, v]] = i$, a brick $S_{u,v}$ is well-defined, so let $\mathsf{sb}(u, v)$ be the subset of $\mathsf{sb}(i)$ consisting of $\mathcal{S}$ with $S_{u,v} \in \mathcal{S}$.

If $\mathcal{S} \in \mathsf{sb}(u, v)$, then any brick $S \in \mathcal{S}$ different from $S_{u,v}$ satisfies $\mathsf{supp}\, S \cap [[u, v]] = \emptyset$ or $\mathsf{supp}\, S \subset [[u, v]] \setminus \{u, v\}$. In particular, $S_{u,v}$ is the unique brick $S \in \mathcal{S}$ satisfying $n \in \mathsf{supp}\, S$ and $\#\,\mathsf{supp}\, S = i$.

Thus, we have the following decomposition into a disjoint union:

$$\mathsf{sb}(i) = \coprod_{n \in [[u,v]],\ \#[[u,v]] = i} \mathsf{sb}(u, v).$$

For a pair (u, v) satisfying $n \in [[u, v]]$ and $\#[[u, v]] = i$, there exists a bijection

$$\begin{aligned} \mathsf{sb}(u, v) &\to \mathsf{sbrick}\, B_{n,l}/\langle e_{[[u,v]]} \rangle \times \mathsf{sbrick}\, B_{n,l}/\langle 1 - e_{[[u,v]] \setminus \{u,v\}} \rangle \\ \mathcal{S} &\mapsto (\{S \in \mathcal{S} \mid \mathsf{supp}\, S \cap [[u, v]] = \emptyset\}, \{S \in \mathcal{S} \mid \mathsf{supp}\, S \subset [[u, v]] \setminus \{u, v\}\}). \end{aligned}$$

The inverse is given by $(\mathcal{S}_1, \mathcal{S}_2) \mapsto \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{S_{u,v}\}$. Since there are isomorphisms of algebras, $B_{n,l}/\langle e_{[[u,v]]} \rangle \cong A_{n-i,l}$ and $B_{n,l}/\langle 1 - e_{[[u,v]] \setminus \{u,v\}} \rangle \cong A_{i-2,l}$ (including $i = 2$), we get that $\#\,\mathsf{sb}(u, v) = a_{i-2,l}a_{n-i,l} = c_{i-1}a_{n-i,l}$. There exist exactly i pairs (u, v) satisfying $n \in [[u, v]]$ and $\#[[u, v]] = i$, so we have $\#\,\mathsf{sb}(i) = i c_{i-1} a_{n-i,l}$.

Now, we have equations

$$b_{n,l} = \#\,\mathsf{sbrick}\,B_{n,l} = \sum_{i=0}^{m} \#\,\mathsf{sb}(i) = 2a_{n-1,l} + \sum_{i=2}^{m} ic_{i-1}a_{n-i,l}.$$

This equation implies the assertion, because $a_{n,l} = 0$ for $n < 0$. □

We can also determine $b_{n,l}$ explicitly if $n \leq l$.

Lemma 3.7. *We have* $b_{n,l} = (n+1)c_n$ *if* $1 \leq n \leq l$.

Proof. By Proposition 3.6 and Lemma 3.4, we have

$$b_{n,l} = 2a_{n-1,l} + \sum_{i=2}^{n} ia_{i-2,l}a_{n-i,l} = 2c_n + \sum_{i=2}^{n} ic_{i-1}c_{n-i+1}.$$

On the other hand, the equalities

$$\sum_{i=2}^{n} ic_{i-1}c_{n-i+1} = \sum_{i=1}^{n-1}(i+1)c_ic_{n-i} = \sum_{i=1}^{n-1}(n-i+1)c_ic_{n-i},$$

hold, so we get that

$$\begin{aligned}2\sum_{i=2}^{n} ic_{i-1}c_{n-i+1} &= \sum_{i=1}^{n-1}(i+1)c_ic_{n-i} + \sum_{i=1}^{n-1}(n-i+1)c_ic_{n-i} \\ &= (n+2)\sum_{i=1}^{n-1} c_ic_{n-i} = (n+2)\left(\sum_{i=0}^{n} c_ic_{n-i} - 2c_n\right) \overset{(*)}{=} (n+2)(c_{n+1} - 2c_n).\end{aligned}$$

Since $(n+2)c_{n+1} = 2(2n+1)c_n$ holds by definition, we have $\sum_{i=2}^{n} ic_{i-1}c_{n-i+1} = (n-1)c_n$; hence, $b_{n,l} = (n+1)c_n$. □

Now, we prove Theorem 3.1 (2).

Proof of Theorem 3.1 (2). We have obtained the first equation in Lemma 3.7. For the second equation, we use the assumption $n \geq l+1$; then Propositions 3.3 and 3.6 imply the assertion. □

Let $l \geq 1$ be an integer. It remains to prove Theorem 3.1 (3). For each $i \in \{0, 1, \ldots, l\}$, we define an integer d_i so that $F_l(X) = \sum_{i=0}^{l} d_iX^{l-i}$. Then we have $d_0 = 1$, $d_1 = -2$, and $d_i = -c_{i-1}$. Recall the following symmetric polynomials $\boldsymbol{e}_n, \boldsymbol{h}_n, \boldsymbol{p}_n \in \mathbb{Z}[X_1, X_2, \ldots, X_l]$ (see [Mac]):

$$\begin{aligned}\boldsymbol{e}_n(X_1, \ldots, X_l) &:= \sum_{J \subset \{1,2,\ldots,l\},\ \#J=n}\ \prod_{j \in J} X_j \quad (n = 0, 1, \ldots, l), \\ \boldsymbol{h}_n(X_1, \ldots, X_l) &:= \sum_{\substack{t_1,t_2\ldots,t_l \in \mathbb{Z}_{\geq 0},\\ t_1+t_2+\cdots+t_l=n}} X_1^{t_1}X_2^{t_2}\cdots X_l^{t_l} \quad (n \in \mathbb{Z}), \\ \boldsymbol{p}_n(X_1, \ldots, X_l) &:= X_1^n + X_2^n + \cdots + X_l^n \quad (n \geq 1).\end{aligned}$$

In particular, we have $\boldsymbol{e}_0 = 1$, $\boldsymbol{h}_0 = 1$, and $\boldsymbol{h}_n = 0$ for $n < 0$. We need a technical lemma on these polynomials.

Lemma 3.8. [Mac] *For any integer* $n \geq 1$, *we have polynomial equations* $\sum_{i=0}^{l}(-1)^i\boldsymbol{e}_i\boldsymbol{h}_{n-i} = 0$ *and* $\sum_{i=1}^{l}(-1)^{i-1}i\boldsymbol{e}_i\boldsymbol{h}_{n-i} = \boldsymbol{p}_n$.

Proof. The first equation is shown in [Mac, I.2, (2.6′)], and the second one is deduced from the second and the third equations in [Mac, I.2, Example 8]. □

Now, we complete the proof of Theorem 3.1.

Proof of Theorem 3.1 (3). We first show the assertion for $a_{n,l}$. Fix $l \geq 1$. For any $n \in \mathbb{Z}$, we set $a'_{n,l} := \boldsymbol{h}_n(\xi_1, \xi_2, \dots, \xi_l)$. It is enough to show that $a'_{n,l} = a_{n,l}$ for all $n \geq 1$.

We claim that $\sum_{i=0}^{l} d_i a'_{n-i,l} = 0$ for $n \geq 1$. Indeed, $\boldsymbol{e}_i(\xi_1, \xi_2, \dots, \xi_l) = (-1)^i d_i$ holds for each $i = 0, 1, \dots, l$. Substituting ξ_j for X_j in the first equation in Lemma 3.8, we have $\sum_{i=0}^{l} d_i a'_{n-i,l} = 0$ for $n \geq 1$.

By Theorem 3.1 (1), we also have $\sum_{i=0}^{l} d_i a_{n-i,l} = 0$ for $n \geq 1$. Because $a_{0,l} = 1 = a'_{0,l}$ and $a_{n,l} = 0 = a'_{n,l}$ for $n < 0$, it can be shown inductively that $a_{n,l} = a'_{n,l}$ for all $n \geq 1$. The assertion for $a_{n,l}$ is now proved.

We next show the assertion for $b_{n,l}$. In the second equation in Lemma 3.8, by setting $X_j = \xi_j$ and using $a'_{n,l} = a_{n,l}$, we have $\sum_{i=1}^{l}(-id_i)a_{n-i,l} = \xi_1^n + \xi_2^n + \cdots + \xi_l^n$. By Proposition 3.6, it is easily seen that $b_{n,l} = \sum_{i=1}^{l}(-id_i)a_{n-i,l}$, which yields $b_{n,l} = \xi_1^n + \xi_2^n + \cdots + \xi_l^n$. ☐

3.2. **Wide subcategories for tilted algebras.** We investigate functorially finiteness of wide subcategories and semibricks mainly for tilted algebras here. Recall from Subsection 1.3 that $\mathsf{f\text{-}wide}\, A$ is the set of functorially finite wide subcategories of $\mathsf{mod}\, A$. For a given algebra A, we consider the following two conditions in this subsection.

Condition 3.9. The equalities $\mathsf{f\text{-}wide}\, A = \mathsf{f_L\text{-}wide}\, A$ and $\mathsf{f\text{-}wide}\, A = \mathsf{f_R\text{-}wide}\, A$ hold.

Note that we always have the inclusions $\mathsf{f_L\text{-}wide}\, A \subset \mathsf{f\text{-}wide}\, A$ and $\mathsf{f_R\text{-}wide}\, A \subset \mathsf{f\text{-}wide}\, A$, see Proposition 1.28.

Condition 3.10. For any $\mathcal{S} \in \mathsf{f_L\text{-}sbrick}\, A$, any subset of $\mathcal{S}$ belongs to $\mathsf{f_L\text{-}sbrick}\, A$.

Ingalls–Thomas studied functorially finite wide subcategories for hereditary algebras [IT]. From their results, it follows that Condition 3.9 holds if A is hereditary.

Proposition 3.11. *If A is hereditary,* $\mathsf{f\text{-}wide}\, A = \mathsf{f_L\text{-}wide}\, A = \mathsf{f_R\text{-}wide}\, A$ *hold.*

Proof. We only show that $\mathsf{f\text{-}wide}\, A = \mathsf{f_L\text{-}wide}\, A$, because the opposite algebra A^{op} is also hereditary.

By Proposition 1.28, it suffices to show that $\mathsf{f\text{-}wide}\, A \subset \mathsf{f_L\text{-}wide}\, A$. Let $\mathcal{W} \in \mathsf{f\text{-}wide}\, A$, then $\mathsf{Fac}\, \mathcal{W} = \mathsf{T}(\mathcal{W})$ holds by [IT, Proposition 2.13]. Then [IT, Corollary 2.17] implies $\mathsf{T}(\mathcal{W}) = \mathsf{Fac}\, \mathcal{W} \in \mathsf{f\text{-}tors}\, A$, which yields $\mathcal{W} \in \mathsf{f_L\text{-}wide}\, A$. Now $\mathsf{f\text{-}wide}\, A \subset \mathsf{f_L\text{-}wide}\, A$ has been proved, and we obtain the assertion. ☐

We also remark relationship between Conditions 3.9 and 3.10.

Proposition 3.12. *If* $\mathsf{f\text{-}wide}\, A = \mathsf{f_L\text{-}wide}\, A$*, then the algebra A satisfies Condition 3.10.*

Proof. Assume $\mathsf{f\text{-}wide}\, A = \mathsf{f_L\text{-}wide}\, A$. Let $\mathcal{S} \in \mathsf{f_L\text{-}sbrick}\, A$ and $\mathcal{S}_1 \subset \mathcal{S}$.

Set $\mathcal{W} := \mathsf{Filt}\, \mathcal{S}$ and $\mathcal{W}_1 := \mathsf{Filt}\, \mathcal{S}_1$. Since $\mathcal{S} \in \mathsf{f_L\text{-}sbrick}\, A$, we have $\mathcal{W} \in \mathsf{f_L\text{-}wide}\, A$, and Proposition 1.28 implies that $\mathcal{W} \in \mathsf{f\text{-}wide}\, A$.

We claim that $\mathcal{W}_1$ is a functorially finite subcategory of $\mathcal{W}$. There exists $M \in \mathsf{s\tau\text{-}tilt}\, A$ such that $\mathsf{ind}(M/\mathsf{rad}_B\, M) = \mathcal{S}$ with $B := \mathsf{End}_A(M)$ by Theorem 1.3. By Theorem 1.27, we obtain an equivalence $\mathsf{Hom}_A(M, ?)\colon \mathcal{W} \to \mathsf{mod}\, C$ for some algebra C sending the elements of $\mathcal{S}$ to the simple C-modules. Thus, $\mathsf{Hom}_A(M, \mathcal{W}_1)$ is equivalent to a Serre subcategory of $\mathsf{mod}\, C$, and it is functorially finite in $\mathsf{mod}\, C$. This implies that $\mathcal{W}_1$ is functorially finite in $\mathcal{W}$.

Because $\mathcal{W}$ belongs to $\mathsf{f\text{-}wide}\, A$, we get that $\mathcal{W}_1 \in \mathsf{f\text{-}wide}\, A$. By assumption, we have $\mathcal{W}_1 \in \mathsf{f_L\text{-}wide}\, A$. Therefore, A satisfies Condition 3.10. ☐

Thus, every hereditary algebra A satisfies both Conditions 3.9 and 3.10. On the other hand, neither of these conditions holds for the following algebra A in Example 3.13.

Example 3.13. Let A be the finite-dimensional K-algebra given by the following quiver with relation:

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \arrow[r, "\gamma"] & 3 \end{tikzcd}, \quad \alpha\gamma = 0.$$

We write e_i for the corresponding idempotent for the vertex $i = 1, 2, 3$.

We consider the indecomposable injective module I_3:

$$I_3 = \left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right), \quad I_3\alpha = 0, \quad I_3\beta = I_3e_2.$$

First, I_3 is a brick, so $\mathsf{Filt}\, I_3 \in \mathsf{wide}\, A$ follows from Proposition 1.24. Because I_3 is injective, we have $\mathsf{Filt}\, I_3 = \mathsf{add}\, I_3$, so it belongs to $\mathsf{f\text{-}wide}\, A$.

We claim that $\mathsf{Filt}\, I_3 \notin \mathsf{f_L\text{-}wide}\, A$, that is, $\mathsf{T}(\mathsf{Filt}\, I_3) = \mathsf{T}(I_3) \notin \mathsf{f\text{-}tors}\, A$. We assume that $\mathsf{T}(I_3) \in \mathsf{f\text{-}tors}\, A$ and deduce a contradiction. We consider the quotient algebra $A' := A/\langle e_3 \rangle$. The algebra A' is isomorphic to the path algebra of the Kronecker quiver $1 \rightrightarrows 2$. Let $M := I_3 \otimes_A A'$. Because $\mathsf{T}(I_3) = \mathsf{Filt}(\mathsf{Fac}\, I_3) \in \mathsf{f\text{-}tors}\, A$, the torsion class $\mathsf{T}(M) = \mathsf{Filt}(\mathsf{Fac}\, M)$ must belong to $\mathsf{f\text{-}tors}\, A'$. We can see $M \in \mathsf{brick}\, A'$, because

$$M = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right), \quad M\alpha = 0, \quad M\beta = Me_2,$$

so we have $\mathsf{Filt}\, M \in \mathsf{wide}\, A'$. Because $\mathsf{T}(M) = \mathsf{T}(\mathsf{Filt}\, M) \in \mathsf{f\text{-}tors}\, A'$, the wide subcategory $\mathsf{Filt}\, M$ must be in $\mathsf{f\text{-}wide}\, A'$ by Proposition 1.28. However, M is in a homogeneous tube in the Auslander–Reiten quiver of $\mathsf{mod}\, A'$, so we obtain that $\mathsf{Filt}\, M \notin \mathsf{f\text{-}wide}\, A$, and it is a contradiction. Thus, the claim $\mathsf{T}(\mathsf{Filt}\, I_3) = \mathsf{T}(I_3) \notin \mathsf{f\text{-}tors}\, A$ is now shown.

Therefore, Condition 3.9 does not hold.

Moreover, we claim $\mathsf{f_L\text{-}wide}\, A \neq \mathsf{f_R\text{-}wide}\, A$. To prove this, we show that $\mathsf{Filt}\, I_3 \in \mathsf{f_R\text{-}wide}\, A$, that is, $\mathsf{F}(I_3) \in \mathsf{f\text{-}torf}\, A$. We use the map $\tau^{-1}\text{-}\mathsf{rigid}\, A \to \mathsf{f_R\text{-}sbrick}\, A$ in Theorem 1.3. Clearly, $I_3 \in \tau^{-1}\text{-}\mathsf{rigid}\, A$ and $I_3 \in \mathsf{brick}\, A$. Thus, the corresponding right finite semibrick is $\{I_3\}$, and we have $\mathsf{F}(I_3) \in \mathsf{f\text{-}torf}\, A$.

For Condition 3.10, we consider a simple module S_2:

$$S_2 = (\, 2\,).$$

We can see $\{I_3, S_2\} \in \mathsf{sbrick}\, A$, and we have shown that $\{I_3\} \notin \mathsf{f_L\text{-}sbrick}\, A$. To show that Condition 3.10 does not hold, it is sufficient to prove that $\{I_3, S_2\} \in \mathsf{f_L\text{-}sbrick}\, A$, that is, $\mathsf{T}(I_3, S_2) \in \mathsf{f\text{-}tors}\, A$. We use the map $\tau\text{-}\mathsf{rigid}\, A \to \mathsf{f_L\text{-}sbrick}\, A$ in Theorem 1.3. Let P_1 be the following indecomposable projective module:

$$P_1 = \left(\begin{smallmatrix} & 1 & & \\ 2 & & 2 & \\ & & & 3 \end{smallmatrix}\right).$$

By straightforward calculation, we obtain that $S_2 \oplus P_1$ is a τ-rigid A-module, and then the corresponding left finite semibrick is $\{I_3, S_2\}$. This implies that $\mathsf{T}(I_3, S_2) \in \mathsf{f\text{-}tors}\, A$. Thus, Condition 3.10 does not hold.

The algebra in Example 3.13 is a tilted algebra of type $\tilde{\mathbb{A}}_2$. The exchange quivers of $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ and $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$ are written in Figure 6 below. The corresponding left finite semibricks and the corresponding right finite semibricks are denoted by red letters. We can see that $\mathsf{f_L\text{-}sbrick}\, A \subset \mathsf{f_R\text{-}sbrick}\, A$ and that $\{I_3\}$ is the unique element in $\mathsf{f_R\text{-}sbrick}\, A \setminus \mathsf{f_L\text{-}sbrick}\, A$.

On Condition 3.9, we prove more general properties for tilted algebras. In the rest, we assume that K is an algebraically closed field.

Theorem 3.14. *Let H be a hereditary algebra, $T \in \mathsf{mod}\, H$ be a tilting module, and $A := \mathsf{End}_H(T)$. Then the following assertions hold.*

1. *If $\mathsf{Sub}_H\, \tau T$ has only finitely many indecomposable H-modules, then* $\mathsf{f\text{-}wide}\, A = \mathsf{f_L\text{-}wide}\, A$.
   If $\mathsf{Fac}_H\, T$ has only finitely many indecomposable H-modules, then $\mathsf{f\text{-}wide}\, A = \mathsf{f_R\text{-}wide}\, A$.
2. *If T is either preprojective or preinjective, then* $\mathsf{f\text{-}wide}\, A = \mathsf{f_L\text{-}wide}\, A = \mathsf{f_R\text{-}wide}\, A$.

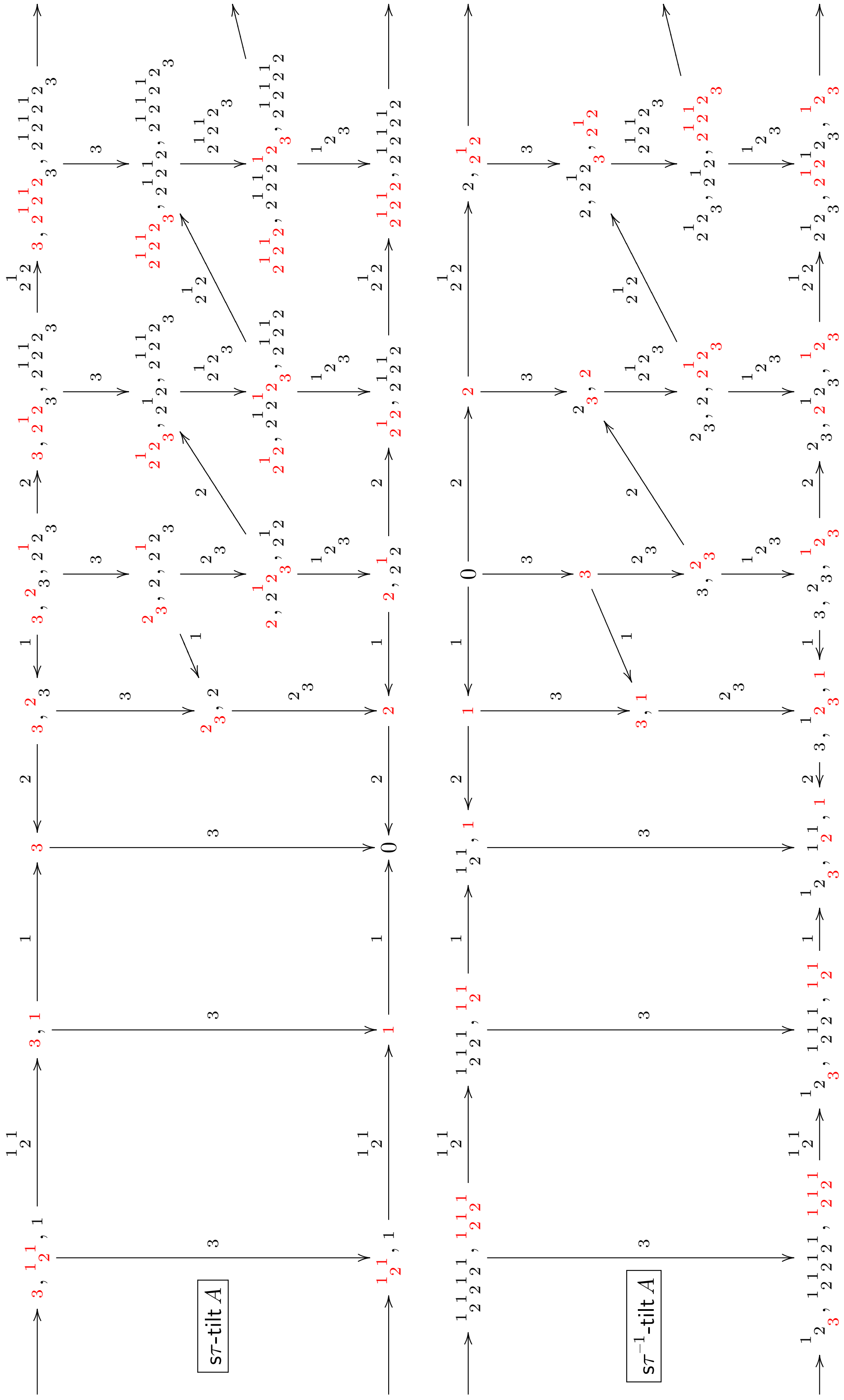


FIGURE 6. The exchange quivers of $\mathsf{s}\tau$-$\mathsf{tilt}\,A$ and $\mathsf{s}\tau^{-1}$-$\mathsf{tilt}\,A$

(3) *Assume that H is a hereditary algebra of extended Dynkin type. We decompose T as $T_{\mathrm{pp}} \oplus T_{\mathrm{reg}} \oplus T_{\mathrm{pi}}$ with a preprojective module T_{pp}, a regular module T_{reg}, and a preinjective module T_{pi}. If $T_{\mathrm{reg}} \neq 0$ and $T_{\mathrm{pi}} = 0$, then we have $\mathsf{f}\text{-}\mathsf{wide}\, A \supsetneq \mathsf{f_R}\text{-}\mathsf{wide}\, A$, and if $T_{\mathrm{reg}} \neq 0$ and $T_{\mathrm{pp}} = 0$, then we have $\mathsf{f}\text{-}\mathsf{wide}\, A \supsetneq \mathsf{f_L}\text{-}\mathsf{wide}\, A$.*

Now, we begin the proof of the theorem. For additive full subcategories $\mathcal{C}_1, \mathcal{C}_2 \subset \mathsf{mod}\, A$, the notation $\mathcal{C}_1 * \mathcal{C}_2$ denotes the full subcategory of $\mathsf{mod}\, A$ consisting of all M such that there exists a short exact sequence $0 \to M_1 \to M \to M_2 \to 0$ with $M_1 \in \mathcal{C}_1$ and $M_2 \in \mathcal{C}_2$.

Lemma 3.15. *Let $(\mathcal{X}, \mathcal{Y})$ be a torsion pair in $\mathsf{mod}\, A$ satisfying the following conditions:*

- *the torsion pair $(\mathcal{X}, \mathcal{Y})$ is splitting,*
- *for any $N_1, N_2 \in \mathcal{Y}$, we have $\mathsf{Ext}^2_A(N_1, N_2) = 0$,*
- *the torsion class $\mathcal{X}$ has only finitely many indecomposable A-modules.*

Then we have $\mathsf{f}\text{-}\mathsf{wide}\, A = \mathsf{f_L}\text{-}\mathsf{wide}\, A$.

Proof. We note that $\mathsf{Ext}^1_A(N, M) = 0$ holds for any $M \in \mathcal{X}$ and any $N \in \mathcal{Y}$, because the torsion pair $(\mathcal{X}, \mathcal{Y})$ is splitting.

Let $\mathcal{W} \in \mathsf{f}\text{-}\mathsf{wide}\, A$ and set $\mathcal{W}_{\mathcal{X}} := \mathcal{W} \cap \mathcal{X}$ and $\mathcal{W}_{\mathcal{Y}} := \mathcal{W} \cap \mathcal{Y}$. It is sufficient to prove that $\mathsf{T}(\mathcal{W}) \in \mathsf{f}\text{-}\mathsf{tors}\, A$.

We first claim that $\mathsf{Fac}\, \mathcal{W}_{\mathcal{X}} * \mathsf{Fac}\, \mathcal{W}_{\mathcal{Y}} \subset \mathsf{Fac}\, \mathcal{W}_{\mathcal{Y}} * \mathsf{Fac}\, \mathcal{W}_{\mathcal{X}}$ holds. Let L belong to $\mathsf{Fac}\, \mathcal{W}_{\mathcal{X}} * \mathsf{Fac}\, \mathcal{W}_{\mathcal{Y}}$. There exists a short exact sequence $0 \to L_1 \xrightarrow{f} L \xrightarrow{g} L_2 \to 0$ such that $L_1 \in \mathsf{Fac}\, \mathcal{W}_{\mathcal{X}}$ and $L_2 \in \mathsf{Fac}\, \mathcal{W}_{\mathcal{Y}}$. Because $L_2 \in \mathsf{Fac}\, \mathcal{W}_{\mathcal{Y}}$, there exists a surjection $h \colon N \to L_2$ with $N \in \mathcal{W}_{\mathcal{Y}}$. We have $N \in \mathcal{Y}$ and $L_1 \in \mathsf{Fac}\, \mathcal{W}_{\mathcal{X}} \subset \mathcal{X}$, because $\mathcal{X}$ is a torsion class. Thus, we have $\mathsf{Ext}^1_A(N, L_1) = 0$, so $\mathsf{Hom}_A(N, g) \colon \mathsf{Hom}_A(N, L) \to \mathsf{Hom}_A(N, L_2)$ is surjective. There exists $h' \colon N \to L$ such that $h = gh'$. We have the following commutative diagram with the rows exact:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & N \arrow[r, equal] \arrow[d, "h'"] & N \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & L_1 \arrow[r, "f"] & L \arrow[r, "g"] & L_2 \arrow[r] & 0 .
\end{tikzcd}$$

We have exact sequences $L_1 \to \mathsf{Coker}\, h' \to \mathsf{Coker}\, h = 0$ by Snake lemma and $0 \to \mathsf{Im}\, h' \to L \to \mathsf{Coker}\, h' \to 0$ by definition. We can easily see $\mathsf{Im}\, h' \in \mathsf{Fac}\, N \subset \mathsf{Fac}\, \mathcal{W}_{\mathcal{Y}}$ and $\mathsf{Coker}\, h' \in \mathsf{Fac}\, L_1 \subset \mathsf{Fac}\, \mathcal{W}_{\mathcal{X}}$. Thus, we have $L \in \mathsf{Fac}\, \mathcal{W}_{\mathcal{Y}} * \mathsf{Fac}\, \mathcal{W}_{\mathcal{X}}$. Therefore, the claim $\mathsf{Fac}\, \mathcal{W}_{\mathcal{X}} * \mathsf{Fac}\, \mathcal{W}_{\mathcal{Y}} \subset \mathsf{Fac}\, \mathcal{W}_{\mathcal{Y}} * \mathsf{Fac}\, \mathcal{W}_{\mathcal{X}}$ is proved.

