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Comparison of theoretical formulae and bootstrap method for statistical error estimation of Feynman-α method

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Abstract

This paper discusses the statistical error of the variance-to-mean ratio, or the Y value in the Feynman-α method, from a single measurement of reactor noise. As a theoretical approach, two practical theoretical formulae are derived to estimate the statistical error of Y: one is based on the propagation of uncertainty with unbiased estimators for the third- and fourth-order central moments; the other is a simplified formula that reuses the Y value under the fundamental mode approximation, where the subcriticality is approximately less than 10,000 pcm. As a numerical approach, the bootstrap method is improved to efficiently estimate the correlations of Y between different counting gate widths, or covariance matrix $\mathbf{\Sigma}_Y$, due to the bunching method. Through an actual reactor noise experiment at the Kyoto University Criticality Assembly, the statistical errors of Y using the theoretical formulae and the bootstrap method are validated by comparing the reference statistical errors that are estimated from the multiple experiments of reactor noise. Furthermore, the impact of $\mathbf{\Sigma}_Y$ on the statistical error of the prompt neutron decay constant α is numerically investigated. Consequently, in the case of this experimental analysis, it was confirmed that the bootstrap method with the correlations of Y seems to be slightly better from the viewpoint of the coverage probability of the estimated confidence intervals of α, although the fitting error method without the correlation of Y could be useful for the order estimation of the statistical error of α.

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Highlights

- ✓ Estimation formulae for statistical error of variance-to-mean ratio Y are derived.
- ✓ Statistical error of Y can be estimated by reusing Y without higher-order moments.
- ✓ Bootstrap method enables covariance estimation of Y between counting gate widths.
- ✓ Covariance is useful in better error-estimation of prompt neutron decay constant.

1. Introduction

The study of subcriticality monitoring is important to achieve safe and efficient operation and management in nuclear fuel-related facilities. It is also important for the Accelerator-Driven System (ADS), where the subcritical state must be maintained during operation [1,2]. Furthermore, in the retrieval of fuel debris from Fukushima Daiichi units 1–3 with the submersion condition, there is a possibility of a positive reactivity insertion event due to the change in the moderation ratio; thus, subcriticality monitoring to prevent recriticality is an important issue [3].

The Feynman-α method, also called the variance-to-mean ratio method, is a practical subcriticality measurement technique based on the zero-power reactor noise analysis [4,5,6,7]. Using the Feynman-α method, the prompt neutron decay constant α can be measured by analyzing the time-series data of neutron counts; then the measurement value of α is converted to the subcriticality $-\rho \equiv (1-k_{\mathrm{eff}})/k_{\mathrm{eff}}$, which is the absolute value of the negative reactivity. In the Feynman-α method, quantification of the statistical error of the variance-to-mean ratio (or Y value) is useful information for clarifying the measurement precision and reconsidering the measurement time if necessary. For example, if the estimated statistical error is unacceptable and should be reduced by half, the central limit theorem implies that four times the measurement time is necessary. Here, one of the simple estimation methods for the statistical error σ_Y is multiple measurements of reactor noise; however, an additional longer measurement time is needed to repeat the multiple times of measurements for the error estimation.

Thus, in our previous study, the statistical error estimation technique using only a single measurement of reactor noise, *i.e.*, without multiple measurements, was proposed using the bootstrap method [8,9]. In the bootstrap method, the statistical error can be simply estimated by a resampling that is based on an experimentally inferred probability distribution of neutron count. It is worth noting that the calculation time is long due to the resampling.

As another approach, we newly propose theoretical formulae to efficiently estimate the statistical error σ_Y for a single measurement. Compared with the previous theoretical investigations [10,11,12,13], the following points are the remarkable features of our present work:

1. In the propagation of uncertainty for the statistical error of Y, the covariance between the sample mean and the unbiased variance is explicitly considered.
2. Unbiased estimators for central moments are utilized.
3. Under the fundamental mode approximation where $-\rho < 10{,}000\ \mathrm{pcm}$, a more simplified formula for σ_Y by reusing Y is also derived.

In the theoretical approach, however, correlations of Y between different counting gate widths T (or covariance matrix $\mathbf{\Sigma}_Y$), which originate from the bunching method for the same time-series data [14], are an unresolved issue. Therefore, to numerically investigate the impact of $\mathbf{\Sigma}_Y$ on the statistical error of α in the fitting process, we improve procedures in the bootstrap method to effectively estimate the bootstrap covariance matrix $\mathbf{\Sigma}_{Y^*}$.

One of the major aims of this study is to validate the statistical errors of Y and α that are obtained from a single measurement of reactor noise, by comparing them with the reference statistical errors or by evaluating the coverage probability of the estimated confidence interval. Here, the reference statistical errors and the coverage probability can be evaluated using the multiple experiments of reactor noise. Through this validation, we aim to confirm whether the estimated statistical errors are reasonable, *i.e.*, neither overestimated nor underestimated. The reduction of statistical errors of Y and α is out of scope of this study.

The rest of the paper is structured as follows. In Section 2, the theory of statistical error σ_Y is described. In Section 3, we explain the improved bootstrap method. In Section 4, the derived estimation formulae and the improved bootstrap method are demonstrated through an experimental analysis for actual reactor noise data that were measured at the Kyoto University

Criticality Assembly (KUCA). In addition, the impact of $\boldsymbol{\Sigma}_Y$ on the statistical error of α in the fitting process is discussed. Finally, in Section 5, concluding remarks are presented.

2. Theory for statistical error of Y value

2.1. Propagation of uncertainty for Y value

Let us assume a steady state of a source-driven subcritical system. In this subcritical system, neutron counts $C_i(T)$ are measured N times ($1 \le i \le N$), where T is the counting gate width and N is the total number of count data. Then, the second-order neutron-correlation value Y is evaluated as the variance-to-mean ratio:

$$Y \equiv \frac{\sigma^2}{\langle C \rangle} - 1 \approx \frac{s^2}{\bar{C}} - 1, \tag{1}$$

$$\sigma^2 \equiv \langle (C - \langle C \rangle)^2 \rangle, \tag{2}$$

where the bracket $\langle \; \rangle$ indicates the expected value; $\langle C \rangle$ and σ^2 are the population mean and variance of neutron counts; and $\bar{C}$ and s^2 represent the sample mean and the unbiased variance of $C_i(T)$, respectively:

$$\bar{C} = \frac{1}{N} \sum_{i=1}^{N} C_i, \tag{3}$$

$$s^2 = \frac{1}{N-1} \sum_{i=1}^{N} (C_i - \bar{C})^2 . \tag{4}$$

Note that the notation T is omitted in Eqs. (1)–(4) for simplicity.

Based on the propagation of uncertainty (or the sandwich rule) for Eq. (1), the statistical error σ_Y can be estimated as follows:

$$\begin{aligned} \sigma_Y &\approx \sqrt{\left(\frac{\partial Y}{\partial \sigma_{\bar{C}}} \sigma_{\bar{C}} \right)^2 + \left(\frac{\partial Y}{\partial s^2} \sigma_{s^2} \right)^2 + 2 \frac{\partial Y}{\partial \sigma_{\bar{C}}} \frac{\partial Y}{\partial s^2} \mathrm{cov}(\bar{C}, s^2)} \\ &= (Y+1) \sqrt{\left(-\frac{\sigma_{\bar{C}}}{\bar{C}} \right)^2 + \left(\frac{\sigma_{s^2}}{s^2} \right)^2 - 2 \frac{\mathrm{cov}(\bar{C}, s^2)}{\bar{C} s^2}}, \end{aligned} \tag{5}$$

where $\sigma_{\bar{C}}$ and σ_{s^2} are the statistical errors of $\bar{C}$ and s^2, respectively; and $\mathrm{cov}(\bar{C}, s^2)$ is the covariance between $\bar{C}$ and s^2. In Eq. (5), the expected values of $\sigma_{\bar{C}}$, σ_{s^2}, and $\mathrm{cov}(\bar{C}, s^2)$ can be derived as follows [15]:

$$\langle \sigma_{\bar{C}} \rangle = \sqrt{\frac{\sigma^2}{N}}, \tag{6}$$

$$\langle \sigma_{s^2} \rangle = \sqrt{\frac{1}{N}\left(\mu_4 - \frac{N-3}{N-1}(\sigma^2)^2\right)}, \tag{7}$$

$$\langle \mathrm{cov}(\bar{C}, s^2) \rangle = \frac{\mu_3}{N}, \tag{8}$$

$$\mu_3 \equiv \langle (C - \langle C \rangle)^3 \rangle, \tag{9}$$

$$\mu_4 \equiv \langle (C - \langle C \rangle)^4 \rangle, \tag{10}$$

where μ_3 and μ_4 correspond to the third- and fourth-order central moments, respectively. If the same time-series data are used for the estimation of both $\bar{C}$ and s^2, the covariance team $\mathrm{cov}(\bar{C}, s^2)$ should not be neglected in Eq. (5).

2.2. Estimation formula using unbiased estimators for central moments

As can be seen from Eqs. (5)–(8), it is necessary for the estimation of the statistical error σ_Y to appropriately evaluate the third- and fourth-order central moments μ_3 and μ_4 from the finite number of neutron count data $C_i(T)$ $(1 \leq i \leq N)$.

