Progress of Theoretical Physics, Vol. 88, No. 2, August 1992

On Existence of Non-Renormalizable Field Theory

—— *Pure SU(2) Lattice Gauge Theory in Five Dimensions* ——

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(Received March 23, 1992)

We have examined whether a non-renormalizable field theory can have a non-trivial fixed point. As a simple example, $SU(2)$ pure Yang-Mills theory in five dimensions is considered with a lattice regularization. Taking into account both fundamental and adjoint representations, we search for a non-trivial second-order phase transition by Monte Carlo techniques. Along the phase boundary for deconfining phase transition, energy discontinuities of hysteresis curves tend to diminish as the coupling constant for adjoint representation decreases to large negative values. However, in order to determine the order of phase transition for such values of coupling constants, more elaborate work will be necessary.

§ 1. Introduction

Renormalizability and gauge invariance of field theories have played a major role as guiding principles in model building. Their importance is most clearly demonstrated by the remarkable success of the standard model of elementary particles. The definition of renormalizability, however, is based merely on perturbation theory, and we should not necessarily assume that a non-renormalizable theory is not fundamental. While non-renormalizability of a theory implies that we have an ill-defined perturbation theory around the Gaussian fixed point, we may still have a non-trivial fixed point in the coupling constant space where we can take a sound continuum limit. The theory is then well-defined around this fixed point. This observation is important since four-dimensional quantum gravity is non-renormalizable.

A well-known example of a non-renormalizable field theory which has a non-trivial fixed point is the three-dimensional non-linear (NL) σ-model. This theory is renormalizable and asymptotically free in two dimensions. Considering an ϵ expansion[1] of the theory near two dimensions, one can deduce that there exists a non-trivial fixed point in greater than two dimensions. We can understand why the three-dimensional NL σ-model has a non-trivial fixed point, since the NL σ-model belongs to the same universality class as the ϕ^4 model in dimensions $2<d<4$ and the ϕ^4 theory is (super-)renormalizable for $d\leq 4$. We can therefore take ϵ even close to 2 as far as the qualitative features of the theory are concerned.

The case for gravity is similar to the above example in the sense that two-dimensional gravity is renormalizable and asymptotically free. Recently, the ϵ-expansion of the theory has been extensively studied.[2] In the limit $\epsilon=2$, the existence of a non-trivial fixed point is yet to be established. More comprehensive numerical searches are needed.[3]

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A simpler case may be the Yang-Mills theory. This theory in four dimensions is renormalizable and asymptotically free. An ϵ-expansion of the theory around $d=4$ was studied some time ago[4)] and the existence of a non-trivial fixed point has been inferred for $d>4$. In the present work we study an $SU(2)$ pure Yang-Mills theory in five dimensions on the lattice. Using Monte Carlo techniques, we search for a non-trivial fixed point as a second-order phase transition point.

The paper is organized as follows. In § 2 we present the results of our Monte Carlo simulation for $d=5$ $SU(2)$ Yang-Mills theory with the fundamental representation. In § 3 we take into account both fundamental and adjoint representations, and present a detailed analysis of the deconfining phase transition. A discussion is given in § 4.

§ 2. $SU(2)$ Yang-Mills theory in five dimensions with fundamental representation

An $SU(N)$ pure Yang-Mills theory in four dimensions is renormalizable and asymptotically free. This theory is non-renormalizable in $d>4$, but the ϵ-expansion in $d=4+\epsilon$ gives the following β function:

$$\beta(g)=\frac{\epsilon}{2}g-\frac{11N}{48\pi^2}g^3+\cdots, \tag{2·1}$$

which, in turn, suggests the existence of a non-trivial fixed point at

$$g^2=g_c^2\equiv\frac{24\pi^2}{11N}\epsilon. \tag{2·2}$$

As is illustrated in Fig. 1, the theory in $d=4+\epsilon$ is in confining phase for $g>g_c$ and in deconfining phase for $g<g_c$. If ϵ is sufficiently small, the phase transition is expected to be of second order, and we have a well-defined theory at $g=g_c$. This idea was pursued by Peskin and the critical exponents were calculated within the framework of the ϵ-expansion.[4)] The question is then whether a well-defined Yang-Mills theory is indeed realized in five or higher dimensions. In the present work we have performed Monte Carlo simulations on the lattice and searched for a second-order deconfining phase transition in $SU(2)$ pure Yang-Mills theory in $d=5$ (i.e., $\epsilon=1$).

