Asymptotic photon propagator and higher-order QED Callan–Symanzik β function

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The QED Callan–Symanzik β function is calculated for a certain class of higher-order photon self-energy diagrams within the framework of perturbative QED. It is found that the formulation of de Rafael and Rosner is valid up to the sixth order but must be modified from the eighth order on. Correction terms are evaluated explicitly. The result, applied to a renormalization group calculation of an eighth-order contribution to the muon anomalous magnetic moment, gives a good agreement with the exact numerical evaluation.

Higher-order Callan–Symanzik β function is not only important for the understanding of the renormalization mechanism but also useful for concrete calculations of certain physical quantities. As a first step toward evaluation of higher-order QED β function in the on-shell scheme, the complete sixth-order term was obtained by de Rafael and Rosner some time ago [1]. Applying the technique developed in ref. [1], Calmet and de Rafael calculated a part of the eighth-order term [2], and Coquereaux extended the analysis further to a class of higher-order terms [3].

In this letter we reexamine the formulation of ref. [1] and show that the particular simplification they have made in which internal photon propagators are replaced by their asymptotic forms is valid only up to the sixth-order term. This implies that the results of refs. [2,3], which go beyond the sixth order, may need corrections. We find that their simplification indeed causes and error in the evaluation of the overall counterterms of the proper photon self-energy. We calculate the correction terms explicitly.

The renormalized photon propagator can be written in the form

$$D_{\mathrm{R}}^{\mu\nu}(q) = -\mathrm{i}\frac{g^{\mu\nu}}{q^2}\, d_{\mathrm{R}}(q^2, m^2, \alpha) + \text{the } q^\mu q^\nu \text{ term}, \quad (1)$$

where d_{R} is related to the proper photon self-energy Π_{R} by

$$d_{\mathrm{R}}(q^2, m^2, \alpha) \equiv \frac{1}{1+\Pi_{\mathrm{R}}(q^2, m^2, \alpha)}\,. \quad (2)$$

The *asymptotic* part of the renormalized propagator d_{R}^{∞} is defined by dropping terms of d_{R} that vanish in the limit $-q^2/m^2 \to \infty$. Thus, it consists only of those terms of d_{R} that are divergent or finite in this limit. This d_{R}^{∞}, defined in the on-shell renormalization scheme, satisfies the homogeneous Callan–Symanzik equation:

$$\left(m\frac{\partial}{\partial m} + \beta(\alpha)\alpha\frac{\partial}{\partial\alpha}\right)\alpha d_{\mathrm{R}}^{\infty}(q^2, m^2, \alpha) = 0\,. \quad (3)$$

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In terms of the asymptotic proper photon self-energy Π_R^∞, which is related to d_R^∞ by (2), this equation can be rewritten as

$$\left[m\frac{\partial}{\partial m}+\beta(\alpha)\left(\alpha\frac{\partial}{\partial\alpha}-1\right)\right]$$
$$\times[1+\Pi_R^\infty(q^2,m^2,\alpha)]=0\,. \qquad (4)$$

Substituting the power series expansions

$$\beta(\alpha)=\frac{\alpha}{\pi}\beta_1+\left(\frac{\alpha}{\pi}\right)^2\beta_2+\left(\frac{\alpha}{\pi}\right)^3\beta_3+\ldots,$$
$$\Pi_R^\infty=\frac{\alpha}{\pi}P_1+\left(\frac{\alpha}{\pi}\right)^2P_2+\left(\frac{\alpha}{\pi}\right)^3P_3+\ldots \qquad (5)$$

in eq. (4), we obtain the following order-by-order relations in powers of α/π:

$$\beta_n=m\frac{\partial}{\partial m}P_n+\sum_{i=1}^{n-2}(n-1-i)\beta_iP_{n-i}\,, \qquad (6)$$

where the summation is empty for $n=1$ and 2. By integrating (6) with respect to m, we obtain

