Editors' Suggestion

Large-scale structure of velocity and passive scalar fields in freely decaying homogeneous anisotropic turbulence

Katsunori Yoshimatsu*
Institute of Materials and Systems for Sustainability, Nagoya University, Nagoya, 464-8601, Japan

Yukio Kaneda†
Aichi Institute of Technology, 1247, Yachikusa, Yakusacho, Toyota, 470-0392, Japan

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We consider freely decaying homogeneous anisotropic turbulence whose energy spectrum $E(k)$ at $k \to 0$ is given by $E(k) = Ck^2 + o(k^2)$ at an initial instant, where k is the wave number and C is a k-independent positive number. An argument is given to show that there are an infinite number of invariants characterizing the large-scale structure of the turbulence. This is a generalization of Saffman's argument, which shows the existence of a finite number of invariants [P. G. Saffman, J. Fluid Mech. **27**, 581 (1967)]. By applying a similar argument to homogeneous anisotropic passive scalar turbulence without any scalar source, we show that there are an infinite number of invariants characterizing the large-scale structure of passive scalar fields. Theoretical analysis based on the invariance and a self-similarity assumption for the large-scale evolution shows that the anisotropy of the velocity and passive scalar fields is persistent at large scales. The decay laws of the velocity and passive scalar fields are derived by a simple dimensional analysis.

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I. INTRODUCTION

Large-scale structure is one of the key properties of turbulence and plays a significant role in the decay of turbulent flows [1–3]. Saffman [4] studied the large-scale structure of freely decaying incompressible homogeneous anisotropic turbulence whose energy spectrum $E(k,t)$ at $k \to 0$ is given by $Ck^2 + o(k^2)$ at an initial instant, where $\boldsymbol{k}$ is a wave vector, $k = |\boldsymbol{k}|$, $C\,(>0)$ is a k-independent constant, and t is time. He showed that the leading order term, the $O(k^0)$ term, of $\hat{R}_{ij}(\boldsymbol{k})$ at $\boldsymbol{k} \to \boldsymbol{0}$ is time independent under appropriate conditions, and that there is a finite number of invariants. Here $\hat{R}_{ij}(\boldsymbol{k},t)$ is the Fourier transform of the second-order two-point velocity correlation tensor $R_{ij}(\boldsymbol{r},t)$ defined by $\langle u_i(\boldsymbol{x})u_j(\boldsymbol{x}+\boldsymbol{r})\rangle$, u_i is the ith velocity component, $\langle\cdot\rangle$ denotes an ensemble average of $\cdot$, $\boldsymbol{x}$ denotes the position, and $\boldsymbol{r}$ is the separation vector. Arguments such as $\boldsymbol{x}$ and t were omitted for brevity. This kind of turbulence is called Saffman turbulence. The tensor $\hat{R}_{ij}(\boldsymbol{k})$ is discontinuous at $\boldsymbol{k} = \boldsymbol{0}$, and $\hat{R}_{ij}(\boldsymbol{k})$ for $\boldsymbol{k} \to \boldsymbol{0}$ depends on the direction $\boldsymbol{k}/k$. He also argued that the integral $\int_{\mathbb{R}^3} R_{ij}(\boldsymbol{r})\,d\boldsymbol{r}$, called here Saffman's integral, is invariant by using the incompressible condition and Gauss's divergence theorem. The invariance of $\int_{\mathbb{R}^3}(R_{11} + R_{22} + R_{33})\,d\boldsymbol{r}$ for isotropic turbulence had been noted by Birkhoff [5]. The decay rate of the total energy and the growth rate of an integral length scale can be derived using the invariants and a self-similarity assumption [6].

*yoshimatsu@nagoya-u.jp
†ykaneda@aitech.ac.jp

By the use of dimensional analysis and direct numerical simulation (DNS), flow anisotropy in the energy-containing range was shown to be persistent for fully developed axisymmetric Saffman turbulence [7]. This persistence was suggested by large eddy simulations [8]. The large-scale structure of decaying axisymmetric turbulence subjected to the external force due to density stratification, the Coriolis force, or a uniform magnetic field was discussed in Ref. [9].

Prior to Saffman turbulence, freely decaying incompressible homogeneous turbulence, whose energy spectrum $E(k,t)$ at $k \to 0$ is given by $C_4 k^4 + o(k^4)$, had been studied, where C_4 (>0) is a k-independent constant proportional to Loitsiansky's integral [3,10]. This kind of turbulence is called Batchelor turbulence. DNS of fully developed isotropic turbulence at sufficiently high Reynolds numbers [11] showed that Loitsiansky's integral is approximately time independent. The DNS is in accordance with Kolmogorov's decay laws [12], which can be derived by a simple dimensional analysis based on the invariance of Loitsiansky's integral and a self-similarity assumption. In Batchelor turbulence, in contrast to Saffman turbulence, the velocity correlation spectral tensor $\hat{R}_{ij}(\boldsymbol{k},t)$ is continuous at $\boldsymbol{k} = \boldsymbol{0}$ and $\hat{R}_{ij}(\boldsymbol{k},t) \to 0$ at $\boldsymbol{k} \to \boldsymbol{0}$.

The large-scale structure of a passive scalar field plays a significant role in the free decay of the passive scalar field in incompressible homogeneous turbulence. The integral $\int_{\mathbb{R}^3} \Theta(\boldsymbol{r},t)\, d\boldsymbol{r}$ for isotropic passive scalar turbulence was shown to be invariant under appropriate conditions [13], where $\Theta(\boldsymbol{r},t) = \langle \theta(\boldsymbol{x})\theta(\boldsymbol{x}+\boldsymbol{r})\rangle$, and $\theta(\boldsymbol{x})$ denotes a passive scalar field with $\langle\theta\rangle = 0$. The invariance implies that if the scalar spectrum $E^\theta(k,t)$ at $k \to 0$ is given by $C^\theta k^2 + o(k^2)$, C^θ is time independent, where C^θ is a k-independent constant. Chasnov [14] considered an isotropic passive scalar field whose scalar spectrum $E^\theta(k,t)$ at $k \to 0$ is given by $C_4^\theta k^4 + o(k^4)$, where C_4^θ (>0) is a k-independent constant. For either case of isotropic passive scalar turbulence, the scalar correlation spectrum $\hat{\Theta}(\boldsymbol{k},t)$, which is defined by the Fourier transform of $\Theta(\boldsymbol{r},t)$, is continuous at $\boldsymbol{k} = \boldsymbol{0}$.

The decay rate of the scalar variance $\langle\theta^2\rangle$ and the growth rate of scalar integral length scales for passive scalar turbulence in general depend on velocity statistics. Corrsin [13] derived the rates of an isotropic passive scalar field with $E^\theta(k) = C^\theta k^2 + o(k^2)$ at $k \to 0$ in Batchelor turbulence. Chasnov [14] derived the rates of isotropic passive scalar fields with $E^\theta(k) = C^\theta k^2 + o(k^2)$ or $E^\theta(k) = C_4^\theta k^4 + o(k^4)$ at $k \to 0$ in Saffman or Batchelor turbulence. These rates were obtained by dimensional analysis under a self-similarity assumption for the velocity and scalar fields.

In this paper, we consider the large-scale structure of (i) a freely decaying incompressible homogeneous anisotropic turbulent velocity field and (ii) a homogeneous anisotropic passive scalar field without any scalar source in incompressible homogeneous turbulence, where the energy spectrum $E(k)$ takes the form $E(k) = Ck^2 + o(k^2)$ at $k \to 0$ for (i), and the scalar spectrum $E^\theta(k)$ takes the form $E^\theta(k) = C^\theta k^2 + o(k^2)$ at $k \to 0$ for (ii). In Sec. II we give a generalization of Saffman's argument and then show the existence of an infinite number of invariants. The methodology is applied to the passive scalar turbulence in Sec. III. The form of the scalar correlation spectrum at $\boldsymbol{k} \to \boldsymbol{0}$ for isotropic passive scalar turbulence is generalized to that for anisotropic passive scalar turbulence. In Secs. IV and V the implications of self-similarity at large scales and invariants for (i) and (ii) are discussed, respectively. The persistence of large-scale flow anisotropy and large-scale scalar anisotropy will be shown. Furthermore, by the use of invariants and a dimensional analysis, the decay laws of Saffman turbulence [6,7] are generalized to (i) in Sec. IV, and the decay laws of (ii) are derived for the velocity field obeying the generalized decay laws in Sec. V. Finally, Sec. VI presents conclusions and discussion.

II. LARGE-SCALE STRUCTURE OF FREELY DECAYING HOMOGENEOUS ANISOTROPIC TURBULENCE

In Sec. II A, we briefly summarize Saffman's invariants [4] in $\boldsymbol{k}$ space. In Sec. II B, by generalizing Saffman's argument, we show that there are an infinite number of invariants.

A. Saffman's invariants

Let $\boldsymbol{u}(\boldsymbol{x}, t)$ be the velocity of an incompressible fluid that obeys the Navier-Stokes (NS) equation

$$\frac{\partial \boldsymbol{u}}{\partial t} = -(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u}, \tag{1}$$

and the incompressibility condition

$$\nabla\cdot\boldsymbol{u} = 0, \tag{2}$$

where ρ is the constant fluid density, $p(\boldsymbol{x}, t)$ is the pressure, ν is the kinematic viscosity, $\boldsymbol{x} = (x_1, x_2, x_3)$ in the Cartesian coordinate system, and $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$.

In homogeneous turbulence, two-point statistics such as $R_{ij}(\boldsymbol{r}, t) = \langle u_i(\boldsymbol{x})u_j(\boldsymbol{x}+\boldsymbol{r})\rangle$ depend on $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{r}$ through the separation vector $\boldsymbol{r}$. The velocity correlation spectral tensor $\hat{R}_{ij}(\boldsymbol{k}, t)$, i.e., the Fourier transform (in the sense of a generalized function, distribution, or hyperfunction) of $R_{ij}(\boldsymbol{r}, t)$, is given by

$$\hat{R}_{ij}(\boldsymbol{k}, t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} R_{ij}(\boldsymbol{r}, t)\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,d\boldsymbol{r}. \tag{3}$$

Here $\hat{\cdot}$ denotes the Fourier transform of $\cdot$. The NS equation (1) and Eq. (2) then give

$$\frac{\partial}{\partial t}\hat{R}_{ij}(\boldsymbol{k}, t) = T_{ij}(\boldsymbol{k}, t) - 2\nu k^2\hat{R}_{ij}(\boldsymbol{k}, t), \tag{4}$$

where

$$T_{ij}(\boldsymbol{k}, t) = \mathrm{i}k_\alpha P_{i\beta}(\tilde{\boldsymbol{k}})\hat{\Gamma}_{\alpha\beta j}(\boldsymbol{k}) - \mathrm{i}k_\alpha P_{j\beta}(\tilde{\boldsymbol{k}})\hat{\Gamma}_{\alpha\beta i}(-\boldsymbol{k}), \tag{5}$$

in which $\hat{\Gamma}_{\alpha\beta j}(\boldsymbol{k})$ is the Fourier transform of $\Gamma_{\alpha\beta j}(\boldsymbol{x}) = \langle N_{\alpha\beta}(\boldsymbol{x})u_j(\boldsymbol{x}+\boldsymbol{r})\rangle$,

$$\hat{\Gamma}_{\alpha\beta j}(\boldsymbol{k}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\langle N_{\alpha\beta}(\boldsymbol{x})u_j(\boldsymbol{x}+\boldsymbol{r})\rangle\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,d\boldsymbol{r}, \tag{6}$$

