Harmonic Analysis for Finite Dimensional Real Frobenius Lie Algebras

by

Edi Kurniadi

A dissertation submitted in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Graduate School of Mathematics

Nagoya University

March 2019

Professor Hideyuki ISHI

I would like to dedicate my thesis to my family that supported me during my study, specially to my beloved wife Ani Nurlaelasari and my beloved son Resal Ahmad Fauzan, thank you so much dears.

Acknowledgements

First of all, I would like to sincerely thank my supervisor Professor Hideyuki Ishi not only for insightful advice for this Ph.D thesis but also for the generous support and guidance during the study. The second, I would also like to thank The Research, Technology Directorate General of Higher Education (RISTEK-DIKTI), Ministry of Research, Technology and Higher Education of Indonesia for the scholarship of my study, and I also sincerely thank Dean of Mathematics and Natural Sciences Faculty (FMIPA) of Universitas Padjadjaran, Head of Department of Mathematics, and my colleagues in Department of Mathematics of FMIPA of Universitas Padjadjaran, and specially thank Professor Sudradjat, Professor Asep K. Supriatna and Professor Budi Nurani for their valuable advice in the beginning of study. Finally, I would like to express many thanks to my friends that can not be mentioned here one by one for their support and encouragement.

Abstract

We first present results of harmonic analysis for real Lie groups whose Lie algebras are 4-dimensional Frobenius. In this context we find square-integrable unitary representations of these groups corresponding to open coadjoint orbits. Concerning square-integrable representations, we compute their Duflo-Moore operators which can be described in terms of their Pfaffians.

Furthermore, we generalize the arguments for the semi-direct product group $G := V \rtimes H$ where V is isomorphic to $\mathbb{R}^n$ and H is a Lie subgroup of $\mathrm{GL}(V)$. We give necessary and sufficient conditions for the coadjoint orbits of G to be open in $\mathfrak{g}^*$. When the coadjoint orbit Ω_{ξ_0} through $\xi_0 = \xi(p_0, \alpha_0) \in \mathfrak{g}^* = V^* \oplus \mathfrak{h}^*$ is open in $\mathfrak{g}^*$, we describe the Duflo-Moore operator C_π for a representation π of G corresponding to the orbit Ω_{ξ_0}. In particular, for the case where the stabilizer H_{p_0} is not trivial, the operator C_π can be written using the Duflo-Moore operator for a representation of H_{p_0}. We apply such general results to the similitude Lie group $\mathrm{Sim}(n) := \mathbb{R}^n \rtimes (\mathbb{R}_+ \times \mathrm{SO}(n))$ and the real connected affine Lie group $\mathrm{Aff}^+(n) := \mathbb{R}^n \rtimes \mathrm{GL}_n^+(\mathbb{R})$.

Contents

Dedication iii
Acknowledgements iv
Abstract v
Introduction 1

1 Preliminaries **11**

1.1 The orbit method 11
1.2 Frobenius Lie algebras 14
1.3 Haar measure 16
1.4 Induced representation 18
1.5 Intertwining Operators 22
1.6 Square-integrable representation 24

2 Harmonic Analysis for Frobenius Lie Algebra of Dimension 4 **27**

2.1 The first type $\mathfrak{g}_I$ 28
2.2 The Second type $\mathfrak{g}_{II}(\tau)$ 38
2.3 The Third type $\mathfrak{g}_{III}(\pm 1)$ 48

3 General Results for $V \rtimes H$ **59**

3.1 Conditions for coadjoint orbits of $G = V \rtimes H$ to be open in $\mathfrak{g}^*$ . . 59
3.2 Duflo-Moore Operators for $V \rtimes H$ with trivial stabilizer case of H 63
3.3 General Formula of Duflo-Moore Operator for $V \rtimes H$ 67

4 Application for similitude and affine Lie group cases **78**

4.1 The Lie algebra $\mathbb{R}^n \rtimes (\mathbb{R} \oplus \mathfrak{so}(n))$ 78
4.2 Duflo-Moore Operator for $\mathrm{Aff}^+(1)$ 82
4.3 $\mathfrak{aff}(n)$ as the Frobenius Lie algebra 86

4.4 The Duflo-Moore operator for $\mathrm{Aff}^{+}(n)$ 91
4.5 Pfaffian of $\mathfrak{aff}(n)$ 92

Bibliography **99**

Introduction

The main object of this thesis is finite dimensional real Frobenius Lie algebras studied in terms of semi-direct products. We present harmonic analysis for real Lie groups whose Lie algebras are Frobenius. In particular, we consider square-integrable unitary representations of the real Lie groups corresponding to open coadjoint orbits and their Duflo-Moore operators.

The notion of Frobenius Lie algebras appeared and was studied at the first time in [47], [48], and [49] in the context to answer what conditions on finite dimensional Lie algebra $\mathfrak{g}$ in order that its universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ has an exact simple module (see [48, p.488]). Frobenius Lie algebras form an important class of Lie algebras having this property. The Lie algebra $\mathfrak{g}$ is called Frobenius if there exists a linear functional f_0 on $\mathfrak{g}$ such that its stabilizer $\mathfrak{g}_{f_0}$ is equal to zero. Let $\{X_i\}_{i=1}^{n}$ be a basis for $\mathfrak{g}$ and $M_{\mathfrak{g}}$ be an $n \times n$ matrix of $\mathfrak{g}$-entry whose (i,j)-component is $[X_i, X_j]$. We define $\det M_{\mathfrak{g}}$ as an element of the symmetric algebra $S(\mathfrak{g})$ which is identified with the polynomial algebra $Pol(\mathfrak{g}^*)$ on $\mathfrak{g}^*$. Then the Lie algebra $\mathfrak{g}$ is Frobenius if $\det M_{\mathfrak{g}}$ is not identically zero. It means that $\mathfrak{g}$ is a Frobenius Lie algebra if and only if $\det M_{\mathfrak{g}}(f_0) = \det \langle f_0, [X_i, X_j]\rangle_{1\leq i,j\leq n} \neq 0$ for a suitable f_0. In other words, the Lie algebra $\mathfrak{g}$ is Frobenius if and only if the alternating bilinear form $B_{f_0} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by $B_{f_0}(X,Y) = \langle f_0, [X,Y]\rangle$ is non-degenerate at some $f_0 \in \mathfrak{g}^*$. In this case, $\mathfrak{g}$ is even dimensional so that we define the Pfaffian $Q_{\mathfrak{g}} \in S(\mathfrak{g})$ as the Pfaffian of the matrix $M_{\mathfrak{g}}$. One can consult further about Frobenius Lie algebras and their properties in [13], [18], [26], [30], [50], [51], and [52].

Let G be a connected Lie group whose Lie algebra is $\mathfrak{g}$. Then G has an open coadjoint orbit if and only if $\mathfrak{g}$ is Frobenius. Keeping in mind the orbit method, we study unitary representations corresponding to open coadjoint orbits. Such representations are expected to be square-integrable. We recall here that for an almost algebraic group G, Lipsman [40] found a one-to-one correspondence between square-integrable representations of G and open orbits in a certain G-space. Lipsman's results were established in a framework of a sophisticated version of the orbit method, and the "open orbits" in [40] did not necessarily mean open coadjoint orbits. Indeed, if G is a compact Lie group, all the irreducible unitary representations are square-integrable, whereas G has no open coadjoint orbit.

In general, we say that an irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ of a locally compact group G is said to be square-integrable if there exist $0 \neq \phi \in \mathcal{H}_\pi$ such that

$$\int_G |(\phi|\pi(g)\phi)_{\mathcal{H}_\pi}|^2 \, dg < \infty. \tag{1}$$

In this case a vector ϕ is called *admissible*. Furthermore, there exists a (not necessarily bounded, densely defined) unique operator $C_\pi : \mathcal{H}_\pi \to \mathcal{H}_\pi$, called Duflo-Moore operator, which is positive self-adjoint and it satisfies (see [15], [27], and [28])

1. ϕ is admissible if and only if $\phi \in \operatorname{dom} C_\pi$, and
2. For $\phi_1, \phi_3 \in \mathcal{H}_\pi$ and $\phi_2, \phi_4 \in \operatorname{dom} C_\pi$, we have

$$\int_G (\phi_1|\pi(g)\phi_2)_{\mathcal{H}_\pi} (\pi(g)\phi_4|\phi_3)_{\mathcal{H}_\pi} \, dg = (\phi_1|\phi_3)_{\mathcal{H}_\pi} (C_\pi\phi_4|C_\pi\phi_2)_{\mathcal{H}_\pi}. \tag{2}$$

The Duflo-Moore operator as well as square-integrable representations are important in the theory of continuous wavelet transform.

The simplest example of Frobenius Lie algebra is $\mathfrak{aff}(1) := \langle X_1, X_2 \rangle$ whose non-zero bracket is given by $[X_1, X_2] = X_2$. This is the Lie algebra of the connected affine group $\mathrm{Aff}^+(1)$ over the real line. There are two open coadjoint orbits $\Omega_{X_2^*}$ and $\Omega_{-X_2^*}$ through X_2^* and $-X_2^*$ respectively. In our case, we construct the unitary representations $\pi_\pm$ of $\mathrm{Aff}^+(1)$ corresponding to the open coadjoint orbits $\Omega_{\pm X_2^*}$ by

$$\begin{aligned} &\pi_\pm(\exp\, \alpha_0 X_1) f(a) = f(e^{-\alpha_0} a), \\ &\pi_\pm(\exp\, \beta_0 X_2) f(a) = e^{\pm 2\pi i \beta_0 a^{-1}} f(a) \quad \left(f \in L^2(\mathbb{R}_+, \frac{da}{a}),\ a > 0, \alpha_0, \beta_0 \in \mathbb{R} \right), \end{aligned} \tag{3}$$

which are square-integrable (see Section 4.2). Moreover, since $\pi_\pm$ is square-integrable, we can compute the Duflo-Moore operator $C_{\pi_\pm}$ for $(\pi_\pm, L^2(\mathbb{R}_+, \frac{da}{a}))$. Even though the harmonic analysis of $\mathrm{Aff}^+(1)$ has been already investigated as in [19], [23], [25], [35], [36], and [54], we note that the representations $\pi_\pm$ are usually realized as sub-representations of the quasi-regular representations on $L^2(\mathbb{R})$ and the Duflo-Moore operator is given in terms of Fourier transforms (see [21], [28]). On the other hand, we have

Proposition (see Proposition 4.3)**.** *The Duflo-Moore operator* $C_{\pi_\pm}$ *for the representation* $(\pi_\pm, L^2(\mathbb{R}_+, \frac{da}{a}))$ *of* $\mathrm{Aff}^+(1)$ *can be written as*

$$C_{\pi_\pm} f(a) = a^{1/2} f(a) \qquad (f \in L^2(\mathbb{R}_+, \frac{da}{a}),\ \ a \in \mathbb{R}_+). \tag{4}$$

Let us note that the Pfaffian $Q_{\mathfrak{aff}(1)} := \mathrm{Pf}\, M_{\mathfrak{aff}(1)} \in S(\mathfrak{g})$ equals X_2. We relate the Duflo-Moore operator $C_{\pi_\pm}$ and $Q_{\mathfrak{aff}(1)}$ as follows.

Proposition (see Proposition 4.4)**.** *The Duflo-Moore operator* $C_{\pi_\pm}$ *for the representation* $(\pi_\pm, L^2(\mathbb{R}_+, \frac{da}{a}))$ *of* $\mathrm{Aff}^+(1)$ *corresponding to* $Q_{\mathfrak{aff}(1)}$ *is written as*

$$C_{\pi_\pm} = \sqrt{2\pi}\ |id\pi(Q_{\mathfrak{aff}(1)})|^{-1/2}. \tag{5}$$

Our explanations above motivate us to study harmonic analysis for the 4-dimensional Frobenius Lie algebras classified in [12] specially for the real case. Here we summarize our preceding work [37], compared with what have done in this thesis. Let $\mathfrak{g}$ be a real Frobenius Lie algebra and $G = \exp(\mathrm{ad}\,\mathfrak{g})$ be a connected Lie subgroup of $\mathrm{GL}(\mathfrak{g})$. Since $\mathfrak{g}$ is Frobenius, the adjoint representation of $\mathfrak{g}$ is faithful, so that we regard $\mathfrak{g}$ as the Lie algebra of G. For $f \in \mathfrak{g}^*$, we denote by Ω_f the coadjoint orbit $\mathrm{Ad}^*(G)f \subset \mathfrak{g}^*$ through f. We pose the following conjectures:

Conjecture. *If* Ω_f *is open in* $\mathfrak{g}^*$*, there exists a polarization* $\mathfrak{p} \subset \mathfrak{g}$ *at* f *such that* $\pi_f := \mathrm{Ind}_{\exp \mathfrak{p}}^G \nu_f$ *is a square-integrable representation, where* ν_f *is a one-dimensional representation of the group* $\exp \mathfrak{p} \subset G$ *defined by* $\nu_f(\exp X) := e^{2\pi i \langle f, X\rangle}$ *for* $X \in \mathfrak{p}$.

