# ENTROPY OF PLANAR RANDOM SURFACES ON THE LATTICE

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We study planar random surfaces on a hypercubic lattice in two and three dimensions by Monte Carlo techniques. Our data are consistent with the formula $n_0(A;C) \sim A^{b_0}\alpha^A$, where $n_0(A;C)$ is the number of planar random surfaces with area $A$ and boundary $C$. We find $b_0 = -1.4 \pm 0.2$, $\alpha = 5.31 \pm 0.03$ (for $d = 2$) and $b_0 = -1.5 \pm 0.2$, $\alpha = 7.13 \pm 0.05$ (for $d = 3$). The values of $b_0$ disagree with those obtained from the Polyakov string model.

There is a well-known correspondence between spin systems and systems of random walks. It is widely believed that an analogous correspondence exists between gauge systems and systems of random surfaces. It is in this sense that study of random surfaces is very important. It has been shown [1,2] that the number of random surfaces having an area $A$, boundary $C$ and fixed topology grows exponentially for large $A$:

$$n_j(A,C) \sim A^{b_j}\alpha^A, \quad j = 0, 1, 2, \ldots, \tag{1}$$

where $j$ denotes the number of handles, $b_j$ is a *universal* constant and $\alpha$ is a regularization-dependent constant. The purpose of this article is to confirm this behavior of $n_j(A,C)$ explicitly by Monte Carlo techniques for planar random surfaces ($j = 0$) and determine the constants $b_0$ and $\alpha$ for the two- and three-dimensional hypercubic lattices. Our results are

$$\begin{aligned} b_0 &= -1.4 \pm 0.2 \quad \text{for } d = 2\,, \\ &= -1.5 \pm 0.2 \quad \text{for } d = 3\,, \end{aligned} \tag{2}$$

$$\begin{aligned} \alpha &= 5.31 \pm 0.03 \quad \text{for } d = 2\,, \\ &= 7.13 \pm 0.05 \quad \text{for } d = 3\,. \end{aligned} \tag{3}$$

Here the estimate of the errors in $\alpha$ is rather crude, since in our method described below we cannot find the precise magnitude of errors of $\lambda_c$ ($= 1/\alpha$). Also the errors of $b_0$ are somewhat correlated to the errors of $\lambda_c$. The values of $\alpha$ in (3) satisfy the bound obtained analytically in ref. [1]:

$$\alpha < 24(d-1)\,. \tag{4}$$

Note that the critical exponents in (2) disagree with those of ref. [3] obtained from the Polyakov string theory [4][‡1]:

$$\begin{aligned} b_0 &= (d-19)/6 = -2.83 \quad \text{for } d = 2\,, \\ &= -2.67 \quad \text{for } d = 3\,. \end{aligned} \tag{5}$$

Our approach is based on the Weingarten model [5] in the large-$N$ limit. The action of this model is given by

$$S = -\lambda N \sum_n \sum_{\substack{\mu,\nu \\ (\mu\neq\nu)}} \mathrm{Tr}\,(A_{n,\mu}A_{n+\hat{\mu},\nu}A^{\dagger}_{n+\hat{\nu},\mu}A^{\dagger}_{n,\nu}) + N\sum_n\sum_\mu \mathrm{Tr}(A_{n,\mu}A^{\dagger}_{n,\mu}), \tag{6}$$

where $A_{n,\mu}$ are complex $N \times N$ matrices and $\lambda = 1/g^2N$. The Wilson loop amplitude in this theory is defined in a similar way as in the usual lattice gauge theory:

$$W(C;\lambda) = \left(\int \prod_{n,\mu} \mathrm{d}A_{n,\mu}\, w(C)\, \mathrm{e}^{-S}\right)\left(\int \prod_{n,\mu} \mathrm{d}A_{n,\mu}\, \mathrm{e}^{-S}\right)^{-1}, \tag{7}$$

[‡1] A similar disagreement between planar random surfaces on the lattice and Polyakov string was noted in ref. [2].

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where

$$w(C) = N^{-1} \operatorname{Tr}(AA \ldots A) . \tag{8}$$

By expanding (7) in powers of $\lambda$, $W(C, \lambda)$ can be formally expressed as

$$W(C;\lambda) = \sum_{j=0}^{\infty} \sum_{A=0}^{\infty} \frac{1}{N^{2j}} n_j(A;C) \lambda^A , \tag{9}$$

where $n_j(A;C)$ ($j = 0, 1, 2, \ldots$) is, as defined before, the number of random surfaces with $A$ plaquettes, fixed boundary $C$, and $j$ handles.