For this purpose, μ_3 and μ_4 can be estimated using the unbiased estimators h_3 and h_4 in the h-statistics, respectively [16], where h_3 and h_4 are obtained by

$$h_3 = \frac{N}{(N-1)(N-2)} \sum_{i=1}^{N} (C_i - \bar{C})^3, \tag{11}$$

$$h_4 = \frac{N^2 - 2N + 3}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N} (C_i - \bar{C})^4 - \frac{3(2N-3)}{N(N-1)(N-2)(N-3)} \left(\sum_{i=1}^{N} (C_i - \bar{C})^2 \right)^2 . \tag{12}$$

Using the unbiased estimators s^2, h_3, and h_4 in Eqs. (5)–(8) instead of σ^2, μ_3, and μ_4, the statistical error $\sigma_{Y,\mathrm{h}}$ can be evaluated by the following formula:

$$\sigma_{Y,\mathrm{h}} \approx (Y+1) \sqrt{\frac{Y+1}{N\bar{C}} + \frac{1}{N}\left(\frac{h_4}{(s^2)^2} - \frac{N-3}{N-1}\right) - \frac{2h_3}{N\bar{C}s^2}} . \tag{13}$$

In the conventional Feynman-α method, the calculations of $\bar{C}$ and s^2 are sufficient for the evaluation of the Y value as defined in Eq. (1). For the estimation of $\sigma_{Y,\mathrm{h}}$ by Eq. (13), additional calculations of h_3 and h_4 are necessary.

2.3. Simplified formula by reusing the second-order neutron-correlation

In general, the magnitude of the nth-order neutron-correlation is proportional to the $(n-1)$th power of the detector importance function, or detection efficiency [17,18,19]. For example, if $Y < 1$, it is expected that the third and fourth-order neutron-correlations are lower than the second-order neutron-correlation. In particular, in the case of $Y \ll 1$, it was clarified that the probability density function of the neutron count can be sufficiently approximated by the negative binomial distribution [7,10,11]. If the negative binominal distribution approximation is applicable, the statistical error σ_Y can be approximately estimated only using $\langle C \rangle$ and Y [10]. Although the negative binominal distribution approximation is useful for estimating σ_Y, the applicability depends on experimental conditions, *e.g.*, low detection efficiency and/or deep subcritical system. Thus, in this subsection, a more simplified formula of the statistical error σ_Y is newly derived without depending on the magnitude of Y, on the basis of the fundamental mode approximation where $0.9 < k_{\mathrm{eff}} < 1$, or $-\rho < 10{,}000$ pcm.

In the steady state of the source-driven subcritical system, a master equation for probability-generating functions of neutron count is described as follows [7,18]:

$$\ln\big(G(Z,T|S)\big) = \int_0^\infty du \int_V dV\, S(\vec{r}) \sum_{q=0}^{\infty} p_\mathrm{s}(q,\vec{r}) \left\{\big(\bar{g}(Z,T|\vec{r},u)\big)^q - 1\right\}, \tag{14}$$

$$G(Z,T|S) \equiv \sum_{C=0}^{\infty} Z^C P(C,T|S)\,, \tag{15}$$

$$\bar{g}(Z,T|\vec{r},u) \equiv \int_0^\infty dE \int_{4\pi} d\Omega \frac{\chi_\mathrm{s}(\vec{r},E)}{4\pi} g\big(Z,T|\vec{r},E,\vec{\Omega},u\big), \tag{16}$$

$$g\big(Z,T|\vec{r},E,\vec{\Omega},u\big) \equiv \sum_{C=0}^{\infty} Z^C p\big(C,T|\vec{r},E,\vec{\Omega},u\big)\,, \tag{17}$$

where

$p\big(C,T|\vec{r},E,\vec{\Omega},u\big)$: probability that C neutrons are detected during the counting gate width T due to a neutron at $(\vec{r},E,\vec{\Omega},u)$, where u is a backward time-variable, *i.e.*, $u \equiv -t$, $u = 0$ corresponds to the counting gate closing time;

$g\big(Z,T|\vec{r},E,\vec{\Omega},u\big)$: probability-generating function for $p\big(C,T|\vec{r},E,\vec{\Omega},u\big)$, where Z is the variable of generating function;

$\bar{g}(Z,T|\vec{r},u)$: weighted mean of probability generating function, of which weighting function is $\frac{\chi_\mathrm{s}(\vec{r},E)}{4\pi}$;

$P(C,T|S)$: probability that C neutrons are detected during the counting gate width T owing to a stationary external neutron source S;

$G(Z,T|S)$: probability-generating function for $P(C,T|S)$.

$S(\vec{r})$: spatial distribution of source strength for the external neutron source;

$\chi_\mathrm{s}(\vec{r},E)$: energy spectrum of the external neutron source;

$p_\mathrm{s}(q,\vec{r})$: probability that q neutrons are emitted per decay of the external source.

Using the mathematical properties of the probability-generating function $G(Z,T|S)$ described by Eq. (14), the nth-order neutron-correlation value $\mathcal{Y}_n$ $(n \geq 2)$ is defined as

$$\mathcal{Y}_n \equiv \frac{1}{\langle C \rangle} \frac{\partial^n}{\partial Z^n} \ln\big(G(Z,T|S)\big)\Big|_{Z=1} , \tag{18}$$

where $\mathcal{Y}_2$ corresponds to the Y value in the Feynman-α method. For example, the first- to fourth-order partial derivatives of $\ln\big(G(Z,T|S)\big)$ with respect to Z are shown below:

$$\frac{\partial(\ln G)}{\partial Z} = \frac{1}{G}\frac{\partial G}{\partial Z} , \tag{19}$$

$$\frac{\partial^2(\ln G)}{\partial Z^2} = \frac{1}{G}\frac{\partial^2 G}{\partial Z^2} - \left(\frac{\partial(\ln G)}{\partial Z}\right)^2 , \tag{20}$$

$$\frac{\partial^3(\ln G)}{\partial Z^3} = \frac{1}{G}\frac{\partial^3 G}{\partial Z^3} - 3\frac{\partial(\ln G)}{\partial Z}\frac{\partial^2(\ln G)}{\partial Z^2} - \left(\frac{\partial(\ln G)}{\partial Z}\right)^3 , \tag{21}$$

$$\frac{\partial^4(\ln G)}{\partial Z^4} = \frac{1}{G}\frac{\partial^4 G}{\partial Z^4} - 4\frac{\partial(\ln G)}{\partial Z}\frac{\partial^3(\ln G)}{\partial Z^3} - 3\left(\frac{\partial^2(\ln G)}{\partial Z^2}\right)^2 - 6\left(\frac{\partial(\ln G)}{\partial Z}\right)^2\frac{\partial^2(\ln G)}{\partial Z^2} - \left(\frac{\partial(\ln G)}{\partial Z}\right)^4 , \tag{22}$$

where arguments $(Z,T|S)$ of $G(Z,T|S)$ are omitted for simplicity. In addition, $G(Z,T|S)$ satisfies the following mathematical properties:

$$G(Z,T|S)|_{Z=1} = \sum_{C=0}^{\infty} P(C,T|S) = 1, \tag{23}$$

$$\frac{\partial^n G}{\partial Z^n}\Big|_{Z=1} = \langle \frac{C!}{(C-n)!} \rangle . \tag{24}$$

Using Eqs. (18)–(24), the second- to fourth-order neutron-correlation values can be expressed as

$$Y \equiv \mathcal{Y}_2 = \frac{1}{\langle C \rangle}\frac{\partial^2(\ln G)}{\partial Z^2}\Big|_{Z=1} = \frac{\langle C(C-1)\rangle - \langle C \rangle^2}{\langle C \rangle} , \tag{25}$$

$$\mathcal{Y}_3 = \frac{1}{\langle C\rangle}\frac{\partial^3(\ln G)}{\partial Z^3}\bigg|_{Z=1} = \frac{\langle C(C-1)(C-2)\rangle - 3Y\langle C\rangle^2 - \langle C\rangle^3}{\langle C\rangle}, \tag{26}$$

$$\begin{aligned}\mathcal{Y}_4 &= \frac{1}{\langle C\rangle}\frac{\partial^4(\ln G)}{\partial Z^4}\bigg|_{Z=1} \\ &= \frac{\langle C(C-1)(C-2)(C-3)\rangle - 4\mathcal{Y}_3\langle C\rangle^2 - 3Y^2\langle C\rangle^2 - 6Y\langle C\rangle^3 - \langle C\rangle^4}{\langle C\rangle},\end{aligned} \tag{27}$$

As reported in our previous studies [18,19], the saturation values of Y, $\mathcal{Y}_3$, and $\mathcal{Y}_4$ in the limit of $T \to \infty$ can be derived by differentiating the right-hand side of Eq. (14) and using the first to fourth-order detector importance functions:

$$Y_\infty \equiv \lim_{T\to\infty} Y = \frac{\int_V S(\vec{r}) \sum_{q=0}^{\infty} p_\mathrm{s}(q,\vec{r}) \left\{ q\bar{I}_{2,\mathrm{s}}^{\dagger}(\vec{r}) + q(q-1)\left(\bar{I}_{1,\mathrm{s}}^{\dagger}(\vec{r})\right)^2 \right\} dV}{\int_V S(\vec{r}) \sum_{q=0}^{\infty} p_\mathrm{s}(q,\vec{r})\, q\bar{I}_{1,\mathrm{s}}^{\dagger}(\vec{r}) dV}, \tag{28}$$

$$\mathcal{Y}_{3,\infty} \equiv \lim_{T\to\infty} \mathcal{Y}_3 = \frac{\int_V S(\vec{r}) \sum_{q=0}^{\infty} p_\mathrm{s}(q,\vec{r}) \begin{Bmatrix} q\bar{I}_{3,\mathrm{s}}^{\dagger}(\vec{r}) + 3q(q-1)\bar{I}_{1,\mathrm{s}}^{\dagger}(\vec{r})\bar{I}_{2,\mathrm{s}}^{\dagger}(\vec{r}) \\ +q(q-1)(q-2)\left(\bar{I}_{1,\mathrm{s}}^{\dagger}(\vec{r})\right)^3 \end{Bmatrix} dV}{\int_V S(\vec{r}) \sum_{q=0}^{\infty} p_\mathrm{s}(q,\vec{r})\, q\bar{I}_{1,\mathrm{s}}^{\dagger}(\vec{r}) dV}, \tag{29}$$