Let $s : S(\mathfrak{g}) \to U(\mathfrak{g})$ be the symmetrization operator. For a unitary representation π of G, $i^{n/2} d\pi(s(Q_\mathfrak{g}))$ is a symmetric operator.

Conjecture. *Let* $(\pi, \mathcal{H}_\pi)$ *be a square-integrable representation of* G*. Then* $i^{n/2} d\pi(s(Q_\mathfrak{g}))$ *is essentially self-adjoint, and the Duflo-Moore operator* C_π *of* π *equals a constant multipe of the operator* $|i^{n/2} d\pi(s(Q_\mathfrak{g}))|^{-1/2}$ *on* $\mathcal{H}_\pi$*. Namely, there exists a positive constant* $c_\pi > 0$ *such that* $C_\pi = c_\pi |i^{n/2} d\pi(s(Q_\mathfrak{g}))|^{-1/2}$.

We remark here that for the case where G is exponential solvable, results similar to both conjectures above were claimed by Duflo and Raïs [16, Théorème 5.3.8]. We also notice that, since a Frobenius Lie algebra is not necessarily almost algebraic, Lipsman's work [40] does not imply our conjectures.

We have already confirmed both Conjectures above for 4-dimensional Frobenius Lie algebras in [37] in terms of group Fourier transforms using [31]. Here we recall the result by Csikós and Verhóczki [12] as follows :

Theorem. *([12, p.448]). Any 4-dimensional Frobenius Lie algebra* $\mathfrak{g}$ *over a field* $\mathbb{F}$ *of characteristic* $\neq 2$ *is isomorphic to one of the following*

1. $\mathfrak{g}_I : [X_1, X_4] = [X_2, X_3] = -X_1, [X_2, X_4] = -\dfrac{X_2}{2}, [X_3, X_4] = -\dfrac{X_3}{2}.$
2. $\mathfrak{g}_{II}(\tau), \tau \in \mathbb{F} : [X_1, X_4] = [X_2, X_3] = -X_1, [X_2, X_4] = -X_3,$
   $[X_3, X_4] = -X_3 + \tau X_2.$
3. $\mathfrak{g}_{III}(\varepsilon)$*, where* $0 \neq \varepsilon \in \mathbb{F} : [X_1, X_3] = [X_2, X_4] = -X_1, [X_1, X_4] = \varepsilon X_2,$
   $[X_2, X_3] = -X_2,$
   The Frobenius Lie algebras $\mathfrak{g}_{III}(\varepsilon)$ *and* $\mathfrak{g}_{III}(\varepsilon')$ *are isomorphic if and only if there exists* $a \in \mathbb{F}$ *for which* $\varepsilon' = a^2\varepsilon$*.*

Contrary to [37], in this thesis we obtain the results of harmonic analysis for the real Lie groups whose Lie algebras are 4-dimensional Frobenius Lie algebras in more concrete realization with more direct computations. Let G_I, $G_{II}(\tau)$, and $G_{III}(\varepsilon)$ be the Lie groups corresponding to 4-dimensional real Frobenius Lie algebras $\mathfrak{g}_I, \mathfrak{g}_{II}(\tau)$, $\mathfrak{g}_{III}(\varepsilon)$ respectively. We declare the results as follows.

Theorem (see Theorem 2.4)**.** *The Duflo Moore operator* C_{π_Ω} *for the representation* $(\pi_\Omega, L^2(\mathbb{R}^2))$ *of* G_I *as in* (2.7) *can be written of the form*

$$C_{\pi_\Omega} f(x, y) = e^{-y} f(x, y) \qquad (f \in L^2(\mathbb{R}^2)). \tag{6}$$

Let us note that the Pfaffian $Q_{\mathfrak{g}_I} := \mathrm{Pf}\, M_{\mathfrak{g}_I} \in S(\mathfrak{g}_I)$ equals X_1^2. We relate the Duflo-Moore operator C_{π_Ω} and $Q_{\mathfrak{g}_I}$ as follows.

Proposition (see Proposition 2.5)**.** *The Duflo-Moore operator* C_{π_Ω} *for the representation* π_Ω *of* G_I *as in* (2.7) *is written in terms of the Pfaffian* $Q_{\mathfrak{g}_I}$ *as*

$$C_{\pi_\Omega} = 2\pi |d\pi(Q_{\mathfrak{g}_I})|^{-1/2}. \tag{7}$$

We can apply exactly the same argument to $\mathfrak{g}_{II}(\tau)$ as the one for $\mathfrak{g}_I$, and we obtain the same result for the Duflo-Moore operator $C_{\pi_{\Omega_\pm}}$ of the representation $\pi_{\Omega_\pm}$ of

$G_{II}(\tau)$ as in (2.41). The latter result for $\mathfrak{g}_{III}(\varepsilon)$ is divided into the cases $\varepsilon = -1$ and $\varepsilon = 1$. For the first case, that is for $\varepsilon = -1$, we have four open coadjoint orbits $\Omega_{\pm X_1^*}$ and $\Omega_{\pm X_2^*}$. We present results only for $\Omega_{\pm X_1^*}$ but the ones for $\Omega_{\pm X_2^*}$ are almost same. The Duflo-Moore operator $C_{\pi_{\Omega_{\pm X_1^*}}}$ for the representation $(\pi_{\Omega_{\pm X_1^*}}, L^2(\mathbb{R}^2))$ of $G_{III}(-1)$ can be written in the following theorem.

Theorem (see Theorem 2.13)**.** *The Duflo-Moore operator for the representation* $\pi_{\Omega_{\pm X_1^*}}$ *of* $G_{III}(-1)$ *as in* (2.60) *can be written of the form*

$$C_{\pi_{\Omega_{\pm X_1^*}}} f(x,y) = e^{-x} f(x,y) \qquad (\ f \in L^2(\mathbb{R}^2)\ ). \tag{8}$$

The Duflo-Moore operator above can be related to the Pfaffian of $\mathfrak{g}_{III}(-1)$. We have

Proposition (see Proposition 2.14)**.** *The Duflo-Moore operator for the representation* $\pi_{\Omega_{\pm X_1^*}}$ *of* $G_{III}(-1)$ *as in* (2.60) *related to the Pfaffian of* $\mathfrak{g}_{III}(-1)$ *is of the form*

$$C_{\pi_{\Omega_{\pm X_1^*}}} = 2\pi |d\pi(Q_{\mathfrak{g}_{III}(-1)})|^{-1/2}, \tag{9}$$

where $Q_{\mathfrak{g}_{III}(-1)} := \mathrm{Pf}(\mathfrak{g}_{III}(-1)) = -X_1^2 + X_2^2$.

Secondly, when $\varepsilon = 1$, we can apply exactly the same argument as the one for $\mathfrak{g}_{III}(-1)$, and we obtain the similar result for Duflo-Moore operator $C_{\pi_{\Omega_{X_1^*}}}$ (see Theorem 2.15 and Proposition 2.16).

The results of harmonic analysis for the real Lie groups, whose Lie algebras are 4-dimensional real Frobenius Lie algebras, motivate us to generalize the arguments further for $G := V \rtimes H$ where V is isomorphic to the n-dimensional vector space $\mathbb{R}^n$ and H is a Lie subgroup of $\mathrm{GL}(V)$. Let $\mathfrak{g} := V \rtimes \mathfrak{h}$ be the Lie algebra of G and $\mathfrak{g}^* = V^* \oplus \mathfrak{h}^*$ be its dual. We shall give conditions for $\mathfrak{g}$ to be Frobenius. Let $\xi_0 := \xi(p_0, \alpha_0)$ be an element of $\mathfrak{g}^*$ with $p_0 \in V^*, \alpha_0 \in \mathfrak{h}^*$ and Ω_{ξ_0} be the coadjoint orbit of G through ξ_0. Moreover, let $\mathfrak{h}_{p_0}$ be a stabilizer of $\mathfrak{h}$ at p_0 and ϖ be the projection map from $\mathfrak{g}^*$ onto V^*. We obtain

Theorem (see Theorem 3.3)**.** Ω_{ξ_0} *is open if and only if the following two conditions are satisfied :*

1. $\varpi(\Omega_{\xi_0})$ *is open in* V^*.

2. $\mathfrak{h}_{p_0} = 0$, *or the coadjoint orbit* $\mathrm{Ad}^*(H_{p_0})(\alpha_0|_{\mathfrak{h}_{p_0}})$ *in* $\mathfrak{h}_{p_0}^*$ *through* $\alpha_0|_{\mathfrak{h}_{p_0}} \in \mathfrak{h}_{p_0}^*$ *is open.*

Corollary. *The Lie algebra* $\mathfrak{g} = V \rtimes \mathfrak{h}$ *is a Frobenius Lie algebra if and only if there exists* $p_0 \in V^*$ *such that* $\mathfrak{h}.p_0 = V^*$ *and the stabilizer* $\mathfrak{h}_{p_0} \subset \mathfrak{h}$ *is zero or a Frobenius Lie algebra.*

For the case that stabilizer $\mathfrak{h}_{p_0} = \{0\}$, we obtain the Duflo-Moore operator as

Theorem (see Theorem 3.7)**.** *The Duflo-Moore operator* $C_\pi : L^2(H) \to L^2(H)$ *for the representation* $\pi := \mathrm{Ind}_V^G \nu_{p_0}$ *as in* (3.13) *and* (3.15) *is described as*

$$C_\pi f_2(h) = C_0^{1/2}\, \Delta_G(h)^{-1/2} f_2(h) \quad (f_2 \in L^2(H)), \tag{10}$$

where $C_0 > 0$ *is a constant given by* (3.11)*.*

For the case $\mathfrak{h}_{p_0} \neq \{0\}$, let us consider some properties of representations of $G = V \rtimes H$ as follows.

Theorem (see Theorems 3.9 and 3.10)**.** *Let* $\mathfrak{m}_0 \subset \mathfrak{h}_{p_0}$ *be a polarization at* $\alpha_0|_{\mathfrak{h}_{p_0}}$ *satisfying Pukanszky condition and let* π_0 *be the induced representation* $\mathrm{Ind}_{\exp \mathfrak{m}_0}^{H_{p_0}} \nu_{\alpha_0}$ *of* H_{p_0}*.*

1. $\mathfrak{p}_0 := V \rtimes \mathfrak{m}_0 \subset \mathfrak{g}$ *is a polarization at* $\xi_0 = \xi(p_0, \alpha_0)$ *satisfying Pukanszky condition.*

2. $\pi := \mathrm{Ind}_{\exp \mathfrak{p}_0}^G \nu_{\xi_0}$ *is irreducible if* π_0 *is irreducible.*

3. π *is square integrable if* π_0 *is square integrable.*

4. *The representation* π *is isomorphic to* $\mathrm{Ind}_{G_{p_0}}^G (\nu_{p_0} \otimes \pi_0)$*, where* $G_{p_0} := \mathbb{R}^n \rtimes H_{p_0}$ *and* $\nu_{p_0} \otimes \pi_0$ *is defined by*

$$\nu_{p_0} \otimes \pi_0(u, h) := \nu_{p_0}(u)\pi_0(h) \quad (\,(u, h) \in G_{p_0}\,).$$

We also obtain the Duflo-Moore operator when the stabilizer $\mathfrak{h}_{p_0} \neq \{0\}$. In this case, C_π is described by using C_{π_0} based on the assertion 4 above.

Theorem (see Theorem 3.14). *The Duflo-Moore operator for the representation* $(\pi, \mathcal{H}_\pi)$ *as in* (3.33) *can be described as*

$$\widetilde{C_\pi \phi}(l) = C_0^{1/2} \Delta_G^{-1/2}(l) C_{\pi_0} \tilde{\phi}(l) \quad (\tilde{\phi}(l) \in \mathcal{H}_{\pi_0}) \tag{11}$$

for almost all $l \in H$.

Furthermore, as an application from our results above, we declare some results for concrete groups as follows.

Theorem (see Thorem 4.1). *The Lie algebra* $\mathfrak{g} := \mathbb{R}^n \rtimes (\mathbb{R} \oplus \mathfrak{so}(n))$ *of the similitude Lie group* $\mathrm{Sim}(n) := \mathbb{R}^n \rtimes (\mathbb{R}_+ \times \mathrm{SO}(n))$ *is not a Frobenius Lie algebra for* $n \geq 3$.