Note that this theory is well-defined only in the large-$N$ limit. Since the action (6) is not bounded from below, the vacuum fluctuating around the origin ($A_\mu \sim 0$) is not stable for finite $N$. The probability that this metastable vacuum tunnels through the potential barrier is proportional to $\exp[-(\text{const.})N^2]$. Hence the vacuum becomes stable in the large-$N$ limit, if $\lambda$ is less than some critical value $\lambda_c$.

In the large-$N$ limit the Wilson loop amplitude in (9) becomes the generating function of the number of planar random surfaces:

$$W(C;\lambda) = \sum_{A=0}^{\infty} n_0(A;C) \lambda^A . \tag{10}$$

From eq. (1) it can be easily shown that a phase transition occurs at $\lambda = 1/\alpha$ ($\equiv \lambda_c$); $W(C;\lambda)$ in (10) converges for $\lambda < \lambda_c$, but does not converge for $\lambda > \lambda_c$. This phase transition can also be seen by observing the development of $W(C;\lambda)$ through Monte Carlo iteration. Namely, if $\lambda > \lambda_c$, the system overflows quickly beyond the potential barrier (see fig. 1a). On the other hand, if $\lambda < \lambda_c$ and $N$ is large enough, the system remains in a metastable state for some time (see fig. 1b). The lifetime of the metastable state is larger when $N$ is larger. In the vicinity of the critical point we can show that

$$W(C;\lambda) \sim \text{const.} - (\text{const.})(\lambda_c - \lambda)^{-(b_0+1)} , \tag{11}$$

by approximating the summation in (10) by integration. Therefore, $b_0$ can be determined by observing the singularity of $W(C;\lambda)$ near the critical point.

The Weingarten model is equivalent to the reduced Weingarten model in the large-$N$ limit [1]. In the actual numerical work we used the reduced Weingarten model in order to save memory space. (Indeed we could increase $N$ to 100.) The action of the reduced Weingarten model is given by

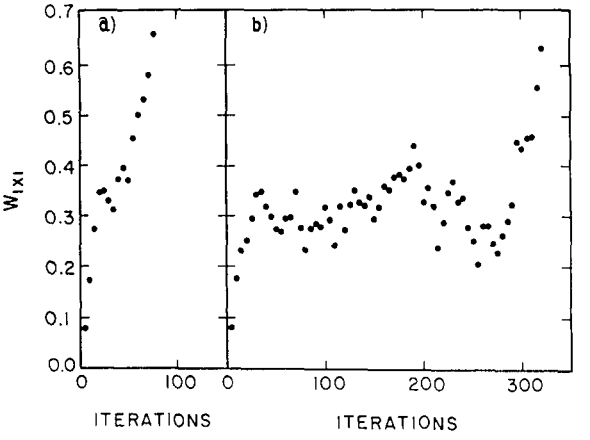


Fig. 1. Typical behavior of the one-by-one Wilson loop as a function of the number of Monte Carlo iterations for $N = 30$ and $d = 2$ when (a) $\lambda$ is just above $\lambda_c$ ($\lambda = 0.19$), and (b) $\lambda$ is just below $\lambda_c$ ($\lambda = 0.18$).

$$S_{\text{reduced}} = -\lambda N \sum_{\substack{\mu,\nu \\ (\mu \neq \nu)}} \operatorname{Tr}(A_\mu A_\nu A_\mu^\dagger A_\nu^\dagger) + N \sum_\mu \operatorname{Tr}(A_\mu A_\mu^\dagger) , \tag{12}$$

where $A_\mu$ ($\mu = 1, \ldots, d$) are complex $N \times N$ matrices. We evaluate the one-by-one Wilson loop amplitude averaged over all the directions

$$W_{1\times1} = \frac{1}{d(d-1)} \sum_{\substack{\mu,\nu \\ (\mu \neq \nu)}} \frac{1}{N} \langle \operatorname{Tr}(A_\mu A_\nu A_\mu^\dagger A_\nu^\dagger) \rangle . \tag{13}$$

