Functorial approaches to Auslander-Reiten theory and Singularity categories

（アウスランダー・ライテン理論及び特異圏に対する関手的手法）

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1 Introduction

Mitchell indicated that most of homological ring theory generalizes to additive category theory [Mit]. In this thesis, we are interested in such functorial generalizations. Especially, a functorial generalization for Auslander algebras contributed to the discovery of Auslander-Reiten theory, which is one of the most important tool in the representation theory of algebra. Auslander showed that the Yoneda embedding $\mathbb{Y} : \mathsf{mod}\,\Lambda \hookrightarrow \mathsf{mod}(\mathsf{mod}\,\Lambda)$ of the module category into the functor category admits an exact left adjoint functor. Hence it induces an equivalence

$$\frac{\mathsf{mod}(\mathsf{mod}\,\Lambda)}{\{F \mid F(\Lambda) = 0\}} \xrightarrow{\sim} \mathsf{mod}\,\Lambda,$$

sometimes called *Auslander's formula* [Aus]. This formula enables us to understand many concepts in the module categories in terms of the functor categories. The notion of *almost split sequence* plays an essential role in the theory, which is an exact sequence in $\mathsf{mod}\,\Lambda$ satisfying some properties. The Yoneda functor sends almost split sequences to the minimal projective resolution of simple functors in $\mathsf{mod}(\mathsf{mod}\,\Lambda)$. Moreover, there exists a one-to-one correspondence between the class of almost split sequences and the class of simple functors of projective dimension two. In this thesis, we report the author's recent results, which generalize classical results for rings and provide some applications.

Part I: The notion of *recollements* is a fundamental tool to study abelian categories and triangulated categories, introduced in [BBD]. It is a pair of full subcategories with some orthogonality conditions, and hence deconstructs a given category to its full subcategories. We should mention that the above Auslander's formula is an ingredient of a recollement of $\mathsf{mod}(\mathsf{mod}\,\Lambda)$. A typical example of recollements is obtained from a finite dimensional k-algebra Λ and its idempotent e, that is, $\mathsf{mod}\,\Lambda$ is decomposed into the pair $(\mathsf{mod}\,\Lambda/\Lambda e\Lambda, \mathsf{mod}\,e\Lambda e)$. We generalize the above recollement by replacing the pair with $(\mathsf{mod}\,\mathcal{A}/[\mathcal{B}], \mathsf{mod}\,\mathcal{B})$ according to a sequence $\mathcal{B} \subseteq \mathcal{A}$ of dualizing k-varieties. As applications, first we mention that Auslander-Bridger sequence is obtained from our recollement. Second application is a contribution to higher Auslander-Reiten theory [Iya07a, Iya07b]. Applying our recollements to n-cluster tilting subcategories of $\mathsf{mod}\,\Lambda$, we get higher version of the classical Auslander's formula.

Part II: Buchweitz introduced the *singularity category* of Iwanaga-Gorenstein rings. He proved that the singularity category is triangle equivalent to the stable category of Cohen-Macaulay modules. We say that two additive categories are *singularly equivalent* if their singularity categories are triangle equivalent. Xiao-Wu Chen provided a sufficient condition for a Noetherian ring Λ and its idempotent subalgebra $e\Lambda e$ so that they are singularly equivalent. Our first result is a functor category version of Chen's theorem. As an application of the result, we consturuct many examples of singularly equivalent categories, where the key is the notion of *Auslander-Buchweitz approximation* [ABu, Has]. For example, our application provides a singular equivalence arising from a cotilting module T in $\mathsf{mod}\,\Lambda$, precisely, $({}^{\perp}T)/[T]$ and $(\mathsf{mod}\,\Lambda)/[T]$ are singularly equivalent. In particular, the canonical module ω over a commutative Noetherian ring R induces a singular equivalence between $(\mathsf{CM}R)/[\omega]$ and $(\mathsf{mod}\,R)/[\omega]$, which generalizes Matsui-Takahashi's equivalence.

Auslander's formulas via Recollements for dualizing k-varieties

The notion of recollements was introduced in [BBD] to get information about quasi-coherent sheaves over a topological space, first in the level of triangulated categories. Recollements of abelian categories also appeared in loc. cit. in the context of gluing t-structures. Afterwards, in the representation theory, Happel provided an application of recollements to the tilting theory [Hap92, Hap93], and then there are various approaches to tilting theory which stem from Happel's work [AKL, Psa]. Moreover, Cline, Parshall and Scott initiated the use of recollements of derived categories to reveal close relation between highest weight categories and quasi-hereditary algebras [CPS].

A recollement of abelian categories is a sequence of abelian categories

$$\mathcal{B} \xrightarrow{\;e\;} \mathcal{A} \xrightarrow{\;q\;} \mathcal{C},$$

where the functor e represents $\mathcal{B}$ as a Serre subcategory of $\mathcal{A}$ and the functor q represents $\mathcal{C}$ as the quotient $\mathcal{A}/\mathcal{B}$, satisfying the additional properties: e admits both a right adjoint e_ρ and a left adjoint e_λ; q admits both a right adjoint q_ρ and a left adjoint q_λ. Throughout the thesis, such a recollement is denoted by a diagram of the form below (see Definition 3.7 for details)

$$\xymatrix{\mathcal{B} \ar[r]|{e} & \mathcal{A} \ar@/_1pc/[l]_{e_\lambda} \ar@/^1pc/[l]^{e_\rho} \ar[r]|{q} & \mathcal{C} \ar@/_1pc/[l]_{q_\lambda} \ar@/^1pc/[l]^{q_\rho}},$$

or $(\mathcal{B}, \mathcal{A}, \mathcal{C})$, for short. It is a benefit of recollements to deconstruct the middle category $\mathcal{A}$ into smaller ones $\mathcal{B}$ and $\mathcal{C}$ which inherit some homological properties of $\mathcal{A}$. One of the most studied example comes from an associative ring (with unit) Λ together with its idempotent e. They induce a recollement

$$\xymatrix{\mathsf{Mod}\,\Lambda/\Lambda e\Lambda \ar[r] & \mathsf{Mod}\,\Lambda \ar@/_1pc/[l] \ar@/^1pc/[l] \ar[r] & \mathsf{Mod}\,e\Lambda e \ar@/_1pc/[l] \ar@/^1pc/[l]}, \tag{1.0.1}$$

where $\mathsf{Mod}\,\Lambda$ denotes the category of right Λ-modules. If Λ is Noetherian, it restricts to a recollement consisting of the categories of finitely presented modules. It is called an *idempotent recollement.* In fact, idempotent recollements appeared in many contexts in the representation theory, e.g. [CS, Eir, FP, Kra17].

Our first aim is to extend idempotent recollements to functor categories over dualizing k-varieties. A *dualizing k-variety* is an analog of the category of finitely generated projective modules over a finite dimensional algebra, but with possibly infinitely many indecomposable objects up to isomorphism [AR74]. It is a Krull-Schmidt Hom-finite k-linear category $\mathcal{A}$ where the standard k-duality $D := \mathrm{Hom}_k(-, k)$ induces the duality between $\mathsf{mod}\,\mathcal{A}$ and $\mathsf{mod}(\mathcal{A}^{\mathsf{op}})$. A typical example of dualizing k-variety is the module category $\mathsf{mod}\,\Lambda$ of a finite dimensional k-algebra Λ.

Theorem 1.1 (Theorem 4.5)**.** *Let $(\mathcal{A}, \mathcal{B})$ be the pair of a dualizing k-variety $\mathcal{A}$ and its functorially finite subcategory $\mathcal{B}$. Then the canonical inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ induces the following recollement*

$$\xymatrix{\mathsf{mod}(\mathcal{A}/[\mathcal{B}]) \ar[r] & \mathsf{mod}\,\mathcal{A} \ar@/_1pc/[l] \ar@/^1pc/[l] \ar[r] & \mathsf{mod}\,\mathcal{B}. \ar@/_1pc/[l] \ar@/^1pc/[l]}$$

Our second result shows that in the functor category of a suitable dualizing k-variety, Auslander-Bridger sequences are nothing other than right-defining exact sequences of recollements (Theorem 5.2).

Our third aim is to approach to higher Auslander-Reiten theory from a viewpoint of functor categories. The theory can be regarded as Auslander-Reiten theory over n-cluster tilting subcategories $\mathcal{B}$ of $\mathsf{mod}\,\mathcal{A}$, where $\mathcal{A}$ is a dualizing k-variety. Using our recollement, we show that there exists a higher analog of Auslander's formula.

Theorem 1.2 (Corollary 4.11)**.** *Suppose $\mathcal{B}$ is an n-cluster tilting subcategory of $\mathsf{mod}\,\Lambda$. Then the composition functor $\mathsf{mod}\,\Lambda \to \mathsf{mod}(\mathsf{mod}\,\Lambda) \to \mathsf{mod}\,\mathcal{B}$ admits an exact right adjoint functor. Here the the first one is the Yoneda embedding and the second one is the natural restriction. Moreover it induces an equivalence*

$$\frac{\mathsf{mod}\,\mathcal{B}}{\mathsf{mod}\,\underline{\mathcal{B}}} \xrightarrow{\sim} \mathsf{mod}\,\Lambda,$$

where the fraction denotes the Serre quotient.

Our recollement enables us to construct an equivalence $\sigma_n : \underline{\mathcal{B}} \xrightarrow{\sim} \overline{\mathcal{B}}$ and bifunctorial isomorphisms

$$\underline{\mathcal{B}}(\sigma_n^- Y, X) \cong D\,\mathrm{Ext}^n_{\mathcal{A}}(X, Y) \cong \overline{\mathcal{B}}(Y, \sigma_n X). \tag{1.2.1}$$

In particular, σ_n coincides with the n-Auslander-Reiten translation τ_n and (1.2.1) gives an n-Auslander-Reiten duality (Theorem 6.12). Moreover, we construct one-to-one correspondence between the isomorphism class of n-almost split sequences and the isomorphism class of simple $\mathcal{B}$-modules of projective dimension $n+1$ (Corollary 6.14).

We should remark that a similar approach to higher Auslander-Reiten theory was given by Jasso and Kvamme [JK] independently, but our approach is slightly different since we do not use an explicit form of τ_n.

Singular equivalences via Auslander-Buchweitz approximations

Let $\mathcal{A}$ be an additive category with weak-kernels. Then the functor category $\mathsf{mod}\,\mathcal{A}$, the category of finitely presented contravariant functors from $\mathcal{A}$ to the category of abelian groups, is abelian. The notion of *singularity category of* $\mathcal{A}$ is defined to be the Verdier quotient

$$\mathsf{D}_{\mathsf{sg}}(\mathcal{A}) := \frac{\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{A})}{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\mathcal{A})},$$

where we denote by $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{A})$ the bounded derived category, and by $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\mathcal{A})$ the homotopy category of bounded complexes whose terms are projective. This concept was introduced as a homological invariant of rings by Buchweitz [Buc86]. Recently it was applied by Orlov to study Landau-Ginzburg models [Orl04]. A lot of studies on singularity categories has been done in various approaches (e.g. [Iya18, KV, Orl09, Ric, Zim]).

For additive categories $\mathcal{A}$ and $\mathcal{A}'$ with weak-kernels, we say that $\mathcal{A}$ is *singularly equivalent* to $\mathcal{A}'$ if there exists a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\mathcal{A}) \simeq \mathsf{D}_{\mathsf{sg}}(\mathcal{A}')$ [ZZ]. If Λ is an Iwanaga-Gorenstein ring, then the singularity category of Λ is triangle equivalent to the stable category of Cohen-Macaulay Λ-modules. Thus the singular equivalence is a generalization of the stable equivalence for Iwanaga-Gorenstein rings.

It is a basic problem to compare homological propeties of a ring Λ with its subalgebra $e\Lambda e$ given by an idempotent $e \in \Lambda$ (e.g. [APT, CPS, DR]). In this context, Xiao-Wu Chen investigated a sufficient condition for a ring Λ and its idempotent subalgebra $e\Lambda e$ so that they induce a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\Lambda) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(e\Lambda e)$ [Che, Thm. 1.3]. The first aim of this article is to provide its functor category version by using the following observations on Serre and Verdier quotients: Let $\mathcal{X}$ be a contravariantly finite subcategory of an additive category $\mathcal{A}$ with weak-kernels. Then $\mathcal{X}$ also admits weak-kernels, hence the canonical functor $Q : \mathsf{mod}\,\mathcal{A} \to \mathsf{mod}\,\mathcal{X}$ induces an equivalence

$$\frac{\mathsf{mod}\,\mathcal{A}}{\mathsf{mod}(\mathcal{A}/[\mathcal{X}])} \xrightarrow{\sim} \mathsf{mod}\,\mathcal{X}, \tag{1.2.2}$$

where the fraction denotes the Serre quotient (e.g. [Buc97, Prop. 3.9]). Moreover, the equivalence (1.2.2) induces a triangle equivalence

$$\frac{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\mathcal{A})}{\mathsf{D}^{\mathrm{b}}_{\mathcal{A}/[\mathcal{X}]}(\mathsf{mod}\,\mathcal{A})} \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\mathcal{X}), \tag{1.2.3}$$

where $\mathsf{D}^{\mathrm{b}}_{\mathcal{A}/[\mathcal{X}]}(\mathsf{mod}\,\mathcal{A})$ is a thick subcategory consisting of objects whose cohomologies belong to $\mathsf{mod}(\mathcal{A}/[\mathcal{X}])$ (see [Miy, Thm. 3.2] and [CPS, Thm. 2.3]). The equivalence (1.2.3) gives the following first result of this paper.

Theorem 1.3 (Lemma 7.1, Theorem 7.2)**.** *Let $\mathcal{A}$ be an additive category with weak-kernels and $\mathcal{X}$ its contravariantly finite full subcategory. Suppose that $\mathsf{pd}_{\mathcal{X}}(\mathcal{A}(-,M)|_{\mathcal{X}}) < \infty$ for any $M \in \mathcal{A}$ and $\mathsf{pd}_{\mathcal{A}}(F) < \infty$ for any $F \in \mathsf{mod}(\mathcal{A}/[\mathcal{X}])$. Then the canonical inclusion $\mathcal{X} \hookrightarrow \mathcal{A}$ induces a triangle equivalence $\bar{Q} : \mathsf{D}_{\mathsf{sg}}(\mathcal{A}) \to \mathsf{D}_{\mathsf{sg}}(\mathcal{X})$.*

Our second result is an application of Theorem 1.3, which provides examples of singularly equivalent categories. We denote by $\widehat{\mathcal{X}}$ the full subcategory of $\mathcal{C}$ consisting of objects M which admit an exact sequence

$$0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0$$

with $X_n, \ldots, X_0 \in \mathcal{X}$ for some $n \in \mathbb{Z}_{\geq 0}$. Our result will be stated under the following condition which is a generalization of the setting appearing in Auslander-Buchweitz theory (see Condition 9.3 for details). A map $f : N \to M$ in $\mathcal{C}$ is called an *$\mathcal{X}$-epimorphism* if the induced map $\mathcal{C}(-,N)|_{\mathcal{X}} \xrightarrow{f\circ-} \mathcal{C}(-,M)|_{\mathcal{X}}$ is surjective.

Condition 1.4. Let $\mathcal{C}$ be an abelian category with enough projectives and let $\mathcal{A} \supseteq \mathcal{X} \supseteq \boldsymbol{\omega}$ be a sequence of full subcategories in $\mathcal{C}$ such that $\mathcal{X}$ and $\boldsymbol{\omega}$ are contravariantly finite in $\mathcal{A}$. We consider the following conditions:

(AB1) If a morphism $f : N \to M$ in $\mathcal{A}$ is an $\boldsymbol{\omega}$-epimorphism, then the kernel of f belongs to $\mathcal{A}$.

(AB2) $\mathrm{Ext}^i_{\mathcal{C}}(X, I) = 0$ for any $X \in \mathcal{X}, I \in \boldsymbol{\omega}$ and $i > 0$.

(AB3) For any $M \in \mathcal{A}$, there exists an exact sequence $0 \to Y_M \to X_M \xrightarrow{f} M$ in $\mathcal{A}$ such that f is a right $\mathcal{X}$-approximation of M and $Y_M \in \widehat{\boldsymbol{\omega}}$.

For example, the classical Auslander-Buchweitz theory (Condition 9.3) provides us with the triple ($\mathcal{C} = \mathcal{A}, \mathcal{X}, \boldsymbol{\omega}$) satisfies the Condition 1.4. Note that, in contrary to Condition 9.3, they are not required that: $\boldsymbol{\omega}$ is a *cogenerator of* $\mathcal{X}$; each morphism f appearing in $0 \to Y_M \to X_M \xrightarrow{f} M$ of (AB3) is surjective.

Since $\mathcal{X}/[\boldsymbol{\omega}]$ can be regarded as an analog of the costable category, we denote by

$$\overline{\mathcal{A}} := \mathcal{A}/[\boldsymbol{\omega}] \quad \text{and} \quad \overline{\mathcal{X}} := \mathcal{X}/[\boldsymbol{\omega}].$$

Our main result is the following:

Theorem 1.5 (Theorem 8.2)**.** *Under Condition 1.4, the canonical inclusion $\overline{\mathcal{X}} \hookrightarrow \overline{\mathcal{A}}$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{A}}) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{X}})$.*

Typical examples satisfying Condition 1.4 come from cotilting theory. Let us recall the notion of cotilting subcategories of $\mathcal{C}$. For a subcategory $\mathcal{X}$ of $\mathcal{C}$, we denote by ${}^{\perp}\mathcal{X}$ the full subcategory of $\mathcal{C}$ of objects M with $\mathrm{Ext}^i_{\mathcal{C}}(M, X) = 0$ for any $i > 0$ and $X \in \mathcal{X}$.

Definition 1.6. Let $\mathcal{C}$ be an abelian category with enough projectives. A full subcategory $\mathcal{T}$ of $\mathcal{C}$ is called a *cotilting subcategory of* $\mathcal{C}$, if it satisfies the following conditions:

- There exists an integer $n \in \mathbb{Z}_{\geq 0}$ such that $\mathsf{id} I \leq n$ for any $I \in \mathcal{T}$;
- $\mathrm{Ext}^i_{\mathcal{C}}(I, J) = 0$ for any $I, J \in \mathcal{T}$ and $i > 0$;
- For each $M \in {}^{\perp}\mathcal{T}$, there exists an exact sequence

$$0 \to M \to I \to M' \to 0$$

  with $I \in \mathcal{T}$ and $M' \in {}^{\perp}\mathcal{T}$.

We call an object $T \in \mathcal{C}$ a *cotilting object* if $\mathsf{add}\, T$ is a cotilting subcategory of $\mathcal{C}$.

The following result is immediate from Theorem 1.5.

Corollary 1.7 (Corollary 8.10)**.** *Let $\mathcal{A}$ be an abelian category with enough projectives and $\mathcal{T}$ its contravariantly finite cotilting subcategory. Then the canonical inclusion $\overline{{}^{\perp}\mathcal{T}} \hookrightarrow \overline{\mathcal{A}}$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{A}}) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\overline{{}^{\perp}\mathcal{T}})$.*

As examples of Corollary 1.7, we have the followings:

Example 1.8. (a) Let Λ be a finite dimensional k-algebra over a field k and T a cotilting Λ-module. Then the canonical inclusion $\overline{{}^{\perp}T} \hookrightarrow \overline{\mathsf{mod}}\Lambda$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\overline{\mathsf{mod}}\Lambda) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\overline{{}^{\perp}T})$.

(b) Let R be a commutative Cohen-Macaulay ring with a canonical R-module ω and $\mathsf{CM}R$ the full subcategory of maximal Cohen-Macaulay R-modules. Then the canonical inclusion $\overline{\mathsf{CM}}R \hookrightarrow \overline{\mathsf{mod}}R$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\overline{\mathsf{mod}}R) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\overline{\mathsf{CM}}R)$.

Theorem 1.5 also provides an alternative proof for Matsui-Takahashi's theorem [MT, Thm. 5.4(3)] (Corollary 8.12): For an Iwanaga-Gorenstein ring Λ, the canonical inclusion $\underline{\mathsf{CM}}\Lambda \hookrightarrow \underline{\mathsf{mod}}\Lambda$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\underline{\mathsf{mod}}\Lambda) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\underline{\mathsf{CM}}\Lambda)$.