Theorem (see Theorem 4.6). *The Lie algebra* $\mathfrak{aff}(n) = \mathbb{R}^n \rtimes \mathfrak{gl}_n(\mathbb{R})$ *of the connected affine automorphism group* $\mathrm{Aff}^+(n)$ *is Frobenius.*

The fact that $\mathfrak{aff}(n)$ is Frobenius was mentioned by Ooms (see [48, p.497]), but we give an alternative proof in this thesis. In more detail, we have criteria for the openness of the coadjoint orbit of $\mathrm{Aff}^+(n)$ as follows. Let Ω_{ξ_1} be a coadjoint orbit of $\mathrm{Aff}^+(n)$ through $\xi_1 := \xi(p_1, \alpha_1) \in \mathfrak{g}^* := \mathfrak{aff}(n)^*$. We denote the centralizer of α_1 in $\mathrm{Mat}_n(\mathbb{R})$ by $\mathrm{Cent}\,(\alpha_1)$ and image of the map $\mathrm{ad}\,(\alpha_1) : \mathrm{Mat}_n(\mathbb{R}) \to \mathrm{Mat}_n(\mathbb{R})$ by $\mathrm{Im}\,\mathrm{ad}\,(\alpha_1)$.

Proposition (see Proposition 4.9). *The orbit* Ω_{ξ_1} *is open in* $\mathfrak{g}^*$ *if and only if the following three conditions are satisfied*

1. $\dim \mathrm{Cent}\,(\alpha_1) = n$.
2. $\mathrm{Im}\,\mathrm{ad}\,(\alpha_1) \cap \{vp_1 \ ; \ v \in \mathbb{R}^n\} = \{0\}$.
3. *If* $A \in \mathrm{Cent}\,(\alpha_1) \setminus \{0\}$, *then* $-p_1 A \neq 0$.

We also investigate the representations of $\mathrm{Aff}^+(n)$ corresponding to open coadjoint orbits and their Duflo-Moore operators. For simplicity, let $G^n := \mathrm{Aff}^+(n) = \mathbb{R}^n \rtimes H^n$ with $H^n := \mathrm{GL}_n^+(\mathbb{R})$, and $\mathfrak{g}^n := \mathfrak{aff}(n) = \mathbb{R}^n \rtimes \mathfrak{h}^n$ with $\mathfrak{h}^n := \mathfrak{gl}_n(\mathbb{R})$. Let $\xi_1^\pm = (\pm 1, 0)$ be an element of $(\mathfrak{g}^1)^*$ and $\xi_n^\pm := \xi(p_n, \alpha_n^\pm)$ be an element of $(\mathfrak{g}^n)^*$ for

$n \geqq 2$ with

$$p_n = (0, 0, \dots, 1), \qquad \alpha_n^{\pm} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ \pm 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}. \tag{12}$$

We observe that $\Omega_{\xi_n^{\pm}} \subset (\mathfrak{g}^n)^*$ are open coadjoint orbits [see Section 4.3]. Moreover, let $\iota_n : \mathfrak{g}^{n-1} \hookrightarrow \mathfrak{h}^n_{p_n}$ be a Lie algebra isomorphism defined by

$$\iota_n(X(v, A)) = \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \ \ (A \in \mathfrak{gl}_{n-1}(\mathbb{R}),\ v \in \mathbb{R}^{n-1}),$$

and let $\mathfrak{p}_n = \mathbb{R}^n \rtimes \mathfrak{m}_n$ be defined inductively by $\mathfrak{m}_n := \iota_n(\mathfrak{p}_{n-1})$ starting from $\mathfrak{p}_1 = \mathbb{R}X_2 \subset \mathfrak{g}^1 = \mathfrak{aff}(1)$. In this case, $\mathfrak{m}_n$ is a polarization of $\mathfrak{h}^n_{p_n}$ satisfying Pukanszky condition and $\xi^{\pm}_{n-1} = \alpha^{\pm}_n \circ \iota_n$. We obtain

Theorem (see Theorem 4.10)**.** *Let $\xi_n^{\pm} = (p_n, \alpha_n^{\pm})$ be the element of $(\mathfrak{g}^n)^* = \mathfrak{aff}(n)^*$ as above.*

1. *$\mathfrak{p}_n$ is a polarization of $\mathfrak{g}^n$ at $\xi_n^{\pm}$ satisfying Pukanszky condition.*
2. *$\pi_{\Omega_{\xi_n^{\pm}}} = \mathrm{Ind}^{G^n}_{\exp \mathfrak{p}_n} \nu_{\xi_n^{\pm}}$ is irreducible and square-integrable.*
3. *Under the identification $H^n_{p_n} \simeq G^{n-1} = \mathrm{Aff}^+(n-1)$, the representation $\pi_{\Omega_{\xi_n^{\pm}}}$ is isomorphic to $\mathrm{Ind}^{G_n}_{\mathbb{R}^n \rtimes G^{n-1}} (\nu_{p_n} \otimes \pi_{\Omega_{\xi^{\pm}_{n-1}}})$.*

Proposition (see Proposition 4.11)**.** *The Duflo-Moore operator of $(\pi_{\Omega_{\xi_n}}, L^2(H^n/H^n_{p_n})$ of $G^n = \mathrm{Aff}^+(n)$ $(n \geq 2)$ can be described as*

$$\widetilde{C_{\pi_{\Omega_{\xi_n}}}} \phi(a) = C_0^{1/2} |\det a|^{1/2} C_{\pi_{\Omega_{\xi_{n-1}}}} \tilde{\phi}(a) \tag{13}$$

for almost all $a \in H^n$.

The general formula for the Pfaffian of $\mathfrak{aff}(n)$ can described as follows.

Proposition (see Proposition 4.12)**.** *Let* $\xi := \xi(p, \alpha)$ *be an element of* $(\mathfrak{g}^n)^*$ *as in* (4.31) *and* Φ_n *be a map given by*

$$\Phi_n : (\mathfrak{g}^n)^* \ni \xi \longmapsto \Phi(\xi) = \mathrm{Ad}^*(g(a))\xi \circ \iota_n \in (\mathfrak{g}^{n-1})^*$$

with

$$a = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \beta_1/\beta_n & \beta_2/\beta_n & \beta_3/\beta_n & \dots & \beta_{n-1}/\beta_n & 1 \end{pmatrix}.$$

Then the Pfaffian of $\mathfrak{g}^n = \mathfrak{aff}(n)$ *is of the form*

$$Q_{\mathfrak{g}^n}(\xi) = \beta_n^n \, Q_{\mathfrak{g}^{n-1}}(\Phi_n(\xi)). \tag{14}$$

Furthermore, we see from Proposition 4.13 and Proposition 4.14 that $\mathrm{Aff}^+(n)$ has exactly two open coadjoint orbits $\Omega_{\xi_n^+}$ and $\Omega_{\xi_n^-}$.

Finally, we explain the organization of this thesis as follows. In Chapter 1, we review the notion of coadjoint orbits which is the main object of the orbit method, Haar measure, and notion of the induced representations based on the Mackey model and the Blattner model. Furthermore, we also review the notion of a Frobenius Lie algebra, whose Lie group has open coadjoint orbits, and the notions of square-integrable representations. Concerning square-integrable representations, we review the notion of Duflo-Moore operator [15]. In Chapter 2, we present the results of harmonic analysis for 4-dimensional real Frobenius Lie algebras. We compute coadjoint orbits of each Lie group of 4-dimensional Frobenius Lie algebras, and we apply the orbits to construction of unitary irreducible representations using the orbit method. In case of 4-dimensional Frobenius Lie algebras, each irreducible unitary representation corresponding to the open coadjoint orbit is square-integrable and we give the formulas of Duflo-Moore operators as in (6) - (9) in Introduction. Differing from [37], our results in Chapter 2 are obtained in more concrete realizations with more direct computations. Chapter 3 consists of three parts. Firstly, we obtain the conditions for the Lie algebra $\mathfrak{g} := V \rtimes \mathfrak{h}$ of Lie group $G := V \rtimes H$ ($V \simeq \mathbb{R}^n, H \subset \mathrm{GL}(V)$) to be Frobenius. Secondly, we assume that the stabilizer $\mathfrak{h}_{p_0} = \{0\}$ and we compute the Duflo-Moore operator formula in this case. Thirdly, for the case of $\mathfrak{h}_{p_0} \neq \{0\}$, we obtain the Duflo-Moore operator for $(\pi, \mathcal{H}_\pi)$ of G using Duflo-Moore operator for $(\pi_0, \mathcal{H}_{\pi_0})$ of the stabilizer H_{p_0}. In Chapter 4, we apply the results to prove that the Lie algebra of similitude Lie group $\mathrm{Sim}(n) := \mathbb{R}^n \rtimes (\mathbb{R} \oplus \mathrm{SO}(n))$ is not Frobenius for $n \geq 3$. Furthermore, we prove that the Lie algebra $\mathfrak{aff}(n) = \mathbb{R}^n \rtimes \mathfrak{gl}_n(\mathbb{R})$ of the connected affine automorphism group $\mathrm{Aff}^+(n)$ is Frobenius. This means that $\mathrm{Aff}^+(n)$ has open coadjoint orbits. In addition, we get in more detail the necessary and sufficient conditions for the coadjoint orbits of $\mathrm{Aff}^+(n)$ to be open in $\mathfrak{aff}(n)^*$ besides the general conditions in Chapter 3. Moreover, these open coadjoint orbits yield square-integrable unitary representations of $\mathrm{Aff}^+(n)$ for which we obtain their Duflo-Moore operators formula. We should mention here that Lipsman and Wolf (see [42]) discussed the Plancherel formulas of parabolic subgroups, while $\mathrm{Aff}^+(n)$ is isomorphic to a maximal parabolic subgroup of $\mathrm{SL}_{n+1}(\mathbb{R})$. But our results are more direct than Lipsman and Wolf's work in [42].

Lastly, we compute the general formula for the Pfaffian of $\mathfrak{aff}(n)$ and we show that $\mathrm{Aff}^+(n)$ has exactly two open coadjoint orbits. We want to find a formula connecting the Duflo-Moore operator and the Pfaffian of $\mathfrak{aff}(n)$, but it remains for future study.

Chapter 1

Preliminaries

In this chapter we introduce some notions contributing in our study. We start from introducing the notion of coadjoint orbits which is the main object of the orbit method, Haar measure, and the notion of induced representation based on the Mackey model and the Blatter model. Furthermore, we also introduce the notion of a Frobenius Lie algebra which has open coadjoint orbits, and square-integrable representations corresponding to open coadjoint orbits. Concerning square-integrable representations, we review the Duflo-Moore operator.

1.1 The orbit method

First of all, let us introduce the notion of adjoint representation as follows.

Definition 1.1. ([38, p. 211]). Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any $g \in G$, the conjugation map given by $C_g : G \ni x \mapsto gxg^{-1} \in G$ is a Lie group homomorphism whose differential is denoted by $\mathrm{Ad}(g)$. The group homomorphism $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is called the **adjoint representation** of G.

Furthermore, the adjoint representation $\mathrm{ad} : \mathfrak{g} \ni X \mapsto \mathrm{ad}\,(X) \in \mathfrak{gl}(\mathfrak{g})$ is defined by $\mathrm{ad}(X)Y = [X, Y] \quad (X, Y \in \mathfrak{g})$. We shall see in the following theorem that both representations are related.