$$P_1=a_1-\tfrac{1}{2}\beta_1L\,,$$
$$P_2=a_2-\tfrac{1}{2}\beta_2L\,,$$
$$P_3=a_3-(\tfrac{1}{2}\beta_3-\tfrac{1}{2}\beta_1a_2)L-\tfrac{1}{8}\beta_1\beta_2L^2\,,$$
$$P_4=a_4-(\tfrac{1}{2}\beta_4-\tfrac{1}{2}\beta_2a_2-\beta_1a_3)L$$
$$-(\tfrac{1}{8}\beta_2^2+\tfrac{1}{4}\beta_1\beta_3-\tfrac{1}{4}\beta_1^2a_2)L^2-\tfrac{1}{24}\beta_1^2\beta_2L^3\,,$$
$$\ldots, \qquad (7)$$

where the a_i's are integration constants and $L\equiv\ln(-q^2/m^2)$. If we write the P_i's in the form

$$P_1=a_1+b_1L\,,$$
$$P_2=a_2+b_2L\,,$$
$$P_3=a_3+b_3L+c_3L^2\,,$$
$$P_4=a_4+b_4L+c_4L^2+d_4L^3\,,$$
$$\ldots, \qquad (8)$$

the β_i's can be expressed in terms of the a_i's and b_i's as

$$\beta_1=-2b_1\,,$$
$$\beta_2=-2b_2\,,$$
$$\beta_3=-2b_3-2a_2b_1\,, \qquad (9)$$
$$\beta_4=-2b_4-2a_2b_2-4a_3b_1\,,$$
$$\ldots. \qquad (9\text{ cont'd})$$

It is useful to express P_i as a sum of $P_i^{[1]}, P_i^{[2]}, \ldots$, where the superscripts [1], [2], ... refer to the number of closed electron loops. We shall likewise decompose β_i, a_i, and b_i. This decomposition in fact corresponds to power series in N, where N is the number of fermion species (we set $N=1$ at the end of the calculation) [3]. In other words, [1], [2], ... correspond to terms of order $N, N^2, \ldots$. The first few of these coefficients obtained previously are as follows: [1,2]:

$$\beta_1=\tfrac{2}{3}\,,\quad \beta_2=\tfrac{1}{2}\,,$$
$$\beta_3^{[1]}=-\tfrac{1}{16}\,,\quad \beta_3^{[2]}=-\tfrac{7}{9}\,,$$
$$\tilde\beta_4^{[3]}=\tfrac{35}{81}+\tfrac{4}{9}\zeta(2)\,. \qquad (10)$$

Similarly, we have (see, for example, ref. [4] and references therein)

$$b_1=-\tfrac{1}{3}\,,\quad a_1=\tfrac{5}{9}\,,$$
$$b_2=-\tfrac{1}{4}\,,\quad a_2=\tfrac{5}{24}-\zeta(3)\,,$$
$$c_3^{[1]}=0\,,\quad c_3^{[2]}=-\tfrac{1}{2}b_1b_2=-\tfrac{1}{24}\,,$$
$$b_3^{[1]}=\tfrac{1}{32}\,,\quad b_3^{[2]}=\tfrac{11}{24}-\tfrac{1}{3}\zeta(3)\,,$$
$$\tilde a_3^{[2]}=-\tfrac{29}{27}+\tfrac{1}{3}\zeta(2)+\tfrac{19}{18}\zeta(3)\,. \qquad (11)$$

As is discussed below, $\tilde\beta_4^{[3]}$ and $\tilde a_3^{[2]}$ given above are actually incorrect. Tildes are attached to distinguish them from the correct $\beta_4^{[3]}$ and $a_3^{[2]}$.

In this paper we concentrate on the contributions to the β function which are the leading terms for large N, i.e., $\beta_n^{[k]}$ with $k=n-1$, although a similar argument holds for other cases, too. For $k=n-1$, eq. (6) reduces to

$$\beta_n^{[n-1]}=m\frac{\partial P_n^{[n-1]}}{\partial m}+(n-2)\beta_1P_{n-1}^{[n-2]}\,, \qquad (12)$$

since β_1 is of order N, whereas β_i and P_i ($i\geqslant 2$) are at most of order N^{i-1}.