$$\begin{aligned}\mathcal{Y}_{4,\infty} &\equiv \lim_{T\to\infty} \mathcal{Y}_4 \\ &= \frac{\int_V S(\vec{r}) \sum_{q=0}^{\infty} p_\mathrm{s}(q,\vec{r}) \begin{Bmatrix} q\bar{I}_{4,\mathrm{s}}^{\dagger}(\vec{r}) + 4q(q-1)\bar{I}_{1,\mathrm{s}}^{\dagger}(\vec{r})\bar{I}_{3,\mathrm{s}}^{\dagger}(\vec{r}) \\ +3q(q-1)\left(\bar{I}_{2,\mathrm{s}}^{\dagger}(\vec{r})\right)^2 \\ +6q(q-1)(q-2)\left(\bar{I}_{1,\mathrm{s}}^{\dagger}(\vec{r})\right)^2 \bar{I}_{2,\mathrm{s}}^{\dagger}(\vec{r}) \\ +q(q-1)(q-2)(q-3)\left(\bar{I}_{1,\mathrm{s}}^{\dagger}(\vec{r})\right)^4 \end{Bmatrix} dV}{\int_V S(\vec{r}) \sum_{q=0}^{\infty} p_\mathrm{s}(q,\vec{r})\, q\bar{I}_{1,\mathrm{s}}^{\dagger}(\vec{r}) dV},\end{aligned} \tag{30}$$

$$\bar{I}_{n,\mathrm{s}}^{\dagger}(\vec{r}) \equiv \int_0^{\infty} dE' \int_{4\pi} d\Omega' \frac{\chi_\mathrm{s}(\vec{r},E')}{4\pi} I_n^{\dagger}\left(\vec{r},E',\vec{\Omega}'\right), \tag{31}$$

where the subscript "∞" indicates the saturation values in the limit of $T \to \infty$; $\bar{I}_{n,\mathrm{s}}^{\dagger}(\vec{r})$ is the weighted mean of the nth-order detector importance function $I_n^{\dagger}\left(\vec{r},E,\vec{\Omega}\right)$ that satisfies the following adjoint neutron transport equations [18,19]:

$$\mathbf{B}^{\dagger} I_1^{\dagger}\left(\vec{r}, E, \vec{\Omega}\right) = \Sigma_{\mathrm{d}}(\vec{r}, E), \tag{32}$$

$$\mathbf{B}^{\dagger} I_2^{\dagger}\left(\vec{r}, E, \vec{\Omega}\right) = \Sigma_{\mathrm{f}}(\vec{r}, E) \sum_{\nu=0}^{\infty} p_{\mathrm{f}}(\nu, \vec{r})\, \nu(\nu-1) \left(\bar{I}_{1,\mathrm{f}}^{\dagger}(\vec{r})\right)^2, \tag{33}$$

$$\mathbf{B}^{\dagger} I_3^{\dagger}\left(\vec{r}, E, \vec{\Omega}\right) = \Sigma_{\mathrm{f}}(\vec{r}, E) \sum_{\nu=0}^{\infty} p_{\mathrm{f}}(\nu, \vec{r}) \begin{pmatrix} 3\nu(\nu-1)\bar{I}_{1,\mathrm{f}}^{\dagger}(\vec{r})\bar{I}_{2,\mathrm{f}}^{\dagger}(\vec{r}) \\ +\nu(\nu-1)(\nu-2)\left(\bar{I}_{1,\mathrm{f}}^{\dagger}(\vec{r})\right)^3 \end{pmatrix}, \tag{34}$$

$$\begin{aligned} &\mathbf{B}^{\dagger} I_4^{\dagger}\left(\vec{r}, E, \vec{\Omega}\right) \\ &= \Sigma_{\mathrm{f}}(\vec{r}, E) \sum_{\nu=0}^{\infty} p_{\mathrm{f}}(\nu, \vec{r}) \begin{pmatrix} 4\nu(\nu-1)\bar{I}_{1,\mathrm{f}}^{\dagger}(\vec{r})\bar{I}_{3,\mathrm{f}}^{\dagger}(\vec{r}) + 3\nu(\nu-1)\left(\bar{I}_{2,\mathrm{f}}^{\dagger}(\vec{r})\right)^2 \\ +6\nu(\nu-1)(\nu-2)\left(\bar{I}_{1,\mathrm{f}}^{\dagger}(\vec{r})\right)^2 \bar{I}_{2,\mathrm{f}}^{\dagger}(\vec{r}) \\ +\nu(\nu-1)(\nu-2)(\nu-3)\left(\bar{I}_{1,\mathrm{f}}^{\dagger}(\vec{r})\right)^4 \end{pmatrix}, \end{aligned} \tag{35}$$

$$\bar{I}_{n,\mathrm{f}}^{\dagger}(\vec{r}) \equiv \int_0^{\infty} dE' \int_{4\pi} d\Omega' \frac{\chi_{\mathrm{f}}(\vec{r}, E')}{4\pi} I_n^{\dagger}\left(\vec{r}, E', \vec{\Omega}'\right), \tag{36}$$

$$\mathbf{B}^{\dagger} \equiv \mathbf{A}^{\dagger} - \mathbf{F}^{\dagger}, \tag{37}$$

$$\mathbf{A}^{\dagger} \equiv -\vec{\Omega}\nabla + \Sigma_{\mathrm{t}}(\vec{r}, E) - \int_0^{\infty} dE' \int_{4\pi} d\Omega'\, \Sigma_{\mathrm{s}}\left(\vec{r}, E \to E', \vec{\Omega} \to \vec{\Omega}'\right), \tag{38}$$

$$\mathbf{F}^{\dagger} \equiv \nu\Sigma_{\mathrm{f}}(\vec{r}, E) \int_0^{\infty} dE' \int_{4\pi} d\Omega' \frac{\chi_{\mathrm{f}}(\vec{r}, E')}{4\pi}, \tag{39}$$

where the superscript "†" indicates the adjoint; $\mathbf{B}^{\dagger}$, $\mathbf{A}^{\dagger}$ and $\mathbf{F}^{\dagger}$ are the adjoint Boltzman, the net neutron-loss, and the neutron-production operators, respectively; $\Sigma_{\mathrm{d}}(\vec{r}, E)$ is the macroscopic neutron-detection cross-section; and $\chi_{\mathrm{f}}(\vec{r}, E)$ is the energy spectrum of fission; $p_{\mathrm{f}}(\nu, \vec{r})$ is the probability that ν neutrons are emitted per fission; other notations maintain their conventional meanings in reactor physics.

Now, let us assume the fundamental mode approximation is applicable to $I_n^{\dagger}\left(\vec{r}, E, \vec{\Omega}\right)$. This approximation is more reasonable under a subcritical condition where the effective neutron multiplication factor k_{eff} is closer to unity. Then, using the fundamental modes of forward and adjoint k_{eff}-eigenfunctions, *i.e.*, $\psi_0(\vec{r},E,\vec{\Omega})$ and $\psi_0^\dagger(\vec{r},E,\vec{\Omega})$, $I_n^\dagger(\vec{r},E,\vec{\Omega})$ can be approximated as follows:

$$I_1^\dagger(\vec{r},E,\vec{\Omega}) \approx \frac{\mathcal{D}_0}{-\rho\mathcal{F}_1}\psi_0^\dagger(\vec{r},E,\vec{\Omega}), \tag{40}$$

$$I_2^\dagger(\vec{r},E,\vec{\Omega}) \approx \left(\frac{\mathcal{D}_0}{-\rho\mathcal{F}_1}\right)^2 \frac{\mathcal{F}_2}{-\rho\mathcal{F}_1}\psi_0^\dagger(\vec{r},E,\vec{\Omega}), \tag{41}$$

$$I_3^\dagger(\vec{r},E,\vec{\Omega}) \approx \left(\frac{\mathcal{D}_0}{-\rho\mathcal{F}_1}\right)^3 \left(3\left(\frac{\mathcal{F}_2}{-\rho\mathcal{F}_1}\right)^2 + \frac{\mathcal{F}_3}{-\rho\mathcal{F}_1}\right)\psi_0^\dagger(\vec{r},E,\vec{\Omega}), \tag{42}$$

$$I_4^\dagger(\vec{r},E,\vec{\Omega}) \approx \left(\frac{\mathcal{D}_0}{-\rho\mathcal{F}_1}\right)^4 \left(15\left(\frac{\mathcal{F}_2}{-\rho\mathcal{F}_1}\right)^3 + 10\,\frac{\mathcal{F}_2\mathcal{F}_3}{(-\rho\mathcal{F}_1)^2} + \frac{\mathcal{F}_4}{-\rho\mathcal{F}_1}\right)\psi_0^\dagger(\vec{r},E,\vec{\Omega}), \tag{43}$$

where parameters $\mathcal{D}_0$ and $\mathcal{F}_n$ are introduced for convenience as follows:

$$\mathcal{D}_0 \equiv \int_V dV \int_0^\infty dE\, \Sigma_{\mathrm{d}}(\vec{r},E)\phi_0(\vec{r},E), \tag{44}$$

$$\mathcal{F}_n \equiv \int_V dV \int_0^\infty dE\, \Sigma_{\mathrm{f}}(\vec{r},E)\phi_0(\vec{r},E) \sum_{\nu=0}^{\infty} \frac{\nu!}{(\nu-n)!}\, p_{\mathrm{f}}(\nu,\vec{r},E)\left(\bar{\psi}_{0,\mathrm{f}}^\dagger(\vec{r})\right)^n, \tag{45}$$

$$\phi_0(\vec{r},E) \equiv \int_{4\pi} \psi_0(\vec{r},E,\vec{\Omega}')d\Omega', \tag{46}$$

$$\bar{\psi}_{0,\mathrm{f}}^\dagger(\vec{r}) \equiv \int_0^\infty dE' \int_{4\pi} d\Omega' \frac{\chi_{\mathrm{f}}(\vec{r},E')}{4\pi}\psi_0^\dagger(\vec{r},E',\vec{\Omega}'). \tag{47}$$

