Asymptotic photon propagator in massive QED and the muon anomalous magnetic moment

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Recently, Faustov et al. have obtained, by a renormalization group technique, the asymptotic contribution to the muon anomalous magnetic moment arising from the eighth-order diagrams containing two electron vacuum-polarization loops, one within the other. Their result disagrees strongly with the value previously obtained by numerical evaluation of exact eighth-order integrals. We have identified the cause of this discrepancy to be an incorrect four-loop Callan–Symanzik β-function. Using the corrected β-function, which takes the effect of finite electron mass into account, we find that the result of Faustov et al. agrees very well with that of numerical integration.

1. Introduction

Since the muon mass M is much larger than the electron mass m ($M/m \approx 207$), the contribution to the muon anomalous magnetic moment a_μ arising from virtual photons including electron vacuum-polarization loops is dominated by the short-distance behavior of the photon propagator. Exploiting the close relationship between this fact and the renormalization procedure, it is possible to determine the $\ln(M/m)$ structure of a large class of diagrams contributing to the muon anomaly [1,2]. Such information is very useful since it enables us to estimate the magnitude of higher-order contributions to a_μ reliably without resorting to extensive numerical work.

In this paper we focus our attention on the diagrams of the type shown in fig. 1, generated from a second-order muon vertex by insertion of the electron vacuum-polarization diagram G. As is well known [2], the large M/m structure, including the constant term, of the magnetic moment contribution of these diagrams can be determined completely by the asymptotic behavior of G. For the eighth-order contribution to a_μ from the diagrams of fig. 2, however, the constant term was left undetermined since

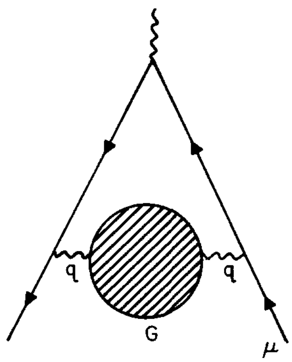

Fig. 1. Diagram generated by inserting an electron vacuum-polarization diagram G in the second-order muon vertex.

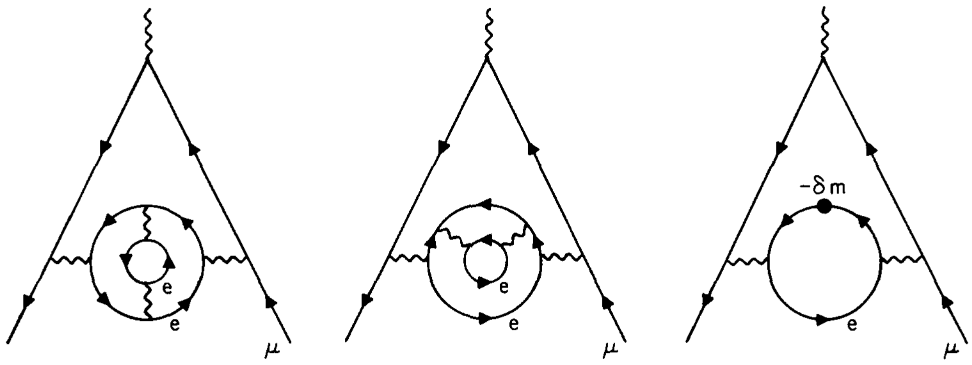

Fig. 2. Muon vertex diagrams containing vacuum-polarization diagrams of fig. 3.

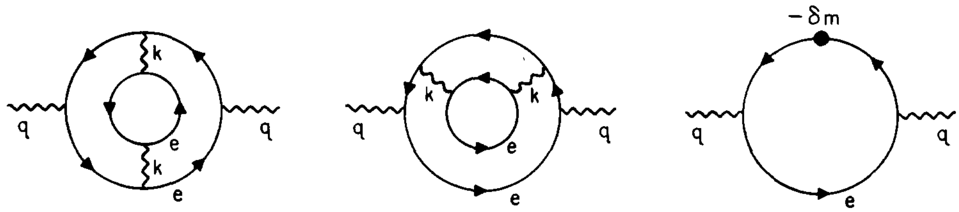

Fig. 3. Sixth-order vacuum polarization diagrams containing two closed electron loops, one within the other, and a diagram containing the corresponding mass counterterm $-\delta m$.

the corresponding constant term of the sixth-order asymptotic photon propagator represented by the diagrams of fig. 3 was not known [2]. Recently, Faustov et al. [3] have attempted to determine this constant term by means of a renormalization group technique. Unfortunately their result, which is expected to be correct within an uncertainty of the order of m/M, disagrees strongly with the value previously obtained [4] by direct numerical evaluation of exact eighth-order integrals.