Since the reduced Weingarten model does not suffer from $U(1)^d$ symmetry breakdown below the critical point $\lambda_c$ [1], we do not need quenching or twisting in principle. The tunnelling effects, however, can be reduced by imposing twisted boundary conditions in the numerical simulation. Actually we used the simplest twisted boundary conditions, which is equivalent to the substitution $\lambda \to -\lambda$ [6]. The effect of changing the sign of $\lambda$ in eq. (12) is to change the sign of Wilson loop amplitudes:

$$W(C;-\lambda) = (-1)^{\min(C)} W(C;\lambda) , \tag{14}$$

where min$(C)$ is the number of plaquettes of the minimal surface bounded by $C$.

Our Monte Carlo simulations are performed as follows. For each set $(i, j)$ of row–column indices of $N \times N$ matrix $A_\mu$ we vary the matrix by

$$(A_\mu)_{ij} \to (A_\mu)_{ij} + \Delta_{ij} , \tag{15}$$

where $\Delta_{ij}$ is an arbitrary complex number. The change is accepted or rejected according to the standard Metropolis scheme. This procedure is applied $N^2$ times (i.e. for all the elements of $A_\mu$) to a given link variable $A_\mu$ before proceeding to the next. We will refer to a sequential pass through all variables $A_\mu$ ($\mu = 1, ..., d$) as one Monte Carlo "iteration". To reduce the computation time, we use the following trick similar to the one in ref. [7]: We define $W_{\mu\nu}$ by $W_{\mu\nu} = A_\mu A_\nu$. Under the variation of $(A_\sigma)_{ij}$ in (15), only changes in

$$(W_{\sigma\nu})_{ik} \to (W_{\sigma\nu})_{ik} + \Delta_{ij}(A_\nu)_{jk} \quad (\nu = 1, ..., d;\ k = 1, ..., N),$$
$$(W_{\mu\sigma})_{kj} \to (W_{\mu\sigma})_{kj} + \Delta_{ij}(A_\mu)_{ki} \quad (\mu = 1, ..., d;\ k = 1, ..., N),$$
(no summation on $i$ and $j$) . (16)

Then by storing both $A_\mu$ and $W_{\mu\nu}$, the computation time of the change of the action $S_{\text{reduced}}$ in (12) grows only linearly in $N$. The size of each variation $\Delta_{ij}$ is selected so that the probability for accepting a change is always greater than 50%.

We now describe our results. Fig. 2 shows the values of the one-by-one Wilson loop $W_{1\times1}$ for $d = 2$ (fig. 2a) and $d = 3$ (fig. 2b) plotted by combining the data for $N = 30$ and $N = 100$ ($d = 2$) and for $N = 32$, $N = 56$, and $N = 100$ ($d = 3$). In fig. 2a each point for $N = 30$ is an average over 500 iterations and that for $N = 100$ is an average over 210 to 240 iterations. In fig. 2b each point for $N = 32$ and $N = 56$ is an average over 210 iterations and that for $N = 100$ is an average over 100 to 150 iterations. The solid curve at small $\lambda$ is $\lambda + 8(d-1)\lambda^3$, the prediction of the strong coupling expansion. The data for $N=30$ ($d=2$) and $N = 32$ ($d = 3$) follow this curve fairly closely in the strong coupling region. As $\lambda$ increases towards $\lambda_c$, however, the quality of data deteriorates, because the system tends to tunnel through the potential barrier quickly in the critical region. This difficulty can be avoided by increasing $N$, since the tunnelling probability is, as mentioned before, proportional to $\exp[-(\text{const.})N^2]$. Hence, the larger $N$ is, the closer to the critical point we can probe. As is shown in fig. 2, we had to increase $N$ to 100 in order to obtain satisfactory data. The data in fig. 2 indicate strongly that the first derivative of $W_{1\times1}$ with respect to $\lambda$ diverges at the critical point for both $d = 2$ and $d = 3$. Divergence of $\mathrm{d}W_{1\times1}/\mathrm{d}\lambda|_{\lambda\to\lambda_c-0}$ means

$$b_0 > -2, \quad \text{for } d = 2 \text{ and } 3 . \tag{17}$$

This bound disagrees with the result (5) derived from the Polyakov string theory.