Theorem 1.1. *([38, p. 529]). Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation of G. Then the adjoint representation $\mathrm{Ad}_* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is given by $\mathrm{Ad}_* = \mathrm{ad}$.*

Moreover, we define a dual representation of Ad, called *the coadjoint representation*, by $\mathrm{Ad}^*(g) := \mathrm{Ad}(g^{-1})^*$. Namely, we have

$$\langle \mathrm{Ad}^*(g)f, X\rangle = \left\langle f, \mathrm{Ad}(g^{-1})X\right\rangle \qquad (X \in \mathfrak{g}, f \in \mathfrak{g}^*). \tag{1.1}$$

Then the above formula implies

$$\left\langle \mathrm{Ad}^*(e^X)f, Y\right\rangle = \left\langle f, e^{\mathrm{ad}(-X)}Y\right\rangle \qquad (X, Y \in \mathfrak{g}). \tag{1.2}$$

We also review the formula for an infinitesimal of coadjoint actions. Namely, for the corresponding representation ad^* of $\mathfrak{g}$ in $\mathfrak{g}^*$ we have

$$\langle \mathrm{ad}^*(X)f, Y\rangle = \langle f, -\mathrm{ad}(X)Y\rangle = \langle f, [Y, X]\rangle \quad (X, Y \in \mathfrak{g}). \tag{1.3}$$

Next, we review a coadjoint orbit. One can find the detailed reading of the properties of the coadjoint orbit for instance in [34],[36], and [46]. The coadjoint orbit of $f \in \mathfrak{g}^*$ is the set $\Omega_f = \{\mathrm{Ad}^*(g)f \ ; \ g \in G\} \subset \mathfrak{g}^*$. First of all, the coadjoint orbit Ω_f has a differential two-form ω_Ω given by $\omega_\Omega(f)(\mathrm{ad}^*(X)f, \mathrm{ad}^*(Y)f) = \langle f, [X, Y]\rangle$. The form ω_Ω is non-degenerate and closed, and this fact implies that the dimension of the coajoint orbit Ω_f is always even. In fact, we know that Ω_f is a symplectic manifold. Secondly, for any $f \in \mathfrak{g}^*$ we define the group stabilizer as

$$G_f = \{g \in G \ ; \ \mathrm{Ad}^*(g)(f) = f\} \subset G, \tag{1.4}$$

and its Lie algebra is denoted by $\mathfrak{g}_f$ and it is given by

$$\mathfrak{g}_f = \{X \in \mathfrak{g} \ ; \ \mathrm{ad}^*(X)f = 0\} \subset \mathfrak{g}. \tag{1.5}$$

One of the important things in our discussion is the orbit method that shall be explained as follows. Let G be a connected Lie group, and $\mathfrak{g}$ the Lie algebra of G. A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called a *polarization* at $f \in \mathfrak{g}^*$ if $\mathfrak{p}$ is a Lagrangian subspace with respect to the alternating form $B_f : \mathfrak{g} \times \mathfrak{g} \ni (X, Y) \longmapsto \langle f, [X, Y]\rangle \in \mathbb{R}$. To be more precise we have

Definition 1.2. ([36, p.26]). A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called a polarization at $f \in \mathfrak{g}^*$ if $f|_{[\mathfrak{p},\mathfrak{p}]} = 0$ and $\mathrm{codim}_{\mathfrak{g}}\, \mathfrak{p} = \frac{1}{2} \dim \Omega_f$.

Let us assume that the coadjoint orbit Ω_f of G is integral, that is, the form ω_Ω

belongs to an integer cohomology class [34, p. 239]. This is known to be equivalent to that there exists a one-dimensional unitary representation $\nu_f : \exp \mathfrak{p} \to \mathbb{C}$ such that $\nu_f(\exp X) := e^{2\pi i \langle f, X\rangle}$ for $X \in \mathfrak{p}$. Now we introduce the notion of *the Pukanszky condition* for a polarization $\mathfrak{p}$ at f.

Definition 1.3. ([3, p. 281 - 286]). A polarization $\mathfrak{p} \subset \mathfrak{g}$ satisfies the Pukanszky condition if

$$p^{-1}(p(f)) = f + \mathfrak{p}^{\perp} \subset \Omega_f, \tag{1.6}$$

where $p : \mathfrak{g}^* \to \mathfrak{p}^*$ is the natural projection map.

The representation π_f of G corresponding to the coadjoint orbit Ω_f is defined by $\pi_f = \mathrm{Ind}_{\exp \mathfrak{p}}^G \nu_f$. The irreducibility of the representation π_f is given by the following theorem.

Theorem 1.2. *([36, p.111]). Let G be an exponential solvable Lie group. The representation π_f is irreducible if only if the polarization $\mathfrak{p} \subset \mathfrak{g}$ satisfies the condition* (1.6).

We recall that a Lie group G is said to be exponential if the exponential map $\exp : \mathfrak{g} \to G$ is diffeomorphism. It is known that an exponential Lie group is necessarily solvable. On the other hand, a Lie algebra $\mathfrak{g}$ is said to be exponential if the corresponding connected and simply connected Lie group is exponential. The following is known.

Proposition 1.3. *(see [36, p. 110]) A Lie algebra $\mathfrak{g}$ is exponential if and only if* $\mathrm{ad}(X)$ *has no non-zero pure imaginary eigenvalues for any $X \in \mathfrak{g}$.*

It was shown in [5] that, if G is exponential, then for each $f \in \mathfrak{g}^*$ there exists a polarization $\mathfrak{p}$ satisfying the Pukanszky condition at f, and the unitary representation $\pi_f := \mathrm{Ind}_{\exp \mathfrak{p}}^G \nu_f$ of G is irreducible. Moreover, the equivalence class $[\pi_f]$ does not depend on the choice of such polarization $\mathfrak{p}$, and the map $f \mapsto [\pi_f]$ induces a one-to-one correspondence from the orbit space $\mathfrak{g}^*/\mathrm{Ad}^*(G)$ onto the unitary dual $\hat{G}$.

Although a Frobenius Lie algebra $\mathfrak{g}$ is not necessarily exponential solvable, we shall consider a unitary representation $\mathrm{Ind}_{\exp \mathfrak{p}}^G \nu_f$ defined from a polarization $\mathfrak{p}$ at $f \in \mathfrak{g}^*$ satisfying the Pukanszky condition when Ω_f is an open coadjoint orbit.

1.2 Frobenius Lie algebras

In the history, a Frobenius Lie algebra was found at first time in [47], [48], and [49] to give conditions for a finite dimensional Lie algebra in order its universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ has an exact simple module. Frobenius Lie algebra is very important in this thesis. Therefore, let us introduce this notion as follows.

Definition 1.4. ([12, p.427]). A Lie algebra $\mathfrak{g}$ over an arbitrary field $\mathbb{F}$ is said to be Frobenius if there exists a linear functional $f_0 \in \mathfrak{g}^*$ such that $\mathfrak{g}_{f_0} = 0$.

We introduce the index of Lie algebra $\mathfrak{g}$ (see for example in [14] and [52]) given by

$$\operatorname{ind}\mathfrak{g} = \min\{\dim \mathfrak{g}_f \ ; \ f \in \mathfrak{g}^*\}. \tag{1.7}$$

We can say that a Lie algebra $\mathfrak{g}$ is Frobenius if $\operatorname{ind}\mathfrak{g} = 0$. Let $\{X_i\}_{i=1}^n$ be a basis for $\mathfrak{g}$ and $M_\mathfrak{g}$ be an $n \times n$ matrix of $\mathfrak{g}$-entry whose (i, j)- component is $[X_i, X_j]$. We define $\det M_\mathfrak{g}$ as an element of the symmetric algebra $S(\mathfrak{g})$ which is identified with the polynomial algebra $\mathrm{Pol}(\mathfrak{g}^*)$ on $\mathfrak{g}^*$ and $\det M_\mathfrak{g}(f_0) = \det \langle f_0, [X_i, X_j]\rangle_{1\leq i,j\leq n},\ f_0 \in \mathfrak{g}^*$. In other words, $\det M_\mathfrak{g}(f_0)$ is equal to the determinant of the alternating bilinear form

$$B_{f_0} : \mathfrak{g} \times \mathfrak{g} \ni (X, Y) \mapsto \langle f_0, [X, Y]\rangle \in \mathbb{F}.$$

Proposition 1.4. *([12, p.428-430], [50, p. 20]). The Lie algebra $\mathfrak{g}$ is Frobenius if one of the following equivalent conditions is satisfied :*

1. *There exists $f_0 \in \mathfrak{g}^*$ so that the stabilizer $\mathfrak{g}_{f_0} = 0$.*
2. $\operatorname{ind}\mathfrak{g} = 0$.
3. $\det M_\mathfrak{g} \neq 0$.
4. $\det M_\mathfrak{g}(f_0) \neq 0$ *for a suitable $f_0 \in \mathfrak{g}^*$.*

We observe that $\det M_\mathfrak{g}(f_0) \neq 0$ if and only if B_{f_0} is non-degenerate. Therefore, the Frobenius Lie algebra has even dimension. Then we define the Pfaffian of $\mathfrak{g}$ in $S(\mathfrak{g})$ as the Pfaffian of the matrix $M_\mathfrak{g}$ and we denote it by $Q_\mathfrak{g}$. Let us recall the notion of Pfaffian for a square alternating matrix. Let $A = (A_{ij})_{1\leq i,j\leq 2n}$ be a square alternating matrix. The Pfaffian of A is defined as follows :

$$\mathrm{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma\in\mathfrak{S}_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} A_{\sigma(2i-1)\sigma(2i)}. \tag{1.8}$$

For example, we see that

$$\mathrm{Pf}\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = af - be + cd.$$

Remark 1. We have $\det A = \mathrm{Pf}(A)^2$.

We go back to the Pfaffian $Q_{\mathfrak{g}} = \mathrm{Pf}\, M_{\mathfrak{g}}$ of a Frobenius Lie algebra $\mathfrak{g}$. We know that $Q_{\mathfrak{g}} \neq 0$. Note that the Pfaffian $Q_{\mathfrak{g}}$ is defined for a fixed basis $\{X_i\}$ of $\mathfrak{g}$, but it is unique up to a constant multiple. Let $\{Y_r\}_{r=1}^n$ be another basis for $\mathfrak{g}$. Then we write $Y_r = \sum_{i=1}^n p_{ir} X_r$ with $p_{ir} \in \mathbb{F}$. We put $P = (p_{ij}) \in \mathrm{GL}_n(\mathbb{F})$. Let $M'_{\mathfrak{g}}$ be an $n \times n$ matrix of $\mathfrak{g}$-entry whose (r, s)-component is $[Y_r, Y_s]$. Then we have $M_{\mathfrak{g}'} = {}^tPM_{\mathfrak{g}}P$, so that

$$\mathrm{Pf}\, M'_{\mathfrak{g}} = (\det P)\, \mathrm{Pf}\, M_{\mathfrak{g}}. \tag{1.9}$$

Therefore, we get

Proposition 1.5. *[50, p.28]. If $\mathfrak{g}$ is a Frobenius Lie algebra with a basis $\{X_i\}_{i=1}^n$, then $Q_{\mathfrak{g}} := \mathrm{Pf}\, M_{\mathfrak{g}} \in S(\mathfrak{g})$ is non-zero and it is determined by $\mathfrak{g}$ up to non-zero scalar multiple.*

Let $\psi : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra automorphism which is naturally extended to an algebra automorphism $\psi : S(\mathfrak{g}) \to S(\mathfrak{g})$. Then we obtain

$$\psi(Q_{\mathfrak{g}}) = (\det \psi) Q_{\mathfrak{g}}. \tag{1.10}$$

To prove (1.10), we take $A_\psi = (a_{ij})$ as the matrix expression of ψ with respect to the basis $\{X_i\}_{i=1}^n$. We get another basis $\{\psi(X_j)\}_{j=1}^n$ of $\mathfrak{g}$ with $\psi(X_j) = \sum_{i=1}^n a_{ij} X_j, j = 1, \ldots, n$. By extension ψ to an algebra automorphism of $S(\mathfrak{g})$ and using (1.9) we obtain

$$\begin{aligned} \psi(Q_{\mathfrak{g}}) &= \mathrm{Pf}\,(\psi[X_i, X_j]) = \mathrm{Pf}\,([\psi(X_i), \psi(X_j)]) \\ &= (\det A_\psi)\, Q_{\mathfrak{g}} = (\det \psi)\, Q_{\mathfrak{g}}. \end{aligned}$$

Moreover, we obtain

Proposition 1.6. *[12, p.430]). Let $\mathfrak{g}$ be n-dimensional Frobenius Lie algebra and let $S(\mathfrak{g})$ be its symmetric algebra of $\mathfrak{g}$. Then we have*

- $\mathrm{D}_{\mathrm{ad}(X)} Q_{\mathfrak{g}} = (\mathrm{tr}\,\mathrm{ad}\,\mathrm{X}) Q_{\mathfrak{g}}$ *where*

  $$\mathrm{D}_{\mathrm{ad}(X)} : S(\mathfrak{g}) \to S(\mathfrak{g})$$

  is an algebra derivation extended from $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$.