By assumption, every indecomposable module in $\mathcal{W}$ belongs to $\mathcal{X}$ or $\mathcal{Y}$. Thus, we have $\mathsf{T}(\mathcal{W}) = \mathsf{Filt}(\mathsf{Fac}\, \mathcal{W}_{\mathcal{X}} \cup \mathsf{Fac}\, \mathcal{W}_{\mathcal{Y}})$. Because $\mathsf{Fac}\, \mathcal{W}_{\mathcal{X}} * \mathsf{Fac}\, \mathcal{W}_{\mathcal{Y}} \subset \mathsf{Fac}\, \mathcal{W}_{\mathcal{Y}} * \mathsf{Fac}\, \mathcal{W}_{\mathcal{X}}$, we have $\mathsf{T}(\mathcal{W}) = \mathsf{T}(\mathcal{W}_{\mathcal{Y}}) * \mathsf{T}(\mathcal{W}_{\mathcal{X}})$. If $\mathsf{T}(\mathcal{W}_{\mathcal{Y}})$ and $\mathsf{T}(\mathcal{W}_{\mathcal{X}})$ are functorially finite, then $\mathsf{T}(\mathcal{W})$ is functorially finite, see [SikS, Theorem 2.6].

We would like to show the functorially finiteness of $\mathsf{T}(\mathcal{W}_{\mathcal{Y}})$.

We prove that $\mathsf{T}(\mathcal{W}_{\mathcal{Y}}) = \mathsf{Fac}\, \mathcal{W}_{\mathcal{Y}}$ by a similar argument to [IT, Proposition 2.13]. Let $0 \to L_1 \to L_2 \to L_3 \to 0$ be a short exact sequence with $L_1, L_3 \in \mathsf{Fac}\, \mathcal{W}_{\mathcal{Y}}$. It is sufficient to show that $L_2 \in \mathsf{Fac}\, \mathcal{W}_{\mathcal{Y}}$. By assumption, there exists a surjection $f_3 \colon N_3 \to L_3$ with $N_3 \in \mathcal{W}_{\mathcal{Y}}$. Taking the pull back, we have the following commutative diagram with the rows exact and f_2 surjective:

$$\begin{tikzcd}
0 \arrow[r] & L_1 \arrow[r] \arrow[d, equal] & L'_2 \arrow[r] \arrow[d, "f_2"] & N_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & L_1 \arrow[r] & L_2 \arrow[r] & L_3 \arrow[r] & 0 .
\end{tikzcd}$$

By assumption again, there exists a surjection $g_1 \colon N_1 \to L_1$ with $N_1 \in \mathcal{W}_{\mathcal{Y}}$. Because $\mathcal{Y}$ is a torsion-free class, we have $\mathsf{Ker}\, g_1 \in \mathcal{Y}$. By assumption, we get $\mathsf{Ext}^2_A(N_3, \mathsf{Ker}\, g_1) = 0$. Thus, we have an exact sequence $\mathsf{Ext}^1_A(N_3, N_1) \to \mathsf{Ext}^1_A(N_3, L_1) \to 0$, so there exists the following commutative diagram with the rows exact and g_2 surjective:

$$\begin{tikzcd}
0 \arrow[r] & N_1 \arrow[r] \arrow[d, "g_1"] & L_2'' \arrow[r] \arrow[d, "g_2"] & N_3 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L_1 \arrow[r] & L_2' \arrow[r] & N_3 \arrow[r] & 0 .
\end{tikzcd}$$

Here, we have $L_2'' \in \mathcal{W}_{\mathcal{Y}}$, because $\mathcal{W}_{\mathcal{Y}}$ is extension closed. Since f_2 and g_2 are surjective, $f_2g_2 \colon L_2'' \to L_2$ is surjective, so we have $L_2 \in \mathsf{Fac}\,\mathcal{W}_{\mathcal{Y}}$. It is now proved that $\mathsf{T}(\mathcal{W}_{\mathcal{Y}}) = \mathsf{Fac}\,\mathcal{W}_{\mathcal{Y}}$.

We claim that $\mathcal{W}_{\mathcal{Y}}$ is covariantly finite in $\mathsf{mod}\,A$. Let $L \in \mathsf{mod}\,A$. There exists a left $\mathcal{W}$-approximation $f \colon L \to M \oplus N$ with $M \in \mathcal{W}_{\mathcal{X}}$ and $N \in \mathcal{W}_{\mathcal{Y}}$, because $\mathcal{W}$ is functorially finite and $(\mathcal{X}, \mathcal{Y})$ is splitting. Compose the projection $p \colon M \oplus N \to N$, then we have a left $\mathcal{W}_{\mathcal{Y}}$-approximation $pf \colon L \to N$, because $\mathsf{Hom}_A(\mathcal{X}, \mathcal{Y}) = 0$. Therefore, $\mathcal{W}_{\mathcal{Y}}$ is covariantly finite.

Now we can take a left $\mathcal{W}_{\mathcal{Y}}$-approximation $A \to N$ of A. Then we get $N \in \mathcal{W}_{\mathcal{Y}} \subset \mathsf{Fac}\,N$ and $\mathsf{Fac}\,\mathcal{W}_{\mathcal{Y}} = \mathsf{Fac}\,N$. By [AS1, Proposition 4.6], $\mathsf{T}(\mathcal{W}_{\mathcal{Y}}) = \mathsf{Fac}\,\mathcal{W}_{\mathcal{Y}} = \mathsf{Fac}\,N$ is functorially finite.

On the other hand, it is clear that $\mathsf{T}(\mathcal{W}_{\mathcal{X}})$ is contained in $\mathcal{X}$, so $\mathsf{T}(\mathcal{W}_{\mathcal{X}})$ has only finitely many indecomposable modules. Thus, $\mathsf{T}(\mathcal{W}_{\mathcal{X}})$ is functorially finite in $\mathsf{mod}\,A$.

We finally get that $\mathsf{T}(\mathcal{W}) = \mathsf{T}(\mathcal{W}_{\mathcal{Y}}) * \mathsf{T}(\mathcal{W}_{\mathcal{X}})$ is functorially finite. □

We can show Theorem 3.14.

Proof of Theorem 3.14. (1) We consider the first statement. The other one is shown by taking K-duals.

The tilting H-module T induces a torsion pair $(\mathsf{Fac}_H\,T, \mathsf{Sub}_H\,\tau T)$ in $\mathsf{mod}\,A$ and a torsion pair $(\mathsf{Fac}_A\,D(\tau T), \mathsf{Sub}_A\,DT)$ in $\mathsf{mod}\,A$. We check that the torsion pair $(\mathsf{Fac}_A\,D(\tau T), \mathsf{Sub}_A\,DT)$ satisfies the conditions in Lemma 3.15.

First, since H is hereditary, $(\mathsf{Fac}_A\,D(\tau T), \mathsf{Sub}_A\,DT)$ is a splitting torsion pair in $\mathsf{mod}\,A$ by [ASS, VI.5.7. Corollary].

Second, by [ASS, VI.4.1. Lemma], the projective dimension of every module in $\mathsf{Sub}_A\,DT$ is at most one. Thus, we have $\mathsf{Ext}^2_A(N_1, N_2) = 0$ for any $N_1, N_2 \in \mathsf{Sub}_A\,DT$.

Third, the torsion-free class $\mathsf{Sub}_H\,\tau T$ in $\mathsf{mod}\,H$ has only finitely many indecomposable H-modules by assumption. Thus, the torsion class $\mathsf{Fac}_A\,D(\tau T)$ in $\mathsf{mod}\,A$ has only finitely many indecomposable A-modules.

Thus, the conditions in Lemma 3.15 hold for $(\mathsf{Fac}_A\,D(\tau T), \mathsf{Sub}_A\,DT)$. Thus, $\mathsf{f\text{-}wide}\,A = \mathsf{f_L\text{-}wide}\,A$.

(2) We consider the case that T is preprojective. The other case is shown similarly.

Since T is preprojective, the torsion-free class $\mathsf{Sub}_H\,\tau T$ has only finitely many indecomposable H-modules [ASS, VIII.2.5. Lemma]. Thus, we get $\mathsf{f\text{-}wide}\,A = \mathsf{f_L\text{-}wide}\,A$ by (1).

On the other hand, there exists some preinjective tilting H-module T' with $A \cong \mathsf{End}_H(T')$. Since T' is preinjective, the torsion class $\mathsf{Fac}_H\,T'$ has only finitely many indecomposable H-modules. Thus, we also have $\mathsf{f\text{-}wide}\,A = \mathsf{f_R\text{-}wide}\,A$ by (1).

(3) We consider the first statement. The other one is shown by taking K-duals. We fully refer to [SimS].

By assumption, there exists a regular stable tube $\mathcal{C}$ in the Auslander–Reiten quiver of $\mathsf{mod}\,H$ containing an indecomposable direct summand of T_{reg}. Let V be the direct sum of the indecomposable direct summands of T in $\mathcal{C}$, and decompose T as $U \oplus V$. We have $\mathsf{Hom}_H(V, U) = 0$, because H is a hereditary algebra of extended Dynkin type.

Now we define the *cone* for each indecomposable H-module in the stable tube $\mathcal{C}$ as in [SimS]. Let r be the rank of the stable tube $\mathcal{C}$. Then $\mathcal{C}$ is isomorphic to $\mathbb{Z}\mathbb{A}_\infty/\langle \tau^r \rangle$ as translation quivers.

We may assume that the set of vertices of $\mathbb{Z}\mathbb{A}_\infty/\langle\tau^r\rangle$ is $\{1,2,3,\ldots\}\times(\mathbb{Z}/r\mathbb{Z})$, and that the set of arrows is

$$\begin{aligned}&\{(a,b+r\mathbb{Z})\to(a+1,b+r\mathbb{Z})\mid a\in\{1,2,3,\ldots\},\ b\in\{0,1,\ldots,r-1\}\}\\ &\cup\{(a+1,b+r\mathbb{Z})\to(a,b+1+r\mathbb{Z})\mid a\in\{1,2,3,\ldots\},\ b\in\{0,1,\ldots,r-1\}\}.\end{aligned}$$

Fix an isomorphism $\mathcal{C}\to\mathbb{Z}\mathbb{A}_\infty/\langle\tau^r\rangle$ and identify $\mathcal{C}$ with $\mathbb{Z}\mathbb{A}_\infty/\langle\tau^r\rangle$ by this isomorphism.

Let W be an indecomposable module in $\mathcal{C}$ and $(a,b+r\mathbb{Z})$ be its position, then the cone $\mathsf{Cone}\,W$ is defined as the set of indecomposable modules in $\mathcal{C}$ located in

$$\{(c,b+d+r\mathbb{Z})\mid c\in\{1,2,\ldots,a\},\ d\in\{0,1,\ldots,a-c\}\}.$$

For example, if $a=3$, this set is pictured as follows:

$$\begin{tikzcd}
 & & (3,b+r\mathbb{Z}) \arrow[dr] & & \\
 & (2,b+r\mathbb{Z}) \arrow[ur] \arrow[dr] & & (2,b+1+r\mathbb{Z}) \arrow[dr] & \\
(1,b+r\mathbb{Z}) \arrow[ur] & & (1,b+1+r\mathbb{Z}) \arrow[ur] & & (1,b+2+r\mathbb{Z})
\end{tikzcd}.$$

Now, let $V_1,\ldots,V_p$ be the distinct elements of $\mathsf{ind}_H\,V$. They are indecomposable H-modules in $\mathcal{C}$. By [SimS, XVII.1.7. Lemma], if $i\neq j$, then $\mathsf{Cone}\,V_i\subsetneq\mathsf{Cone}\,V_j$ or $\mathsf{Cone}\,V_i\supsetneq\mathsf{Cone}\,V_j$ or $\mathsf{Cone}\,V_i\cap\mathsf{Cone}\,V_j=\emptyset$ holds. Thus, we may assume that $\mathsf{Cone}\,V_1\supsetneq\mathsf{Cone}\,V_j$ or $\mathsf{Cone}\,V_1\cap\mathsf{Cone}\,V_j=\emptyset$ holds for $j=2,\ldots,p$.

We define an idempotent $e\in A$ as $U\oplus V\to V\to U\oplus V$. Since $\mathsf{Hom}_H(V,U)=0$, we have $A/\langle e\rangle\cong\mathsf{End}_H(U)$ as K-algebras. By [SimS, XVII.2.3. Theorem] (c), the indecomposable A-modules in $\mathsf{Hom}_H(T,\mathcal{C}\cap\mathsf{Fac}_H\,T\cap V^\perp)$, where $V^\perp$ is considered in $\mathsf{mod}\,H$, forms a standard stable tube $\mathcal{C}'$ in the Auslander–Reiten quiver of $\mathsf{mod}\,A/\langle e\rangle$. Let $M_1,\ldots,M_m$ be all the distinct $A/\langle e\rangle$-modules in the mouth of the stable tube $\mathcal{C}'$.

In this setting, $P_1:=\mathsf{Hom}_H(T,V_1)$ is a projective A-module. If V_1 is located on $(a,b+r\mathbb{Z})$ in $\mathcal{C}$, we consider the H-module W_1 corresponding to the vertex $(a+1,b-1+r\mathbb{Z})$. From the proof of [SimS, XVII.2.3. Theorem], we obtain the following properties:

- $\mathsf{Hom}_H(T,W_1)$ is an $A/\langle e\rangle$-module lying in the mouth of the stable tube $\mathcal{C}'$,
- there exists an $A/\langle 1-e\rangle$-module N such that $\mathsf{rad}_A\,P_1=\mathsf{Hom}_H(T,W_1)\oplus N$.

Therefore, we may assume $M_1=\mathsf{Hom}_H(T,W_1)$, and we obtain that $M_1=\mathsf{Hom}_H(T,W_1)$ coincides with the maximum A-submodule X of P_1 satisfying $X\in\mathsf{mod}\,A/\langle e\rangle$, or equivalently, $M_1=\mathsf{Hom}_A(A/\langle e\rangle,P_1)$.

Now, we claim that the set $\mathcal{S}:=\{P_1,M_2,\ldots,M_m\}$ is a semibrick in $\mathsf{mod}\,A$. First, by a property of standard stable tubes, we have $\{M_2,\ldots,M_m\}\in\mathsf{sbrick}\,A$. We also have $\mathsf{Hom}_A(P_1,M_i)=0$, since M_i is an $A/\langle e\rangle$-module. Next, $\mathsf{Hom}_A(M_i,P_1)\cong\mathsf{Hom}_A(M_i,M_1)=0$ holds for $i=2,\ldots,m$, because $M_i\in\mathsf{mod}\,A/\langle e\rangle$ and $M_1=\mathsf{Hom}_A(A/\langle e\rangle,P_1)$. It remains to show $P_1\in\mathsf{brick}\,A$. Since V_1 is a direct summand of V, it satisfies $\mathsf{Ext}^1_H(V_1,V_1)=0$; hence, $V_1\in\mathsf{brick}\,H$ by [SimS, XVII.1.6. Lemma]. Thus, $P_1\in\mathsf{brick}\,A$. Therefore, we have $\mathcal{S}\in\mathsf{sbrick}\,A$. We can consider a wide subcategory $\mathcal{W}=\mathsf{Filt}_A\,\mathcal{S}$ of $\mathsf{mod}\,A$.

We prove that $\mathcal{W}\in\mathsf{f\text{-}wide}\,A$. We have $\mathcal{W}=\mathsf{add}_A\,P_1*\mathsf{Filt}_A(M_2,\ldots,M_m)$, since P_1 is projective. By construction, $M_2,\ldots,M_m$ are $A/\langle e\rangle$-modules in the mouth of $\mathcal{C}'$ and there also exists M_1 in the mouth. Thus, a wide subcategory $\mathsf{Filt}_A(M_2,\ldots,M_m)$ belongs to $\mathsf{f\text{-}wide}\,A/\langle e\rangle$; hence, $\mathsf{Filt}_A(M_2,\ldots,M_m)\in\mathsf{f\text{-}wide}\,A$. On the other hand, $\mathsf{add}_A\,P_1$ is obviously functorially finite in $\mathsf{mod}\,A$. Thus, we get that $\mathcal{W}\in\mathsf{f\text{-}wide}\,A$ by [SikS, Theorem 2.6].

We finally prove that $\mathcal{W}\notin\mathsf{f_R\text{-}wide}\,A$, or equivalently, $\mathsf{F}_A(\mathcal{W})\notin\mathsf{f\text{-}torf}\,A$. We assume that $\mathsf{F}_A(\mathcal{W})\in\mathsf{f\text{-}torf}\,A$, and deduce a contradiction. This assumption implies that $\mathsf{F}_A(\mathcal{W})\cap\mathsf{mod}\,A/\langle e\rangle$ is functorially finite in $\mathsf{mod}\,A/\langle e\rangle$. The bricks $M_2,\ldots,M_m$ already belong to $\mathsf{mod}\,A/\langle e\rangle$, and $M_1=\mathsf{Hom}_A(A/\langle e\rangle,P_1)$ holds. Therefore, we have $\mathsf{F}_A(M_1,\ldots,M_m)=\mathsf{F}_A(\mathcal{W})\cap\mathsf{mod}\,A/\langle e\rangle\in$

$\mathsf{f}\text{-}\mathsf{torf}\, A/\langle e\rangle$; hence, a wide subcategory $\mathsf{Filt}_A(M_1,\dots,M_m)$ must belong to $\mathsf{f}\text{-}\mathsf{wide}\, A/\langle e\rangle$ by Proposition 1.28. However, this is a contradiction, because $M_1,\dots,M_m$ are all the $A/\langle e\rangle$-modules in the mouth of the stable tube $\mathcal{C}'$. Thus, we obtain that $\mathsf{F}_A(\mathcal{W}) \notin \mathsf{f}\text{-}\mathsf{torf}\, A$.

The proof is now complete. □

Finally, we obtain the following result for tilted algebras A of extended Dynkin type.

Corollary 3.16. *We use the setting of Theorem 3.14 (3). Then, the tilting H-module T satisfies one of the conditions (1)–(5) in the following table, which shows whether* $\mathsf{f}\text{-}\mathsf{wide}\, A = \mathsf{f_L}\text{-}\mathsf{wide}\, A$ *or* $\mathsf{f}\text{-}\mathsf{wide}\, A = \mathsf{f_R}\text{-}\mathsf{wide}\, A$ *holds in each case.*

No.	T_{pp}	T_{reg}	T_{pi}	$\mathsf{f}\text{-}\mathsf{wide}\, A = \mathsf{f_L}\text{-}\mathsf{wide}\, A$	$\mathsf{f}\text{-}\mathsf{wide}\, A = \mathsf{f_R}\text{-}\mathsf{wide}\, A$
(1)	$\neq 0$	$= 0$	$= 0$	Yes	Yes
(2)	$= 0$	$= 0$	$\neq 0$	Yes	Yes
(3)	$\neq 0$	$\neq 0$	$= 0$	Yes	No
(4)	$= 0$	$\neq 0$	$\neq 0$	No	Yes
(5)	$\neq 0$		$\neq 0$	Yes	Yes

Proof. Since there never exists a regular tilting H-module [SimS, XVII.3.4. Lemma], T satisfies one of the conditions (1)–(5).

In the cases (1) and (2), both of the conditions $\mathsf{f}\text{-}\mathsf{wide}\, A = \mathsf{f_L}\text{-}\mathsf{wide}\, A$ and $\mathsf{f}\text{-}\mathsf{wide}\, A = \mathsf{f_R}\text{-}\mathsf{wide}\, A$ hold by Theorem 3.14 (2).

Next, we consider the case (3). By Step 3° of the proof of [SimS, XVII.3.5. Theorem], $\mathsf{Sub}_H\, \tau T$ has only finitely many indecomposable H-modules. Thus, we can apply Theorem 3.14 (1) and obtain $\mathsf{f}\text{-}\mathsf{wide}\, A = \mathsf{f_L}\text{-}\mathsf{wide}\, A$. On the other hand, Theorem 3.14 (3) implies $\mathsf{f}\text{-}\mathsf{wide}\, A \neq \mathsf{f_R}\text{-}\mathsf{wide}\, A$.

Similarly, in the case (4), we get $\mathsf{f}\text{-}\mathsf{wide}\, A = \mathsf{f_R}\text{-}\mathsf{wide}\, A$ and $\mathsf{f}\text{-}\mathsf{wide}\, A \neq \mathsf{f_L}\text{-}\mathsf{wide}\, A$.

It remains to deal with the case (5). Then, A is representation-finite [SimS, XVII.3.3. Lemma]. Thus, we have both $\mathsf{f}\text{-}\mathsf{wide}\, A = \mathsf{f_L}\text{-}\mathsf{wide}\, A$ and $\mathsf{f}\text{-}\mathsf{wide}\, A = \mathsf{f_R}\text{-}\mathsf{wide}\, A$. □

Part 2. Bricks over preprojective algebras and join-irreducible elements of Coxeter groups

0. INTRODUCTION

The representation theory of *preprojective algebras* Π of Dynkin type Δ has been developed by investigating their relationship with the *Coxeter groups* $W = W(\Delta)$ associated to Δ. In particular, the ideal $I(w)$ of Π associated to each element $w \in W$ introduced by [IR, BIRS] plays an important role. For example, see [AM, AIRT, BKT, GLS, Kim, Miz2, ORT, Tho].

The Coxeter group W has a partial order $\le$ called *the right weak order*. The partially ordered set $(W, \le)$ is a *lattice* [BB], that is, W admits the two binary operations called the *join* $x \vee y$ and the *meet* $x \wedge y$ for any $x, y \in W$. In our study, we efficiently use *join-irreducible* elements in a lattice L. We write $\mathsf{j\text{-}irr}\, L$ for the set of join-irreducible elements in L.

Reading [Rea] introduced the important notion of canonical join representations. For a given element $x \in L$, a subset $U = \{u_1, u_2, \dots, u_m\} \subset L$ is called the *canonical join representation* if U satisfies $x = \bigvee_{i=1}^{m} u_i$ and some additional minimal conditions. In this case, $U \subset \mathsf{j\text{-}irr}\, L$ holds.

Any element in a Coxeter group of Dynkin type has a unique canonical join representation, since the Coxeter group is a *semidistributive* lattice, see [IRRT] for the detail. One of the aims of this part is to show that the canonical join representations of the elements in the Coxeter group W are strongly related to the representation theory of Π. We will explain the details later in this section.

We will show some of our results in a more general setting. Let A be a finite-dimensional algebra over a field K. We write $\mathsf{torf}\, A$ for the set of *torsion-free classes* in the category $\mathsf{mod}\, A$ of finite-dimensional A-modules. The set $\mathsf{torf}\, A$ has a natural partial order $\subset$ defined by inclusion relations, and then, the partially ordered set $(\mathsf{torf}\, A, \subset)$ is also a lattice.

In the rest, we assume that A is τ-tilting finite, that is, $\mathsf{torf}\, A$ is a finite set. There are many bijections between $\mathsf{torf}\, A$ and many important objects in $\mathsf{mod}\, A$ or in its bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ obtained in [AIR, BY, KY, MS] and Part 1. In particular, we have a bijection F from the set $\mathsf{sbrick}\, A$ of *semibricks* in $\mathsf{mod}\, A$ to the set $\mathsf{torf}\, A$, where $\mathsf{F}(S)$ is defined as the minimum torsion-free class containing a semibrick S. Here, a semibrick S is defined as a module in $\mathsf{mod}\, A$ which admits a decomposition $S = \bigoplus_{i=1}^{s} S_i$ with $\mathsf{End}_A(S_i)$ a division K-algebra (that is, S_i is a *brick*) and with $\mathsf{Hom}_A(S_i, S_j) = 0$ for $i \neq j$. The sets $\mathsf{torf}\, A$ and $\mathsf{sbrick}\, A$ have bijections from the set $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$ of *support τ^{-1}-tilting A-modules* satisfying the following commutative diagram by [AIR] and Part 1 in this thesis:

$$\begin{tikzcd}
\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A \arrow[r, "\mathsf{Sub}"] \arrow[rr, bend right, "M \mapsto \mathsf{soc}_{\mathsf{End}_A(M)} M"'] & \mathsf{torf}\, A & \mathsf{sbrick}\, A \arrow[l, "\mathsf{F}"'] .
\end{tikzcd}$$

Moreover, the bijection F is restricted to a bijection from the set $\mathsf{brick}\, A$ of bricks in $\mathsf{mod}\, A$ to the set $\mathsf{j\text{-}irr}(\mathsf{torf}\, A)$, and we have the following commutative diagram of bijections:

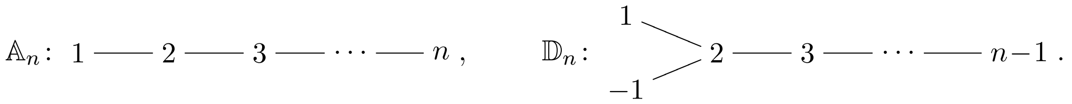
$$\begin{tikzcd}
\mathrm{i}\tau^{-1}\text{-}\mathsf{rigid}\, A \arrow[r, "\mathsf{Sub}"] \arrow[rr, bend right, "M \mapsto \mathsf{soc}_{\mathsf{End}_A(M)} M"'] & \mathsf{j\text{-}irr}(\mathsf{torf}\, A) & \mathsf{brick}\, A \arrow[l, "\mathsf{F}"'] .
\end{tikzcd}$$

Here, $\mathrm{i}\tau^{-1}\text{-}\mathsf{rigid}\, A$ denotes the set of indecomposable τ^{-1}-rigid modules in $\mathsf{mod}\, A$.

As the first step, we will show that the canonical join representation of a torsion-free class is given by the decomposition of the corresponding semibrick as a direct sum of bricks. This fact was independently obtained also in [BCZ].

Theorem 0.1 (Theorem 1.8)**.** *Let $\mathcal{F} \in \mathsf{torf}\, A$, take the unique semibrick $S \in \mathsf{sbrick}\, A$ satisfying $\mathcal{F} = \mathsf{F}(S)$, and decompose S as $\bigoplus_{i=1}^{m} S_i$ with $S_i \in \mathsf{brick}\, A$. Then, the representation $\mathcal{F} = \bigvee_{i=1}^{m} \mathsf{F}(S_i)$ is the canonical join representation.*

For the preprojective algebra $\varPi$, Mizuno [Miz1] proved that the two lattices $(W, \leq)$ and $(\mathsf{torf}\,\varPi, \subset)$ are isomorphic by the correspondence $w \mapsto \mathsf{Sub}(\varPi/I(w))$ and that $\varPi/I(w)$ is a support τ^{-1}-tilting $\varPi$-module. Therefore, we obtain a bijection $S(?)\colon W \to \mathsf{sbrick}\,\varPi$ given by $S(w) := \mathsf{soc}_{\mathsf{End}_\varPi(\varPi/I(w))}(\varPi/I(w))$. The main aim of this part is to describe the semibrick $S(w)$ for each element $w \in W$ as a quiver representation in the case $\varDelta = \mathbb{A}_n$ or $\mathbb{D}_n$:



If $\varDelta = \mathbb{A}_n$, then W is the symmetric group $\mathfrak{S}_{n+1}$, and if $\varDelta = \mathbb{D}_n$, then W is the subgroup of the automorphism group on the set $\{\pm1, \pm2, \ldots, \pm n\}$ consisting of all elements w such that $w(-i) = -w(i)$ holds for each i and that the number $\#\{i > 0 \mid w(i) < 0\}$ is even. Thus, we can express every $w \in W$ in the form $(w(1), w(2), \ldots, w(m))$, and our description of the semibrick $S(w)$ is constructed by this expression.

Mizuno's isomorphism $W \to \mathsf{torf}\,\varPi$ of lattices is restricted to a bijection $\mathsf{j\text{-}irr}\,W \to \mathsf{j\text{-}irr}(\mathsf{torf}\,\varPi)$ between the join-irreducible elements, so we also obtain a bijection $S(?)\colon \mathsf{j\text{-}irr}\,W \to \mathsf{brick}\,\varPi$. By [IRRT] (types $\mathbb{A}_n$ and $\mathbb{D}_n$) and [Dem] (type $\mathbb{E}_n$, calculated by a computer program), the cardinality of each set is

$$\begin{cases} 2^{n+1} - n - 2 & (\varDelta = \mathbb{A}_n) \\ 3^n - n \cdot 2^{n-1} - n - 1 & (\varDelta = \mathbb{D}_n) \\ 1272 & (\varDelta = \mathbb{E}_6) \\ 17635 & (\varDelta = \mathbb{E}_7) \\ 881752 & (\varDelta = \mathbb{E}_8) \end{cases}.$$

Moreover, we obtain the following property immediately from Theorem 0.1.