Note that $\psi_0(\vec{r},E,\vec{\Omega})$ and $\psi_0^\dagger(\vec{r},E,\vec{\Omega})$ satisfy the following forward and adjoint k_{eff}-eigenvalue equations, respectively:

$$\mathbf{A}\psi_0(\vec{r},E,\vec{\Omega}) = \frac{1}{k_{\mathrm{eff}}}\mathbf{F}\psi_0(\vec{r},E,\vec{\Omega}), \tag{48}$$

$$\mathbf{A}^\dagger\psi_0^\dagger(\vec{r},E,\vec{\Omega}) = \frac{1}{k_{\mathrm{eff}}}\mathbf{F}^\dagger\psi_0^\dagger(\vec{r},E,\vec{\Omega}), \tag{49}$$

$$\mathbf{A} \equiv \vec{\Omega}\nabla + \Sigma_{\mathrm{t}}(\vec{r}, E) - \int_0^\infty dE' \int_{4\pi} d\Omega' \, \Sigma_{\mathrm{s}}\left(\vec{r}, E' \to E, \vec{\Omega}' \to \vec{\Omega}\right), \tag{50}$$

$$\mathbf{F} \equiv \frac{\chi_{\mathrm{f}}(\vec{r}, E)}{4\pi} \int_0^\infty dE' \int_{4\pi} d\Omega' \, \nu\Sigma_{\mathrm{f}}(\vec{r}, E'). \tag{51}$$

By substituting Eqs. (40)–(43) into Eqs. (28)–(30), the saturation values based on the fundamental mode approximation can be rewritten as follows:

$$Y_\infty \approx \left(\frac{\mathcal{D}_0}{-\rho\mathcal{F}_1}\right)\left(\frac{\mathcal{F}_2}{-\rho\mathcal{F}_1} + \frac{\mathcal{S}_2}{\mathcal{S}_1}\right), \tag{52}$$

$$\mathcal{Y}_{3,\infty} \approx \left(\frac{\mathcal{D}_0}{-\rho\mathcal{F}_1}\right)^2 \left(\frac{\mathcal{F}_3}{-\rho\mathcal{F}_1} + \frac{\mathcal{S}_3}{\mathcal{S}_1} + 3\frac{\mathcal{F}_2}{-\rho\mathcal{F}_1}\left(\frac{\mathcal{F}_2}{-\rho\mathcal{F}_1} + \frac{\mathcal{S}_2}{\mathcal{S}_1}\right)\right), \tag{53}$$

$$\mathcal{Y}_{4,\infty} \approx \left(\frac{\mathcal{D}_0}{-\rho\mathcal{F}_1}\right)^3 \left(\begin{array}{c} \dfrac{\mathcal{F}_4}{-\rho\mathcal{F}_1} + \dfrac{\mathcal{S}_4}{\mathcal{S}_1} + 6\dfrac{\mathcal{F}_2}{-\rho\mathcal{F}_1}\left(\dfrac{\mathcal{F}_3}{-\rho\mathcal{F}_1} + \dfrac{\mathcal{S}_3}{\mathcal{S}_1}\right) \\ + \left(4\dfrac{\mathcal{F}_3}{-\rho\mathcal{F}_1} + 15\left(\dfrac{\mathcal{F}_2}{-\rho\mathcal{F}_1}\right)^2\right)\left(\dfrac{\mathcal{F}_2}{-\rho\mathcal{F}_1} + \dfrac{\mathcal{S}_2}{\mathcal{S}_1}\right) \end{array}\right), \tag{54}$$

$$\mathcal{S}_n \equiv \int_V dV \, S(\vec{r}) \sum_{q=0}^{\infty} \frac{q!}{(q-n)!} p_{\mathrm{s}}(q, \vec{r}) \left(\bar{\psi}_{0,\mathrm{s}}^\dagger(\vec{r})\right)^n, \tag{55}$$

$$\bar{\psi}_{0,\mathrm{s}}^\dagger(\vec{r}) \equiv \int_0^\infty dE' \int_{4\pi} d\Omega' \frac{\chi_{\mathrm{s}}(\vec{r}, E')}{4\pi} \psi_0^\dagger\left(\vec{r}, E', \vec{\Omega}'\right). \tag{56}$$

From Eqs. (52)–(54), if $-\rho < 0.1$ or $0.9 < k_{\mathrm{eff}} < 1$, ratios of $\mathcal{Y}_{3,\infty}/Y_\infty^2$ and $\mathcal{Y}_{4,\infty}/Y_\infty^3$ can be further approximated:

$$\frac{\mathcal{Y}_{3,\infty}}{Y_\infty^2} \approx 3 + \frac{\mathcal{F}_1}{\mathcal{F}_2}\left(\frac{\mathcal{F}_3}{\mathcal{F}_2} - 3\frac{\mathcal{S}_2}{\mathcal{S}_1}\right)(-\rho), \tag{57}$$

$$\frac{\mathcal{Y}_{4,\infty}}{Y_\infty^3} \approx 15 + 10\frac{\mathcal{F}_1}{\mathcal{F}_2}\left(\frac{\mathcal{F}_3}{\mathcal{F}_2} - 3\frac{\mathcal{S}_2}{\mathcal{S}_1}\right)(-\rho), \tag{58}$$

where the magnitude of $\frac{\mathcal{F}_1}{\mathcal{F}_2}\left(\frac{\mathcal{F}_3}{\mathcal{F}_2} - 3\frac{\mathcal{S}_2}{\mathcal{S}_1}\right)$ is approximately less than 1 [17,20]. It is interestingly noted that these ratios converge to constant values without depending on $\mathcal{F}_n$ and $\mathcal{S}_n$, as k_{eff} approaches to unity:

$$\lim_{-\rho \to +0} \frac{\mathcal{Y}_{3,\infty}}{Y_\infty^2} \approx 3, \tag{59}$$

$$\lim_{-\rho \to +0} \frac{\mathcal{Y}_{4,\infty}}{Y_\infty^3} \approx 15. \tag{60}$$

Finally, on the basis of the above-mentioned discussion, it is assumed that the magnitudes of $\mathcal{Y}_3$ and $\mathcal{Y}_4$ can be roughly approximated by the second and the third power of Y, if $-\rho < 0.1 = 10{,}000$ pcm:

$$\mathcal{Y}_3 \approx 3Y^2, \tag{61}$$

$$\mathcal{Y}_4 \approx 15Y^3, \tag{62}$$

Although errors exist between the true values of $\mathcal{Y}_3$ and $\mathcal{Y}_4$ and their approximations, using the approximations seems to be better than completely neglecting $\mathcal{Y}_3$ and $\mathcal{Y}_4$, or $\mathcal{Y}_3 \approx \mathcal{Y}_4 \approx 0$. Thus, Eqs. (61) and (62) are utilized to discuss the importance of the second-order neutron-correlation for estimating σ_Y. By substituting Eqs. (61) and (62) into Eqs. (26) and (27), the third- and fourth-order factorial moments can be approximated using $\langle C \rangle$ and the second-order neutron-correlation value Y:

$$\langle C(C-1)(C-2) \rangle \approx \langle C \rangle^3 + 3Y\langle C \rangle^2 + 3Y^2\langle C \rangle, \tag{63}$$

$$\langle C(C-1)(C-2)(C-3) \rangle \approx \langle C \rangle^4 + 6Y\langle C \rangle^3 + 15Y^2\langle C \rangle^2 + 15Y^3\langle C \rangle. \tag{64}$$

From Eqs. (63) and (64), the following approximation formulae for the third- and fourth-order central moments μ_3 and μ_4 can be obtained:

$$\mu_3 \approx (3Y^2 + 3Y + 1)\langle C \rangle, \tag{65}$$

$$\mu_4 \approx 3(Y+1)^2\langle C \rangle^2 + (15Y^3 + 18Y^2 + 7Y + 1)\langle C \rangle. \tag{66}$$

By substituting Eqs. (65) and (66) into Eqs. (5)–(8) and approximating as $\langle C \rangle \approx \bar{C}$, the simplified formula for the statistical error $\sigma_{Y,\mathrm{2nd}}$ can be finally derived as follows:

$$\sigma_{Y,\mathrm{2nd}} \approx (Y+1)\sqrt{\frac{Y(2Y+1)(5Y+2)}{N(Y+1)^2\bar{C}}+\frac{2}{N-1}}. \tag{67}$$

If $Y \approx 0$ in Eq. (67), *i.e.*, the probability distribution of neutron counts is sufficiently approximated by the Poisson distribution, the statistical error $\sigma_{Y,\mathrm{P}}$ is guessed by the following simple formula:

$$\sigma_{Y,\mathrm{P}} \approx \sqrt{\frac{2}{N-1}}. \tag{68}$$

By comparing Eq. (67) with Eq. (68), the statistical error σ_Y is corrected due to the second-order neutron-correlation value Y.

Using Eq. (67), the statistical error $\sigma_{Y,\mathrm{2nd}}$ can be approximately estimated by reusing the Y value without calculation of h_3 and h_4, *i.e.*, the calculations of $\bar{C}$ and s^2 are sufficient for the error estimation. In addition, Eq. (67) provides useful knowledge about the statistical error of Y. For example, if the Feynman-α experiment is conducted under a situation where the sample mean $\bar{C}$ is large enough to satisfy $\bar{C} \gg \frac{(N-1)Y(2Y+1)(5Y+2)}{2N(Y+1)^2}$, $\sigma_{Y,\mathrm{2nd}}$ can be mainly reduced by increasing N or total measurement time NT. Under such a condition, a high strength external neutron source, or large $\bar{C}$, contributes little to improving the statistical error of Y, although the relative statistical error of mean $\langle\frac{\sigma_{\bar{C}}}{\bar{C}}\rangle$ can be reduced. Note that the relative statistical error $\langle\frac{\sigma_{Y_{\mathrm{2nd}}}}{Y}\rangle$ can be reduced by increasing Y using a detector with higher efficiency, as the absolute value of Y is proportional to the detection efficiency. Hence, improvement of the detection efficiency is important for reducing the relative error of Y value.