- *If $\mathfrak{g}$ is non-zero with* $\mathrm{char}\,\mathbb{F} = 0$ *then $\mathfrak{g}$ is non-unimodular.*

We shall consider the case $\mathbb{F} = \mathbb{R}$. Namely, let $\mathfrak{g}$ be a real Frobenius Lie algebra and G be a real Lie group whose Lie algebra is $\mathfrak{g}$. We see that the Lie algebra $\mathfrak{g}$ is Frobenius if and only if G has open coadjoint orbits. Moreover, substituting $\psi = \mathrm{Ad}(g)$ $(g \in G)$ to (1.10) we get

Proposition 1.7.

$$Q_{\mathfrak{g}}(\mathrm{Ad}^*(g)^{-1}\xi) = (\det \mathrm{Ad}(g))\ Q_{\mathfrak{g}}(\xi) \quad (g \in G,\ \xi \in \mathfrak{g}^*). \tag{1.11}$$

We shall apply the proposition above later, particularly in Section 4.5 to derive the general formula for the Pfaffian of $\mathfrak{aff}(n)$.

1.3 Haar measure

In this subsection we shall introduce the notion of an invariant measure over a locally compact topological group G.

Definition 1.5. (see [4, p. 67-70] & [23, p. 29-34]). Let $C_c(G)$ be the space of all continuous functions on G with a compact support. A norm on $C_c(G)$ is defined by

$$||f||_\infty := \sup_{g \in G} |f(g)|.$$

The support of $f \in C_c(G)$ is denoted by $\mathrm{supp}(f)$ and defined as the closure of $\{g \in G\ ;\ f(g) \neq 0\}$.

Definition 1.6. (see [4, p. 67-70] & [23, p. 29-34]). A linear functional μ on $C_c(G)$ is said to be a Borel measure if for each compact subset $M \subset G$, there exists a constant C_M such that

$$|\mu(f)| \leqq C_M ||f||_\infty \quad (f \in C_c(G),\ \mathrm{supp}(f) \subset M).$$

Moreover, a linear functional μ is said to be positive if $\mu(f) \geqq 0$ for any non-negative function $f \in C_c(G)$.

For $f \in C_c(G)$ and $a \in G$, we define a left translation $L_a f$ and a right translation $R_a f$ in $C_c(G)$ by $L_a f(g) := f(a^{-1}g)$ and $R_a f(g) := f(ga)$ respectively.

A positive Borel measure μ_l is called a **left Haar measure** if μ_l is left-invariant, that is,

$$\mu_l(L_a f) = \mu_l(f) \qquad (f \in C_c(G), a \in G).$$

We also define a right Haar measure μ_r by

$$\mu_r(R_a f) = \mu_r(f) \qquad (f \in C_c(G), a \in G).$$

Theorem 1.8. *(see [4, p. 67]). Every locally compact group G has a unique left Haar measure μ_l up to multiplication by a positive number.*

For each $a \in G$, let us define $\mu_{l,a}(f) = \mu_l(R_a f)$. Since $L_x \circ R_a = R_a \circ L_x$ we have

$$\mu_{l,a}(L_x f) = \mu_l(R_a L_x f) = \mu_l(L_x R_a f) = \mu_{l,a}(f).$$

Therefore, $\mu_{l,a}(f)$ is a left Haar measure. By Theorem 1.8, there exists a positive number denoted by $\Delta_G(a)$ such that

$$\mu_{l,a} = \Delta_G(a)^{-1}\mu_l. \tag{1.12}$$

The function $\Delta_G : G \ni a \mapsto \Delta_G(a) \in \mathbb{R}_+$ is called the **modular function** of G which is continuous homomorphism. We note that some authors denote the modular function of G by Δ_G and others denote it by Δ_G^{-1}. Furthermore, we shall use the left Haar measure in the Blattner model in Chapter 3.

Let μ' be a measure on G given by

$$\int_G f(g)\, d\mu'(g) := \int_G f(g)\Delta_G(g)^{-1}\, d\mu_l(g).$$

Then we have

$$\begin{aligned}\int_G R_a f(g)\,d\mu'(g) &= \int_G f(ga)\Delta_G(g)^{-1}\,d\mu_l(g)\\ &= \Delta_G(a)\int_G f(ga)\Delta_G((ga)^{-1})\,d\mu_l(g)\\ &= \Delta_G(a)\int_G f(g')\Delta_G(g')^{-1}\,d\mu_l(g'a^{-1})\quad (g'=ga)\\ &= \Delta_G(a)\int_G f(g')\Delta_G(g')^{-1}\,d\mu_{l,a}(g')\\ &= \Delta_G(a)\int_G f(g')\Delta_G(g')^{-1}\Delta_G(a)^{-1}\,d\mu_l(g')\\ &= \int_G f(g')\Delta_G(g')^{-1}\,d\mu_l(g') = \int_G f(g')\,d\mu'(g').\end{aligned}$$

Therefore, μ' is a right Haar measure and we can write

$$\mu_r = \Delta_G^{-1}\,\mu_l. \tag{1.13}$$

By Theorem 1.8 and (1.13), we can obtain

$$\mu_{r,a} = \Delta_G(a)\,\mu_r. \tag{1.14}$$

We shall use the right Haar measure in the Mackey model in Chapter 2. In this thesis, we usually write $dg\ \ (g\in G)$ for the left Haar measure $d\mu_l(g)$.

1.4 Induced representation

We now review briefly the notion of induced representations of Lie groups as an essential summary of [6], [8], [11], [17], [23], [24], [33], [34], [36], [39], and [41]. Our setting devides into two parts. The first is the Mackey model as in [36] which is applied to Chapter 2, and the second is the Blattner model as devoted in [8], [9], and [23] which is applied to Chapter 3.

For the first setting, let H be a closed subgroup of a Lie group G, and let $L^2(X)$, where $X := H\backslash G = \{Hg\ ;\ g\in G\}$, $\dim X = n$, be what so-called natural Hilbert space consisting of square integrable sections of line bundle L of half-densities on X. Although the Mackey model can be considered for locally compact groups, we discuss it in the Lie groups category following [36]. For a given coordinate chart U on X with the component $(x^1, x^2, \ldots, x^n)$, we can write the section f by $f(x)\sqrt{d^n x}$ and the the inner product on $L^2(X)$ is given by

$$(f_1|f_2)_{L^2(X)} = \int_X f_1(x)\overline{f_2(x)}\, d^n x \qquad (f_1, f_2 \in L^2(X)). \tag{1.15}$$

Let ρ be a unitary representation of H in a Hilbert space $\mathcal{H}_\rho$. We shall define a unitary representation π of G through an extension of $(\rho, \mathcal{H}_\rho)$. This representation is called *the induced representation* and is denoted by $\mathrm{Ind}_H^G \rho$.

Let $s : U \to G$ be a section of natural projection $p : g \mapsto Hg$. We assume that almost all $g \in G$ can be written as $g = hs(x)$ ($h \in H, x \in U$). We emphasize here that in general the fiber bundle $H \to G \to X$ is non-trivial, so that there is no smooth or even continuous section $s : X \to G$ of the projection map p on the whole X (see [36, p.380]). Let $d_l^G(g)$ and $d_l^H(h)$ be left invariant volume forms for G and H respectively. In terms $(h, x) \in H \times U$ we obtain

$$d_l^G(g) = r(h, x)\, d_l^H(h)\, d^n x \quad (\text{ for some smooth function } r \text{ on } H \times U \ ).$$

Indeed, since $d_l^G(h'g) = d_l^G(g)$ and $d_l^H(h'h) = d_l^H(h)$ then $r(h'h, x) = r(h, x)$. This implies $d_l^G(g) = r(x)\, d_l^H(h)\, d^n x$. Now let us define a measure $d\mu_s$ on U as follows :

$$d\mu_s(x) = r(x)\, \Delta_G(s(x))^{-1}\, d^n x.$$

Therefore, we have

$$d_l^G(g) = \Delta_G(s(x))\, d_l^H(h)\, d\mu_s(x). \tag{1.16}$$

Furthermore, since $d_r^G(g) = \Delta_G(g)^{-1}\, d_l^G(g)$ and $d_r^H(h) = \Delta_H(h)^{-1}\, d_l^H(h)$, then we have

$$d_r^G(g) = d_r^G(hs(x)) = \Delta_{G,H}(h)^{-1}\, d_r^H(h)\, d\mu_s(x), \tag{1.17}$$

where $\Delta_{G,H}(h) = \dfrac{\Delta_G(h)}{\Delta_H(h)}$. Moreover, an important property of measure μ_s is given as follows:

Lemma 1.9. *([36, p. 381]). We have a measure relation of the form*

$$d\mu_s(x.g) = \Delta_{G,H}(h_s(x, g))\ d\mu_s(x), \tag{1.18}$$

where ***the master equation***

$$s(x)g = h_s(x,g)s(x.g) \tag{1.19}$$

defines $h_s(x,g) \in H$.

The explanations above suggest us to define **the unitary induced representation** $\mathrm{Ind}_H^G \rho$ on the representation space

$$\begin{aligned}\mathcal{H}_\pi &= L^2(X, \mathcal{H}_\rho, \mu_s) \\ &= \left\{ f : X \to \mathcal{H}_\rho \ ; \ \ ||f||^2_{\mathcal{H}_\pi} = \int_X |f(x)|^2_{\mathcal{H}_\rho} \, d\mu_s(x) < \infty \quad \right\}\end{aligned} \tag{1.20}$$

by

$$\pi(g)f(x) := \Delta_{G,H}(h_s(x,g))^{1/2}\rho(h_s(x,g))f(x \cdot g). \tag{1.21}$$

Indeed, this representation is unitary since

$$\begin{aligned}||\pi(g)f(x)|| &= \int_G \Delta_{G,H}(h_s(x,g))||\rho(h_s(x,g))f(x\cdot g)||^2 \, d\mu_s(x) \\ &= \int_G ||f(x\cdot g)||^2 \, \Delta_{G,H}(h_s(x,g)) \, d\mu_s(x) \quad (\rho \text{ is unitary}) \\ &= \int_G ||f(x\cdot g)||^2 \, d\mu_s(x\cdot g) \quad (d\mu_s(x\cdot g) = \Delta_{G,H}(h_s(x,g)) \, d\mu_s(x)) \\ &= \int_G ||f(x')||^2 \, d\mu_s(x') = ||f||^2 \quad (x' = x\cdot g).\end{aligned}$$

We state the useful property for us the so-called **the induction by stages**.

Proposition 1.10 (see [23], [36])**.** *Let G be a Lie group and $G_1 \subset G_2 \subset G$ be two Lie subgroups of G. Let ρ be a unitary representation of G_1 on $\mathcal{H}_\rho$. Then the induced representations* $\mathrm{Ind}_{G_1}^G \rho$ *and* $\mathrm{Ind}_{G_2}^G (\mathrm{Ind}_{G_1}^{G_2} \rho)$ *are unitarily equivalent.*

For the second setting, we review the Blattner model like as in [8], [9], and [23]. Let G be a locally compact group and H be a closed subgroup of G. It is well known that a G-invariant Borel measure on G/H exists if and only if $\Delta_H = \Delta_G|H$.

Definition 1.7. ([23, p.54]). Let G be a locally compact group and $H \subset G$ be a closed subgroup of G. We define the space $\mathcal{E}(G/H)$ as follows :

$$\mathcal{E}(G/H) := \{\xi : G \to \mathbb{C} \ ; \ \xi(gh) = \Delta_{H,G}(h)\xi(g), \quad \forall g \in G, h \in H,$$
$$\xi \text{ is continuous with compact support modulo } H\}.$$

Let us see a useful proposition as follows.

Proposition 1.11. *([23, p. 55]). Let G be a locally compact group and $H \subset G$ be a closed subgroup of G. There exists a unique (up to multiplication by a positive number) G-invariant positive linear functional on the space $\mathcal{E}(G/H)$, denoted by*

$$\mu_{G,H}(\xi) = \oint_{G/H} \xi(x)\, d\mu_{G,H}(x). \tag{1.22}$$

Furthermore, we have

$$\int_G f(x)\, dx = \oint_{G/H} \{\int_H f(xh)\Delta_{G,H}(h)\, dh\} d\mu_{G,H}(x). \tag{1.23}$$

Let K be a closed subgroup of H. By the transitivity of $\mu_{G,H}$, we obtain

$$\oint_{G/K} \phi(g)\, d\dot{g} = \oint_{G/H} \{\oint_{H/K} \phi(gh)\Delta_{G,H}(h)\, d\mu_{G,H}(h)\}\, d\mu_{G,H}(g) \quad (\phi \in \mathcal{E}(G/K)). \tag{1.24}$$

Definition 1.8. ([23, p. 59]). Let G be a locally compact group and H be its closed subgroup. We assume ρ is a unitary representation of H on $\mathcal{H}_\rho$. The space $\mathcal{E}(G/H, \rho)$ is defined by

$$\mathcal{E}(G/H, \rho) := \{f : G \to \mathcal{H}_\rho \ ; \ f(gh) = \Delta_{G,H}(h)^{-1/2}\rho(h)^{-1}f(g), \quad \forall g \in G, h \in H,$$
$$f \text{ is continuous with compact support modulo } H\}.$$

We define a scalar product on $\mathcal{E}(G/H, \rho)$ by

$$(f_1|f_2)_{\mathrm{Ind}_\rho} := \oint_{G/H} (f_1(g)|f_2(g))\, d\dot{g} \quad (f_1, f_2 \in \mathcal{E}(G/H, \rho)).$$

and its norm given by $||f||_{\mathrm{Ind}_\rho} := \sqrt{(f|f)_{\mathrm{Ind}_\rho}}$.