Corollary 0.2 (Corollary 2.3)**.** *Let $w \in W$ and take $w_1, w_2, \ldots, w_m \in \mathsf{j\text{-}irr}\,W$ such that $S(w) = \bigoplus_{i=1}^m S(w_i)$. Then, $w = \bigvee_{i=1}^m w_i$ holds, and it is the canonical join representation of w in W.*

In this part, we will give a description of the semibrick $S(w)$ by the following two steps:

(a) we find the canonical join representation $\bigvee_{i=1}^m w_i$ of w; and
(b) we explicitly describe the brick $S(w_i)$ for each $w_i \in \mathsf{j\text{-}irr}\,W$.

There is a combinatorial "Young diagram-like" description by Iyama–Reading–Reiten–Thomas [IRRT] of the module $J(w) := (\varPi/I(w))e_l$ for $w \in \mathsf{j\text{-}irr}\,W$ in the case $\varDelta$ is $\mathbb{A}_n$ or $\mathbb{D}_n$, where l is the unique *descent* of $w \in \mathsf{j\text{-}irr}\,W$ and e_l is the primitive idempotent of $\varPi$ corresponding to l. The module $J(w)$ is an indecomposable direct summand of $\varPi/I(w) \in \mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A$ satisfying $\mathsf{Sub}\,J(w) = \mathsf{Sub}(\varPi/I(w))$, so $S(w) = \mathsf{soc}_{\mathsf{End}_\varPi(J(w))}\,J(w)$ follows.

For example, let $w := (2,5,8,1,3,4,6,7,9) \in W(\mathbb{A}_8)$ and $w' := (6,9,-7,-4,1,2,3,5,8) \in W(\mathbb{D}_9)$. Then, the modules $J(w)$, $J(w')$, $S(w)$, and $S(w')$ are expressed by the following figures:

$$J(w) = \begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline 4 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 5 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{2}{c}{} \\ \cline{1-1} 7 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}, \qquad S(w) = \begin{array}{ccc} \cline{2-3} & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{1} \\ \cline{2-3} & \multicolumn{1}{|c|}{3} & \\ \cline{1-2} \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{4} & \\ \cline{1-2} \multicolumn{1}{|c|}{6} & & \\ \cline{1-1} \multicolumn{1}{|c|}{7} & & \\ \cline{1-1} \end{array},$$

$$J(w') = \begin{array}{|c|c|c|c|c|c|c|}
\hline
2 & {}^{-1}_{1} & -2 & -3 & -4 & -5 & -6 \\ \hline
3 & 2 & {}^{1}_{-1} & -2 & -3 \\ \cline{1-5}
4 & 3 & 2 & {}_{1} \\ \cline{1-4}
5 & 4 & 3 & 2 \\ \cline{1-4}
6 & 5 & 4 & 3 \\ \cline{1-4}
7 & 6 & 5 \\ \cline{1-3}
8 \\ \cline{1-1}
\end{array}, \qquad
S(w') = \begin{array}{ccccccc}
 & & & & (4) & (5) & -6 \\
 & & {}_{-1} & (2) & (3) & & \\
 & & (2) & {}_{1} & & & \\
 & (4) & (3) & 2 & & & \\
 & (5) & 4 & 3 & & & \\
7 & 6 & 5 & & & & \\
8 & & & & & &
\end{array}$$

Here, for each module M above, each square $\boxed{i}$ in the figure for M denotes a one-dimensional subspace of e_iM if $i \geq -1$; and of $e_{|i|}M$ if $i \leq -2$. As a K-vector space, M is the direct sum of these one-dimensional subspaces. In the figure for $S(w')$, for each $i = 2, 3, 4, 5$, the two squares $\boxed{(i)}$ together denote a certain one-dimensional subspace of the two-dimensional vector space corresponding to the two squares $\boxed{i}$ and $\boxed{-i}$ in the figure for $J(w')$.

We can check that $w \in W(\mathbb{A}_n)$ consists of two strictly increasing sequences, and that the right-most entry of each row in the figure for $J(w)$ appears in the latter increasing sequence. Similarly, $w' \in W(\mathbb{D}_n)$ also consists of two strictly increasing sequences. If i is the right-most entry of some row in the figure for $J(w')$, then i appears in the latter increasing sequence if $i \geq -1$, and $i - 1$ appears there if $i \leq -2$.

The bricks $S(w)$ and $S(w')$ can be expressed more simply by using quiver representations:

$$S(w) = 1 \leftarrow 2 \rightarrow 3 \rightarrow 4 \leftarrow 5 \rightarrow 6 \rightarrow 7,$$

$$S(w') = \begin{tikzcd}
-1 \arrow[r] & -2 \arrow[r] \arrow[dl] & -3 \arrow[dl] \arrow[dr] & -4 \arrow[l] \arrow[r] & -5 \arrow[r] \arrow[dl] \arrow[dr] & -6 & & \\
1 \arrow[r] & 2 \arrow[r] & 3 & 4 \arrow[l] \arrow[r] & 5 & 6 \arrow[l] & 7 \arrow[l] \arrow[r] & 8
\end{tikzcd}.$$

In this part, we give a combinatorial algorithm to obtain a quiver representation of the brick $S(w)$ for each $w \in \mathsf{j\text{-}irr}\, W$ in the case $\mathit{\Delta} = \mathbb{A}_n$ or $\mathbb{D}_n$; then, the step (b) is done.

First, if $\mathit{\Delta} = \mathbb{A}_n$, then we have obtained the following result.

Theorem 0.3 (Theorem 3.1, Corollary 3.3)**.** *Let $w \in \mathsf{j\text{-}irr}\, W(\mathbb{A}_n)$ with its unique descent l. Then, the brick $S(w)$ is given as follows.*

- *Set $R := w([l+1, n+1])$, $a := w(l)$, $b := w(l+1)$, and $V := [b, a-1]$.*
- *The brick $S(w)$ has a K-basis $(\langle i \rangle)_{i \in V}$, where $\langle i \rangle$ belongs to $e_iS(w)$.*
- *For each $i \in V$, place a symbol i denoting the K-vector subspace $K\langle i \rangle$.*
- *For each $i \in V \setminus \{\max V\}$, we write exactly one arrow between i and $i+1$, where the orientation is $i \rightarrow i+1$ if $i+1 \in R$ and $i \leftarrow i+1$ if $i+1 \notin R$.*

Second, if $\mathit{\Delta} = \mathbb{D}_n$, then the bricks are obtained from the following procedure.

Theorem 0.4 (Theorem 3.7, Corollary 3.10)**.** *Let $w \in \mathsf{j\text{-}irr}\, W(\mathbb{D}_n)$ with its unique descent l. Then, the brick $S(w)$ is given as follows.*

- *Set $R := w([|l|+1, n])$, $a := w(l)$, $b := w(|l|+1)$, and*

$$r := \max\{k \geq 0 \mid [1,k] \subset \pm R\}, \quad c := \begin{cases} w^{-1}(|w(1)|) & (r \geq 1) \\ 1 & (r = 0) \end{cases},$$

$$(V_-, V_+) := \begin{cases} (\emptyset, [b, a-1]) & (b \geq 2) \\ (\emptyset, \{c\} \cup [2, a-1]) & (b = \pm 1) \\ ([b+1, -2] \cup \{-c\}, \{c\} \cup [2, a-1]) & (b \leq -2) \end{cases}, \quad V := V_+ \amalg V_-.$$

- *The brick* $S(w)$ *has a* K*-basis* $(\langle i\rangle)_{i\in V}$*, where* $\langle i\rangle$ *belongs to* $e_iS(w)$ *if* $i \geq -1$*, and* $e_{|i|}S(w)$ *if* $i \leq -2$*.*
- *For each* $i \in V$*, place a symbol* i *denoting the* K*-vector subspace* $K\langle i\rangle$*.*
- *We write the following arrows.*
  - (i) *For each* $i \in V_+ \setminus \{\mathsf{max}\, V_+\}$*, draw an arrow* $i \to |i|+1$ *if* $|i|+1 \in R$*; and* $i \leftarrow |i|+1$ *otherwise.*
  - (ii) *For each* $i \in V_- \setminus \{\mathsf{min}\, V_-\}$*, draw an arrow* $i \leftarrow -(|i|+1)$ *if* $-(|i|+1) \in R$*; and* $i \to -(|i|+1)$ *otherwise.*
  - (iii) *If* $r \geq 1$*, for each* $i \in V_-$ *with* $|i| \leq r$*, draw an arrow* $-i \leftarrow -(|i|+1)$ *if* $|i|+1 \in R$*; and* $i \to |i|+1$ *otherwise.*
  - (iv) *If* $r = 0$*, draw an arrow* $-c \leftarrow 2$ *if* $c \leftarrow 2$ *exists in* (i)*, and draw an arrow* $c \to -2$ *if* $-c \to -2$ *exists in* (ii)*.*

Consequently, we obtain that the bricks over the preprojective algebra of type $\mathbb{A}_n$ is a module over some path algebra of type $\mathbb{A}_n$. On the other hand, the preprojective algebra of type $\mathbb{D}_n$ does not have the corresponding property.

Finally, we consider an arbitrary element $w \in W$. In Propositions 4.4 and 4.8, we will explicitly determine the canonical join representation $\bigvee_{i=1}^{m} w_i$ of $w \in W$ by using the characterization of canonical join representations in the Coxeter groups of Dynkin type given by Reading [Rea]. Then, in Theorems 4.6 and 4.10, we explicitly write down the semibrick $S(w) = \bigoplus_{i=1}^{m} S(w_i)$ by using the description of bricks. This is what we desire in this part.

For example, let $\mathit{\Delta} := \mathbb{A}_8$ and $w := (4, 9, 3, 6, 2, 8, 5, 1, 7)$. Then, its canonical join representation is $w_2 \vee w_4 \vee w_6 \vee w_7$, where

$$\begin{aligned} w_2 &:= (1, 2, 4, 9, 3, 5, 6, 7, 8), & w_4 &:= (1, 3, 4, 6, 2, 5, 7, 8, 9), \\ w_6 &:= (1, 2, 3, 4, 6, 8, 5, 7, 9), & w_7 &:= (2, 3, 4, 5, 1, 6, 7, 8, 9). \end{aligned}$$

Thus, the semibrick $S(w)$ is the direct sum of the following bricks:

$$\begin{array}{lccccccccccccccl} S(w_2) = & & & & & 3 & \leftarrow & 4 & \rightarrow & 5 & \rightarrow & 6 & \rightarrow & 7 & \rightarrow & 8, \\ S(w_4) = & & & 2 & \leftarrow & 3 & \leftarrow & 4 & \rightarrow & 5 & & & & & & , \\ S(w_6) = & & & & & & & & & 5 & \leftarrow & 6 & \rightarrow & 7 & & , \\ S(w_7) = & 1 & \leftarrow & 2 & \leftarrow & 3 & \leftarrow & 4 & & & & & & & & . \end{array}$$

0.1. **Notation.** The composition of two maps $f\colon X \to Y$ and $g\colon Y \to Z$ is denoted by gf.

We define the multiplication on the automorphism group on a finite set X by $(\sigma\tau)(i) := \sigma(\tau(i))$ for $i \in X$. For $a, b \in X$, the notation $(a \quad b)$ means the transposition which exchanges a and b.

For integers $a, b \in \mathbb{Z}$, we define $[a, b] := \{i \in \mathbb{Z} \mid a \leq i \leq b\}$. For a set $X \subset \mathbb{Z}$, we set $-X := \{-i \mid i \in \mathbb{Z}\}$ and $\pm X := X \cup (-X)$.

Throughout this part, K is a field and A is a finite-dimensional K-algebra. Unless otherwise stated, A-modules are finite-dimensional left A-modules, and we write $\mathsf{mod}\, A$ for the category of finite-dimensional left A-modules. Let $M \in \mathsf{mod}\, A$, and decompose M as $M \cong \bigoplus_{i=1}^{m} M_i^{\oplus l_i}$ with $M_i \not\cong M_j$ for $i \neq j$ and with $l_i \geq 1$ for each i. Then, we define the number $|M| := m$, and we say that M is *basic* if $l_i = 1$ for any i. We set the multiplication on the endomorphism algebra $\mathsf{End}_A(M)$ as $g \cdot f := gf$. Thus, M is also a left $\mathsf{End}_A(M)$-module by $fx := f(x)$ for $f \in \mathsf{End}_A(M)$ and $x \in M$.

For a quiver Q, the composition of the two arrows $\alpha\colon i \to j$ and $\beta\colon j \to k$ in Q is denoted by $\alpha\beta$, which is a path from i to k.

1. General observations of τ-tilting finite algebras

In this section, we observe some general properties holding for τ-tilting finite algebras A over a field K.

1.1. **Lattices.** First, we recall the notion of lattices.

Definition 1.1. Let $(L, \leq)$ be a partially ordered set.

(1) For $x, y, z \in L$, the element z is called the *meet* of x and y if z is the maximum element satisfying $z \leq x$ and $z \leq y$. In this case, z is denoted by $x \wedge y$.
(2) For $x, y, z \in L$, the element z is called the *join* of x and y if z is the minimum element satisfying $z \geq x$ and $z \geq y$. In this case, z is denoted by $x \vee y$.
(3) The set L is called a *lattice* if L admits the meet $x \wedge y$ and the join $x \vee y$ for any $x, y \in L$.
(4) The set L is called a *finite lattice* if L is a finite set and a lattice.

The operations join and meet clearly satisfy the associative relations, so we may use the expressions $x \wedge y \wedge z$ and $x \vee y \vee z$. If $L \neq \emptyset$ is a finite lattice, there exist the maximum element $\mathsf{max}\, L$ and the minimum element $\mathsf{min}\, L$. In this case, we define $\bigwedge_{x \in \emptyset} x := \mathsf{max}\, L$ and $\bigvee_{x \in \emptyset} x := \mathsf{min}\, L$.

Later in this part, we will consider the decomposition of an element in a lattice with respect to the operation join, so we recall the notion of join-irreducible elements.

Definition 1.2. Let L be a lattice. An element $x \in L$ is called a *join-irreducible* element if the following conditions hold:

- x is not the minimum element of L; and
- for any $y, z \in L$, if $x = y \vee z$, then $y = x$ or $z = x$.

We write $\mathsf{j\text{-}irr}\, L$ for the set of join-irreducible elements in W.

We remark that $x \in \mathsf{j\text{-}irr}\, L$ is equivalent to that there exists a unique maximal element of the set $\{y \in W \mid y < x\}$ if L is a finite lattice. This fails if we drop the assumption that L is finite [BCZ, Remark 3.1.2].

1.2. **Torsion-free classes.** Let A be a finite-dimensional algebra.

A full subcategory $\mathcal{F}$ of $\mathsf{mod}\, A$ is called a *torsion-free class* in $\mathsf{mod}\, A$ if $\mathcal{F}$ is closed under submodules and extensions, and we write $\mathsf{torf}\, A$ for the set of torsion-free classes in $\mathsf{mod}\, A$. For a full subcategory $\mathcal{C} \subset \mathsf{mod}\, A$, we define

$$\begin{aligned} \mathsf{add}\, \mathcal{C} &:= \{M \in \mathsf{mod}\, A \mid M \text{ is a direct summand of } \textstyle\bigoplus_{i=1}^{s} C_i \text{ for some } C_1, C_2, \ldots, C_s \in \mathcal{C}\}, \\ \mathsf{Filt}\, \mathcal{C} &:= \{M \in \mathsf{mod}\, A \mid \text{there exists } 0 = M_0 \subset M_1 \subset \cdots \subset M_l = M \text{ with } M_i/M_{i-1} \in \mathsf{add}\, \mathcal{C}\}, \\ \mathsf{Sub}\, \mathcal{C} &:= \{M \in \mathsf{mod}\, A \mid M \text{ is a submodule of some object in } \mathsf{add}\, \mathcal{C}\}, \\ \mathsf{F}(\mathcal{C}) &:= \mathsf{Filt}(\mathsf{Sub}\, \mathcal{C}). \end{aligned}$$

Then, $\mathsf{F}(\mathcal{C})$ is the smallest torsion-free class containing $\mathcal{C}$, see [MS, Lemma 3.1].

The set $\mathsf{torf}\, A$ has a natural partial order defined by inclusions, and then, the partially ordered set $(\mathsf{torf}\, A, \subset)$ is a finite lattice with $\mathcal{F}_1 \wedge \mathcal{F}_2 = \mathcal{F}_1 \cap \mathcal{F}_2$ and $\mathcal{F}_1 \vee \mathcal{F}_2 = \mathsf{F}(\mathcal{F}_1 \cup \mathcal{F}_2)$. The notion of *torsion classes* is dually defined.

A torsion-free class in $\mathsf{mod}\, A$ is not necessarily functorially finite in $\mathsf{mod}\, A$. Demonet–Iyama–Jasso [DIJ] introduced the notion of *τ-tilting finiteness*, which is equivalent to that $\mathsf{torf}\, A$ is a finite set. In their paper, they proved that A is τ-tilting finite if and only if every torsion-free class is functorially finite. In the rest, A is assumed to be τ-tilting finite.

Functorially finite torsion-free classes are strongly connected with support τ^{-1}-tilting A-modules, which were introduced by Adachi–Iyama–Reiten [AIR]. They proved that the set $\mathsf{torf}\, A$ has a bijection from the set $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$ of support τ^{-1}-tilting A-modules.

Let $M \in \mathsf{mod}\, A$ and I be an injective A-module in $\mathsf{mod}\, A$. Then, M is called a *τ^{-1}-rigid module* if $\mathsf{Hom}_A(\tau^{-1}M, M) = 0$, and the pair (M, I) is called a *τ^{-1}-rigid pair* if M is τ^{-1}-rigid and $\mathsf{Hom}_A(M, I) = 0$. If a τ^{-1}-rigid pair (M, I) satisfies $|M| + |I| = |A|$, the pair (M, I) is called a *support τ^{-1}-tilting pair*, and an A-module M is called a *support τ^{-1}-tilting module* if there exists some injective module I such that (M, I) is a support τ-tilting pair. We write $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\, A$

for the set of basic support τ^{-1}-tilting modules in $\mathsf{mod}\,A$. The notion of *support τ-tilting modules* is dually defined.

If M is τ^{-1}-rigid, then the full subcategory $\mathsf{Sub}\,M$ is a torsion-free class [AS2, Theorem 5.10]. Adachi–Iyama–Reiten proved that this correspondence $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A \ni M \mapsto \mathsf{Sub}\,M \in \mathsf{torf}\,A$ is a bijection.

Proposition 1.3. [AIR, Theorem 2.7] *The correspondence* $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A \ni M \mapsto \mathsf{Sub}\,M \in \mathsf{torf}\,A$ *is a bijection.*

Thus, we induce a partial order $\leq$ on the set $\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A$ from inclusion relations on $\mathsf{torf}\,A$; namely, $M \leq N$ holds if and only if $\mathsf{Sub}\,M \subset \mathsf{Sub}\,N$. Then, $(\mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A, \leq)$ is clearly a lattice.

1.3. **Semibricks.** We assume that A is τ-tilting finite as in the previous subsection. We define the notions of bricks and semibricks as follows.

Definition 1.4. Let S be an A-module.

(1) The module S is called a *brick* if the endomorphism ring $\mathsf{End}_A(S)$ is a division ring. We write $\mathsf{brick}\,A$ for the set of bricks.

(2) The module S is called a *semibrick* if S is decomposed as the direct sum $\bigoplus_{i=1}^{m} S_i$ of bricks $S_1, S_2, \ldots, S_m \in \mathsf{brick}\,A$ satisfying $\mathsf{Hom}_A(S_i, S_j) = 0$ if $i \neq j$. We write $\mathsf{sbrick}\,A$ for the set of semibricks in $\mathsf{mod}\,A$.

The notion of semibricks is originally defined as sets of Hom-orthogonal bricks in Part 1, but it does not matter here, since A is assumed to be τ-tilting finite by Corollary 1.10 in Part 1. Then, Proposition 1.9 in Part 1 tells us that there is a bijection $\mathsf{F}\colon \mathsf{sbrick}\,A \to \mathsf{torf}\,A$ taking the minimum torsion-free class $\mathsf{F}(S)$ containing each semibrick S. Moreover, it satisfies the property below.

Proposition 1.5 (Proposition 1.9 in Part 1)**.** *We have the following commutative diagram of bijections:*

$$\begin{tikzcd} \mathsf{s}\tau^{-1}\text{-}\mathsf{tilt}\,A \arrow[r, "\mathsf{Sub}"] \arrow[rr, bend right, "M\mapsto \mathsf{soc}_{\mathsf{End}_A(M)} M"'] & \mathsf{torf}\,A & \mathsf{sbrick}\,A \arrow[l, "\mathsf{F}"'] \end{tikzcd} .$$

Now, we set $\mathrm{i}\tau^{-1}\text{-}\mathsf{rigid}\,A$ as the set of indecomposable τ^{-1}-rigid A-modules in $\mathsf{mod}\,A$. Then, we also have another commutative diagram.

Proposition 1.6. *We have the following commutative diagram of bijections:*

$$\begin{tikzcd} \mathrm{i}\tau^{-1}\text{-}\mathsf{rigid}\,A \arrow[r, "\mathsf{Sub}"] \arrow[rr, bend right, "M\mapsto \mathsf{soc}_{\mathsf{End}_A(M)} M"'] & \mathsf{j}\text{-}\mathsf{irr}(\mathsf{torf}\,A) & \mathsf{brick}\,A \arrow[l, "\mathsf{F}"'] \end{tikzcd} .$$

Proof. The map $\mathsf{Sub}\colon \mathrm{i}\tau^{-1}\text{-}\mathsf{rigid}\,A \to \mathsf{j}\text{-}\mathsf{irr}(\mathsf{torf}\,A)$ is bijective by [IRRT, Theorem 2.7]. On the other hand, it follows from [DIJ, Theorem 4.2, Lemma 4.3] that the map $\mathrm{i}\tau^{-1}\text{-}\mathsf{rigid}\,A \ni M \mapsto \mathsf{soc}_{\mathsf{End}_A(M)} M \in \mathsf{brick}\,A$ is a bijection satisfying $\mathsf{F}(\mathsf{soc}_{\mathsf{End}_A(M)} M) = \mathsf{Sub}\,M$. Thus, we have the desired commutative diagram of bijections. $\square$

1.4. **Canonical join representations.** Now that the bijection $\mathsf{F}\colon \mathsf{sbrick}\,A \to \mathsf{torf}\,A$ is restricted to a bijection $\mathsf{brick}\,A \to \mathsf{j}\text{-}\mathsf{irr}(\mathsf{torf}\,A)$, the following natural question occurs:

> Let $\mathcal{F} \in \mathsf{torf}\,A$, take the unique semibrick $S \in \mathsf{sbrick}\,A$ satisfying $\mathcal{F} = \mathsf{F}(S)$, and decompose S as $\bigoplus_{i=1}^{m} S_i$ with $S_i \in \mathsf{brick}\,A$. Then, what is the relationship between $\mathsf{F}(S) \in \mathsf{torf}\,A$ and $\mathsf{F}(S_1), \mathsf{F}(S_2), \ldots, \mathsf{F}(S_m) \in \mathsf{j}\text{-}\mathsf{irr}(\mathsf{torf}\,A)$?

Clearly, $\mathsf{F}(S) = \bigvee_{i=1}^{m} \mathsf{F}(S_i)$ holds, since $\mathsf{F}(S)$ is the minimum torsion-free class containing all $\mathsf{F}(S_i)$. Actually, this will turn out to be a canonical join representation. Here, the notion of canonical join representations was introduced by Reading [Rea], and defined as follows.

Definition 1.7. Let L be a finite lattice, $x \in L$, and $U \subset L$. Then, we say that U is a *canonical join representation* if

(a) $x = \bigvee_{u \in U} u$ holds; and
(b) for any proper subset $U' \subsetneq U$, the join $\bigvee_{u \in U'} u$ never coincides with x; and
(c) if $V \subset L$ satisfies the properties (a) and (b), then, for every $u \in U$, there exists $v \in V$ such that $u \leq v$.

In this case, we also say $x = \bigvee_{u \in U} u$ is a canonical join representation.

If $x \in L$ has a canonical join representation U, then we can easily check that it is the unique canonical join representation for each $x \in L$, and that U is a subset of $\mathsf{j\text{-}irr}\, L$. The existence of a canonical join representation of each element is not guaranteed for a general finite lattice. In the case that $L = \mathsf{torf}\, A$, every $\mathcal{F} \in \mathsf{torf}\, A$ has a canonical join representation given by the indecomposable decomposition of semibricks.

Theorem 1.8. *Let $\mathcal{F} \in \mathsf{torf}\, A$, take the unique semibrick $S \in \mathsf{sbrick}\, A$ satisfying $\mathcal{F} = \mathsf{F}(S)$, and decompose S as $\bigoplus_{i=1}^{m} S_i$ with $S_i \in \mathsf{brick}\, A$. Then, the representation $\mathcal{F} = \bigvee_{i=1}^{m} \mathsf{F}(S_i)$ is the canonical join representation.*

Proof. We have seen the property (a): $\mathcal{F} = \mathsf{F}(S) = \bigvee_{i=1}^{m} \mathsf{F}(S_i)$.

We show the property (b). Let I be a proper subset of $[1, m]$. Take $j \in [1, m] \setminus I$. Then, the brick S_j cannot belong to $\mathsf{F}(\{S_i\}_{i \in I}) = \mathsf{F}(\bigcup_{i \in I} \mathsf{F}(S_i)) = \bigvee_{i \in I} \mathsf{F}(S_i)$, since $\mathsf{Hom}_A(S_j, S_i) = 0$ holds for each $i \in I$. This implies that $\mathsf{F}(S) \neq \bigvee_{i \in I} \mathsf{F}(S_i)$.

Next, we show the property (c). Let $\mathcal{F}_1, \ldots, \mathcal{F}_{m'} \in \mathsf{torf}\, A$ satisfy $\mathcal{F} = \bigvee_{j=1}^{m'} \mathcal{F}_j$ and the property (a). For each $i \in [1, m]$, the brick S_i belongs to $\mathsf{F}(S) = \mathcal{F}$, which coincides with $\bigvee_{j=1}^{m'} \mathcal{F}_j = \mathsf{F}(\bigcup_{j=1}^{m'} \mathcal{F}_j)$. Thus, there must exist some $j \in [1, m']$ such that $\mathsf{Hom}_A(S_i, \mathcal{F}_j) \neq 0$. We take a semibrick S' such that $\mathcal{F}_j = \mathsf{F}(S')$, then there exists a nonzero homomorphism $f \colon S_i \to S'$. By Lemma 1.7 in Part 1, f is injective, since $S_i, S' \in \mathcal{F} = \mathsf{F}(S)$ and S_i is a direct summand of S. This implies that $\mathsf{F}(S_i) \subset \mathsf{F}(S') = \mathcal{F}_j$. ☐

In particular, the partially ordered set $\mathsf{torf}\, A$ admits a canonical join representation for any $\mathcal{F} \in \mathsf{torf}\, A$.

The notion of canonical join representations is defined in a fully combinatorial way, but decomposing semibricks into direct sums of bricks is a purely representation-theoritic problem. These two are related by Theorem 1.8.

The relationship between semibricks and torsion classes are independently discussed by Barnard–Carroll–Zhu [BCZ] and Demonet–Iyama–Reiten–Reading–Thomas [DIRRT] in the setting that the algebra A is not necessarily τ-tilting finite. In particular, our Theorem 1.8 is generalized in [BCZ, Proposition 3.2.5].

2. PRELIMINARIES FOR PREPROJECTIVE ALGEBRAS

In this section, we recall some properties on Coxeter groups and preprojective algebras of Dynkin type.

2.1. Coxeter groups.

Coxeter groups of Dynkin type are strongly related to the corresponding preprojective algebras. In this subsection, we state the definition of Coxeter groups of Dynkin type, and prepare some basic terms on the combinatorics of Coxeter groups. For more information, see [BB].

Let $\mathit{\Delta}$ be a Dynkin diagram whose vertices set is $\mathit{\Delta}_0$. Then, the *Coxeter group W* for $\mathit{\Delta}$ is the group defined by the generators $\{s_i \mid i \in \mathit{\Delta}_0\}$ and the relations

- $s_i^2 = 1$ for each i;

- $s_is_j = s_js_i$ if there is no edge between i and j in $\varDelta$; and
- $s_is_js_i = s_js_is_j$ if there is exactly one edge between i and j in $\varDelta$.

It is well-known that the Coxeter group W associated to a Dynkin diagram $\varDelta$ is a finite group.