The action is given by

$$S=\sum_{\square}\beta_F S_{\square}^F, \tag{2·3}$$

$$S_{\square}^F=1-\frac{1}{2}\mathrm{Tr}\,U_F, \tag{2·4}$$

where U_F is an ordered product of the four group elements in fundamental

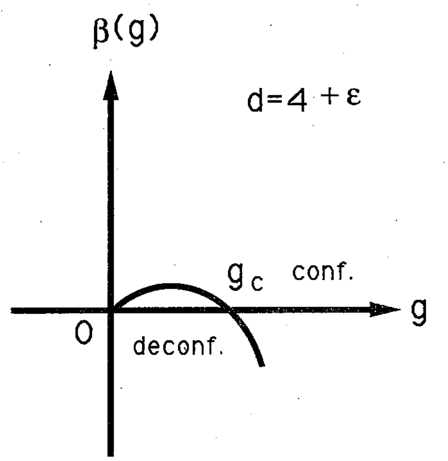
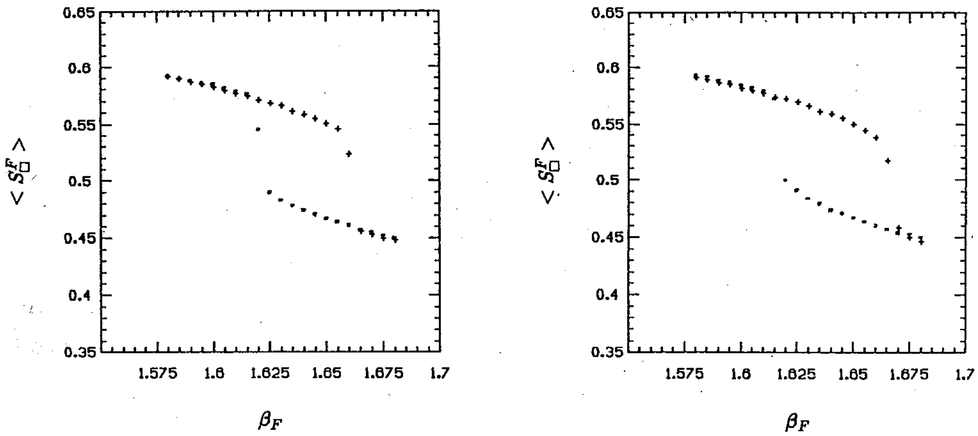


Fig. 1. The β function for $SU(N)$ Yang-Mills theory in $d=4+\epsilon$.



(a) (b)

Fig. 2. Thermal cycles of $\langle S_\square{}^F \rangle$ with lattice sizes 6^5 (a) and 8^5 (b).

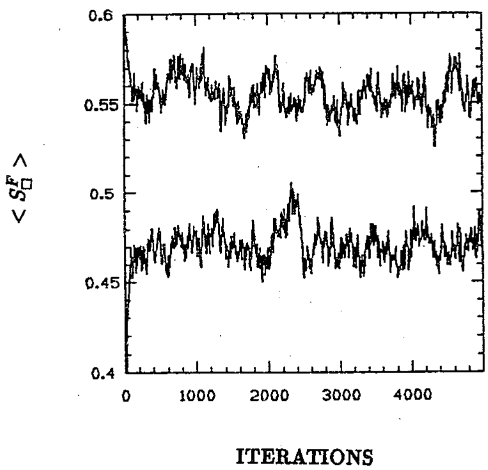


Fig. 3. Time history of $\langle S_\square{}^F \rangle$ at $\beta_F=1.64$ with lattice size 4^5. The upper curve is the evolution of a random configuration, and the lower is that of an ordered configuration.

representation around a plaquette $\square$, and $\sum_\square$ indicates the summations over plaquettes. Throughout the present work, periodic boundary conditions are imposed. The coupling constant β_F and the ordinary coupling constant g are related by

$$\frac{4}{g^2}=\beta_F\,. \qquad (2\cdot5)$$

As an order parameter, $\langle S_\square{}^F \rangle$ is calculated. The heat bath algorithm has been adopted here. One Monte Carlo iteration consists of applying this algorithm five times to each link. In Figs. 2(a) and (b) we show respectively a thermal cycle in β_F for lattice sizes 6^5 and 8^5. Each point represents an average over 150 iterations. They both exhibit clear hysteresis curves around

$$\beta_F=1.64\,, \qquad (2\cdot6)$$

and the transition appears of first order. To confirm this, we made long runs of 5000 iterations on a 4^5 lattice with both random and ordered initial configurations at $\beta_F=1.64$. The time history is displayed in Fig. 3. The two states are separated throughout the runs, and the transition is indeed of first order. All these results confirm Creutz' simulation with 40 iterations on a 4^5 lattice.[5)]