3. Bootstrap method

3.1. Bootstrap statistical error of Y value

In our previous study, the statistical error estimation for the Feynman-α method using the bootstrap method was proposed [9]. The bootstrap method enables us to easily estimate statistical errors of both Y and the prompt neutron decay constant α, by resampling using an experimentally inferred probability distribution of neutron count. In this study, the procedures are improved to effectively calculate the covariance of $Y(T)$ between different gate widths T, or the "bootstrap covariance matrix $\mathbf{\Sigma}_{Y^*}$." The improved procedures of the bootstrap method are explained below:

1. Original time-series data of neutron counts $\vec{C}(T_0) = \{C_1, C_2, \cdots, C_{N_0}\}$ are provided by a single measurement of reactor noise, where the basic counting gate width is T_0; the total number of count data is N_0.
2. An upper limit value of bunching is set to K, where $1 < K < N_0$.
3. An empty vector $\vec{C}^*(T_0) = \{\}$ is prepared ($i = 1$).
4. The "resampling position ξ_i" is determined using a uniform random integer number, $1 \le \xi_i \le (N_0 - K + 1)$. Then, successive time-series data $\vec{C}_{\xi_i} = \{C_{\xi_i}, C_{\xi_i+1}, \cdots, C_{\xi_i+K-1}\}$ are extracted from the original time series data, and added at the end of the vector $\vec{C}^*(T_0)$. This extraction of successive data is important to estimate the covariance of $Y(T)$.
5. As shown in Fig. 1, a "bootstrap sample of time-series data $\vec{C}^*(T_0)$" is newly generated by repeating $L = \lceil N_0/K \rceil$ times of random-resampling described in step 4:

$$\vec{C}^*(T_0) = \{\vec{C}_{\xi_1}, \vec{C}_{\xi_2}, \cdots, \vec{C}_{\xi_L}\}. \quad (69)$$

Note that extra data in $\vec{C}_{\xi_L}$ is removed so that the total number of count data in $\vec{C}^*(T_0)$ is equal to N_0, if necessary.

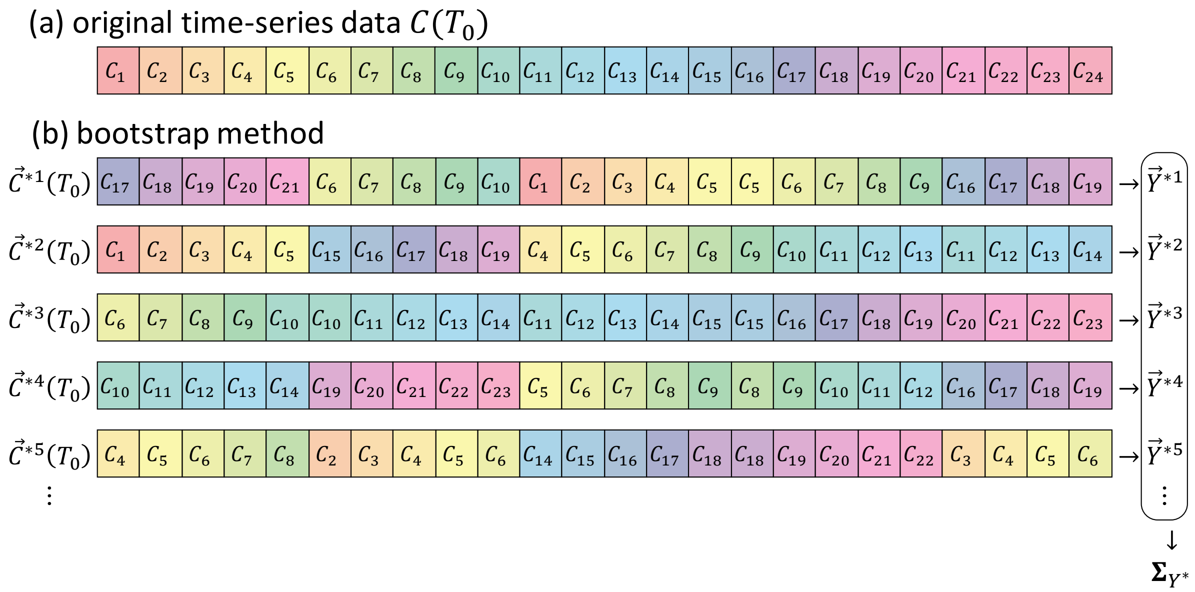


Figure 1. Example of bootstrap method for $\mathbf{\Sigma}_{Y^*}$ $(N_0 = 24, K = 5)$

6. By using Eq. (1) and applying an efficient bunching method to the bootstrap sample $\vec{C}^*(T_0)$ in step 5, the variation in "bootstrap replicate $Y^*(kT_0)$" is evaluated for the bunching gate width kT_0, where k is the bunching number. To recursively apply the bunching method to an already-bunched data, the bunching number k is given by $k = p \times 2^j$ $(j = 0,1,\cdots)$, where p is an initial bunching number (*e.g.*, $p = 2,3,5,7,9,11,13,15,17,19$). As shown in Fig. 2, $\vec{C}^*(2kT_0)$ is effectively produced by combining a pair of successive elements in $\vec{C}^*(kT_0)$. Consequently, a row vector $\vec{Y}^* = \{Y^*(T_0), Y^*(2T_0), \cdots, Y^*(KT_0)\}$ is obtained by this recursive bunching method.

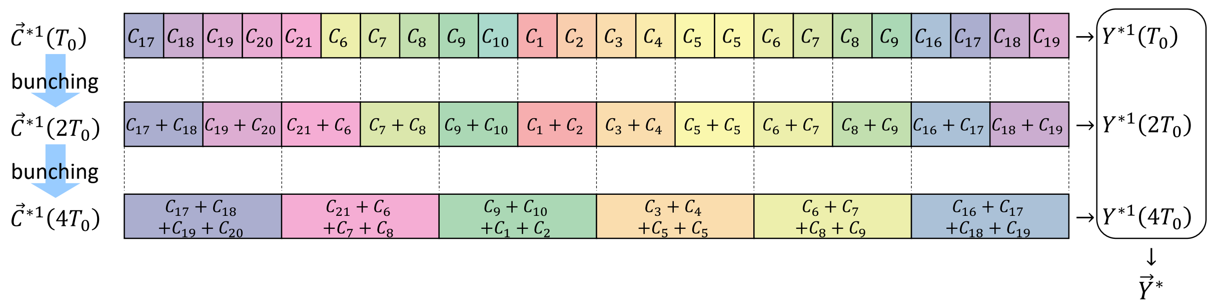


Figure 2. Example of recursive bunching method $(p = 2)$

7. To estimate the covariance matrix of the bootstrap replicate $\vec{Y}^*$, steps 3–6 are repeated B times. Consequently, a set of bootstrap replicates $\vec{Y}^{*b}$ are obtained for $b = 1,2,\cdots,B$, where B is the total number of bootstrap replicates.
8. Using the row vectors $\vec{Y}^{*b}$, the bootstrap covariance matrix $\mathbf{\Sigma}_{Y^*}$ is calculated as follows:

$$\mathbf{\Sigma}_{Y^*} = \frac{1}{B-1}\sum_{b=1}^{B}\left(\vec{Y}^{*b} - \vec{Y}^*_{\mathrm{ave}}\right)^T\left(\vec{Y}^{*b} - \vec{Y}^*_{\mathrm{ave}}\right), \tag{70}$$

$$\vec{Y}^*_{\mathrm{ave}} = \frac{1}{B}\sum_{b=1}^{B}\vec{Y}^{*b}, \tag{71}$$

where the bootstrap standard deviation $\sigma_{Y^*}(kT_0)$ corresponds to the square root of the diagonal element in $\mathbf{\Sigma}_{Y^*}$.

3.2. Bootstrap statistical error of prompt neutron decay constant

After step 8, the following additional procedures are necessary to estimate the statistical error of the prompt neutron decay constant α:

9. Using the bootstrap covariance matrix $\mathbf{\Sigma}_{Y^*}$ in the least squares fitting process, the prompt neutron decay constant α^{*b} is evaluated by fitting a model function of $Y(T)$ to each value of $\vec{Y}^{*b}$. Consequently, bootstrap replicates α^{*b} are obtained for $b = 1,2,\cdots,B$.
10. As the result of step 9, a frequency distribution of α^* is obtained. On the basis of this "bootstrap frequency distribution," the percentile confidence interval (or 2.5 and 97.5 percentile points) can be simply estimated to evaluate the range of the statistical error of α. Namely, the B bootstrap replicates α^{*b} are sorted in ascending order. From the $(0.025 \times B)$th and $(0.975 \times B)$th smallest values of sorted α^{*b}, the lower and upper limits of 95% bootstrap confidence interval are simply estimated, respectively. If

necessary, the bootstrap standard deviation σ_{α^*} can also be estimated as an indicator of the statistical error of α:

$$\sigma_{\alpha^*} = \frac{1}{B-1}\sqrt{\sum_{b=1}^{B}(\alpha^{*b} - \bar{\alpha}^*)^2}\,. \tag{72}$$

$$\bar{\alpha}^* = \frac{1}{B}\sum_{b=1}^{B}\alpha^{*b}\,. \tag{73}$$

Note that the 95% bootstrap confidence interval differs from the intervals of $[\bar{\alpha}^* - 1.96\sigma_{\alpha^*}, \bar{\alpha}^* + 1.96\sigma_{\alpha^*}]$, if the bootstrap frequency distribution is not well approximated by a normal distribution.

4. Experimental analysis

4.1. Experimental conditions

In our previous study, reactor noise experiments were conducted in the A-core (A3/8"p36EU-NU) at the KUCA [9]. The experimental conditions are briefly explained below.