Definition 1.9. ([23, p. 61]). Let $L^2(G/H,\rho)$ be a completion of $\mathcal{E}(G/H,\rho)$ with $||\cdot||_{\mathrm{Ind}_\rho}$. The action of G on $L^2(G/H,\rho)$ is denoted by $\pi := \mathrm{Ind}_H^G\,\rho$, and given by

$$\pi(g)f(x) := f(g^{-1}x) \quad (f \in L^2(G/H,\rho)\ g, x \in G). \tag{1.25}$$

We call π an induced representation of G.

Furthermore, we also introduce the notion of unitary representations of semi-direct products (for detail one can see in [43], [44], and [45]). Let $G := N \rtimes H$ be a semi-direct product of separable, locally compact groups N and H where N is an abelian group. We define a usual product in G by $(n,h)(n',h') := (n\tau(h)(n'), hh')$, and $(n,h)^{-1} := (\tau(h^{-1})(n^{-1}), h^{-1})$, where $\tau : H \to \mathrm{Aut}(N)$ is a group homomorphism. The regularity of G must be satisfied. Namely, we assume that we can find an analytic subset $\hat{N}_1$ of the set of characters $\hat{N}$ of N which intersects each G-orbit exactly once. Furthermore, the construction of this representation follows constructions in [4] and rewritten as follows.

Theorem 1.12. *(see [4, p.508-509]). Let $G := N \rtimes H$ be a regular semi-direct product of separable, locally compact groups N and H with N is abelian. Then every irreducible unitary representation π of G is induced from an irreducible representation ν of $N \rtimes H_{\hat{n}_0}$ with $H_{\hat{n}_0}$ is a stabilizer of H at a point $\hat{n}_0 \in \hat{N}$ such that $\nu|_N$ equals $\hat{n}_0\mathrm{Id}$ and $\nu = \hat{n}_0 \otimes L$ $(L \in \hat{H}_{\hat{n}_0})$. Namely, $\pi = \mathrm{Ind}_{N \rtimes H_{\hat{n}_0}}^G\,\nu$.*

1.5 Intertwining Operators

Let π and π' be unitary representations of G in the representation spaces $\mathcal{H}_\pi$ and $\mathcal{H}_{\pi'}$ respectively. We introduce the notion of **an intertwining operator**.

Definition 1.10. ([6, p. 9]). A bounded linear operator $T : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ is called **an intertwining operator** from π to π', if it satisfies $T \circ \pi(g) = \pi'(g) \circ T$ for any $g \in G$. The representations π and π' are said to be **equivalent** if the intertwining operator $T : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ is unitary.

We recall that the induced representation $\pi_M := \pi_{\mathrm{Mackey}}$ in the Mackey model is defined on the representation space $L^2(X, \mathcal{H}_\rho, \mu_s)$ given by

$$\pi_M(g_0)f(x) := \Delta_{G,H}(h_s(x,g_0))^{1/2}\rho(h_s(x,g_0))f(x\cdot g_0) \quad (f \in L^2(X,\mathcal{H}_\rho,\mu_s)).$$

On the other hand, the induced representation $\pi_B := \pi_{\text{Blattner}}$ in the Blattner model is defined on the representation space $L^2(G/H, \rho)$ given by

$$\pi_B(g_0)\phi(g) := \phi(g_0^{-1}g) \quad (\phi \in L^2(G/H, \rho)).$$

In this section, we shall see that π_M and π_B is equivalent.

First of all, let us introduce a linear isomorphism

$$R : \mathcal{E}(G/H) \to C_c(X)$$

given by $R\xi(x) := \xi(s(x)^{-1}) \quad (\xi \in \mathcal{E}(G/H), x \in X)$ with its invers is given by $R^{-1}f(g) = \Delta_{G,H}(h)f(x) \quad (f \in C_c(X), g^{-1} = hs(x))$. Let us define for each $a \in G$, $\xi_a(g) := \xi(a^{-1}g)$. We see that

$$\begin{aligned} R\xi_a(x) &= \xi_a(s(x)^{-1}) = \xi(a^{-1}s(x)^{-1}) \\ &= \xi(h_s(x,a)^{-1}s(x \cdot a)^{-1}) \quad (s(x)a = h_s(x,a)s(x \cdot a)) \\ &= \Delta_{G,H}(h_s(x,a))R\xi(x \cdot a). \end{aligned}$$

It follows that

$$\begin{aligned} \int_X R\xi_a(x)\, d\mu_s(x) &= \int_X R\xi(x \cdot a)\Delta_{G,H}(h_s(x,a))\, d\mu_s(x) \\ &= \int_X R\xi(x \cdot a)\, d\mu_s(x \cdot a) \quad \text{(by (1.18))} \\ &= \int_X R\xi(x')\, d\mu_s(x') \quad (x' = x \cdot a). \end{aligned}$$

Therefore a linear functional $\mu' : \mathcal{E}(G/H) \to \mathbb{R}$ given by $\mu'(\xi) = \int_X R\xi(x)\, d\mu_s(x)$ is left G-invariant. By Proposition 1.11, there exist $C_0 > 0$ such that $\mu'(\xi) = C_0\mu_{G,H}(\xi)$. In other words we have

$$\int_X \xi(s(x)^{-1})\, d\mu_s(x) = C_0 \oint_{G/H} \xi(g)\, d\mu_{G,H}(g). \tag{1.26}$$

Now we are ready to prove the following proposition.

Proposition 1.13. *The unitary representations π_B and π_M are equivalent.*

Proof. Let us recall that $s : X \supset U \to G$ is a section of of natural projection

$p : G \ni g \mapsto Hg \in X$ and almost all $g \in G$ can be expressed as $g = hs(x)$ ($h \in H, x \in U$). We shall show that the linear isomorphism

$$T : L^2(G/H, \rho) \to L^2(X, \mathcal{H}_\rho, \mu_s)$$

given by $T\phi(x) := \frac{1}{\sqrt{C_0}}\phi(s(x)^{-1})$ (C_0 is defined in (1.26)) is an intertwining operator which is unitary. Indeed, the invers map of T is given by $T^{-1}f(g) = \Delta_{G,H}(h)^{1/2}\rho(h)f(x)$ $(g^{-1} = hs(x))$. Using (1.26), we have

$$\begin{aligned}
||T\phi||^2_{L^2(X,\mathcal{H}_\rho,\mu_s)} &= \int_X |T\phi(x)|^2 \, d\mu_s(x) \\
&= \int_X \frac{1}{C_0}|\phi(s(x)^{-1})|^2 \, d\mu_s(x) \\
&= \oint_{G/H} |\phi(g)|^2 \, d\mu_{G,H}(g) = ||\phi||^2_{L^2(G/H,\rho)}.
\end{aligned}$$

This means that T is unitary. Moreover, we see that

$$\begin{aligned}
T \circ \pi_B(g_0)\phi(x) &= \pi_B(g_0)\phi(s(x)^{-1}) \\
&= \phi(g_0^{-1}s(x)^{-1}) = \phi((s(x)g_0)^{-1}) \\
&= \phi((h_s(x,g_0)s(x \cdot g_0))^{-1}) = \phi(s(x \cdot g_0)^{-1}h_s(x,g_0)^{-1}) \\
&= \Delta_{G,H}(h_s(x,g_0))^{1/2}\rho(h_s(x,g_0))\phi(s(x \cdot g_0)^{-1}) \\
&= \pi_M(g_0) \circ T\phi(x).
\end{aligned}$$

In other words $T \circ \pi_B(g_0) = \pi_M(g_0) \circ T$. Therefore, T is a unitary intertwining operator. Thus, π_B and π_M are equivalent as desired. □

1.6 Square-integrable representation

It is well known that the study of square-integrable representations corresponds to continuous wavelet transform (see for examples in [21], [22]). First of all, let us introduce the notion of a square-integrable representation as follows :

Definition 1.11 (see [15]). Let $(\pi, \mathcal{H}_\pi)$ be an irreducible unitary representation of a locally compact group G. Then π is said to be square-integrable if there exists a non-zero vector $\phi \in \mathcal{H}_\pi$ such that

$$\int_G |(\phi|\pi(g)\phi)_{\mathcal{H}_\pi}|^2 \, dg < \infty. \tag{1.27}$$

If this happens then a vector ϕ is called *admissible*. Moreover, Duflo and Moore [15] showed that there exists a (not necessarily bounded, densely defined) unique operator C_π on $\mathcal{H}_\pi$ and it is called Duflo-Moore operator which is positive self-adjoint. This operator satisfies (see [15] or [27] for detail)

1. ϕ is admissible if and only if $\phi \in \operatorname{dom} C_\pi$, and
2. For $\phi_1, \phi_3 \in \mathcal{H}_\pi$ and $\phi_2, \phi_4 \in \operatorname{dom} C_\pi$, we have

$$\int_G (\phi_1|\pi(g)\phi_2)_{\mathcal{H}_\pi} (\pi(g)\phi_4|\phi_3)_{\mathcal{H}_\pi} \, dg = (\phi_1|\phi_3)_{\mathcal{H}_\pi} (C_\pi\phi_4|C_\pi\phi_2)_{\mathcal{H}_\pi}. \tag{1.28}$$

By changing variable $g' = ga$ with a fixed $a \in G$ at the integral in (1.28), we obtain

$$\begin{aligned} &(\phi_1|\phi_3)_{\mathcal{H}_\pi} (C_\pi\phi_4|C_\pi\phi_2)_{\mathcal{H}_\pi} \\ &\quad = \int_G (\phi_1|\pi(g')\pi(a)\phi_2)_{\mathcal{H}_\pi} (\pi(g')\pi(a)\phi_4|\phi_3)_{\mathcal{H}_\pi} \Delta_G(a) \, dg' \\ &\quad = \Delta_G(a) (\phi_1|\phi_3)_{\mathcal{H}_\pi} (C_\pi\pi(a)\phi_4|C_\pi\pi(a)\phi_3)_{\mathcal{H}_\pi}, \end{aligned}$$

where Δ_G denotes the modular function of G. Therefore, if $\phi_2 \in \operatorname{dom} C_\pi$, then $\pi(a)\phi_2 \in \operatorname{dom} C_\pi$, and we have

$$C_\pi^2 \phi_2 = \Delta_G(a) \, \pi(a)^{-1} \circ C_\pi^2 \circ \pi(a)\phi_2$$

thanks to the self-adjointness of C_π. Moreover, since C_π is positive, we obtain

$$\pi(a) \circ C_\pi \circ \pi(a)^{-1} = \Delta_G(a)^{1/2} C_\pi \qquad (a \in G). \tag{1.29}$$

Taking an admissible vector $v_0 \in \operatorname{dom} C_\pi$, we have an isometric embedding $W_{v_0} : \mathcal{H}_\pi \to L^2(G)$ defined by

$$W_{v_0} v(g) := (v|\pi(g)v_0)/||C_\pi v_0||^2 \quad (v \in \mathcal{H}_\pi, \, g \in G).$$

We observe that the map W_{v_0}, called *a continuous wavelet transform*, is an intertwining operator from π into the left-regular representation. In this way, we see that a square-integrable representation is a subrepresentation of the left-regular representation, and vice-versa. For more detail, the works of square-integrable representations for instance can be read in [7], [10], [29], [31], [32], and [53].