Let $\tilde P_n$ be the expression obtained from P_n by replacing all its internal photon self-energy parts by their asymptotic forms, where by internal self-energy parts we mean those which are not directly attached to the external momentum q. We define ΔP_n by

$$\Delta P_n\equiv P_n-\tilde P_n\,. \qquad (13)$$

Let $\tilde{\beta}_n$ be the "β function" that is obtained by the replacement $P_n \to \tilde{P}_n$ in (12). As is readily seen $\tilde{\beta}_n$ and β_n are related by

$$\beta_n^{[n-1]} = \tilde{\beta}_n^{[n-1]} + m \frac{\partial \Delta P_n^{[n-1]}}{\partial m} + (n-2)\beta_1 \Delta P_{n-1}^{[n-2]} . \tag{14}$$

Note that it was $\tilde{\beta}_n$ and not β_n that was actually obtained in ref. [1]. The crucial message of this letter is that $\tilde{\beta}_n$ is in general different from β_n, since the external photon momentum q taking large values provides no constraint at all on the values taken by the internal photon momenta.

Our remaining task is to evaluate explicitly the correction terms in β_n due to ΔP. In fig. 1 we show the diagrams including counterterms that contribute to $P_n^{[n-1]}$. We have omitted the electron wavefunction renormalization counterterm and the vertex renormalization counterterm, since they cancel each other by the Ward identity. There are $n-2$ internal electron loops in each diagram. The corresponding diagrams for $\tilde{P}_n^{[n-1]}$ are likewise shown in fig. 2. According to the Weinberg theorem [5] the difference $\Delta P_n^{[n-1]}$ comes only from the overall counterterms shown in fig. 3. Furthermore it is a finite constant independent of m or q. Its derivative with respect to m therefore vanishes, and we have

$$\beta_n^{[n-1]} = \tilde{\beta}_n^{[n-1]} + (n-2)\beta_1 \Delta P_{n-1}^{[n-2]} . \tag{15}$$

Since P_1 and P_2 have no internal photon self-energy part, we have

$$\Delta P_1 = \Delta P_2 = 0 , \tag{16}$$

while higher order ΔP_i's are nonvanishing. Thus the

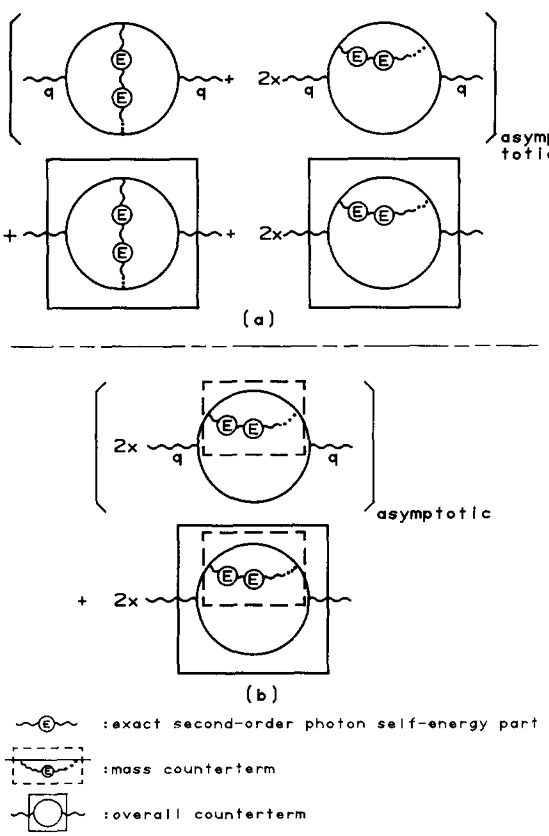

Fig. 1. Diagrams contributing to $P_n^{[n-1]}$. Diagrams in (1a) consist of asymptotic (large q) limits and overall counterterms (defined at $q=0$). Diagrams in (1b) are those containing electron mass counterterms corresponding to the diagrams of (1a).