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Notation and convention

The symbols $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ always denote additive categories, and the set of morphisms $A \to B$ in $\mathcal{A}$ is denoted by $\mathcal{A}(A, B)$. We consider only additive functors between additive categories; that is functors F which satisfy $F(f+g) = F(f) + F(g)$ whenever $f+g$ is defined. For a given category $\mathcal{A}$, we denote its opposite category by $\mathcal{A}^{\mathsf{op}}$. For a functor $F : \mathcal{A} \to \mathcal{C}$, its *image* and *kernel* are defined as the full subcategories of $\mathcal{A}$

$$\operatorname{Im} F := \{C \in \mathcal{C} \mid {}^{\exists} A \in \mathcal{A},\ F(A) \cong C\} \quad \text{and} \quad \operatorname{Ker} F := \{A \in \mathcal{A} \mid F(A) = 0\},$$

respectively. Let $\mathcal{B}$ be a subcategory of $\mathcal{A}$. We denote by $\mathcal{A}/[\mathcal{B}]$ the ideal quotient category of $\mathcal{A}$ modulo the (two-sided) ideal $[\mathcal{B}]$ in $\mathcal{A}$ consisting of all morphisms having a factorization through an object in $\mathcal{B}$. If there exists a fully faithful functor $\mathcal{B} \hookrightarrow \mathcal{A}$, we often regard $\mathcal{B}$ as a full subcategory of $\mathcal{A}$. For each $B \in \mathcal{A}$, we denote by $\mathsf{add}\, B$ the full subcategory consisting of direct summands of finite direct sums of B and we abbreviate $\mathcal{A}/[B]$ to indicate $\mathcal{A}/[\mathsf{add}\, B]$.

Throughout k always denotes a field. The additive category is said to be k-linear if each morphism-space $\mathcal{A}(A, B)$ is a k-module and the composition $\mathcal{A}(B, C) \times \mathcal{A}(A, B) \to \mathcal{A}(A, C)$ is k-bilinear. The additive functor is said to be k-linear if it gives a k-linear maps between morphism-spaces. In the case that given categories are k-linear, we consider only additive k-linear functors.

The word ring and algebra always mean ring with a unit and finite dimensional algebra over a field k, respectively. Throughout Λ denotes a ring. The category of finitely presented right Λ-modules and its full subcategory of projective (resp. injective) Λ-modules will be denoted by $\mathsf{mod}\, \Lambda$ and $\mathsf{proj}\Lambda$ (resp. $\mathsf{inj}\Lambda$), respectively. The stable (resp. costable) category of $\mathsf{mod}\, \Lambda$ will be denoted by $\underline{\mathsf{mod}}\Lambda := \mathsf{mod}\, \Lambda/[\mathsf{proj}\, \Lambda]$ (resp. $\overline{\mathsf{mod}}\Lambda := \mathsf{mod}\, \Lambda/[\mathsf{inj}\, \Lambda]$).

Part I

Auslander's formulas via Recollements for dualizing k-varieties

This part is based on the paper [Oga17].

3 Preliminaries

This thesis is concerned with replacing theorems about rings by theorems about additive categories. So we firstly recall some basic facts on functor categories of additive categories. We denote by Ab the category of abelian groups. For an essentially small category $\mathcal{A}$, a *(right) $\mathcal{A}$-module* is defined to be a contravariant functor $\mathcal{A} \to \mathsf{Ab}$ and a *morphism* $X \to Y$ is a natural transformation. Thus we define an abelian category $\mathsf{Mod}\,\mathcal{A}$ of $\mathcal{A}$-modules, where we call this the *functor category of* $\mathcal{A}$. In the functor category $\mathsf{Mod}\,\mathcal{A}$, the morphism-space $(\mathsf{Mod}\,\mathcal{A})(X, Y)$ is denoted by $\mathrm{Hom}_{\mathcal{A}}(X, Y)$. In the case that a given category $\mathcal{A}$ is k-linear, it is natural to consider, instead of $\mathsf{Mod}\,\mathcal{A}$; the equivalent category of k-linear functors from $\mathcal{A}$ to $\mathsf{Mod}\,k$, which is denoted by the same symbol.

An $\mathcal{A}$-module X is *finitely generated* if there exists an epimorphism $\mathcal{A}(-, A) \twoheadrightarrow X$ for some $A \in \mathcal{A}$. An $\mathcal{A}$-module X is said to be *finitely presented* if there exists an exact sequence

$$\mathcal{A}(-, B) \to \mathcal{A}(-, A) \to X \to 0$$

for some $A, B \in \mathcal{A}$. We denote by $\mathsf{mod}\,\mathcal{A}$ the full subcategory of finitely presented $\mathcal{A}$-modules. The subcategory $\mathsf{mod}\,\mathcal{A}$ is closed under cokernels and extensions in $\mathsf{Mod}\,\mathcal{A}$. However it is not always abelian since it is not necessarily closed under kernels. Let $f : B \to A$ be a morphism in $\mathcal{A}$. We call a morphism $g : C \to B$ a *weak-kernel for f* if the induced sequence

$$\mathcal{A}(-, C) \xrightarrow{g\circ -} \mathcal{A}(-, B) \xrightarrow{f\circ -} \mathcal{A}(-, A)$$

is exact. We say $\mathcal{A}$ *admits weak-kernels* if every morphism in $\mathcal{A}$ has a weak-kernel. The notion of *weak-cokernel* is defined dually. We recall the following well-known fact.

Lemma 3.1. [Fre, Thm. 1.4] *The following are equivalent for a category $\mathcal{A}$.*

(i) *The category $\mathcal{A}$ admits weak-kernels.*

(ii) *The full subcategory $\mathsf{mod}\,\mathcal{A}$ is an exact abelian subcategory in $\mathsf{Mod}\,\mathcal{A}$, that is, it is abelian and the canonical inclusion $\mathsf{mod}\,\mathcal{A} \hookrightarrow \mathsf{Mod}\,\mathcal{A}$ is exact.*

The functor category $\mathsf{mod}\,\mathcal{A}$ over an additive category $\mathcal{A}$ with weak-kernels is a generalization of the module category $\mathsf{mod}\,\Lambda$ over a Noetherian ring Λ in the following sense.

Lemma 3.2. *Let Λ be a Noetherian ring. Then we have an equivalence $\mathsf{mod}(\mathsf{proj}\,\Lambda) \xrightarrow{\sim} \mathsf{mod}\,\Lambda$ which evaluates the functor $X \in \mathsf{mod}(\mathsf{proj}\,\Lambda)$ on Λ.*

Like the case for module categories, $\mathsf{proj}\,\mathcal{A}$ (resp. $\mathsf{inj}\,\mathcal{A}$) denotes the full subcategory of projective (resp. injective) $\mathcal{A}$-modules in $\mathsf{mod}\,\mathcal{A}$, and the stable (resp. costable) category will be denoted by $\underline{\mathsf{mod}}\mathcal{A} := (\mathsf{mod}\,\mathcal{A})/[\mathsf{proj}\,\mathcal{A}]$ (resp. $\overline{\mathsf{mod}}\mathcal{A} := (\mathsf{mod}\,\mathcal{A})/[\mathsf{inj}\,\mathcal{A}]$). For a full subcategory $\mathcal{B} \subseteq \mathsf{mod}\,\mathcal{A}$ which contains $\mathsf{proj}\,\mathcal{A}$ (resp. $\mathsf{inj}\,\mathcal{A}$), we also use the symbol $\underline{\mathcal{B}} := \mathcal{B}/[\mathsf{proj}\,\mathcal{A}]$ (resp. $\overline{\mathcal{B}} := \mathcal{B}/[\mathsf{inj}\,\mathcal{A}]$).

3.1 Dualizing k-varieties

In this subsection we recall from [AR74] the notion of dualizing k-varieties. Let $\mathcal{A}$ be a *Krull-Schmidt* k-linear category, that is, each object $A \in \mathcal{A}$ admits a decomposition $A \cong \coprod_{i=1}^{n} A_i$ with $\mathcal{A}(A_i, A_i)$ a local k-algebra for any $i \in \{1, \dots, n\}$. We also assume that $\mathcal{A}$ is *Hom-finite*, that is, each morphism-space $\mathcal{A}(A, B)$ is a finite dimensional k-module. We denote by $D := \mathrm{Hom}_k(-, k) : \mathsf{mod}\,k \to \mathsf{mod}\,k$ the standard k-duality.

Definition 3.3. [AR74, Section 2] A Krull-Schmidt Hom-finite k-linear category $\mathcal{A}$ is a *dualizing k-variety* if the standard k-duality $D : \mathsf{Mod}\,\mathcal{A} \to \mathsf{Mod}(\mathcal{A}^{\mathsf{op}})$, $X \mapsto D \circ X$ induces a duality $D : \mathsf{mod}\,\mathcal{A} \xrightarrow{\sim} \mathsf{mod}(\mathcal{A}^{\mathsf{op}})$.

It is obvious that $\mathcal{A}$ is a dualizing k-variety if and only if so is $\mathcal{A}^{\mathsf{op}}$. If $\mathcal{A}$ is a dualizing k-variety, due to the duality between $\mathsf{mod}\,\mathcal{A}$ and $\mathsf{mod}(\mathcal{A}^{\mathsf{op}})$, $\mathsf{mod}\,\mathcal{A}$ is closed under kernels in $\mathsf{Mod}\,\mathcal{A}$. Thus $\mathsf{mod}\,\mathcal{A}$ is an exact abelian subcategory in $\mathsf{Mod}\,\mathcal{A}$. By Lemma 3.1, $\mathcal{A}$ admits weak-kernels and weak-cokernels. The following proposition gives us basic examples of dualizing k-varieties.

Proposition 3.4. [AR74, Prop. 2.6, Prop. 3.4] *Suppose $\mathcal{A}$ is a dualizing k-variety. Then $\mathsf{mod}\,\mathcal{A}$ is a dualizing k-variety. Moreover, $\mathsf{mod}\,\mathcal{A}$ admits injective hulls and projective covers.*

Next we recall the definition of functorially finite subcategories. The symbol $X|_{\mathcal{B}}$ denotes the restricted functor of an $\mathcal{A}$-module X onto a subcategory $\mathcal{B}$. Especially, for a functor category $\mathsf{mod}\,\mathcal{A}$ and its full subcategory $\mathcal{B}$ we also write $\mathrm{Ext}^i_{\mathcal{A}}(\mathcal{B}, X) := \mathrm{Ext}^i_{\mathcal{A}}(-, X)|_{\mathcal{B}}$, where $X \in \mathsf{mod}\,\mathcal{A}$ and $i \in \mathbb{Z}_{\geq 0}$.

Definition 3.5. Let $\mathcal{A}$ be an arbitrary category and $\mathcal{B}$ a full subcategory in $\mathcal{A}$.

(a) The full subcategory $\mathcal{B}$ is *contravariantly finite* if the functor $\mathcal{A}(-, A)|_{\mathcal{B}}$ is a finitely generated $\mathcal{B}$-module for each $A \in \mathcal{A}$.

(b) The full subcategory $\mathcal{B}$ is *covariantly finite* if the functor $\mathcal{A}(A, -)|_{\mathcal{B}}$ is a finitely generated $\mathcal{B}^{\mathsf{op}}$-module for each $A \in \mathcal{A}$.

(c) We call $\mathcal{B}$ a *functorially finite* subcategory if it is contravariantly finite and covariantly finite.

If a $\mathcal{B}$-module $\mathcal{A}(-, A)|_{\mathcal{B}}$ is finitely generated, then there exists an epimorphism

$$\mathcal{B}(-, B) \xrightarrow{\alpha \circ -} \mathcal{A}(-, A)|_{\mathcal{B}} \to 0$$

in $\mathsf{Mod}\,\mathcal{B}$ for some $B \in \mathcal{B}$. Then we call the induced morphism $\alpha : B \to A$ a *right $\mathcal{B}$-approximation of A*. Dually we define the notion of *left $\mathcal{B}$-approximation*.

It is basic that the subcategories $\mathsf{proj}\mathcal{A}$ and $\mathsf{inj}\mathcal{A}$ are functorially finite in $\mathsf{mod}\,\mathcal{A}$ if $\mathcal{A}$ is a dualizing k-variety. The following result gives a criterion for a given subcategory to be a dualizing k-variety.

Proposition 3.6. [AS, Thm. 2.3][Iya07a, Prop. 1.2] *Let $\mathcal{B}$ be a functorially finite subcategory in a dualizing k-variety $\mathcal{A}$. Then $\mathcal{B}$ is a dualizing k-variety.*

3.2 Recollements of abelian categories

In this subsection we recall the definition of recollements of abelian categories, as well as some basic properties which are needed in this paper. Let us start with introducing basic terminology. Throughout this subsection, $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are assumed to be abelian. A pair of functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ is said to be an *adjoint pair* if there exists a bifunctorial isomorphism $\mathcal{A}(A, RB) \cong \mathcal{B}(LA, B)$ in $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We simply denote this adjoint pair by $(L \dashv R)$. For a functor $\Phi : \mathcal{A} \to \mathcal{B}$, we often denote its right (resp. left) adjoint by Φ_ρ (resp. Φ_λ). If Φ admits a right adjoint Φ_ρ as well as a left adjoint Φ_λ, we denote this situation by $\mathcal{A} \underset{\Phi_\rho \,\leftarrow}{\overset{\Phi_\lambda\,\leftarrow}{\xrightarrow{\ \Phi\ }}} \mathcal{B}$.

We recall the definition of recollement, following [FP, Psa] (see also [Pop, Ch. 4]).

Definition 3.7. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be abelian categories. A *recollement of $\mathcal{A}$ relative to $\mathcal{B}$ and $\mathcal{C}$* is given by six functors

$$\mathcal{B} \underset{e_\rho\,\leftarrow}{\overset{e_\lambda\,\leftarrow}{\xrightarrow{\ e\ }}} \mathcal{A} \underset{q_\rho\,\leftarrow}{\overset{q_\lambda\,\leftarrow}{\xrightarrow{\ q\ }}} \mathcal{C}$$

such that

(R1) They form four adjoint pairs $(e_\lambda \dashv e), (e \dashv e_\rho), (q_\lambda \dashv q)$ and $(q \dashv q_\rho)$.

(R2) The functors q_λ, q_ρ and e are fully faithful.

(R3) $\operatorname{Im} e = \operatorname{Ker} q$.

We denote this recollement by $(\mathcal{B}, \mathcal{A}, \mathcal{C})$ for short.

Notice that the functors q and e are exact, since each of them admits a right adjoint and a left adjoint. The following proposition shows that a recollement is a special case of Serre quotients.

Proposition 3.8. [Pop, Thm. 4.9] *Let $q : \mathcal{A} \to \mathcal{C}$ be an exact functor. If it admits a fully faithful right adjoint q_ρ (resp. left adjoint q_λ), the functor q induces an equivalence between $\mathcal{C}$ and the Serre quotient $\mathcal{A}/\operatorname{Ker} q$ of $\mathcal{A}$ with respect to the Serre subcategory* $\operatorname{Ker} q$.

Thus, given a recollement $(\mathcal{B}, \mathcal{A}, \mathcal{C})$, we have an equivalence $\mathcal{A}/\mathcal{B} \xrightarrow{\sim} \mathcal{C}$, since $\operatorname{Ker} q = \mathcal{B}$. The following notions play a central role in Section 5.

Proposition 3.9. [Psa, Prop. 2.6] *For a given recollement $(\mathcal{B}, \mathcal{A}, \mathcal{C})$ and an object $A \in \mathcal{A}$,*

(a) *we have an exact sequence $0 \to ee_\rho(A) \xrightarrow{\eta} A \xrightarrow{\varepsilon} q_\rho q(A) \to B \to 0$, where η and ε are the counit and the unit of the adjoint pairs, respectively. We call it the* right-defining exact sequence *of A.*

(b) *we have an exact sequence $0 \to B' \to q_\lambda q(A) \xrightarrow{\eta'} A \xrightarrow{\varepsilon'} ee_\lambda(A) \to 0$, where η' and ε' are the counit and unit of the adjoint pairs, respectively. We call it the* left-defining exact sequence *of A.*

Moreover, if there exists an exact sequence $0 \to A' \to A \to A'' \to A''' \to 0$ with $A', A''' \in \operatorname{Im} e$ and $A'' \in \operatorname{Im} q_\rho$, then it is isomorphic to the right-defining exact sequence of A. The dual statement holds for the left-defining exact sequences.

Proof. We only prove the latter statement, that is, an exact sequence $0 \to A' \xrightarrow{f} A \xrightarrow{g} A'' \to A''' \to 0$ with $A', A''' \in \operatorname{Im} e$ and $A'' \in \operatorname{Im} q_\rho$ is isomorphic to the right-defining exact sequence of A. By considering the right-defining exact sequences of A and A', we show that a given morphism $A' \xrightarrow{f} A$ induces the following commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & ee_\rho(A) & \xrightarrow{\eta} & A & \longrightarrow & q_\rho q(A) & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle ee_\rho(f)} & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle q_\rho q(f)} & & \uparrow & & \\
0 & \longrightarrow & ee_\rho(A') & \longrightarrow & A' & \longrightarrow & q_\rho q(A') & \longrightarrow & B' & \longrightarrow & 0.
\end{array}$$

Since $A' \in \operatorname{Im} e$, the counit $ee_\rho(A') \xrightarrow{\sim} A'$ is an isomorphism. Since $A'' \in \operatorname{Im} q_\rho$ and e_ρ is left exact, the induced morphism $e_\rho(f) : e_\rho(A') \xrightarrow{\sim} e_\rho(A)$ is an isomorphism. Thus we deduce the commutative diagram below:

$$\begin{array}{ccc}
ee_\rho(A) & \xrightarrow{\eta} & A \\
{\scriptstyle ee_\rho(f)}\uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle f} \\
ee_\rho(A') & \xrightarrow[\cong]{} & A'
\end{array}$$

Hence the morphism $A' \xrightarrow{f} A$ is isomorphic to the counit $ee_\rho(A) \xrightarrow{\eta} A$.

Similarly we show that a given morphism $A \xrightarrow{g} A''$ is isomorphic to the unit $A \xrightarrow{\epsilon} q_\rho q(A)$ under isomorphisms $A'' \xrightarrow{\sim} q_\rho q(A'')$ and $q_\rho q(g) : q_\rho q(A) \xrightarrow{\sim} q_\rho q(A'')$. A natural isomorphism $B \to A'''$ is induced from the universality of cokernels. We have thus obtained desired isomorphisms. □

4 Recollements over dualizing k-varieties

We start with introducing basic terminology. Let us recall the notion of the *tensor product* $\mathcal{B} \otimes \mathcal{A}$ of two additive categories. The objects of $\mathcal{B} \otimes \mathcal{A}$ are the pairs (B, A) with $B \in \mathcal{B}$

and $A \in \mathcal{A}$ and the morphisms from (B, A) to (B', A') is the tensor product of abelian groups $\mathcal{B}(B, B') \otimes \mathcal{A}(A, A')$. In the case that given categories $\mathcal{A}$ and $\mathcal{B}$ are k-linear, the morphism space $(\mathcal{B} \otimes \mathcal{A})((B, A), (B', A'))$ is defined to be the tensor product of k-modules $\mathcal{B}(B, B') \otimes_k \mathcal{A}(A, A')$. We define *$\mathcal{B}$-$\mathcal{A}$-bimodule* to be a contravariant additive functor from $\mathcal{B}^{\mathsf{op}} \otimes \mathcal{A}$ to Ab. Given a $\mathcal{B}$-$\mathcal{A}$-bimodule X, we regard it as a contravariant functor from $\mathcal{A}$ to $\mathsf{Mod}(\mathcal{B}^{\mathsf{op}})$ as follows: For each $A \in \mathcal{A}$, we define a covariant functor $X(-, A) : \mathcal{B} \to \mathsf{Mod}\, k$ by setting $X(g^{\mathsf{op}}, A) := X(g^{\mathsf{op}} \otimes 1_A) : X(B'^{\mathsf{op}}, A) \to X(B^{\mathsf{op}}, A)$ for a morphism $g : B' \to B$ in $\mathcal{B}$. Let $f : A \to A'$ be a morphism in $\mathcal{A}$. We define a natural transformation $X(-, f) : X(-, A') \to X(-, A)$ by setting $X(B^{\mathsf{op}}, f) := X(1_{B^{\mathsf{op}}} \otimes f)$ for each $B \in \mathcal{B}$. These assignments give rise to a contravariant functor from $\mathcal{A}$ to $\mathsf{Mod}(\mathcal{B}^{\mathsf{op}})$. Similarly, we regard a $\mathcal{B}$-$\mathcal{A}$-bimodule X as a covariant functor from $\mathcal{B}$ to $\mathsf{Mod}\, \mathcal{A}$.