Chapter 2

Harmonic Analysis for Frobenius Lie Algebra of Dimension 4

Based on the work [12] by Csikós and Verhóczki about classification of isomorphism classes of Frobenius Lie algebras with dimension ≤ 6 over a field with characteristic not equals 2, we present some results of harmonic analysis of real Lie groups whose Lie algebras are Frobenius of dimension 4. Particularly, we work on real case. The isomorphism classes of Frobenius Lie algebras of dimension 4 are given by

Theorem 2.1. *(see [12, p.448]) For any Frobenius Lie algebra of dimension 4 over a field $\mathbb{F}$ of characteristic $\neq 2$ is isomorphic to one of the following*

1. $\mathfrak{g}_I : [X_1, X_4] = [X_2, X_3] = -X_1, [X_2, X_4] = -\dfrac{X_2}{2}, [X_3, X_4] = -\dfrac{X_3}{2},$
2. $\mathfrak{g}_{II}(\tau), \tau \in \mathbb{F} : [X_1, X_4] = [X_2, X_3] = -X_1, [X_2, X_4] = -X_3,$
   $[X_3, X_4] = -X_3 + \tau X_2,$
3. $\mathfrak{g}_{III}(\varepsilon)$, *where* $0 \neq \varepsilon \in \mathbb{F} : [X_1, X_3] = [X_2, X_4] = -X_1, [X_1, X_4] = \varepsilon X_2,$
   $[X_2, X_3] = -X_2.$
   The Frobenius Lie algebras $\mathfrak{g}_{III}(\varepsilon)$ and $\mathfrak{g}_{III}(\varepsilon')$ are isomorphic if and only if ε/ε' is the square element of an element of $\mathbb{F}$.

From Theorem 2.1 above, we get a list of 4-dimensional real Frobenius Lie algebras as follows :

1. $\mathfrak{g}_I : [X_1, X_4] = [X_2, X_3] = -X_1, [X_2, X_4] = -\dfrac{X_2}{2}, [X_3, X_4] = -\dfrac{X_3}{2}$,
2. $\mathfrak{g}_{II}(\tau), \tau \in \mathbb{R} : [X_1, X_4] = [X_2, X_3] = -X_1, [X_2, X_4] = -X_3$, $[X_3, X_4] = -X_3 + \tau X_2$,
3. $\mathfrak{g}_{III}(\varepsilon), \varepsilon = \pm 1 : [X_1, X_3] = [X_2, X_4] = -X_1, [X_1, X_4] = \varepsilon X_2$, $[X_2, X_3] = -X_2$.

Remark 2. For the third type, for $\varepsilon = -1$, we know that $\mathfrak{g}_{III}(-1)$ is exponential solvable and it is isomorphic to a direct sum $\mathfrak{aff}(1) \oplus \mathfrak{aff}(1)$ where $\mathfrak{aff}(1)$ is a Lie algebra of $ax + b$ group. On the other hand, for $\varepsilon = 1$, the Lie algebra $\mathfrak{g}_{III}(1)$ is solvable but not exponential and $\mathfrak{g}_{III}(1)$ is isomorphic to the Lie algebra $\mathbb{R}^2 \rtimes (\mathbb{R} \times \mathfrak{so}(2))$ of a similitude group $\mathrm{Sim}(2) := \mathbb{R}^2 \rtimes (\mathbb{R}_+ \times \mathrm{SO}(2))$ (see Section 4.1).

2.1 The first type $\mathfrak{g}_I$

We observe that the first type of real Frobenius Lie algebra $\mathfrak{g}_I$ is exponential solvable. Let G_I be an exponential solvable Lie group whose Lie algebra is $\mathfrak{g}_I$. We start by computing coadjoint orbits of G_I. Let $U = aX_1 + bX_2 + cX_3 + qX_4 \in \mathfrak{g}_I$ and $F = \alpha X_1^* + \beta X_2^* + \gamma X_3^* + \delta X_4^* \in \mathfrak{g}_I^*$. We obtain $\mathrm{ad}(-U) = \begin{pmatrix} -q & -c & b & a \\ 0 & -q/2 & 0 & b/2 \\ 0 & 0 & -q/2 & c/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ with respect to the basis $\{X_i\}_{i=1}^4$. In addition, for the case $q \neq 0$ we use identity

$$\begin{pmatrix} -q & -c & b & a \\ 0 & -q/2 & 0 & b/2 \\ 0 & 0 & -q/2 & c/2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = P \begin{pmatrix} -q & 0 & 0 & 0 \\ 0 & -q/2 & 0 & 0 \\ 0 & 0 & -q/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P^{-1} \tag{2.1}$$

for a suitable non-singular matrix P, to compute $\exp(\mathrm{ad}(-U))$. The result is

$$\exp(\mathrm{ad}(-U)) = \begin{pmatrix} \lambda^2 & \frac{2c\lambda}{q}(\lambda-1) & \frac{2b\lambda}{q}(1-\lambda) & \frac{a}{q}(1-\lambda^2) \\ 0 & \lambda & 0 & \frac{b}{q}(1-\lambda) \\ 0 & 0 & \lambda & \frac{c}{q}(1-\lambda) \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.2}$$

where $\lambda := \exp(-q/2) > 0$. When $q = 0$, $\exp(\mathrm{ad}(-U))$ is given by the limit of $d \to 0$ in the expression (2.2), that is

$$\begin{pmatrix} 1 & -c & -b & a \\ 0 & 1 & 0 & \frac{b}{2} \\ 0 & 0 & 1 & \frac{c}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.3}$$

Since $\mathfrak{g}_I$ is exponential, we compute the coadjoint orbit $\Omega_F = \Omega_{(\alpha,\beta,\gamma,\delta)}$ of G_I as the set of $\mathrm{Ad}^*(\exp U)F = xX_1^* + yX_2^* + zX_3^* + tX_4^*$. We obtain the following from (2.2)

$$\begin{aligned} x &= \lambda^2\alpha, \\ y &= \frac{2c\lambda}{q}(\lambda-1)\alpha + \lambda\beta, \\ z &= \frac{2b\lambda}{q}(1-\lambda)\alpha + \lambda\gamma, \\ t &= \frac{a}{q}(1-\lambda^2)\alpha + \frac{b}{q}(1-\lambda)\beta + \frac{c}{q}(1-\lambda)\gamma + \delta. \end{aligned} \tag{2.4}$$

Again the case $q = 0$ is obtained by taking limit. From (2.4) we can determine all coadjoint orbits of G_I. It is easy to see for $\alpha = \beta = \gamma = 0$ we have 0-dimensional coadjoint orbits. Also for $\alpha = 0$ and $(\beta, \gamma) \neq (0, 0)$ we have 2-dimensional coadjoint orbits of the form $\Omega_{(0,\cos\theta,\sin\theta,0)}$ and for the last we have 4-dimensional coadjoint orbits if $\alpha \neq 0$. Thus we have the following theorem

Theorem 2.2. *Let G_I be the exponential solvable Lie group whose Lie algebra is the 4-dimensional real Frobenius Lie algebra $\mathfrak{g}_I$. Then the set $\mathcal{O}(G_I)$ of all coadjoint orbits for the group G_I consists of*

1. *the 0-dimensional coadjoint orbit $\Omega_{(0,0,0,\delta)} = \{(0, 0, 0, \delta)\}$ for $\delta \in \mathbb{R}$.*

2. *the 2-dimensional coadjoint orbits* $\Omega_{(0,\cos\theta,\sin\theta,0)} = \{(0, e^{-q/2}\sin\theta, e^{-q/2}\cos\theta, t)\ ;\ q,t \in \mathbb{R}\}$ *where* $\theta \in [0, 2\pi)$.

3. *the 4-dimensional coadjoint orbit* $\Omega_{(\pm 1,0,0,0)} = \{(x,y,z,t)\ ;\ \pm x > 0\}$.

Irreducible unitary representations corresponding to the coadjoint orbits obtained in Theorem 2.2 can be stated as follows.

Theorem 2.3. 1. *The irreducible unitary representation* π_Ω *of* G_I *corresponding to the coadjoint orbit* $\Omega_{(0,0,0,\delta)} = \{(0,0,0,\delta)\}$ *is one-dimensional given by*

$$\pi_\Omega(g(a,b,c,q)) = e^{2\pi i\delta q}, \tag{2.5}$$

where $g(a,b,c,q) = \exp U \in G$ *with* $U = aX_1 + bX_2 + cX_3 + qX_4$.

2. *The irreducible unitary representation* π_Ω *of* G_I *corresponding to the coadjoint orbit* $\Omega_{(0,\cos\theta,\sin\theta,0)}$ *is realized on* $L^2(\mathbb{R})$ *by*

$$(\pi_\Omega(g(a,b,c,q))f)(x) = e^{4\pi i(e^{d/2}-1)(b\cos\theta + c\sin\theta)\frac{e^{x/2}}{q}} f(x+q), \tag{2.6}$$

where $f \in L^2(\mathbb{R}, dx)$.

3. *The irreducible unitary representation* π_Ω *of* G_I *corresponding to the coadjoint orbit* $\Omega_{(\pm 1,0,0,0)}$ *is realized on* $L^2(\mathbb{R}^2, dxdy)$ *by*

$$\begin{aligned}
(\pi_\Omega(\exp aX_1)f)(x,y) &= e^{\pm 2\pi i a e^{y}} f(x,y),\\
(\pi_\Omega(\exp bX_2)f)(x,y) &= e^{\pm 2\pi i b x e^{y/2}} f(x,y),\\
(\pi_\Omega(\exp cX_3)f)(x,y) &= f(x + ce^{y/2}, y),\\
(\pi_\Omega(\exp qX_4)f)(x,y) &= f(x, y+q),
\end{aligned} \tag{2.7}$$

where $f \in L^2(\mathbb{R}^2, dxdy)$.

Proof. For the first assertion, we see that the symmetric bilinear form B_F is identically zero since $\operatorname{rank} B_F = \dim\Omega = 0$. Hence, a polarization $\mathfrak{p} = \mathfrak{g}_I, \exp\mathfrak{p} = G_I$ and $\pi_\Omega = \nu_F$. Therefore, the irreducible unitary representations of G_I corresponding to this orbit can be written in the simple formula $\pi_\Omega(g(a,b,c,q)) = e^{2\pi i\delta q}$.

For the second assertion, we construct the irreducible unitary representations of G_I corresponding to 2-dimensional coadjoint orbits $\Omega_{(0,\cos\theta,\sin\theta,0)}$ as follows. Let $\mathfrak{p} = \mathbb{R}X_1 \oplus \mathbb{R}X_2 \oplus \mathbb{R}X_3$ be a polarization of $\mathfrak{g}_I$ at $F = \cos\theta X_2^* + \sin\theta X_3^*$ satisfying the Pukanszky condition. We have the 1-dimensional irreducible unitary representation ν_F of $\exp\mathfrak{p}$ of the form $\nu_F(\exp(aX_1+bX_2+cX_3)) = e^{2\pi i(b\cos\theta+c\sin\theta)}$. Identifying the coset space $\exp\mathfrak{p}\backslash G_I$ with $\mathbb{R}$ by

$$\mathbb{R} \ni x \mapsto \exp\mathfrak{p}\exp xX_4 \in \exp\mathfrak{p}\backslash G_I,$$

we have a section

$$s : \exp\mathfrak{p}\backslash G_I \simeq \mathbb{R} \ni x \mapsto \exp xX_4 \in G_I,$$

then the master equation

$$s(x)g(a,b,c,q) = h_s(x, g(a,b,c,q))s(x \cdot g(a,b,c,q)),$$
$$(\, x \in \exp\mathfrak{p}\backslash G_I,\ g(a,b,c,q) \in G_1,\ h_s(x, g(a,b,c,q)) \in \exp\mathfrak{p}\,)$$

becomes in our case

$$\begin{pmatrix} e^x & 0 & 0 & 0 \\ 0 & e^{x/2} & 0 & 0 \\ 0 & 0 & e^{x/2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^q & 2ce^{q/2}/q(e^{q/2}-1) & -2be^{q/2}/d(e^{q/2}-1) & -\frac{a}{q}(e^q-1 \\ 0 & e^{q/2} & 0 & -\frac{b}{q}(e^{q/2}-1) \\ 0 & 0 & e^{q/2} & -\frac{c}{q}(e^{q/2}-1) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & x_3 & -x_2 & -x_1 \\ 0 & 1 & 0 & -x_2/2 \\ 0 & 0 & 1 & -x_3/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^y & 0 & 0 & 0 \\ 0 & e^{y/2} & 0 & 0 \\ 0 & 0 & e^{y/2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.8}$$

where we see that

$$y = x + q,\ \ x_1 = \frac{ae^x}{q}(e^q - 1),$$
$$x_2 = \frac{2be^{x/2}}{q}(e^{q/2} - 1),\ \ x_3 = \frac{2ce^{x/2}}{q}(e^{d/2} - 1).$$

Therefore, the formula of the irreducible unitary representation of G_I is

$$(\pi_\Omega(g(a,b,c,q))f)(x) = e^{4\pi i(e^{q/2}-1)(b\cos\theta + c\sin\theta)\frac{e^{x/2}}{q}} f(x+q) \quad (f \in L^2(\mathbb{R}, dx)).$$

We can also compute this representation with respect to its basis $\{X_1, X_2, X_3, X_4\}$ as follows.