§ 3. Addition of adjoint representation

3.1. *Phase diagram*

In the previous section we examined the phase transition of $SU(2)$ Yang-Mills theory with fundamental representation in five dimensions. The transition, however, turned out to be of first order and we cannot take a well-defined continuum limit there. We now add the adjoint representation to the theory and search for a second-order doconfining phase transition in the two-parameter space (β_F, β_A) where β_A is the coupling constant for adjoint representation. The action is given by

$$S=\sum_{\Box}(\beta_F S_{\Box}{}^F+\beta_A S_{\Box}{}^A)\,, \tag{3·1}$$

where $S_{\Box}{}^F$ is taken from Eq. (2.4) and $S_{\Box}{}^A$ is defined by

$$S_{\Box}{}^A=1-\frac{1}{3}\mathrm{Tr}\,U_A\,. \tag{3·2}$$

Here, U_A is an ordered product of four link variables in adjoint representation around a plaquette $\Box$. The ordinary coupling constant g is now written in terms of β_F and β_A as*)

$$\frac{4}{g^2}=\beta_F+\frac{8}{3}\beta_A\,. \tag{3·3}$$

For numerical work it is convenient to rewrite the trace of adjoint representation as

$$\mathrm{Tr}\,U_A=|\mathrm{Tr}\,U_F|^2-1\,. \tag{3·4}$$

The Metropolis algorithm has been employed here. In Fig. 4 we show a phase diagram of the theory in β_F-β_A space which was obtained from simulations on a 4^5 lattice. The phase space is divided into three regions, I, II and III. One of the distinct features of this diagram compared with the one for $d=4$ case[6] is that the boundary curve between regions II and III extends without any sign of termination, while the corresponding curve for $d=4$ ends at $(\beta_F, \beta_A)=(1.48, 0.9)$ before reaching the β_F axis. This difference presumably arises from the fact that there is only confining phase in $d=4$ but there are confining and deconfining phases for $d>4$, according to the ϵ-expansion (see Fig. 1). Hence, the

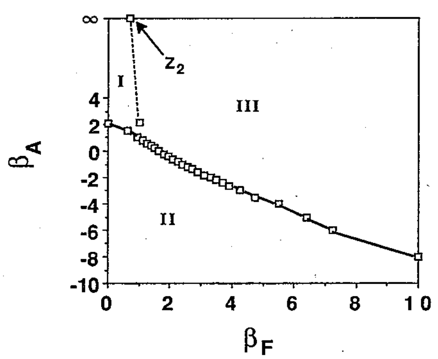


Fig. 4. Phase diagram of $SU(2)$ Yang-Mills theory in five dimensions.

*) When $\beta_A<-(3/8)\beta_F$, g^2 is negative, indicating that the positivity is violated at the classical level. Therefore, one has to check the positivity in the continuum limit, if a second-order phase transition is found around this region.