For later use, we recall in Proposition 4.2 that a $\mathcal{B}$-$\mathcal{A}$-bimodule X induces a Hom-tensor adjunctions: Given a $\mathcal{B}$-$\mathcal{A}$-bimodule X, we define a covariant functor $\mathrm{Hom}_{\mathcal{A}}(X, -) : \mathsf{Mod}\, \mathcal{A} \to \mathsf{Mod}\, \mathcal{B}$ which sends $Y \in \mathsf{Mod}\, \mathcal{A}$ to the functor $\mathrm{Hom}_{\mathcal{A}}(X, Y) : \mathcal{B} \to \mathsf{Mod}\, k$ given by $B \mapsto \mathrm{Hom}_{\mathcal{A}}(X(B^{\mathsf{op}}, -), Y)$ for $B \in \mathcal{B}$. In the next lemma, we define a covariant functor $- \otimes_{\mathcal{B}} X : \mathsf{Mod}\, \mathcal{B} \to \mathsf{Mod}\, \mathcal{A}$.

Lemma 4.1. *Let X be a $\mathcal{B}$-$\mathcal{A}$-bimodule. Then, there exists a unique right-exact functor $\widetilde{X} : \mathsf{Mod}\, \mathcal{B} \to \mathsf{Mod}\, \mathcal{A}$ up to isomorphism which preserves coproducts and makes the following diagram commutative up to isomorphism*

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{X} & \mathsf{Mod}\, \mathcal{A} \\ {\scriptstyle \mathbb{Y}}\downarrow & \nearrow_{\widetilde{X}} & \\ \mathsf{Mod}\, \mathcal{B} & & \end{array}$$

where we regard X as a covariant functor from $\mathcal{B}$ to $\mathsf{Mod}\, \mathcal{A}$ and $\mathbb{Y}$ denotes the Yoneda functor sending $B \in \mathcal{B}$ to $\mathcal{B}(-, B) \in \mathsf{Mod}\, \mathcal{B}$. We write $- \otimes_{\mathcal{B}} X$ instead of $\widetilde{X}$.

Proof. This is well-known for experts but we recall a construction of $\widetilde{X}$ for the convenience of the reader (e.g. [Kra00, Section 2] for details). Any $\mathcal{B}$-module M admits a projective presentation

$$\bigoplus_{j \in J} \mathcal{B}(-, B_j) \to \bigoplus_{i \in I} \mathcal{B}(-, B_i) \to M \to 0.$$

Thanks to the Yoneda Lemma, we obtain a set of morphisms $\{\beta_{ji} : B_j \to B_i\}_{j \in J, i \in I}$ in $\mathcal{B}$. The induced set of morphisms $\{X(\beta_{ji}^{\mathsf{op}}, -) : X(B_j^{\mathsf{op}}, -) \to X(B_i^{\mathsf{op}}, -)\}_{j \in J, i \in I}$ in $\mathsf{Mod}\, \mathcal{A}$ gives a canonical morphism $f : \bigoplus_{j \in J} X(B_j^{\mathsf{op}}, -) \to \bigoplus_{i \in I} X(B_i^{\mathsf{op}}, -)$. Put $\widetilde{X}(M) := \mathrm{Cok}\, f$. We omit a remaining part of the proof. $\square$

Proposition 4.2. *Let X be a $\mathcal{B}$-$\mathcal{A}$-bimodule. Then, the induced functors $\mathrm{Hom}_{\mathcal{A}}(X, -) : \mathsf{Mod}\, \mathcal{A} \to \mathsf{Mod}\, \mathcal{B}$ and $- \otimes_{\mathcal{B}} X : \mathsf{Mod}\, \mathcal{B} \to \mathsf{Mod}\, \mathcal{A}$ form an adjoint pair $(- \otimes_{\mathcal{B}} X \dashv \mathrm{Hom}_{\mathcal{A}}(X, -))$.*

Proof. We shall show the bifunctorial isomorphism $\mathrm{Hom}_{\mathcal{A}}(M \otimes_{\mathcal{B}} X, Y) \cong \mathrm{Hom}_{\mathcal{B}}(M, \mathrm{Hom}_{\mathcal{A}}(X, Y))$ in $M \in \mathsf{Mod}\, \mathcal{B}$ and $Y \in \mathsf{Mod}\, \mathcal{A}$. Take a projective presentation

$$\bigoplus_{j \in J} \mathcal{B}(-, B_j) \to \bigoplus_{i \in I} \mathcal{B}(-, B_i) \to M \to 0 \tag{4.2.1}$$

of $M \in \mathsf{Mod}\,\mathcal{B}$. By Lemma 4.1, applying $- \otimes_{\mathcal{B}} X$ to the above yields an exact sequence

$$\bigoplus_{j\in J} X(B_j^{\mathsf{op}}, -) \to \bigoplus_{i\in I} X(B_i^{\mathsf{op}}, -) \to M \otimes_{\mathcal{B}} X \to 0$$

in $\mathsf{Mod}\,\mathcal{A}$. Since $\mathrm{Hom}_{\mathcal{A}}(-, Y)$ is left-exact and sends coproducts to products, we have an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(M \otimes_{\mathcal{B}} X, Y) \to \prod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(X(B_i^{\mathsf{op}}, -), Y) \to \prod_{j\in J} \mathrm{Hom}_{\mathcal{A}}(X(B_j^{\mathsf{op}}, -), Y) \tag{4.2.2}$$

in $\mathsf{Mod}\,k$. On the other hand, applying $\mathrm{Hom}_{\mathcal{B}}(-, \mathrm{Hom}_{\mathcal{A}}(X, Y))$ to the above presentation (4.2.1), we have an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{B}}(M, \mathrm{Hom}_{\mathcal{A}}(X, Y)) \to \prod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(X(B_i^{\mathsf{op}}, -), Y) \to \prod_{j\in J} \mathrm{Hom}_{\mathcal{A}}(X(B_j^{\mathsf{op}}, -), Y) \tag{4.2.3}$$

in $\mathsf{Mod}\,k$. By comparing (4.2.2) and (4.2.3), we have a desired isomorphism. □

We often regard $\mathcal{A}$ as an $\mathcal{A}$-$\mathcal{A}$-bimodule by the following way

$${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} := \mathcal{A}(-, +) : \mathcal{A}^{\mathsf{op}} \otimes \mathcal{A} \to \mathsf{Ab},\ (A^{\mathsf{op}}, A') \mapsto \mathcal{A}(A', A).$$

Consider a full subcategory $\mathcal{B}$ in $\mathcal{A}$. The canonical inclusion $i : \mathcal{B} \hookrightarrow \mathcal{A}$ gives a natural $\mathcal{A}$-$\mathcal{B}$-bimodule structure on $\mathcal{A}$ by

$${}_{\mathcal{A}}\mathcal{A}_{\mathcal{B}} := \mathcal{A}(i(-), +) : \mathcal{A}^{\mathsf{op}} \otimes \mathcal{B} \to \mathsf{Ab},\ (A^{\mathsf{op}}, B) \mapsto \mathcal{A}(i(B), A) = \mathcal{A}(B, A).$$

Similarly, we define a $\mathcal{B}$-$\mathcal{A}$-bimodule ${}_{\mathcal{B}}\mathcal{A}_{\mathcal{A}} := \mathcal{A}(-, i(+))$.

The first step is to show the following elementary proposition, which is a categorical analog of the recollement (1.0.1) and well-known for experts. We denote by $i^* : \mathsf{Mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathcal{B}$, $X \mapsto X|_{\mathcal{B}}$ the natural restriction functor induced from $i : \mathcal{B} \to \mathcal{A}$ and also denote by $p^* : \mathsf{Mod}(\mathcal{A}/[\mathcal{B}]) \to \mathsf{Mod}\,\mathcal{A}$, $X \mapsto X \circ p$ the natural restriction functor induced from $p : \mathcal{A} \to \mathcal{A}/[\mathcal{B}]$.

Proposition 4.3. [Psa, Example 2.13] *Let $(\mathcal{A}, \mathcal{B})$ be the pair of an additive category $\mathcal{A}$ and its full subcategory $\mathcal{B}$. Then we have the following recollement:*

$$\begin{tikzcd} \mathsf{Mod}(\mathcal{A}/[\mathcal{B}]) \arrow[r, "p^*"] & \mathsf{Mod}\,\mathcal{A} \arrow[l, bend right, "p^*_\lambda"'] \arrow[l, bend left, "p^*_\rho"] \arrow[r, "i^*"] & \mathsf{Mod}\,\mathcal{B}. \arrow[l, bend right, "i^*_\lambda"'] \arrow[l, bend left, "i^*_\rho"] \end{tikzcd} \tag{4.3.1}$$

Proof. (i) We shall construct the adjoint pairs on the right side in (4.3.1). Note that there exist isomorphisms $i^* \cong \mathrm{Hom}_{\mathcal{A}}({}_{\mathcal{B}}\mathcal{A}_{\mathcal{A}}, -) \cong - \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathcal{A}_{\mathcal{B}})$. Thus it admits a left adjoint $i^*_\lambda :=$

$-\otimes_{\mathcal{B}} ({}_{\mathcal{B}}\mathcal{A}_{\mathcal{A}})$ and a right adjoint $i^*_\rho := \mathrm{Hom}_{\mathcal{B}}({}_{\mathcal{A}}\mathcal{A}_{\mathcal{B}}, -)$. By easy calculation, we show that i^*_λ and i^*_ρ are fully faithful. In fact, we have the following isomorphisms:

$$\begin{aligned} i^* \circ i^*_\lambda &\cong - \otimes_{\mathcal{B}} ({}_{\mathcal{B}}\mathcal{A}_{\mathcal{A}}) \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathcal{A}_{\mathcal{B}}) \cong - \otimes_{\mathcal{B}} \mathcal{B} \ \cong \ \mathrm{id}_{\mathsf{Mod}\,\mathcal{B}}, \\ i^* \circ i^*_\rho &\cong \mathrm{Hom}_{\mathcal{A}}({}_{\mathcal{B}}\mathcal{A}_{\mathcal{A}}, \mathrm{Hom}_{\mathcal{B}}({}_{\mathcal{A}}\mathcal{A}_{\mathcal{B}}, -)) \\ &\cong \mathrm{Hom}_{\mathcal{B}}(({}_{\mathcal{B}}\mathcal{A}_{\mathcal{A}}) \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathcal{A}_{\mathcal{B}}), -) \\ &\cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{B}, -) \ \cong \ \mathrm{id}_{\mathsf{Mod}\,\mathcal{B}}. \end{aligned}$$

Thus we have constructed the right side of (4.3.1).

(ii) We shall construct the adjoint pairs on the left side in (4.3.1). By a similar argument to the above, the restriction functor p^* is a fully faithful exact functor which admits a left adjoint p^*_λ and a right adjoint p^*_ρ. Thus we have obtained the left side of (4.3.1).

(iii) It remains to show that $\mathrm{Im}\, p^* = \mathrm{Ker}\, i^*$. This follows from the following obvious lemma.

Lemma 4.4. *Let X be an object in $\mathsf{Mod}\,\mathcal{A}$. Then X belongs to $\mathrm{Im}\, p^*$ if and only if X vanishes on objects in $\mathcal{B}$. In particular, we have $\mathrm{Im}\, p^* = \mathrm{Ker}\, i^*$.*

We have thus proved Proposition 4.3. □

Now we state our main theorem.

Theorem 4.5. *Let $(\mathcal{A}, \mathcal{B})$ be the pair of a dualizing k-variety $\mathcal{A}$ and its functorially finite subcategory $\mathcal{B}$. Then the recollement (4.3.1) restricts to the following one:*

$$\mathsf{mod}(\mathcal{A}/[\mathcal{B}]) \xrightarrow{e} \mathsf{mod}\,\mathcal{A} \xrightarrow{q} \mathsf{mod}\,\mathcal{B}, \qquad e_\lambda, e_\rho : \mathsf{mod}\,\mathcal{A} \to \mathsf{mod}(\mathcal{A}/[\mathcal{B}]), \quad q_\lambda, q_\rho : \mathsf{mod}\,\mathcal{B} \to \mathsf{mod}\,\mathcal{A}. \tag{4.5.1}$$

In particular, we have an equivalence $\dfrac{\mathsf{mod}\,\mathcal{A}}{\mathsf{mod}(\mathcal{A}/[\mathcal{B}])} \simeq \mathsf{mod}\,\mathcal{B}$.

We call this the *recollement arising from the pair $(\mathcal{A}, \mathcal{B})$ of a dualizing k-variety $\mathcal{A}$ and a functorially finite subcategory $\mathcal{B}$ in $\mathcal{A}$.*

In the rest of this section, we give a proof of Theorem 4.5. Assume that $\mathcal{A}$ and $\mathcal{B}$ are Krull-Schmidt Hom-finite k-linear categories. First we consider the right part of the recollement (4.3.1). The functor i^*_λ preserves indecomposable projectives because $i^*_\lambda(\mathcal{B}(-, B)) = \mathcal{B}(-, B) \otimes_{\mathcal{B}} ({}_{\mathcal{B}}\mathcal{A}_{\mathcal{A}}) \cong \mathcal{A}(-, B)$. Since i^*_λ is right-exact, we have the restricted functor $i^*_\lambda : \mathsf{mod}\,\mathcal{B} \to \mathsf{mod}\,\mathcal{A}$, which is denoted by the same symbol. In general i^* and i^*_ρ do not restrict to the subcategories of finitely presented functors. However, if $(\mathcal{A}, \mathcal{B})$ is a pair of a dualizing k-variety $\mathcal{A}$ and its functorially finite subcategory $\mathcal{B}$, i^* and i^*_ρ restrict to the subcategories.

Second we consider the left part of the recollement (4.3.1). Like the case for the canonical inclusion i, although the left adjoint p^*_λ preserves finitely presented functors, p^* and p^*_ρ do not necessarily preserve finitely presented functors.

The next lemma shows a necessary and sufficient condition so that i^* and p^* preserves finitely presented functors, see [Buc97, Prop. 3.9] for the equivalence (i) and (iii) below.

Lemma 4.6. *Let $\mathcal{A}$ be a category with weak-kernels and $\mathcal{B}$ a full subcategory in $\mathcal{A}$. Then the following are equivalent for the recollement (4.3.1).*

(i) *The category $\mathcal{B}$ is contravariantly finite.*

(ii) *The functor i^* restricts to the functor $i^* : \mathsf{mod}\,\mathcal{A} \to \mathsf{mod}\,\mathcal{B}$.*

(iii) *The functor p^* restricts to the functor $p^* : \mathsf{mod}(\mathcal{A}/[\mathcal{B}]) \to \mathsf{mod}\,\mathcal{A}$.*

Proof. (i) $\Rightarrow$ (ii): Since i^* is exact, we have only to show that $i^*(\mathcal{A}(-,A))$ is finitely presented for any $A \in \mathcal{A}$. Since $\mathcal{B}$ is contravariantly finite, there exists a right $\mathcal{B}$-approximation $\alpha_0 : B_0 \to A$. The morphism α_0 induces an epimorphism $\mathcal{B}(-,B_0) \xrightarrow{\alpha_0\circ-} \mathcal{A}(-,A)|_{\mathcal{B}} \to 0$, that is, the $\mathcal{B}$-module $\mathcal{A}(-,A)|_{\mathcal{B}}$ is finitely generated. Since $\mathsf{mod}\,\mathcal{A}$ is abelian, we have the kernel-sequence $0 \to X \to \mathcal{A}(-,B_0) \xrightarrow{\alpha_0\circ-} \mathcal{A}(-,A)$ in $\mathsf{mod}\,\mathcal{A}$ induced from the morphism α_0. Since $X \in \mathsf{mod}\,\mathcal{A}$, there exists an epimorphism $\mathcal{A}(-,A') \to X \to 0$ and thus we have an exact sequence

$$\mathcal{A}(-,A')|_{\mathcal{B}} \to \mathcal{B}(-,B_0) \to \mathcal{A}(-,A)|_{\mathcal{B}} \to 0$$

in $\mathsf{mod}\,\mathcal{B}$. The fact that $\mathcal{A}(-,A')|_{\mathcal{B}}$ is finitely generated shows that $\mathcal{A}(-,A)|_{\mathcal{B}}$ is finitely presented.

(ii) $\Rightarrow$ (i): For any $A \in \mathcal{A}$, the functor $i^*(\mathcal{A}(-,A)) = \mathcal{A}(-,A)|_{\mathcal{B}}$ is finitely presented. This shows that $\mathcal{B}$ is contravariantly finite by definition. □

In the rest, we assume that $\mathcal{B}$ is contravariantly finite in $\mathcal{A}$. In this case, we have the following lemma which is a "finitely presented" version of Lemma 4.4.

Lemma 4.7. *Assume that $\mathcal{B}$ is contravariantly finite in $\mathcal{A}$. Then a finitely presented $\mathcal{A}$-module X belongs to $\mathsf{mod}(\mathcal{A}/[\mathcal{B}])$ if and only if X vanishes on objects in $\mathcal{B}$. In particular, we have $\mathsf{mod}(\mathcal{A}/[\mathcal{B}]) = \operatorname{Ker} q$.*

Combining Lemmas 4.6 and 4.7, we have the following proposition, which gives a part of recollement (4.5.1) without the right adjoints q_ρ and e_ρ.

Lemma 4.8. *Let $\mathcal{B}$ be a contravariantly finite full subcategory of $\mathcal{A}$. Then the functors i^*, p^*, i^*_λ and p^*_λ in the recollement (4.3.1) restrict to the following diagram*

$$\mathsf{mod}(\mathcal{A}/[\mathcal{B}]) \overset{\xleftarrow{\;p^*_\lambda\;}}{\xrightarrow{\;p^*\;}} \mathsf{mod}\,\mathcal{A} \overset{\xleftarrow{\;i^*_\lambda\;}}{\xrightarrow{\;i^*\;}} \mathsf{mod}\,\mathcal{B},$$

with $\operatorname{Ker} i^* = \operatorname{Im} p^*$.

In the rest, *we assume that $\mathcal{A}$ is a dualizing k-variety and $\mathcal{B}$ is functorially finite in $\mathcal{A}$.* Note that, in this case, $\mathcal{A}/[\mathcal{B}]$ is also a dualizing k-variety. In fact, for any $X \in \mathsf{mod}(\mathcal{A}/[\mathcal{B}])$, DX is $\mathcal{A}^{\mathsf{op}}$-module which vanishes on $\mathcal{B}$, hence $DX \in \mathsf{mod}(\mathcal{A}/[\mathcal{B}])^{\mathsf{op}}$. Dually we have that $DX' \in \mathsf{mod}(\mathcal{A}/[\mathcal{B}])$ for any $X' \in \mathsf{mod}(\mathcal{A}/[\mathcal{B}])^{\mathsf{op}}$.

To show Theorem 4.5, it remains to show that the functors i^* and p^* admit right adjoints, respectively. The following proposition is our key observation.

Proposition 4.9. *There exist the following adjoint pairs for the pair $(\mathcal{A}, \mathcal{B})$ of a dualizing k-variety $\mathcal{A}$ and its functorially finite full subcategory $\mathcal{B}$:*

$$\mathsf{mod}\,\mathcal{A} \xrightarrow{\ q\ } \mathsf{mod}\,\mathcal{B}, \qquad q_\lambda, q_\rho : \mathsf{mod}\,\mathcal{B} \to \mathsf{mod}\,\mathcal{A}$$

where $q := i^*$ is the restriction functor induced by the canonical inclusion $i : \mathcal{B} \hookrightarrow \mathcal{A}$. Moreover, we have isomorphisms $q_\rho \cong \mathrm{Hom}_{\mathcal{B}}({}_{\mathcal{A}}\mathcal{A}_{\mathcal{B}}, -)$ and $q_\lambda \cong - \otimes_{\mathcal{B}} ({}_{\mathcal{B}}\mathcal{A}_{\mathcal{A}})$.