$N_{\alpha\beta}(\boldsymbol{x}, t) = u_\alpha(\boldsymbol{x}, t)u_\beta(\boldsymbol{x}, t)$, $P_{ij}(\tilde{\boldsymbol{k}}) = \delta_{ij} - \tilde{k}_i\tilde{k}_j$, and $\tilde{\boldsymbol{k}} = \boldsymbol{k}/k$. The summation convention is applied to repeated Greek indices but not to Roman indices, unless otherwise stated. Because of the incompressibility condition (2), the Fourier transform $\hat{R}_{ij}(\boldsymbol{k}, t)$ can be written as

$$\hat{R}_{ij}(\boldsymbol{k}, t) = P_{i\alpha}(\tilde{\boldsymbol{k}})P_{j\beta}(\tilde{\boldsymbol{k}})\mathcal{M}_{\alpha\beta}(\boldsymbol{k}, t), \tag{7}$$

without loss of generality, where $\mathcal{M}_{\alpha\beta}(\boldsymbol{k}, t)$ is an appropriate function of $\boldsymbol{k}$ and t. Note that $\mathcal{M}_{\alpha\beta}(\boldsymbol{k}, t)$ does not need to be generally symmetric in α and β. Saffman [4] considered turbulence where $\mathcal{M}_{\alpha\beta}(\boldsymbol{k}, t)$ at an initial instant, for example, $t = t_0$, has the expansion

$$\mathcal{M}_{\alpha\beta}(\boldsymbol{k}, t_0) = M_{\alpha\beta} + o(1) \;\text{ as }\; \boldsymbol{k}\to\boldsymbol{0}, \tag{8}$$

where $M_{\alpha\beta}$ is $\boldsymbol{k}$-independent and a nonzero constant that is symmetric in α and β. The six constants $M_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$ are Saffman's invariants. It was argued in Ref. [4] that the leading order term of $\mathcal{M}_{\alpha\beta}(\boldsymbol{k}, t)$ at $\boldsymbol{k}\to\boldsymbol{0}$ in Eq. (7) is $O(k^0)$ and is a dynamical invariant, such that $\hat{R}_{ij}(\boldsymbol{k}, t)$ can be written as

$$\hat{R}_{ij}(\boldsymbol{k}, t) = P_{i\alpha}(\tilde{\boldsymbol{k}})P_{j\beta}(\tilde{\boldsymbol{k}})M_{\alpha\beta} + o(1) \;\text{ as }\; \boldsymbol{k}\to\boldsymbol{0}, \tag{9}$$

for any $t \geqslant t_0$, where $M_{\alpha\beta}$ is invariant and is given by the initial condition (8). The first term of the right-hand side in Eq. (9) has reflectional symmetry. The energy spectrum $E(k, t)$ defined by $(1/2)\int\hat{R}_{\alpha\alpha}(\boldsymbol{q}, t)\,dS_k$ has the form $E(k, t) = Ck^2 + o(k^2)$, where C is a positive constant, and $\int\cdot\,dS_k$ denotes the integration of $\cdot$ over the spherical surface with the radius $|\boldsymbol{q}| = k$ and center at $\boldsymbol{q} = \boldsymbol{0}$.

B. Generalization of Saffman's invariants

Now, we consider a generalization of Eq. (8). Similar to Eq. (8), $\mathcal{M}_{\alpha\beta}(\boldsymbol{k}, t_0)$ for $\boldsymbol{k} \to \boldsymbol{0}$ is $O(k^0)$; however, in contrast to Eq. (8), $M_{\alpha\beta}$ may depend on the direction of $\tilde{\boldsymbol{k}}$,

$$\mathcal{M}_{\alpha\beta}(\boldsymbol{k}, t_0) = M_{\alpha\beta}(\tilde{\boldsymbol{k}}) + o(1), \tag{10}$$

where $M_{\alpha\beta}(\tilde{\boldsymbol{k}})$ can be regarded as a function of only the angles ϑ and φ of the spherical polar coordinates such that $\boldsymbol{k} = (k \sin\vartheta \cos\varphi, k \sin\vartheta \sin\varphi, k \cos\vartheta)$,

$$M_{\alpha\beta}(\tilde{\boldsymbol{k}}) = M_{\alpha\beta}(\vartheta, \varphi), \tag{11}$$

in which $M_{\alpha\beta}(\vartheta, \varphi)$ is an appropriate function of ϑ and φ, but independent of k. Generally, $M_{\alpha\beta}(\vartheta, \varphi)$ does not need to be symmetric in α and β. Also, $M_{\alpha\beta}(\vartheta, \varphi)$ does not need to have reflectional symmetry in general.

In the following, we assume $M_{\alpha\beta}(\vartheta, \varphi) \in L^2(S^2)$, that is, $\int_{S^2} |M_{\alpha\beta}(\vartheta, \varphi)|^2 \, d\vartheta \, d\varphi < \infty$. Here $L^2(S^2)$ is the space of square integrable functions on the unit spherical surface. As is well known, any $M_{\alpha\beta}(\vartheta, \varphi) \in L^2(S^2)$ can be expanded by Laplace's spherical harmonics $Y_n^m(\vartheta, \varphi)$ as

$$M_{\alpha\beta}(\vartheta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{mn}^{\alpha\beta} Y_n^m(\vartheta, \varphi), \tag{12}$$

where $Y_n^m(\vartheta, \varphi) \propto P_n^m(\cos\vartheta) \exp(im\varphi)$ and $P_n^m(\cos\vartheta)$ is the Legendre polynomials. The expansion (12) is equivalent to the following expansion in the powers of $\tilde{k}_1$, $\tilde{k}_2$, and $\tilde{k}_3$ (see e.g., Ref. [15]):

$$M_{\alpha\beta}(\tilde{\boldsymbol{k}}) = \sum_{a,b,c=0}^{\infty} B_{abc}^{\alpha\beta} \tilde{k}_1^a \tilde{k}_2^b \tilde{k}_3^c. \tag{13}$$

If $M_{\alpha\beta}(\tilde{\boldsymbol{k}})$ is $\tilde{\boldsymbol{k}}$ independent, then $A_{mn}^{\alpha\beta} = B_{abc}^{\alpha\beta} = 0$ for $(m, n) \neq (0, 0)$ and $(a, b, c) \neq (0, 0, 0)$, and $A_{00}^{\alpha\beta} (= B_{000}^{\alpha\beta})$ is Saffman's invariant. We now show the invariance of $M_{\alpha\beta}(\tilde{\boldsymbol{k}})$, $A_{mn}^{\alpha\beta}$ and $B_{abc}^{\alpha\beta}$. Hereafter, the turbulence with the invariants, $A_{mn}^{\alpha\beta}$ and $B_{abc}^{\alpha\beta}$, is called "generalized Saffman turbulence." Suppose that an infinitesimally small localized disturbance centered at $\boldsymbol{x}$ is added to the turbulence field. Then it immediately affects the convective term in the disturbed region, and this results in the change of the pressure gradient as well as the velocity field at $\boldsymbol{x} + \boldsymbol{r}$, and the changes are $O(r^{-3})$ at $r \to \infty$, where $r = |\boldsymbol{r}|$. This suggests that the interaction between two points are nonlocal, so that the correlation $\langle N_{\alpha\beta}(\boldsymbol{x}) u_j(\boldsymbol{x} + \boldsymbol{r}) \rangle$ in Eq. (6) decays algebraically in r at $r \to \infty$. We assume here that this correlation decays algebraically as $O(r^{-s})$ with $s = 3$ at $r \to \infty$. Then the integral (6) is finite in the limit of $\boldsymbol{k} \to \boldsymbol{0}$, so that

$$\hat{\Gamma}_{\alpha\beta j}(\boldsymbol{k}) = O(k^0) \quad \text{as} \quad \boldsymbol{k} \to \boldsymbol{0}. \tag{14}$$

Since $k_\alpha P_{i\beta} = O(k)$, Eqs. (5) and (14) imply

$$T_{ij}(\boldsymbol{k}) = O(k) \quad \text{as} \quad \boldsymbol{k} \to \boldsymbol{0}. \tag{15}$$

The estimate (14) need not be limited to $s = 3$, and we have $T_{ij}(\boldsymbol{k}) = O(k^{s-2})$ for $s > 2$ instead of the estimate (14). We then obtain

$$T_{ij}(\boldsymbol{k}) = o(k^0) \quad \text{as} \ \boldsymbol{k} \to \boldsymbol{0}. \tag{16}$$

For incompressible homogeneous axisymmetric turbulence subjected to external force with $E(k) \propto k^2$ at $k \to 0$ [9] as well as Saffman turbulence [2,9], Davidson presented an argument in accordance with the assumption $\langle N_{\alpha\beta}(\boldsymbol{x}) u_j(\boldsymbol{x} + \boldsymbol{r}) \rangle = O(r^{-3})$ at $r \to \infty$ and the estimate (15). It is seen that the estimates (15) and (16) are in accordance with a class of spectral closure theories including eddy-damped quasinormal Markovian (EDQNM) closures (see e.g., Ref. [1]), abridged Lagrangian history direct interaction approximation (see, e.g., Ref. [16]) and Lagrangian renormalized approximation (see, e.g., Ref. [17]). Another argument for the estimate (15) is given in Appendix A.

Equations (4) and (15) give

$$\frac{\partial}{\partial t}\hat{R}_{ij}(\boldsymbol{k},t) = O(k) \quad \text{as} \quad \boldsymbol{k} \to \boldsymbol{0}. \tag{17}$$

Equation (17) implies that the $O(k^0)$ term in $\hat{R}_{ij}(\boldsymbol{k},t)$ for $\boldsymbol{k} \to \boldsymbol{0}$ is time independent. The same is true for $\mathcal{M}_{\alpha\beta}(\boldsymbol{k},t)$, that is, if $\mathcal{M}_{\alpha\beta}(\boldsymbol{k},t)$ at $t = t_0$ is given by Eq. (10), then $\mathcal{M}_{\alpha\beta}(\boldsymbol{k},t)$ at any time t ($\geqslant t_0$) is given by

$$\mathcal{M}_{\alpha\beta}(\boldsymbol{k},t) = M_{\alpha\beta}(\tilde{\boldsymbol{k}}) + o(1) \quad \text{as} \quad \boldsymbol{k} \to \boldsymbol{0}, \tag{18}$$

where $M_{\alpha\beta}(\tilde{\boldsymbol{k}})$ is a time-independent constant determined by the initial condition (10). This means that $A^{\alpha\beta}_{mn}$ and $B^{\alpha\beta}_{abc}$ in Eqs. (12) and (13) are dynamical invariants. Based on Eqs. (7), (10), and (18), we obtain

$$\hat{R}_{ij}(\boldsymbol{k},t) = P_{i\alpha}(\tilde{\boldsymbol{k}})P_{j\beta}(\tilde{\boldsymbol{k}})M_{\alpha\beta}(\tilde{\boldsymbol{k}}) + o(1) \quad \text{as} \quad \boldsymbol{k} \to \boldsymbol{0}, \tag{19}$$

for any t ($\geqslant t_0$), and the energy spectrum $E(k,t) = Ck^2 + o(k^2)$ with C being an invariant. Saffman [4] reported the invariance of $M_{\alpha\beta}$ in Eq. (9) under the assumption of the analyticity of $\hat{\boldsymbol{u}}(\boldsymbol{k},t)$ in time. A discussion about the time analyticity is given in Appendix B.

There are representations of $\hat{R}_{ij}(\boldsymbol{k})$ different from Eq. (7), for example, the so-called $\mathcal{E}$-$\mathcal{Z}$-$\mathcal{H}$ decomposition used in Ref. [18]. Here we use Eq. (7), because its relation to Eq. (9) is clear. The relation between the decomposition and Eq. (7) is discussed in Appendix C. A computational tool using the spherical harmonics expansion of $\hat{R}_{ij}(\boldsymbol{k})$ was proposed for the linear inviscid flow dynamics [21]. The application of vectorial spherical harmonics (see e.g., Ref. [1]) to $\hat{R}_{ij}(\boldsymbol{k},t)$, which is a function of k as well as $\tilde{\boldsymbol{k}}$, is given in Refs. [19,20].