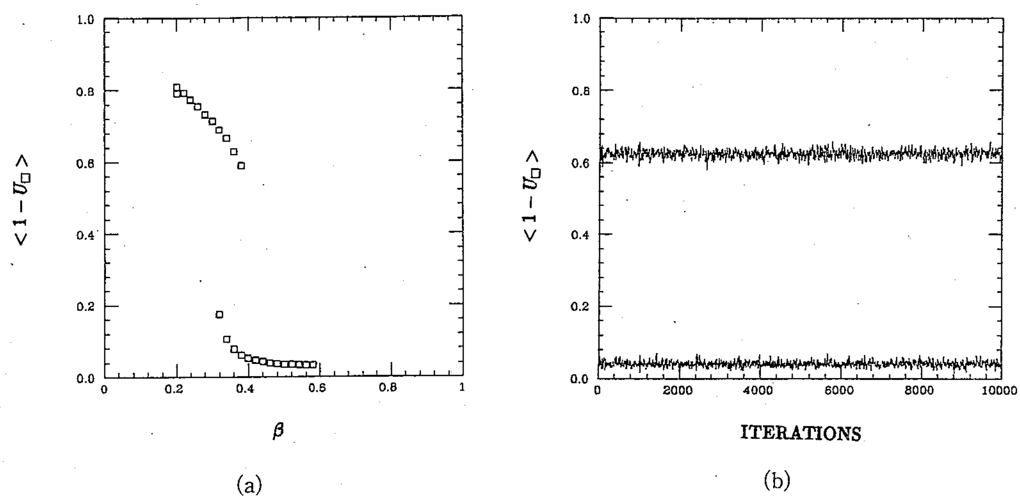


Fig. 5. Thermal cycle (a) and time history (b) of $\langle 1-U_\Box \rangle$ for Z_2 gauge theory in five dimensions. In (a) each point represents an average over 100 iterations. In (b) the upper curve is the evolution of a random configuration, and the lower is that of an ordered configuration.

boundary between regions II and III in Fig. 4 marks the deconfining phase transition. This transition is investigated further in the next subsection. We also remark that at $\beta_F=0$, there is a clear first-order phase transition around $\beta_A=2.1$ (see Fig. 6(a) below) in agreement with the results for $d=5$ $SO(3)$ Yang-Mills theory.[7]

The phase boundaries between regions I and II and between regions II and III are clearly observed. However, the boundary between I and III turned out not as clear due to slow thermalization in the region with large positive β_A. An exception is the case where we take the limit $\beta_A \to \infty$, in which the theory approaches Z_2 gauge theory. The action for Z_2 gauge theory is given by

$$S=\sum_{\Box}\beta(1-U_\Box)\,, \tag{3·5}$$

where $U_\Box$ is an ordered product of the four link variables with value $+1$ or -1 around a plaquette $\Box$. Our simulation of this theory with Metropolis algorithm on a 4^5 lattice has exhibited a clear first-order phase transition around

$$\beta=0.35\,. \tag{3·6}$$

In Figs. 5(a) and (b) we show a thermal cycle of $\langle 1-U_\Box \rangle$ in β and a time history of 10000 iterations at $\beta=0.35$, respectively. Hence, in Fig. 4 we have the Z_2 limit for $\beta_A \to \infty$ at

$$\beta_F=2\beta=0.7\,. \tag{3·7}$$

By analogy with the results for $d=4$,[6] we expect that the point where the three boundary curves meet lies near $(\beta_F, \beta_A)=(1.0, 1.0)$. We did not find the exact location of this point. However, phase transitions are clearly of first order in this region with $\beta_A>0$. As is discussed in the next subsection, our search of second-order phase transition was therefore focused on the area with large negative values of β_A along the