Each element $w \in W$ has the minimum number l such that w can be written as a product $s_{i_1}s_{i_2}\cdots s_{i_l}$ of l generators. Such number is called the *length* of w, and is denoted by $l(w)$. If $l = l(w)$ and $w = s_{i_1}s_{i_2}\cdots s_{i_l}$, then $s_{i_1}s_{i_2}\cdots s_{i_l}$ is called a *reduced expression* of w, which is not necessarily unique.

If an element $w \in W$ has the maximum length among the elements of W, then w is called a *longest element* of W. Actually, such an element uniquely exists, and it is often denoted by w_0.

We can consider several partial orders on the Coxeter group W, but in this part, we only use the *right weak order*: for $w, w' \in W$, the inequality $w \leq w'$ holds if and only if $l(w') = l(w) + l(w^{-1}w')$. Then, the poset $(W, \leq)$ is a lattice.

We write $\mathsf{j\text{-}irr}\, W$ for the set of join-irreducible elements of the partially ordered set $(W, \leq)$. For $w \in W$, the maximal elements of the set $\{w' \in W \mid w' < w\}$ are ws_i for all $i \in \varDelta_0$ satisfying $l(w) > l(ws_i)$. Therefore, $w \in W$ is join-irreducible if and only if there uniquely exists $i \in \varDelta_0$ such that $l(w) > l(ws_i)$. In this case, we say that w is a join-irreducible element of *type* i.

When we consider the right weak order of the Coxeter group, the notion of inversions is useful. We call an element $t \in W$ a *reflection* of W if there exist some $w \in W$ and $i \in \varDelta_0$ satisfying $t = ws_iw^{-1}$. Fix $w \in W$, then a reflection t of W is called an *inversion* if $l(tw) < l(w)$, and the set of inversions of w is denoted by $\mathsf{inv}(w)$. It is well-known that, for two elements $w, w' \in W$, the inequality $w \leq w'$ holds if and only if $\mathsf{inv}(w) \subset \mathsf{inv}(w')$.

2.2. **Bijections.** Now that the preparation on Coxeter groups of Dynkin type is done, let us see how they are related to the corresponding preprojective algebras.

We quickly recall the definition of preprojective algebras of Dynkin type. Let $\varDelta$ be a Dynkin diagram. We define the *double quiver* Q for $\varDelta$, that is, the set Q_0 of vertices of Q is $\varDelta_0$, and the set Q_1 of arrows of Q consists of $i \to j$ and $j \to i$ for each edge between i and j of $\varDelta$. For each arrow $\alpha\colon i \to j$ in Q_1, we write α^* for the reversed arrow $j \to i$. There is a subset $Q'_1 \subset Q_1$ such that, for each $\alpha \in Q_1$, the condition $\alpha \in Q'_1$ holds if and only if $\alpha^* \notin Q'_1$. Then, the *preprojective algebra* $\varPi$ corresponding to $\varDelta$ is given by $KQ/\langle \sum_{\alpha \in Q'_1}(\alpha\alpha^* - \alpha^*\alpha)\rangle$. Here, the choice of the subset Q'_1 is not unique in general, but $\varPi$ is uniquely defined up to isomorphisms, since $\varDelta$ is Dynkin. It is well-known that $\varPi$ is self-injective. For each vertex $i \in Q_0$, we write e_i for the idempotent of $\varPi$ corresponding to the vertex i.

Let $\varPi$ be the preprojective algebra of Dynkin type $\varDelta$, and set $I_i := \varPi(1 - e_i)\varPi$, which is a maximal ideal of $\varPi$. We write $\langle I_i \mid i \in \varDelta_0\rangle$ for the set of ideals of the form $I_{i_1}I_{i_2}\cdots I_{i_k}$.

There is an important ideal $I(w)$ of $\varPi$ associated to each element w of the Coxeter group W for $\varDelta$. The ideal $I(w)$ is defined as follows: take a reduced expression of $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ and set $I(w) := I_{i_1}I_{i_2}\cdots I_{i_k}$. Clearly, $I(w)$ belongs to the set $\langle I_i \mid i \in \varDelta_0\rangle$.

By [Miz1, Theorem 2.14], $I(w)$ does not depend on the choice of a reduced expression of w, and the well-defined correspondence $w \mapsto I(w)$ gives a bijection $W \to \langle I_i \mid i \in \varDelta_0\rangle$. We remark that a similar bijection exists for a preprojective algebra of non-Dynkin type, see [BIRS, Theorem III.1.9].

Moreover, Mizuno proved the set $\langle I_i \mid i \in \varDelta_0\rangle$ coincides with the set $\mathsf{s\tau\text{-}tilt}\,\varPi$ of support τ-tilting $\varPi$-modules. He also proved that the bijection $W \ni w \mapsto I(w) \in \mathsf{s\tau\text{-}tilt}\,\varPi$ is an isomorphism $(W, \leq) \to (\mathsf{s\tau\text{-}tilt}\,\varPi, \geq)$ of lattices [Miz1, Theorem 2.30].

In our convention, we need the dual version of this isomorphism. The torsion-free class corresponding to the torsion class $\mathsf{Fac}\, I(w)$ is $\mathsf{Sub}(\varPi/I(w))$, and it follows from Mizuno's isomorphism and [ORT, Proposition 6.4] that the module $\varPi/I(w)$ is a support τ^{-1}-tilting module. Thus, we obtain the following isomorphism of lattices.

Proposition 2.1. *There exists an isomorphism* $(W, \leq) \to (\mathsf{s\tau^{-1}\text{-}tilt}\,\varPi, \leq)$ *of lattices given by* $w \mapsto \varPi/I(w)$.

In this map, the longest element $w_0 \in W$ corresponds to the injective cogenerator Π, and the identity element id_W corresponds to 0.

Since the Coxeter group W for the Dynkin diagram Δ is a finite group, Π is τ-tilting finite. Therefore, we obtain the following bijections from Propositions 1.5, 1.6, and 2.1.

Proposition 2.2. *There exists a bijection $S(?)\colon W \to \mathsf{sbrick}\,\Pi$ defined by the formula $S(w) := \mathsf{soc}_{\mathsf{End}(\Pi/I(w))}(\Pi/I(w))$. As a restriction, we have another bijection $S(?)\colon \mathsf{j\text{-}irr}\,W \to \mathsf{brick}\,\Pi$.*

The aim of this part is to describe the semibrick $S(w)$ for each $w \in W$ explicitly.

Since the partially ordered sets $(W, \leq)$ and $(\mathsf{torf}\,A, \subset)$ are isomorphic, we obtain the following property immediately from Theorem 1.8.

Corollary 2.3. *Let $w \in W$ and take $w_1, w_2, \ldots, w_m \in \mathsf{j\text{-}irr}\,W$ such that $S(w) = \bigoplus_{i=1}^m S(w_i)$. Then, $w = \bigvee_{i=1}^m w_i$ holds, and it is the canonical join representation of w in W.*

We will explicitly determine the canonical join representation for each $w \in W$ in Section 4. It is a purely combinatorial problem.

Then, the remaining task is to describe the brick $S(w)$ for each join-irreducible element $w \in \mathsf{j\text{-}irr}\,W$. For this purpose, we use the following bijection by Iyama–Reading–Reiten–Thomas [IRRT].

Proposition 2.4. [IRRT, Theorem 4.1] *For each $w \in \mathsf{j\text{-}irr}\,W$ of type l, we set a module $J(w) := (\Pi/I(w))e_l$, which is a direct summand of $\Pi/I(w)$. Then, $\mathsf{Sub}\,J(w) = \mathsf{Sub}(\Pi/I(w))$ holds, and this induces a bijection $J(?)\colon \mathsf{j\text{-}irr}\,W \to \mathsf{i}\tau^{-1}\text{-}\mathsf{rigid}\,\Pi$.*

Thus, by Proposition 1.6, we obtain the following formula.

Proposition 2.5. *Let $w \in \mathsf{j\text{-}irr}\,W$ be of type l, and set $J(w) := (\Pi/I(w))e_l$. Then, the brick $S(w)$ is equal to $\mathsf{soc}_{\mathsf{End}_\Pi(J(w))}\,J(w)$.*

Moreover, they have already given a combinatorial "Young diagram–like" description of $J(w)$ for $\Delta = \mathbb{A}_n, \mathbb{D}_n$. This will be cited in the following subsections. By using this and Proposition 1.6, we will write down the explicit structure of the brick $S(w)$ for each $w \in \mathsf{j\text{-}irr}\,W$ in Section 3.

Now, we have recalled some properties holding for any preprojective algebra of Dynkin type. In the next two subsections, we will observe the preprojective algebras of type $\mathbb{A}_n$ and $\mathbb{D}_n$ in detail.

2.3. **Type $\mathbb{A}_n$.** Let $\Delta := \mathbb{A}_n$ in this subsection. The preprojective algebra Π of type $\mathbb{A}_n$ is given by the following quiver and relations:

$$\begin{tikzcd}
1 \arrow[r, shift left, "\alpha_1"] & 2 \arrow[l, shift left, "\beta_2"] \arrow[r, shift left, "\alpha_2"] & 3 \arrow[l, shift left, "\beta_3"] \arrow[r, shift left, "\alpha_3"] & \cdots \arrow[l, shift left, "\beta_4"] \arrow[r, shift left, "\alpha_{n-1}"] & n \arrow[l, shift left, "\beta_n"] ;
\end{tikzcd}$$

$$\alpha_1\beta_2 = 0, \quad \alpha_i\beta_{i+1} = \beta_i\alpha_{i-1}\ (2 \leq i \leq n-1), \quad \beta_n\alpha_{n-1} = 0.$$

The Coxeter group W of type $\mathbb{A}_n$ is isomorphic to the symmetric group $\mathfrak{S}_{n+1}$ by sending each s_i to the transposition $(i \quad i+1)$. We identify the Coxeter group with $\mathfrak{S}_{n+1}$ by this isomorphism, and we express $w \in W$ as $(w(1), w(2), \ldots, w(n+1))$.

The reflections of W are precisely the transpositions $(a \quad b)$ with $a, b \in [1, n+1]$ and $a > b$, and the set $\mathsf{inv}(w)$ of inversions of $w \in W$ is

$$\{(a \quad b) \mid a, b \in [1, n+1],\ a > b,\ w^{-1}(a) < w^{-1}(b)\}.$$

An element $w \in W$ is a join-irreducible element of type l if and only if l is the unique element in $[1, n]$ satisfying $w(l) > w(l+1)$. In this case, we have $w(l) \geq 2$.

We set a basis of each indecomposable projective module Πe_l as follows. Let $i, j, l \in Q_0 = [1, n]$ with $i \leq j \geq l$. We define a path $p(i, j, l)$ in Q as

$$p(i, j, l) := (\alpha_i\alpha_{i+1}\cdots\alpha_{j-1}) \cdot (\beta_j\beta_{j-1}\cdots\beta_{l+1}).$$

This is the shortest path starting from i, going through j, and ending at l. As an element in $\varPi$, the path $p(i,j,l)$ is not zero in $\varPi$ if and only if $i \geq j-l+1$, so set

$$\varGamma[l] := \{(i,j) \in Q_0 \times Q_0 \mid j-l+1 \leq i \leq j \geq l\}.$$

We obtain the following assertion from straightforward calculation.

Lemma 2.6. *The set* $\{p(i,j,l) \mid (i,j) \in \varGamma[l]\}$ *forms a* K*-basis of* $\varPi e_l$*.*

This basis allows us to express $\varPi e_l$ as

(2.1)
$$\begin{tikzcd}
l \arrow[r] \arrow[d] & l-1 \arrow[r] \arrow[d] & \cdots \arrow[r] & 1 \arrow[d] \\
l+1 \arrow[r] \arrow[d] & l \arrow[r] \arrow[d] & \cdots \arrow[r] & 2 \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] \\
n \arrow[r] & n-1 \arrow[r] & \cdots \arrow[r] & n-l+1
\end{tikzcd} .$$

Here, each number i in the row starting at j denotes a one-dimensional vector space $Kp(i,j,l)$ with a basis $p(i,j,l)$, and each arrow stands for the identity map $K \to K$ with respect to these bases.

In examples later, we sometimes write $\varPi e_l$ like a Young diagram by enclosing each entry with a square and omitting arrows: for example, if $n=8$ and $l=3$, then $\varPi e_l$ is denoted by

(2.2)
$$\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 4 & 3 & 2 \\ \hline 5 & 4 & 3 \\ \hline 6 & 5 & 4 \\ \hline 7 & 6 & 5 \\ \hline 8 & 7 & 6 \\ \hline \end{array}.$$

We use similar notation for subfactor modules of $\varPi e_l$.

Under this preparation, we recall the result of [IRRT] for type $\mathbb{A}_n$.

Proposition 2.7. [IRRT, Theorem 6.1] *Let* $w \in \mathsf{j\text{-}irr}\, W$ *be a join-irreducible element of type* l*. Then, the module* $J(w) \in \mathsf{i}\tau^{-1}\text{-}\mathsf{rigid}\, \varPi$ *is expressed as follows.*

- *Consider the diagram* (2.1)*.*
- *For each* $j \in [l,n]$*, in the row starting at* j*, keep the entries* i *satisfying* $i \geq w(j+1)$ *and delete the others.*

2.4. **Type $\mathbb{D}_n$.** Let $\varDelta := \mathbb{D}_n$ in this subsection. The preprojective algebra $\varPi$ of type $\mathbb{D}_n$ is given by the following quiver and relations:

$$\begin{tikzcd}
1 \arrow[dr, shift left, "\alpha_1^+"] & & & & \\
& 2 \arrow[ul, shift left, "\beta_2^+"] \arrow[dl, shift left, "\beta_2^-"] \arrow[r, shift left, "\alpha_2"] & 3 \arrow[l, shift left, "\beta_3"] \arrow[r, shift left, "\alpha_3"] & \cdots \arrow[l, shift left, "\beta_4"] \arrow[r, shift left, "\alpha_{n-2}"] & n-1 \arrow[l, shift left, "\beta_{n-1}"] \\
-1 \arrow[ur, shift left, "\alpha_1^-"] & & & &
\end{tikzcd} ;$$

$$\alpha_1^+\beta_2^+ = 0, \quad \alpha_1^-\beta_2^- = 0, \quad \alpha_2\beta_3 = \beta_2^+\alpha_1^+ + \beta_2^-\alpha_1^-,$$
$$\alpha_i\beta_{i+1} = \beta_i\alpha_{i-1} \ (3 \leq i \leq n-2), \quad \beta_{n-1}\alpha_{n-2} = 0.$$

To avoid complicated notation, we set $\alpha_1 := \alpha_1^+ + \alpha_1^-$ and $\beta_2 := \beta_2^+ + \beta_2^-$.

The Coxeter group W of type $\mathbb{D}_n$ is isomorphic to the group consisting of all automorphisms w on the set $\pm[1,n]$ satisfying the following conditions:

- $w(-i) = -w(i)$ holds for each $i \in [1, n]$; and
- the number of elements in $\{i \in [1, n] \mid w(i) < 0\}$ is even.

Here, $s_i \in W$ is sent to $(-1 \quad 2)(-2 \quad 1)$ if $i = -1$; and $(-i \quad -(i+1))(i \quad i+1)$ if $i \neq -1$. We identify W with the group above by this isomorphism. Since $w(-i) = -w(i)$ holds, we express $w \in W$ as $(w(1), w(2), \dots, w(n))$.

The reflections of W are precisely the elements of the form $(-a \quad -b)(a \quad b)$ with $a, b \in \pm[1, n]$ and $a > |b|$, and the set $\mathsf{inv}(w)$ of inversions of $w \in W$ is

$$\{(-a \quad -b)(a \quad b) \mid a, b \in \pm[1, n],\ a > |b|,\ w^{-1}(a) < w^{-1}(b)\}.$$

An element $w \in W$ is a join-irreducible element of type l if and only if l is the unique element in $\{-1\} \cup [1, n-1] = Q_0$ such that $w(l) > w(|l| + 1)$ holds.

We fix one or two bases of each indecomposable projective module Πe_l as follows. We divide the argument by whether $l = \pm 1$ or not.

We consider the case $l = \pm 1$ first. Let $i, j \in Q_0 = \{-1\} \cup [1, n-1]$ with $i \leq j \neq -l$. We define a path $p(i, j, l)$ by

$$p(i, j, \pm 1) := \begin{cases} (\alpha_i \alpha_{i+1} \cdots \alpha_{j-1}) \cdot (\beta_j \beta_{j-1} \cdots \beta_3) \beta_2^{\pm} & (i \geq 2) \\ \alpha_1^{+} p(2, j, \pm 1) & (i = 1) \\ \alpha_1^{-} p(2, j, \pm 1) & (i = -1) \end{cases}.$$

This is a shortest path starting from i, going through j, and ending at l. As an element in Π, the path $p(i, j, l)$ is not zero in Π if and only if $i \neq (-1)^j l$, so set

$$\Gamma[l] := \{(i, j) \in Q_0 \times Q_0 \mid (-1)^j l \neq i \leq j \neq -l\}.$$

We obtain the following assertion from straightforward calculation.

Lemma 2.8. *The set* $\{p(i, j, l) \mid (i, j) \in \Gamma[l]\}$ *forms a* K*-basis of* Πe_l.

This basis allows us to express Πe_l as

$$\begin{tikzcd} l \arrow[d] & & & & \\ 2 \arrow[r] \arrow[d] & -l \arrow[d] & & & \\ \vdots \arrow[d] & \vdots \arrow[d] & & & \\ n-2 \arrow[r] \arrow[d] & n-3 \arrow[r] \arrow[d] & \cdots \arrow[r] & (-1)^{n-3} l \arrow[d] & \\ n-1 \arrow[r] & n-2 \arrow[r] & \cdots \arrow[r] & 2 \arrow[r] & (-1)^{n-2} l \end{tikzcd} \tag{2.3}$$

.

Here, each number i in the row starting at j denotes a one-dimensional vector space $Kp(i, j, l)$ with a basis $p(i, j, l)$, and each arrow stands for the identity map $K \to K$ with respect to these bases.

If we use the “Young diagram-like” notation as (2.2) for the case $n = 9$ and $l = 1$, then Πe_l is denoted by

(2.4)

1							
2	−1						
3	2	1					
4	3	2	−1				
5	4	3	2	1			
6	5	4	3	2	−1		
7	6	5	4	3	2	1	
8	7	6	5	4	3	2	−1

.

The indecomposable τ^{-1}-rigid module $J(w)$ for $w \in \mathsf{j\text{-}irr}\, W$ of type $l = \pm 1$ is given as follows.

Proposition 2.9. [IRRT, Theorem 6.5] *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type $l = \pm 1$. Then, the module $J(w) \in \mathsf{i}\tau^{-1}\text{-}\mathsf{rigid}\, \Pi$ is expressed as follows.*

- *Consider the diagram (2.3).*
- *For each $j \in \{l\} \cup [2, n-1]$, in the row starting at j, keep the entries i satisfying $i \geq w(|j|+1)$ and delete the others.*

Next, we consider the case $l \geq 2$. Let $i \in \pm Q_0 = \pm[1, n-1]$ and $j \in Q_0 = \{-1\} \cup [1, n-1]$ with $i \leq j \geq l$. Set $t := (-1)^{j-l+1}$. We define two paths $p_1(i, j, l)$ and $p_{-1}(i, j, l)$ in Q by

$$p_\varepsilon(i,j,l) := \begin{cases} (\alpha_i \alpha_{i-1} \cdots \alpha_{j-1}) \cdot (\beta_j \beta_{j-1} \cdots \beta_{l+1}) & (i \geq 2) \\ \alpha_1^+ p_\varepsilon(2,j,l) & (i = 1) \\ \alpha_1^- p_\varepsilon(2,j,l) & (i = -1) \\ \beta_2 p_\varepsilon(\varepsilon t, j, l) & (i = -2) \\ (\beta_{-i} \beta_{-i-1} \cdots \beta_3) p_\varepsilon(-2, j, l) & (i \leq -3) \end{cases}.$$

This is a shortest path

- starting from i, going through j, and ending at l if $i \geq -1$; and
- starting from $|i|$, going through εt and then j, and ending at l if $i \leq -2$.

As an element in Π, the path $p_\varepsilon(i, j, l)$ is not zero in Π if and only if $i \geq j - (n-1) - l$, so set

$$\Gamma[l] := \{(i, j) \in \pm Q_0 \times Q_0 \mid j - (n-1) - l \leq i \leq j \geq l\}.$$

We obtain the following assertion from straightforward calculation.

Lemma 2.10. *Let $\varepsilon = \pm 1$. Then, the set $\{p_\varepsilon(i, j, l) \mid (i, j) \in \Gamma[l]\}$ forms a K-basis of Πe_l.*

Each basis above allows us to express Πe_l as

(2.5)

$$\begin{tikzcd}
 & & & & -\varepsilon \arrow[dr] & & & & & & & \\
l \arrow[r] \arrow[ddd] & l-1 \arrow[r] \arrow[ddd] & \cdots \arrow[r] & 2 \arrow[ur] \arrow[dr] \arrow[ddd] & & -2 \arrow[r] \arrow[dd] & \cdots \arrow[r] & -m \arrow[r] \arrow[ddd] & -m-1 \arrow[r] \arrow[ddd] & \cdots \arrow[r] & -n+2 \arrow[r] \arrow[ddd] & -n+1 \arrow[ddd] \\
 & & & & \varepsilon \arrow[ur, swap, "-1"] \arrow[dd] & & & & & & & \\
 & & & & & \varepsilon \arrow[dr] & & & & & & \\
l+1 \arrow[r] \arrow[dd] & l \arrow[r] \arrow[dd] & \cdots \arrow[r] & 3 \arrow[r] \arrow[dd] & 2 \arrow[ur] \arrow[dr] \arrow[dd] & & \cdots \arrow[r] & -m+1 \arrow[r] \arrow[dd] & -m \arrow[r] \arrow[dd] & \cdots \arrow[r] & -n+3 \arrow[r] \arrow[dd] & -n+2 \arrow[dd] \\
 & & & & & -\varepsilon \arrow[ur, swap, "-1"] \arrow[d] & & & & & & \\
\vdots \arrow[dd] & \vdots \arrow[dd] & & \vdots \arrow[dd] & \vdots \arrow[dd] & \vdots \arrow[dd] & & \vdots \arrow[d] & \vdots \arrow[dd] & & \vdots \arrow[dd] & \vdots \arrow[dd] \\
 & & & & & & & -\varepsilon t \arrow[dr] & & & & \\
n-1 \arrow[r] & n-2 \arrow[r] & \cdots \arrow[r] & m+1 \arrow[r] & m \arrow[r] & m-1 \arrow[r] & \cdots \arrow[ur] \arrow[dr] & & -2 \arrow[r] & \cdots \arrow[r] & -l+1 \arrow[r] & -l \\
 & & & & & & & \varepsilon t \arrow[ur, swap, "-1"] & & & &
\end{tikzcd},$$

where $m := n - l$, $t = (-1)^{m-1}$. Here, each number i in the row starting at j denotes a one-dimensional vector space $Kp(i, j, l)$ with a basis $p(i, j, l)$. Each arrow with the label “-1”

stands for the map $K \ni x \mapsto -x \in K$, and each of the other arrows stands for the identity map $K \to K$, with respect to these bases.

If we use the "Young diagram-like" notation as (2.2) for the case $n = 9$, $l = 2$, and $\varepsilon = 1$, then $\mathit{\Pi} e_l$ is denoted by

$$\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 2 & \substack{-1\\1} & -2 & -3 & -4 & -5 & -6 & -7 & -8 \\ \hline 3 & 2 & \substack{1\\-1} & -2 & -3 & -4 & -5 & -6 & -7 \\ \hline 4 & 3 & 2 & \substack{-1\\1} & -2 & -3 & -4 & -5 & -6 \\ \hline 5 & 4 & 3 & 2 & \substack{1\\-1} & -2 & -3 & -4 & -5 \\ \hline 6 & 5 & 4 & 3 & 2 & \substack{-1\\1} & -2 & -3 & -4 \\ \hline 7 & 6 & 5 & 4 & 3 & 2 & \substack{1\\-1} & -2 & -3 \\ \hline 8 & 7 & 6 & 5 & 4 & 3 & 2 & \substack{-1\\1} & -2 \\ \hline \end{array}. \tag{2.6}$$

We use similar notation for subfactor modules of $\mathit{\Pi} e_l$.

The indecomposable τ^{-1}-rigid module $J(w)$ for $w \in \mathsf{j\text{-}irr}\, W$ of type $l \neq \pm 1$ is given as follows.

Proposition 2.11. [IRRT, Theorem 6.12] *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type $l \neq \pm 1$. If $w(l+1) \leq 1$, then set*

$$m := \mathsf{max}\{k \in [l+1, n] \mid w(k) \leq 1\}, \quad \varepsilon := \begin{cases} (-1)^{m-(l+1)} & (w(m) \leq -2) \\ (-1)^{m-(l+1)} w(m) & (w(m) = \pm 1) \end{cases};$$

otherwise, set $\varepsilon := 1$. Then, the module $J(w) \in \mathsf{i}\tau^{-1}\text{-}\mathsf{rigid}\, \mathit{\Pi}$ is expressed as follows.

- *Consider the diagram (2.5).*
- *For each $j \in [l, n-1]$, in the row starting at j, keep the entries i satisfying*

$$\begin{cases} i \geq w(j+1) & (w(j+1) \geq 2) \\ i \geq 2 \quad \text{or} \quad i = w(j+1) & (w(j+1) = \pm 1) \\ i \geq w(j+1) + 1 & (w(j+1) \leq -2) \end{cases}$$

and delete the others.

3. DESCRIPTION OF BRICKS

In this section, we describe the bricks over the preprojective algebras $\mathit{\Pi}$ of Dynkin type $\mathit{\Delta} = \mathbb{A}_n, \mathbb{D}_n$. For $w \in \mathsf{j\text{-}irr}\, W$, we have obtained that the brick $S(w)$ is $\mathsf{soc}_{\mathsf{End}_{\mathit{\Pi}}(J(w))} J(w)$ in Proposition 2.5, and the module $J(w) \in \mathsf{i}\tau^{-1}\text{-}\mathsf{rigid}\, \mathit{\Pi}$ is combinatorially determined in Propositions 2.7, 2.9, and 2.11.

We remark that the bricks in $\mathsf{mod}\, \mathit{\Pi}$ coincide with the layers of $\mathit{\Pi}$ [IRRT, Theorem 1.2]. Thus, the dimension vector of each brick in $\mathsf{mod}\, \mathit{\Pi}$ is a positive root by [AIRT, Theorem 2.7]. Here, a module L in $\mathsf{mod}\, \mathit{\Pi}$ is called a *layer* if there exist some $w \in W$ and some vertex i in $\mathit{\Delta}$ such that $w < ws_i$ and $L \cong I(w)/I(ws_i)$ [AIRT, Section 2].

3.1. Type $\mathbb{A}_n$.

We state the result and give an example first.

Theorem 3.1. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type l. Set*

$$R := w([l+1, n+1]), \quad a := w(l), \quad b := w(l+1), \quad V := [b, a-1].$$

Then, the brick $S(w)$ is isomorphic to the $\mathit{\Pi}$-module $S'(w)$ defined as follows.

(a) *The brick $S'(w)$ has a K-basis $(\langle i \rangle)_{i \in V}$, and if $j = i$, then $e_j \langle i \rangle := \langle i \rangle$; otherwise, $e_j \langle i \rangle := 0$.*

(b) *Let $i \in V$. If $j \neq i - 1$, then $\alpha_j \langle i \rangle := 0$. If $j \neq i + 1$, then $\beta_j \langle i \rangle := 0$.*

(c) *If* $i \in V \setminus \{\max V\}$*, then*

$$\alpha_i\langle i+1\rangle := \begin{cases} \langle i\rangle & (i+1 \notin R) \\ 0 & (i+1 \in R) \end{cases}, \quad \beta_{i+1}\langle i\rangle := \begin{cases} 0 & (i+1 \notin R) \\ \langle i+1\rangle & (i+1 \in R) \end{cases}.$$

Example 3.2. Let $n := 8$ and $w = (2,5,8,1,3,4,6,7,9)$. Then, we have $l = 3$, $a = 8$, $b = 1$, and $V = [1,7]$. The module $S(w)$ has a K-basis $\langle 1\rangle, \langle 2\rangle, \ldots, \langle 7\rangle$ and its structure as a Π-module can be written as

$$\langle 1\rangle \xleftarrow{\alpha_1} \langle 2\rangle \xrightarrow{\beta_3} \langle 3\rangle \xrightarrow{\beta_4} \langle 4\rangle \xleftarrow{\alpha_4} \langle 5\rangle \xrightarrow{\beta_6} \langle 6\rangle \xrightarrow{\beta_7} \langle 7\rangle.$$

In an abbreviated form, the brick $S(w)$ is denoted by

$$1 \leftarrow 2 \rightarrow 3 \rightarrow 4 \leftarrow 5 \rightarrow 6 \rightarrow 7. \tag{3.1}$$

If we use the notation as (2.2), then by Proposition 2.7, the module $J(w)$ and the "position" of a submodule $S(w)$ in $J(w)$ are described as follows:

$$J(w) = \begin{array}{|c|c|c|}\hline 3&2&1\\\hline 4&3\\\cline{1-2} 5&4\\\cline{1-2} 6\\\cline{1-1} 7\\\cline{1-1}\end{array}, \quad S(w) = \begin{array}{c|c|c|}\cline{2-3} &2&1\\\cline{2-3} &3\\\cline{1-2} \multicolumn{1}{|c|}{5}&4\\\cline{1-2} \multicolumn{1}{|c|}{6}\\\cline{1-1} \multicolumn{1}{|c|}{7}\\\cline{1-1}\end{array}.$$

Compare this expression of the brick $S(w)$ to (3.1). If we use such abbreviated expressions of bricks as (3.1), then the theorem can be restated as follows.