In this paper we analyze this discrepancy and show that it is caused by the use in ref. [3] of the incomplete four-loop Callan–Symanzik β-function obtained by Calmet and de Rafael [5]. We have found a correction term to their β-function, taking into account the effect of finite electron mass properly. The disagreement disappears when this term is included. The derivation of the correct β-function is described in detail in a separate paper [6].

2. Asymptotic photon propagator and the muon anomaly

The general expression for the renormalized photon propagator has the form

$$D_R^{\mu\nu}(q) = -\mathrm{i}\frac{g^{\mu\nu}}{q^2} d_R(q^2, m^2, \alpha) + \text{the } q^\mu q^\nu \text{ term}\,, \tag{1}$$

where d_R can be written as

$$d_R(q^2, m^2, \alpha) \equiv \frac{1}{1+\alpha\Pi_R(q^2, m^2, \alpha)} \tag{2}$$

and Π_R is the proper photon self-energy on the on-shell (OS) renormalization scheme; it is defined by

$$\Pi_R(q^2=0, m^2, \alpha) = 0\,. \tag{3}$$

We are interested in the asymptotic behavior of the photon propagator and the corresponding muon anomaly. The asymptotic part of the renormalized photon propagator d_R^∞ is defined as follows: At each order of perturbation theory drop terms that vanish in the limit $-q^2/m^2 \to \infty$ while keeping divergent and constant terms. The asymptotic photon propagator d_R^∞ satisfies the Callan–Symanzik equation with the inhomogeneous term dropped [7]:

$$\left(m\frac{\partial}{\partial m} + \beta(\alpha)\alpha\frac{\partial}{\partial\alpha}\right)\alpha d_R^\infty(q^2, m^2, \alpha) = 0\,, \tag{4}$$

where

$$\beta(\alpha) \equiv Z_3^{-1} m \frac{dZ_3}{dm},$$

$$Z_3 = 1 - \alpha \Pi(\Lambda^2; 0, m^2, \alpha) . \quad (5)$$

As is shown in ref. [2], the contribution to the muon anomaly from the diagrams of fig. 1 can be expressed in terms of d_R as

$$a_\mu(M/m, \alpha) = \frac{\alpha}{\pi} \int_0^1 dx(1-x)\left[d_R\left(\frac{-x^2}{1-x}\frac{M^2}{m^2}, \alpha\right) - 1\right]. \quad (6)$$

The asymptotic muon anomaly a_μ^∞, which is obtained by dropping terms of a_μ that vanish in the limit $M/m \to \infty$, is a polynomial in $\ln(M/m)$ in any finite order in α, satisfies a Callan–Symanzik equation of the form (4), and can be expressed in terms of d_R^∞ as

$$a_\mu^\infty(M/m, \alpha) = \frac{a}{\pi} \int_0^1 dx(1-x)\left[d_R^\infty\left(\frac{-x^2}{1-x}\frac{M^2}{m^2}, \alpha\right) - 1\right]. \quad (7)$$

The error caused by this approximation is $O(m/M)$ (see ref. [2]).

The function d_R^∞ is related to the asymptotic proper photon self-energy Π_R^∞ as

$$d_R^\infty = (1 + \alpha \Pi_R^\infty)^{-1} . \quad (8)$$

Π_R^∞ can be written as a power series in α/π as

$$\alpha \Pi_R^\infty \equiv P_1 \frac{\alpha}{\pi} + P_2\left(\frac{\alpha}{\pi}\right)^2 + P_3\left(\frac{\alpha}{\pi}\right)^3 + \ldots , \quad (9)$$