Proof. Recall that every dualizing k-variety admits weak-kernels and weak-cokernels. As we have seen in Lemma 4.6, there exists an adjoint pair $(q_\lambda \dashv q)$ between $\mathsf{mod}\,\mathcal{A}$ and $\mathsf{mod}\,\mathcal{B}$. Since $\mathcal{A}^{\mathsf{op}}$ is also a dualizing k-variety and $\mathcal{B}^{\mathsf{op}}$ is functorially finite in $\mathcal{A}^{\mathsf{op}}$, by Lemma 4.6, we have an adjoint pair $(q'_\lambda \dashv q')$ between $\mathsf{mod}(\mathcal{A}^{\mathsf{op}})$ and $\mathsf{mod}(\mathcal{B}^{\mathsf{op}})$, where q' is the restriction functor induced by the inclusion $\mathcal{B}^{\mathsf{op}} \hookrightarrow \mathcal{A}^{\mathsf{op}}$. Since $\mathcal{A}$ and $\mathcal{B}$ are dualizing k-varieties, we have the following functors:

$$\begin{array}{ccc} \mathsf{mod}\,\mathcal{A} & \underset{q_\lambda}{\overset{q}{\rightleftarrows}} & \mathsf{mod}\,\mathcal{B} \\ D \updownarrow & & \updownarrow D \\ \mathsf{mod}(\mathcal{A}^{\mathsf{op}}) & \underset{q'_\lambda}{\overset{q'}{\rightleftarrows}} & \mathsf{mod}(\mathcal{B}^{\mathsf{op}}). \end{array}$$

First we notice that $q \cong Dq'D$ holds by definition. Put $q_\rho := Dq'_\lambda D : \mathsf{mod}\,\mathcal{B} \to \mathsf{mod}\,\mathcal{A}$. It is easy to check that q and q_ρ form an adjoint pair $(q \dashv q_\rho)$.

In the remaining part of the proof, we shall verify the latter statement, namely, an isomorphism $q_\rho \cong \mathrm{Hom}_{\mathcal{B}}({}_{\mathcal{A}}\mathcal{A}_{\mathcal{B}}, -)$. This can be verified by the following calculations. Since q_ρ is left-exact and preserves injective objects, we have only to check the values of q_ρ on injective $\mathcal{B}$-modules. Due to the duality $D : \mathsf{mod}\,\mathcal{B} \to \mathsf{mod}(\mathcal{B}^{\mathsf{op}})$, each injective $\mathcal{B}$-module is isomorphic to $D\mathcal{B}(B, -)$ for some $B \in \mathcal{B}$.

$$\begin{aligned} q_\rho(D\mathcal{B}(B,-)) &= Dq'_\lambda D(D\mathcal{B}(B,-)) \\ &\cong D(({}_{\mathcal{A}}\mathcal{A}_{\mathcal{B}}) \otimes_{\mathcal{B}} \mathcal{B}(B,-)) \\ &\cong \mathrm{Hom}_{\mathcal{B}^{\mathsf{op}}}(\mathcal{B}(B,-), D({}_{\mathcal{A}}\mathcal{A}_{\mathcal{B}})) \\ &\cong \mathrm{Hom}_{\mathcal{B}}({}_{\mathcal{A}}\mathcal{A}_{\mathcal{B}}, D\mathcal{B}(B,-)). \end{aligned}$$

Therefore $q_\rho \cong \mathrm{Hom}_{\mathcal{B}}({}_{\mathcal{A}}\mathcal{A}_{\mathcal{B}}, -)$ on $\mathsf{mod}\,\mathcal{B}$, and hence it is fully faithful. □

By a similar argument in the proof of Proposition 4.9, we obtain the following.

Proposition 4.10. *There exist the following adjoint pairs for the pair $(\mathcal{A}, \mathcal{B})$ of a dualizing k-variety $\mathcal{A}$ and its functorially finite full subcategory $\mathcal{B}$:*

$$\mathsf{mod}(\mathcal{A}/[\mathcal{B}]) \xrightarrow{\ e\ } \mathsf{mod}\,\mathcal{A}, \qquad e_\lambda, e_\rho : \mathsf{mod}\,\mathcal{A} \to \mathsf{mod}(\mathcal{A}/[\mathcal{B}])$$

where $e := p^*$ is the restriction functor induced by the canonical projection $p : \mathcal{A} \to \mathcal{A}/[\mathcal{B}]$. Moreover, we have isomorphisms $e_\rho \cong \mathrm{Hom}_{\mathcal{A}}({}_{\mathcal{A}/[\mathcal{B}]}(\mathcal{A}/[\mathcal{B}])_{\mathcal{A}}, -)$ and $e_\lambda \cong - \otimes_{\mathcal{A}} ({}_{\mathcal{A}}(\mathcal{A}/[\mathcal{B}])_{\mathcal{A}/[\mathcal{B}]})$.

Now we are ready to prove Theorem 4.5.

Proof of Theorem 4.5. By Proposition 4.9 and Proposition 4.10, we have four adjoint pairs $(q_\lambda \dashv q), (q \dashv q_\rho), (e_\lambda \dashv e)$ and $(e \dashv e_\rho)$ with q_ρ, q_λ and e fully faithful. By definition, $\mathrm{Ker}\, q$ is a full subcategory in $\mathsf{mod}\,\mathcal{A}$ consisting of functors which vanishes on $\mathcal{B}$. Due to Lemma 4.7, we have $\mathrm{Ker}\, q = \mathsf{mod}(\mathcal{A}/[\mathcal{B}])$. Hence they form a recollement. □

We end this section with applying Theorem 4.5 to the following special setting. Let Λ be a finite dimensional k-algebra and $\mathcal{B}$ a functorially finite subcategory of $\mathsf{mod}\,\Lambda$ containing Λ. Applying Theorem 4.5 to the pair $(\mathcal{B}, \mathsf{proj}\Lambda)$ yields the following recollement

$$\mathsf{mod}\,\underline{\mathcal{B}} \xrightarrow{\;e\;} \mathsf{mod}\,\mathcal{B} \xrightarrow{\;q\;} \mathsf{mod}\,\Lambda, \qquad e_\lambda, e_\rho : \mathsf{mod}\,\mathcal{B} \to \mathsf{mod}\,\underline{\mathcal{B}}, \quad q_\lambda, q_\rho : \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\mathcal{B}$$

where we identify $\mathsf{mod}(\mathsf{proj}\,\Lambda)$ with $\mathsf{mod}\,\Lambda$ via the equivalence $\mathsf{mod}(\mathsf{proj}\,\Lambda) \xrightarrow{\sim} \mathsf{mod}\,\Lambda, X \mapsto X(\Lambda)$. By Proposition 3.8, this recollement induces the following.

Corollary 4.11. *The functor q_ρ is one composed the Yoneda embedding $\mathsf{mod}\,\Lambda \to \mathsf{mod}(\mathsf{mod}\,\Lambda)$ with the natural restriction $\mathsf{mod}(\mathsf{mod}\,\Lambda) \to \mathsf{mod}\,\mathcal{B}$ and admits an exact right adjoint functor q. Moreover it induces an equivalence*

$$\frac{\mathsf{mod}\,\mathcal{B}}{\mathsf{mod}\,\underline{\mathcal{B}}} \xrightarrow{\sim} \mathsf{mod}\,\Lambda.$$

We call this generalized Auslander's formula*.*

By setting $\mathcal{B} = \mathsf{mod}\,\Lambda$, we recover classical Auslander's formula (see [Aus, p. 205] and [Len, p. 1] for definition).

Corollary 4.12. *The Yoneda embedding $\mathsf{mod}\,\Lambda \to \mathsf{mod}(\mathsf{mod}\,\Lambda)$ admits an exact right adjoint functor. Moreover it induces an equivalence* $\dfrac{\mathsf{mod}(\mathsf{mod}\,\Lambda)}{\mathsf{mod}(\underline{\mathsf{mod}}\Lambda)} \xrightarrow{\sim} \mathsf{mod}\,\Lambda$.

5 Application to Auslander-Bridger sequences

The aim of this section is to show a close relationship between recollements and Auslander-Bridger sequences. Throughout this section, we fix a dualizing k-variety $\mathcal{A}$. Let $\mathcal{B}$ be a functorially finite subcategory in $\mathsf{mod}\,\mathcal{A}$ which contains $\mathsf{proj}\,\mathcal{A}$ and $\mathsf{inj}\,\mathcal{A}$.

Firstly, we recall the definition of Auslander-Bridger sequence, following [IJ, Prop. 2.7]. For the category $\mathcal{B}$, we denote the $\mathcal{B}$-duality by $(-)^* := \mathrm{Hom}_{\mathcal{B}}(-, \mathcal{B})$. Note that the $\mathcal{B}$-duality yields a duality $(-)^* : \mathsf{proj}\mathcal{B} \xrightarrow{\sim} \mathsf{proj}(\mathcal{B}^{\mathsf{op}}),\ \mathcal{B}(-, B) \mapsto \mathcal{B}(B, -)$. Let $X \in \mathsf{mod}\,\mathcal{B}$ with a minimal projective presentation $\mathcal{B}(-, B_1) \xrightarrow{\alpha} \mathcal{B}(-, B_0) \to X \to 0$ and set

$$\mathrm{Tr}X := \mathrm{Cok}\,\alpha^*$$

in $\mathsf{mod}(\mathcal{B}^{\mathsf{op}})$, see [ABr].

Definition-Proposition 5.1. *For each object $X \in \mathsf{mod}\,\mathcal{B}$, there exists an exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathcal{B}^{\mathsf{op}}}(\mathrm{Tr}X, \mathcal{B}^{\mathsf{op}}) \to X \xrightarrow{\epsilon} X^{**} \to \mathrm{Ext}^2_{\mathcal{B}^{\mathsf{op}}}(\mathrm{Tr}X, \mathcal{B}^{\mathsf{op}}) \to 0,$$

which is called the Auslander-Bridger sequence of X.

For the convenience of the reader, we recall from [Aus, Prop. 6.3] and [IJ, Prop. 2.7] the construction of the Auslander-Bridger sequence of X. Take a minimal projective presentation $\mathcal{B}(-, B_0) \to \mathcal{B}(-, B_1) \to X \to 0$ of X. Taking a left $\mathcal{B}$-approximation of a cokernel of $B_0 \to B_1$ yields an exact sequence $B_0 \to B_1 \to B_2$. Again, by taking a left $\mathcal{B}$-approximation of a cokernel of $B_1 \to B_2$, we get an exact sequence

$$B_0 \to B_1 \to B_2 \to B_3 \tag{5.1.1}$$

in $\mathsf{mod}\,\mathcal{A}$. By the construction, the sequence (5.1.1) induces the exact sequence

$$\mathcal{B}(B_3, -) \xrightarrow{h} \mathcal{B}(B_2, -) \xrightarrow{g} \mathcal{B}(B_1, -) \xrightarrow{f} \mathcal{B}(B_0, -) \tag{5.1.2}$$

in $\mathsf{mod}(\mathcal{B}^{\mathsf{op}})$. Note that $X^* = \mathrm{Ker}\, f$ and $\mathrm{Tr}X = \mathrm{Cok}\, f$. Taking the $\mathcal{B}^{\mathsf{op}}$-duality of (5.1.2) yields the following sequence

$$0 \to \mathrm{Hom}_{\mathcal{B}^{\mathsf{op}}}(\mathrm{Tr}X, \mathcal{B}^{\mathsf{op}}) \to \mathcal{B}(-, B_0) \xrightarrow{f^*} \mathcal{B}(-, B_1) \xrightarrow{g^*} \mathcal{B}(-, B_2) \xrightarrow{h^*} \mathcal{B}(-, B_3) \tag{5.1.3}$$

Note that $\mathrm{Cok}\, f^* \cong X$ and $\mathrm{Ker}\, h^* = X^{**}$. Since (5.1.3) is a complex, we have a canonical inclusion $i : \mathrm{Im}\, g^* \hookrightarrow \mathrm{Ker}\, h^*$ and a unique canonical epimorphism $\epsilon' : X \twoheadrightarrow \mathrm{Im}\, g^*$. It is readily verified that there exists a commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Im}\, f^* & \longrightarrow & \mathcal{B}(-, B_1) & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle \epsilon'} & & \\
0 & \longrightarrow & \mathrm{Ker}\, g^* & \longrightarrow & \mathcal{B}(-, B_1) & \longrightarrow & \mathrm{Im}\, g^* & \longrightarrow & 0 \\
 & & & & & & \downarrow{\scriptstyle i} & & \\
 & & & & & & X^{**} & &
\end{array}$$

We set $\epsilon := i \circ \epsilon'$. By the Snake Lemma, we have $\mathrm{Ker}\, \epsilon \cong \mathrm{Ker}\, \epsilon' \cong \mathrm{Ker}\, g^* / \mathrm{Im}\, f^*$. It is easy to verify that $\mathrm{Cok}\, \epsilon \cong \mathrm{Cok}\, i = \mathrm{Ker}\, h^* / \mathrm{Im}\, g^*$. Since (5.1.3) is the $\mathcal{B}^{\mathsf{op}}$-duality of the projective resolution of $\mathrm{Tr}X$, we have the isomorphisms $\mathrm{Ker}\, g^* / \mathrm{Im}\, f^* \cong \mathrm{Ext}^1_{\mathcal{B}^{\mathsf{op}}}(\mathrm{Tr}X, \mathcal{B}^{\mathsf{op}})$ and $\mathrm{Ker}\, h^* / \mathrm{Im}\, g^* \cong \mathrm{Ext}^2_{\mathcal{B}^{\mathsf{op}}}(\mathrm{Tr}X, \mathcal{B}^{\mathsf{op}})$. We have thus obtained the Auslander-Bridger sequence

$$0 \to \mathrm{Ext}^1_{\mathcal{B}^{\mathsf{op}}}(\mathrm{Tr}X, \mathcal{B}^{\mathsf{op}}) \to X \xrightarrow{\epsilon} X^{**} \to \mathrm{Ext}^2_{\mathcal{B}^{\mathsf{op}}}(\mathrm{Tr}X, \mathcal{B}^{\mathsf{op}}) \to 0.$$

We give an interpretation of the Auslander-Bridger sequences via the recollement appearing below. Due to Theorem 4.5, the pair $(\mathcal{B}, \mathsf{proj}\,\mathcal{A})$ induces the following recollement:

$$\mathsf{mod}\,\underline{\mathcal{B}} \xrightarrow{e} \mathsf{mod}\,\mathcal{B} \xrightarrow{q} \mathsf{mod}\,\mathcal{A}, \quad e_\lambda, e_\rho : \mathsf{mod}\,\mathcal{B} \to \mathsf{mod}\,\underline{\mathcal{B}}, \quad q_\lambda, q_\rho : \mathsf{mod}\,\mathcal{A} \to \mathsf{mod}\,\mathcal{B} \tag{5.1.4}$$

where we identify $\mathsf{mod}(\mathsf{proj}\,\mathcal{A})$ with $\mathsf{mod}\,\mathcal{A}$ via the equivalence $\mathsf{mod}(\mathsf{proj}\,\mathcal{A}) \xrightarrow{\sim} \mathsf{mod}\,\mathcal{A}$.

Theorem 5.2. *Let* $(\mathsf{mod}\,\underline{\mathcal{B}}, \mathsf{mod}\,\mathcal{B}, \mathsf{mod}\,\mathcal{A})$ *be a recollement (5.1.4). Then the right-defining exact sequence*

$$0 \to (ee_\rho)X \to X \to (q_\rho q)X \to X' \to 0$$

of $X \in \mathsf{mod}\,\mathcal{B}$ *is isomorphic to the Auslander-Bridger sequence of* X.

In the rest, we give a proof of Theorem 5.2. By Lemma 4.7, $\mathsf{mod}\,\underline{\mathcal{B}}$ is a full subcategory in $\mathsf{mod}\,\mathcal{B}$ consisting of objects X which admits a projective presentation

$$\mathcal{B}(-, B_1) \to \mathcal{B}(-, B_0) \to X \to 0$$

with $B_1 \to B_0$ an epimorphism in $\mathcal{B}$. Proposition 4.9 gives an explicit description of the functor q_ρ.

Lemma 5.3. *The functor* $q_\rho : \mathsf{mod}\,\mathcal{A} \to \mathsf{mod}\,\mathcal{B}$ *sends* A *to* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{B}, A)$.

We define the *2nd syzygy category* $\Omega^2(\mathsf{mod}\,\mathcal{B})$ *of* $\mathsf{mod}\,\mathcal{B}$ to be the full subcategory of $\mathsf{mod}\,\mathcal{B}$ consisting of objects X which admits an exact sequence $0 \to X \to \mathcal{B}(-, B_0) \to \mathcal{B}(-, B_1)$ for some $B_0, B_1 \in \mathcal{B}$.

Lemma 5.4. *We have the equality* $\mathrm{Im}\, q_\rho = \Omega^2(\mathsf{mod}\,\mathcal{B})$.

Proof. We show that $\mathrm{Im}\, q_\rho \subseteq \Omega^2(\mathsf{mod}\,\mathcal{B})$. Let $M \in \mathsf{mod}\,\mathcal{A}$. Due to $\mathsf{inj}\,\mathcal{A} \subseteq \mathcal{B}$, there exists an exact sequence $0 \to M \to B_0 \to B_1$ in $\mathsf{mod}\,\mathcal{A}$ with $B_0, B_1 \in \mathcal{B}$. Applying q_ρ to the above exact sequence gives an exact sequence $0 \to q_\rho M \to q_\rho B_0 \to q_\rho B_1$. Since $q_\rho(B_i) \cong \mathcal{B}(-, B_i)$ for $i = 0, 1$, we have $q_\rho M \in \Omega^2(\mathsf{mod}\,\mathcal{B})$.

To show the converse, take an object $X \in \Omega^2(\mathsf{mod}\,\mathcal{B})$ with an exact sequence $0 \to X \to \mathcal{B}(-, B_0) \xrightarrow{f\circ -} \mathcal{B}(-, B_1)$. Then $X \cong q_\rho(\mathrm{Ker}\, f) \in \mathrm{Im}\, q_\rho$ follows. This finishes the proof. □

Now we are ready to prove Theorem 5.2.

Proof. Due to Proposition 3.9, it is enough to show that $X^{**} \in \mathrm{Im}\, q_\rho$ and $q(\epsilon)$ is an isomorphism.

(i) Since we get an exact sequence $0 \to X^{**} \to \mathcal{B}(-, B_2) \xrightarrow{h^*} \mathcal{B}(-, B_3)$ from (5.1.3), $X^{**} \in \Omega^2(\mathsf{mod}\,\mathcal{B})$ holds.

(ii) Since q is a restriction functor with respect to the subcategory $\mathsf{proj}\mathcal{A}$, we evaluate the sequence (5.1.3) on $P \in \mathsf{proj}\mathcal{A}$. Since the sequence $B_0 \to B_1 \to B_2 \to B_3$ is exact, we have the following commutative diagram

$$\begin{array}{ccccccc} \mathcal{B}(P, B_0) & \xrightarrow{f^*(P)} & \mathcal{B}(P, B_1) & \xrightarrow{g^*(P)} & \mathcal{B}(P, B_2) & \xrightarrow{h^*(P)} & \mathcal{B}(P, B_3) \\ & & \downarrow & & \uparrow & & \\ & & X(P) & \xrightarrow{\epsilon(P)} & \mathrm{Im}\, g^*(P), & & \end{array}$$

with the first row exact. Since $\mathrm{Cok}\, f^* \cong X$, $\epsilon(P) = q(\epsilon)$ is an isomorphism. □

6 Application to Auslander-Reiten theory

Throughout this section let $\mathcal{A}$ be a dualizing k-variety and n a positive integer. We recall the notion of n-cluster tilting subcategory in $\mathsf{mod}\,\mathcal{A}$. Let $\mathcal{B}$ be a subcategory of $\mathsf{mod}\,\mathcal{A}$. For convenience, we define the full subcategories $\mathcal{B}^{\perp_n}$ and ${}^{\perp_n}\mathcal{B}$ by

$$\begin{aligned} \mathcal{B}^{\perp_n} &:= \{M \in \mathsf{mod}\,\mathcal{A} \mid i \in \{1, \dots, n\}\, \mathrm{Ext}^i_{\mathcal{A}}(\mathcal{B}, M) = 0\}, \\ {}^{\perp_n}\mathcal{B} &:= \{M \in \mathsf{mod}\,\mathcal{A} \mid i \in \{1, \dots, n\}\, \mathrm{Ext}^i_{\mathcal{A}}(M, \mathcal{B}) = 0\}. \end{aligned}$$

Definition 6.1. [Iya07b, Def. 2.2] A functorially finite subcategory $\mathcal{B}$ in $\mathsf{mod}\,\mathcal{A}$ together with $n \in \mathbb{N}$ is said to be *n-cluster-tilting* if the equalities $\mathcal{B} = {}^{\perp_{n-1}}\mathcal{B} = \mathcal{B}^{\perp_{n-1}}$ hold.