Corollary 3.3. *Let* $w \in \mathsf{j\text{-}irr}\, W$ *be a join-irreducible element of type* l*, and use the setting of Theorem 3.1. We express the brick* $S(w)$ *in the following abbreviation rules.*

- *For each* $i \in V$*, the* K*-vector subspace* $K\langle i\rangle$ *is denoted by the symbol* i*.*
- *If the action of some* $\gamma \in Q_1$ *on* $S(w)$ *induces a nonzero* K*-linear map* $K\langle i\rangle \to K\langle j\rangle$*, then we draw an arrow from the symbol* i *to the symbol* j*.*

Then, for each $i \in V \setminus \{\max V\}$*, there exists exactly one arrow between* i *and* $i+1$*, and the orientation is* $i \to i+1$ *if* $i+1 \in R$ *and* $i \leftarrow i+1$ *if* $i+1 \notin R$*.*

It is easy to see that there exists some path algebra A of type $\mathbb{A}_n$ such that the brick $S(w)$ is an A-module, and that any 2-cycle in Q annihilates all the bricks in Π. Let I be the ideal of Π generated by all the 2-cycles in Q, then [DIRRT, Corollary 5.20] implies that $\mathsf{torf}\,\Pi \cong \mathsf{torf}(\Pi/I)$ as lattices. Thus, there is an isomorphism from W to $\mathsf{torf}(\Pi/I)$ as lattices by Propositions 1.5 and 2.1. The relationship between W and Π/I is investigated from another point of view in [BCZ, Section 4].

Now we start the proof of Theorem 3.1. For this purpose, we restate Proposition 2.7 as follows.

Lemma 3.4. *Let* $w \in \mathsf{j\text{-}irr}\, W$ *be a join-irreducible element of type* l*.*

(1) *Assume* $(i,j) \in \Gamma[l]$*. Then,* $p(i,j,l) \notin I(w)$ *holds if and only if* $i \geq w(j+1)$*.*
(2) *Define* $\Gamma(w) \subset \Gamma[l]$ *as the subset consisting of the elements* $(i,j) \in \Gamma[l]$ *with* $p(i,j,l) \notin I(w)$*. Then, the set* $\{p(i,j,l) \mid (i,j) \in \Gamma(w)\}$ *induces a* K*-basis of* $J(w)$*.*

To express $S(w)$, we define the following set for $k \geq 1$:

$$\Gamma_k(w) := \{(i,j) \in \Gamma(w) \mid \min\{x \geq 1 \mid (i,j+x) \notin \Gamma(w)\} = k\}.$$

It is easy to see that $\Gamma(w)$ is the disjoint union of the $\Gamma_k(w)$'s. Moreover, we extend the definition of the path $p(i,j,l)$ to $\tilde{\Gamma}[l] := \{(i,j) \in Q_0 \times \mathbb{Z} \mid i \leq j \geq l\}$ by setting $p(i,j,l) := 0$ if $j \geq n+1$, and define $w(k) := k$ if $k \geq n+2$.

Lemma 3.5. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type l. Consider the endomorphism $f := (\cdot p(l, l+1, l))\colon J(w) \to J(w)$.*

(1) *We have $S(w) = \mathsf{Ker}\, f$.*

(2) *Let $(i, j) \in \mathit{\Gamma}(w)$. Then, $p(i, j, l) \in \mathsf{Ker}\, f$ holds if and only if $(i, j) \in \mathit{\Gamma}_1(w)$.*

(3) *The set $\{p(i, j, l) \mid (i, j) \in \mathit{\Gamma}_1(w)\}$ induces a K-basis of $\mathsf{Ker}\, f$.*

Proof. (1) For every nonisomorphic endomorphism $g\colon J(w) \to J(w)$, it is clear that there exists $h\colon J(w) \to J(w)$ such that $g = hf$. Thus, $S(w) = \mathsf{Ker}\, f$ holds.

(2) As an element in $\mathit{\Pi}$, we have $f(p(i, j, l)) = p(i, j, l)p(l, l + 1, l) = p(i, j + 1, l)$. Then, Lemma 3.4 implies the assertion.

(3) From Lemma 3.4, recall that the set $\{p(i, j, l) \mid (i, j) \in \mathit{\Gamma}(w)\}$ induces a basis of $J(w)$, so this set is linearly independent in $J(w)$.

Thus, the subset $\{p(i, j, l) \mid (i, j) \in \mathit{\Gamma}_1(w)\}$ is linearly independent in $J(w)$, and is contained in $\mathsf{Ker}\, f$ by (2).

On the other hand, in the proof of (2), we got $f(p(i, j, l)) = p(i, j+1, l)$. If $(i, j) \in \mathit{\Gamma}(w) \setminus \mathit{\Gamma}_1(w)$, then $(i, j + 1, l) \in \mathit{\Gamma}(w)$. The set $\{p(i, j + 1, l) \mid (i, j) \in \mathit{\Gamma}(w) \setminus \mathit{\Gamma}_1(w)\}$ is linearly independent in $J(w)$. Thus, the set $\{p(i, j, l) \mid (i, j) \in \mathit{\Gamma}_1(w)\}$ generates $\mathsf{Ker}\, f$ as a K-vector space in $J(w)$.

Therefore, we conclude that the set $\{p(i, j, l) \mid (i, j) \in \mathit{\Gamma}_1(w)\}$ induces a K-basis of $\mathsf{Ker}\, f$. □

Lemma 3.6. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type l, and define V as in Theorem 3.1. Then, there exists a bijection $\mathit{\Gamma}_1(w) \to V$ given by $(i, j) \mapsto i$.*

Proof. In the proof, we fully use the notation in Theorem 3.1.

We first show the well-definedness of the map $\mathit{\Gamma}_1(w) \to V$.

We remark that, for $k \in [l + 1, n + 1]$, the condition $w(k) = k$ holds if and only if $k > a$, and that this condition is also equivalent to $w(k) > a$. Lemma 3.4 and $(i, j) \in \mathit{\Gamma}(w)$ give $j \geq i \geq w(j + 1)$. Thus, $w(j + 1) \leq j$ holds, so we get $j + 1 \leq a$, or equivalently, $j < a$. Therefore, we obtain $i \leq j < a$.

On the other hand, Lemma 3.4 and $(i, j) \in \mathit{\Gamma}(w)$ also imply $j \geq i \geq w(j+1) \geq w(l+1) = b$.

These imply that the map $\mathit{\Gamma}_1(w) \to V$ is well-defined. It is clearly injective by Lemma 3.4.

We next prove that the map $\mathit{\Gamma}_1(w) \to V$ is also surjective. Let $i \in V$. Then, $i < a$ holds, so there exists some $j \in [l, n]$ such that $(i, j) \in \mathit{\Gamma}(w)$ by Lemma 3.4. Take the maximum j, then it is easy to obtain $(i, j) \in \mathit{\Gamma}_1(w)$ from Lemma 3.4.

Hence, the map $\mathit{\Gamma}_1(w) \to V$ is also surjective, and thus, bijective. □

Now, we show Theorem 3.1.

Proof. By Lemma 3.5, we can define a map $\rho\colon V \to Q_0$ as follows: $\rho(i)$ is the unique element $j \in Q_0$ such that $(i, j) \in \mathit{\Gamma}_1(w)$. Set $\langle i \rangle := p(i, \rho(i), l)$ for each $i \in V$. It suffices to show that $(\langle i \rangle)_{i \in V}$ satisfies the properties (a), (b), and (c), since the three properties are enough to define a $\mathit{\Pi}$-module.

First, $(\langle i \rangle)_{i \in V}$ is a K-basis of $S(w)$ by Lemma 3.6, and $K\langle i \rangle$ is clearly a subspace of $e_i S(w)$. Thus, the property (a) holds, and (b) follows from (a).

We begin the proof of (c).

Let $i \in V \setminus \{\max V\}$ and set $j := \rho(i + 1)$. Then,

$$\alpha_i \langle i + 1 \rangle = \alpha_i p(i + 1, j, l) = p(i, j, l) = \begin{cases} \langle i \rangle & (\text{if } i + 1 \notin R, \text{ since } (i, j) \in \mathit{\Gamma}_1(w)) \\ 0 & (\text{if } i + 1 \in R, \text{ since } (i, j) \notin \mathit{\Gamma}(w)) \end{cases}.$$

Next, let $i \in V \setminus \{\max V\}$ and set $j := \rho(i)$. Then,

$$\begin{aligned} \beta_{i+1} \langle i \rangle &= \beta_{i+1} p(i, j, l) = p(i + 1, j + 1, l) \\ &= \begin{cases} 0 & (\text{if } i + 1 \in R, \text{ since } (i + 1, j + 1) \notin \mathit{\Gamma}(w)) \\ \langle i + 1 \rangle & (\text{if } i + 1 \notin R, \text{ since } (i + 1, j + 1) \in \mathit{\Gamma}_1(w)) \end{cases}. \end{aligned}$$

From these, we have the property (c). □

3.2. **Type $\mathbb{D}_n$.** We state the result and give some examples first. Recall $\alpha_1 = \alpha_1^+ + \alpha_1^-$ and $\beta_2 = \beta_2^+ + \beta_2^-$.

Theorem 3.7. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type l. Set*

$$R := w([|l|+1, n]), \quad a := w(l), \quad b := w(|l|+1),$$

$$r := \max\{k \geq 0 \mid [1,k] \subset \pm R\}, \quad c := \begin{cases} w^{-1}(|w(1)|) & (r \geq 1) \\ 1 & (r = 0) \end{cases},$$

$$(V_-, V_+) := \begin{cases} (\emptyset, [b, a-1]) & (b \geq 2) \\ (\emptyset, \{c\} \cup [2, a-1]) & (b = \pm 1) \\ ([b+1, -2] \cup \{-c\}, \{c\} \cup [2, a-1]) & (b \leq -2) \end{cases}, \quad V := V_+ \amalg V_-.$$

Then, the brick $S(w)$ is isomorphic to the Π-module $S'(w)$ defined as follows.

(a) *The brick $S'(w)$ has a K-basis $(\langle i \rangle)_{i \in V}$, and if $j = |i| \geq 2$ or $j = i \in \{\pm 1\}$, then $e_j \langle i \rangle := \langle i \rangle$; otherwise $e_j \langle i \rangle := 0$.*

(b) *Let $i \in V$. If $j \neq |i| - 1$, then $\alpha_j \langle i \rangle := 0$. If $j \neq |i|+1$, then $\beta_j \langle i \rangle := 0$.*

(c) *The remaining actions of arrows are given as follows, where we set $\langle j \rangle := 0$ if $j \notin V$ (in this case, the coefficient ξ_i^+, ξ_i^-, η_i^+, or η_i^- of $\langle j \rangle$ is set as zero below).*

  (i) *For $i \in V_+ \setminus \{\max V_+\}$, we have $\alpha_{|i|} \langle |i|+1 \rangle := \xi_i^+ \langle i \rangle + \xi_i^- \langle -i \rangle$, where*

$$\xi_i^+ := \begin{cases} 1 & (|i|+1 \notin R) \\ 0 & (|i|+1 \in R) \end{cases}, \quad \xi_i^- := \begin{cases} 1 & (|i| = 1,\ r = 0,\ 2 \notin R) \\ 0 & (\text{otherwise}) \end{cases}.$$

  (ii) *For $i \in V_+ \setminus \{\max V_+\}$, we have $\beta_{|i|+1} \langle i \rangle := \eta_i^+ \langle |i|+1 \rangle + \eta_i^- \langle -(|i|+1) \rangle$, where*

$$\eta_i^+ := \begin{cases} 1 & (|i|+1 \in R) \\ 0 & (|i|+1 \notin R) \end{cases}, \quad \eta_i^- := \begin{cases} -1 & (|i| = 1,\ r = 0,\ -2 \notin R) \\ 0 & (\text{otherwise}) \end{cases}.$$

  (iii) *For $i \in V_- \setminus \{\min V_-\}$, we have $\alpha_{|i|} \langle -(|i|+1) \rangle := \xi_i^+ \langle -i \rangle + \xi_i^- \langle i \rangle$, where*

$$\xi_i^+ := \begin{cases} 1 & (|i| \leq r,\ |i|+1 \in R) \\ 0 & (\text{otherwise}) \end{cases}, \quad \xi_i^- := \begin{cases} 1 & (-(|i|+1) \in R) \\ 0 & (-(|i|+1) \notin R) \end{cases}.$$

  (iv) *For $i \in V_-$, we have $\beta_{|i|+1} \langle i \rangle := \eta_i^+ \langle |i|+1 \rangle + \eta_i^- \langle -(|i|+1) \rangle$, where*

$$\eta_i^+ := \begin{cases} 1 & (|i| \leq r,\ |i|+1 \notin R) \\ 0 & (\text{otherwise}) \end{cases}, \quad \eta_i^- := \begin{cases} 1 & (|i| \neq r,\ -(|i|+1) \notin R) \\ -1 & (|i| = r) \\ 0 & (\text{otherwise}) \end{cases}.$$

The proof of the theorem given in later depends on whether the type l of the join-irreducible element $w \in \mathsf{j\text{-}irr}\, W$ is ±1 or not, because the description of the indecomposable τ^{-1}-rigid module $J(w)$ does so. The following examples show the difference of the calculation of the brick $S(w)$ in these two cases.

Example 3.8. Let $n := 9$, $w := (9, -7, -6, -4, -1, 2, 3, 5, 8)$. Then, we have $l = 1$, $a = 9$, $b = -7$, $r = 8$, and $c = -1$. Thus, $(V_-, V_+) = ([-6, -2] \cup \{1\}, \{-1\} \cup [2, 8])$, and the desired brick $S(w)$ is written as

$$\begin{tikzcd}
\langle 1 \rangle \arrow[r, "\beta_2^+"] & \color{red}\langle -2 \rangle \arrow[r, "\beta_3"] \arrow[dl, "\alpha_1^-"'] & \color{red}\langle -3 \rangle \arrow[dl, "\alpha_2"'] \arrow[dr, "\beta_4"] & \color{red}\langle -4 \rangle \arrow[l, "\alpha_3"'] \arrow[r, "\beta_5"] & \color{red}\langle -5 \rangle \arrow[dl, "\alpha_4"'] \arrow[dr, "\beta_6"] & \color{red}\langle -6 \rangle \arrow[l, "\alpha_5"'] \arrow[dr, "\beta_7"] & & \\
\langle -1 \rangle \arrow[r, "\beta_2^-"] & \langle 2 \rangle \arrow[r, "\beta_3"] & \langle 3 \rangle & \langle 4 \rangle \arrow[l, "\alpha_3"'] \arrow[r, "\beta_5"] & \langle 5 \rangle & \langle 6 \rangle \arrow[l, "\alpha_5"'] & \langle 7 \rangle \arrow[l, "\alpha_6"'] \arrow[r, "\beta_8"] & \langle 8 \rangle
\end{tikzcd}.$$

By omitting the labels of the arrows, the brick $S(w)$ can be written in the following abbreviated way, which is enough to determine $S(w)$ up to isomorphisms:

(3.2)
$$\begin{tikzcd}
1 \arrow[r] & -2 \arrow[r] \arrow[dl] & -3 \arrow[dl] \arrow[dr] & -4 \arrow[l] \arrow[r] & -5 \arrow[dl] \arrow[dr] & -6 \arrow[l] \arrow[dr] & & \\
-1 \arrow[r] & 2 \arrow[r] & 3 & 4 \arrow[l] \arrow[r] & 5 & 6 \arrow[l] & 7 \arrow[l] \arrow[r] & 8
\end{tikzcd}.$$

If we use the notation as (2.4), then by Proposition 2.9, the module $J(w)$ and the "position" of a submodule $S(w)$ in $J(w)$ are described as follows:

$$J(w) = \begin{array}{cccc}
\boxed{1} & & & \\
\boxed{2} & \boxed{-1} & & \\
\boxed{3} & \boxed{2} & \boxed{1} & \\
\boxed{4} & \boxed{3} & \boxed{2} & \boxed{-1} \\
\boxed{5} & \boxed{4} & \boxed{3} & \boxed{2} \\
\boxed{6} & \boxed{5} & \boxed{4} & \boxed{3} \\
\boxed{7} & \boxed{6} & \boxed{5} & \\
\boxed{8} & & &
\end{array}, \quad S(w) = \begin{array}{cccc}
 & & & \\
 & & & \\
 & & \boxed{1} & \\
 & & \boxed{2} & \boxed{-1} \\
 & \boxed{4} & \boxed{3} & \boxed{2} \\
\boxed{6} & \boxed{5} & \boxed{4} & \boxed{3} \\
\boxed{7} & \boxed{6} & \boxed{5} & \\
\boxed{8} & & &
\end{array}.$$

In the figure for $S(w)$, every square $\boxed{i}$ with a red letter denotes $K\langle -i\rangle$, which is a subspace of $e_iS(w)$. There are five such squares $\boxed{2}$, $\boxed{3}$, $\boxed{4}$, $\boxed{5}$, $\boxed{6}$. Every other square $\boxed{i}$ denotes $K\langle i\rangle$, and it is a subspace of $e_iS(w)$. Compare this expression of the brick $S(w)$ to (3.2).

Example 3.9. Let $n := 9$, $w := (-6, 9, -7, -4, -1, 2, 3, 5, 8)$. Then, we have $l = 2$, $a = 9$, $b = -7$, $r = 5$, and $c = -1$. Thus, $(V_-, V_+) = ([-6, -2] \cup \{1\}, \{-1\} \cup [2, 8])$, and the desired brick $S(w)$ is written as

$$\begin{tikzcd}
\langle 1\rangle \arrow[r, "\beta_2^+"] & \langle -2\rangle \arrow[r, "\beta_3"] \arrow[dl, "\alpha_1^-"'] & \langle -3\rangle \arrow[dl, "\alpha_2"'] \arrow[dr, "\beta_4"] & \langle -4\rangle \arrow[l, "\alpha_3"'] \arrow[r, "\beta_5"] & \langle -5\rangle \arrow[dl, "\alpha_4"'] \arrow[dr, "\beta_6"] \arrow[r, "-\beta_6"] & \langle -6\rangle & & \\
\langle -1\rangle \arrow[r, "\beta_2^-"] & \langle 2\rangle \arrow[r, "\beta_3"] & \langle 3\rangle & \langle 4\rangle \arrow[l, "\alpha_3"'] \arrow[r, "\beta_5"] & \langle 5\rangle & \langle 6\rangle \arrow[l, "\alpha_5"'] & \langle 7\rangle \arrow[l, "\alpha_6"'] \arrow[r, "\beta_8"] & \langle 8\rangle
\end{tikzcd}.$$

The brick $S(w)$ can be written in the following abbreviated way:

(3.3)
$$\begin{tikzcd}
1 \arrow[r] & -2 \arrow[r] \arrow[dl] & -3 \arrow[dl] \arrow[dr] & -4 \arrow[l] \arrow[r] & -5 \arrow[dl] \arrow[dr] \arrow[r] & -6 & & \\
-1 \arrow[r] & 2 \arrow[r] & 3 & 4 \arrow[l] \arrow[r] & 5 & 6 \arrow[l] & 7 \arrow[l] \arrow[r] & 8
\end{tikzcd}.$$

Now we use the notation as (2.6), then by Proposition 2.11, the module $J(w)$ and the "position" of a submodule $S(w)$ in $J(w)$ are described as follows:

$$J(w) = \begin{array}{ccccccc} 2 & \substack{1\\-1} & -2 & -3 & -4 & -5 & -6 \\ 3 & 2 & \substack{-1\\1} & -2 & -3 & & \\ 4 & 3 & 2 & {}_{-1} & & & \\ 5 & 4 & 3 & 2 & & & \\ 6 & 5 & 4 & 3 & & & \\ 7 & 6 & 5 & & & & \\ 8 & & & & & & \end{array}, \qquad S(w) = \begin{array}{ccccccc} & & & & (4) & (5) & -6 \\ & & {}_{1} & (2) & (3) & & \\ & & (2) & {}_{-1} & & & \\ & (4) & (3) & 2 & & & \\ & (5) & 4 & 3 & & & \\ 7 & 6 & 5 & & & & \\ 8 & & & & & & \end{array}.$$

In the figure for $S(w)$, for each $i = 2, 3, 4, 5$, the two squares $\boxed{(i)}$ together denote a certain one-dimensional subspace of the two-dimensional vector space corresponding to the two squares $\boxed{i}$ and $\boxed{-i}$ in the figure for $J(w)$. This one-dimensional vector space is actually $K\langle -i\rangle$, which is a subspace of $e_iS(w)$. Every other square $\boxed{i}$ in the figure for $S(w)$ denotes $K\langle i\rangle$, which is a subspace of $e_{|i|}S(w)$ if $i \leq -2$, and of $e_iS(w)$ if $i \geq -1$. We can check that this expression of the brick $S(w)$ can be rewritten as (3.3).

We mainly use such abbreviated expressions of bricks as (3.2) and (3.3) in the rest. Theorem 3.7 can be restated as follows by using the abbreviated expressions.

Corollary 3.10. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type l, and use the setting of Theorem 3.7. We express the brick $S(w)$ in the same abbreviation rules as Corollary 3.3. Then, there exist the following arrows, and no other arrows exist.*

- (i) *For each $i \in V_+ \setminus \{\max V_+\}$, there exists an arrow $i \to |i|+1$ if $|i|+1 \in R$; and $i \leftarrow |i|+1$ otherwise.*
- (ii) *For each $i \in V_- \setminus \{\min V_-\}$, there exists an arrow $i \leftarrow -(|i|+1)$ if $-(|i|+1) \in R$; and $i \to -(|i|+1)$ otherwise.*
- (iii) *If $r \geq 1$, then for each $i \in V_-$ with $|i| \leq r$, there exists an arrow $-i \leftarrow -(|i|+1)$ if $|i|+1 \in R$; and $i \to |i|+1$ otherwise.*
- (iv) *If $r = 0$, then there exists an arrow $-c \leftarrow 2$ if $c \leftarrow 2$ exists in* (i)*, and an arrow $c \to -2$ if $-c \to -2$ exists in* (ii)*.*

Proof. We remark that, for $i \in [1, r]$, the condition $-i \in R$ is equivalent to $i \notin R$. Then, Theorem 3.7 yields the assertion. $\square$

Unlike the case of type $\mathbb{A}_n$, for $w \in \mathsf{j\text{-}irr}\, W$, there may not exist a path algebra A of type $\mathbb{D}_n$ such that the brick $S(w)$ is an A-module. For example, the bricks obtained in Examples 3.8 and 3.9 cannot be modules over any path algebra of type $\mathbb{D}_n$, since the 2-cycle $\alpha_2\beta_3$ annihilates none of them. Our results imply that, if an element in Π is the product of some two 2-cycles, then it annihilates all the bricks in $\mathsf{brick}\, \Pi$.

We give more examples.

Example 3.11. In these examples, assume $n := 9$.

(1) Let $w := (3, 5, 8, -7, -4, 1, 2, 6, 9)$. Then, we have $l = 3$, $a = 8$, $b = -7$, $r = 2$, and $c = 1$. Thus, $(V_-, V_+) = ([-6, -1], [1, 7])$, and the desired brick $S(w)$ is written as

$$\begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] \arrow[dl] \arrow[dr] & -3 & -4 \arrow[l] \arrow[r] & -5 \arrow[r] & -6 \\ 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] & 5 \arrow[l] \arrow[r] & 6 & 7 \arrow[l] \end{tikzcd}.$$

(2) Let $w := (1, 3, 5, 8, -7, -4, 2, 6, 9)$. Then, we have $l = 4$, $a = 8$, $b = -7$, $r = 0$, and $c = 1$. Thus, $(V_-, V_+) = ([-6, -1], [1, 7])$, and the desired brick $S(w)$ is written as

$$\begin{tikzcd}
-1 \arrow[r] & -2 \arrow[r] & -3 & -4 \arrow[l] \arrow[r] & -5 \arrow[r] & -6 & \\
1 \arrow[r] \arrow[ur] & 2 & 3 \arrow[l] & 4 \arrow[l] & 5 \arrow[l] \arrow[r] & 6 & 7 \arrow[l]
\end{tikzcd} .$$

(3) Let $w := (1, 2, 3, 5, 8, -7, -4, 6, 9)$. Then, we have $l = 5$, $a = 8$, $b = -7$, $r = 0$, and $c = 1$. Thus, $(V_-, V_+) = ([-6, -1], [1, 7])$, and the desired brick $S(w)$ is written as

$$\begin{tikzcd}
-1 \arrow[r] & -2 \arrow[r] & -3 & -4 \arrow[l] \arrow[r] & -5 \arrow[r] & -6 & \\
1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] & 4 \arrow[l] & 5 \arrow[l] \arrow[r] & 6 & 7 \arrow[l]
\end{tikzcd} .$$

We also have the list of bricks in the case $\varDelta = \mathbb{D}_5$ in Appendix.

Now, we start the proof of Theorem 3.7. We divide the argument by whether the type l of $w \in \mathsf{j\text{-}irr}\, W$ is ± 1 or not.

We first assume that $l = \pm 1$. We can restate Proposition 2.9 as follows.

Lemma 3.12. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type $l = \pm 1$.*

1. *Assume $(i, j) \in \varGamma[l]$. Then, $p(i, j, l) \notin I(w)$ holds if and only if $i \geq w(|j| + 1)$.*
2. *Consider the subset $\varGamma(w) \subset \varGamma[l]$ consisting of the elements $(i, j) \in \varGamma[l]$ with $p(i, j, l) \notin I(w)$. Then, the set $\{p(i, j, l) \mid (i, j) \in \varGamma(w)\}$ induces a K-basis of $J(w)$.*

In this lemma, we can replace the condition $i \geq w(|j| + 1)$ by $|i| \geq w(|j| + 1)$ in (1), since $w(m) = (-1)^{m-1} l$ holds for the number $m := |w^{-1}(1)|$.

To express $S(w)$, we define the following set for $k \geq 1$:

$$\varGamma_k(w) := \{(i, j) \in \varGamma(w) \mid \min\{x \geq 1 \mid ((-1)^x i, |j| + x) \notin \varGamma(w)\} = k\}.$$

It is easy to see that $\varGamma(w)$ is the disjoint union of the $\varGamma_k(w)$'s. Moreover, we extend the definition of the path $p(i, j, l)$ to $\widetilde{\varGamma}[l] := \{(i, j) \in Q_0 \times \mathbb{Z} \mid i \leq j \geq l\}$ by setting $p(i, j, l) := 0$ if $j \geq n$, and also define $w(k) := k$ if $k \geq n + 1$. In Example 3.8, the squares with black letters denote $\varGamma_1(w)$, and the squares with red letters denote $\varGamma_2(w)$.

Lemma 3.13. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type $l = \pm 1$. Consider the endomorphism $f := (\cdot p(l, 3, l)) \colon J(w) \to J(w)$.*

1. *We have $S(w) = \mathsf{Ker}\, f$.*
2. *Let $(i, j) \in \varGamma(w)$. Then, $p(i, j, l) \in \mathsf{Ker}\, f$ holds if and only if $(i, j) \in \varGamma_1(w) \amalg \varGamma_2(w)$.*
3. *The set $\{p(i, j, l) \mid (i, j) \in \varGamma_1(w) \amalg \varGamma_2(w)\}$ induces a K-basis of $\mathsf{Ker}\, f$.*

Proof. Similar argument to Lemma 3.5 works. We remark that $f(p(i, j, l)) = p(i, j, l)p(l, 3, l) = p(i, |j| + 2, l)$ hold in $\varPi$. □

Lemma 3.14. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type $l = \pm 1$. Define V_+ and V_- as in Theorem 3.1.*

1. *There exists a bijection $\varGamma_1(w) \to V_+$ given by $(i, j) \mapsto i$.*
2. *There exists a bijection $\varGamma_2(w) \to V_-$ given by $(i, j) \mapsto i$ if $|i| = 1$; and $(i, j) \mapsto -i$ otherwise.*

Proof. We use the notation in Theorem 3.7 in the proof.