where the coefficients are of the form

$$P_1 = a_1 + b_1 L , \quad P_2 = a_2 + b_2 L ,$$

$$P_3 = a_3 + b_3 L + c_3 L^2 ,$$

$$P_4 = a_4 + b_4 L + c_4 L^2 + d_4 L^3, \quad \ldots , \quad (10)$$

with

$$L \equiv \ln - \frac{q^2}{m^2} . \quad (11)$$

P_1 and P_2 receive contributions from only one electron loop and need no further indices. For $i \geqslant 3$ it is useful to express P_i as a sum of $P_i^{[1]}$, $P_i^{[2]}$, etc., where [1], [2], etc. refer to the number of closed electron loops. We shall decompose a_i, b_i, ... similarly. The first few coefficients of (10) are known (see (4.14)–(4.19) of ref. [2]):

$$a_1 = \tfrac{5}{9} , \quad b_1 = -\tfrac{1}{3} ,$$

$$a_2 = \tfrac{5}{24} - \zeta(3) , \quad b_2 = -\tfrac{1}{4} ,$$

$$b_3^{[1]} = \tfrac{1}{32} , \quad b_3^{[2]} = \tfrac{11}{24} - \tfrac{1}{3}\zeta(3) ,$$

$$c_3^{[1]} = 0 , \quad c_3^{[2]} = -\tfrac{1}{24} , \quad (12)$$

where $\zeta(n)$ is the Riemann ζ-function of argument n.

In this paper we concentrate on the diagrams of fig. 3, which have two closed electron loops, one within the other, and thus contribute to $P_3^{[2]}$, or equivalently to $a_3^{[2]}$, $b_3^{[2]}$, $c_3^{[2]}$. It is seen from (8) and (9) that their contribution to d_R^∞ is

$$(-a_3^{[2]} - b_3^{[2]} L - c_3^{[2]} L^2)\left(\frac{\alpha}{\pi}\right)^3 . \quad (13)$$

Inserting (13) in (7) we find the contribution of the diagrams of fig. 2 to the eighth-order muon anomaly to be [2]

$$a_\mu^\infty[\text{fig. } 2] = \left(\frac{\alpha}{\pi}\right)^4 \Big(-a_3^{[2]} I_0 - b_3^{[2]} I_1 - c_3^{[2]} I_2 - (2b_3^{[2]} I_0 + 4c_3^{[2]} I_1) \ln \frac{M}{m} - 4c_3^{[2]} I_0 \ln^2 \frac{M}{m}\Big) , \quad (14)$$

where

$$I_m \equiv \int_0^1 dx(1-x) \ln^m \frac{x^2}{1-x}, \quad m = 0, 1, 2, \ldots ,$$

$$I_0 = \tfrac{1}{2} , \quad I_1 = -\tfrac{5}{4} , \quad I_2 = \tfrac{13}{4} + 2\zeta(2) , \quad \text{etc.} \quad (15)$$

Since $b_3^{[2]}$ and $c_3^{[2]}$ are known, the evaluation of a_μ^∞ [fig. 2] is reduced to that of $a_3^{[2]}$.

3. Sketch of derivation of $a_3^{[2]}$ in ref. [3]

In ref. [3] the coefficient $a_3^{[2]}$ is obtained, not directly, but by a roundabout way applying the renormalization scheme independence of the invariant charge $\bar{\alpha} \equiv \alpha/(1 + \alpha \Pi)$. Specifically they compare $\bar{\alpha}$ in the MOM scheme and the on-shell scheme.

In the MOM scheme, one has

$$\Pi_{\text{MOM}}(q^2, \lambda^2, \alpha_{\text{MOM}}) = 0 \quad \text{at } q^2 = -\lambda^2 . \quad (16)$$

Thus

$$\bar{\alpha}(q^2=-\lambda^2)=\alpha_{\rm MOM}(\lambda)\,. \tag{17}$$

From the invariance of $\bar{\alpha}$ and the asymptotic limit of (17) one finds

$$\alpha_{\rm MOM}(\lambda)=\frac{\alpha_{\rm OS}}{1+\alpha_{\rm OS}\Pi_{\rm R}^{\infty}(-\lambda^2, m^2, \alpha_{\rm OS})}\,, \tag{18}$$

where the left-hand side is the value for the massless QED in the MOM scheme. $\alpha_{\rm MOM}$ and $\alpha_{\rm OS}$ are related by

$$\psi(\alpha_{\rm MOM})=\frac{\partial\alpha_{\rm MOM}}{\partial\alpha_{\rm OS}}\beta_{\rm OS}(\alpha_{\rm OS})\,, \tag{19}$$

ψ and $\beta_{\rm OS}$ being the Gell-Mann–Low and Callan–Symanzik functions, respectively.