The large-scale structure of the velocity field is characterized by $P_{i\alpha}(\tilde{\boldsymbol{k}})P_{j\beta}(\tilde{\boldsymbol{k}})M_{\alpha\beta}(\tilde{\boldsymbol{k}})$, the $O(k^0)$ term, in Eq. (19). Taking the limit of Eq. (3) at $\boldsymbol{k} \to \boldsymbol{0}$ for any fixed $\tilde{\boldsymbol{k}}$, and noting that the leading term of Eq. (19) is $O(k^0)$, we see that the far-field term of $R_{ij}(\boldsymbol{r})$ which is denoted by $R^{\infty}_{ij}(\boldsymbol{r})$, the leading order term of $R_{ij}(\boldsymbol{r})$ at sufficiently large r of $R_{ij}(\boldsymbol{r})$, decays as $O(r^{-3})$, and $R^{\infty}_{ij}(\boldsymbol{r})$ is completely determined by $M_{\alpha\beta}(\tilde{\boldsymbol{k}})$, the coefficients $A^{\alpha\beta}_{mn}$ and $B^{\alpha\beta}_{abc}$. For isotropic Saffman turbulence, $R_{\alpha\alpha}(\boldsymbol{r}) = R_{11}(\boldsymbol{r}) + R_{22}(\boldsymbol{r}) + R_{33}(\boldsymbol{r}) = o(r^{-3})$, but $R_{11}(\boldsymbol{r}) = R_{22}(\boldsymbol{r}) = R_{33}(\boldsymbol{r}) = O(r^{-3})$ at $r \to \infty$ [4]. Let us consider the integral $\int_{\mathbb{R}^3} R_{ij}(\boldsymbol{r})\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,d\boldsymbol{r}$ at $\boldsymbol{k} \to \boldsymbol{0}$. One might think that the contribution of $R^{\infty}_{ij}(\boldsymbol{r})$ at $r \gg 1$ to $\int_{\mathbb{R}^3} R_{ij}(\boldsymbol{r})\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,d\boldsymbol{r}$ may vanish owing to the long-wavelength oscillations at $\boldsymbol{k} \to \boldsymbol{0}$. However, the oscillations eliminate the contribution from the far-field satisfying the condition $r \gg 1/k$ ($\gg$1), but they do not eliminate the far-field contribution from $r \sim 1/k$ ($\gg$1). Therefore, the contribution of $R^{\infty}_{ij}(\boldsymbol{r})$ at $\boldsymbol{k} \to \boldsymbol{0}$ does not vanish. It is to be noted that statistics of turbulence at $k \to 0$ are in general not necessarily uniquely determined by statistics of turbulence in the far field, because they may be generally affected by not only far-field statistics but also non-far-field statistics.

Llor and Soulard [22] gave a discussion on the relation between the two-point second-order longitudinal correlation function of velocity at $r \to \infty$ and the energy spectrum $E(k)$ at $k \to 0$, which is given by $E(k) = C_s k^s + o(k^s)$ ($1 \leqslant s \leqslant 4$), for incompressible homogeneous isotropic turbulence. Here C_s is a k-independent constant.

III. LARGE-SCALE STRUCTURE OF A FREELY DECAYING PASSIVE SCALAR FIELD

A freely decaying passive scalar field in incompressible homogeneous turbulence is now considered. The passive scalar field $\theta(\boldsymbol{x},t)$ obeys the scalar advection diffusion equation,

$$\frac{\partial \theta}{\partial t} = -(\boldsymbol{u}\cdot\nabla)\theta + \kappa\nabla^2\theta, \tag{20}$$

where κ is the molecular or thermal diffusivity, and $\langle\theta\rangle = 0$. The spectral correlation $\hat{\Theta}(\boldsymbol{k}, t)$ is defined by the Fourier transform of the two-point scalar correlation; $\Theta(\boldsymbol{r}, t) = \langle\theta(\boldsymbol{x}, t)\theta(\boldsymbol{x} + \boldsymbol{r}, t)\rangle$.

Corrsin [13] considered freely decaying isotropic passive scalar fluctuations where $\Theta(\boldsymbol{r}, t) = o(r^{-3})$ and third-order scalar-velocity correlations $\langle N^\theta_\alpha(\boldsymbol{x}, t)\theta(\boldsymbol{x} + \boldsymbol{r}, t)\rangle = o(r^{-3})$ at $r \to \infty$, in which $N^\theta_\alpha(\boldsymbol{x}) = u_\alpha(\boldsymbol{x})\theta(\boldsymbol{x})$. In the turbulence, the $O(k^0)$ term of $\hat{\Theta}(\boldsymbol{k}, t)$ is independent of the direction of $\tilde{\boldsymbol{k}}$ and time. Corrsin's argument is here generalized to anisotropic scalar fluctuations in incompressible homogeneous anisotropic turbulence. We assume that

$$\hat{\Theta}(\boldsymbol{k}, t_0) = \chi(\tilde{\boldsymbol{k}}) + o(1) \quad \text{as} \quad \boldsymbol{k} \to \boldsymbol{0}, \tag{21}$$

at an initial time instant t_0. Below, it is shown that for any $t\ (\geqslant t_0)$

$$\hat{\Theta}(\boldsymbol{k}, t) = \chi(\tilde{\boldsymbol{k}}) + o(1) \quad \text{as} \quad \boldsymbol{k} \to \boldsymbol{0} \tag{22}$$

under certain assumptions. The large-scale structure of the passive scalar field is characterized by $\chi(\tilde{\boldsymbol{k}})$. In Eq. (20), the velocity field is assumed to obey Eqs. (1) and (2), but a term representing external forces can be added to the right-hand side of Eq. (1).

Equation (20) gives

$$\frac{\partial}{\partial t}\hat{\Theta}(\boldsymbol{k}, t) = T^\theta(\boldsymbol{k}, t) - 2\kappa k^2\hat{\Theta}(\boldsymbol{k}, t), \tag{23}$$

where

$$T^\theta(\boldsymbol{k}, t) = \mathrm{i}k_\alpha\Gamma^\theta_\alpha(\boldsymbol{k}, t) - \mathrm{i}k_\alpha\Gamma^\theta_\alpha(-\boldsymbol{k}, t), \tag{24}$$

in which $\Gamma^\theta_\alpha(\boldsymbol{k}, t)$ is defined by

$$\Gamma^\theta_\alpha(\boldsymbol{k}, t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\left\langle N^\theta_\alpha(\boldsymbol{x})\theta(\boldsymbol{x} + \boldsymbol{r})\right\rangle \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,d\boldsymbol{r}. \tag{25}$$

We assume here that $\langle N^\theta_\alpha(\boldsymbol{x})\theta(\boldsymbol{x} + \boldsymbol{r})\rangle$ decays as $O(r^{-s})$ for $s = 3$ at $r \to \infty$ so that $\Gamma^\theta_\alpha(\boldsymbol{k}, t)$ is finite in the limit of $\boldsymbol{k} \to \boldsymbol{0}$. We have

$$\Gamma^\theta_\alpha(\boldsymbol{k}, t) = O(k^0) \quad \text{as} \quad \boldsymbol{k} \to \boldsymbol{0}, \tag{26}$$

which implies

$$T^\theta(\boldsymbol{k}, t) = O(k) \quad \text{as} \quad \boldsymbol{k} \to \boldsymbol{0}. \tag{27}$$

We have $T^\theta(\boldsymbol{k}) = o(k^0)$ for $s > 2$ instead of the estimate (27), as discussed in the derivation of Eq. (16). The estimate (27) can be also obtained by an argument similar to that in Appendix A. These estimates of $T^\theta(\boldsymbol{k})$ are in accordance with spectral closure theories.

Equations (23) and (27) give

$$\frac{\partial}{\partial t}\hat{\Theta}(\boldsymbol{k}, t) = O(k) \quad \text{as} \quad \boldsymbol{k} \to \boldsymbol{0}. \tag{28}$$

Equation (28) implies that the $O(k^0)$ term in $\hat{\Theta}(\boldsymbol{k}, t)$ for $\boldsymbol{k} \to \boldsymbol{0}$ is time independent, i.e., Eq. (22) holds for any time $t\ (\geqslant t_0)$ and $\chi(\tilde{\boldsymbol{k}})$ is a time-independent constant determined by the initial condition (21). By the use of the expansion by Laplace's spherical harmonics or in the powers of $\tilde{k}_1$, $\tilde{k}_2$, and $\tilde{k}_3$, such as Eqs. (12) or (13), we obtain

$$\chi(\vartheta, \varphi) = \sum_{\substack{n=0\\(n=\text{even})}}^{\infty}\sum_{m=-n}^{n} A^\theta_{mn}Y^m_n(\vartheta, \varphi) \tag{29}$$

or

$$\chi(\tilde{\boldsymbol{k}}) = \sum_{\substack{a,\,b,\,c=0 \\ (a+b+c=\text{even})}}^{\infty} B^{\theta}_{abc}\tilde{k}_1^a\tilde{k}_2^b\tilde{k}_3^c, \tag{30}$$

noting that $\hat{\Theta}(\boldsymbol{k},t)$ is reflectionally symmetric, $\hat{\Theta}(-\boldsymbol{k},t) = \hat{\Theta}(\boldsymbol{k},t)$, by definition. The time independence of $\chi(\tilde{\boldsymbol{k}})$ for any $\tilde{\boldsymbol{k}}$ implies that A^{θ}_{mn} and B^{θ}_{abc} are dynamical invariants. These invariants are called here generalized Corrsin's invariants. Note that $\chi(\tilde{\boldsymbol{k}})$ is generally discontinuous at $\boldsymbol{k} = \boldsymbol{0}$. Moreover, it is shown that the scalar spectrum $E^{\theta}(k)$, defined by $(1/2)\int \hat{\Theta}(\boldsymbol{q},t)dS_k$, takes the time-independent form of $E^{\theta}(k) = C^{\theta}k^2 + o(k^2)$ at $k \to 0$ with C^{θ} being a positive constant.

As is the case of $R_{ij}(\boldsymbol{r},t)$, taking the limit of $\boldsymbol{k} \to \boldsymbol{0}$ for $\int_{\mathbb{R}^3} \Theta(\boldsymbol{r},t)\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,d\boldsymbol{r}$, and using Eq. (22), we generally obtain $\langle\theta(\boldsymbol{x})\theta(\boldsymbol{x}')\rangle = O(r^{-3})$ at $r \to \infty$, where $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{r}$. The $O(r^{-3})$ term of $\langle\theta(\boldsymbol{x})\theta(\boldsymbol{x}')\rangle$ corresponds to the anisotropic part of $\chi(\tilde{\boldsymbol{k}})$, $\chi(\tilde{\boldsymbol{k}}) - B^{\theta}_{000}$. The integral of $\langle\theta(\boldsymbol{x})\theta(\boldsymbol{x}')\rangle$ without the $O(r^{-3})$ term over the whole $\boldsymbol{r}$ domain is absolutely convergent, because $\langle\theta(\boldsymbol{x})\theta(\boldsymbol{x}')\rangle$ without the $O(r^{-3})$ term is $o(r^{-3})$ at $r \to \infty$. Hence, the Fourier transform of the $o(r^{-3})$ term becomes a constant that is independent of $\tilde{\boldsymbol{k}}$ for $k \ll 1$. This shows that the $o(r^{-3})$ term corresponds to the isotropic part of $\chi(\tilde{\boldsymbol{k}})$, B^{θ}_{000}.