(1) We see the well-definedness of the map $\varGamma_1(w) \to V_+$.

We first show that every $(i, j) \in \varGamma_1(w)$ satisfies that $i < a$. We remark that, for $k \in [2, n]$, the condition $w(k) = k$ holds if and only if $k > a$, and that this condition is also equivalent to $w(k) > a$. Lemma 3.12 and $(i, j) \in \varGamma(w)$ give $j \geq i \geq w(|j| + 1)$. Thus, $w(|j| + 1) \leq j$ holds, so we get $|j| + 1 \leq a$, or equivalently, $|j| < a$. Therefore, we obtain $i \leq |j| < a$.

We also prove that, if $(i,j) \in \Gamma_1(w)$ and $|i| = 1$, then $i = c$ (*). Since $(i,j) \in \Gamma(w)$, we get $i = (-1)^{|j|-1}$. In this case, Lemma 3.12 and $(i,j) \in \Gamma_1(w)$ yield $w(|j|+2) \geq 2$ and $i \geq w(|j|+1)$. Since $|w^{-1}(1)| \geq 2$, we have $w(|j|+1) = \pm 1$. Thus, $|j|+1 = |w^{-1}(1)|$ holds; hence, we have $(-1)^{|j|-1}l = (-1)^{|w^{-1}(1)|}l = c$. Therefore, $i = c$.

Moreover, Lemma 3.12 and $(i,j) \in \Gamma(w)$ imply $j \geq i \geq w(|j|+1) \geq w(2) = b$.

These imply that the map $\Gamma_1(w) \to V_+$ is well-defined. By Lemma 3.12, it is clearly injective.

We next prove that the map $\Gamma_1(w) \to V_+$ is also surjective. Let $i \in V_+$, then $|i| < a$ holds. Thus, the first remark yields $w(|i|+1) < |i|+1$, so there exists some $j \in \{l\} \cup [2, n-1]$ such that $(i,j) \in \Gamma(w)$. Take the maximum j among such j's.

If $i \geq 2$, then (i,j) belongs to $\Gamma_1(w)$ by Lemma 3.12.

If $i = c$, then $(i,j) \in \Gamma(w)$ and $(i, |j|+2) \notin \Gamma(w)$ hold. On the other hand, we obtain $(-i, |j|+1) \notin \Gamma_1(w)$ from (*). From these, (i,j) must be in $\Gamma_1(w)$.

Therefore, $(i,j) \in \Gamma_1(w)$ holds, so the map $\Gamma_1(w) \to V_+$ is also surjective, and thus, bijective.

(2) We can check the following properties.

- For $k \in [2,n]$, the condition $w(k) \geq k-1$ holds if and only if $k > |b|+1$, and this condition is also equivalent to $w(k) > |b|$.
- If $i \in V_-$, then $w(|i|+1) < |i|$ and $w(|i|+2) < |i|+2$ hold.

Then, similar argument to (1) works. □

Now, we show Theorem 3.7 in the case $l = \pm 1$.

Proof. By Lemma 3.14, we can define a map $\rho\colon V \to Q_0$ as follows.

- If $i \in V_+$, then $\rho(i)$ is the unique element $j \in Q_0$ such that $(i,j) \in \Gamma_1(w)$.
- If $i \in V_-$ and $i = \pm 1$, then $\rho(i)$ is the unique element $j \in Q_0$ such that $(i,j) \in \Gamma_2(w)$.
- If $i \in V_-$ and $i \leq -2$, then $\rho(i)$ is the unique element $j \in Q_0$ such that $(-i,j) \in \Gamma_2(w)$.

Set $\langle i \rangle := p(i, \rho(i), l)$ for each $i \in V$. It suffices to show that $(\langle i \rangle)_{i \in V}$ satisfies the properties (a), (b), and (c), since the three properties are enough to define a Π-module.

First, $(\langle i \rangle)_{i \in V}$ is a K-basis of $S(w)$ by Lemma 3.13, and $K\langle i \rangle$ is clearly a subspace of $e_i S(w)$ if $i \geq -1$; and of $e_{|i|}S(w)$ if $i \leq -2$. Thus, the property (a) has been proved, and the property (b) follows from (a).

In the following observation, we fully use Lemma 3.12.

We begin the proof of (c)(i). First, we assume $2 \in V_+$, and set $j := \rho(2)$.

- If $2 \notin R$, then $w(j+1) = c$ and $w(j+2) \geq 3$ hold, so we have $(c,j) \in \Gamma_1(w)$ and $(-c,j) \notin \Gamma(w)$.
- If $2 \in R$, then $w(j+1) = 2$ holds, so we have $(c,j), (-c,j) \notin \Gamma(w)$.

These imply

$$\alpha_1\langle 2 \rangle = \alpha_1 p(2,j,l) = p(c,j,l) + p(-c,j,l) = \begin{cases} \langle c \rangle & (2 \notin R) \\ 0 & (2 \in R) \end{cases}.$$

Second, let $i \in V_+ \setminus \{\max V_+\}$ and $i \geq 2$, and set $j := \rho(i+1)$. Then,

$$\alpha_i\langle i+1 \rangle = \alpha_i p(i+1,j,l) = p(i,j,l) = \begin{cases} \langle i \rangle & (\text{if } i+1 \notin R, \text{ since } (i,j) \in \Gamma_1(w)) \\ 0 & (\text{if } i+1 \in R, \text{ since } (i,j) \notin \Gamma(w)) \end{cases}.$$

Since $l = \pm 1$, we have $r \geq 1$. Thus, we have proved (c)(i).

Next, we begin the proof of (c)(ii). Let $i \in V_+ \setminus \{\max V_+\}$, and set $j := \rho(i)$. Then,

$$\begin{aligned} \beta_{|i|+1}\langle i \rangle &= \beta_{|i|+1} p(i,j,l) = p(|i|+1, j+1, l) \\ &= \begin{cases} 0 & (\text{if } i+1 \notin R, \text{ since } (|i|+1, j+1) \notin \Gamma(w)) \\ \langle |i|+1 \rangle & (\text{if } i+1 \in R, \text{ since } (|i|+1, j+1) \in \Gamma_1(w)) \end{cases}. \end{aligned}$$

Since $r \geq 1$, this implies (c)(ii).

Before continuing the proof, we remark the following: every $i \in V_-$ satisfies $|i| < r$, since $l = \pm 1$. Thus, if $i \in V_-$, then $|i| + 1 \notin R$ is equivalent to $-(|i| + 1) \in R$.

We proceed to the proof of (c)(iii). First, assume $-c \in V_- \setminus \{\mathsf{min}\, V_-\}$, and set $j := \rho(-2)$.

- If $-2 \notin R$, then $w(j + 1) = c$ and $w(j + 2) = 2$ hold, so we have $(c, j) \in \Gamma_1(w)$ and $(-c, j) \notin \Gamma(w)$.
- If $-2 \in R$, then $w(j + 1) \le -2$ and $w(j + 2) = c$ hold, so we have $(-c, j) \in \Gamma_1(w)$ and $(c, j) \notin \Gamma(w)$.

Thus,

$$\alpha_1\langle -2\rangle = \alpha_1 p(2, j, l) = p(c, j, l) + p(-c, j, l) = \begin{cases} \langle c\rangle & (-2 \notin R) \\ \langle -c\rangle & (-2 \in R) \end{cases}.$$

Second, let $i \in V_- \setminus \{\mathsf{min}\, V_-\}$ and $|i| \ge 2$, and set $j := \rho(-(|i| + 1))$. Then,

$$\begin{aligned} \alpha_{|i|}\langle -(|i| + 1)\rangle &= \alpha_{|i|} p(|i| + 1, j, l) = p(|i|, j, l) \\ &= \begin{cases} \langle |i|\rangle = \langle -i\rangle & (\text{if } |i| + 1 \in R, \text{ since } (|i|, j) \in \Gamma_1(w)) \\ \langle -|i|\rangle = \langle i\rangle & (\text{if } |i| + 1 \notin R, \text{ since } (|i|, j) \in \Gamma_2(w)) \end{cases}. \end{aligned}$$

These observations yield (c)(iii).

The remaining task is to check (c)(iv). Assume $i \in V_-$, and set $j := \rho(i)$. Then,

$$\begin{aligned} \beta_{|i|+1}\langle i\rangle &= \beta_{|i|+1} p(i, j, l) = p(|i| + 1, j + 1, l) \\ &= \begin{cases} \langle -(|i| + 1)\rangle & (\text{if } |i| + 1 \in R, \text{ since } (|i| + 1, j + 1) \in \Gamma_2(w)) \\ \langle |i| + 1\rangle & (\text{if } |i| + 1 \notin R, \text{ since } (|i| + 1, j + 1) \in \Gamma_1(w)) \end{cases}. \end{aligned}$$

Thus, we have obtained (c)(iv).

Now, all the desired properties have been proved. □

We next assume that the type l is not ±1. We can restate Proposition 2.11 as follows.

Lemma 3.15. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type $l \ne \pm 1$, and set c as in Theorem 3.7. If $w(l+1) \le 1$, then set $m := \mathsf{max}\{k \in [l+1, n] \mid w(k) \le 1\}$ and $\varepsilon := (-1)^{m-(l+1)} c$; otherwise, set $\varepsilon := 1$.*

1. *Assume $(i, j) \in \Gamma[l]$. Then, $p_\varepsilon(i, j, l) \notin I(w)$ holds if and only if*

$$\begin{cases} i \ge w(j + 1) & (w(j + 1) \ge 2) \\ i \ge 2 \quad \text{or} \quad i = w(j + 1) & (w(j + 1) = \pm 1) \\ i \ge w(j + 1) + 1 & (w(j + 1) \le -2) \end{cases}.$$

2. *Consider the subset $\Gamma(w) \subset \Gamma[l]$ consisting of the elements $(i, j) \in \Gamma[l]$ with $p_\varepsilon(i, j, l) \notin I(w)$. Then, the set $\{p_\varepsilon(i, j, l) \mid (i, j) \in \Gamma(w)\}$ induces a K-basis of $J(w)$.*

To express $S(w)$, we define the following set for $k \ge 1$:

$$\Gamma_k(w) := \{(i, j) \in \Gamma(w) \mid \mathsf{min}\{x \ge 1 \mid (i, j + x) \notin \Gamma(w)\} = k\}.$$

Moreover, we extend the definition of the path $p_\varepsilon(i, j, l)$ to $\tilde{\Gamma}[l] := \{(i, j) \in \pm Q_0 \times \mathbb{Z} \mid i \le j \ge l\}$ by setting $p_\varepsilon(i, j, l) := 0$ if $j \ge n$, and define $w(k) := k$ if $k \ge n + 1$.

Then, straightforward calculation yields the relation

$$p_{-\varepsilon}(i, j, l) = -p_\varepsilon(i, j, l) + p_\varepsilon(|i|, j + |i| - 1, l) \tag{3.4}$$

for $(i, j) \in \tilde{\Gamma}[l]$ with $i \le -2$.

Lemma 3.16. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type $l \ne \pm 1$. Set R, a, b, r, c as in Theorem 3.7, and ε as in Lemma 3.15.*

(1) *Consider the endomorphisms*

$$f_1 := (\cdot p_\varepsilon(l,l+1,l))\colon J(w) \to J(w) \quad \textit{and} \quad f_2 := (\cdot p_\varepsilon(-l,l,l))\colon J(w) \to J(w).$$

Then, $S(w) = \operatorname{Ker} f_1 \cap \operatorname{Ker} f_2$ *holds.*

(2) *Let* $(i,j) \in \Gamma(w)$*. Then,* $p_{-\varepsilon}(i,j,l) \in \operatorname{Ker} f_1$ *holds if and only if* $(i,j) \in \Gamma_1(w)$*.*

(3) *The set* $\{p_{-\varepsilon}(i,j,l) \mid (i,j) \in \Gamma_1(w)\}$ *induces a* K*-basis of* $\operatorname{Ker} f_1$*.*

(4) *Set* $\Lambda_1(w) := \{(i,j) \in \Gamma_1(w) \mid a-1 \geq i\}$*. The set* $\{p_{-\varepsilon}(i,j,l) \mid (i,j) \in \Lambda_1(w)\}$ *induces a* K*-basis of* $S(w)$*.*

(5) *Assume* $b \leq -2$ *and* $r \geq 1$*, and let* $(i,j) \in \Gamma_1(w)$ *with* $-2 \geq i \geq b+1$*. Then,* $p_{-\varepsilon}(|i|, j+|i|-1, l) \notin \operatorname{Ker} f_1$ *holds if and only if* $|i| \leq r$*. In this case,* $(|i|, j+|i|-1)$ *belongs to* $\Gamma_2(w)$*.*

(6) *The submodule* $\operatorname{Ker} f_1 \cap \operatorname{Ker} f_2$ *has a basis formed by*
  - $p_\varepsilon(i,j,l)$ *for each* $(i,j) \in \Lambda_1(w)$ *with* $i \geq -1$ *and* $-r-1 \geq i$*; and*
  - $p_{-\varepsilon}(i,j,l)$ *for each* $(i,j) \in \Lambda_1(w)$ *with* $-2 \geq i \geq -r$*.*

Proof. The part (1) is clear.

The proofs of (2) and (3) are similar to Lemma 3.5. We remark that $f_1(p_{-\varepsilon}(i,j,l)) = p_\varepsilon(i,j+1,l)$ holds in Π.

(4) Let $(i,j) \in \Gamma_1(w)$. We show that $p_{-\varepsilon}(i,j,l) \in \operatorname{Ker} f_2$ holds if and only if $i \leq w(l) - 1$.

We first assume that $i \geq 2$. In this case, $f_2(p_{-\varepsilon}(i,j,l)) = p_{-\varepsilon}(i,j,l)p_\varepsilon(-l,l,l) = p_\varepsilon(-i,l+j-i,l)$ hold. Thus, $f_2(p_{-\varepsilon}(i,j,l)) = 0$ in $J(w)$ holds if and only if $p_\varepsilon(-i,l+j-i,l) \in \Pi$ belongs to $I(w)$. This is equivalent to $w(l+j-i+1)+1 > -i$ by Lemma 3.15, and also to $\#(R \cap [-n,-i-1]) < j-i+1$.

On the other hand, $(i,j) \in \Gamma_1(w)$ gives $w(j+2)-1 \geq i \geq w(j+1)$, because $i \geq 2$. This implies that $\#(R \cap [-n,i]) = j+1-l$.

Therefore, $f_2(p_{-\varepsilon}(i,j,l)) = 0$ in $J(w)$ holds if and only if $\#(R \cap [-i,i]) > i-l$. This condition is equivalent to that $\#(w([1,l]) \cap [-i,i]) < l$. This exactly means that there exists some $k \in [1,l]$ such that $|w(k)| > i$, and it is equivalent to $a > i$.

Now, the proof for $i \geq 2$ is complete.

Next, we assume that $i = \pm 1$. We must show $p_{-\varepsilon}(i,j,l) \in \operatorname{Ker} f_2$. In this case,

$$\begin{aligned} f_2(p_{-\varepsilon}(i,j,l)) = p_{-\varepsilon}(i,j,l)p_\varepsilon(-l,l,l) &= \alpha_i p_\varepsilon(2,j,l)p_\varepsilon(-l,l,l) = \alpha_i p_\varepsilon(-2,l+j-2,l) \\ &= \begin{cases} p_\varepsilon(i,l+j-1,l) & (i = \varepsilon(-1)^j) \\ 0 & (i = -\varepsilon(-1)^j) \end{cases}, \end{aligned}$$

since $p_\varepsilon(-2,l+j-2,l)$ factors through $\varepsilon(-1)^{j-1}$.

Thus, we may assume $i = \varepsilon(-1)^j$. First, $p_\varepsilon(i,l+j-1,l) \in I(w)$ is equivalent to that ($w(l+j) \geq 2$ or $w(l+j) = -i$) by Lemma 3.15. On the other hand, $(i,j) \in \Gamma_1(w)$ implies that $w(j+2) \geq 2$ or $w(j+2) = -i$. Since $l \geq 2$, we have ($w(l+j) \geq 2$ or $w(l+j) = -i$), and $p_\varepsilon(i,l+j-1,l) \in I(w)$.

Consequently, $i = \pm 1$ implies that $p_{-\varepsilon}(i,j,l) \in \operatorname{Ker} f_2$.

Finally, we assume that $i \leq -2$. Then, $p_{-\varepsilon}(i,j,l) \in \operatorname{Ker} f_2$ holds, because the path $p_{-\varepsilon}(i,j,l)$ has $p_{-\varepsilon}(1,j,l)$ or $p_{-\varepsilon}(-1,j,l)$ in its ending.

Now, we have proved that $p_{-\varepsilon}(i,j,l) \in \operatorname{Ker} f_2$ holds if and only if $i \leq a-1$, and obtained that $f_2(p_{-\varepsilon}(i,j,l)) = p_\varepsilon(-i,l+j-i,l) \neq 0$ in $J(w)$ if $(i,j) \in \Gamma_1(w)$ and $i \geq a$.

Thus, the set $\{f_2(p_{-\varepsilon}(i,j,l)) \mid (i,j) \in \Gamma_1(w) \setminus \Lambda_1(w)\}$ is linearly independent in $J(w)$, so $\{p_{-\varepsilon}(i,j,l) \mid (i,j) \in \Lambda_1(w)\}$ generates $\operatorname{Ker} f_1 \cap \operatorname{Ker} f_2$. This set is clearly linearly independent in $J(w)$. Therefore, we obtain the assertion from (1).

(5) Let $(i,j) \in \Gamma_1(w)$ with $-2 \geq i \geq b+1$.

For the first statement, it is easy to see that $f_1(p_{-\varepsilon}(|i|,j+|i|-1,l)) = p_\varepsilon(|i|,j+|i|,l)$ in Π, so $p_{-\varepsilon}(|i|,j+|i|-1,l) \notin \operatorname{Ker} f_1$ precisely means $p_\varepsilon(|i|,j+|i|,l) \notin I(w)$ in Π. Lemma 3.15 yields that this holds if and only if $w(j+|i|+1) \leq |i|$, because $|i| \geq 2$. It is equivalent to $\#(R \cap [-n,|i|]) \geq j+|i|+1-l$.

On the other hand, $(i,j) \in \Gamma_1(w)$ gives $w(j+2) \geq i = -|i| \geq w(j+1)+1$, because $i \leq -2$. This implies that $\#(R \cap [-n, -|i|-1]) = j+1-l$.

Therefore, $f_1(p_{-\varepsilon}(|i|, j+|i|-1, l)) \neq 0$ in $J(w)$ holds if and only if $\#(R \cap [-|i|, |i|]) \geq |i|$. This exactly means $[1, |i|] \subset \pm R$. By the definition of the number r, it is equivalent to $|i| \leq r$. In this case, $\#(R \cap [-|i|, |i|]) = |i|$.

The first statement has been proved.

Next, we show the second statement, so we assume $|i| \leq r$. It suffices to prove $(|i|, j+|i|) \in \Gamma(w)$ and $(|i|, j+|i|+1) \notin \Gamma(w)$. We already have $\#(R \cap [-|i|, |i|]) = |i|$, and by the argument above, this yields $\#(R \cap [-n, |i|]) = j+|i|+1-l$. Thus, we have $w(j+|i|+1) \leq |i|$ and $w(j+|i|+2) > |i|$. Since $|i| \geq 2$, Lemma 3.15 implies that $(|i|, j+|i|) \in \Gamma(w)$ and $(|i|, j+|i|+1) \notin \Gamma(w)$. Thus, $(|i|, j+|i|-1)$ belongs to $\Gamma_2(w)$.

(6) In (4), $p_{-\varepsilon}(i,j,l) = p_\varepsilon(i,j,l)$ holds for each $(i,j) \in \Lambda_1(w)$ with $i \geq -1$.

On the other hand, let $(i,j) \in \Lambda_1(w)$ with $i < -r$ and $i \leq -2$. Clearly, $p_{-\varepsilon}(i,j,l) \in \mathsf{Ker}\, f_1$. By (5), we have $p_{-\varepsilon}(|i|, j+|i|-1, l) \in \mathsf{Ker}\, f_1$.

If $p_{-\varepsilon}(|i|, j+|i|-1, l) \neq 0$ in $J(w)$, then $\#(R \cap [-|i|, |i|]) \geq |i|-1$ follows from similar argument to the proof of the first statement of (5). This implies $|i| < a$, since $l \geq 2$. We have $(|i|, j+|i|-1, l) \in \Lambda_1(w)$. Thus, in the K-basis of $\mathsf{Ker}\, f_1 \cap \mathsf{Ker}\, f_2$ given in (4), we can replace $p_{-\varepsilon}(i,j,l)$ to $p_\varepsilon(i,j,l)$ to obtain another K-basis of $S(w)$.

If $p_{-\varepsilon}(|i|, j+|i|-1, l) = 0$ in $J(w)$, then $p_{-\varepsilon}(i,j,l) = p_\varepsilon(i,j,l)$ holds.

We repeat this procedure, and get that the elements in the statement form a K-basis of $\mathsf{Ker}\, f_1 \cap \mathsf{Ker}\, f_2$. ☐

The next assertion follows from the definition of $\Lambda_1(w)$.

Lemma 3.17. *Let $w \in \mathsf{j\text{-}irr}\, W$ be a join-irreducible element of type $l \neq \pm 1$. Then, there exists a bijection $\Lambda_1(w) \to V$ given by $(i,j) \mapsto i$.*

Proof. The well-definedness can be checked by Lemma 3.16.

We clearly have $\max\{k \in [l+1, n] \mid w(k) < k\} - 1 \geq a-1$. Then, Lemma 3.15 and the definition of V yield that, for any $i \in V$, there exists some j such that $(i,j) \in \Gamma(w)$. Thus, the definition of $\Lambda_1(w)$ and $i \leq a-1$ imply that there uniquely exists j such that $(i,j) \in \Lambda_1(w)$. This means that the map $\Lambda_1(w) \to V$ is bijective. ☐

Now, we show Theorem 3.7 in the case $l \neq \pm 1$.

Proof. By Lemma 3.16, we can define a map $\rho\colon V \to Q_0$ as follows: $\rho(i)$ is the unique element $j \in Q_0$ such that $(i,j) \in \Gamma_1(w)$. Set $\langle i \rangle := p(i, \rho(i), l)$ for each $i \in V$. It suffices to show that $(\langle i \rangle)_{i \in V}$ satisfies the properties (a), (b), and (c), since the three properties are enough to define a Π-module.

First, $(\langle i \rangle)_{i \in V}$ is a K-basis of $S(w)$ by Lemma 3.17, and $K\langle i \rangle$ is clearly a subspace of $e_i S(w)$ if $i \geq -1$; and of $e_{|i|} S(w)$ if $i \leq -2$. Thus, the property (a) has been obtained, and the property (b) follows from (a).

In the rest, we fully use Lemma 3.15.

We begin the proof of (c)(i). First, we assume $2 \in V_+$, and set $j := \rho(2)$.

- If $2 \notin R$ and $r \geq 1$, then $w(j+1) = c$. Thus, $(c,j) \in \Lambda_1(w)$ and $(-c,j) \notin \Gamma(w)$ follow.
- If $2 \notin R$ and $r = 0$, then $w(j+1) \leq -2$. Thus, $(c,j), (-c,j) \in \Lambda_1(w)$ follows.
- If $2 \in R$, then $w(j+1) = 2$. Thus, $(c,j), (-c,j) \notin \Gamma(w)$ follows.

Therefore,

$$\alpha_1 \langle 2 \rangle = \alpha_1 p_\varepsilon(2,j,l) = p_\varepsilon(c,j,l) + p_\varepsilon(-c,j,l) = \begin{cases} \langle c \rangle & (2 \notin R,\ r \geq 1) \\ \langle c \rangle + \langle -c \rangle & (2 \notin R,\ r = 0) \\ 0 & (2 \in R) \end{cases}.$$

Second, we assume $i \in V_+ \setminus \{\max V_+\}$ and $i \geq 2$, and set $j := \rho(i+1)$. Then,

$$\alpha_i\langle i+1\rangle = \alpha_i p_\varepsilon(i+1,j,l) = p_\varepsilon(i,j,l) = \begin{cases} \langle i\rangle & (\text{if } i+1 \notin R, \text{ since } (i,j) \in \mathit{\Gamma}_1(w)) \\ 0 & (\text{if } i+1 \in R, \text{ since } (i,j) \notin \mathit{\Gamma}(w)) \end{cases}.$$

Thus, we have the property (c)(i).

We begin the proof of (c)(ii). First, let $c \in V_+ \setminus \{\max V_+\}$, and set $j := \rho(c)$. In this case, $w(j+1) \leq 1$, $w(j+2) \geq 2$, and $\varepsilon = (-1)^{j-l}c$ hold, so the path $p_\varepsilon(-2,j,l)$ factors through $-c$. We observe the following properties.

- If $-2 \notin R$ and $r = 0$, then $(-2,j) \in \mathit{\Lambda}_1(w)$; otherwise $(-2,j) \notin \mathit{\Gamma}(w)$.
- If $2 \in R$, then $(2,j+1) \in \mathit{\Lambda}_1(w)$; otherwise $(2,j+1) \notin \mathit{\Gamma}(w)$.

Thus, we have

$$\beta_2\langle c\rangle = \beta_2 p_\varepsilon(c,j,l) = p_{-\varepsilon}(-2,j,l) = -p_\varepsilon(-2,j,l) + p_\varepsilon(2,j+1,l) = \eta_c^-\langle -2\rangle + \eta_c^+\langle 2\rangle.$$

Second, let $i \in V_+ \setminus \{\max V_+\}$ and $i \geq 2$, and set $j := \rho(i)$. Then,

$$\begin{aligned}\beta_{|i|+1}\langle i\rangle &= \beta_{|i|+1} p_\varepsilon(i,j,l) = p_\varepsilon(|i|+1,j+1,l) \\ &= \begin{cases} 0 & (\text{if } i+1 \notin R, \text{ since } (|i|+1,j+1) \notin \mathit{\Gamma}(w)) \\ \langle |i|+1\rangle & (\text{if } i+1 \in R, \text{ since } (|i|+1,j+1) \in \mathit{\Gamma}_1(w)) \end{cases}.\end{aligned}$$

These observations imply the property (c)(ii).

We next consider the elements in V_-. In order to observe the actions of the arrows to $\langle -i\rangle$ ($i \in [2,r]$), we define sets $\mathit{\Omega}(w)$ and $\mathit{\Lambda}_2(w)$ as

$$\mathit{\Omega}(w) := \{(i,j) \in \mathit{\Lambda}_1(w) \mid i \in [-r,-2]\}, \quad \mathit{\Lambda}_2(w) := \{(i,j) \in \mathit{\Gamma}_2(w) \mid i \in [2,r]\}.$$

The element $\langle -i\rangle$ is equal to the path $p_{-\varepsilon}(i,j,l)$ with $(i,j) \in \mathit{\Omega}(w)$, but we want to deal with the paths of the form $p_\varepsilon(i',j',l)$. In the formula (3.4), $p_{-\varepsilon}(i,j,l)$ is a linear combination of $p_\varepsilon(i,j,l)$ and $p_\varepsilon(|i|,j+|i|-1,l)$. By Lemma 3.16 (5), $(|i|,j+|i|-1)$ belongs to $\mathit{\Lambda}_2(w)$. Moreover, $\varphi\colon \mathit{\Omega}(w) \ni (i,j) \mapsto (|i|,j+|i|-1) \in \mathit{\Lambda}_2(w)$ is a bijection.

In the figure for $J(w)$ in Example 3.9, the squares with positive blue numbers are the elements of $\mathit{\Lambda}_2(w)$, and that the squares with negative blue numbers are the elements of $\mathit{\Omega}(w)$.

Now, we begin the proof of (c)(iii). We first assume $-c \in V_- \setminus \{\min V_-\}$, and set $j := \rho(-2)$.