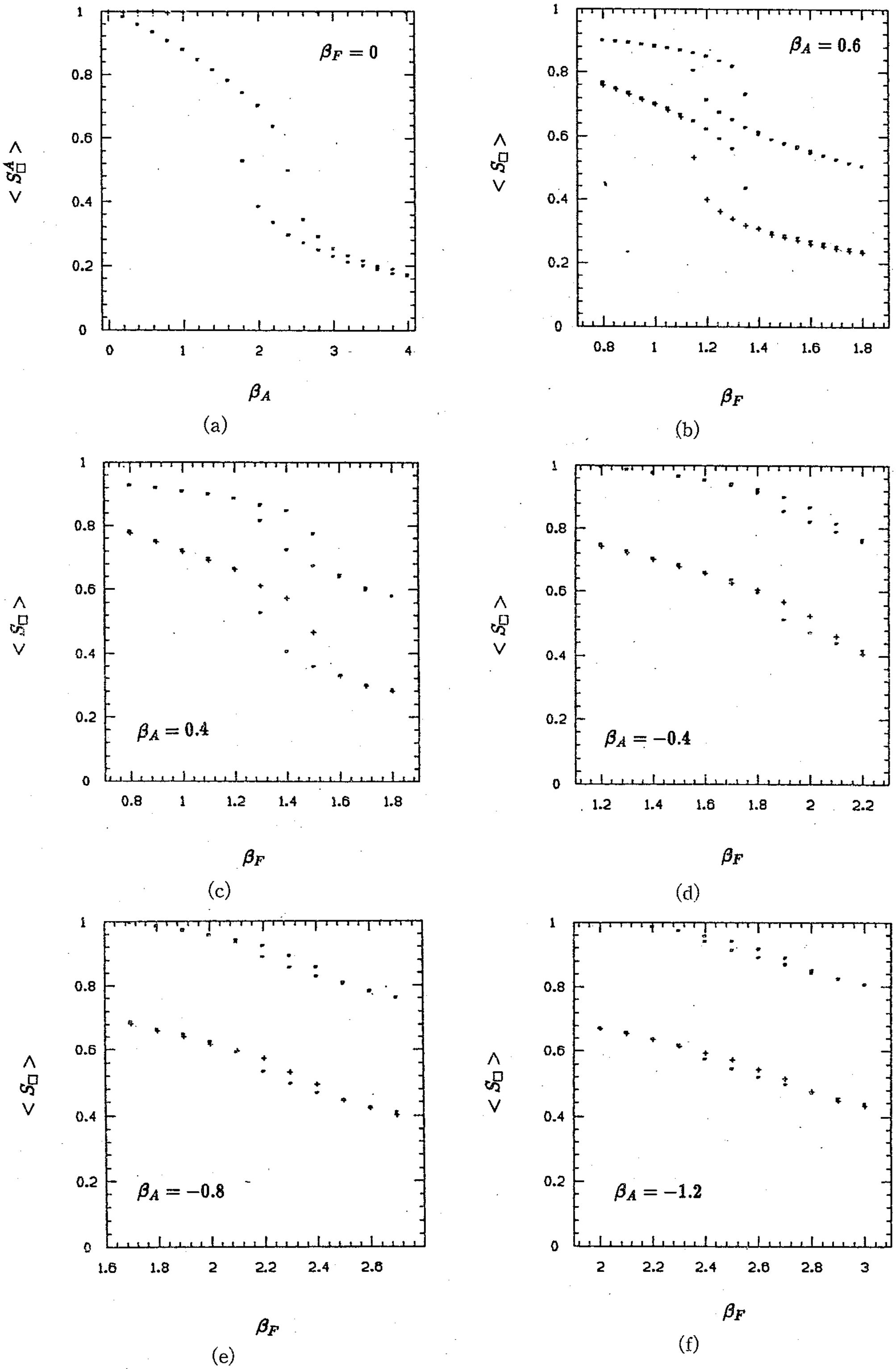


Fig. 6. Thermal cycles in β_A (a) and β_F (the rest) on a 4^5 lattice. The upper hysteresis curves represent $\langle S_\square{}^A \rangle$ and the lower $\langle S_\square{}^F \rangle$. In (a) only $\langle S_\square{}^A \rangle$ is plotted.

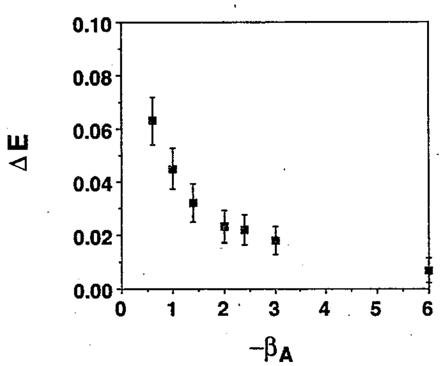


Fig. 7. Discontinuity ΔE of $\langle \beta_F S_\Box{}^F + \beta_A S_\Box{}^A \rangle$ at phase transition points as a function of $-\beta_A$.

deconfining-phase boundary curve between regions II and III.

3.2. *Deconfining phase transition*

In this subsection we give the details of our search of a second-order phase transition along the phase boundary between regions II and III in Fig. 4. In Fig. 6 we show thermal cycles in β_A and β_F on a 4^5 lattice. Each point represents an average over 150 iterations. Conspicuous hysteresis curves mark this phase boundary and indicate that the phase transition is of first order. As we can see from Fig. 6, however, the discontinuities at phase transition points tend to decrease as β_A is decreased. This is more explicitly shown in Fig. 7 where we plot discontinuities ΔE of $\langle \beta_F S_\Box{}^F + \beta_A S_\Box{}^A \rangle$ at phase transition points as a function of $-\beta_A$. Each point is an average over 3000 iterations on a 4^5 lattice. It should be noted that we may have a second-order phase transition for large negative values of β_A along this phase boundary of first-order phase transition.