We also introduce $D(\boldsymbol{k},t)$, the Fourier transform of $\langle(\partial/\partial x_\alpha)\theta(\boldsymbol{x})(\partial/\partial x'_\alpha)\theta(\boldsymbol{x}')\rangle$, which will be used in Sec. V. Since $D(\boldsymbol{k},t) = k^2\hat{\Theta}(\boldsymbol{k},t)$,

$$D(\boldsymbol{k},t) = k^2\chi(\tilde{\boldsymbol{k}}) + o(k^2) \quad \text{as} \quad \boldsymbol{k} \to \boldsymbol{0}. \tag{31}$$

Therefore, because of Eq. (22), we find that Eq. (31) holds for any time t ($\geqslant t_0$).

IV. SELF-SIMILARITY OF GENERALIZED SAFFMAN TURBULENCE

We first discuss the self-similarity of the generalized Saffman turbulence for sufficiently small k. By the use of the self-similarity and invariance of $M_{\alpha\beta}(\tilde{\boldsymbol{k}})$, the persistence of flow anisotropy at large scales is shown. Subsequently, the decay laws of the fully developed generalized Saffman turbulence are obtained by using dimensional analysis.

A. Self-similarity and invariance of $M_{\alpha\beta}(\tilde{\boldsymbol{k}})$

We assume that large eddies evolve in accordance with the self-similar form,

$$\hat{R}_{ij}(\boldsymbol{k},t) = c_{ij}(t)f_{ij}(k_1\ell_1, k_2\ell_2, k_3\ell_3) = c_{ij}(t)f_{ij}(\boldsymbol{\zeta}), \tag{32}$$

in a certain time range and domain of the wave vector space $\boldsymbol{k}$ including small enough k range, where $\boldsymbol{\zeta}$ is a self-similar variable defined by

$$\boldsymbol{\zeta} = (k_1\ell_1, k_2\ell_2, k_3\ell_3), \tag{33}$$

$c_{ij}(t)$ is an appropriate function of time t, and $\ell_m(t)$ is an appropriate length scale in the mth Cartesian direction. The length scale ℓ_m may depend on i and j. Therefore, it can be written as ℓ_m^{ij}. However, for the convenience of writing and reading, we simply write it as ℓ_m. Note that $\hat{R}_{ij}(\boldsymbol{k},t)$ depends on time t only through $c_{ij}(t)$, $\boldsymbol{\ell}(t) = (\ell_1(t), \ell_2(t), \ell_3(t))$, and $f_{ij}(k_1\ell_1, k_2\ell_2, k_3\ell_3) = f_{ij}(\boldsymbol{\zeta})$ is time independent at any fixed $\boldsymbol{\zeta}$. The function $f_{ij}(\boldsymbol{\zeta})$ is dimensionless. Below we show that $c_{ij}(t)$ is constant and $c_{ij} \simeq \text{constant} \times \langle u_iu_j\rangle\ell_1\ell_2\ell_3$ [see Eq. (51)].

Based on Eqs. (19) and (32), we obtain

$$c_{ij}(t)f_{ij}(\boldsymbol{\zeta}) = \Phi_{ij}(\tilde{\boldsymbol{k}}) + o(1) \quad \text{as} \quad \boldsymbol{k} \to \boldsymbol{0} \tag{34}$$

for $t \geqslant t_0$, where

$$\Phi_{ij}(\tilde{\boldsymbol{k}}) = P_{i\alpha}(\tilde{\boldsymbol{k}})P_{j\beta}(\tilde{\boldsymbol{k}})M_{\alpha\beta}(\tilde{\boldsymbol{k}}). \tag{35}$$

Because

$$\boldsymbol{k} = \left(\frac{\zeta_1}{\ell_1}, \frac{\zeta_2}{\ell_2}, \frac{\zeta_3}{\ell_3}\right), \tag{36}$$

Eq. (34) means

$$c_{ij}(t) f_{ij}(\boldsymbol{\zeta}) = \Phi_{ij}(\tilde{\boldsymbol{k}}) + o(1) \quad \text{as} \quad \boldsymbol{\zeta} \to \boldsymbol{0}. \tag{37}$$

Equation (37) implies that

$$c_{ij}(t) f^0_{ij}(\tilde{\boldsymbol{\zeta}}) = \Phi_{ij}(\tilde{\boldsymbol{k}}), \tag{38}$$

where $\tilde{\boldsymbol{\zeta}} = \boldsymbol{\zeta}/|\boldsymbol{\zeta}|$, $f^0_{ij}(\tilde{\boldsymbol{\zeta}})$ is the leading order term of $f_{ij}(\boldsymbol{\zeta})$ in the limit of $\boldsymbol{\zeta} \to \boldsymbol{0}$, and $f^0_{ij}(\tilde{\boldsymbol{\zeta}})$ is independent of time at any fixed $\tilde{\boldsymbol{\zeta}}$. As shown in Sec. II B, $\Phi_{ij}(\tilde{\boldsymbol{k}})$ must be time independent at any fixed $\tilde{\boldsymbol{k}}$. This time independence at fixed $\tilde{\boldsymbol{k}}$ should not be confused with the time independence at fixed $\boldsymbol{\zeta}$. In fact, $\Phi_{ij}(\tilde{\boldsymbol{k}})$ may be time independent in general at fixed $\boldsymbol{\zeta}$, because $\tilde{\boldsymbol{k}}$ may be time dependent at fixed $\boldsymbol{\zeta}$, as seen in Eq. (36).

To get an idea about the possible time dependence of $\Phi_{ij}(\tilde{\boldsymbol{k}})$ at fixed $\boldsymbol{\zeta}$ or $\tilde{\boldsymbol{\zeta}}$, it may be of help to note that

$$\tilde{k}_i = \frac{k_i}{k} = \frac{\zeta_i/\ell_i}{\{(\zeta_1/\ell_1)^2 + (\zeta_2/\ell_2)^2 + (\zeta_3/\ell_3)^2\}^{1/2}} = \frac{\tilde{\zeta}_i/\tilde{\ell}_i}{\{(\tilde{\zeta}_1/\tilde{\ell}_1)^2 + (\tilde{\zeta}_2/\tilde{\ell}_2)^2 + (\tilde{\zeta}_3/\tilde{\ell}_3)^2\}^{1/2}}, \tag{39}$$

where $\tilde{\boldsymbol{\ell}} = \boldsymbol{\ell}/\ell$, and $\ell = |\boldsymbol{\ell}|$. Although, as is clear from Eq. (39), $\tilde{\boldsymbol{k}}$ may be time dependent in general at fixed $\boldsymbol{\zeta}$ or $\tilde{\boldsymbol{\zeta}}$, it must be time independent at some particular points of $\boldsymbol{\zeta}$ or $\tilde{\boldsymbol{\zeta}}$. For example, $\tilde{\boldsymbol{k}} = (1, 0, 0)$ for $\tilde{\boldsymbol{\zeta}} = (1, 0, 0)$, so that $\tilde{\boldsymbol{k}}$ is time independent for $\tilde{\boldsymbol{\zeta}} = (1, 0, 0)$. This and the time independence of $\Phi_{ij}(\tilde{\boldsymbol{k}})$ at fixed $\tilde{\boldsymbol{k}}$ imply that $\Phi_{ij}(\tilde{\boldsymbol{k}})$ at $\tilde{\boldsymbol{\zeta}} = (1, 0, 0)$ is time independent. Therefore, Eq. (38) for $\tilde{\boldsymbol{k}} = (1, 0, 0)$ results in

$$c_{ij}(t) = \text{const}, \tag{40}$$

provided that $f^0_{ij}(1, 0, 0) \neq 0$.

Note that $\Phi_{ij}(\tilde{\boldsymbol{k}})$ on the right-hand side in Eq. (38) is time independent at any fixed $\tilde{\boldsymbol{k}}$ as a consequence of the dynamics under consideration, whereas $f^0_{ij}(\tilde{\boldsymbol{\zeta}})$ is time independent at any fixed $\tilde{\boldsymbol{\zeta}}$ as a consequence of the self-similarity assumption (32). Since c_{ij} is time independent, as shown above, $\Phi_{ij}(\tilde{\boldsymbol{k}})$ must be time independent not only at any fixed $\tilde{\boldsymbol{k}}$ but also at any fixed $\tilde{\boldsymbol{\zeta}}$.

Because of Eq. (39), $\tilde{\boldsymbol{k}}$ is time independent at any fixed $\tilde{\boldsymbol{\zeta}}$, if

$$\tilde{\ell}_j = \frac{\ell_j}{\ell} = \text{const for} \quad j = 1, 2, 3. \tag{41}$$

Therefore, if Eq. (41) holds, $\Phi_{ij}(\tilde{\boldsymbol{k}})$ can be time independent not only at any fixed $\tilde{\boldsymbol{k}}$ but also at any fixed $\tilde{\boldsymbol{\zeta}}$. Thus, Eq. (41) is a sufficient condition for the compatibility of (i) the time independence of $\Phi_{ij}(\tilde{\boldsymbol{k}})$ at any fixed $\tilde{\boldsymbol{k}}$ and (ii) the time independence at any fixed $\tilde{\boldsymbol{\zeta}}$, i.e., the time independence of $\tilde{\ell}_j$ $(j = 1, 2, 3)$ as shown in Eq. (41) is a sufficient condition for the compatibility of the dynamics and the self-similarity assumption (32).

Equation (39) suggests that Eq. (41) is not only a sufficient condition but also a necessary condition for the compatibility, although it is not trivial. As a matter of fact, under weak assumptions, this can be confirmed to be true. In other words, Eq. (41) is an inevitable consequence of the compatibility of the invariance of $\Phi_{ij}(\tilde{\boldsymbol{k}})$ at any fixed $\tilde{\boldsymbol{k}}$ and the self-similarity assumption (32), as shown below. This implies that we do not need to introduce any other extra assumption to verify the time independence (41).

Since $|\tilde{\boldsymbol{k}}| = 1$, at least one of the components $\tilde{k}_1$, $\tilde{k}_2$, or $\tilde{k}_3$ is nonzero. Let us assume that $\tilde{k}_1 > 0$ and consider the expansion of $\Phi_{ij}(\tilde{\boldsymbol{k}})$ in the powers of $\tilde{k}_2/\tilde{k}_1$ for small $\tilde{k}_2/\tilde{k}_1$ at $\tilde{k}_3 = 0$. For $\tilde{k}_1 \neq 1$

and $\tilde{k}_3 = 0$, we obtain

$$\tilde{k}_1 = \frac{1}{\sqrt{1 + (k_2/k_1)^2}}, \quad \tilde{k}_2 = \frac{k_2/k_1}{\sqrt{1 + (k_2/k_1)^2}}, \tag{42}$$

where $k_2/k_1 = \tilde{k}_2/\tilde{k}_1$. Equation (13) implies that $\Phi_{ij}(\tilde{\boldsymbol{k}})$ can be expanded as

$$\Phi_{ij}(\tilde{\boldsymbol{k}}) = \sum_{a,b,c=0}^{\infty} C^{ij}_{abc} \tilde{k}_1^a \tilde{k}_2^b \tilde{k}_3^c, \tag{43}$$

where C^{ij}_{abc} is an appropriate constant independent of time and $\tilde{\boldsymbol{k}}$, and C^{ij}_{abc} is expressed by the linear combination of $B^{\alpha\beta}_{abc}$ in Eq. (13). By substituting $\tilde{k}_3 = 0$ and Eq. (42) into Eq. (43), and expanding $\Phi_{ij}(\tilde{k}_1, \tilde{k}_2, 0)$ in the powers of $\sigma = \tilde{k}_2/\tilde{k}_1 = k_2/k_1$ for small σ, we obtain

$$\Phi_{ij}(\tilde{k}_1, \tilde{k}_2, 0) = \sum_{m=0}^{\infty} \Xi_m \sigma^m, \tag{44}$$

where Ξ_m is an appropriate constant independent of time and $\tilde{\boldsymbol{k}}$. This implies that $\Phi_{ij}(\tilde{k}_1, \tilde{k}_2, 0)$ depends on $\tilde{k}_1$ and $\tilde{k}_2$ only through $\sigma = \tilde{k}_2/\tilde{k}_1 = k_2/k_1 = (\zeta_2/\ell_2)/(\zeta_1/\ell_1) = \gamma\lambda$, where $\gamma = \ell_1/\ell_2 = \tilde{\ell}_1/\tilde{\ell}_2$ and $\lambda = \zeta_2/\zeta_1 = \tilde{\zeta}_2/\tilde{\zeta}_1$.