Note that 1-cluster tilting subcategory is nothing other than $\mathsf{mod}\,\mathcal{A}$. It is obvious that every n-cluster tilting subcategory contains $\mathsf{proj}\,\mathcal{A}$ and $\mathsf{inj}\,\mathcal{A}$, since $\mathrm{Ext}^i_{\mathcal{A}}(\mathsf{proj}\,\mathcal{A}, -)$ and $\mathrm{Ext}^i_{\mathcal{A}}(-, \mathsf{inj}\,\mathcal{A})$ is zero for any $i > 0$. This fact forces each right $\mathcal{B}$-approximation $B \to M$ of M to be an epimorphism in $\mathsf{mod}\,\mathcal{A}$, for every $M \in \mathsf{mod}\,\mathcal{A}$. Dually each left $\mathcal{B}$-approximation is a monomorphism.

In this section we always assume that $\mathcal{B}$ *denotes an n-cluster-tilting subcategory in* $\mathsf{mod}\,\mathcal{A}$. We collect some facts for later use. The following notion is instrumental in this section.

Definition 6.2. [Jas, Def. 2.4] Let $\mathcal{B}$ be an n-cluster-tilting subcategory in $\mathsf{mod}\,\mathcal{A}$. A complex $\delta : 0 \to B_{n+1} \to B_n \to \cdots \to B_0 \to 0$ in $\mathcal{B}$ is said to be

(b) *right n-exact* if the induced sequence

$$0 \to \mathcal{B}(B_0, -) \to \cdots \to \mathcal{B}(B_n, -) \to \mathcal{B}(B_{n+1}, -)$$

are exact in $\mathsf{mod}(\mathcal{B}^{\mathsf{op}})$;

(a) *left n-exact* if the induced sequence

$$0 \to \mathcal{B}(-, B_{n+1}) \to \mathcal{B}(-, B_n) \to \cdots \to \mathcal{B}(-, B_0)$$

are exact in $\mathsf{mod}\,\mathcal{B}$;

(c) *n-exact* if it is right n-exact and left n-exact.

Recall the following basic properties of n-cluster tilting subcategory $\mathcal{B}$.

Lemma 6.3. [Jas, Thm. 3.16] *The following hold for an n-cluster-tilting subcategory $\mathcal{B}$.*

(a) *Each morphism $B_{n+1} \to B_n$ in $\mathcal{B}$ can be embedded in a right n-exact sequence $\delta : B_{n+1} \to B_n \to \cdots \to B_0 \to 0$.*

(b) *Each monomorphism $B_{n+1} \to B_n$ in $\mathcal{B}$ can be embedded in an n-exact sequence $\delta : 0 \to B_{n+1} \to B_n \to \cdots \to B_0 \to 0$.*

(a') *Each morphism $B_1 \to B_0$ in $\mathcal{B}$ can be embedded in a left n-exact sequence $\delta : 0 \to B_{n+1} \to \cdots \to B_1 \to B_0$.*

(b') *Each epimorphism $B_1 \to B_0$ in $\mathcal{B}$ can be embedded in an n-exact sequence $\delta : 0 \to B_{n+1} \to \cdots \to B_1 \to B_0 \to 0$.*

Moreover, δ is uniquely determined up to homotopy in every case.

As an immediate consequence, we have the following lemma.

Lemma 6.4. *Let $\mathcal{B}$ be an n-cluster-tilting subcategory in $\mathsf{mod}\,\mathcal{A}$. Then we have $\mathsf{pd}_{\mathcal{B}}(X) \le n+1$ for any $X \in \mathsf{mod}\,\mathcal{B}$.*

Proof. Let X be an object in $\mathsf{mod}\,\mathcal{B}$ with a projective presentation $\mathcal{B}(-, B_1) \xrightarrow{\beta\circ-} \mathcal{B}(-, B_0) \to X \to 0$. By Lemma 6.3, there exists a left n-exact sequence $0 \to B_{n+1} \to \cdots \to B_1 \xrightarrow{\beta} B_0$. This gives rise to a projective resolution of X which shows that $\mathsf{pd}_{\mathcal{B}}(X) \le n+1$. $\square$

As a generalization of Auslander's defect introduced in [Aus] (see also Section IV. 4 in [ARS]), we define the following concepts, which were introduced by Jasso and Kvamme independently.

Definition 6.5. [JK] Let $\delta : 0 \to B_{n+1} \to B_n \to \cdots \to B_0 \to 0$ be an n-exact sequence in $\mathcal{B}$. The *contravariant n-defect* δ^{*n} and the *covariant n-defect* δ_{*n} are defined by the exactness of the following sequences:

$$0 \to \mathcal{B}(-, B_{n+1}) \to \mathcal{B}(-, B_n) \to \cdots \to \mathcal{B}(-, B_0) \to \delta^{*n} \to 0,$$
$$0 \to \mathcal{B}(B_0, -) \to \cdots \to \mathcal{B}(B_n, -) \to \mathcal{B}(B_{n+1}, -) \to \delta_{*n} \to 0.$$

We give the following characterization of n-defects.

Proposition 6.6. *The full subcategory of contravariant n-defects equals to $\mathsf{mod}\,\underline{\mathcal{B}}$ in $\mathsf{mod}\,\mathcal{B}$. Dually the full subcategory of covariant n-defects equals to $\mathsf{mod}(\overline{\mathcal{B}}^{\mathsf{op}})$ in $\mathsf{mod}(\mathcal{B}^{\mathsf{op}})$.*

Proof. We only prove the former assertion. Consider $X \in \mathsf{mod}\,\mathcal{B}$ with a projective presentation $\mathcal{B}(-, B_1) \xrightarrow{\mathcal{B}(-,f_0)} \mathcal{B}(-, B_0) \to X \to 0$. Assume that X belongs to $\mathsf{mod}\,\underline{\mathcal{B}}$. Since X vanishes on $P \in \mathsf{proj}\,\mathcal{A}$, the map f_0 is an epimorphism in $\mathsf{mod}\,\mathcal{A}$. By Lemma 6.3, the map f_0 can be embedded in an n-exact sequence $\delta : 0 \to B_{n+1} \to \cdots \to B_1 \xrightarrow{f_0} B_0 \to 0$. Hence $\delta^{*n} \cong X$.

Conversely, we shall show that contravariant n-defect δ^{*n} belongs to $\mathsf{mod}\,\underline{\mathcal{B}}$. The corresponding n-exact sequence $\delta : 0 \to B_{n+1} \to \cdots \to B_1 \to B_0 \to 0$ induces an exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(P, B_1) \to \mathrm{Hom}_{\mathcal{A}}(P, B_0) \to \delta^{*n}(P) \to 0$$

for any $P \in \mathsf{proj}\,\mathcal{A}$. This concludes that $\delta^{*n}(P) = 0$, hence $\delta^{*n} \in \mathsf{mod}\,\underline{\mathcal{B}}$. $\square$

6.1 Auslander-Reiten duality

We shall construct the n-Auslander-Reiten duality from a viewpoint of functor category. First we shall show that there exists a duality between $\mathsf{mod}\,\underline{\mathcal{B}}$ and $\mathsf{mod}(\overline{\mathcal{B}}^{\mathsf{op}})$. We denote $\mathsf{C}(\mathcal{B})$ the category of complexes in $\mathcal{B}$. For convenience, we consider the homotopy category $\mathsf{K}(\mathcal{B})$ of $\mathsf{C}(\mathcal{B})$ and its full subcategory $\mathsf{K}^{n\text{-ex}}(\mathcal{B})$ consisting of n-exact sequences $\delta : 0 \to B_{n+1} \to \cdots \to B_1 \to b_0 \to 0$ with the degree of b_0 being zero. The following proposition is a key to construct n-Auslander-Reiten dualiy.

Proposition 6.7. *For n-exact sequences $\delta : 0 \to B_{n+1} \to \cdots \to B_1 \to B_0 \to 0$ and $\delta' : 0 \to B'_{n+1} \to \cdots \to B'_1 \to B'_0 \to 0$, the following are equivalent.*

(i) *The sequence δ is homotopy equivalent to δ'.*

(ii) *There exists an isomorphism $\delta^{*n} \cong \delta'^{*n}$.*

(iii) *There exists an isomorphism $\delta_{*n} \cong \delta'_{*n}$.*

Moreover, we have a duality $\Phi : \mathsf{mod}(\overline{\mathcal{B}}^{\mathsf{op}}) \simeq \mathsf{mod}\,\underline{\mathcal{B}}$ sending δ_{*n} to δ^{*n}.

Proof. (i)⇔(ii): We assume that δ is homotopy equivalent to δ', that is, there exists chain maps $\phi : \delta \to \delta'$ and $\psi : \delta' \to \delta$:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & B_{n+1} & \xrightarrow{\beta_{n+1}} & \cdots & \longrightarrow & B_1 & \xrightarrow{\beta_1} & B_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_{n+1}} & & & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \\
0 & \longrightarrow & B'_{n+1} & \longrightarrow & \cdots & \longrightarrow & B'_1 & \longrightarrow & B'_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_{n+1}} & & & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_0} & & \\
0 & \longrightarrow & B_{n+1} & \longrightarrow & \cdots & \longrightarrow & B_1 & \longrightarrow & B_0 & \longrightarrow & 0
\end{array}$$

with $1 - \psi\phi$ and $1 - \phi\psi$ null-homotopic. By a standard argument on the above diagram, we have an isomorphism $\delta^{*n} \cong \delta'^{*n}$. The converse is obvious. The implications (i)⇔(iii) can be proved by a dual argument of the above.

We shall show the later assertion. By Proposition 6.6, the functor $\mathrm{Cok} : \mathsf{K}^{n\text{-ex}}(\mathcal{B}) \to \mathsf{mod}\,\underline{\mathcal{B}}$ sending δ to $\mathrm{Cok}\,\mathrm{Hom}_{\mathcal{B}}(-, \beta_1) = \delta^{*n}$ is full and dense. To show it is faithful, we take a morphism $\phi : \delta \to \delta'$ in $\mathsf{K}^{n\text{-ex}}(\mathcal{B})$ such that $\mathrm{Cok}\,\phi = 0$. The condition $\mathrm{Cok}\,\phi = 0$ forces that ϕ is null-homotopic as follows: Via the Yoneda embedding $\mathcal{B} \to \mathsf{mod}\,\mathcal{B}$ the morphism ϕ induces a chain map $\mathrm{Hom}_{\mathcal{B}}(-, \phi) : \mathrm{Hom}_{\mathcal{B}}(-, \delta) \to \mathrm{Hom}_{\mathcal{B}}(-, \delta')$ of complexes in $\mathsf{mod}\,\mathcal{B}$. Since $\mathrm{Cok}\,\mathrm{Hom}_{\mathcal{B}}(-, \phi) = 0$, it follows that $\mathrm{Hom}_{\mathcal{B}}(-, \phi)$ is zero in homology. Since $\mathrm{Hom}_{\mathcal{B}}(-, \delta)$ and $\mathrm{Hom}_{\mathcal{B}}(-, \delta')$ are complexes with projective components, we get that $\mathrm{Hom}_{\mathcal{B}}(-, \phi)$ must be null-homotopic. Hence, since the Yoneda embedding $\mathcal{B} \to \mathsf{mod}\,\mathcal{B}$ is full and faithful, it follows that ϕ is null-homotopic. Therefore Cok gives an equivalence. Dually we have a duality $\mathsf{K}^{n\text{-ex}}(\mathcal{B}) \to \mathsf{mod}(\overline{\mathcal{B}}^{\mathsf{op}})$ sending δ to $\mathrm{Cok}\,\mathrm{Hom}_{\mathcal{B}}(\beta_{n+1}, -) = \delta_{*n}$. It is obvious that the composed functor

$$\Phi : \mathsf{mod}(\overline{\mathcal{B}}^{\mathsf{op}}) \xrightarrow{\sim} \mathsf{K}^{n\text{-ex}}(\mathcal{B}) \xrightarrow{\sim} \mathsf{mod}\,\underline{\mathcal{B}}, \quad \delta_{*n} \mapsto \delta \mapsto \delta^{*n}$$

gives a desired duality. □

As we have seen in Section 4, the category $\underline{\mathcal{B}}$ is a dualizing k-variety and thus we have the duality $D : \mathsf{mod}\,\underline{\mathcal{B}} \xrightarrow{\sim} \mathsf{mod}(\underline{\mathcal{B}}^{\mathsf{op}})$. By composing the duality Φ in Proposition 6.7 with the duality D, we have the following equivalence.

Proposition 6.8. *There exists an equivalence $\sigma_n : \underline{\mathcal{B}} \xrightarrow{\sim} \overline{\mathcal{B}}$ which makes the following diagram commutative up to isomorphism:*

$$
\begin{array}{ccc}
\mathsf{mod}\,\underline{\mathcal{B}} & \xleftarrow{\ \Phi\ } & \mathsf{mod}(\overline{\mathcal{B}}^{\mathsf{op}}) \\
{\scriptstyle D}\downarrow & \swarrow{\scriptstyle -\circ\sigma_n} & \\
\mathsf{mod}(\underline{\mathcal{B}}^{\mathsf{op}}) & &
\end{array}
$$

Proof. It is clear that $D \circ \Phi$ gives the equivalence from $\mathsf{mod}(\overline{\mathcal{B}}^{\mathsf{op}})$ to $\mathsf{mod}(\underline{\mathcal{B}}^{\mathsf{op}})$. We restrict this onto their projective objects, that is, $\mathsf{proj}(\overline{\mathcal{B}}^{\mathsf{op}}) \simeq \mathsf{proj}(\underline{\mathcal{B}}^{\mathsf{op}})$. Thus we have the equivalence $\sigma_n : \underline{\mathcal{B}} \xrightarrow{\sim} \overline{\mathcal{B}}$ which makes the above diagram commutative up to isomorphisms. $\square$

By the dual argument, we have the equivalence $\sigma_n^- : \overline{\mathcal{B}} \to \underline{\mathcal{B}}$ which makes the following diagram commutative up to isomorphisms:

$$
\begin{array}{ccc}
\mathsf{mod}\,\underline{\mathcal{B}} & \xrightarrow{\ \Phi^{-1}\ } & \mathsf{mod}(\overline{\mathcal{B}}^{\mathsf{op}}) \\
& \searrow{\scriptstyle -\circ\sigma_n^-} & \downarrow{\scriptstyle D} \\
& & \mathsf{mod}\,\overline{\mathcal{B}}
\end{array}
$$

As an immediate consequence of the above diagrams, we have the higher defect formula. Moreover, as a special case of the higher defect formula we obtain the higher Auslander-Reiten duality by using a modification of Krause's proof of the classical formula (see [Kra03]).

Theorem 6.9. *There exist the following formulas.*

(a) (Higher defect formula) *functorial isomorphisms $D\delta^{*n} \cong \delta_{*n} \circ \sigma_n$ and $D\delta_{*n} \cong \delta^{*n} \circ \sigma_n^-$.*

(b) (Higher Auslander-Reiten duality) *bifunctorial isomorphisms $\underline{\mathcal{B}}(\sigma_n^- y, x) \cong D\,\mathrm{Ext}^n_{\mathcal{A}}(x, y) \cong \overline{\mathcal{B}}(y, \sigma_n x)$ in $x, y \in \mathcal{B}$.*

Proof. (a) It directly follows from Proposition 6.8 the fact that the duality $\Phi : \mathsf{mod}(\overline{\mathcal{B}}^{\mathsf{op}}) \to \mathsf{mod}\,\underline{\mathcal{B}}$ sends δ_{*n} to δ^{*n} (Proposition 6.7).

(b) We only prove the second isomorphism. Fix an object $C \in \mathcal{B}$. Let $C \hookrightarrow I(C)$ be an injective hull of C in $\mathsf{mod}\,\mathcal{A}$. Complete the n-exact sequence $\delta : 0 \to C \hookrightarrow I(C) \to B_{n-1} \to \cdots \to B_0 \to 0$. By Proposition 6.8, we have the isomorphisms $D\delta^{*n} \cong D\Phi(\delta_{*n}) \cong \delta_{*n} \circ \sigma_n$. By [Iya11, Lem. 3.5], we have the exact sequence

$$0 \to \mathcal{B}(-, C) \to \mathcal{B}(-, I(C)) \to \mathcal{B}(-, B_{n-1}) \to \cdots \to \mathcal{B}(-, B_0) \to \mathrm{Ext}^n_{\mathcal{A}}(-, C) \to \mathrm{Ext}^n_{\mathcal{A}}(-, I(C))$$

on $\mathcal{B}$. Since $\mathrm{Ext}^n_{\mathcal{A}}(-, I(C)) = 0$, we conclude $\delta^{*n} \cong \mathrm{Ext}^n_{\mathcal{A}}(-, C)$. Since $C \hookrightarrow I(C)$ is an injective hull, the exact sequence

$$0 \to \mathcal{B}(B_0, -) \to \cdots \to \mathcal{B}(B_{n-1}, -) \to \mathcal{B}(I(C), -) \to \mathcal{B}(C, -) \to \delta_{*n} \to 0$$

shows the isomorphism $\delta_{*n} \cong \overline{\mathcal{B}}(C, -)$. Therefore we obtain the desired isomorphism $D\,\mathrm{Ext}^n_{\mathcal{A}}(-, C) \cong \overline{\mathcal{B}}(C, \sigma_n(-))$. $\square$

The isomorphisms in Theorem 6.9(b) are nothing other than n-Auslander-Reiten duality. In particular, the functor σ_n (resp. σ^n) coincides with the n-Auslander-Reiten translation τ_n (resp. τ^n).

We recall the notion of the n-Auslander-Reiten duality. Let

$$\tau : \underline{\mathsf{mod}}\mathcal{A} \to \overline{\mathsf{mod}}\mathcal{A} \text{ and } \tau^- : \overline{\mathsf{mod}}\mathcal{A} \to \underline{\mathsf{mod}}\mathcal{A}$$

be the Auslander-Reiten translations. As a higher version of the Auslander-Reiten translation, the notion of n-Auslander-Reiten translation is defined as follows. We denote the n-th syzygy (resp. n-th cosyzygy) functor by $\Omega^n : \underline{\mathsf{mod}}\mathcal{A} \to \underline{\mathsf{mod}}\mathcal{A}$ (resp. $\Omega^{-n} : \overline{\mathsf{mod}}\mathcal{A} \to \overline{\mathsf{mod}}\mathcal{A}$).

Definition-Theorem 6.10. [Iya07b, Thm. 1.4.1] *The* n-Auslander-Reiten translations *are defined to be the functors*

$$\begin{aligned} \tau_n &:= \tau\Omega^{n-1} : \underline{\mathsf{mod}}\mathcal{A} \xrightarrow{\Omega^{n-1}} \underline{\mathsf{mod}}\mathcal{A} \xrightarrow{\tau} \overline{\mathsf{mod}}\mathcal{A}, \\ \tau_n^- &:= \tau^-\Omega^{-(n-1)} : \overline{\mathsf{mod}}\mathcal{A} \xrightarrow{\Omega^{-(n-1)}} \overline{\mathsf{mod}}\mathcal{A} \xrightarrow{\tau^-} \underline{\mathsf{mod}}\mathcal{A}. \end{aligned}$$

These functors induce mutually quasi-inverse equivalences

$$\tau_n : \underline{\mathcal{B}} \to \overline{\mathcal{B}} \text{ and } \tau_n^- : \overline{\mathcal{B}} \to \underline{\mathcal{B}}.$$

We have the following analog of Theorem 6.9(ii).

Proposition 6.11. [Iya07b, Thm. 1.5] *There exist bifunctorial isomorphisms* $\underline{\mathcal{B}}(\tau_n^- C, B) \cong D\operatorname{Ext}^n_A(B, C) \cong \overline{\mathcal{B}}(C, \tau_n B)$ *in* $B, C \in \mathcal{B}$.

Combining results above, we obtain the following explicit form of σ_n and σ_n^-.