If the phase transition between regions II and III in Fig. 4 corresponds to the one which is suggested by the ϵ-expansion, it has to be a deconfining phase transition. Around phase transition points we thus calculated Creutz ratios

$$\chi(I, J) = -\ln\left(\frac{W(I, J)\,W(I-1, J-1)}{W(I, J-1)\,W(I-1, J)}\right), \tag{3·8}$$

where $W(I, J)$ stands for an average of I by J Wilson loop. We also evaluated Polyakov lines

$$\Omega = \frac{1}{5}\sum_{\mu=1}^{5} \Omega_\mu , \tag{3·9}$$

where

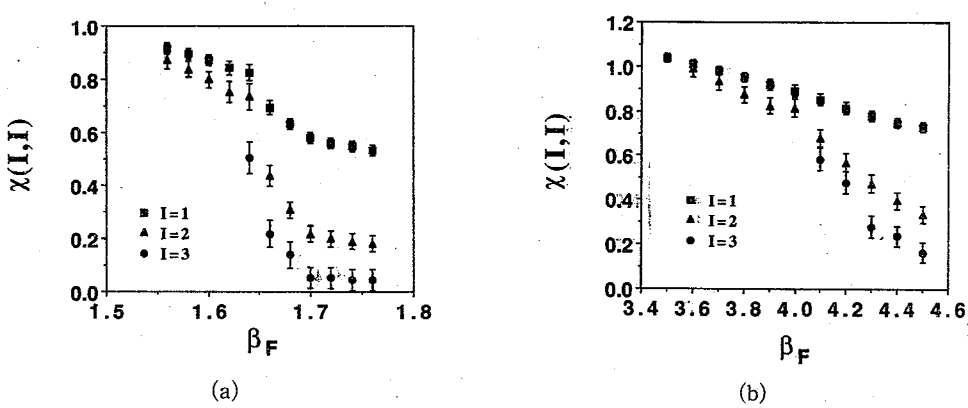


Fig. 8. Creutz ratio $\chi(I, I)$ as a function of β_F at $\beta_A=0$ (a) and -3.0 (b).

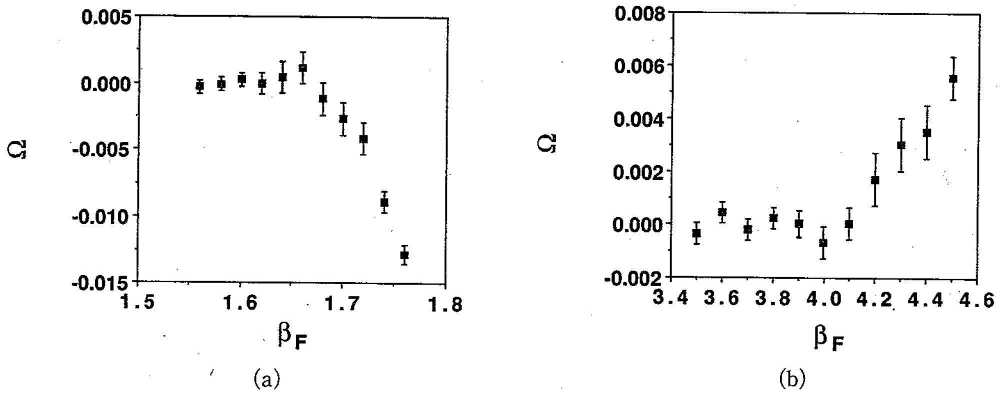


Fig. 9. Polyakov line Ω as a function of β_F at $\beta_A=0$ (a) and -3.0 (b).