Since $\tilde{k}_3 = 0$ for $\tilde{\zeta}_3 = 0$, Eq. (38) gives

$$c_{ij} f^0_{ij}(\tilde{\zeta}_1, \tilde{\zeta}_2, 0) = \Phi_{ij}(\tilde{k}_1, \tilde{k}_2, 0), \tag{45}$$

where c_{ij} is time independent as shown in Eq. (40). Since Eq. (44) means that $\Phi_{ij}(\tilde{k}_1, \tilde{k}_2, 0)$ is a function of only σ ($=\gamma\lambda$), and since Eqs. (44) and (45) show that $f^0_{ij}(\tilde{\zeta}_1, \tilde{\zeta}_2, 0)$ is a function of only λ for small σ, hereafter we simply write $\Phi_{ij}(\tilde{k}_1, \tilde{k}_2, 0)$ and $f^0_{ij}(\tilde{\zeta}_1, \tilde{\zeta}_2, 0)$ as $\Phi_{ij}(\gamma\lambda)$ and $f^0_{ij}(\lambda)$, respectively. Because of the time independence of $f^0_{ij}(\tilde{\boldsymbol{\zeta}})$ at any $\tilde{\boldsymbol{\zeta}}$, $(d/d\lambda)^N f^0_{ij}(\lambda)$ must be also time independent for any positive integer N at any $\lambda = \tilde{\zeta}_2/\tilde{\zeta}_1$, in particular at $\lambda = 0$. Therefore, Eqs. (44) and (45) give

$$\left[\frac{d^N}{d\lambda^N} c_{ij} f^0_{ij}(\lambda)\right]_{\lambda=0} = \left[\frac{\partial^N}{\partial\lambda^N} \Phi_{ij}(\gamma\lambda)\right]_{\lambda=0} = N!\, \Xi_N \gamma^N = \text{time independent}. \tag{46}$$

Since Ξ_N is constant, Eq. (46) implies

$$\frac{\ell_1}{\ell_2} = \text{const}, \tag{47}$$

provided that there is any positive integer N such that $\Xi_N \neq 0$.

By considering the expansion for small $\tau = \tilde{k}_3/\tilde{k}_1$ at $\tilde{k}_2 = 0$, in the same way as the derivation of Eqs. (44) and (47), we can show

$$\Phi_{ij}(\tilde{k}_1, 0, \tilde{k}_3) = \sum_{m=0}^{\infty} \Xi'_m \tau^m \tag{48}$$

and

$$\frac{\ell_1}{\ell_3} = \text{const}, \tag{49}$$

if there is any positive integer N such that $\Xi'_N \neq 0$. Here Ξ'_m is an appropriate constant independent of time and $\tilde{\boldsymbol{k}}$. Equations (47) and (49) imply that

$$\frac{\ell_j}{\ell} = \text{const for} \quad j = 1, 2, 3. \tag{50}$$

Equation (50) holds, even if $M_{\alpha\beta}(\vartheta, \varphi)$ is independent of ϑ and φ.

B. Persistence of flow anisotropy in the energy-containing range

We assume that Eq. (32) is a good approximation for $\hat{R}_{ij}(\boldsymbol{k},t)$ in the $(\boldsymbol{k},t)$ domain including a small enough k range. In a strict sense, we do not impose self-similarity on the $\boldsymbol{k}$ range including the viscosity-dominant wave vector range. The contribution of $\hat{R}_{ij}(\boldsymbol{k},t)$ in the $(\boldsymbol{k},t)$ domain including a small enough k range is dominant in certain integrals such as $\int_{\mathbb{R}^3}\hat{R}_{ij}(\boldsymbol{k},t)\,d\boldsymbol{k}$. On the basis of Eq. (32), we obtain

$$\langle u_i u_j\rangle(t) = \int_{\mathbb{R}^3}\hat{R}_{ij}(\boldsymbol{k},t)\dot{\boldsymbol{k}} \simeq \int_{\mathbb{R}^3} c_{ij} f_{ij}(k_1\ell_1,k_2\ell_2,k_3\ell_3)\,d\boldsymbol{k}$$
$$= \frac{c_{ij}}{\ell_1\ell_2\ell_3}\int_{\mathbb{R}^3} f_{ij}(\boldsymbol{\zeta})\,d\boldsymbol{\zeta}. \tag{51}$$

Then, we have $c_{ij}\simeq \text{const}\times\langle u_i u_j\rangle\ell_1\ell_2\ell_3$ and thus

$$\langle u_i u_j\rangle\ell_1\ell_2\ell_3 \simeq \text{const}, \tag{52}$$

where this constant value may depend on i and j.

In this paper, we consider only freely decaying homogeneous turbulence, and assume that the time dependence of the length scales ℓ_1, ℓ_2, and ℓ_3 is independent from the "component" (i,j) under the self-similarity. Using Eq. (52), we then obtain

$$\frac{\langle u_i^2\rangle}{\langle u_j^2\rangle} \simeq \text{const for } i,j=1,2,3\,(i\neq j). \tag{53}$$

The independence is called "componential independence." This componential independence was also assumed in Ref. [7]. However, this assumption is not trivial, and it is not surprising if it does not hold for other types of turbulence under external forces, such as rotating turbulence, stably stratified turbulence, and magnetohydrodynamic turbulence subjected to a uniform magnetic field. Hence, this assumption needs to be examined. The DNS results in Ref. [7] are consistent with this assumption.

Now, we consider the relation between the length scales ℓ_i $(i=1,2,3)$ and the length scales measured in DNS such as

$$L_j^{(n)}(t) = \frac{\int_0^\infty \langle u_n(\boldsymbol{x}+r\boldsymbol{i}_j)u_n(\boldsymbol{x})\rangle\,dr}{\langle u_n^2\rangle}, \tag{54}$$

where $\boldsymbol{i}_j$ is the unit vector in the jth Cartesian direction. The length scale $L_1^{(n)}$ is given by

$$L_1^{(n)} = \pi\frac{\int_{\mathbb{R}^2}\hat{R}_{nn}(0,k_2,k_3,t)\,dk_2\,dk_3}{\int_{\mathbb{R}^3}\hat{R}_{nn}(\boldsymbol{k},t)\,d\boldsymbol{k}} \simeq \gamma_1^{(n)}\ell_1, \tag{55}$$

where $\gamma_1^{(n)}$ is constant. Substitution of Eq. (32) into Eq. (55) gives

$$\gamma_1^{(n)} = \pi\frac{\int_{\mathbb{R}^2} f_{nn}(0,\zeta_2,\zeta_3)\,d\zeta_2\,d\zeta_3}{\int_{\mathbb{R}^3} f_{nn}(\boldsymbol{\zeta})\,d\boldsymbol{\zeta}}. \tag{56}$$

Similarly, we obtain

$$L_2^{(n)} \simeq \text{const}\times\ell_2,\quad L_3^{(n)}\simeq\text{const}\times\ell_3. \tag{57}$$

Therefore, Eq. (50) yields

$$\frac{L_i^{(n)}}{L_j^{(n)}} \simeq \text{const for } i,j=1,2,3\,(i\neq j). \tag{58}$$

Note that $\langle u_j^2\rangle$ and $L_j^{(n)}$ are representative quantities in the energy-containing range, whereas the invariants are representative quantities in the far field. Therefore, the self-similarity we assumed here is the self-similarity of flow ranging from the far field to the energy-containing range.

Based on the invariance of $M_{\alpha\beta}(\tilde{\boldsymbol{k}})$ and the flow self-similarity in freely decaying turbulence without any external force, we showed the persistence of the flow anisotropy measured by $\langle u_i^2\rangle/\langle u_j^2\rangle$ in Eq. (53) and $L_i^{(n)}/L_j^{(n)}$ in Eq. (58). Therefore, one might think that $L_i^{(n)}/L_j^{(n)}$ for $i \neq j$ is also constant for turbulence under external forces. The analysis in Sec. II is only valid for turbulence without any external force. It does not exclude the possibility that the length-scale ratios are not constant for turbulence under external forces (see Appendix D).

C. Decay laws for generalized Saffman turbulence

We consider the decay of generalized anisotropic Saffman turbulence. Because of Eqs. (53) and (58), we need to use only one length scale and the intensity of one velocity component. Thus, we choose $L_3^{(3)}$ and $\langle u_3^2\rangle$. Under the assumption that the flux of energy to small scales is controlled by the energy-containing range, dimensional analysis leads to

$$\frac{d\langle u_3^2\rangle}{dt} = -\Lambda \frac{\langle u_3^2\rangle^{3/2}}{L_3^{(3)}}, \tag{59}$$

where Λ is constant for fully developed turbulence at sufficiently high Reynolds number. The integration of Eq. (59) with Eqs. (52), (55), and (57) yields the isotropic-like decay laws:

$$\langle u_j^2\rangle \propto t^{-6/5}, \quad L_j^{(n)} \propto t^{2/5} \quad \text{for} \quad j = 1, 2, 3. \tag{60}$$

These decay laws include the decay laws in Ref. [6] showing the decay rate of total energy and the growth rate of an integral length scale that are in good agreement with observations made in laboratory experiments at sufficiently high Reynolds numbers [23]. Equation (60) includes also the decay laws for fully developed homogeneous axisymmetric Saffman turbulence [7].

EDQNM closures have been used in the study of the decay of incompressible homogeneous turbulence whose energy spectrum $E(k)$ is given by $C_s k^s + o(k^s)\,(1 \leqslant s \leqslant 4)$ at $k \to 0$ (see, e.g., Ref. [1]). The persistence of large-scale anisotropy and the return to isotropy at small scales have been observed in homogeneous axisymmetric Saffman turbulence [24] and in homogeneous anisotropic Saffman turbulence [25]. It was shown that non-self-similar decay of isotropic turbulence does not directly related to the asymptotic behavior of $E(k)$ at $k \to 0$ [26,27]. Complete self-similarity including the dissipative range in the decay of isotropic turbulence was found only for $s = 1$ [28].

V. SELF-SIMILARITY OF A PASSIVE SCALAR FIELD IN HOMOGENEOUS TURBULENCE

We now apply the methodology of Sec. IV to a passive scalar field with the invariance of $\chi(\tilde{\boldsymbol{k}})$ in incompressible homogeneous turbulence.