Theorem 6.12. *The functor* σ_n *and* σ_n^- *are isomorphic to the* n*-Auslander-Reiten translations* τ_n *and* τ_n^-*, respectively.*

Proof. Theorem 6.9 and Theorem 6.11 gives an isomorphism $\overline{\mathcal{B}}(C, \sigma_n B) \cong \overline{\mathcal{B}}(C, \tau_n B)$. By Yoneda Lemma, we have an isomorphism $\sigma_n \cong \tau_n$. □

Note that Theorem 6.9 is independently obtained by Jasso and Kvamme in [JK, Theorem 3.7, Corollary 3.8]. The proof is different since we proved the higher defect formula in Theorem 6.9 without using the explicit form of τ_n.

6.2 Almost split sequences

As an application of the duality $\Phi : \mathsf{mod}\,\underline{\mathcal{B}} \xrightarrow{\sim} \mathsf{mod}(\overline{\mathcal{B}}^{\mathsf{op}})$, we give a characterization fo almost split sequences. Recall that a morphism $f : M \to L$ in $\mathcal{B}$ is said to be *right almost split* if it is a non-split epimorphism and each non-split epimorphism $g : N \to L$ factors through f. The notion of *left almost split* is defined dually. We firstly show the following well-known lemma, e.g. [Iya08, Prop. 2.10].

Lemma 6.13. *Let*

$$\delta: 0 \to N \xrightarrow{\beta_{n+1}} B_n \to \cdots \to B_1 \xrightarrow{\beta_1} L \to 0 \tag{6.13.1}$$

be an n-exact sequence in $\mathcal{B}$ with morphisms β_i lying in the Jacobson radical of $\mathsf{mod}\,\mathcal{A}$ *for all* $1 \le i \le n+1$. *Then the following are equivalent:*

(i) β_1 *is right almost split;*

(ii) β_{n+1} *is left almost split.*

Proof. (i) $\Rightarrow$(ii): Since β_1 is right almost split, we get that L is indecomposable. Again, since β_1 is right almost split, the morphism $\beta_1 : B_1 \to L$ induces an epimorphism $\mathcal{B}(-, B_1) \xrightarrow{\beta_1 \circ -} \mathrm{rad}_{\mathcal{B}}(-, L) \to 0$, where the symbol $\mathrm{rad}_{\mathcal{B}}(M, M')$ denotes the subspace of $\mathcal{B}(M, M')$ consisting of morphisms lying in the Jacobson radical of $\mathsf{mod}\,\mathcal{A}$ for any $M, M' \in \mathcal{B}$. Thus we get an isomorphism $\delta^{*n} \cong \mathcal{B}(-, L)/\,\mathrm{rad}_{\mathcal{B}}(-, L)$. In particular, the contravariantly n-defect δ^{*n} is a simple $\mathcal{B}$-module.

Due to the duality $\Phi : \mathsf{mod}\,\underline{\mathcal{B}} \to \mathsf{mod}(\overline{\mathcal{B}}^{\mathsf{op}})$, the covariant n-defect δ_{*n} is a simple $\mathcal{B}^{\mathsf{op}}$-module. The induced projective resolution

$$0 \to \mathcal{B}(L, -) \to \mathcal{B}(B_1, -) \to \cdots \to \mathcal{B}(B_n, -) \xrightarrow{- \circ \beta_{n+1}} \mathcal{B}(N, -) \to \delta_{*n} \to 0$$

of the $\mathcal{B}^{\mathsf{op}}$-module δ_{*n} guarantees $\delta_{*n}(N) \neq 0$. Since δ_{*n} is simple and β_{n+1} lies in $\mathrm{rad}_{\mathcal{B}}(N, B_n)$, we have that N is indecomposable. Thus δ_{*n} is isomorphic to $\mathcal{B}(N, -)/\,\mathrm{rad}_{\mathcal{B}}(N, -)$, which shows that β_{n+1} is left almost split.

The converse can be proved dually. □

When the equivalent conditions in Lemma 6.13 are satisfied, we call the sequence (6.13.1) an *n-almost split sequence*. As a corollary, we have the following one-to-one correspondence.

Corollary 6.14. *Let $\delta : 0 \to N \xrightarrow{\beta_{n+1}} B_n \to \cdots \to B_1 \xrightarrow{\beta_1} L \to 0$ be an n-exact sequence in $\mathcal{B}$. Then the assignment $\delta \mapsto \delta^{*n}$ gives a one-to-one correspondence between the isomorphism class of n-almost split sequences and the isomorphism class of simple $\mathcal{B}$-modules of projective dimension $n+1$.*

Proof. By the proof of Lemma 6.13, the n-almost split sequence δ corresponds to the simple $\mathcal{B}$-module δ^{*n}. Since each β_i belongs to the Jacobson radical of $\mathsf{mod}\,\mathcal{A}$, the induced projective resolution of δ^{*n} is minimal. This guarantees that the assignment is injective.

To show the assignment is surjective, let S be a simple $\mathcal{B}$-module of projective dimension $n+1$. There uniquely exists an indecomposable $\mathcal{A}$-module L such that $S \cong \mathcal{B}(-, L)/\,\mathrm{rad}_{\mathcal{B}}(-, L)$. Hence, on the minimal projective presentation $\mathcal{B}(-, B_1) \xrightarrow{\beta_1 \circ -} \mathcal{B}(-, L) \to S \to 0$, β_1 is right almost split. We shall show that the morphism $\beta_1 : B_1 \to L$ is an epimorphism in $\mathsf{mod}\,\mathcal{A}$. It suffices to show that $\beta_1 \circ - : \mathcal{B}(L, B) \to \mathcal{B}(B_1, B)$ is a monomorphism for any B in $\mathcal{B}$. Applying $\mathrm{Hom}_{\mathcal{B}}(-, \mathcal{B}(-, B))$ to the projective presentation of S provides an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{B}}(S, \mathcal{B}(-, B)) \to \mathcal{B}(L, B) \xrightarrow{- \circ \beta_1} \mathcal{B}(B_1, B).$$

We have $\mathrm{Hom}_{\mathcal{B}}(S, \mathcal{B}(-, B)) = 0$, indeed otherwise S is a submodule of $\mathcal{B}(-, B)$. By Lemma 6.4, this implies $\mathsf{pd}S \leq n$ which contradicts to the assumption $\mathsf{pd}S = n + 1$. Thus we conclude that β_1 is an epimorphism. Due to Lemma 6.3, there exists an n-almost split sequence

$$\delta : 0 \to N \to B_n \to \cdots \to B_1 \to L \to 0$$

such that $\delta^{*n} \cong S$. □

Part II

Singular equivalences via Auslander-Buchweitz approximations

This part is based on the paper [Oga18].

7 A functor category version of Chen's theorem

The aim of this section to provide a sufficient condition for an additive category $\mathcal{A}$ and its subcategory $\mathcal{X}$ so that the canonical inclusion $\mathcal{X} \hookrightarrow \mathcal{A}$ induces a triangle equivalence $\mathsf{D_{sg}}(\mathcal{A}) \simeq \mathsf{D_{sg}}(\mathcal{X})$, which generalizes Xiao-Wu Chen's theorem.

The category $\mathsf{mod}\,\mathcal{A}$ is not necessarily abelian, however, if every morphism in $\mathcal{A}$ has weak-kernels, then $\mathsf{mod}\,\mathcal{A}$ is abelian (Lemma 3.1). Since we are interested in the case that $\mathsf{mod}\,\mathcal{A}$ is abelian, *throughout this section, let $\mathcal{A}$ be an additive category with weak-kernels and $\mathcal{X}$ its contravariantly finite full subcategory.* Due to Proposition 4.8, the canonical functor $Q : \mathsf{mod}\,\mathcal{A} \to \mathsf{mod}\,\mathcal{X}$ induces an equivalence

$$\frac{\mathsf{mod}\,\mathcal{A}}{\mathsf{mod}(\mathcal{A}/[\mathcal{X}])} \xrightarrow{\sim} \mathsf{mod}\,\mathcal{X}. \tag{7.0.1}$$

Moreover, by [Miy, Thm. 3.2], it induces a triangle equivalence

$$\frac{\mathsf{D^b}(\mathsf{mod}\,\mathcal{A})}{\mathsf{D^b_{\mathcal{A}/[\mathcal{X}]}}(\mathsf{mod}\,\mathcal{A})} \xrightarrow{\sim} \mathsf{D^b}(\mathsf{mod}\,\mathcal{X}).$$

Then we have the following commutative diagram

$$\begin{array}{ccccc} \mathsf{mod}(\mathcal{A}/[\mathcal{X}]) & \hookrightarrow & \mathsf{mod}\,\mathcal{A} & \xrightarrow{Q} & \mathsf{mod}\,\mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{D^b_{\mathcal{A}/[\mathcal{X}]}}(\mathsf{mod}\,\mathcal{A}) & \hookrightarrow & \mathsf{D^b}(\mathsf{mod}\,\mathcal{A}) & \xrightarrow{Q'} & \mathsf{D^b}(\mathsf{mod}\,\mathcal{X}) \end{array}$$

where the arrows of the shape $\hookrightarrow$ denote canonical inclusions, and Q' is the functor induced from Q. Note that $\mathsf{D^b_{\mathcal{A}/[\mathcal{X}]}}(\mathsf{mod}\,\mathcal{A})$ is the thick subcategory of $\mathsf{D^b}(\mathsf{mod}\,\mathcal{A})$ containing $\mathsf{mod}(\mathcal{A}/[\mathcal{X}])$. The following lemma gives a natural sufficient condition so that the canonical functor $\mathsf{D^b}(\mathsf{mod}\,\mathcal{A}) \to \mathsf{D^b}(\mathsf{mod}\,\mathcal{X})$ induces a triangle functor $\mathsf{D_{sg}}(\mathcal{A}) \to \mathsf{D_{sg}}(\mathcal{X})$.

Lemma 7.1. *The following conditions are equivalent:*

(i) $\mathsf{pd}_{\mathcal{X}}(\mathcal{A}(-, M)|_{\mathcal{X}}) < \infty$ *for any* $M \in \mathcal{A}$*;*

(ii) *The canonical functor* $Q' : \mathsf{D^b}(\mathsf{mod}\,\mathcal{A}) \to \mathsf{D^b}(\mathsf{mod}\,\mathcal{X})$ *restricts to* $Q' : \mathsf{K^b}(\mathsf{proj}\,\mathcal{A}) \to \mathsf{K^b}(\mathsf{proj}\,\mathcal{X})$.

If this is the case, we have an induced triangle functor $\bar{Q} : \mathsf{D_{sg}}(\mathcal{A}) \to \mathsf{D_{sg}}(\mathcal{X})$.

Proof. (i) $\Leftrightarrow$ (ii): Since the functor $Q' : \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{A}) \to \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{X})$ restricts to $Q'|_{\mathsf{mod}\,\mathcal{A}} = Q : \mathsf{mod}\,\mathcal{A} \to \mathsf{mod}\,\mathcal{X}$, the condition (i) holds if and only if $Q'(\mathsf{proj}\,\mathcal{A}) \subseteq \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\mathcal{X})$ if and only if the condition (ii) holds.

The latter statement follows from the universality of the Verdier quotient. $\square$

Since our aim is to compare the singularity categories $\mathsf{D}_{\mathsf{sg}}(\mathcal{A})$ and $\mathsf{D}_{\mathsf{sg}}(\mathcal{X})$, it is natural to assume that the equivalent conditions in Lemma 7.1 are satisfied. Our main result gives a necessary and sufficient condition so that the canonical inclusion $\mathcal{X} \hookrightarrow \mathcal{A}$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\mathcal{A}) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\mathcal{X})$.

Theorem 7.2. *We assume that* $\mathsf{pd}_{\mathcal{X}}(\mathcal{A}(-,M)|_{\mathcal{X}}) < \infty$ *for any* $M \in \mathcal{A}$. *Then the following conditions are equivalent:*

(i) $\mathsf{pd}_{\mathcal{A}}(F) < \infty$ *for any* $F \in \mathsf{mod}(\mathcal{A}/[\mathcal{X}])$*;*

(ii) *The induced functor* $\bar{Q} : \mathsf{D}_{\mathsf{sg}}(\mathcal{A}) \to \mathsf{D}_{\mathsf{sg}}(\mathcal{X})$ *is a triangle equivalence.*

To prove Theorem 7.2, we firstly show Proposition 7.3 in a more general framework: Let $\mathcal{T}$ be a triangulated category with a translation [1]. For a class $\mathcal{S}$ of objects in $\mathcal{T}$, we denote by $\mathsf{tri}\,\mathcal{S}$ the smallest triangulated full subcategory of $\mathcal{T}$ containing $\mathcal{S}$. For two classes $\mathcal{U}$ and $\mathcal{V}$ of objects in $\mathcal{T}$, we denote by $\mathcal{U} * \mathcal{V}$ the class of objects X occurring in a triangle $U \to X \to V \to U[1]$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Note that the operation $*$ is associative by the octahedral axiom.

Proposition 7.3. *Let* $\mathcal{U}$ *and* $\mathcal{V}$ *be triangulated full subcategories of* $\mathcal{T}$ *and consider the Verdier quotients with respect to them:*

$$\mathcal{U} \to \mathcal{T} \xrightarrow{Q_1} \mathcal{T}/\mathcal{U} \quad \text{and} \quad \mathcal{V} \to \mathcal{T} \xrightarrow{Q_2} \mathcal{T}/\mathcal{V}.$$

Then, there exist natural triangle equivalences

$$\frac{\mathcal{T}/\mathcal{U}}{\mathsf{tri}(Q_1\mathcal{V})} \simeq \frac{\mathcal{T}}{\mathsf{tri}(\mathcal{U},\mathcal{V})} \simeq \frac{\mathcal{T}/\mathcal{V}}{\mathsf{tri}(Q_2\mathcal{U})},$$

where $Q_1\mathcal{V}$ *is the full subcategory of* $\mathcal{T}/\mathcal{U}$ *consisting of objects isomorphic to* Q_1V *for some* $V \in \mathcal{V}$*, and the symbol* $Q_2\mathcal{U}$ *is used in a similar meaning.*

Proof. We shall show an equality $\mathsf{tri}(Q_1\mathcal{V}) = \mathsf{tri}(\mathcal{U},\mathcal{V})/\mathcal{U}$, where $\mathsf{tri}(\mathcal{U},\mathcal{V})$ denotes the smallest triangulated full subcategory of $\mathcal{T}$ containing $\mathcal{U}$ and $\mathcal{V}$. We set $\mathcal{S} := \mathcal{U} \cup \mathcal{V}$. Obviously we have $Q_1\mathcal{S} = Q_1\mathcal{V}$. Since $\mathsf{tri}(\mathcal{U},\mathcal{V}) = \bigcup_{n\geq 0} \mathcal{S}^{*n}$, we have the following equalities:

$$\mathsf{tri}(\mathcal{U},\mathcal{V})/\mathcal{U} = Q_1\left(\bigcup_{n\geq 0} \mathcal{S}^{*n}\right) = \bigcup_{n\geq 0} (Q_1\mathcal{S})^{*n} = \bigcup_{n\geq 0} (Q_1\mathcal{V})^{*n} = \mathsf{tri}(Q_1\mathcal{V}).$$

Hence we have a desired triangle equivalence $\dfrac{\mathcal{T}/\mathcal{U}}{\mathsf{tri}(Q_1\mathcal{V})} = \dfrac{\mathcal{T}/\mathcal{U}}{\mathsf{tri}(\mathcal{U},\mathcal{V})/\mathcal{U}} \xrightarrow{\sim} \dfrac{\mathcal{T}}{\mathsf{tri}(\mathcal{U},\mathcal{V})}$. $\square$

Now we are ready to prove Theorem 7.2.

Proof of Theorem 7.2. Apply Proposition 7.3 for $\mathcal{T} = \mathsf{D^b}(\mathsf{mod}\,\mathcal{A}), \mathcal{U} = \mathsf{D^b}_{\mathsf{mod}(\mathcal{A}/[\mathcal{X}])}(\mathsf{mod}\,\mathcal{A})$ and $\mathcal{V} = \mathsf{K^b}(\mathsf{proj}\,\mathcal{A})$. Then $\mathcal{T}/\mathcal{U} = \mathsf{D^b}(\mathsf{mod}\,\mathcal{X})$ and $\mathcal{T}/\mathcal{V} = \mathsf{D_{sg}}(\mathcal{A})$. The assumption gives $Q_1\mathcal{V} = \mathsf{K^b}(\mathsf{proj}\,\mathcal{X})$. Hence $\frac{\mathcal{T}/\mathcal{U}}{Q_1\mathcal{V}} = \mathsf{D_{sg}}(\mathcal{X})$. Thus we have a triangle equivalence $\mathsf{D_{sg}}(\mathcal{X}) \simeq \frac{\mathsf{D_{sg}}(\mathcal{A})}{\mathsf{tri}(Q_2\mathcal{U})}$. This shows the condition (i) is equivalent to $Q_2\mathcal{U} = 0$, namely $\mathcal{U} \subset \mathcal{V}$, which is nothing but the condition (ii). □

We end this section with recovering the following Chen's theorem as a special case of Theorem 7.2 and Lemma 7.1.

Example 7.4. [Che, Thm. 1.3] (see also [PSS, Thm. 5.2], [KY, Prop. 3.3]) Let Λ be a Noetherian ring and e its idempotent. Assume that $\mathsf{pd}_{e\Lambda e}(\Lambda e) < \infty$. Then the canonical inclusion $e\Lambda e \hookrightarrow \Lambda$ induces a triangle functor $\bar{Q} : \mathsf{D_{sg}}(\Lambda) \to \mathsf{D_{sg}}(e\Lambda e)$, and the following are equivalent:

(i) $\mathsf{pd}_{\Lambda}(M) < \infty$ for any $M \in \mathsf{mod}(\Lambda/\Lambda e\Lambda)$;

(ii) The induced functor $\bar{Q} : \mathsf{D_{sg}}(\Lambda) \xrightarrow{\sim} \mathsf{D_{sg}}(e\Lambda e)$ is a triangle equivalence.

8 Sufficient conditions for singular equivalence

The aim of this section is to construct a singular equivalence from our generalized Auslander-Buchweitz condition (Condition 1.4). First we introduce some terminology. Let $\mathcal{C}$ denote an abelian category and let $\mathcal{C} \supseteq \mathcal{A} \supseteq \mathcal{B}$ be a sequence of full subcategories of $\mathcal{C}$. We call the kernel of a $\mathcal{B}$-epimorphism the *$\mathcal{B}$-epikernel*, for short. *We assume that $\mathcal{A}$ is closed under $\mathcal{B}$-epikernels and $\mathcal{B}$ is contravariantly finite in $\mathcal{A}$.* Then the ideal-quotient category $\mathcal{A}/[\mathcal{B}]$ admits weak-kernels. In fact, for a morphism $\alpha : M \to L$ of $\mathcal{A}$, we obtain its weak-kernel as follows: We take a right $\mathcal{B}$-approximation $\beta : B_L \to L$ of L, and consider an induced exact sequence

$$0 \to N \xrightarrow{(\gamma\ \delta)} M \oplus B_L \xrightarrow{\binom{\alpha}{\beta}} L$$

in $\mathcal{C}$. Since $\mathcal{A}$ is closed under $\mathcal{B}$-epikernels and the morphism $\binom{\alpha}{\beta}$ is an $\mathcal{B}$-epimorphism, we have $N \in \mathcal{A}$. It is basic that the morphism γ is a weak-kernel of α in $\mathcal{A}/[\mathcal{B}]$.

8.1 Singular equivalences from Auslander-Buchweitz approximation

In this subsection we give a proof of the following main theorem. First we recall our set-up:

Condition 8.1. Let $\mathcal{A} \supseteq \mathcal{X} \supseteq \boldsymbol{\omega}$ be a sequence of full subcategories in $\mathcal{C}$ such that $\mathcal{X}$ and $\boldsymbol{\omega}$ are contravariantly finite in $\mathcal{A}$. We consider the following conditions:

(AB1) $\mathcal{A}$ is closed under $\boldsymbol{\omega}$-epikernels.

(AB2) $\mathcal{X} \subseteq {}^{\perp}\boldsymbol{\omega}$.

(AB3) For any $M \in \mathcal{A}$, there exists an exact sequence $0 \to Y_M \to X_M \xrightarrow{f} M$ in $\mathcal{A}$ such that f is a right $\mathcal{X}$-approximation of M and $Y_M \in \widehat{\boldsymbol{\omega}}$.