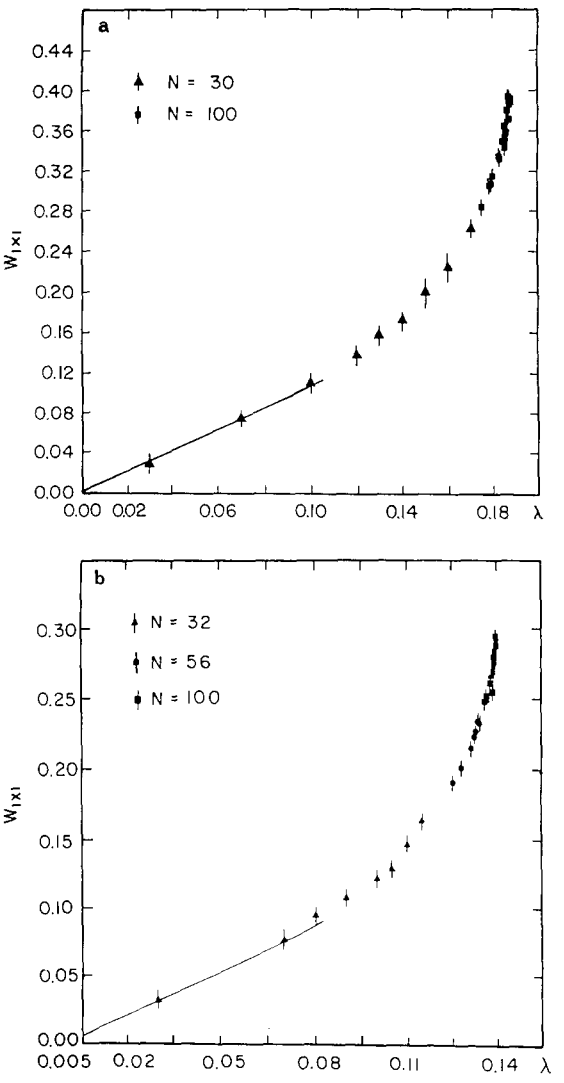


Fig. 2. The one-by-one Wilson loop as a function of $\lambda$ for (a) $N = 30$ and 100 in $d = 2$ dimensions, and (b) $N = 32$, 56, and 100 in $d = 3$ dimensions. The solid curve at small $\lambda$ in each graph is $\lambda + 8(d-1)\lambda^3$, the leading terms of the strong coupling expansion.