A. Self-similarity and invariance of $\chi(\tilde{\boldsymbol{k}})$

Here we employ $D(\boldsymbol{k}, t) = k^2\hat{\Theta}(\boldsymbol{k}, t)$, instead of $\hat{\Theta}(\boldsymbol{k}, t)$. The reason for the use of $D(\boldsymbol{k}, t)$ will be discussed at the end of this subsection. We assume that $D(\boldsymbol{k}, t)$ evolves in accordance with the self-similar form,

$$D(\boldsymbol{k}, t) = d(t) f^{\theta}\left(k_1\ell_1^{\theta}, k_2\ell_2^{\theta}, k_3\ell_3^{\theta}\right) = d(t) f^{\theta}(\boldsymbol{\zeta}^{\theta}), \tag{61}$$

in a certain time range and domain of the $\boldsymbol{k}$ space including small enough k range, where

$$\boldsymbol{\zeta}^{\theta} = \left(k_1\ell_1^{\theta}, k_2\ell_2^{\theta}, k_3\ell_3^{\theta}\right), \tag{62}$$

$d(t)$ is an appropriate function of time t, and $\ell_j^\theta(t)$ is an appropriate length scale of a passive scalar in the jth Cartesian direction. Similar to the case of Eq. (32), $f^\theta(k_1\ell_1^\theta, k_2\ell_2^\theta, k_3\ell_3^\theta) = f^\theta(\boldsymbol{\zeta}^\theta)$ is time independent at any fixed $\boldsymbol{\zeta}^\theta$, and $D(\boldsymbol{k}, t)$ depends on time t only through $d(t)$ and $\boldsymbol{\ell}^\theta(t) = (\ell_1^\theta(t), \ell_2^\theta(t), \ell_3^\theta(t))$, according to Eq. (61). The function $f^\theta(\boldsymbol{\zeta}^\theta)$ is dimensionless. Next we show that $d(t)[\ell^\theta(t)]^2$ is constant and $d(t)[\ell^\theta(t)]^2 \simeq \text{constant} \times \langle\theta^2\rangle[\ell^\theta(t)]^3$. We assume that Eq. (61) is a good approximation for $\hat{\Theta}(\boldsymbol{k}, t)$ in the $(\boldsymbol{k}, t)$ domain including a small enough k range. Note that the self-similarity is not imposed on the $\boldsymbol{k}$ range including the scalar-diffusion dominant wave vector range.

The comparison of Eq. (61) with Eq. (31) yields

$$D(\boldsymbol{k}, t) = d(t)f^\theta(\boldsymbol{\zeta}^\theta) = k^2\chi(\tilde{\boldsymbol{k}}) + o(k^2), \tag{63}$$

as $\boldsymbol{k} \to \boldsymbol{0}$. Equations (61) and (63) imply that

$$d(t)f^\theta(\boldsymbol{\zeta}^\theta) = \left(\frac{\zeta^\theta}{\ell^\theta}\right)^2 \left\{\left(\frac{\tilde{\zeta}_1^\theta}{\tilde{\ell}_1^\theta}\right)^2 + \left(\frac{\tilde{\zeta}_2^\theta}{\tilde{\ell}_2^\theta}\right)^2 + \left(\frac{\tilde{\zeta}_3^\theta}{\tilde{\ell}_3^\theta}\right)^2\right\}\chi(\tilde{\boldsymbol{k}}) + o([\zeta^\theta]^2), \tag{64}$$

as $\boldsymbol{\zeta}^\theta \to \boldsymbol{0}$, where we used $\zeta^\theta = |\boldsymbol{\zeta}^\theta|$, $\ell^\theta = |\boldsymbol{\ell}^\theta|$, $\tilde{\boldsymbol{\xi}}^\theta = \boldsymbol{\zeta}^\theta/\zeta^\theta$, $\tilde{\boldsymbol{\ell}}^\theta = \boldsymbol{\ell}^\theta/\ell^\theta$, and

$$k^2 = \left(\frac{\zeta_1^\theta}{\ell_1^\theta}\right)^2 + \left(\frac{\zeta_2^\theta}{\ell_2^\theta}\right)^2 + \left(\frac{\zeta_3^\theta}{\ell_3^\theta}\right)^2 = \left(\frac{\zeta^\theta}{\ell^\theta}\right)^2 \left\{\left(\frac{\tilde{\zeta}_1^\theta}{\tilde{\ell}_1^\theta}\right)^2 + \left(\frac{\tilde{\zeta}_2^\theta}{\tilde{\ell}_2^\theta}\right)^2 + \left(\frac{\tilde{\zeta}_3^\theta}{\tilde{\ell}_3^\theta}\right)^2\right\}. \tag{65}$$

Since $f^\theta(\boldsymbol{\zeta}^\theta)$ in Eq. (64) is time independent at any fixed $\boldsymbol{\zeta}^\theta$,

$$\left\{\left(\frac{\tilde{\zeta}_1^\theta}{\tilde{\ell}_1^\theta}\right)^2 + \left(\frac{\tilde{\zeta}_2^\theta}{\tilde{\ell}_2^\theta}\right)^2 + \left(\frac{\tilde{\zeta}_3^\theta}{\tilde{\ell}_3^\theta}\right)^2\right\}\frac{\chi(\tilde{\boldsymbol{k}})}{d(t)[\ell^\theta(t)]^2} \quad \text{too must be time independent} \tag{66}$$

at any fixed $\boldsymbol{\zeta}^\theta$. Based on Eq. (62), we obtain

$$\tilde{k}_i = \frac{k_i}{k} = \frac{\tilde{\zeta}_i^\theta/\tilde{\ell}_i^\theta}{\left\{\left(\tilde{\zeta}_1^\theta/\tilde{\ell}_1^\theta\right)^2 + \left(\tilde{\zeta}_2^\theta/\tilde{\ell}_2^\theta\right)^2 + \left(\tilde{\zeta}_3^\theta/\tilde{\ell}_3^\theta\right)^2\right\}^{1/2}}. \tag{67}$$

Then, for $\tilde{\boldsymbol{\xi}}^\theta = (1, 0, 0)$, Eq. (67) gives $\tilde{\boldsymbol{k}} = (1, 0, 0)$, so that Eq. (66) yields

$$\left(\frac{1}{\tilde{\ell}_1^\theta}\right)^2 \frac{\chi(1, 0, 0)}{d(t)[\ell^\theta(t)]^2} = \text{const.} \tag{68}$$

Because of the time independence of $\chi(1, 0, 0)$, Eq. (68) implies that

$$\left(\frac{1}{\tilde{\ell}_1^\theta}\right)^2 \frac{1}{d(t)[\ell^\theta(t)]^2} = \text{const,} \tag{69}$$

where we assumed $\chi(1, 0, 0) \neq 0$. Similarly, we obtain

$$\left(\frac{1}{\tilde{\ell}_2^\theta}\right)^2 \frac{1}{d(t)[\ell^\theta(t)]^2} = \text{const,} \tag{70}$$

$$\left(\frac{1}{\tilde{\ell}_3^\theta}\right)^2 \frac{1}{d(t)[\ell^\theta(t)]^2} = \text{const.} \tag{71}$$

Equations (69)–(71) imply that

$$\tilde{\ell}_j^\theta = \frac{\ell_j^\theta}{\ell^\theta} = \text{const for} \quad j = 1, 2, 3, \tag{72}$$

and then

$$d(t)[\ell^\theta(t)]^2 = \text{const}, \tag{73}$$

where the constant value in Eq. (72) may also depend on j.

Since $\hat{\Theta}(\boldsymbol{k}) = D(\boldsymbol{k})/k^2$, Eq. (61) gives

$$\begin{aligned}\langle\theta^2(t)\rangle &= \int_{\mathbb{R}^3}\hat{\Theta}(\boldsymbol{k},t)\,d\boldsymbol{k} = \int_{\mathbb{R}^3}\frac{D(\boldsymbol{k},t)}{k^2}\,d\boldsymbol{k} \simeq \int_{\mathbb{R}^3}\frac{d(t)f^\theta\left(k_1\ell_1^\theta, k_2\ell_2^\theta, k_3\ell_3^\theta\right)}{k^2}\,d\boldsymbol{k}\\ &= \frac{d(t)}{\ell_1^\theta\ell_2^\theta\ell_3^\theta}\int_{\mathbb{R}^3}\frac{f^\theta(\boldsymbol{\zeta}^\theta)}{k^2}\,d\boldsymbol{\zeta}^\theta = \frac{d(t)}{\ell_1^\theta\ell_2^\theta\ell_3^\theta}\int_{\mathbb{R}^3}\frac{f^\theta(\boldsymbol{\zeta}^\theta)}{\left(\zeta_1^\theta/\ell_1^\theta\right)^2+\left(\zeta_2^\theta/\ell_2^\theta\right)^2+\left(\zeta_3^\theta/\ell_3^\theta\right)^2}\,d\boldsymbol{\zeta}^\theta\\ &\simeq \text{const}\times\frac{d(t)}{\ell^\theta(t)},\end{aligned} \tag{74}$$

where we used Eqs. (62) and (72). Equations (73) and (74) lead to

$$\langle\theta^2(t)\rangle[\ell^\theta(t)]^3 \simeq \text{const}. \tag{75}$$

The length scale ℓ_j^θ $(j = 1, 2, 3)$ can be regarded as the integral length scale $L_j^\theta(t)$ defined by

$$L_j^\theta(t) = \frac{\int_0^\infty\langle\theta(\boldsymbol{x})\theta(\boldsymbol{x}+r\boldsymbol{i}_j)\rangle\,dr}{\langle\theta^2(t)\rangle}. \tag{76}$$

Similar to the derivation of Eq. (55) in Sec. IV B, we find that

$$L_j^\theta(t) \simeq \gamma_j^\theta\ell_j^\theta(t) \quad \text{for } j = 1, 2, 3, \tag{77}$$

where γ_j^θ is constant. Thus, based on Eq. (72), we obtain

$$\frac{L_i^\theta}{L_j^\theta} \simeq \text{const for } i, j = 1, 2, 3\,(i \neq j). \tag{78}$$

We applied the self-similar assumption (61) to $D(\boldsymbol{k},t)$ but not $\hat{\Theta}(\boldsymbol{k},t)$ in order to derive Eq. (72) in the case that $\chi(\tilde{\boldsymbol{k}})$ is isotropic, where $\chi(\tilde{\boldsymbol{k}})$ is $\tilde{\boldsymbol{k}}$ independent. If we use $\hat{\Theta}(\boldsymbol{k},t)$ in the case, we obtain no information about the ratio $\ell_i^\theta/\ell_j^\theta$ $(i \neq j)$. Passive scalar fields can be anisotropic due to flow anisotropy, even if $\chi(\tilde{\boldsymbol{k}})$ is isotropic. Let the scalar field be isotropic at an initial time $t = t_0$, then $L_1^\theta(t_0) = L_2^\theta(t_0) = L_3^\theta(t_0)$. The self-similarity assumption may not hold for an initial transient or premature stage. Thus, $L_i^\theta(t)/L_j^\theta(t)\,(i \neq j)$ can evolve with time t during the transient time period. Therefore, $L_i^\theta(t)/L_j^\theta(t)\,(i \neq j)$ does not need to be 1, after the passive scalar turbulence becomes fully developed. For the fully developed state, the self-similarity shown in Eq. (61) may be expected.