- If $-2 \in R$ and $r \geq 2$, then $w(j+1) \leq -3$, $w(j+2) = -2$, $w(j+3) = c$, $w(j+4) \geq 3$, and $\varepsilon = (-1)^{j+2-l}c$ hold, so the path $p_\varepsilon(-2,j,l)$ factors through $-c$, and $(-c,j+1) \in \mathit{\Lambda}_1(w)$ follows. Thus,

$$\begin{aligned}\alpha_1\langle -2\rangle &= \alpha_1 p_{-\varepsilon}(-2,j,l) = -\alpha_1 p_\varepsilon(-2,j,l) + \alpha_1 p_\varepsilon(2,j+1,l) \\ &= -p_\varepsilon(c,j+1,l) + (p_\varepsilon(c,j+1,l) + p_\varepsilon(-c,j+1,l)) \\ &= p_\varepsilon(-c,j+1,l) = \langle -c\rangle.\end{aligned}$$

- If $-2 \in R$ and $r = 0$, then $w(j+1) \leq -3$, $w(j+2) = -2$, $w(j+3) \geq 3$, and $\varepsilon = (-1)^{j+1-l}c$ hold, so the path $p_\varepsilon(-2,j,l)$ factors through c, and $(-c,j+1) \in \mathit{\Lambda}_1(w)$ follows. Thus,

$$\alpha_1\langle -2\rangle = \alpha_1 p_\varepsilon(-2,j,l) = p_\varepsilon(-c,j+1,l) = \langle -c\rangle.$$

- If $-2 \notin R$ and $r \geq 2$, then $w(j+1) \leq -3$, $w(j+2) = c$, $w(j+3) = 2$, and $\varepsilon = (-1)^{j+1-l}c$ hold, so the path $p_\varepsilon(-2,j,l)$ factors through c, and $(c,j+1) \in \mathit{\Lambda}_1(w)$ follows. Thus,

$$\begin{aligned}\alpha_1\langle -2\rangle &= \alpha_1 p_{-\varepsilon}(-2,j,l) = -\alpha_1 p_\varepsilon(-2,j,l) + \alpha_1 p_\varepsilon(2,j+1,l) \\ &= -p_\varepsilon(-c,j+1,l) + (p_\varepsilon(c,j+1,l) + p_\varepsilon(-c,j+1,l)) \\ &= p_\varepsilon(c,j+1,l) = \langle c\rangle.\end{aligned}$$

- If $-2 \notin R$ and $r = 1$, then $w(j+1) \leq -3$, $w(j+2) = c$, $w(j+3) \geq 3$, and $\varepsilon = (-1)^{j+1-l}c$ hold, so the path $p_\varepsilon(-2, j, l)$ factors through c, and $(-c, j+1) \notin \mathit{\Gamma}(w)$ follows. Thus,
$$\alpha_1\langle -2\rangle = \alpha_1 p_\varepsilon(-2, j, l) = p_\varepsilon(-c, j+1, l) = 0.$$
- If $-2 \notin R$ and $r = 0$, then $w(j+1) \leq -3$, $w(j+2) \geq 2$, and $\varepsilon = (-1)^{j-l}c$ hold, so the path $p_\varepsilon(-2, j, l)$ factors through $-c$, and $(c, j+1) \notin \mathit{\Gamma}(w)$ follows. Thus,
$$\alpha_1\langle -2\rangle = \alpha_1 p_\varepsilon(-2, j, l) = p_\varepsilon(c, j+1, l) = 0.$$

Second, we assume $i \in V_- \setminus \{\min V_-\}$ and $|i| \geq 2$, and take the unique j such that $(-(|i|+1), j) \in \mathit{\Lambda}_1(w)$, and observe the action of the arrow α_i for $\langle -(|i|+1)\rangle \in S(w)$.

- If $|i| < r$, then $(-(|i|+1), j) \in \mathit{\Omega}(w)$ and $\varphi(-(|i|+1), j) = (|i|+1, j+|i|) \in \mathit{\Lambda}_2(w)$ hold, and
$$\begin{aligned}\alpha_{|i|}\langle -(|i|+1)\rangle &= \alpha_{|i|} p_{-\varepsilon}(-(|i|+1), j, l) = -\alpha_{|i|} p_\varepsilon(-(|i|+1), j, l) + \alpha_{|i|} p_\varepsilon(|i|+1, j+|i|, l)\\ &= -p_\varepsilon(-|i|, j+1, l) + p_\varepsilon(|i|, j+|i|, l)\\ &= \begin{cases}\langle -|i|\rangle & (\text{if } -(|i|+1) \in R, \text{ since } (-|i|, j+1) \in \mathit{\Omega}(w))\\ p_\varepsilon(|i|, j+|i|, l) & (\text{if } -(|i|+1) \notin R, \text{ since } (-|i|, j+1) \notin \mathit{\Gamma}(w))\end{cases}\\ &= \begin{cases}\langle i\rangle & (\text{if } -(|i|+1) \in R)\\ \langle -i\rangle & (\text{if } -(|i|+1) \notin R, \text{ since } |i|+1 \in R \text{ and } (|i|, j+|i|) \in \mathit{\Lambda}_1(w))\end{cases}.\end{aligned}$$
- If $|i| \geq r$, then
$$\begin{aligned}\alpha_i\langle -(|i|+1)\rangle &= \alpha_i p_\varepsilon(-(|i|+1), j, l) = p_\varepsilon(-|i|, j+1, l)\\ &= \begin{cases}\langle -|i|\rangle = \langle i\rangle & (\text{if } -(|i|+1) \in R, \text{ since } (-|i|, j+1) \in \mathit{\Lambda}_1(w))\\ 0 & (\text{if } -(|i|+1) \notin R, \text{ since } (-|i|, j+1) \notin \mathit{\Gamma}(w))\end{cases}.\end{aligned}$$

These observations and the definition of r tell us that (c)(iii) holds.

Finally, we would like to show the property (c)(iv). First, we assume $-c \in V_-$, and set $j := \rho(-c)$.

- If $r \geq 2$, then $w(j+1) \leq -2$, $w(j+2) = c$, $w(j+3) \geq 2$, and $\varepsilon = (-1)^{j+1-l}c$ hold, so the path $p_\varepsilon(-2, j, l)$ factors through c. Thus,
$$\begin{aligned}\beta_2\langle -c\rangle &= \beta_2 p_\varepsilon(-c, j, l) = -p_\varepsilon(-2, j, l) + p_\varepsilon(2, j+1, l)\\ &= \begin{cases}p_\varepsilon(2, j+1, l) & (\text{if } -2 \in R, \text{ since } (-2, j) \notin \mathit{\Gamma}(w))\\ p_{-\varepsilon}(-2, j, l) & (\text{if } -2 \notin R)\end{cases}\\ &= \begin{cases}\langle 2\rangle & (\text{if } -2 \in R, \text{ since } 2 \notin R \text{ and } (2, j+1) \in \mathit{\Lambda}_1(w))\\ \langle -2\rangle & (\text{if } -2 \notin R, \text{ since } (-2, j) \in \mathit{\Omega}(w))\end{cases}.\end{aligned}$$
- If $r = 1$, the path $p_\varepsilon(-2, j, l)$ factors through c by the same reason. Since $r = 1$, we have $-2, 2 \notin R$, so $(-2, j), (2, j+1) \in \mathit{\Lambda}_1(w)$. Thus,
$$\beta_2\langle -c\rangle = \beta_2 p_\varepsilon(-c, j, l) = -p_\varepsilon(-2, j, l) + p_\varepsilon(2, j+1, l) = -\langle -2\rangle + \langle 2\rangle.$$
- If $r = 0$, then $w(j+1) \leq -2$, $w(j+2) \geq 2$, and $\varepsilon = (-1)^{j-l}c$ hold, so the path $p_\varepsilon(-2, j, l)$ factors through $-c$. Thus,
$$\begin{aligned}\beta_2\langle -c\rangle &= \beta_2 p_\varepsilon(-c, j, l) = p_\varepsilon(-2, j, l)\\ &= \begin{cases}0 & (\text{if } -2 \in R, \text{ since } (-2, j) \notin \mathit{\Gamma}(w))\\ \langle -2\rangle & (\text{if } -2 \notin R, \text{ since } (-2, j) \in \mathit{\Lambda}_1(w))\end{cases}.\end{aligned}$$

Second, we assume $i \in V_-$, $|i| \geq 2$, and set $j := \rho(2)$. We observe the action of the element β_2 for $\langle -i\rangle \in S(w)$.

- If $|i| < r$, then $(i,j) \in \Omega(w)$ and $\varphi(i,j) = (|i|, j+|i|-1) \in \Lambda_2(w)$ hold, so

$$\begin{aligned}\beta_{|i|+1}\langle i\rangle &= \beta_{|i|+1}p_{-\varepsilon}(i,j,l) = -\beta_{|i|+1}p_{\varepsilon}(i,j,l) + \beta_{|i|+1}p_{\varepsilon}(|i|, j+|i|-1, l)\\ &= -p_{\varepsilon}(-(|i|+1), j, l) + p_{\varepsilon}(|i|+1, j+|i|, l)\\ &= \begin{cases} p_{\varepsilon}(|i|+1, j+|i|, l) & (\text{if } -(|i|+1) \in R, \text{ since } (-(|i|+1), j) \notin \Gamma(w))\\ p_{-\varepsilon}(-(|i|+1), j, l) & (\text{if } -(|i|+1) \notin R)\end{cases}\\ &= \begin{cases} \langle |i|+1\rangle & (\text{if } -(|i|+1) \in R, \text{ since } |i|+1 \notin R \text{ and } (|i|+1, j+|i|) \in \Lambda_1(w))\\ \langle -(|i|+1)\rangle & (\text{if } -(|i|+1) \notin R, \text{ since } (-(|i|+1), j) \in \Omega(w))\end{cases}.\end{aligned}$$

- If $|i| = r$, then $(i,j) \in \Omega(w)$ and $\varphi(i,j) = (|i|, j+|i|-1) \in \Lambda_2(w)$ hold. Since $|i| = r$, we have $-(|i|+1), |i|+1 \notin R$, so $(|i|+1, j+|i|), (-(|i|+1), j) \in \Lambda_1(w)$ hold. Thus,

$$\begin{aligned}\beta_{|i|+1}\langle i\rangle &= \beta_{|i|+1}p_{-\varepsilon}(i,j,l) = -\beta_{|i|+1}p_{\varepsilon}(i,j,l) + \beta_{|i|+1}p_{\varepsilon}(|i|, j+|i|-1, l)\\ &= -p_{\varepsilon}(-(|i|+1), j, l) + p_{\varepsilon}(|i|+1, j+|i|, l) = -\langle -(|i|+1)\rangle + \langle |i|+1\rangle.\end{aligned}$$

- If $|i| > r$, then

$$\begin{aligned}\beta_{|i|+1}\langle i\rangle &= \beta_{|i|+1}p_{\varepsilon}(i,j,l) = p_{\varepsilon}(-(|i|+1), j, l)\\ &= \begin{cases} 0 & (\text{if } -(|i|+1) \in R, \text{ since } (-(|i|+1), j) \notin \Gamma(w))\\ \langle -(|i|+1)\rangle & (\text{if } -(|i|+1) \notin R, \text{ since } (-(|i|+1), j) \in \Lambda_1(w))\end{cases}.\end{aligned}$$

The property (c)(iv) follows from these observations and the definition of r.

Now, all the proof is complete. □

4. Description of semibricks

4.1. **Canonical join representations in Coxeter groups.** Let Δ be a Dynkin diagram $\mathbb{A}_n$ or $\mathbb{D}_n$, and Π and W be the corresponding preprojective algebra and the Coxeter group, respectively. We obtained a canonical bijection $S(?)\colon W \to \mathsf{sbrick}\,\Pi$ in Proposition 2.2. The aim of this section is to give the explicit description of this map. In the previous section, this aim has been achieved for the restricted bijection $S(?)\colon \mathsf{j\text{-}irr}\,W \to \mathsf{brick}\,\Pi$. To extend this to all elements in W, it is enough to determine the canonical join representations in W for $\Delta = \mathbb{A}_n, \mathbb{D}_n$.

It would be difficult to prove that a set of join-irreducible elements gives a canonical join representation of a given element in W by directly checking the conditions in Definition 1.7. Fortunately, Reading [Rea] has obtained a nice property characterizing canonical join representations in finite Coxeter groups. To explain this, we prepare some notation.

Let Δ_0 be the vertices set of Δ. Then, W has the canonical generators $\{s_i \mid i \in \Delta_0\}$. For each $w \in W$, set $\mathsf{des}(w)$ and $\mathsf{cov}(w)$ as the set of *descents* and the set of *cover reflections* of w, respectively: that is,

$$\mathsf{des}(w) := \{i \in \Delta_0 \mid ws_i < w\}, \quad \mathsf{cov}(w) := \{ws_iw^{-1} \mid i \in \mathsf{des}(w)\}.$$

There exists a natural bijection $\mathsf{des}(w) \to \mathsf{cov}(w)$ defined by $i \mapsto ws_iw^{-1}$. By using the set $\mathsf{cov}(w)$, we can write the canonical join representation of w as follows.

Proposition 4.1. [Rea, Theorem 10-3.9] *Let $w \in W$. For each $t \in \mathsf{cov}(w)$, the set $\{v \in W \mid v \leq w,\ t \in \mathsf{inv}(v)\}$ has a unique minimal element w_t. Moreover, $\bigvee_{t \in \mathsf{cov}(w)} w_t$ is the canonical join representation of w.*

Hence, we have the following way to find canonical join representations.

Proposition 4.2. *Let $w \in W$. Assume that, for each $d \in \mathsf{des}(w)$, there exists a join-irreducible element $w_d \in \mathsf{j\text{-}irr}\,W$ satisfying $w_d \leq w$ and $\mathsf{cov}(w_d) = \{ws_dw^{-1}\}$. Then, $\bigvee_{d \in \mathsf{des}(w)} w_d$ is the canonical join representation of w.*

Proof. Let $d \in \mathsf{des}(w)$ and set $t := ws_dw^{-1} \in \mathsf{cov}(w)$. By Proposition 4.1, it suffices to show that w_d is a minimal element of $V := \{v \in W \mid v \leq w,\ t \in \mathsf{inv}(v)\}$. We assume that $v \in V$ satisfies $v < w_d$ and deduce a contradiction. Take the unique descent d' of $w_d \in \mathsf{j\text{-}irr}\, W$, then $t = w_d s_{d'} w_d^{-1}$ holds.

Since $w_d s_{d'} = tw_d$ and $d' \in \mathsf{des}(w_d)$, we get $l(t \cdot w_d s_{d'}) = l(t \cdot tw_d) = l(w_d) > l(w_d s_{d'})$. Thus, $t \notin \mathsf{inv}(w_d s_{d'})$.

On the other hand, the inequality $v \leq w_d s_{d'}$ holds, since w_d is a join-irreducible element with its unique descent d'. Thus, we have $\mathsf{inv}(v) \subset \mathsf{inv}(w_d s_{d'})$. By assumption, t belongs to $\mathsf{inv}(v)$, so t must be in $\mathsf{inv}(w_d s_{d'})$.

These two results contradict to each other. Thus, there exists no $v \in V$ such that $v < w_d$. This exactly means that w_d is a minimal element of V. $\square$

Before proceeding to the next subsection, we give an example of canonical join representations. We recall that the *Hasse quiver* of W is defined as follows.

- The vertices are the elements of W.
- For any $w, w' \in W$, we write an arrow $w \to w'$ if and only if $w > w'$ holds and there exists no $v \in W$ such that $w > v > w'$.

Example 4.3. Let $\mathit{\Delta} = \mathbb{A}_3$. Then, the Hasse quiver of W is

$$\begin{tikzcd}[column sep=tiny]
 & & & & & (4,3,2,1) \arrow[dll] \arrow[d] \arrow[drr] & & & & & \\
 & & & (4,3,1,2)^{\color{red}*}_{\color{blue}*} \arrow[dll] \arrow[drr] & & (4,2,3,1) \arrow[dll] \arrow[drr] & & (3,4,2,1) \arrow[dll] \arrow[drr] & & & \\
 & (4,1,3,2)^{\color{red}*} \arrow[dl] \arrow[dr] & & (4,2,1,3) \arrow[dlll] \arrow[dr] & & (3,4,1,2)_{\color{blue}*} \arrow[dr] & & (2,4,3,1) \arrow[dlll] \arrow[drrr] & & (3,2,4,1) \arrow[dl] \arrow[dr] & \\
(4,1,2,3)^{\color{red}*} \arrow[dr] & & (1,4,3,2)^{\color{red}*} \arrow[dl] \arrow[dr] & & (2,4,1,3) \arrow[dr] & & (3,1,4,2)_{\color{blue}*} \arrow[dlll] \arrow[dr] & & (3,2,1,4) \arrow[dl] \arrow[dr] & & (2,3,4,1)\ . \arrow[dl] \\
 & (1,4,2,3)^{\color{red}*} \arrow[drr] & & (1,3,4,2) \arrow[drr] & & (2,1,4,3) \arrow[dll] \arrow[drr] & & (3,1,2,4)_{\color{blue}*} \arrow[dll] & & (2,3,1,4) \arrow[dll] & \\
 & & & (1,2,4,3)^{\color{red}*} \arrow[drr] & & (1,3,2,4) \arrow[d] & & (2,1,3,4) \arrow[dll] & & & \\
 & & & & & (1,2,3,4) & & & & &
\end{tikzcd}$$

We determine the canonical join representation of the element $w := (4,3,1,2)$ from the Hasse quiver. In this case, we have $\mathsf{des}(w) = \{1,2\}$ and $\mathsf{cov}(w) = \{(4\quad 3), (3\quad 1)\}$. Thus, we consider the following sets:

- $\{v \in W \mid v \leq w,\ (4\quad 3) \in \mathsf{inv}(v)\}$, whose elements are indicated by ${}^{\color{red}*}$; and
- $\{v \in W \mid v \leq w,\ (3\quad 1) \in \mathsf{inv}(v)\}$, whose elements are indicated by ${}_{\color{blue}*}$.

These sets have $(1,2,4,3)$ and $(3,1,2,4)$ as their unique minimum elements, respectively. By Proposition 4.1, the canonical join representation of w is $(1,2,4,3) \vee (3,1,2,4)$. We also remark that $\mathsf{cov}((1,2,4,3)) = \{(4\quad 3)\}$ and $\mathsf{cov}((3,1,2,4)) = \{(3\quad 1)\}$ hold.

4.2. **Type $\mathbb{A}_n$.** Let $\mathit{\Delta} = \mathbb{A}_n$. For each element w in $\mathsf{j\text{-}irr}\, W$ of type l, we set

$$L(w) := w([1,l]), \quad R(w) := w([l+1, n+1]).$$

It is easy to see that the correspondence $w \mapsto R(w)$ is injective.

The following procedure gives the canonical join representation of a given element of the Coxeter group W. This coincides with [Rea, Theorem 10-5.6].

Proposition 4.4. *Let $w \in W$, and set $a_d := w(d)$, $b_d := w(d+1)$ for each $d \in \mathsf{des}(w)$. Then, the canonical join representation of w is $\bigvee_{d \in \mathsf{des}(w)} w_d$, where $w_d \in \mathsf{j\text{-}irr}\, W$ is the unique join-irreducible element such that $R(w_d)$ coincides with R_d defined as follows:*

$$X_d := w([d+1, n+1]), \quad R_d := ([b_d, a_d - 1] \cap X_d) \cup [a_d + 1, n+1].$$

Proof. Let $d \in \mathsf{des}(w)$. It is easy to see that there uniquely exists $w_d \in \mathsf{j\text{-}irr}\, W$ with $R(w_d) = R_d$. Then, $L(w_d) = [1, b_d - 1] \cup ([b_d + 1, a_d] \setminus X_d)$. From this, we can straightforwardly check that $\mathsf{inv}(w_d) \subset \mathsf{inv}(w)$, which is equivalent to $w_d \le w$. Moreover, the unique cover reflection of w_d is $(a_d\ \ b_d)$, and it is equal to ws_dw^{-1}. Therefore, the assertion follows from Proposition 4.2. □

Example 4.5. Let $n := 8$ and $w := (4, 9, 3, 6, 2, 8, 5, 1, 7)$. Then, we have $\mathsf{des}(w) = \{2, 4, 6, 7\}$. The canonical join representation of w is $\bigvee_{d \in \mathsf{des}(w)} w_d$, where w_d is given as follows for each $d \in \mathsf{des}(w)$.

d	a_d	b_d	$R(w_d)$	w_d
2	9	3	$\{3, 5, 6, 7, 8\}$	$(1, 2, 4, 9, 3, 5, 6, 7, 8)$
4	6	2	$\{2, 5, 7, 8, 9\}$	$(1, 3, 4, 6, 2, 5, 7, 8, 9)$
6	8	5	$\{5, 7, 9\}$	$(1, 2, 3, 4, 6, 8, 5, 7, 9)$
7	5	1	$\{1, 6, 7, 8, 9\}$	$(2, 3, 4, 5, 1, 6, 7, 8, 9)$

Combining Corollary 3.3 and Proposition 4.4, we can obtain the semibrick $S(w)$ directly.

Theorem 4.6. *Let $w \in W$. Then, the semibrick $S(w)$ is $\bigoplus_{d \in \mathsf{des}(w)} S_d$, where S_d is the brick whose abbreviated description as in Corollary 3.3 is given as follows.*

- *Set R_d as in Proposition 4.4, and $a_d := w(d)$, $b_d := w(d+1)$, $V_d := [b_d, a_d - 1]$.*
- *The brick S_d has a K-basis $(\langle i \rangle_d)_{i \in V_d}$, where $\langle i \rangle_d$ belongs to $e_i S_d$.*
- *For each $i \in V_d$, place a symbol i denoting the K-vector subspace $K\langle i \rangle_d$.*
- *For each $i \in V_d \setminus \{\max V_d\}$, we write exactly one arrow between i and $i+1$, and the orientation is $i \to i+1$ if $i+1 \in R_d$ and $i \leftarrow i+1$ if $i+1 \notin R_d$.*

Proof. For each $d \in \mathsf{des}(w)$, let w_d be the join-irreducible element in the canonical join representation given in Proposition 4.4. Then, we can check that the abbreviated description of $S(w_d)$ in Corollary 3.3 coincides with the statement. □

In Theorem 4.6, we remark that R_d can be replaced by $R_d \cap V_d = [b_d, a_d - 1] \cap X_d$.

Example 4.7. Let $n := 8$ and $w := (4, 9, 3, 6, 2, 8, 5, 1, 7)$ as in Example 4.5. Then, the semibrick $S(w)$ is the direct sum of the following bricks:

$$\begin{array}{lccccccccccccccl}
S_2 = & & & & & 3 & \leftarrow & 4 & \to & 5 & \to & 6 & \to & 7 & \to 8 & , \\
S_4 = & & & 2 & \leftarrow & 3 & \leftarrow & 4 & \to & 5 & & & & & & , \\
S_6 = & & & & & & & & & 5 & \leftarrow & 6 & \to & 7 & & , \\
S_7 = & 1 & \leftarrow & 2 & \leftarrow & 3 & \leftarrow & 4 & & & & & & & & .
\end{array}$$

4.3. **Type $\mathbb{D}_n$.** Let $\Delta = \mathbb{D}_n$. For each element w in $\mathsf{j\text{-}irr}\, W$ of type l, we set

$$L(w) := \{|w(k)| \mid k \in [1, |l|]\}, \quad R(w) := w([|l| + 1, n]).$$

As in the case of type $\mathbb{A}_n$, it is easy to see that the correspondence $w \mapsto R(w)$ is injective.

The canonical join representations of the elements of the Coxeter group W are given by the following procedure.

Proposition 4.8. *Let $w \in W$, and set $a_d := w(d)$, $b_d := w(|d|+1)$, $X_d := w([|d|+1, n])$ for each $d \in \mathsf{des}(w)$. Then, the canonical join representation of w is $\bigvee_{d \in \mathsf{des}(w)} w_d$, where $w_d \in \mathsf{j\text{-}irr}\, W$ is the unique join-irreducible element such that $R(w_d)$ coincides with R_d defined as follows.*

(A) *If* $a_d + b_d < 0$ *and* $w([1, |d|]) \subset \pm[a_d, n]$, *then*

$$R_d = \begin{cases} \{-a_d\} \cup (\pm[1, a_d - 1] \cap X_d) \cup ([a_d + 1, -b_d - 1] \setminus (-X_d)) \cup [-b_d + 1, n] & (a_d > 0) \\ ([-a_d, -b_d - 1] \setminus (-X_d)) \cup [-b_d + 1, n] & (a_d < 0) \end{cases}.$$

(B) *Otherwise,*

$$R_d = \begin{cases} ([b_d, a_d - 1] \cap X_d) \cup [a_d + 1, n] & (a_d + b_d > 0) \\ ([b_d, a_d - 1] \cap X_d) \cup ([a_d + 1, -b_d - 1] \setminus (-X_d)) \cup [-b_d + 1, n] & (a_d + b_d < 0) \end{cases}$$

Proof. The proof is similar to the one for type $\mathbb{A}_n$. In this case, the set $L(w_d)$ is given as follows.

(A) If $a_d + b_d < 0$ and $w([1, |d|]) \subset \pm[a_d, n]$, then

$$L(w_d) = \begin{cases} [a_d + 1, -b_d] \cap (-X_d) & (a_d > 0) \\ [1, -a_d - 1] \cup ([-a_d + 1, -b_d] \cap (-X_d)) & (a_d < 0) \end{cases}.$$

(B) Otherwise,

$$L(w_d) = \begin{cases} [1, b_d - 1] \cup ([b_d + 1, a_d] \setminus X_d) & (b_d > 0) \\ ([1, -b_d - 1] \setminus (\pm X_d)) \cup ([-b_d + 1, a_d] \setminus X_d) & (b_d < 0,\ a_d + b_d > 0) \\ [1, a_d] \setminus (\pm X_d) & (a_d + b_d < 0) \end{cases}.$$

By using these, we can check $w_d \leq w$ and $\mathsf{cov}(w_d) = \{w s_d w^{-1}\}$. □

In the rest, the symbols (A) and (B) mean the conditions (A) and (B) in Proposition 4.8, respectively.

Example 4.9. Let $n := 9$ and $w := (5, 3, -7, 4, -6, -8, 9, -1, 2)$. Then, we have $\mathsf{des}(w) = \{1, 2, 4, 5, 7\}$. The canonical join representation of w is $\bigvee_{d \in \mathsf{des}(w)} w_d$, where w_d is given as follows for each $d \in \mathsf{des}(w)$.

d	a_d	b_d	(A) or (B)	$R(w_d)$	w_d
1	5	3	(B)	$\{3, 4, 6, 7, 8, 9\}$	$(1, 2, 5, 3, 4, 6, 7, 8, 9)$
2	3	-7	(A)	$\{-3, -1, 2, 4, 5, 8, 9\}$	$(6, 7, -3, -1, 2, 4, 5, 8, 9)$
4	4	-6	(B)	$\{-6, -1, 2, 5, 7, 8, 9\}$	$(3, 4, -6, -1, 2, 5, 7, 8, 9)$
5	-6	-8	(A)	$\{6, 7, 9\}$	$(1, 2, 3, 4, 5, 8, 6, 7, 9)$
7	9	-1	(B)	$\{-1, 2\}$	$(-3, 4, 5, 6, 7, 8, 9, -1, 2)$

Now, by combining Corollary 3.10 and Proposition 4.8, we can obtain the semibrick $S(w)$ from $w \in W$ directly. We need to define a few notations: for integers $a > b$ and $c = \pm 1$, we set

$$(V_-(a, b, c), V_+(a, b, c)) := \begin{cases} (\emptyset, [b, a - 1]) & (b \geq 2) \\ (\emptyset, \{c\} \cup [2, a - 1]) & (b = \pm 1) \\ ([b + 1, -2] \cup \{-c\}, \{c\} \cup [2, a - 1]) & (b \leq -2) \end{cases}.$$

Theorem 4.10. *Let* $w \in W$. *Then, the semibrick* $S(w)$ *is* $\bigoplus_{d \in \mathsf{des}(w)} S_d$, *where* S_d *is the brick whose abbreviated description as in Corollary 3.3 is given as follows.*

- *Set* R_d *as in Proposition 4.8, and* $a_d := w(d)$, $b_d := w(d + 1)$,

$$r_d := \max\{k \geq 0 \mid [1, k] \subset \pm R_d\}, \quad c_d := \begin{cases} w^{-1}(|w(1)|) & (r_d \geq 1) \\ 1 & (r_d = 0) \end{cases},$$

$$((V_-)_d, (V_+)_d) := \begin{cases} (V_-(-b, -a, c), V_+(-b, -a, c)) & ((\mathrm{A})) \\ (V_-(a, b, c), V_+(a, b, c)) & ((\mathrm{B})) \end{cases}, \quad V_d := (V_+)_d \amalg (V_-)_d.$$

- *The brick* S_d *has a* K*-basis* $(\langle i \rangle_d)_{i \in V_d}$, *where* $\langle i \rangle_d$ *belongs to* $e_i S_d$ *if* $i \geq -1$, *and* $e_{|i|} S_d$ *if* $i \leq -2$.
- *For each* $i \in V_d$, *place a symbol* i *denoting the* K*-vector subspace* $K\langle i \rangle_d$.