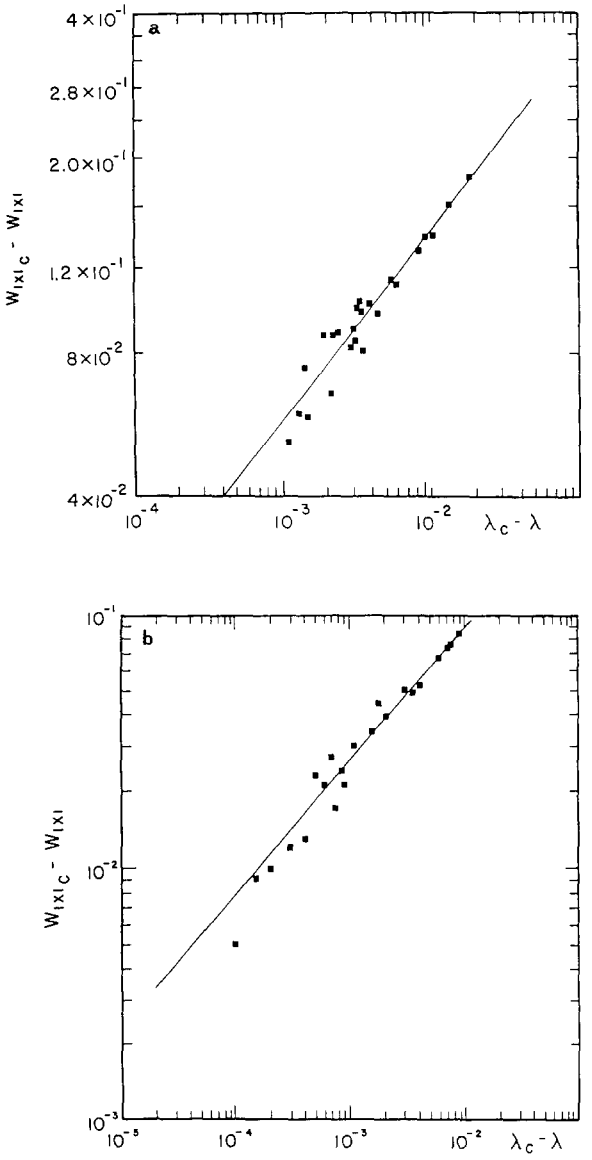


Fig. 3. Plots of $\log(W_{1\times1_c} - W_{1\times1})$ versus $\log(\lambda_c - \lambda)$ near the critical point for (a) $N = 100$ in $d = 2$ dimensions, and (b) $N = 100$ in $d = 3$ dimensions.

Finally, the constants $b_0$ and $\alpha$ are obtained as follows: First, we determine $\lambda_c$ ($=1/\alpha$) according to whether the vacuum is metastable or not (see fig. 1). Although this determination of $\lambda_c$ is rather subtle because of the tunnelling effects, we can determine $\lambda_c$ fairly reliably if $N$ is large enough. Secondly, after obtaining $\lambda_c$, we fit the plot of $\log(W_{1\times1_c} - W_{1\times1})$ versus $\log(\lambda_c - \lambda)$ by a straight line according to eq. (11). The best fit determines the constants $b_0$ and $W_{1\times1_c}$. The best results for $d = 2$ and $d = 3$ are shown in fig. 3. In this fitting we used the data only from the vicinity of the critical point (i.e. data for $N = 100$). The values of $b_0$ and $\alpha$ are given in (2) and (3), respectively. $W_{1\times1_c}$ is found to be

$$W_{1\times1_c} = 0.44 \pm 0.04 \quad \text{for } d = 2\,, \qquad = 0.33 \pm 0.03 \quad \text{for } d = 3\,. \tag{18}$$

In this article we have reported the numerical calculation of the critical exponent $b_0$ and the critical point $\alpha$ ($=1/\lambda_c$) of planar random surfaces for $d = 2$ and $d = 3$. The divergence of $dW_{1\times1}/d\lambda|_{\lambda\to\lambda_c-0}$ mentioned above indicates that the zero-momentum two-point Green's function diverges at the critical point. This fact suggests strongly that the glueball mass vanishes at the critical point and that the continuum limit of the large-$N$ Weingarten model exists. (Unfortunately, we could not exclude the alternative possibility that the residue of the Green function diverges, while the glueball mass remains finite [‡2]. Besides numerical work we also examined planar random surfaces analytically in the limit $d \to \infty$ [9]. In the large-$d$ limit we found that $b_0$ is exactly equal to $-1.5$ and that the glueball mass vanishes at the critical point, excluding the alternative possibility mentioned above. Furthermore, since the SU($N$) lattice gauge theory and the Weingarten model have the same gauge symmetry, it is natural to conjecture that the continuum limit of the large-$N$ Weingarten theory belongs to the same universality class as the weak coupling limit of the large-$N$ SU($N$) lattice gauge theory, as is the case for non-linear $\sigma$ model and $\phi^4$ model. If this conjecture is correct, we can confirm the equivalence between the large-$N$ gauge theory and the string theory, because the large-$N$ Weingarten model is equivalent to the string theory on the lattice.

Our Monte Carlo program ran on a Floating Point Systems FPS-164 Array Processor.

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‡2 A more detailed analysis has been done in ref. [8].

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*References*

[1] T. Eguchi and H. Kawai, Phys. Lett. 114B (1982) 247.
[2] B. Durhuus, J. Fröhlich and T. Jonsson, Self-avoiding and planar random surfaces on the lattice, NORDITA preprint 83/15.
[3] A.B. Zamolodchikov, Phys. Lett. 117B (1982) 87.
[4] A.M. Polyakov, Phys. Lett. 103B (1981) 207.
[5] D. Weingarten, Phys. Lett. 90B (1980) 280.
[6] A. González-Arroyo and M. Okawa, Phys. Lett. 120B (1983) 174; Phys. Rev. D27 (1983) 2397.
[7] M. Okawa, Phys. Rev. Lett. 49 (1982) 353.
[8] B. Durhuus, J. Fröhlich and T. Johnsson, private communication.
[9] H. Kawai and Y. Okamoto, in preparation.