These equations enable us to set up relations among the expansion coefficients of $\alpha\Pi_{\rm R}^{\infty}$ of (9), ψ, and $\beta_{\rm OS}$, where

$$\psi(z)=z\left[\psi_1\frac{z}{2\pi}+\psi_2\left(\frac{z}{2\pi}\right)^2+\psi_3\left(\frac{z}{2\pi}\right)^3+\ldots\right], \tag{20}$$

$$\beta_{\rm OS}(\alpha)=\beta_1\frac{\alpha}{\pi}+\beta_2\left(\frac{\alpha}{\pi}\right)^2+\beta_3\left(\frac{\alpha}{\pi}\right)^3+\ldots\,. \tag{21}$$

Substituting these expansions in (18) and (19) one finds

$$\psi_4^{[3]}=8\beta_4^{[3]}+16\beta_3^{[2]}a_1+8\beta_2a_1^2$$
$$-16\beta_1(a_3^{[2]}+a_1a_2)\,, \tag{22}$$

which was first obtained in ref. [3]. The coefficients known from previous works are [8,9]

$$\psi_1=\tfrac{2}{3}\,,\quad \psi_2=1\,,\quad \psi_3^{[1]}=-\tfrac{1}{4}\,, \tag{23}$$

$$\psi_3^{[2]}=-\tfrac{23}{9}+\tfrac{8}{3}\zeta(3),\quad \psi_4^{[3]}=8-\tfrac{16}{3}\zeta(3), \tag{23cont'd}$$

and [5,10]

$$\beta_1=\tfrac{2}{3}\,,\quad \beta_2=\tfrac{1}{2}\,,\quad \beta_3^{[1]}=-\tfrac{1}{16}\,,$$
$$\beta_3^{[2]}=-\tfrac{7}{9}\,,\quad \beta_4^{[3]}=\tfrac{35}{81}+\tfrac{4}{9}\zeta(2)\,, \tag{24}$$

where $\psi_4^{[3]}$ and $\beta_4^{[3]}$ are the parts of ψ_4 and β_4 corresponding to the diagrams shown in fig. 4.

Making use of (23) and (24) one finds from (22) that

$$a_3^{[2]}=-\tfrac{29}{27}+\tfrac{1}{3}\zeta(2)+\tfrac{19}{18}\zeta(3)\simeq 0.743\,075\ldots\,, \tag{25}$$

which unfortunately disagrees with the value

$$a_3^{[2]}\simeq -0.293\,, \tag{26}$$

obtained from (12), (14), and the numerical evaluation of the exact formula for a_μ [fig. 2] reported in ref. [4] [see (41) below].

We have carefully examined the derivation of (22) and convinced ourselves that (22) contains no error. We have noted, however, that ref. [3] makes use of $\beta_4^{[3]}$ obtained in ref. [5] and listed in (24), in which an additional term required by the finiteness of the electron mass is overlooked. We shall show in the next section that the correct $\beta_4^{[3]}$ indeed differs by a finite amount from that of ref. [5].

4. Correction to $\beta_4^{[3]}$ of ref. [5]

In order to examine how the coefficients β_i of (21) are determined, let us rewrite (4) as

$$\beta(\alpha)=\left[m\frac{\partial}{\partial m}+\beta(\alpha)\left(\alpha\frac{\partial}{\partial\alpha}-1\right)\right]\alpha\Pi_{\rm R}^{\infty}\,. \tag{27}$$