The experimental core and the loaded fuel assembly are shown in Figs. 3 and 4, respectively. The core-average ^{235}U enrichment was 5.4 wt%. Using MCNP6.2 [21] with ENDF-B/VII.1 [22], core characteristics parameters were numerically estimated as follows: Effective neutron multiplication factor $k_{\mathrm{eff}} = 0.93716 \pm 0.00003$; effective delayed neutron fraction $\beta_{\mathrm{eff}} = 775 \pm 6$ [pcm]; and neutron generation time $\Lambda = 42.04 \pm 0.04$ [μs]. Consequently, subcriticality $-\rho = 6705 \pm 3$ [pcm] and $\frac{\beta_{\mathrm{eff}}-\rho}{\Lambda} = 1779 \pm 2$ [1/s].

In this experiment, ^{3}He detectors (#1–4) were placed at axially center positions of excore reflector assemblies. Using these detectors, the time-series data of neutron counts were successively measured. At the shutdown state, the reactor noise was measured without any external neutron source such as Am-Be or Cf source, *i.e.*, using only the inherent neutron source, which mainly consists of spontaneous fission of ^{238}U and (α,n) reactions of ^{27}Al due to α-decay of uranium isotopes [23]. Detector#2 was used for the present reactor noise analysis, where the neutron count rate $R = \bar{C}/T$ was 4.444 ± 0.011 [count/s].

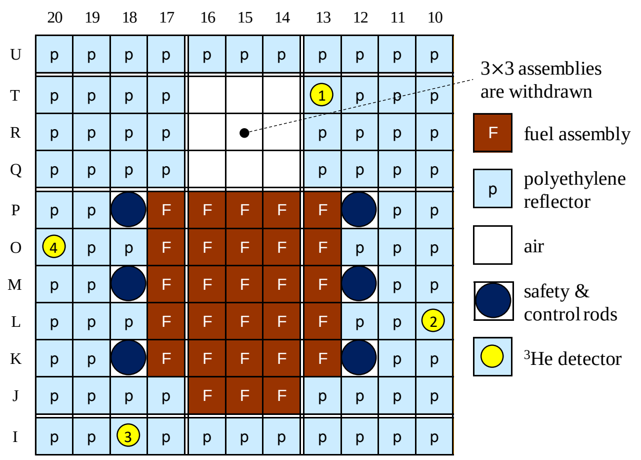


Figure 3. Top view of experimental core (A3/8”p36EU-NU).

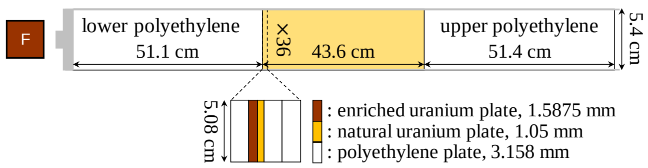


Figure 4. Fuel assembly loaded in experimental core.

To measure the reference value of the statistical error $\sigma_{Y,\mathrm{ref}}$, reactor noise measurements were repeated 93 times. Measurement time was 10 min for each measurement. Using the recursive bunching method, the variation in $Y(kT_0)$ was independently evaluated for each 10 min-measurement, where $T_0 = 10^{-4}$ [s], $N_0 = 6{,}000{,}000$, and $K = 1024$. Thus, if the bunching counting gate width is kT_0, the number of counting gate N_k corresponds to $N_k = \lfloor 6{,}000{,}000/k \rfloor$. Using 93 sets of $\vec{Y}_m = \{Y_m(T_0), Y_m(2T_0), \cdots Y_m(KT_0)\}$, the reference covariance matrix $\boldsymbol{\Sigma}_{Y,\mathrm{ref}}$ was estimated as

$$\boldsymbol{\Sigma}_{Y,\mathrm{ref}} = \frac{1}{93-1} \sum_{m=1}^{93} \left(\vec{Y}_m - \vec{Y}_{\mathrm{ave}}\right)^T \left(\vec{Y}_m - \vec{Y}_{\mathrm{ave}}\right), \tag{74}$$

$$\vec{Y}_{\mathrm{ave}} = \frac{1}{93}\sum_{m=1}^{93}\vec{Y}_m\,, \tag{75}$$

where the statistical error $\sigma_{Y,\mathrm{ref}}$ corresponds to the square root of the diagonal element in $\mathbf{\Sigma}_{Y,\mathrm{ref}}$ Figures 5 and 6 shows $\vec{Y}_{\mathrm{ave}}$ with the reference statistical error $\sigma_{Y,\mathrm{ref}}$ and the correlations of $\mathbf{\Sigma}_{Y,\mathrm{ref}}$ (*i.e.*, $\left(\mathrm{diag}\left(\mathbf{\Sigma}_{Y,\mathrm{ref}}\right)\right)^{-1/2}\mathbf{\Sigma}_{Y,\mathrm{ref}}\left(\mathrm{diag}\left(\mathbf{\Sigma}_{Y,\mathrm{ref}}\right)\right)^{-1/2}$), respectively. The measured Y values are less than approximately 0.1. In addition, $Y(kT_0)$ are positively correlated between different gate widths kT_0 due to the bunching method. The overall trend of Fig. 6 indicates that the correlations of $Y(kT_0)$ become smaller as the difference between kT_0 and $k'T_0$ increases. Because of the recursive bunching method, as shown in Fig. 2, the correlations between kT_0 and $2kT_0$ tend to become stronger.

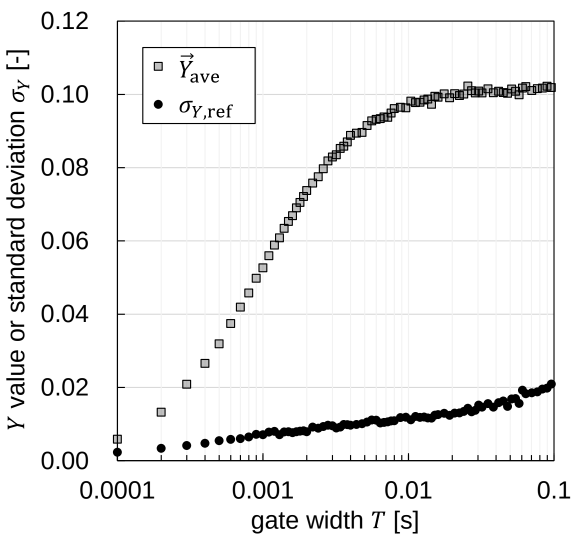


Figure 5. $\vec{Y}_{\mathrm{ave}}$ with reference statistical error $\sigma_{Y,\mathrm{ref}}$.

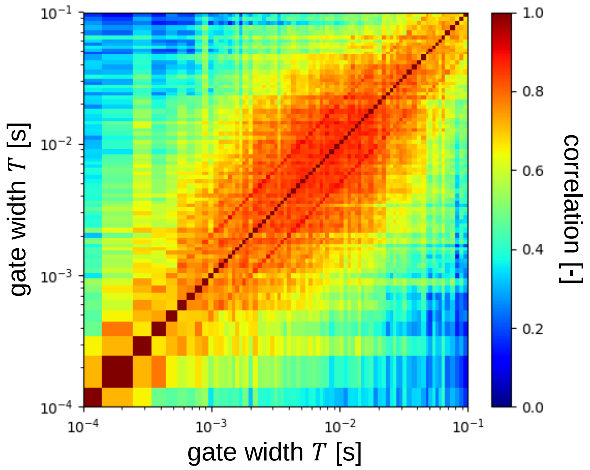


Figure 6. Correlations of the reference covariance matrix $\mathbf{\Sigma}_{Y,\mathrm{ref}}$.

To confirm the validity of the error estimation formulae, one of the 10 min-measurements was selected. Then, the statistical errors $\sigma_{Y,\mathrm{h}}$ and $\sigma_{Y,\mathrm{2nd}}$ were estimated by Eqs. (13) and (67), respectively. To better understand the second-order neutron-correlation effect, the approximated statistical error $\sigma_{Y,\mathrm{P}}$ based on the Poisson distribution was evaluated using Eq. (68). Furthermore, as an alternative error estimation technique, the bootstrap standard deviation σ_{Y^*} was also evaluated by the bootstrap method with $B = 1000$.

4.2. Results of statistical error of Y

Figure 7 shows the reference $\sigma_{Y,\mathrm{ref}}$ and the following statistical errors for the 50th trial ($m = 50$) of 10 min-measurement: (1) $\sigma_{Y,\mathrm{P}}$ based on the Poisson distribution, (2) $\sigma_{Y,\mathrm{2nd}}$ using the simplified formula by reusing the Y vaule, (3) $\sigma_{Y,\mathrm{h}}$ using the unbiased estimators for the third- and fourth-order central moments, and (4) σ_{Y^*} by the bootstrap method. The summary of statistical errors for $1 \leq m \leq 93$ is shown in the attached mp4 file (sigmaY.mp4).

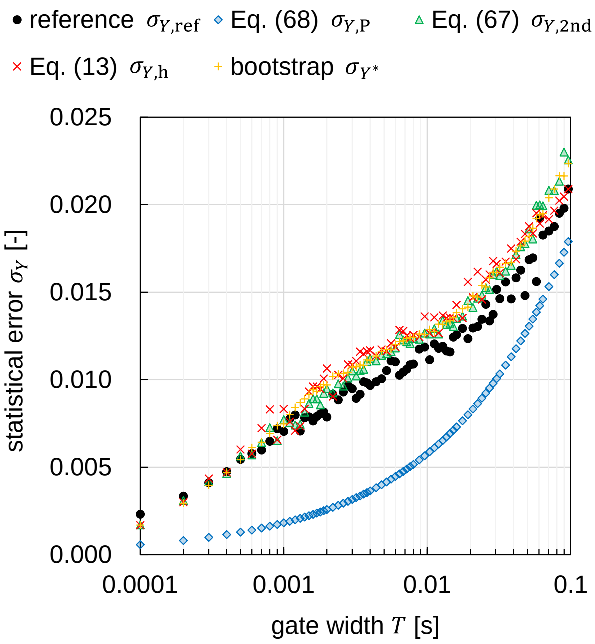


Figure 7. Estimation results of the statistical error of Y value ($m = 50$).