Theorem 8.2. *Under Condition 8.1, the canonical inclusion $\overline{\mathcal{X}} \hookrightarrow \overline{\mathcal{A}}$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{A}}) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{X}})$.*

Let $\mathcal{C}$ be an abelian category with enough projectives and consider a sequence $\mathcal{A} \supseteq \mathcal{X} \supseteq \boldsymbol{\omega}$ of full subcategories in $\mathcal{C}$ such that $\mathcal{X}$ and $\boldsymbol{\omega}$ are contravariantly finite in $\mathcal{A}$. We always assume (AB1) in Condition 8.1.

Proposition 8.3. *The ideal-quotient $\overline{\mathcal{A}}$ admits weak-kernels and $\overline{\mathcal{X}}$ is its contravariantly finite full subcategory. Moreover, the canonical inclusion $\overline{\mathcal{X}} \hookrightarrow \overline{\mathcal{A}}$ induces the following equivalence*

$$\frac{\mathsf{mod}\,\overline{\mathcal{A}}}{\mathsf{mod}(\mathcal{A}/[\mathcal{X}])} \xrightarrow{\sim} \mathsf{mod}\,\overline{\mathcal{X}}.$$

Proof. Since $\mathcal{A}$ is closed under $\boldsymbol{\omega}$-epikernels, $\overline{\mathcal{A}}$ admits weak-kernels. Since $\mathcal{X}$ is contravariantly finite in $\mathcal{A}$, so is $\overline{\mathcal{X}}$ in $\overline{\mathcal{A}}$. Note that there exists an equivalence $\mathcal{A}/[\mathcal{X}] \simeq \overline{\mathcal{A}}/[\overline{\mathcal{X}}]$. By (7.0.1), we have a desired equivalence. □

To prove that the inclusion $\overline{\mathcal{X}} \hookrightarrow \overline{\mathcal{A}}$ induces a triangle functor $\mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{A}}) \to \mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{X}})$, we shall check a sufficient condition given in Lemma 7.1.

Lemma 8.4. *Assume (AB2) and (AB3). Let $X \in \mathcal{X}$ be given. Then,*

(a) *One has $\operatorname{Ext}^i_{\mathcal{C}}(X, I) = 0$ for any $I \in \widehat{\boldsymbol{\omega}}$ and $i > 0$.*

(b) *Every morphism $f : X \to I$ with $I \in \widehat{\boldsymbol{\omega}}$ factors through an object in $\boldsymbol{\omega}$.*

Proof. We only show the assertion (b). Since $I \in \widehat{\boldsymbol{\omega}}$, there exists an exact sequence $0 \to I' \to W \to I \to 0$ with $W \in \boldsymbol{\omega}$ and $I' \in \widehat{\boldsymbol{\omega}}$. Applying $\mathcal{C}(X, -)$, by (a), we conclude that f factors through W. □

Proposition 8.5. *Assume (AB2) and (AB3). Then the canonical inclusion* $\operatorname{inc} : \overline{\mathcal{X}} \hookrightarrow \overline{\mathcal{A}}$ *admits a right adjoint R. Moreover, we have $\mathsf{pd}_{\overline{\mathcal{X}}}(\overline{\mathcal{A}}(-, M)|_{\overline{\mathcal{X}}}) = 0$ for any $M \in \mathcal{A}$.*

Proof. The proof is similar to one given in [BR, Ch. V Prop. 1.2], but our situation is slightly different from that in loc. cit. So we include a detailed proof. By (AB3), for each $M \in \mathcal{A}$, there exists an exact sequence in $\mathcal{A}$

$$0 \to Y_M \to X_M \xrightarrow{\alpha} M$$

with α a right $\mathcal{X}$-approximation of M and $Y_M \in \widehat{\boldsymbol{\omega}}$. We shall show that the morphism $\overline{\mathcal{X}}(X, X_M) \xrightarrow{\alpha\circ -} \overline{\mathcal{A}}(X, M)$ is a functorial isomorphism in $X \in \mathcal{X}$. Its surjectivity is clear, since α is a right $\mathcal{X}$-approximation. To show its injectivity, take a morphism $h \in \mathcal{X}(X, X_M)$ such that $\alpha \circ h$ factors through an object I of $\boldsymbol{\omega}$. Thus we have the following commutative diagram:

$$\begin{array}{ccccccc} & & & & X & \xrightarrow{h'} & I \\ & & & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h''} \\ 0 & \longrightarrow & Y_M & \longrightarrow & X_M & \xrightarrow{\alpha} & M \end{array}$$

Since α is a right $\mathcal{X}$-approximation, there exists a morphism $\alpha' : I \to X_M$ such that $\alpha\alpha' = h''$. The morphism $h - \alpha' h'$ factors through $Y_M \in \widehat{\boldsymbol{\omega}}$. By Lemma 8.4(ii), this implies that $h - \alpha' h'$ factors through $\boldsymbol{\omega}$. Hence h factors through $\boldsymbol{\omega}$. By the Yoneda lemma, the assignment $M \mapsto X_M$ gives rise to a functor $R : \mathcal{A} \to \mathcal{X}$. The bifunctorial isomorphism $\overline{\mathcal{X}}(X, R(M)) \xrightarrow{\alpha\circ -} \overline{\mathcal{A}}(X, M)$ says the pair of functors (inc, R) forms an adjoint pair.

The latter statement is obvious. □

Proposition 8.6. *Let $\mathcal{B}$ be a contravariantly finite full subcategory of $\mathcal{A}$ and assume that $\mathcal{A}$ is closed under $\mathcal{B}$-epikernels. Let $F \in \mathsf{mod}(\mathcal{A}/[\mathcal{B}])$ be given. Then there exists an exact sequence*

$$0 \to N \xrightarrow{g} M \xrightarrow{f} L \tag{8.6.1}$$

in $\mathcal{A}$ which satisfies the following conditions:

(a) *The morphism f is a $\mathcal{B}$-epimorphism;*

(b) *The induces sequence*

$$0 \to \mathcal{A}(-, N) \xrightarrow{g\circ -} \mathcal{A}(-, M) \xrightarrow{f\circ -} \mathcal{A}(-, L) \to F \to 0$$

is exact.

In particular, $\mathsf{pd}_{\mathcal{A}}(F) \leq 2$.

Proof. First F is a finitely presented $\mathcal{A}$-module. Indeed, a right $\mathcal{B}$-approximation $B_Y \to Y$ of any $Y \in \mathcal{A}$ induces a projective presentation

$$\mathcal{A}(-, B_Y) \to \mathcal{A}(-, Y) \to \mathcal{A}/[\mathcal{B}](-, Y) \to 0$$

of the $\mathcal{A}$-module $\mathcal{A}/[\mathcal{B}](-, Y)$. This shows that $\mathcal{A}/[\mathcal{B}](-, Y)$ belongs to $\mathsf{mod}\,\mathcal{A}$, hence so does F.

Thus we have a projective presentation $\mathcal{A}(-, M) \xrightarrow{f\circ -} \mathcal{A}(-, L) \to F \to 0$ of the $\mathcal{A}$-module F. Since F vanishes on $\mathcal{B}$, the induced morphism f is a $\mathcal{B}$-epimorphism. Thus we have an exact sequence $0 \to N \xrightarrow{g} M \xrightarrow{f} L$ in $\mathcal{A}$. Applying the Yoneda embedding, we have a projective resolution $0 \to \mathcal{A}(-, N) \to \mathcal{A}(-, M) \to \mathcal{A}(-, L) \to F \to 0$ of the $\mathcal{A}$-module F. □

Let $M \in \mathcal{A}$ and $f : B_M \to M$ be a right $\mathcal{B}$-approximation of M. Then we write $\Omega_{\mathcal{B}}(M) := \operatorname{Ker} f$. We define $\Omega_{\mathcal{B}}^n(M)$ inductively for $n \geq 1$. We prove the following key-proposition which generalizes the well-known result given in [AR74, Prop. 4.1, 4.2] and [AR96, Prop. 1.2]. The proof is similar but a bit different from the original ones.

Proposition 8.7. *For $F \in \mathsf{mod}(\mathcal{A}/[\mathcal{B}])$, the exact sequence (8.6.1) in Proposition 8.6 induces a projective resolution*

$$\begin{aligned} \cdots &\longrightarrow \mathcal{A}/[\mathcal{B}](-, \Omega_{\mathcal{B}}^2(N)) \xrightarrow{\Omega_{\mathcal{B}}^2 g\circ -} \mathcal{A}/[\mathcal{B}](-, \Omega_{\mathcal{B}}^2(M)) \xrightarrow{\Omega_{\mathcal{B}}^2 f\circ -} \mathcal{A}/[\mathcal{B}](-, \Omega_{\mathcal{B}}^2(L)) \\ &\longrightarrow \mathcal{A}/[\mathcal{B}](-, \Omega_{\mathcal{B}}(N)) \xrightarrow{\Omega_{\mathcal{B}} g\circ -} \mathcal{A}/[\mathcal{B}](-, \Omega_{\mathcal{B}}(M)) \xrightarrow{\Omega_{\mathcal{B}} f\circ -} \mathcal{A}/[\mathcal{B}](-, \Omega_{\mathcal{B}}(L)) \\ &\longrightarrow \mathcal{A}/[\mathcal{B}](-, N) \xrightarrow{g\circ -} \mathcal{A}/[\mathcal{B}](-, M) \xrightarrow{f\circ -} \mathcal{A}/[\mathcal{B}](-, L) \longrightarrow F \longrightarrow 0 \end{aligned} \tag{8.7.1}$$

of the $\mathcal{A}/[\mathcal{B}]$-module F.

Proof. For the sequence (8.6.1), we take right $\mathcal{B}$-approximations $\alpha_L : B_L \to L$ and $\alpha_N : B_N \to N$. Since the morphism f is $\mathcal{B}$-epimorphism, we have a morphism $\beta : B_L \to M$ such that $\alpha_L = f \circ \beta$. The induced morphism $\alpha_M := \binom{\beta}{g\alpha_N} : B_M := B_L \oplus B_N \to M$ is a right $\mathcal{B}$-approximation of M. Since $\mathcal{A}$ is closed under $\mathcal{B}$-epikernels, we have the following commutative diagram in $\mathcal{A}$:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_{\mathcal{B}}(N) & \xrightarrow{\Omega_{\mathcal{B}} g} & \Omega_{\mathcal{B}}(M) & \xrightarrow{\Omega_{\mathcal{B}} f} & \Omega_{\mathcal{B}}(L) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_N & \longrightarrow & B_M & \longrightarrow & B_L & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_N} & & \downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha_L} & & \\
0 & \longrightarrow & N & \xrightarrow{g} & M & \xrightarrow{f} & L & &
\end{array}
$$

where all columns and rows are exact, and the middle row splits. Applying the Yoneda embedding and the Snake Lemma, we have the following commutative diagram in $\mathsf{mod}\,\mathcal{A}$.

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{A}(-,\Omega_{\mathcal{B}}(N)) & \longrightarrow & \mathcal{A}(-,\Omega_{\mathcal{B}}(M)) & \longrightarrow & \mathcal{A}(-,\Omega_{\mathcal{B}}(L)) & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{A}(-,B_N) & \longrightarrow & \mathcal{A}(-,B_M) & \longrightarrow & \mathcal{A}(-,B_L) & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle \alpha_N \circ -} & & \downarrow{\scriptstyle \alpha_M \circ -} & & \downarrow{\scriptstyle \alpha_L \circ -} & & & & \\
0 & \longrightarrow & \mathcal{A}(-,N) & \xrightarrow{g\circ -} & \mathcal{A}(-,M) & \xrightarrow{f\circ -} & \mathcal{A}(-,L) & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
 & & \mathcal{A}/[\mathcal{B}](-,N) & \longrightarrow & \mathcal{A}/[\mathcal{B}](-,M) & \longrightarrow & \mathcal{A}/[\mathcal{B}](-,L) & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}
$$

In particular, we have an exact sequence

$$
\begin{array}{l}
0 \longrightarrow \mathcal{A}(-,\Omega_{\mathcal{B}}(N)) \longrightarrow \mathcal{A}(-,\Omega_{\mathcal{B}}(M)) \longrightarrow \mathcal{A}(-,\Omega_{\mathcal{B}}(L)) \\
\quad \xrightarrow{\delta} \mathcal{A}/[\mathcal{B}](-,N) \longrightarrow \mathcal{A}/[\mathcal{B}](-,M) \longrightarrow \mathcal{A}/[\mathcal{B}](-,L) \longrightarrow F \longrightarrow 0
\end{array}
$$

in $\mathsf{mod}\,\mathcal{A}$. We have an exact sequence $0 \to \Omega_{\mathcal{B}}(N) \to \Omega_{\mathcal{B}}(M) \xrightarrow{\Omega_{\mathcal{B}} f} \Omega_{\mathcal{B}}(L)$ such that $\Omega_{\mathcal{B}} f$ is an $\mathcal{B}$-epimorphism. Inductively, we have a desired projective resolution of the $\mathcal{A}/[\mathcal{B}]$-module F. □

Lemma 8.8. *Under Condition 8.1,*

(a) *For any $L \in \mathcal{A}$, there exists $n \geq 0$ such that $\Omega^n_{\mathcal{X}}(L) \in \mathcal{X}$.*

(b) *For each $F \in \mathsf{mod}(\mathcal{A}/[\mathcal{X}])$, we have $\mathsf{pd}_{\mathcal{A}/[\mathcal{X}]}(F) < \infty$.*

Proof. (a) For an object $L \in \mathcal{A}$, due to (AB3), we get an exact sequence

$$0 \to Y \to X_0 \xrightarrow{f_0} L$$

such that f_0 is a right $\mathcal{X}$-approximation of L and $Y \in \widehat{\boldsymbol{\omega}}$. Since $Y \in \widehat{\boldsymbol{\omega}}$, we get an exact sequence

$$0 \to I_n \to I_{n-1} \xrightarrow{f_{n-1}} \cdots \to I_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} L$$

with $I_i \in \boldsymbol{\omega}$ for $1 \le i \le n$. By Lemma 8.4, each morphism $f_i : I_i \to \operatorname{Im} f_i$ is a right $\mathcal{X}$-approximation of $\operatorname{Im} f_i$ for each $1 \le i \le n$ hence $I_n = \Omega^n_{\mathcal{X}}(L) \in \mathcal{X}$.

(b) We consider the projective resolution (8.7.1) of the $\mathcal{A}/[\mathcal{X}]$-module F given in Proposition 8.7 by setting $\mathcal{B} := \mathcal{X}$. Then the assertion follows from (a), since $\mathcal{A}/[\mathcal{X}](-, \Omega^n_{\mathcal{X}}(L)) = 0$. □

Proposition 8.9. *Under Condition 8.1, for each $F \in \mathsf{mod}(\mathcal{A}/[\mathcal{X}])$, one has $\mathsf{pd}_{\overline{\mathcal{A}}}(F) < \infty$.*

Proof. Since $\mathsf{pd}_{\mathcal{A}/[\mathcal{X}]}(F) < \infty$ by Lemma 8.8 and the canonical inclusion $\iota : \mathsf{mod}(\mathcal{A}/[\mathcal{X}]) \hookrightarrow \mathsf{mod}(\overline{\mathcal{A}})$ is exact, it is enough to check the case of $F = \mathcal{A}/[\mathcal{X}](-, M)$ for some $M \in \mathcal{A}$. By (AB3), there exists an exact sequence $0 \to Y_M \xrightarrow{g} X_M \xrightarrow{f} M$ in $\mathcal{A}$ with f a right $\mathcal{X}$-approximation of M and $Y_M \in \widehat{\boldsymbol{\omega}}$. Applying the Yoneda embedding yields a projective resolution

$$0 \to \mathcal{A}(-, Y_M) \xrightarrow{g\circ-} \mathcal{A}(-, X_M) \xrightarrow{f\circ-} \mathcal{A}(-, M) \to \mathcal{A}/[\mathcal{X}](-, M) \to 0$$

of the $\mathcal{A}$-module $\mathcal{A}/[\mathcal{X}](-, M)$. Applying Proposition 8.7 to $\mathcal{B} := \boldsymbol{\omega}$, we have a projective resolution of the $\overline{\mathcal{A}}$-module $\mathcal{A}/[\mathcal{X}](-, M)$:

$$\cdots \longrightarrow \overline{\mathcal{A}}(-, \Omega_{\boldsymbol{\omega}}(Y_M)) \xrightarrow{\Omega_{\boldsymbol{\omega}} g\circ-} \overline{\mathcal{A}}(-, \Omega_{\boldsymbol{\omega}}(X_M)) \xrightarrow{\Omega_{\boldsymbol{\omega}} f\circ-} \overline{\mathcal{A}}(-, \Omega_{\boldsymbol{\omega}}(M))$$
$$\longrightarrow \overline{\mathcal{A}}(-, Y_M) \xrightarrow{g\circ-} \overline{\mathcal{A}}(-, X_M) \xrightarrow{f\circ-} \overline{\mathcal{A}}(-, M) \longrightarrow \mathcal{A}/[\mathcal{X}](-, M) \longrightarrow 0.$$

Since $Y_M \in \widehat{\boldsymbol{\omega}}$, one has $\Omega^n_{\boldsymbol{\omega}}(Y_M) \in \boldsymbol{\omega}$ for some $n \ge 0$. Thus $\overline{\mathcal{A}}(-, \Omega^n_{\mathcal{X}}(Y_M)) = 0$ and hence $\mathsf{pd}_{\overline{\mathcal{A}}}(\mathcal{A}/[\mathcal{X}](-, M)) < \infty$. □

We are ready to prove Theorem 8.2.

Proof of Theorem 8.2. By Lemma 7.1 and Proposition 8.5, the canonical inclusion $\overline{\mathcal{X}} \hookrightarrow \overline{\mathcal{A}}$ induces a triangle functor $\bar{Q} : \mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{A}}) \to \mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{X}})$. By Theorem 7.2 and Proposition 8.9, the triangle functor $\bar{Q}$ is an equivalence. □

8.2 Singular equivalences from cotilting objects

In this subsection we construct a singular equivalence from a given cotilting subcategory, using Theorem 8.2. We denote by $\mathsf{P}(\mathcal{C})$ (resp. $\mathsf{GP}(\mathcal{C})$) the full subcategory of $\mathcal{C}$ consisting of projective (resp. Gorenstein projective) objects. We abbreviate $\Omega M := \Omega_{\mathsf{P}(\mathcal{C})} M$ for each $M \in \mathcal{C}$ and denote by $\Omega^n \mathcal{A}$ the full subcategory of $\mathcal{C}$ consisting of objects isomorphic to $\Omega^n M$ for some $M \in \mathcal{A}$. Moreover we define $\Omega^- M$ to be the kernel of a left $\mathsf{P}(\mathcal{C})$-approximation of M. Inductively we define $\Omega^{-n} M$ for any $n \ge 1$.

Corollary 8.10. *Let $\mathcal{A}$ be an abelian category with enough projectives and $\mathcal{T}$ its contravariantly finite cotilting subcategory. Then the canonical inclusion $\overline{{}^{\perp}\mathcal{T}} \hookrightarrow \overline{\mathcal{A}}$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\overline{\mathcal{A}}) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\overline{{}^{\perp}\mathcal{T}})$.*

Proof. Setting $\mathcal{X} := {}^{\perp}\mathcal{T}$ and $\boldsymbol{\omega} := \mathcal{T}$, we shall show that the sequence $\mathcal{A} \supseteq \mathcal{X} \supseteq \boldsymbol{\omega}$ satisfies conditions (AB1)-(AB3). The condition (AB1) is obvious, because $\mathcal{A} = \mathcal{C}$. The condition (AB2) holds by definition.

(AB3): By [ABu, Thm. 1.1], for any $M \in \widehat{\mathcal{X}}$, there exists an exact sequence

$$0 \to Y_M \to X_M \to M \to 0$$

with $Y_M \in \widehat{\omega}$ and $X_M \in \mathcal{X}$. It remains to show $\widehat{\mathcal{X}} = \mathcal{A}$. Since there exists an integer $n \geq 0$ such that $\mathsf{id} I \leq n$ for all $I \in \boldsymbol{\omega}$, it follows that $\Omega^n M \in \mathcal{X}$ holds for all $M \in \mathcal{A}$. This shows $\widehat{\mathcal{X}} = \mathcal{A}$. Thanks to Theorem 8.2, we have a desired triangle equivalence. □

8.3 Matsui-Takahashi's Singular equivalence

We provide an alternative proof for Matsui-Takahashi's singular equivalence.