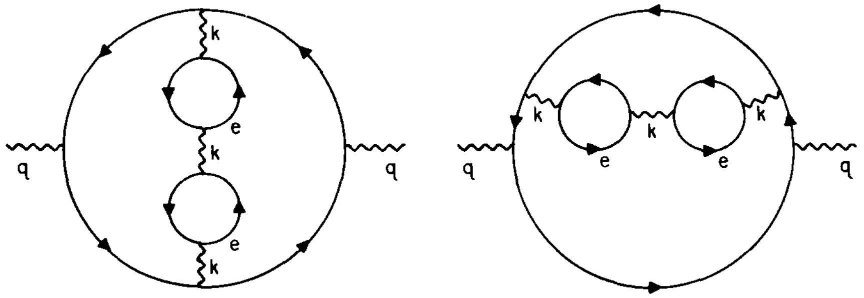

Fig. 4. Eighth-order vacuum-polarization diagrams containing two internal closed electron loops in the same photon line. Corresponding counterterm diagrams are not shown.

Substituting (9) and (21) in (27), one finds

$$\beta_1 = m\frac{\partial}{\partial m}P_1\,, \quad \beta_2 = m\frac{\partial}{\partial m}P_2\,,$$

$$\beta_3 = m\frac{\partial}{\partial m}P_3 + \beta_1 P_2\,,$$

$$\beta_4 = m\frac{\partial}{\partial m}P_4 + 2\beta_1 P_3 + \beta_2 P_2\,,$$

$$\beta_5 = m\frac{\partial}{\partial m}P_5 + 3\beta_1 P_4 + 2\beta_2 P_3 + \beta_3 P_2\,, \quad \ldots . \tag{28}$$

These equations can be decomposed further into component equations by expressing P_i and β_i as sums of terms classified according to the number of closed electron loops. (This decomposition can be mechanized by introducing N fermions [11]. We shall not bother to do so here.) In particular, $P_3^{[2]}$ is a part of P_3 represented by the diagrams of fig. 3. Similarly, $P_4^{[3]}$ is represented by the diagrams of fig. 4. With $\beta_3^{[2]}$ and $\beta_4^{[3]}$ defined correspondingly, we obtain from (28) the equations

$$\beta_3^{[2]} = m\frac{\partial}{\partial m}P_3^{[2]} + \beta_1 P_2\,,$$

$$\beta_4^{[3]} = m\frac{\partial}{\partial m}P_4^{[3]} + 2\beta_1 P_3^{[2]}\,, \quad \ldots . \tag{29}$$

It is convenient to introduce here the concept of internal electron mass m_i and external electron mass m_e. Let us call a photon self-energy subdiagram *internal* if it is not attached to photon lines carrying the external momentum q. The mass in any electron loop which is left after all internal photon self-energy parts in a vacuum-polarization diagram are shrunk down to points will be called external. All other electron masses, in the initial Feynman diagrams, will be called internal.

Let $\tilde{P}_l$ be the expression obtained from P_l by replacing all its internal photon self-energy parts by their asymptotic forms. Let

$$\Delta P_l \equiv P_l - \tilde{P}_l\,. \tag{30}$$

With the help of the Weinberg theorem it is seen that $\Delta P_3^{[2]}$ and $\Delta P_4^{[3]}$ are finite constants in the limit $-q^2 \to \infty$. Thus we obtain

$$m\frac{\partial}{\partial m}\Delta P_3^{[2]} = 0\,, \quad m\frac{\partial}{\partial m}\Delta P_4^{[3]} = 0\,. \tag{31}$$

Referring back to (10), $\Delta P_3^{[2]}$ and $\Delta P_4^{[3]}$ contribute only to $a_3^{[2]}$ and $a_4^{[3]}$, respectively.

It is important to note that the determination of β_l worked out in ref. [10] applies to $\tilde{P}_l$ but not to P_l. For instance, the equation (which corresponds to (4.17) of ref. [10])

$$m_i\frac{\partial}{\partial m_i}\tilde{P}_4^{[3]} + 2\beta_1 \tilde{P}_3^{[2]} = 0\,, \tag{32}$$

where m_i is the internal mass, holds for $\tilde{P}_l$ but not for P_l. Let us denote the β-function obtained in ref. [5] as $\tilde{\beta}_4^{[3]}$ which satisfies the relation (eq. (5.3) of ref. [10])

$$\tilde{\beta}_4^{[3]} = m_e\frac{\partial}{\partial m_e}\tilde{P}_4^{[3]}\,, \tag{33}$$