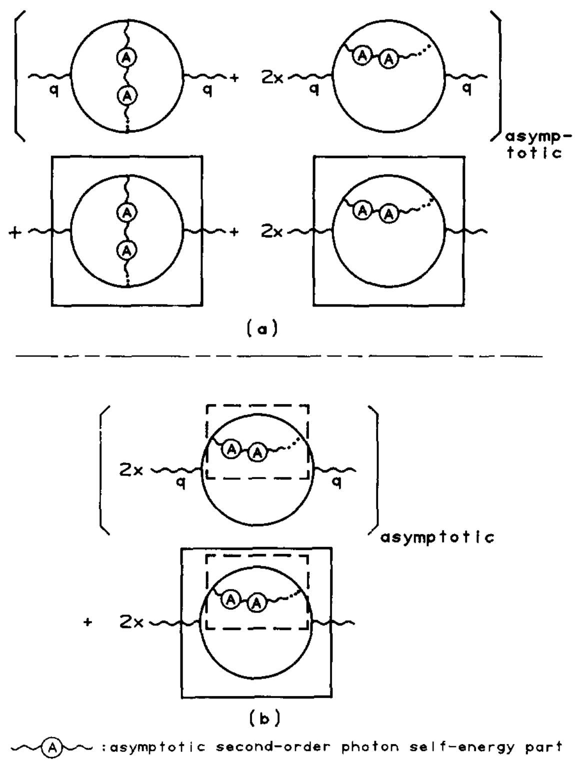

Fig. 2. Diagrams contributing to $\tilde{P}_n^{[n-1]}$, which are obtained from the diagrams of fig. 1 by replacing the exact second-order photon self-energy parts by their asymptotic forms.

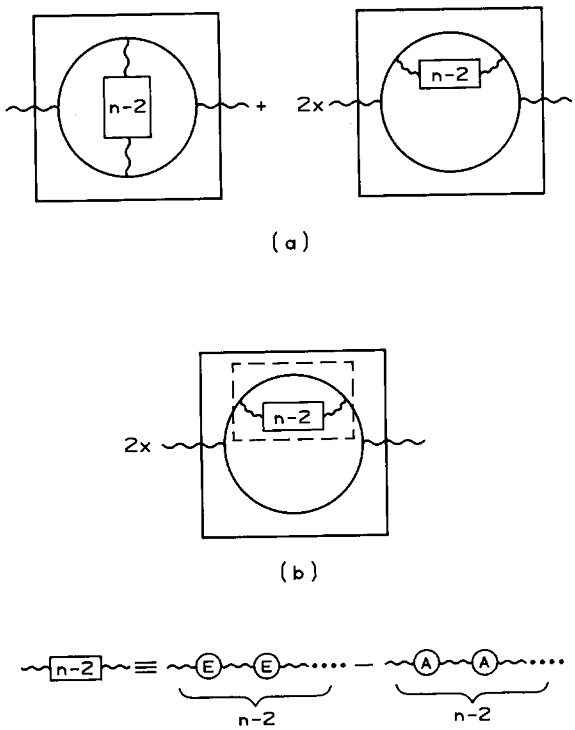

Fig. 3. Diagrams representing $\Delta P_n^{[n-1]}$.

formulation of ref. [1] gives correct results up to $n=3$, whereas correction terms are needed for $n \geqslant 4$.