As shown in Fig. 7, $\sigma_{Y,\mathrm{2nd}}$, $\sigma_{Y,\mathrm{h}}$, and σ_{Y^*} agree well with reference $\sigma_{Y,\mathrm{ref}}$. Compared with these results, $\sigma_{Y,\mathrm{P}}$ is significantly underestimated, although $\sigma_{Y,\mathrm{P}}$ is the simplest way to roughly guess the statistical error using only N without the measured neutron count data. By comparing $\sigma_{Y,\mathrm{P}}$ and $\sigma_{Y,\mathrm{2nd}}$, it is confirmed that the second-order neutron-correlation effect is important for improving the estimation of the statistical error σ_Y. As $\sigma_{Y,\mathrm{2nd}}$ is nearly equal to $\sigma_{Y,\mathrm{h}}$, it is demonstrated that the simplified formula of Eq. (67) is applicable to the experimental results in this study. Therefore, it seems to be reasonable that the third- and fourth-order neutron-correlation values can be approximated estimated by Eqs. (61) and (62). The advantage of $\sigma_{Y,\mathrm{2nd}}$ is convenience, *i.e.*, the statistical error can be obtained from the measurement values $\bar{C}$ and Y only. Although $\sigma_{Y,\mathrm{h}}$ requires the additional calculation for h_3 and h_4, the calculation cost is insignificant; thus $\sigma_{Y,\mathrm{h}}$ is also one of the practical estimation methods. Note that the statistical fluctuation of $\sigma_{Y,\mathrm{h}}$ seems to be larger owing to the use of higher-order central moments h_3 and h_4.

As previously reported in reference [9], the bootstrap method enables reasonable estimation of statistical errors, such as the bootstrap standard deviation σ_{Y^*}. As shown in Fig. 7, σ_{Y^*} and $\sigma_{Y,\mathrm{h}}$ are almost the same. The statistical fluctuation in σ_{Y^*} is relatively small because the total number of bootstrap replicates B is sufficiently large. The disadvantage of the bootstrap method is that calculation cost is relatively high owing to the resampling procedures. In the present analysis, the bootstrap replicates Y^* were randomly resampled 1000 times to obtain σ_{Y^*} with high precision, thus the total calculation time of the bootstrap method is approximately at least 1000 times higher than that of $\sigma_{Y,\mathrm{2nd}}$ and $\sigma_{Y,\mathrm{h}}$. Because of this calculation time, the bootstrap method may be unsuitable for real-time statistical error estimation in the on-line monitoring system.

4.3. Discussion on the statistical error of the prompt neutron decay constant

In the present study, theoretical formulae only for σ_Y were derived. The theoretical derivation for the statistical error of the prompt neutron decay constant α (which is denoted by σ_α) is a challenging issue, because the derivation of σ_α is more complicated owing to the fitting procedure for α. In addition, as shown in Fig. 6, the Y values have correlations between different gate widths because of the bunching method. The derivation of a theoretical formula for the estimation of the covariance matrix $\mathbf{\Sigma}_Y$ is not straightforward.

Using the improved bootstrap method, however, the bootstrap covariance matrix $\mathbf{\Sigma}_{Y^*}$ can be numerically evaluated as shown in Fig. 8. The summary of correlations of $\mathbf{\Sigma}_{Y^*}$ for $1 \leq m \leq 93$ is shown in the attached mp4 file (correlationY.mp4). By comparing $\mathbf{\Sigma}_{Y^*}$ with the reference correlations of $\mathbf{\Sigma}_{Y,\mathrm{ref}}$ in Fig. 6, we can see that $\mathbf{\Sigma}_{Y^*}$ seems to be useful as an alternative to $\mathbf{\Sigma}_{Y,\mathrm{ref}}$.

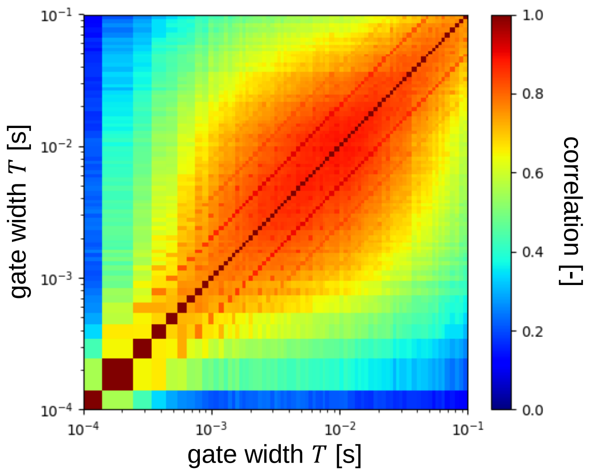


Figure 8. Correlations of bootstrap covariance matrix $\mathbf{\Sigma}_{Y^*}$ $(m = 50)$.

As a numerical study, the following two fitting procedures for α were compared for each of the 10 min-measurements:

(a) Fitting error method without correlation: Using the inverse of the statistical error $\sigma_{Y,\mathrm{h}}$ described by Eq. (13), *i.e.*, $\frac{1}{\sigma_{Y,\mathrm{h}}}$, as the weight, the least squares fitting was performed to estimate the α value. As will be discussed later, the absolute value of $\sigma_{Y,\mathrm{h}}$ was just used as is, *i.e.*, $\sigma_{Y,\mathrm{h}}$ was not scaled according to the χ^2 value after the fitting. Then, the estimated fitting error of α (which is denoted as $\sigma_{\alpha,\mathrm{fit}}$) was used as an alternative to σ_α. If the probability distribution of Y is approximated by a normal distribution and the correlations (off-diagonal elements) of $\mathbf{\Sigma}_Y$ do not have a large impact on the estimation of the fitting error, it is expected that the magnitude of the fitting error $\sigma_{\alpha,\mathrm{fit}}$ is reasonable as a candidate of the statistical error σ_α because $\sigma_{\alpha,\mathrm{fit}}$ is evaluated by the propagation of uncertainty using the Jacobian matrix with the statistical error $\sigma_{Y,\mathrm{h}}$ [24]. By assuming a normal distribution for the probability distribution of α, the 95% confidence intervals were approximated as the range of $[\alpha - 1.96\sigma_{\alpha,\mathrm{fit}}, \alpha + 1.96\sigma_{\alpha,\mathrm{fit}}]$.

(b) Bootstrap method with correlation: Using the bootstrap covariance matrix $\mathbf{\Sigma}_{Y^*}$, the 95% bootstrap confidence intervals were estimated to express the range of statistical error of α without the assumption of normality for Y, as described in Section 3.2. The bootstrap standard deviation σ_{α^*} was also estimated and compared with $\sigma_{\alpha,\mathrm{fit}}$.

To mainly evaluate α, the following model function was used in the least squares fitting [9]:

$$Y(T) \approx Y_\infty \left(1 - \frac{1 - \exp(-\alpha T)}{\alpha T}\right) + A\,T + B, \qquad (76)$$

where Y_∞ is the saturation value, and A and B are supplemental fitting parameters to correct the spatial and neutron-energetic higher-order modes, the delayed neutron and the dead-time effects, respectively. Owing to their effects, a more rigorous theoretical expression of $Y(T)$ can be expressed as

$$Y(T) \approx \sum_{n=0}^{\infty} Y_{\mathrm{p},n,\infty} \left(1 - \frac{1 - \exp(-\alpha_n T)}{\alpha_n T}\right) + \sum_{i=1}^{6} Y_{\mathrm{d},i,\infty} \left(1 - \frac{1 - \exp(-\omega_i T)}{\omega_i T}\right) - 2R\tau, \qquad (77)$$

where the first, second, and third terms on the right-hand side correspond to terms due to spatial and neutron-energetic modes [17], delayed neutrons [25], and dead-time τ [26], respectively; α_n is the nth mode of the prompt neutron decay constant (the fundamental mode corresponds to $n = 0$); ω_i indicates the decay constant due to a delayed neutron; $Y_{\mathrm{p},n,\infty}$ and $Y_{\mathrm{d},i,\infty}$ are saturation values for each component with respect to α_n and ω_i. Note that the decay constants satisfy the following conditions: $\omega_i \ll \alpha_0$ and $\alpha_0 < \alpha_1 < \alpha_2 < \cdots$. In the fitting process, a complicated formula does not necessarily yield a good fitting result owing to the overfitting issue. In this study, thus, we utilized the simplified fitting formula of Eq. (76) instead of Eq. (77). To simplify the fitting formula, let us assume $\omega_i T \ll 1$ and $\alpha_n T \gg 1$. Then Eq. (77) can be approximated as

$$Y(T) \approx Y_{\mathrm{p},0,\infty} \left(1 - \frac{1 - \exp(-\alpha_0 T)}{\alpha_0 T}\right) + \sum_{i=1}^{6} \left(\frac{1}{2} Y_{\mathrm{d},i,\infty} \omega_i\right) T + \sum_{n=1}^{\infty} Y_{\mathrm{p},n,\infty} - 2R\tau. \qquad (78)$$

Thus, the fitting parameters A and B in Eq. (76) correspond to $\sum_{i=1}^{6}\left(\frac{1}{2}Y_{\mathrm{d},i,\infty}\omega_i\right)$ and $\sum_{n=1}^{\infty}Y_{\mathrm{p},n,\infty} - 2R\tau$, respectively.