- *We write the following arrows.*
  - (i) *For each $i \in (V_d)_+ \setminus \{\mathsf{max}(V_d)_+\}$, draw an arrow $i \to |i|+1$ if $|i|+1 \in R_d$; and $i \leftarrow |i|+1$ otherwise.*
  - (ii) *For each $i \in (V_d)_- \setminus \{\mathsf{min}(V_d)_-\}$, draw an arrow $i \leftarrow -(|i|+1)$ if $-(|i|+1) \in R_d$; and $i \to -(|i|+1)$ otherwise.*
  - (iii) *If $r_d \geq 1$, then for each $i \in (V_d)_-$ with $|i| \leq r_d$, draw an arrow $-i \leftarrow -(|i|+1)$ if $|i|+1 \in R_d$; and $i \to |i|+1$ otherwise.*
  - (iv) *If $r_d = 0$, then draw an arrow $-c \leftarrow 2$ if $c \leftarrow 2$ exists in* (i)*, and an arrow $c \to -2$ if $-c \to -2$ exists in* (ii)*.*

Proof. Apply Corollary 3.10 to the element $w_d \in \mathsf{j\text{-}irr}\, W$ defined in Proposition 4.8 for each $d \in \mathsf{des}(w)$. □

Example 4.11. Let $n := 9$ and $w := (5, 3, -7, 4, -6, 8, 9, -1, 2)$ as in Example 4.9. Then, the semibrick $S(w)$ is the direct sum of the following bricks:

$$S_1 = \begin{tikzcd} 3 \arrow[r] & 4 \end{tikzcd} ,$$

$$S_2 = \begin{tikzcd} 1 \arrow[r] & -2 \arrow[dl] \arrow[dr] & & & & \\ -1 \arrow[r] & 2 & 3 \arrow[l] \arrow[r] & 4 \arrow[r] & 5 & 6 \arrow[l] \end{tikzcd} ,$$

$$S_4 = \begin{tikzcd} 1 \arrow[r] & -2 \arrow[r] \arrow[dl] \arrow[dr] & -3 \arrow[r] & -4 \arrow[r] & -5 \\ -1 \arrow[r] & 2 & 3 \arrow[l] & & \end{tikzcd} ,$$

$$S_5 = \begin{tikzcd} 6 \arrow[r] & 7 \end{tikzcd} ,$$

$$S_7 = \begin{tikzcd} -1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] & 5 \arrow[l] & 6 \arrow[l] & 7 \arrow[l] & 8 \arrow[l] \end{tikzcd} .$$

Appendix A. Example: The bricks over the preprojective algebra of type $\mathbb{D}_5$

In this section, we give the list of bricks over the preprojective algebra of type $\mathbb{D}_5$.

For the preparation, we first define two notions denoted by $\sigma(w)$ and $\chi(w)$ associated to each join-irreducible element $w \in \mathsf{j\text{-}irr}\, W$ in the Coxeter group $W = W(\mathbb{D}_n)$ of type $\mathbb{D}_n$ (it is not needed to assume $n = 5$ here), and then list all the join-irreducible elements and the corresponding bricks by using these notions in the case $n = 5$.

First, we define $\sigma(w)$. Recall that we have defined the integers a, b, r in Subsection 3.2 for w. By using these integers, we define $\sigma(w)$ of w as the triple $(a, b, r') \in \mathbb{Z}^3$, where

$$r' := \begin{cases} 0 & (b \geq -1) \\ \mathsf{min}\{r, |b|-1\} & (b \leq -2) \end{cases} ,$$

and call $\sigma(w)$ the *shape* of w. For any $\sigma \in \mathbb{Z}^3$, we write $(\mathsf{j\text{-}irr}\, W)_\sigma \subset \mathsf{j\text{-}irr}\, W$ for the subset of elements in $\mathsf{j\text{-}irr}\, W$ whose shapes are σ. It is easy to see that $(\mathsf{j\text{-}irr}\, W)_\sigma \neq \emptyset$ if and only if σ is a triple (a, b, r') satisfying one of the following conditions (a), (b), (c):

- (a) $2 \leq a \leq n$, $-1 \leq b < a$, $b \neq 0$, $r' = 0$; or
- (b) $2 \leq a \leq n$, $-a < b \leq -2$, $0 \leq r' \leq |b| - 1$; or
- (c) $2 \leq a \leq n$, $-n \leq b < -a$, $0 \leq r' \leq |a| - 2$.

Next, we define the other notion $\chi(w)$ by using R defined in Subsection 3.2 for w. We set $\chi(w)$ as the sequence $(x(1), x(2), \ldots, x(n)) \in \{0, 1, 2\}^n$ whose terms are given by

$$x(i) := \begin{cases} 0 & (-j, j \notin R) \\ 1 & (-j \in R) \\ 2 & (j \in R) \end{cases} .$$

We have a map $\chi\colon \textsf{j-irr}\, W \to \{0,1,2\}^n$, which is clearly injective. For $\sigma = (a, b, r')$ satisfying the condition above, we can straightforwardly check $\chi((\textsf{j-irr}\, W)_\sigma) = \prod_{i=1}^n X_i$, where X_i is defined as follows in each of the three cases (a), (b), and (c):

(a)

i	$i < \lvert b\rvert$	$i = \lvert b\rvert$	$\lvert b\rvert < i < a$	$i = a$	$i > a$
X_i	$\{0\}$	$\{1\}$ if $b = -1$; $\{2\}$ otherwise	$\{0,2\}$	$\{0\}$	$\{2\}$

,

(b)

i	$i \le r'$	$i = r'+1 \ne \lvert b\rvert$	$r'+1 < i < \lvert b\rvert$	$i = \lvert b\rvert$	$\lvert b\rvert < i < a$	$i = a$	$i > a$
X_i	$\{1,2\}$	$\{0\}$	$\{0,1,2\}$	$\{1\}$	$\{0,2\}$	$\{0\}$	$\{2\}$

,

(c)

i	$i \le r'$	$i = r'+1$	$r'+1 < i < a$	$i = a$	$a < i < \lvert b\rvert$	$i = \lvert b\rvert$	$i > a$
X_i	$\{1,2\}$	$\{0\}$	$\{0,1,2\}$	$\{0\}$	$\{1,2\}$	$\{1\}$	$\{2\}$

.

Therefore, by setting $x := \mathsf{max}\{a, \lvert b\rvert\}$ and $y := \mathsf{min}\{a, \lvert b\rvert\}$, we have

$$\#(\textsf{j-irr}\, W)_\sigma = \begin{cases} 2^{x-y-1} & (b \ge -1) \\ 2^{r'} \cdot 3^{\mathsf{max}\{y-r'-2,0\}} \cdot 2^{x-y-1} & (b \le -2) \end{cases}.$$

From now on, we consider $\mathbb{D}_5$, so let $n := 5$. For σ satisfying the condition above, the following lists show all the elements w in $(\textsf{j-irr}\, W)_\sigma$ and the corresponding bricks $S(w)$ over the preprojective algebra Π of type $\mathbb{D}_5$. The elements in $(\textsf{j-irr}\, W)_\sigma$ are arranged so that w comes before w' if and only if $\chi(w) < \chi(w')$ in the lexicographical order of $\{0,1,2\}^n$, and each w is shortly denoted by a string $j_1 j_2 \cdots j_n$, where $j_i := w(i)$ if $w(i) > 0$; $j_i := \underline{w(i)}$ if $w(i) < 0$. For example, $\underline{1}2\underline{5}34$ means $(-1, 2, -5, 3, 4)$. The join-irreducible elements and the bricks are explicitly described as follows by Corollary 3.10:

- $\sigma = (2, -5, 0)$ (4 elements):

$$S(\underline{1}2\underline{543}) = \begin{tikzcd} -1 \arrow[r] & -2 & -3 \arrow[l] & -4 \arrow[l] \\ 1 \arrow[ur] \end{tikzcd}, \qquad S(12\underline{53}4) = \begin{tikzcd} -1 \arrow[r] & -2 & -3 \arrow[l] \arrow[r] & -4 \\ 1 \arrow[ur] \end{tikzcd},$$

$$S(12\underline{54}3) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 & -4 \arrow[l] \\ 1 \arrow[ur] \end{tikzcd}, \qquad S(\underline{1}2\underline{5}34) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 \arrow[ur] \end{tikzcd};$$

- $\sigma = (2, -4, 0)$ (2 elements):

$$S(12\underline{43}5) = \begin{tikzcd} -1 \arrow[r] & -2 & -3 \arrow[l] \\ 1 \arrow[ur] \end{tikzcd}, \qquad S(\underline{1}2\underline{4}35) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \\ 1 \arrow[ur] \end{tikzcd};$$

- $\sigma = (2, -3, 0)$ (1 element):

$$S(\underline{1}2\underline{3}45) = \begin{tikzcd} -1 \arrow[r] & -2 \\ 1 \arrow[ur] \end{tikzcd};$$

- $\sigma = (2, -1, 0)$ (1 element):

$$S(\underline{21}345) = -1\,;$$

- $\sigma = (2, 1, 0)$ (1 element):

$$S(21345) = 1\,;$$

- $\sigma = (3, -5, 0)$ (6 elements):

$$S(123\underline{54}) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 & -4 \arrow[l] \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] \end{tikzcd}, \qquad S(\underline{1}23\underline{5}4) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] \end{tikzcd},$$

$$S(\underline{1}3\underline{542}) = \begin{tikzcd} -1 & -2 \arrow[l] \arrow[r] & -3 & -4 \arrow[l] \\ 1 & 2 \arrow[l] \arrow[ul] \end{tikzcd}, \qquad S(13\underline{52}4) = \begin{tikzcd} -1 & -2 \arrow[l] \arrow[r] & -3 \arrow[r] & -4 \\ 1 & 2 \arrow[l] \arrow[ul] \end{tikzcd},$$

$$S(13\underline{54}2) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 & -4 \arrow[l] \\ 1 \arrow[ur] \arrow[r] & 2 \end{tikzcd}, \qquad S(\underline{1}3\underline{5}24) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 \arrow[ur] \arrow[r] & 2 \end{tikzcd};$$

- $\sigma = (3, -5, 1)$ (4 elements):

$$S(\underline{2}3\underline{541}) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 & -4 \arrow[l] \\ -1 & 2 \arrow[l] \end{tikzcd}, \qquad S(23\underline{51}4) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ -1 & 2 \arrow[l] \end{tikzcd},$$

$$S(23\underline{54}1) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 & -4 \arrow[l] \\ 1 & 2 \arrow[l] \end{tikzcd}, \qquad S(\underline{2}3\underline{5}14) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 & 2 \arrow[l] \end{tikzcd};$$

- $\sigma = (3, -4, 0)$ (3 elements):

$$S(\underline{1}23\underline{4}5) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] \end{tikzcd}, \qquad S(13\underline{42}5) = \begin{tikzcd} -1 & -2 \arrow[l] \arrow[r] & -3 \\ 1 & 2 \arrow[l] \arrow[ul] \end{tikzcd},$$

$$S(\underline{1}3\underline{4}25) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \\ 1 \arrow[ur] \arrow[r] & 2 \end{tikzcd};$$

- $\sigma = (3, -4, 1)$ (2 elements):

$$S(23\underline{41}5) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \\ -1 & 2 \arrow[l] \end{tikzcd}, \qquad S(\underline{2}3\underline{4}15) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \\ 1 & 2 \arrow[l] \end{tikzcd};$$

- $\sigma = (3, -2, 0)$ (1 element):

$$S(\underline{1}3\underline{2}45) = \begin{tikzcd} -1 & \\ 1 & 2 \arrow[l] \arrow[ul] \end{tikzcd};$$

- $\sigma = (3, -2, 1)$ (2 elements):

$$S(3\underline{21}45) = \begin{tikzcd} 1 \arrow[dr] & \\ -1 & 2 \arrow[l] \end{tikzcd}, \qquad S(\underline{32}145) = \begin{tikzcd} -1 \arrow[dr] & \\ 1 & 2 \arrow[l] \end{tikzcd};$$

- $\sigma = (3, -1, 0)$ (2 elements):

$$S(\underline{2}3\underline{1}45) = -1 \longleftarrow 2, \qquad S(\underline{31}245) = -1 \longrightarrow 2;$$

- $\sigma = (3, 1, 0)$ (2 elements):

$$S(23145) = 1 \longleftarrow 2, \qquad S(31245) = 1 \longrightarrow 2;$$

- $\sigma = (3, 2, 0)$ (1 element):

$$S(13245) = 2;$$

- $\sigma = (4, -5, 0)$ (9 elements):

$$S(\underline{1}234\underline{5}) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \end{tikzcd}, \qquad S(124\underline{53}) = \begin{tikzcd} -1 \arrow[r] & -2 & -3 \arrow[l] \arrow[r] & -4 \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \end{tikzcd},$$

$$S(\underline{1}24\underline{5}3) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] \arrow[r] & 3 \end{tikzcd}, \qquad S(134\underline{52}) = \begin{tikzcd} -1 & -2 \arrow[l] \arrow[r] & -3 \arrow[r] & -4 \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \end{tikzcd},$$

$$S(\underline{1}4\underline{532}) = \begin{tikzcd} -1 & -2 \arrow[l] & -3 \arrow[l] \arrow[r] & -4 \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \end{tikzcd}, \qquad S(14\underline{52}3) = \begin{tikzcd} -1 & -2 \arrow[l] \arrow[r] & -3 \arrow[r] & -4 \\ 1 & 2 \arrow[l] \arrow[ul] \arrow[r] & 3 \end{tikzcd},$$

$$S(\underline{1}34\underline{5}2) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 \arrow[ur] \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd}, \qquad S(14\underline{53}2) = \begin{tikzcd} -1 \arrow[r] & -2 & -3 \arrow[l] \arrow[r] & -4 \\ 1 \arrow[ur] \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd},$$

$$S(\underline{1}4\underline{5}23) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 \arrow[ur] \arrow[r] & 2 \arrow[r] & 3 \end{tikzcd};$$

- $\sigma = (4, -5, 1)$ (6 elements):

$$S(234\underline{51}) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ -1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}, \qquad S(\underline{2}4\underline{531}) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 & -3 \arrow[l] \arrow[r] & -4 \\ -1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd},$$

$$S(24\underline{51}3) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ -1 & 2 \arrow[l] \arrow[r] & 3 \end{tikzcd}, \qquad S(\underline{2}34\underline{5}1) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd},$$

$$S(24\underline{53}1) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 & -3 \arrow[l] \arrow[r] & -4 \\ 1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}, \qquad S(\underline{2}4\underline{5}13) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \arrow[r] & -4 \\ 1 & 2 \arrow[l] \arrow[r] & 3 \end{tikzcd};$$

- $\sigma = (4, -5, 2)$ (4 elements):

$$S(\underline{3}4\underline{521}) = \begin{tikzcd} 1 \arrow[dr] & -2 \arrow[l] \arrow[r] \arrow[dr] & -3 \arrow[r] & -4 \\ -1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}, \qquad S(34\underline{51}2) = \begin{tikzcd} 1 \arrow[r] & -2 \arrow[r] \arrow[dl] \arrow[dr] & -3 \arrow[r] & -4 \\ -1 \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd},$$

$$S(34\underline{52}1) = \begin{tikzcd} -1 \arrow[dr] & -2 \arrow[l] \arrow[r] \arrow[dr] & -3 \arrow[r] & -4 \\ 1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}, \qquad S(\underline{3}4\underline{5}12) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] \arrow[dl] \arrow[dr] & -3 \arrow[r] & -4 \\ 1 \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd};$$

- $\sigma = (4, -3, 0)$ (3 elements):

$$S(\underline{1}24\underline{3}5) = \begin{tikzcd} -1 \arrow[r] & -2 & \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \end{tikzcd}, \qquad S(14\underline{32}5) = \begin{tikzcd} -1 & -2 \arrow[l] & \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \end{tikzcd},$$

$$S(\underline{1}4\underline{3}25) = \begin{tikzcd} -1 \arrow[r] & -2 & \\ 1 \arrow[r] \arrow[ur] & 2 & 3 \arrow[l] \end{tikzcd};$$

- $\sigma = (4, -3, 1)$ (2 elements):

$$S(24\underline{31}5) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 & \\ -1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}, \qquad S(\underline{2}4\underline{3}15) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 & \\ 1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd};$$

- $\sigma = (4, -3, 2)$ (4 elements):

$$S(\underline{4321}5) = \begin{tikzcd} 1 \arrow[dr] & -2 \arrow[l] \arrow[dr] & \\ -1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}, \qquad S(4\underline{31}25) = \begin{tikzcd} 1 \arrow[r] & -2 \arrow[dl] \arrow[dr] & \\ -1 \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd},$$

$$S(4\underline{32}15) = \begin{tikzcd} -1 \arrow[dr] & -2 \arrow[l] \arrow[dr] & \\ 1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}, \qquad S(\underline{43}125) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[dl] \arrow[dr] & \\ 1 \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd};$$

- $\sigma = (4, -2, 0)$ (2 elements):

$$S(\underline{1}34\underline{2}5) = \begin{tikzcd} -1 & & \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \end{tikzcd}, \qquad S(\underline{1}4\underline{2}35) = \begin{tikzcd} -1 & & \\ 1 & 2 \arrow[l] \arrow[ul] \arrow[r] & 3 \end{tikzcd};$$

- $\sigma = (4, -2, 1)$ (4 elements):

$$S(34\underline{21}5) = \begin{tikzcd} 1 \arrow[dr] & & \\ -1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}, \qquad S(4\underline{21}35) = \begin{tikzcd} 1 \arrow[dr] & & \\ -1 & 2 \arrow[l] \arrow[r] & 3 \end{tikzcd},$$

$$S(\underline{3}4\underline{2}15) = \begin{tikzcd} -1 \arrow[dr] & & \\ 1 & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}, \qquad S(\underline{42}135) = \begin{tikzcd} -1 \arrow[dr] & & \\ 1 & 2 \arrow[l] \arrow[r] & 3 \end{tikzcd};$$

- $\sigma = (4, -1, 0)$ (4 elements):

$$S(\underline{2}34\underline{1}5) = -1 \longleftarrow 2 \longleftarrow 3\,, \qquad S(\underline{2}4\underline{1}35) = -1 \longleftarrow 2 \longrightarrow 3\,,$$

$$S(\underline{3}4\underline{1}25) = -1 \longrightarrow 2 \longleftarrow 3\,, \qquad S(\underline{41}235) = -1 \longrightarrow 2 \longrightarrow 3\,;$$

- $\sigma = (4, 1, 0)$ (4 elements):

$$S(23415) = 1 \longleftarrow 2 \longleftarrow 3\,, \qquad S(24135) = 1 \longleftarrow 2 \longrightarrow 3\,,$$

$$S(34125) = 1 \longrightarrow 2 \longleftarrow 3\,, \qquad S(41235) = 1 \longrightarrow 2 \longrightarrow 3\,;$$

- $\sigma = (4, 2, 0)$ (2 elements):

$$S(13425) = 2 \longleftarrow 3\,, \qquad S(14235) = 2 \longrightarrow 3\,;$$

- $\sigma = (4,3,0)$ (1 element):

$$S(12435) = 3\ ;$$

- $\sigma = (5,-4,0)$ (9 elements):

$$S(\underline{1}235\underline{4}) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(125\underline{43}) = \begin{tikzcd} -1 \arrow[r] & -2 & -3 \arrow[l] \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{1}25\underline{4}3) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd}, \qquad S(135\underline{42}) = \begin{tikzcd} -1 & -2 \arrow[l] \arrow[r] & -3 \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{1}5\underline{432}) = \begin{tikzcd} -1 & -2 \arrow[l] & -3 \arrow[l] \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(15\underline{42}3) = \begin{tikzcd} -1 & -2 \arrow[l] \arrow[r] & -3 \\ 1 & 2 \arrow[l] \arrow[ul] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{1}35\underline{4}2) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \\ 1 \arrow[ur] \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(15\underline{43}2) = \begin{tikzcd} -1 \arrow[r] & -2 & -3 \arrow[l] \\ 1 \arrow[ur] \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{1}5\underline{4}23) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] & -3 \\ 1 \arrow[ur] \arrow[r] & 2 \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd};$$

- $\sigma = (5,-4,1)$ (6 elements):

$$S(235\underline{41}) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \\ -1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{2}5\underline{431}) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 & -3 \arrow[l] \\ -1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(25\underline{41}3) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \\ -1 & 2 \arrow[l] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{2}35\underline{4}1) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \\ 1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(25\underline{43}1) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 & -3 \arrow[l] \\ 1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{2}5\underline{4}13) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \arrow[r] & -3 \\ 1 & 2 \arrow[l] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd};$$

- $\sigma = (5,-4,2)$ (4 elements):

$$S(\underline{3}5\underline{421}) = \begin{tikzcd} 1 \arrow[dr] & -2 \arrow[l] \arrow[r] \arrow[dr] & -3 \\ -1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(35\underline{41}2) = \begin{tikzcd} 1 \arrow[r] & -2 \arrow[r] \arrow[dl] \arrow[dr] & -3 \\ -1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(35\underline{42}1) = \begin{tikzcd} -1 \arrow[dr] & -2 \arrow[l] \arrow[r] \arrow[dr] & -3 \\ 1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{3}5\underline{4}12) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] \arrow[dl] \arrow[dr] & -3 \\ 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd};$$

- $\sigma = (5,-4,3)$ (8 elements):

$$S(5\underline{4321}) = \begin{tikzcd} 1 \arrow[dr] & -2 \arrow[l] \arrow[dr] & -3 \arrow[l] \arrow[dr] \\ -1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{5421}3) = \begin{tikzcd} 1 \arrow[dr] & -2 \arrow[l] \arrow[r] & -3 \arrow[dl] \arrow[dr] \\ -1 & 2 \arrow[l] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{5431}2) = \begin{tikzcd} 1 \arrow[r] & -2 \arrow[dl] \arrow[dr] & -3 \arrow[l] \arrow[dr] \\ -1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(5\underline{41}23) = \begin{tikzcd} 1 \arrow[r] & -2 \arrow[r] \arrow[dl] & -3 \arrow[dl] \arrow[dr] \\ -1 \arrow[r] & 2 \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{5432}1) = \begin{tikzcd} -1 \arrow[dr] & -2 \arrow[l] \arrow[dr] & -3 \arrow[l] \arrow[dr] \\ 1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(5\underline{42}13) = \begin{tikzcd} -1 \arrow[dr] & -2 \arrow[l] \arrow[r] & -3 \arrow[dl] \arrow[dr] \\ 1 & 2 \arrow[l] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd},$$

$$S(5\underline{43}12) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[dl] \arrow[dr] & -3 \arrow[l] \arrow[dr] \\ 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{54}123) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[r] \arrow[dl] & -3 \arrow[dl] \arrow[dr] \\ 1 \arrow[r] & 2 \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd};$$

- $\sigma = (5,-3,0)$ (6 elements):

$$S(\underline{1}245\underline{3}) = \begin{tikzcd} -1 \arrow[r] & -2 \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{1}25\underline{3}4) = \begin{tikzcd} -1 \arrow[r] & -2 \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd},$$

$$S(145\underline{32}) = \begin{tikzcd} -1 & -2 \arrow[l] \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(15\underline{32}4) = \begin{tikzcd} -1 & -2 \arrow[l] \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd},$$

$$S(\underline{1}45\underline{3}2) = \begin{tikzcd} -1 \arrow[r] & -2 \\ 1 \arrow[ur] \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad S(\underline{1}5\underline{3}24) = \begin{tikzcd} -1 \arrow[r] & -2 \\ 1 \arrow[ur] \arrow[r] & 2 & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd};$$

- $\sigma = (5, -3, 1)$ (4 elements):

$$S(245\underline{31}) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \\ -1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(25\underline{31}4) = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & -2 \\ -1 & 2 \arrow[l] & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd},$$

$$S(\underline{2}45\underline{3}1) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \\ 1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{2}5\underline{3}14) = \begin{tikzcd} -1 \arrow[r] \arrow[dr] & -2 \\ 1 & 2 \arrow[l] & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd};$$

- $\sigma = (5, -3, 2)$ (8 elements):

$$S(\underline{4}5\underline{321}) = \begin{tikzcd} 1 \arrow[dr] & -2 \arrow[l] \arrow[dr] \\ -1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{5321}4) = \begin{tikzcd} 1 \arrow[dr] & -2 \arrow[l] \arrow[dr] \\ -1 & 2 \arrow[l] & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd},$$

$$S(45\underline{31}2) = \begin{tikzcd} 1 \arrow[r] & -2 \arrow[dl] \arrow[dr] \\ -1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(5\underline{31}24) = \begin{tikzcd} 1 \arrow[r] & -2 \arrow[dl] \arrow[dr] \\ -1 \arrow[r] & 2 & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd},$$

$$S(45\underline{32}1) = \begin{tikzcd} -1 \arrow[dr] & -2 \arrow[l] \arrow[dr] \\ 1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(5\underline{32}14) = \begin{tikzcd} -1 \arrow[dr] & -2 \arrow[l] \arrow[dr] \\ 1 & 2 \arrow[l] & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd},$$

$$S(\underline{4}5\underline{3}12) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[dl] \arrow[dr] \\ 1 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{53}124) = \begin{tikzcd} -1 \arrow[r] & -2 \arrow[dl] \arrow[dr] \\ 1 \arrow[r] & 2 & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd};$$

- $\sigma = (5, -2, 0)$ (4 elements):

$$S(\underline{1}345\underline{2}) = \begin{tikzcd} -1 \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{1}35\underline{2}4) = \begin{tikzcd} -1 \\ 1 & 2 \arrow[l] \arrow[ul] & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd},$$

$$S(\underline{1}45\underline{2}3) = \begin{tikzcd} -1 \\ 1 & 2 \arrow[l] \arrow[ul] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{1}5\underline{2}34) = \begin{tikzcd} -1 \\ 1 & 2 \arrow[l] \arrow[ul] \arrow[r] & 3 \arrow[r] & 4 \end{tikzcd};$$

- $\sigma = (5, -2, 1)$ (8 elements):

$$S(345\underline{21}) = \begin{tikzcd} 1 \arrow[dr] \\ -1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(35\underline{21}4) = \begin{tikzcd} 1 \arrow[dr] \\ -1 & 2 \arrow[l] & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd},$$

$$S(45\underline{21}3) = \begin{tikzcd} 1 \arrow[dr] \\ -1 & 2 \arrow[l] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd},$$

$$S(5\underline{21}34) = \begin{tikzcd} 1 \arrow[dr] \\ -1 & 2 \arrow[l] \arrow[r] & 3 \arrow[r] & 4 \end{tikzcd},$$

$$S(\underline{3}45\underline{2}1) = \begin{tikzcd} -1 \arrow[dr] \\ 1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{3}5\underline{2}14) = \begin{tikzcd} -1 \arrow[dr] \\ 1 & 2 \arrow[l] & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd},$$

$$S(\underline{4}5\underline{2}13) = \begin{tikzcd} -1 \arrow[dr] \\ 1 & 2 \arrow[l] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd},$$

$$S(\underline{52}134) = \begin{tikzcd} -1 \arrow[dr] \\ 1 & 2 \arrow[l] \arrow[r] & 3 \arrow[r] & 4 \end{tikzcd};$$

- $\sigma = (5, -1, 0)$ (8 elements):

$S(\underline{2}345\underline{1}) = -1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4\,,$ $\quad S(\underline{2}35\underline{1}4) = -1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4\,,$

$S(\underline{2}45\underline{1}3) = -1 \longleftarrow 2 \longrightarrow 3 \longleftarrow 4\,,$ $\quad S(\underline{2}5\underline{1}34) = -1 \longleftarrow 2 \longrightarrow 3 \longrightarrow 4\,,$

$S(\underline{3}45\underline{1}2) = -1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4\,,$ $\quad S(\underline{3}5\underline{1}24) = -1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4\,,$

$S(\underline{4}5\underline{1}23) = -1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4\,,$ $\quad S(\underline{51}234) = -1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4\,;$

- $\sigma = (5, 1, 0)$ (8 elements):

$S(23451) = 1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4\,,$ $\quad S(23514) = 1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4\,,$

$S(24513) = 1 \longleftarrow 2 \longrightarrow 3 \longleftarrow 4\,,$ $\quad S(25134) = 1 \longleftarrow 2 \longrightarrow 3 \longrightarrow 4\,,$

$S(34512) = 1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4\,,$ $\quad S(35124) = 1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4\,,$

$S(45123) = 1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4\,,$ $\quad S(51234) = 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4\,;$

- $\sigma = (5, 2, 0)$ (4 elements):

$S(13452) = 2 \longleftarrow 3 \longleftarrow 4\,,$ $\quad S(13524) = 2 \longleftarrow 3 \longrightarrow 4\,,$

$S(14523) = 2 \longrightarrow 3 \longleftarrow 4\,,$ $\quad S(15234) = 2 \longrightarrow 3 \longrightarrow 4\,;$

- $\sigma = (5,3,0)$ (2 elements):

$$S(12453) = \begin{tikzcd} 3 & 4 \arrow[l] \end{tikzcd} , \qquad S(12534) = \begin{tikzcd} 3 \arrow[r] & 4 \end{tikzcd} ;$$

- $\sigma = (5,4,0)$ (1 element):

$$S(12354) = 4 .$$

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