We now calculate the correction term $\Delta P_n^{[n-1]}$ in (15). We first consider the "direct" contribution of fig. 3a. Following the notation of ref. [1], we write this contribution as

$$\left(\frac{\alpha}{\pi}\right)^n \Delta P_n^{[n-1]}(\text{fig. 3a}) = \int \frac{d^4k}{(2\pi)^4}(-i)(\Omega_{\rho\sigma}^{[n-2]} - \Omega_{\rho\sigma}^{\infty[n-2]}) \times \tfrac{1}{24} g_{\mu\nu} g_{\alpha\beta} \left.\frac{\partial^2 \Pi^{\mu\nu\rho\sigma}}{\partial q_\alpha\, \partial q_\beta}\right|_{q=0}, \quad (17)$$

where

$$\Omega_{\rho\sigma}^{[0]} = \Omega_{\rho\sigma}^{\infty[0]} = \frac{g_{\rho\sigma}}{k^2 + i\epsilon},$$

$$\Omega_{\rho\sigma}^{[1]} = -\left(g_{\rho\sigma} - \frac{k_\rho k_\sigma}{k^2}\right)\frac{1}{k^2+i\epsilon}\frac{\alpha}{\pi}\varpi, \quad (18)$$

$\vdots$

$$\Omega_{\rho\sigma}^{[n-2]} = (-1)^{n-2}\left(g_{\rho\sigma} - \frac{k_\rho k_\sigma}{k^2}\right)\frac{1}{k^2+i\epsilon}\left(\frac{\alpha}{\pi}\varpi\right)^{n-2}, \quad (18\text{ cont'd})$$

with

$$\varpi = a_1 + b_1\left(1 - \frac{2m^2}{-k^2}\right)\sqrt{1 + \frac{4m^2}{-k^2}} \times \ln\frac{\sqrt{1+4m^2/(-k^2)}+1}{\sqrt{1+4m^2/(-k^2)}-1} - \frac{4}{3}\frac{m^2}{-k^2}, \quad (19)$$

a_1 and b_1 being given by (11). $\Omega_{\rho\sigma}^{\infty}$ is obtained from $\Omega_{\rho\sigma}$ by replacing ϖ with ϖ^∞ where ϖ^∞ is identical with P_1 defined in (5) and can be written as

$$\varpi^\infty = P_1 = a_1 + b_1 \ln -\frac{k^2}{m^2}. \quad (20)$$

$\Pi^{\mu\nu\rho\sigma}(q, k, m^2)$ is the lowest-order light-by-light scattering amplitude. Introducing Ξ by

$$-\frac{\alpha^2}{k^2}\Xi\left(\frac{m^2}{-k^2}\right) \equiv \tfrac{1}{24} g_{\mu\nu} g_{\rho\sigma} g_{\alpha\beta} \left.\frac{\partial^2 \Pi^{\mu\nu\rho\sigma}(q,k,m^2)}{\partial q_\alpha\, \partial q_\beta}\right|_{q=0}, \quad (21)$$

we can rewrite (17) as

$$\Delta P_n^{[n-1]}(\text{fig. 3a}) = (-1)^{n-1} \times \tfrac{1}{16}\int_0^1 d\theta\, [\varpi^{n-2} - (\varpi^\infty)^{n-2}]\frac{1+\theta}{\theta(1-\theta)}\Xi(\theta), \quad (22)$$

where the integration variable θ is related to $-k^2$ by $-k^2 = [(1-\theta)^2/\theta]m^2$. In terms of θ we have [3]

$$\varpi = a_1 + b_1\frac{4\theta(1-\theta) - (1-4\theta+\theta^2)(1+\theta)\ln\theta}{(1-\theta)^3},$$

$$\varpi^\infty = a_1 + b_1 \ln\frac{(1-\theta)^2}{\theta},$$

$$\Xi(\theta) = -48\frac{\theta^2(1+\theta^2)\ln\theta}{(1+\theta)^5(1-\theta)} - 4\frac{1+10\theta^2+\theta^4}{(1+\theta)^4}. \quad (23)$$