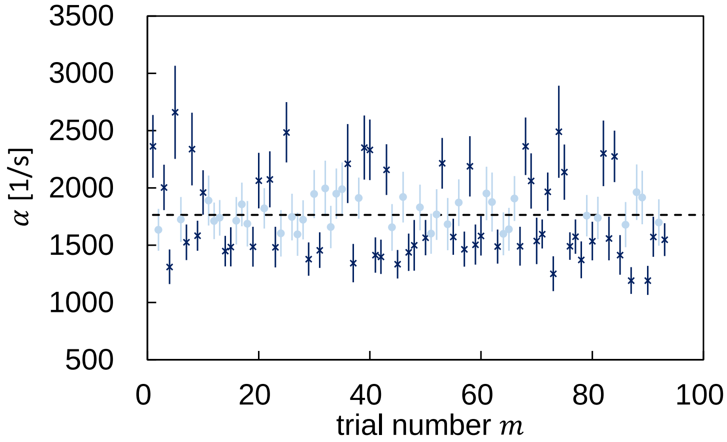
In this study, "scipy.optimize.curve_fit" was utilized for the non-linear least squares fitting [27]. In the module of "scipy.optimize.curve_fit," the option of "absolute_sigma" was explicitly set to "True" in order to simply use the absolute value of $\sigma_{Y,\mathrm{h}}$ as in the estimation of $\sigma_{\alpha,\mathrm{fit}}$. In the default setting of "absolute_sigma=False," the χ^2 value after the fitting was normalized to be equal to the number of freedoms, which corresponds to $\left(\text{total number of } Y(kT_0)\right) - (\text{total number of fitting parameters})$. The option of "absolute_sigma" influences the fitting error and has no impact on procedure (b) because the 95% bootstrap confidence interval and σ_{α^*} are calculated from 1000 fitting values of α^* in the bootstrap method.

Figure 9 shows the estimated 95% confidence intervals obtained by using the two fitting procedures (a) and (b). In Fig. 9, the black dashed horizontal line represents the sample mean of 93 trial values of α; and a dark-colored plot indicates that the sample mean exists out of the estimated 95% confidence interval. As results of procedures (a) and (b) for 93 trials ($1 \leq m \leq 93$), the sample means of the 93 fitting values α_m (which is denoted as $\bar{\alpha}$) were 1764 and 1772 [1/s], respectively, while the standard deviations of the 93 fitting values α_m (which is denoted as $\sigma_{\alpha,\mathrm{ref}}$) were 324 and 252 [1/s] (which are regarded as the reference values of σ_α for each procedure), respectively. Owing to the difference in the fitting procedure, there are differences in the fitting results between procedures (a) and (b). The sample means $\bar{\alpha}$ are nearly equal to the numerical results of $\frac{\beta_{\mathrm{eff}}-\rho}{\Lambda} = 1779 \pm 2$ [1/s] using MCNP6.2 with ENDF-B/VII.1; thus, it was confirmed that the simplified fitting formula of Eq. (76) was reasonably applicable to the estimation of α in this experimental analysis.

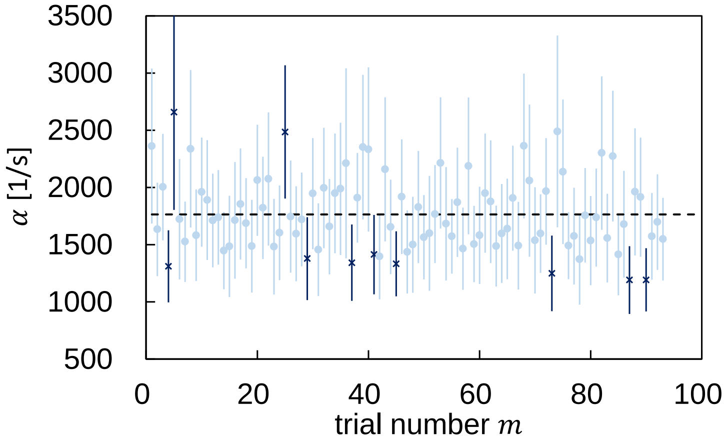
As can be seen from Fig. 9, the widths of 95% confidence intervals statistically fluctuate. For procedures (a) and (b), the sample means of 93 fitting errors $\sigma_{\alpha,\mathrm{fit},m}$ and 93 bootstrap standard deviations $\sigma_{\alpha^*,m}$ were 238 and 272 [1/s], respectively. By comparing it with $\sigma_{\alpha,\mathrm{ref}}$, the statistical error of α estimated by procedure (a) seems to be slightly underestimated. Note that, if "absolute_sigma" is set to "False" in procedure (a), the estimated statistical error $\sigma_{\alpha,\mathrm{fit}}$ is significantly underestimated (sample mean of $\sigma_{\alpha,\mathrm{fit},m} \approx 102$ [1/s]), as shown in the Fig. 9-(i). Consequently, it was confirmed that $\sigma_{Y,\mathrm{h}}$ should not be scaled according to the χ^2 value after the fitting in procedure (a) to avoid the underestimation of the statistical error $\sigma_{\alpha,\mathrm{fit}}$. This fact implies that the χ^2 test without correlation is useless to judge the goodness of fit.

To discuss the validity of the estimated 95% confidence intervals, we considered the coverage probability that the sample mean $\bar{\alpha}$ exists within the range of the estimated 95% confidence interval. The coverage probability of procedures (a) and (b) were $83/93 \approx 89\%$ and $87/93 \approx 94\%$, respectively. The coverage probability of procedure (b) was slightly closer to 95% than that of procedure (a). Thus, to accurately evaluate the confidence interval for the statistical error of α, it seemed that procedure (b) was slightly better than procedure (a), although procedure (a) could approximately guess the order of magnitude of statistical error σ_α in this experimental analysis.

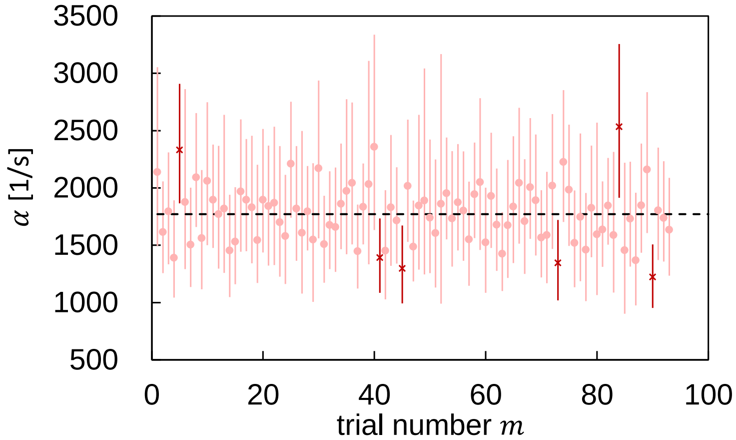
Consequently, a comparison of the fitting procedures (a) and (b) implied that the correlations of $\mathbf{\Sigma}_Y$ could be negligible for the order estimation of the statistical error σ_α. To improve the coverage probability of the estimated confidence intervals, a more rigorous treatment of the correlations of $\mathbf{\Sigma}_Y$ might be important. However, the theoretical approach for the estimation of $\mathbf{\Sigma}_Y$ is still an open problem. On the other hand, although the bootstrap method is based on the numerical approach, $\mathbf{\Sigma}_Y$ can be numerically estimated from a single measurement of reactor noise; furthermore, the 95% confidence interval can be reasonably estimated without the assumption of normality.



(i) Fitting error method without correlation (absolute_sigma=False)



(ii) Fitting error method without correlation (absolute_sigma=True)



(iii) Bootstrap method with correlation

Figure 9 Estimated 95% confidence interval of α.

5. Conclusion

For the statistical error estimation of the Y value from a single measurement of reactor noise, the practical estimation formulae of $\sigma_{Y,\mathrm{h}}$ and $\sigma_{Y,\mathrm{2nd}}$ were derived by the propagation of uncertainty where the covariance between the sample mean and the unbiased variance is explicitly considered. The estimation of $\sigma_{Y,\mathrm{h}}$ requires additional calculations for the unbiased estimations of the third- and the fourth-order central moments, h_3 and h_4. On the other hand, $\sigma_{Y,\mathrm{2nd}}$ can be estimated by reusing the Y value without calculation of h_3 and h_4. Note that the simplified formula of $\sigma_{Y,\mathrm{2nd}}$ is applicable under the fundamental mode approximation where the subcriticality is approximately less than 10,000 pcm.

In addition, the bootstrap method was improved to efficiently estimate the bootstrap covariance matrix $\mathbf{\Sigma}_{Y^*}$. Using the improved bootstrap method, both the statistical error of Y and the correlation of Y between different counting gate widths T due to the bunching method could be numerically obtained by the resampling based on a probability distribution that is experimentally inferred from a single measurement of reactor noise.

Through the reactor noise analysis for the actual KUCA experiment, it was validated that the statistical errors $\sigma_{Y,\mathrm{h}}$ and $\sigma_{Y,\mathrm{2nd}}$, and the bootstrap standard deviation σ_{Y^*} agree well with the reference value of $\sigma_{Y,\mathrm{ref}}$, which was obtained from multiple measurements of reactor noise. Furthermore, it was confirmed that the second-order neutron-correlation effect is important in the estimation of the statistical error of the Y value by comparison between $\sigma_{Y,\mathrm{2nd}}$ and the approximated statistical error $\sigma_{Y,\mathrm{P}}$ using the Poisson distribution. Compared with the bootstrap method, the practical estimation formulae of $\sigma_{Y,\mathrm{h}}$ and $\sigma_{Y,\mathrm{2nd}}$ are more advantageous owing to their lower calculation cost.

To investigate the impact of correlations of $\mathbf{\Sigma}_{Y}$ on the statistical error of the prompt neutron decay constant α, the following two fitting procedures were compared: the fitting error method using $\sigma_{Y,\mathrm{h}}$ without correlation and the bootstrap method with correlation using $\mathbf{\Sigma}_{Y^*}$. As a result, in the case of this experimental analysis, it was confirmed that the fitting error method without correlation could be useful for the order estimation of the statistical error of α. To avoid significant underestimation of statistical error, $\sigma_{Y,\mathrm{h}}$ should not be scaled according to the χ^2 value after the fitting. Compared with the fitting error method without correlation, the bootstrap method with correlation seems to be slightly better from the viewpoint of the coverage probability of the estimated confidence intervals. Thus, consideration of the correlations of $\mathbf{\Sigma}_Y$ might be important to improve the coverage probability; however, the theoretical approach for the estimation of $\mathbf{\Sigma}_Y$ is still an open problem.

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