B. Decay laws for passive scalar fluctuations

Next we discuss the decay of passive scalar fluctuations with the invariance of $\chi(\tilde{\boldsymbol{k}})$ in fully developed generalized anisotropic Saffman turbulence, called here fully developed Saffman-Corrsin passive scalar turbulence. Dimensional analysis suggests that

$$\langle(\boldsymbol{u}\cdot\nabla)\theta^2\rangle \sim \left(\frac{\langle u_1^2\rangle^{1/2}}{L_1^\theta}+\frac{\langle u_2^2\rangle^{1/2}}{L_2^\theta}+\frac{\langle u_3^2\rangle^{1/2}}{L_3^\theta}\right)\langle\theta^2\rangle. \tag{79}$$

Equation (78) holds for fully developed passive scalar turbulence with generalized Corrsin's invariants. Moreover, we here consider the self-similar decay of generalized Saffman turbulence, which shows the persistence of flow anisotropy, Eqs. (53) and (58). Then Eq. (79) results in

$$\langle(\boldsymbol{u}\cdot\nabla)\theta^2\rangle \sim \frac{\langle u_3^2\rangle^{1/2}\langle\theta^2\rangle}{L_3^\theta}. \tag{80}$$

Therefore, dimensional analysis yields

$$\frac{d}{dt}\langle\theta^2\rangle = -\Lambda^\theta \frac{\langle u_3^2\rangle^{1/2}\langle\theta^2\rangle}{L_3^\theta}, \tag{81}$$

where Λ^θ is constant. The integration of Eq. (81) with Eqs. (60), (75), and (77) gives

$$\langle\theta^2\rangle \propto t^{-6/5}, \quad L_j^\theta \propto t^{2/5} \quad \text{for} \quad j = 1, 2, 3, \tag{82}$$

irrespective of the Prandtl number or Schmidt number ν/κ. These laws are in accordance with the prediction for fully developed isotropic Saffman-Corrsin passive scalar turbulence [14]. Numerical simulations based on EDQNM models have been performed for the decay of homogeneous passive scalar turbulence [29,30]. It was shown that neither the Prandtl number nor initial scalar integral length scale affects the passive scalar decay for sufficiently high Reynolds and Péclet numbers [29]. The persistence of scalar anisotropy was predicted for passive scalar decay without any scalar gradient under shear-released homogeneous turbulence [30].

VI. CONCLUSIONS AND DISCUSSION

We considered the large-scale structure of incompressible homogeneous anisotropic turbulence without any external force and homogeneous anisotropic passive scalar turbulence without any scalar source. A generalization of Saffman's argument suggests that the large-scale structure of the velocity field whose energy spectrum $E(k)$ is given by $Ck^2 + o(k^2)$ at $k \to 0$, is characterized by an infinite number of invariants that include Saffman's invariants. Applying a similar argument to a passive scalar field, we showed that an infinite number of invariants characterize the large-scale structure of the scalar field with $E^\theta(k) = C^\theta k^2 + o(k^2)$ at $k \to 0$. The contributions of the nonlinear terms become negligible in the time derivative of $\hat{R}_{ij}(\boldsymbol{k}, t)$ and the time derivative of $\hat{\Theta}(\boldsymbol{k}, t)$ at any t ($\geqslant t_0$) at $\boldsymbol{k} \to \boldsymbol{0}$, when $\hat{R}_{ij}(\boldsymbol{k}, t_0) = M_{\alpha\beta}(\boldsymbol{k}/k) + o(1)$ and $\hat{\Theta}(\boldsymbol{k}, t_0) = \chi(\boldsymbol{k}/k) + o(1)$ at $\boldsymbol{k} \to \boldsymbol{0}$, respectively, where t_0 is an appropriate initial time. This implies the invariance of $M_{\alpha\beta}(\boldsymbol{k}/k)$ and $\chi(\boldsymbol{k}/k)$.

Implications of the invariance and self-similarity were examined. By the use of the invariants and a self-similarity assumption for $\hat{R}_{ij}(\boldsymbol{k})$ in appropriate k and time ranges, it was analytically derived that flow anisotropy is persistent at large scales. The anisotropy is measured using the ratio of the intensity of each velocity component $\langle u_i^2\rangle/\langle u_j^2\rangle$ for $i \neq j$, and the ratio of the integral length scales of an nth velocity component $L_i^{(n)}/L_j^{(n)}$ for $i \neq j$. The persistent anisotropy $\langle u_i^2\rangle/\langle u_j^2\rangle$ for $i \neq j$ was obtained under the assumption that the length scales are independent of velocity components. Note that under the assumption, the persistence holds in the self-similar states of any freely decaying incompressible homogeneous turbulence with $E(k) = Ck^2 + o(k^2)$ at $k \to 0$. Moreover, the scalar anisotropy is preserved, which is measured by the ratio of the scalar integral length scales L_i^θ/L_j^θ for $i \neq j$, when self-similarity of $\hat{\Theta}(\boldsymbol{k})$ holds in appropriate k and time ranges. The self-similarity is not imposed on the range in which viscous dissipation or scalar diffusion dominates. The self-similarity assumed here may be expected, if homogeneous turbulence becomes fully developed or mature. However, the assumption may not hold for premature states of turbulence.

In deriving the time independence of L_i^θ/L_j^θ ($i \neq j$), it is not necessary to assume any particular dynamics for the velocity field, as long as the self-similarity, Eq. (61), and a certain decay rate in r of $\langle u_j(\boldsymbol{x})\theta(\boldsymbol{x})\theta(\boldsymbol{x}+\boldsymbol{r})\rangle$ are satisfied. This implies that the time independence is not a consequence of the time independence of $L_i^{(n)}/L_j^{(n)}$ ($i \neq j$), i.e., the former holds independently from the latter.

Using these analytical results and dimensional analysis, we found the decay laws of the fully developed generalized Saffman turbulence, and the decay laws of the fully developed passive scalar turbulence with generalized Corrsin's invariants in fully developed generalized Saffman turbulence. The DNS examination of theoretical predictions will be reported elsewhere.

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APPENDIX A: THE ESTIMATE (16)

Let g and h be any two observables in statistically homogeneous turbulence. In order to avoid complexity associated with dealing with the Fourier transforms of functions that are not integrable in general, it is convenient to introduce a damping factor, say, $d_\varepsilon(\boldsymbol{x})$, and define $g_\varepsilon(\boldsymbol{x})$ and $h_\varepsilon(\boldsymbol{x})$ as $g_\varepsilon(\boldsymbol{x}) = g(\boldsymbol{x})d_\varepsilon(\boldsymbol{x})$ and $h_\varepsilon(\boldsymbol{x}) = h(\boldsymbol{x})d_\varepsilon(\boldsymbol{x})$. We put $d_\varepsilon(\boldsymbol{x}) = \exp(-\varepsilon^2 x^2)$ in this Appendix, where $x = |\boldsymbol{x}|$ and $\varepsilon > 0$.

In statistically homogeneous turbulence, the correlation $\langle g(\boldsymbol{x})h(\boldsymbol{x}+\boldsymbol{r})\rangle$ is independent of $\boldsymbol{x}$, so that we have

$$\begin{aligned}\langle \hat{g}_\varepsilon(-\boldsymbol{k})\hat{h}_\varepsilon(\boldsymbol{k})\rangle &= \frac{1}{(2\pi)^6}\left\langle\left\{\int_{\mathbb{R}^3} g_\varepsilon(\boldsymbol{x})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\,d\boldsymbol{x}\right\}\left\{\int_{\mathbb{R}^3} h_\varepsilon(\boldsymbol{y})\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y})\,d\boldsymbol{y}\right\}\right\rangle \\ &= \frac{1}{(2\pi)^6}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\langle g(\boldsymbol{x})f(\boldsymbol{y})\rangle\exp[-\varepsilon^2(x^2+y^2)]\exp[-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})]\,d\boldsymbol{x}\,d\boldsymbol{y} \\ &= \frac{1}{8(2\pi)^6}\int_{\mathbb{R}^3}\exp(-\varepsilon^2 q^2/2)\,d\boldsymbol{q}\int_{\mathbb{R}^3} R_\varepsilon^{gh}(\boldsymbol{r})\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,d\boldsymbol{r} \\ &= \frac{1}{(2\sqrt{2\pi}\,\varepsilon)^3}\hat{R}_\varepsilon^{gh}(\boldsymbol{k}),\end{aligned} \tag{A1}$$

where $\boldsymbol{r} = \boldsymbol{y}-\boldsymbol{x}$, $\boldsymbol{q} = \boldsymbol{y}+\boldsymbol{x}$, $y = |\boldsymbol{y}|$, $q = |\boldsymbol{q}|$, $R_\varepsilon^{gh}(\boldsymbol{r}) = \langle g(\boldsymbol{x})h(\boldsymbol{x}+\boldsymbol{r})\rangle\exp(-\varepsilon^2 r^2/2)$, $\hat{R}_\varepsilon^{gh}(\boldsymbol{k})$ is the Fourier transform of $R_\varepsilon^{gh}(\boldsymbol{r})$ with respect to $\boldsymbol{r}$, and we used $x^2+y^2 = (r^2+q^2)/2$.

Since

$$\langle \hat{g}_\varepsilon(-\boldsymbol{k})\hat{h}_\varepsilon(\boldsymbol{k})\rangle \leqslant \langle|\hat{g}_\varepsilon(-\boldsymbol{k})|^2\rangle^{1/2}\langle|\hat{h}_\varepsilon(\boldsymbol{k})|^2\rangle^{1/2}, \tag{A2}$$

because of the Cauchy-Schwarz inequality, Eq. (A1) gives

$$\hat{R}_\varepsilon^{gh}(\boldsymbol{k}) \leqslant \left\{\hat{R}_\varepsilon^{gg}(\boldsymbol{k})\right\}^{1/2}\left\{\hat{R}_\varepsilon^{hh}(\boldsymbol{k})\right\}^{1/2}. \tag{A3}$$

Let $g(\boldsymbol{x})$ and $h(\boldsymbol{x})$ be given by $g(\boldsymbol{x}) = N_{\alpha\beta}(\boldsymbol{x}) - \langle N_{\alpha\beta}(\boldsymbol{x})\rangle$ and $h(\boldsymbol{x}) = u_j(\boldsymbol{x})$. Then $\langle g(\boldsymbol{x})\rangle = \langle h(\boldsymbol{x})\rangle = 0$. If the correlations $R^{gg}(\boldsymbol{r}) = \langle g(\boldsymbol{x})g(\boldsymbol{x}+\boldsymbol{r})\rangle$ and $R^{hh}(\boldsymbol{r}) = \langle h(\boldsymbol{x})h(\boldsymbol{x}+\boldsymbol{r})\rangle$ decays as $O(r^{-3})$ at $r\to\infty$, then in the same way as in the derivation of Eq. (14), we obtain

$$\hat{R}^{gg}(\boldsymbol{k}) = O(k^0), \tag{A4}$$

$$\hat{R}^{hh}(\boldsymbol{k}) = O(k^0), \tag{A5}$$

as $\boldsymbol{k}\to\boldsymbol{0}$. Equation (A3) in the limit of $\boldsymbol{k}\to\boldsymbol{0}$ and $\varepsilon\to 0$ then implies the estimate (15).

The estimates (A4) and (A5) are closely related to the squares $(\int_V g\,d\boldsymbol{x})^2$ and $(\int_V h\,d\boldsymbol{x})^2$, where the symbol $\int_V d\boldsymbol{x}$ denotes the integral over a certain large domain with the volume V. The average of the former is given by

$$\langle(\bar{g})^2\rangle = \frac{1}{V}\int_V \langle g(\boldsymbol{x})g(\boldsymbol{x}+\boldsymbol{r})\rangle d\boldsymbol{r} = \frac{1}{V}\int_V R^{gg}(\boldsymbol{r})\,d\boldsymbol{r}, \tag{A6}$$

where $\langle g\rangle = 0$ and $\bar{g}$ is defined by $\bar{g} = (\int_V g\,d\boldsymbol{x})/V$. If one can assume that the random field $g(\boldsymbol{x})$ at any position is statistically independent from $g(\boldsymbol{x}+\boldsymbol{r})$ at any different position with sufficiently large r in an appropriate sense so that $\langle(\bar{g})^2\rangle = O(1/V)$ in the limit of $V \to \infty$, as is assumed for the angular momentum in Ref. [31], then Eq. (A6) implies that the integral $\int_V R^{gg}(\boldsymbol{r})\,d\boldsymbol{r}$ must be finite. (But, it is not trivial whether the statistical independence assumption is well satisfied in turbulence.) We then have the estimate (A4) for $\boldsymbol{k} \to \boldsymbol{0}$. The estimate (A5) can be obtained similarly.