Let us next consider the contribution to $\Delta P_n^{[n-1]}$

from the electron mass counterterm (see fig. 3b). Since the mass derivative of the second-order photon self-energy is given by

$$\frac{\partial \Pi^{(2)}}{\partial m} = -\frac{\alpha}{\pi}\int_0^1 dx \frac{4mx(1-x)}{m^2 - x(1-x)q^2}, \tag{24}$$

we have

$$\left(\frac{\alpha}{\pi}\right)^n \Delta P_n^{[n-1]}(\text{fig. 3b})$$
$$= -\Delta m^{[n-2]} \frac{\partial \Pi^{(2)}}{\partial m}\bigg|_{q^2=0} = \frac{2}{3}\frac{\alpha}{\pi}\frac{\Delta m^{[n-2]}}{m}, \tag{25}$$

where $\Delta m^{[n-2]}$ is the difference of the electron self-mass arising from the diagram of fig. 4. $\Delta m^{[n-2]}$ can be calculated as follows: Since the second-order mass counterterm with photon mass squared t is given by $\delta m_t^{(2)} = \Sigma_t^{(2)}(\not p = m)$, we have

$$\Delta m^{[n-2]} = \frac{\alpha}{\pi}\frac{m}{2}\int_0^\infty dt \int_0^1 d\theta\,(2-\theta)$$
$$\times \ln[m^2(1-\theta)^2 + \theta t]\, R(t), \tag{26}$$

where the spectral function $R(t)$ is defined by [see (18)]

$$\int_0^\infty dt \frac{R(t)}{k^2 - t} \equiv (-1)^{n-2}\frac{1}{k^2 + i\epsilon}$$
$$\times \left[\left(\frac{\alpha}{\pi}\varpi(k^2)\right)^{n-2} - \left(\frac{\alpha}{\pi}\varpi^\infty(k^2)\right)^{n-2}\right]. \tag{27}$$

By writing

$$\int_0^\infty dt \frac{\rho(t)}{k^2 - t}$$
$$\equiv (-1)^{n-2}\{[\varpi(k^2)]^{n-2} - [\varpi^\infty(k^2)]^{n-2}\}, \tag{28}$$

we can express $R(t)$ in terms of $\rho(t)$ as

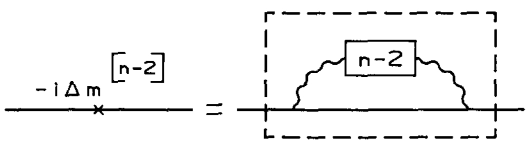

Fig. 4. Diagrams representing $\Delta m^{[n-2]}$. See fig. 3 to find the meaning of the symbols.

$$R(t) = \left(\frac{\alpha}{\pi}\right)^{n-2}\left(\frac{\rho(t)}{t} - \delta(t)\int_0^\infty \frac{dt'}{t'}\rho(t')\right). \tag{29}$$

Substituting (29) into (26), we find

$$\Delta m^{[n-2]} = \left(\frac{\alpha}{\pi}\right)^{n-1}\frac{m}{2}\int_0^1 d\theta\,(2-\theta)$$
$$\times \int_0^\infty dt \frac{\rho(t)}{t}\ln\left(1 + \frac{\theta t}{m^2(1-\theta)^2}\right). \tag{30}$$

Note that the limit at $t=0$ is unambiguous in (30), although it is not properly exhibited in (27)-(29) to avoid excessive notation.

Integrating by parts in θ and using (28) with $k^2 = -[(1-\theta)^2/\theta]m^2$, we can write $\Delta m^{[n-2]}$ as a single integral over θ. Substituting this in (25), we finally have

$$\Delta P_n^{[n-1]}(\text{fig. 3b}) = (-1)^{n-1}$$
$$\times \tfrac{1}{6}\int_0^1 d\theta\,[\varpi^{n-2} - (\varpi^\infty)^{n-2}]\frac{(1+\theta)(3-\theta)}{\theta}, \tag{31}$$

where ϖ and ϖ^∞ are given by (23).