m_e being the external mass. Now, rewriting $\beta_4^{[3]}$ of (29) as

$$\beta_4^{[3]} = m_e\frac{\partial}{\partial m_e}\tilde{P}_4^{[3]} + m_i\frac{\partial}{\partial m_i}\tilde{P}_4^{[3]}$$

$$+\, 2\beta_1 \tilde{P}_3^{[2]} + 2\beta_1 \Delta P_3^{[2]} + m\frac{\partial}{\partial m}\Delta P_4^{[3]}\,, \tag{34}$$

and making use of (31), (32), and (33), we obtain

$$\beta_4^{[3]} = \tilde{\beta}_4^{[3]} + 2\beta_1 \Delta P_3^{[2]}\,. \tag{35}$$

Thus $\tilde{\beta}_4^{[3]}$ calculated in ref. [5] requires a correction term. The appearance of the $\Delta P_3^{[2]}$ term in (35) can be traced to the fact that the cancellation of $\ln(-q^2/m^2)$ between the $\partial/\partial m$ and $\partial/\partial\alpha$ terms in (27), of which (32) is an example, does not extend to the constant term: $\partial/\partial m$ drops the constant term while $\partial/\partial\alpha$ does not.

In order to understand why $\Delta P_3^{[2]} \neq 0$ for massive QED, note that, while we are interested in the large q behavior of the photon propagator, internal photon momentum k is not constrained at all by q and free to take any value. (See fig. 3 for the notation.) For $|k| \gg m$ the presence of internal photon self-energy part can be ignored and the contribution of this region to $P_3^{[2]}$ and $\tilde{P}_3^{[2]}$ are identical. On the other hand, the internal photon self-energy part has quite different effects on $P_3^{[2]}$ and $\tilde{P}_3^{[2]}$ in the $|k| = O(m)$ region. This leads to non-vanishing $\Delta P_3^{[2]}$. Note that the second region is absent in massless QED. In other words, the large q behavior and $m=0$ behavior of a photon propagator are different when internal photon self-energy parts are included.

To confirm $\Delta P_3^{[2]} \neq 0$, we have evaluated it directly by analytic means [6]:

$$\Delta P_3^{[2]} = \tfrac{23}{32} - \zeta(2) - \tfrac{7}{64}\zeta(3) . \tag{36}$$

Adding this to the result of ref. [3], we find $a_3^{[2]}$ for massive QED to be

$$a_3^{[2]} = -\tfrac{307}{864} - \tfrac{2}{3}\zeta(2) + \tfrac{545}{576}\zeta(3)$$
$$= -0.314\,583\,9... , \tag{37}$$

which agrees with (26) obtained from numerical means. From (35), (36), and the value of $\tilde{\beta}_4^{[3]}$ reported in ref. [5], we also obtain

$$\beta_4^{[3]} = \tfrac{901}{648} - \tfrac{8}{9}\zeta(2) - \tfrac{7}{48}\zeta(3)$$
$$= -0.247\,031\,481.... \tag{38}$$

5. Asymptotic contribution of diagrams of fig. 2 to a_μ

Now that we have $a_3^{[2]}$ that takes care of non-vanishing electron mass, it is easy to evaluate (14). We find

$$a_\mu^\infty[\text{fig. } 2] = \Bigg(\tfrac{1}{12} \ln^2 \frac{M}{m} + [\tfrac{1}{3}\zeta(3) - \tfrac{2}{3}] \ln \frac{M}{m}$$
$$+ \tfrac{1531}{1728} + \tfrac{5}{12}\zeta(2) - \tfrac{1025}{1152}\zeta(3) \Bigg) \left(\frac{\alpha}{\pi}\right)^4$$
$$= 1.452\,570... \left(\frac{\alpha}{\pi}\right)^4 , \tag{39}$$

where we used the value [12]

$$M/m = 206.768\,262(30) . \tag{40}$$

The value (39) is in agreement with the numerical result ((2.14) of ref. [4])

$$a_\mu[\text{fig. } 2] = 1.441\,6(18) \left(\frac{\alpha}{\pi}\right)^4 \tag{41}$$

within the uncertainty of $O(m/M)$.

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