APPENDIX B: ANALYTICITY IN TIME

Saffman [4] presented an argument to show that the invariance of $M_{\alpha\beta}$ in Eq. (8), based on the assumption that the solution $\hat{\boldsymbol{u}}(\boldsymbol{k},t)$ of the NS equation (1) together with the incompressibility condition (2) is analytic in time at $t = t_0$. Moreover, he presented an example of the breakdown of the analyticity in time. He presented another example in Ref. [32]. One might think that these examples provide counter-examples invalidating the analyticity used in the derivation of the invariance of $M_{\alpha\beta}$ in Eq. (8). However, recall that a certain initial spectrum at $k \gg 1$ was assumed to decrease sufficiently fast in the derivation, and the velocity in the examples does not satisfy these conditions. In general, examples violating certain conditions do not invalidate a statement assuming the conditions. To see this point, consider the vorticity $\boldsymbol{\omega}(\boldsymbol{x},t)$ obeying the Stokes dynamics:

$$\frac{\partial}{\partial t}\boldsymbol{\omega}(\boldsymbol{x},t) = \nu\nabla^2\boldsymbol{\omega}(\boldsymbol{x},t), \tag{B1}$$

for which one can easily construct an example that is in conflict with analyticity in time. For example, suppose that $\boldsymbol{\omega}(\boldsymbol{x},t_0) = \boldsymbol{0}$ in a certain compact domain, say $\mathcal{D}$, but nonzero outside $\mathcal{D}$. Then, at $t = t_0$, $(\partial/\partial t)^n\boldsymbol{\omega}(\boldsymbol{x},t) = \boldsymbol{0}$ for any positive integer n in $\mathcal{D}$, while $\boldsymbol{\omega}(\boldsymbol{x},t)$ may be nonzero at $t\,(>t_0)$. Therefore, this example conflicts with the assumption of the analyticity of $\boldsymbol{\omega}(\boldsymbol{x},t)$ in t. However, in $\boldsymbol{k}$ space, this does not invalidate the assumption of the analyticity in time of $\hat{\boldsymbol{\omega}}(\boldsymbol{k},t) = \hat{\boldsymbol{\omega}}(\boldsymbol{k},t_0)\exp\{-\nu k^2(t-t_0)\}$.

APPENDIX C: DECOMPOSITION OF $\mathcal{M}_{\alpha\beta}(\boldsymbol{k})$

Let $\boldsymbol{v}(\boldsymbol{x})$ be given by $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\eta}$, where $\boldsymbol{\eta}$ is any statistically homogeneous vector satisfying $P_{i\alpha}\hat{\eta}_\alpha(\boldsymbol{k}) = 0$. Then we have $P_{i\alpha}\hat{v}_\alpha(\boldsymbol{k}) = \hat{u}_i(\boldsymbol{k})$, and $\hat{R}_{ij}(\boldsymbol{k}) = P_{i\alpha}P_{j\beta}\mathcal{M}_{\alpha\beta}(\boldsymbol{k})$, i.e., Eq. (7), where $\langle\hat{v}_\alpha(\boldsymbol{p})\hat{v}_\beta(\boldsymbol{k})\rangle = \mathcal{M}_{\alpha\beta}(\boldsymbol{k})\delta(\boldsymbol{k}+\boldsymbol{p})$. Substitution of $\hat{\boldsymbol{v}} = \hat{\boldsymbol{u}} + \hat{\boldsymbol{\eta}}$ into $\langle\hat{v}_\alpha(\boldsymbol{p})\hat{v}_\beta(\boldsymbol{k})\rangle$ gives

$$\mathcal{M}_{\alpha\beta}(\boldsymbol{k}) = \mathcal{M}^I_{\alpha\beta}(\boldsymbol{k}) + \mathcal{M}^C_{\alpha\beta}(\boldsymbol{k}), \tag{C1}$$

where $\mathcal{M}^I_{\alpha\beta}(\boldsymbol{k}) = \hat{R}_{\alpha\beta}(\boldsymbol{k})$ and $P_{i\alpha}P_{j\beta}\mathcal{M}^C_{\alpha\beta}(\boldsymbol{k}) = 0$. Here $\mathcal{M}^I_{\alpha\beta}(\boldsymbol{k})\delta(\boldsymbol{k}+\boldsymbol{p}) = \langle\hat{u}_\alpha(\boldsymbol{p})\hat{u}_\beta(\boldsymbol{k})\rangle$, and $\mathcal{M}^C_{\alpha\beta}(\boldsymbol{k})\delta(\boldsymbol{k}+\boldsymbol{p}) = \langle\hat{\eta}_\alpha(\boldsymbol{p})\hat{\eta}_\beta(\boldsymbol{k})\rangle + \langle\hat{\eta}_\alpha(\boldsymbol{p})\hat{u}_\beta(\boldsymbol{k})\rangle + \langle\hat{u}_\alpha(\boldsymbol{p})\hat{\eta}_\beta(\boldsymbol{k})\rangle$. It is seen that Eq. (7) is redundant in the sense that it holds for any $\mathcal{M}^C_{\alpha\beta}(\boldsymbol{k})$ satisfying $P_{i\alpha}P_{j\beta}\mathcal{M}^C_{\alpha\beta}(\boldsymbol{k}) = 0$.

The $\mathcal{E}$-$\mathcal{Z}$-$\mathcal{H}$ decomposition of $\hat{R}_{\alpha\beta}(\boldsymbol{k})$ implies that

$$\mathcal{M}^I_{\alpha\beta}(\boldsymbol{k}) = \hat{R}_{\alpha\beta}(\boldsymbol{k}) = \mathcal{E}(\boldsymbol{k})P_{\alpha\beta}(\tilde{\boldsymbol{k}}) + \Re[\mathcal{Z}(\boldsymbol{k})N_\alpha(\tilde{\boldsymbol{k}})N_\beta(\tilde{\boldsymbol{k}})] + \mathrm{i}\mathcal{H}(\boldsymbol{k})\epsilon_{\alpha\beta\mu}\tilde{k}_\mu, \tag{C2}$$

where $\epsilon_{\alpha\beta\mu}$ is the alternating third-order tensor, $\mathcal{E}(\boldsymbol{k})$ and $\mathcal{H}(\boldsymbol{k})$ are real, and $\mathcal{Z}(\boldsymbol{k})$ is complex [1,18]. Here $\boldsymbol{N}(\tilde{\boldsymbol{k}}) = \boldsymbol{e}^{(2)}(\tilde{\boldsymbol{k}}) - \mathrm{i}\boldsymbol{e}^{(1)}(\tilde{\boldsymbol{k}})$, in which $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$ are unit vectors defined by

$$\boldsymbol{e}^{(1)}(\tilde{\boldsymbol{k}}) = \frac{\tilde{\boldsymbol{k}}\times\boldsymbol{i}_3}{|\tilde{\boldsymbol{k}}\times\boldsymbol{i}_3|}, \quad \boldsymbol{e}^{(2)}(\tilde{\boldsymbol{k}}) = \frac{\tilde{\boldsymbol{k}}\times\boldsymbol{e}^{(1)}}{|\tilde{\boldsymbol{k}}\times\boldsymbol{e}^{(1)}|}, \tag{C3}$$

and $\boldsymbol{i}_3$ is taken as here $\boldsymbol{i}_3 = (0,0,1)$.

When Eq. (19) holds, $\mathcal{E}(\boldsymbol{k})$, $\mathcal{Z}(\boldsymbol{k})$ and $\mathcal{H}(\boldsymbol{k})$ can be written as $\mathcal{E}(\boldsymbol{k}) = \mathcal{E}_0(\tilde{\boldsymbol{k}}) + o(1)$, $\mathcal{Z}(\boldsymbol{k}) = \mathcal{Z}_0(\tilde{\boldsymbol{k}}) + o(1)$, and $\mathcal{H}(\boldsymbol{k}) = \mathcal{H}_0(\tilde{\boldsymbol{k}}) + o(1)$, as $\boldsymbol{k} \to \boldsymbol{0}$, where $\mathcal{E}_0(\tilde{\boldsymbol{k}})$, $\mathcal{Z}_0(\tilde{\boldsymbol{k}})$ and $\mathcal{H}_0(\tilde{\boldsymbol{k}})$ are $O(k^0)$. Equations (7) and (10) give

$$M_{\alpha\beta}(\tilde{\boldsymbol{k}}) = \mathcal{E}_0(\tilde{\boldsymbol{k}})P_{\alpha\beta}(\tilde{\boldsymbol{k}}) + \Re[\mathcal{Z}_0(\tilde{\boldsymbol{k}})N_\alpha(\tilde{\boldsymbol{k}})N_\beta(\tilde{\boldsymbol{k}})] + \mathrm{i}\mathcal{H}_0(\tilde{\boldsymbol{k}})\epsilon_{\alpha\beta\mu}\tilde{k}_\mu. \tag{C4}$$

The right-hand side of Eq. (C4) is expanded as Eqs. (12) and (13).

APPENDIX D: TURBULENCE DECAY UNDER EXTERNAL FORCES

Let us consider homogeneous incompressible turbulence under the Coriolis force as a representative example of turbulence under external forces. Its fluid motion obeys Eqs. (1) and (2), but a term representing the force $-2\boldsymbol{\Omega} \times \boldsymbol{u}$ is added to the right-hand side of Eq. (1). Here we set $\boldsymbol{\Omega} = (0, 0, \Omega)$ and use the so-called helical decomposition [1,18]:

$$\hat{\boldsymbol{u}}(\boldsymbol{k}, t) = \xi_1(\boldsymbol{k}, t)\boldsymbol{N}(\tilde{\boldsymbol{k}}) + \xi_{-1}(\boldsymbol{k}, t)\boldsymbol{N}(-\tilde{\boldsymbol{k}}). \tag{D1}$$

Based on the field defined by

$$a_s(\boldsymbol{k}, t) = \exp(-2\mathrm{i}\Omega s\tilde{k}_3 t)\xi_s(\boldsymbol{k}, t) \quad (s = \pm 1), \tag{D2}$$

and in almost the same way as in the derivation of Eq. (19), it is shown that

$$\frac{\partial}{\partial t}A_{ss'}(\boldsymbol{k}, t) = o(1) \quad \text{as} \quad \boldsymbol{k} \to \boldsymbol{0}, \tag{D3}$$

under certain assumptions, where $A_{ss'}(\boldsymbol{k}, t)\delta(\boldsymbol{k} + \boldsymbol{p}) = \langle a_s(\boldsymbol{p}, t)a_{s'}(\boldsymbol{k}, t)\rangle$ and

$$A_{ss'}(\boldsymbol{k}, t) = \chi^a_{ss'}(\tilde{\boldsymbol{k}}) + o(1) \quad \text{for} \quad t \geqslant t_0, \tag{D4}$$

in which t_0 is the initial time. Here $\chi^a_{ss'}(\tilde{\boldsymbol{k}})$ is a time-independent constant and may depend on the direction $\tilde{\boldsymbol{k}}$. Note that the leading order term of $A_{ss'}(\boldsymbol{k}, t)$ at $\boldsymbol{k} \to \boldsymbol{0}$ is time independent, but that of $\hat{R}_{ij}(\boldsymbol{k}, t)$ at $\boldsymbol{k} \to \boldsymbol{0}$ can be in general time dependent. Because of the prefactor $\exp(-2\mathrm{i}\Omega st\tilde{k}_3)$ in Eq. (D2), it is unlikely that the self-similarity assumption can hold as in Eq. (32). Therefore, the analysis presented in this paper is in general not applicable to $\hat{R}_{ij}(\boldsymbol{k}, t)$ for homogeneous turbulence under external forces.

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