Definition 8.11. Let $\mathcal{C}$ be an abelian category with enough projectives. A full subcategory $\mathcal{A}$ of $\mathcal{C}$ is called *quasi-resolving* if it is closed under kernels of epimorphisms and contains all projectives. A quasi-resolving subcategory is called *resolving* if it is closed under extensions and direct summands.

Corollary 8.12. [MT, Thm. 5.4(3)] *Let $\mathcal{A}$ be a quasi-resolving subcategory of an abelian category $\mathcal{C}$ with enough projectives. Assume that $\mathcal{A}$ together with an integer $n \in \mathbb{Z}_{\geq 0}$ satisfies the condition*

$$\Omega^n\mathcal{A} \textit{ is contained in } \mathsf{GP}(\mathcal{C}) \textit{ and closed under cosyzygies} \qquad (*)$$

and set $\mathcal{X} := \Omega^n\mathcal{A}$. Then the canonical inclusion $\underline{\mathcal{X}} \hookrightarrow \underline{\mathcal{A}}$ induces a triangle equivalence $\mathsf{D_{sg}}(\underline{\mathcal{A}}) \xrightarrow{\sim} \mathsf{D_{sg}}(\underline{\mathcal{X}})$.

Proof. Setting $\mathcal{X} := \Omega^n\mathcal{A}$ and $\boldsymbol{\omega} := \mathsf{P}(\mathcal{C})$, we shall show that the sequence $\mathcal{A} \supseteq \mathcal{X} \supseteq \boldsymbol{\omega}$ of subcategories in $\mathcal{C}$ satisfies the conditions (AB1)-(AB3). (AB1): Since $\mathsf{P}(\mathcal{C})$-epikernels are epimorphisms, the condition (AB1) follows from the definition of quasi-resolving subcategories.

(AB2): Since $\mathcal{X} \subseteq \mathsf{GP}(\mathcal{C})$, we have $\mathcal{X} \subseteq {}^{\perp}\boldsymbol{\omega}$.

(AB3): Let $M \in \mathcal{A}$. By the condition $(*)$, we have an exact sequence

$$0 \to G \to P_{n-1} \to \cdots P_0 \to M \to 0$$

with $G \in \mathcal{X}$ and $P_{n-1}, \cdots, P_0 \in \mathsf{P}(\mathcal{C})$. Since $G \in \mathsf{GP}(\mathcal{C})$, we have an exact sequence

$$0 \to G \xrightarrow{g_n} Q_{n-1} \xrightarrow{g_{n-1}} \cdots \to Q_0 \xrightarrow{g_0} \Omega^{-n}(G) \to 0$$

with the canonical morphisms $\operatorname{Im} g_i \to Q_i$ being left $\mathsf{P}(\mathcal{C})$-approximations for each $1 \leq i \leq n$. Thus we have the following chain map, where $\Omega^{-n}(G) \in \Omega^n\mathcal{A} = \mathcal{X}$ by the condition $(*)$.

$$\begin{array}{ccccccccccccc} 0 & \to & G & \to & Q_{n-1} & \to & \cdots & \to & Q_0 & \to & \Omega^{-n}(G) & \to & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \to & G & \to & P_{n-1} & \to & \cdots & \to & P_0 & \to & M & \to & 0 \end{array}$$

By taking the mapping cone of the above chain map, we have an exact sequence

$$0 \to G \to Q_{n-1} \oplus G \to Q_{n-2} \oplus P_{n-1} \to \cdots \to Q_0 \oplus P_1 \to \Omega^{-n}(G) \oplus P_0 \to M \to 0.$$

Since the left-most morphism $G \to Q_{n-1} \oplus G$ is a split-monomorphism, we have the following exact sequence

$$0 \to Q_{n-1} \to Q_{n-2} \oplus P_{n-1} \to \cdots \to Q_0 \oplus P_1 \to \Omega^{-n}(G) \oplus P_0 \xrightarrow{f} M \to 0. \qquad (8.12.1)$$

Obviously $\operatorname{Ker} f \in \widehat{\omega}$ holds. The exact sequence $0 \to \operatorname{Ker} f \to \Omega^{-n}(G) \oplus P_0 \xrightarrow{f} M \to 0$ is a desired one. Indeed, f is a right $\mathcal{X}$-approximation by Lemma 8.4. □

Recall that an additive category $\mathcal{A}$ with weak-kernels is said to be *Iwanaga-Gorenstein* if $\mathsf{id}_{\mathcal{A}}(\mathcal{A}(-, M)), \mathsf{id}_{\mathcal{A}^{\mathsf{op}}}(\mathcal{A}(M, -)) < \infty$ for any $M \in \mathcal{A}$. Typical examples of Iwanaga-Gorenstein rings are finite dimensional selfinjective algebras over a field k and commutative Gorenstein rings of finite Krull dimension. As an obvious consequence of Corollary 8.10 or 8.12, we have:

Example 8.13. Let Λ be an Iwanaga-Gorenstein ring with $\mathsf{id}_\Lambda(\Lambda) = n$ and $\mathsf{CM}\Lambda := {}^\perp\Lambda$. Then the canonical inclusion $\underline{\mathsf{CM}}\Lambda \hookrightarrow \underline{\mathsf{mod}}\Lambda$ induces a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\underline{\mathsf{mod}}\Lambda) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(\underline{\mathsf{CM}}\Lambda)$.

9 More Results and Examples

In this section, we provide further investigations on Condition 8.1. First we give sufficient conditions so that $\mathcal{X}/[\omega]$ is Iwanaga-Gorenstein and of finite global dimension, respectively.

Theorem 9.1. *Let Λ be a finite dimensional algebra and $T \in \mathsf{mod}\,\Lambda$ a cotilting module. We set $\underline{{}^\perp T} := {}^\perp T/[\Lambda]$ and $\overline{{}^\perp T} := {}^\perp T/[T]$. Then the followings hold:*

(a) *If Λ is Iwanaga-Gorenstein, then so is $\overline{{}^\perp T}$. Moreover, one has $\mathsf{id}_{(\overline{{}^\perp T})} F \leq 3 \max\{\mathsf{pd}_\Lambda T, \mathsf{id}_\Lambda \Lambda\}$ for any projective $(\overline{{}^\perp T})$-module F.*

(b) *If $\mathsf{gl.dim}\Lambda = n$, then we have $\mathsf{gl.dim}(\overline{{}^\perp T}) \leq 3n - 1$.*

The assertion (b) can be found in [Kim, Thm. 6.1]. Let us recall from [INP, Thm. 3.4] (see also [Eno, Jia]), there exist Auslander-Reiten translations on ${}^\perp T$, that is, mutually equivalences

$$\tau : \underline{{}^\perp T} \xrightarrow{\sim} \overline{{}^\perp T} \quad \text{and} \quad \tau^- : \overline{{}^\perp T} \xrightarrow{\sim} \underline{{}^\perp T}.$$

Moreover, they induce functorial isomorphisms

$$D \operatorname{Ext}^1_\Lambda(M, N) \cong \underline{{}^\perp T}(\tau^- N, M) \cong \overline{{}^\perp T}(N, \tau M)$$

in $M, N \in {}^\perp T$ which are known as Auslander-Reiten dualities, where $D := \operatorname{Hom}_k(-.k)$.

Proof of Theorem 9.1. (a) Since there exists an equivalence $\overline{{}^{\perp}T} \xrightarrow{\sim} \underline{{}^{\perp}T}$, we shall show that $\underline{{}^{\perp}T}$ is Iwanaga-Gorenstein. Thanks to Auslander-Reiten duality, every injective $(\underline{{}^{\perp}T})$-module is of the form $\mathrm{Ext}^1_\Lambda(-, M)$ for some $M \in {}^{\perp}T$. Since T is a cotilting module, we get an exact sequence $0 \to M \to T' \to N \to 0$ with $T' \in \mathsf{add}\, T$ and $N \in {}^{\perp}T$. The induced sequence

$$0 \to \mathrm{Hom}_\Lambda(-, M) \to \mathrm{Hom}_\Lambda(-, T') \to \mathrm{Hom}_\Lambda(-, N) \to \mathrm{Ext}^1_\Lambda(-, M) \to 0$$

gives a projective resolution of $({}^{\perp}T)$-module $\mathrm{Ext}^1_\Lambda(-, M)$. By Proposition 8.7, we have a projective resolution

$$\begin{array}{l} \cdots \longrightarrow \underline{{}^{\perp}T}(-, \Omega_\Lambda(M)) \longrightarrow \underline{{}^{\perp}T}(-, \Omega_\Lambda(T')) \longrightarrow \underline{{}^{\perp}T}(-, \Omega_\Lambda(N)) \\ \xrightarrow{\delta} \underline{{}^{\perp}T}(-, M) \longrightarrow \underline{{}^{\perp}T}(-, T') \longrightarrow \underline{{}^{\perp}T}(-, N) \longrightarrow \mathrm{Ext}^1_\Lambda(-, M) \longrightarrow 0 \end{array} \tag{9.1.1}$$

of the $(\underline{{}^{\perp}T})$-module $\mathrm{Ext}^1_\Lambda(-, M)$. Since Λ is Iwanaga-Gorenstein, T is a tilting module, in particular $\mathsf{pd}_\Lambda(T) < \infty$. Thus there exists an integer $n \geq 0$ such that $\Omega^n_\Lambda(T') \in \mathsf{proj}\, \Lambda$. Hence every injective $(\underline{{}^{\perp}T})$-module $\mathrm{Ext}^1_\Lambda(-, M)$ is of finite projective dimension. Next we shall show that every projective $(\underline{{}^{\perp}T})$-module $\underline{{}^{\perp}T}(-, M)$ is of finite injective dimension. Considering the first syzygy of M, namely an exact sequence $0 \to \Omega_\Lambda M \to P \to M \to 0$ with $P \in \mathsf{proj}\, \Lambda$, we get an injective resolution

$$0 \to \underline{{}^{\perp}T}(-, M) \to \mathrm{Ext}^1_\Lambda(-, \Omega_\Lambda M) \to \mathrm{Ext}^1_\Lambda(-, P) \to \mathrm{Ext}^1_\Lambda(-, M) \to \cdots \tag{9.1.2}$$

of the $(\underline{{}^{\perp}T})$-module $\underline{{}^{\perp}T}(-, M)$. Since Λ is Iwanaga-Gorenstein, we have $\mathsf{id}_\Lambda P < \infty$. We have thus concluded that $\overline{{}^{\perp}T}$ is Iwanaga-Gorenstein. The latter formula follows from the sequence (9.1.1) and (9.1.2).

(b) We shall show that $\mathsf{gl.dim}(\underline{{}^{\perp}T}) \leq 3n-1$. Let $F \in \mathsf{mod}(\underline{{}^{\perp}T})$ with a projective presentation $\underline{{}^{\perp}T}(-, M) \to \underline{{}^{\perp}T}(-, L) \to F \to 0$. Since F vanishes on $\mathsf{proj}\, \Lambda$, the corresponding morphism $f : M \to L$ is an epimorphism in $\mathsf{mod}\, \Lambda$. Since ${}^{\perp}T$ is closed under epimorphisms, we have an exact sequence $0 \to N \to M \to L \to 0$ in ${}^{\perp}T$ which induces a projective resolution

$$\begin{array}{l} \cdots \longrightarrow \underline{{}^{\perp}T}(-, \Omega_\Lambda(N)) \longrightarrow \underline{{}^{\perp}T}(-, \Omega_\Lambda(M)) \longrightarrow \underline{{}^{\perp}T}(-, \Omega_\Lambda(L)) \\ \longrightarrow \underline{{}^{\perp}T}(-, N) \longrightarrow \underline{{}^{\perp}T}(-, M) \longrightarrow \underline{{}^{\perp}T}(-, L) \longrightarrow F \longrightarrow 0 \end{array}$$

of the $(\underline{{}^{\perp}T})$-module F. The assumption $\mathsf{gl.dim}\Lambda = n$ implies $\Omega^n_\Lambda(L) \in \mathsf{proj}\, \Lambda$. Hence $\mathsf{pd}_{(\underline{{}^{\perp}T})} F \leq 3n - 1$. □

Theorem 9.1 contains the following well-known result.

Example 9.2. [AR74, Prop. 10.2] Let Λ be a finite dimensional algebra with $\mathsf{gl.dim}\Lambda = n$. Then we have $\mathsf{gl.dim}(\underline{\mathsf{mod}}\Lambda) \leq 3n - 1$.

Next we explain that (AB1)-(AB3) in Condition 8.1 are satisfied in the classical Auslander-Buchweitz theory: Let $\mathcal{C}$ be an abelian category with enough projectives and $\mathcal{X} \supseteq \boldsymbol{\omega}$ a sequence of full subcategories in $\mathcal{C}$. We say that $\boldsymbol{\omega}$ is a *cogenerator of* $\mathcal{X}$ if, for each $X \in \mathcal{X}$, there exists an exact sequence $0 \to X \to I \to X' \to 0$ with $I \in \boldsymbol{\omega}, X' \in \mathcal{X}$.

Condition 9.3. [ABu, p. 9, 17] For a sequence $\mathcal{X} \supseteq \boldsymbol{\omega}$ of full subcategories in $\mathcal{C}$, we consider the following conditions:

- $\widehat{\mathcal{X}} = \mathcal{C}$;
- $\mathcal{X}$ is closed under direct summands and extension;
- $\mathrm{Ext}^i_{\mathcal{C}}(X, I) = 0$ for any $X \in \mathcal{X}, I \in \boldsymbol{\omega}$ and $i > 0$;
- $\boldsymbol{\omega}$ is a cogenerator of $\mathcal{X}$ which is closed under direct summands.

Under these conditions, it is known that, for each $M \in \mathcal{C}$, there exists an exact sequence

$$0 \to Y_M \to X_M \to M \to 0 \tag{9.3.1}$$

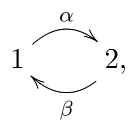
with $X_M \in \mathcal{X}, Y_M \in \widehat{\boldsymbol{\omega}}$ [ABu, Thm. 1.1]. The sequence (9.3.1) is called the *Auslander-Buchweitz approximation of* M. As a benefit of our *generalized Auslander-Buchweitz approximation* in (AB3), we shall show Proposition 9.4. Notice that, in the proposition, the subcategory ω is not necessarily a cogenerator of $\mathcal{X}$, and right $\mathcal{X}$-approximations of objects of $\mathcal{A}$ appearing in (AB3) are not necessarily surjective.

Proposition 9.4. *Let $\mathcal{A}$ be an abelian category with enough projectives and $\mathcal{X} \supseteq \boldsymbol{\omega}$ a sequence of full subcategories of $\mathcal{A}$. Suppose that $\mathcal{X}$ is a torsion class of $\mathcal{A}$ and $\boldsymbol{\omega}$ is contravariantly finite in $\mathcal{A}$ and satisfies* $\mathrm{Ext}^i_{\mathcal{A}}(X, I) = 0$ *for any $X \in \mathcal{X}, I \in \boldsymbol{\omega}$ and $i > 0$. Then the sequence $\mathcal{A} \supseteq \mathcal{X} \supseteq \boldsymbol{\omega}$ satisfies (AB1)-(AB3).*

Proof. The conditions (AB1) and (AB2) are obvious. Since $\mathcal{X}$ is a torsion class, for any $M \in \mathcal{A}$ there exists an exact sequence $0 \to X \to M$ with $X \in \mathcal{X}$, hence (AB3) holds. □

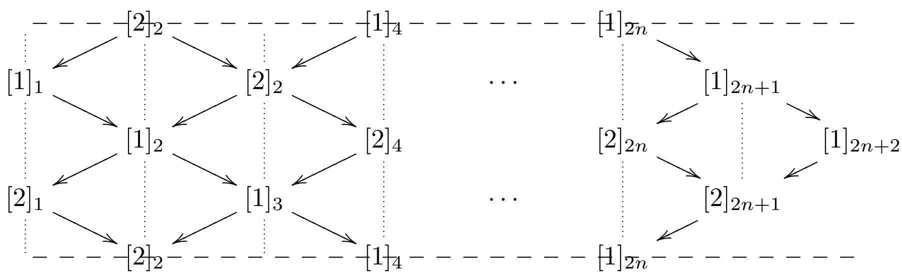
We end this section by giving examples of singularly equivalent categories using Corollary 8.10.

Example 9.5. Fix an integer $n \in \mathbb{Z}_{>0}$. Let Λ be the algebra defined by the following quiver with relations.

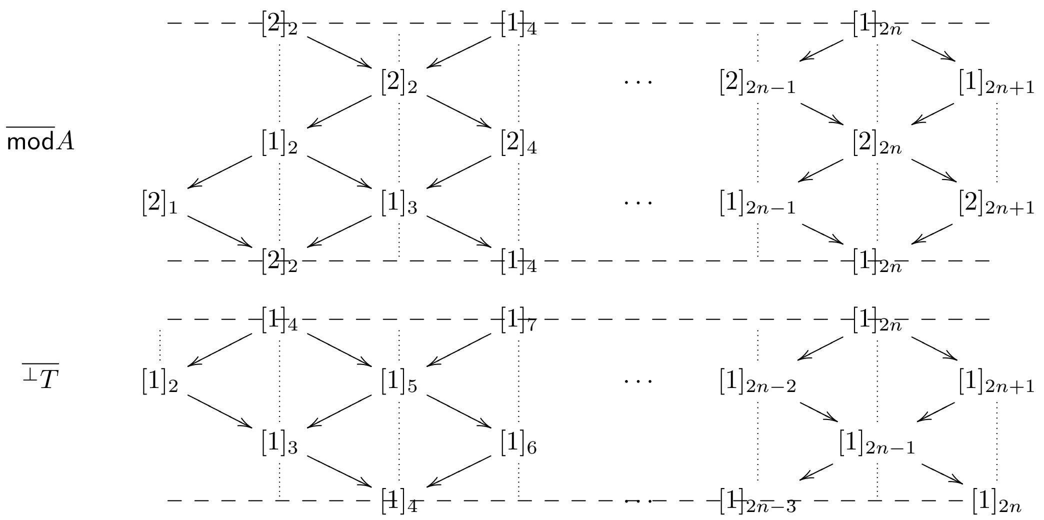


$\langle(\alpha\beta)^n\alpha\rangle$

We describe the Auslander-Reiten quiver of Λ. Since Λ is a Nakayama algebra, an indecomposable module is determined by the pair (m, l) of the socle l and the Loewy length l. We shall denote the module by $[m]_l$.



We can easily check that the module $T := [1]_1 \oplus [1]_{2n+2}$ is a cotilting module of $\mathsf{id}_\Lambda(T) = 1$. Due to Corollary 8.10, we conclude that $\overline{\mathsf{mod}}\Lambda := (\mathsf{mod}\,\Lambda)/[T]$ is singularly equivalent to $\overline{{}^\perp T} := ({}^\perp T)/[T]$. Their Auslander-Reiten quivers are described as follows:



where the dotted lines stand for natural mesh relations.

Claim. *If $n = 1$, both $\overline{\mathsf{mod}}\Lambda$ and $\overline{{}^\perp T}$ are of finite global dimension, otherwise they are non Iwanaga-Gorenstein.*

Proof. We only check the case of $n \geq 2$. By calculations, the injective $(\overline{{}^\perp T})$-module $D\overline{{}^\perp T}([1]_3, -)$ has the following projective resolution:

$$\cdots \to P_5 \to P_3 \to P_{2n+1} \to P_{2n-1} \to P_{2n+1} \to P_3 \to P_4 \to P_{2n+1} \to I_3 \to 0,$$

where we set $I_3 := D\overline{{}^\perp T}([1]_3, -)$ and $P_l := \overline{{}^\perp T}(-, [1]_l)$ for each $1 \leq l \leq 2n+1$. We notice that $\Omega^2 I_3 \cong \Omega^8 I_3$. Hence $\overline{{}^\perp T}$ is non Iwanaga-Gorenstein. It remains to check the assertion for $\overline{\mathsf{mod}}\Lambda$. We denote by $Q : \mathsf{mod}(\overline{\mathsf{mod}}\Lambda) \to \mathsf{mod}(\overline{{}^\perp T})$ the canonical functor. There exists an injective object $J \in \mathsf{inj}(\overline{\mathsf{mod}}\Lambda)$ such that $QJ \cong I_3$. If $\overline{\mathsf{mod}}\Lambda$ is Iwanaga-Gorenstein, then J is of finite projective dimension. Moreover, since Q is exact and preserves projectives, it turns out that I_3 is of finite projective dimension. This is a contradiction. $\square$

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