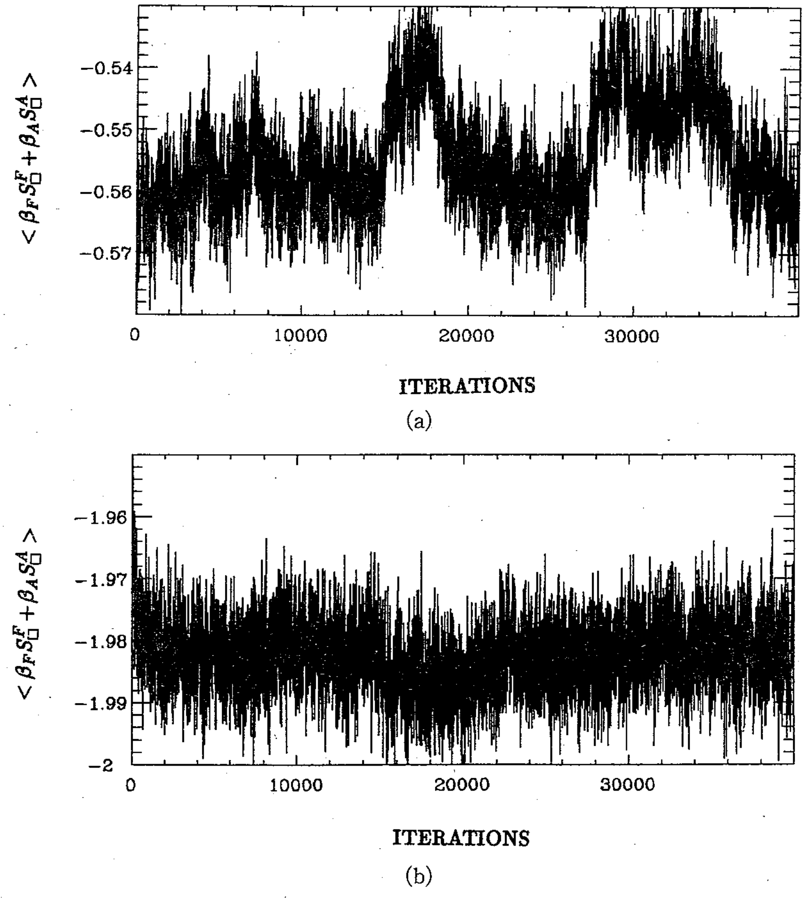


Fig. 10. Time history of $\langle \beta_F S_\square{}^F + \beta_A S_\square{}^A \rangle$ at $(\beta_F, \beta_A)=(4.25, -3.0)$ (a) and $(7.25, -6.0)$ (b).

$$\Omega_\mu = \langle \mathrm{Tr}(\prod_{n_\mu=1}^{L} U_{(n_1,\cdots n_5),\mu}) \rangle \,. \tag{3·10}$$

Here, L is the lattice size in one direction. The results on a 6^5 lattice around two points on the boundary curve, $(\beta_F, \beta_A)=(1.64, 0)$ and $(4.25, -3.0)$, are displayed in

Figs. 8 and 9. Each point represents an average over 1000 iterations. The figures clearly indicate that the transition is indeed a deconfining one. In fact, Drouffe has suggested in a mean field approximation that the order of this deconfining transition turns from first order to second order as β_A decreases.[8] Hence, we made long runs of 40000 iterations on a 4^5 lattice at $(\beta_F, \beta_A)=(4.25, -3.0)$ and $(7.25, -6.0)$ where we observe very small hysteresis effects (see Fig. 7). The results are shown in Fig. 10. The results for $(\beta_F, \beta_A)=(4.25, -3.0)$ show a flip-flop behavior, suggesting that the transition is still of first order. On the other hand, the results for $(\beta_F, \beta_A)=(7.25, -6.0)$ do not show any clear flip-flop. We have applied the finite-size scaling[9] at the transition points $(\beta_F, \beta_A)=(7.25, -6.0)$ and $(10.0, -8.0)$ to see whether the specific heat scales with lattice size as in a second-order phase transition. However, we could not find any clear sign of divergence in specific heat. This is because not only the latent heat but also the energy fluctuation is vanishingly small in this region. More elaborate work with more iterations and larger lattice is necessary in order to determine conclusively whether or not the order of the deconfining phase transition changes from first order to second order.

§ 4. Discussion

In the present work we have searched for a second-order phase transition in a non-renormalizable field theory. Unlike the NL σ-model in three dimensions which has a corresponding (super-)renormalizable "partner" (ϕ^4 model) in the same universality class, Yang-Mills theory in five dimensions is a simple but non-trivial example without any apparent renormalizable counterpart in the same universality class. Even though it turned out difficult to find a second-order phase transition in five-dimensional $SU(2)$ Yang-Mills theory, it is certainly worthwhile to explore other possibilities, such as introducing another coupling, considering an $SU(N)$ model with larger N, and so forth.

Acknowledgements

We are grateful to M. Fukugita, M. Okawa and A. Ukawa for useful discussions. We are particularly indebted to M. Okawa for his advice on numerical work and for providing us with a copy of his $SU(2)$ heat bath program. We thank the National Laboratory for High Energy Physics for computation time on HITAC S820/80. This work was supported in part by a Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture.

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