From (22) and (31) we have calculated $\Delta P_3^{[2]}$ analytically:

$$\Delta P_3^{[2]}(\text{fig. 3a}) = \tfrac{7}{32} - \tfrac{7}{64}\zeta(3),$$
$$\Delta P_3^{[2]}(\text{fig. 3b}) = \tfrac{1}{2} - \zeta(2). \tag{32}$$

Using (15), (32), and $\tilde\beta_4^{[3]}$ given in (10), we have

$$\beta_4^{[3]} = \tfrac{901}{648} - \tfrac{8}{9}\zeta(2) - \tfrac{7}{48}\zeta(3). \tag{33}$$

We list in table 1 the values of $\beta_n^{[n-1]}$ for $n \leqslant 10$ [see (15)], which were obtained by numerically integrating $\tilde\beta_n^{[n-1]}$ (eqs. (5.6) and (5.7) of ref. [3]), (22), and (31). Note that the magnitudes of these values stay within order 1, although the $n=10$ result shows a sign of eventual growth. (Our calculation for $n > 10$ suffers from roundoff errors and is not accurate enough to show clearly the onset of exponential growth of perturbation series.) This result is in sharp contrast to the behavior of $\tilde\beta_n^{[n-1]}$ found in ref. [3] which grows much more rapidly with increasing n (see table I of ref. [3]).

In order to understand the cause of this difference,

Table 1
Numerical values for $\beta_n^{[n-1]}$.

n	$\beta_n^{[n-1]}$ (fig. 1a)	$\beta_n^{[n-1]}$ (fig. 1b)
2	−0.5	1.0
3	−0.0555...	−0.7222...
4	−0.000916...	−0.2461...
5	0.018	−0.21
6	0.038	−0.27
7	0.076	−0.44
8	0.17	−0.87
9	0.41	−2.0
10	1.1	−5.4

it is useful to examine the large n behavior of the contribution to $\tilde{\beta}_n^{[n-1]}$ from fig. 2b, which is given by [3]

$$\tilde{\beta}_n^{[n-1]}(\text{fig. 2b}) = n!\left[(-1)^n(\tfrac{2}{3})^{n+1}\,e^{5/6}+4(\tfrac{1}{3})^{n+1}\,e^{-5/3}\right] \times\{1+O(n^{-1})\}\,. \qquad (34)$$

The first and second terms within the brackets reflect the singularities of ϖ^∞ at $\theta=1$ (or $k^2=0$) and $\theta=0$ (or $k^2=\infty$), respectively [see (23)]. The first term dominates the second in magnitude and is the main cause of rapid growth of $\tilde{\beta}_n^{[n-1]}$. On the other hand, the large n formula for $\tilde{\beta}_n^{[n-1]}$(fig. 1b) has only the second term of (34) since the exact photon propagator ϖ has no singularity at $\theta=1$. This is why $\beta_n^{[n-1]}$ in table 1 has not started growing yet. Clearly, the rapid growth of $\tilde{\beta}_n^{[n-1]}$ is nothing but an artifact of the unphysical singularity of the asymptotic photon propagator ϖ^∞.

It is amusing to note that, for positive α/π, the perturbation series

$$\sum_{n=0}^{\infty}\tilde{\beta}_n^{[n-1]}\left(\frac{\alpha}{\pi}\right)^n \qquad (35)$$

is eventually dominated by the second term of (34) since the first term is Borel-summable while the second is not. Of course, the series obtained by replacing $\tilde{\beta}_n^{[n-1]}$ with $\beta_n^{[n-1]}$ in (35) is also non-Borel-summable for positive α/π.

Finally, by adding the correction terms (32) to the value $\tilde{a}_3^{[2]}$ in (11), which was recently obtained in ref. [4], we arrive at

$$a_3^{[2]}=-\tfrac{307}{864}-\tfrac{2}{3}\zeta(2)+\tfrac{545}{576}\zeta(3)\,. \qquad (36)$$

As is well known [6,7], a large class of diagrams contributing to the muon anomalous magnetic moment can be evaluated analytically by the renormalization group technique within an accuracy of order m/M, where M is the muon mass. Our result (36) enables us to extend this feature to an additional set of muon vertex diagrams. The analytic value following from (36) is found to be in excellent agreement [8] with the exact numerical evaluation [9]

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