- the master equation with respect to e^{aX_1} is of the form

$$\begin{aligned}\exp xX_4 \, \exp aX_1 &= \exp(e^{\mathrm{ad}\,xX_4} aX_1) \, \exp xX_4 \\ &= \exp(ae^{x} X_1) \, \exp xX_4.\end{aligned}$$

- the master equation with respect to e^{bX_2} is of the form

$$\begin{aligned}\exp xX_4 \, \exp bX_2 &= \exp(e^{\mathrm{ad}\,xX_4} bX_2) \, \exp xX_4 \\ &= \exp(be^{x/2} X_2) \, \exp xX_4.\end{aligned}$$

- the master equation with respect to e^{cX_3} is of the form

$$\begin{aligned}\exp xX_4 \, \exp cX_3 &= \exp(e^{\mathrm{ad}\,xX_4} cX_3) \, \exp xX_4 \\ &= \exp(ce^{x/2} X_3) \, \exp xX_4.\end{aligned}$$

- the master equation with respect to e^{qX_4} is of the form

$$\exp xX_4 \, \exp qX_4 = \exp(x+q)X_4,$$

so that we obtain simpler formula as follows :

$$\begin{aligned}(\pi_\Omega(\exp aX_1)f)(x) &= f(x), \\ (\pi_\Omega(\exp bX_2)f)(x) &= e^{2\pi i b e^{x/2}\cos\theta} f(x), \\ (\pi_\Omega(\exp cX_3)f)(x) &= e^{2\pi i c e^{x/2}\sin\theta} f(x), \\ (\pi_\Omega(\exp qX_4)f)(x) &= f(x+q) \qquad (f \in L^2(\mathbb{R}, dx)).\end{aligned} \tag{2.9}$$

For the third assertion, the irreducible unitary representations corresponding to 4-dimensional coadjoint orbits $\Omega_{(\pm 1,0,0,0)}$ through $F = \pm X_1^*$ can be computed by considering a real algebraic polarization $\mathfrak{p} = \langle X_1, X_2 \rangle$ satisfying the Pukanszky condition. Identifying the coset space $\exp \mathfrak{p} \backslash G_I$ with $\mathbb{R}^2$ by

$$\mathbb{R}^2 \ni (x, y) \mapsto \exp \mathfrak{p} \exp xX_3 \exp yX_4 \in \exp \mathfrak{p} \backslash G_I,$$

we have a section

$$s : \exp \mathfrak{p} \backslash G_I \simeq \mathbb{R}^2 \ni (x, y) \mapsto \exp xX_3 \exp yX_4 \in G_I,$$

then finishing the master equation

$$s(\dot{x})g = h_s(\dot{x}, g)s(\dot{x} \cdot g), \quad (\dot{x} \in \exp \mathfrak{p} \backslash G_I,\ g \in G_1, h_s(\dot{x}, g) \in \exp \mathfrak{p})$$

with respect to the basis $\{X_1, X_2, X_3, X_4\}$ as follows

- the master equation with respect to e^{aX_1} is of the form

$$\begin{aligned} \exp xX_3 \ \exp yX_4 \ \exp aX_1 &= \exp xX_3 \ \exp(e^{\mathrm{ad} yX_4} aX_1) \ \exp yX_4 \\ &= \exp xX_3 \ \exp(ae^y X_1) \ \exp yX_4 \\ &= \exp(e^{\mathrm{ad} xX_3} \, (ae^y X_1)) \ \exp xX_3 \ \exp yX_4 \\ &= \exp(ae^y X_1) \ \exp xX_3 \ \exp yX_4. \end{aligned}$$

- the master equation with respect to e^{bX_2} is of the form

$$\begin{aligned} \exp xX_3 \ \exp yX_4 \ \exp bX_2 &= \exp xX_3 \ \exp(e^{\mathrm{ad} yX_4} bX_2) \ \exp yX_4 \\ &= \exp xX_3 \ \exp(be^{y/2} X_2) \ \exp yX_4 \\ &= \exp(e^{\mathrm{ad} xX_3} \, (be^{y/2} X_2)) \ \exp xX_3 \ \exp yX_4 \\ &= \exp(bxe^{y/2} X_1 + be^{y/2} X_2) \ \exp xX_3 \ \exp yX_4. \end{aligned}$$

- the master equation with respect to e^{cX_3} is of the form

$$\exp xX_3 \ \exp yX_4 \ \exp cX_3 = \exp(x + ce^{y/2})X_3 \ \exp yX_4.$$

- the master equation with respect to e^{qX_4} is of the form

$$\exp xX_3 \ \exp yX_4 \ \exp qX_4 = \exp xX_3 \ \exp(y + q)X_4.$$

Therefore, we obtain the formulas below.

$$\begin{aligned}(\pi_\Omega(e^{aX_1})f)(x,y) &= e^{\pm 2\pi i a e^y} f(x,y),\\ (\pi_\Omega(e^{bX_2})f)(x,y) &= e^{\pm 2\pi i b x e^{y/2}} f(x,y),\\ (\pi_\Omega(e^{cX_3})f)(x,y) &= f(x+ce^{y/2},y),\\ (\pi_\Omega(e^{qX_4})f)(x,y) &= f(x,y+q),\end{aligned}$$

where $f \in L^2(\mathbb{R}^2, dxdy)$. □

Furthermore, we shall compute the Duflo-Moore operator of representation $\pi := \pi_\Omega$ of G_I given by (2.7) corresponding to $\Omega_{(\pm 1,0,0,0)}$ directly. To do that, we note that for $\phi \in C_c(G_I)$, we have

$$\int_{G_I} \phi(g)\,dg = \int_{\mathbb{R}^4} \phi(e^{aX_1}e^{bX_2}e^{cX_3}e^{qX_4})\,\frac{dadbdcdq}{e^{2q}}. \tag{2.10}$$

We compute for $f_1, f_2 \in L^2(\mathbb{R}^2)$ the integral

$$\int_{G_I} |(f_1|\pi(g)f_2)|^2\,dg. \tag{2.11}$$

Now we put $g = e^{aX_1}g'$ where $g' = e^{bX_2}e^{cX_3}e^{qX_4}$. We obtain

$$\begin{aligned}(f_1|\pi(g)f_2) &= \int_{\mathbb{R}^2} f_1(x,y)\overline{e^{2\pi i a e^y}\pi(g')f_2(x,y)}\,dxdy\\ &= \int_{\mathbb{R}} e^{-2\pi i a e^y}\{\int_{\mathbb{R}} f_1(x,y)\overline{\pi(g')f_2(x,y)}\,dx\}\,dy\\ &= \int_0^\infty e^{-2\pi i a\eta}\{\int_{\mathbb{R}} f_1(x,\log\eta)\overline{\pi(g')f_2(x,\log\eta)}\,dx\}\,\frac{d\eta}{\eta}\\ &\quad (\eta = e^y,\ dy = \frac{d\eta}{\eta}).\end{aligned} \tag{2.12}$$

Using Plancherel formula, we have

$$\int_{\mathbb{R}} |(f_1|\pi(e^{aX_1})\pi(g')f_2)|^2\,da = \int_0^\infty \left|\int_{\mathbb{R}} f_1(x,\log\eta)\overline{\pi(g')f_2(x,\log\eta)}\,dx\right|^2 \frac{d\eta}{\eta^2}. \tag{2.13}$$

We put $g' = e^{bX_2}g''$, where $g'' = e^{cX_3}e^{qX_4}$. Thus, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} f_1(x,\log\eta)\overline{\pi(g')f_2(x,\log\eta)}\,dx \\
&\qquad = \int_{\mathbb{R}} e^{-2\pi ibx\sqrt{\eta}} f_1(x,\log\eta)\overline{\pi(g'')f_2(x,\log\eta)}\,dx \\
&\qquad = \int_{\mathbb{R}} e^{-2\pi ibx'} f_1(\frac{x'}{\sqrt{\eta}},\log\eta)\overline{\pi(g'')f_2(\frac{x'}{\sqrt{\eta}},\log\eta)}\,\frac{dx'}{\sqrt{\eta}} \\
&\qquad (x' = x\sqrt{\eta}\ , dx = \frac{dx'}{\sqrt{\eta}}). \qquad (2.14)
\end{aligned}$$

Again by Plancherel formula we have

$$\begin{aligned}
&\int_{\mathbb{R}} \left|\int_{\mathbb{R}} f_1(x,\log\eta)\overline{\pi(g')f_2(x,\log\eta)}\,dx\right|^2\,db \\
&\qquad = \frac{1}{\eta}\int_{\mathbb{R}} |f_1(\frac{x'}{\sqrt{\eta}},\log\eta)\overline{\pi(g'')f_2(\frac{x'}{\sqrt{\eta}},\log\eta)}|^2\,dx'. \qquad (2.15)
\end{aligned}$$

Combining (2.13) and (2.15) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} |(f_1|\pi(e^{aX_1})\pi(e^{bX_2})\pi(g'')f_2)|^2\,dadb \\
&\qquad = \int_0^\infty \int_{\mathbb{R}} |f_1(\frac{x'}{\sqrt{\eta}},\log\eta)|^2\,|\pi(g'')f_2(\frac{x'}{\sqrt{\eta}},\log\eta)|^2\,dx'\frac{d\eta}{\eta^3}. \qquad (2.16)
\end{aligned}$$

We can see

$$\begin{aligned}
\pi(g'')f_2(\frac{x'}{\sqrt{\eta}},\log\eta) &= \pi(e^{cX_3})\pi(e^{qX_4})f_2(\frac{x'}{\sqrt{\eta}},\log\eta) \\
&= \pi(e^{qX_4})f_2(\frac{x'}{\sqrt{\eta}} + c\sqrt{\eta},\log\eta), \qquad (2.17)
\end{aligned}$$

then we have

$$
\begin{aligned}
&\int_{\mathbb{R}} |\pi(e^{qX_4}) f_2(\frac{x'}{\sqrt{\eta}} + c\sqrt{\eta}, \log\eta)|^2 \, dc \\
&\qquad = \int_{\mathbb{R}} |\pi(e^{qX_4}) f_2(c', \log\eta)|^2 \, \frac{dc'}{\sqrt{\eta}} \\
&\qquad (c' = \frac{x'}{\sqrt{\eta}} + c\sqrt{\eta}, \;\; dc' = \sqrt{\eta}\, dc) \\
&\qquad = \int_{\mathbb{R}} |f_2(c', \log\eta + q)|^2 \, \frac{dc'}{\sqrt{\eta}}.
\end{aligned} \tag{2.18}
$$

Therefore, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^2} |\pi(e^{cX_3})\pi(e^{qX_4}) f_2(\frac{x'}{\sqrt{\eta}}, \log\eta)|^2 \, \frac{dcdq}{e^{2q}} \\
&\qquad = \int_{\mathbb{R}^2} |f_2(c', \log\eta + q)|^2 \, \frac{dc'}{\sqrt{\eta}} \; \frac{dq}{e^{2q}} \\
&\qquad = \int_{\mathbb{R}^2} |f_2(c', q')|^2 \; \frac{dc'}{\sqrt{\eta}} \; \frac{\eta^2 dq'}{e^{2q'}} \\
&\qquad (q' = \log\eta + q).
\end{aligned} \tag{2.19}
$$

Combining (2.16) and (2.19), then the formula (2.11) becomes

$$
\begin{aligned}
\int_{G_I} |(f_1|\pi(g) f_2)|^2 \, dg &= \int_{\mathbb{R}^4} |(f_1|\pi(e^{aX_1})\pi(e^{bX_2})\pi(e^{cX_3})\pi(e^{dX_4}) f_2|^2 \, \frac{dadbdcdq}{e^{2q}} \\
&= \int_0^\infty \int_{\mathbb{R}} |f_1(\frac{x'}{\sqrt{\eta}}, \log\eta)|^2 \, dx' \frac{d\eta}{\eta\sqrt{\eta}} \int_{\mathbb{R}^2} |f_2(c', q')|^2 \, \frac{dc'dq'}{e^{2q'}} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} |f_1(r,s)|^2 \, dr\, ds \cdot \int_{\mathbb{R}^2} |e^{-q'} f_2(c', q')|^2 \, dc'dq' \\
&(r = \frac{x'}{\sqrt{\eta}}, \; s = \log\eta) \\
&= ||f_1||^2_{L^2(\mathbb{R}^2)} \cdot \int_{\mathbb{R}^2} |e^{-q'} f_2(c', q')|^2 \, dc'dq'.
\end{aligned} \tag{2.20}
$$

Therefore, we obtain the following theorem.

Theorem 2.4. *The Duflo Moore operator* C_{π_Ω} *for the representation* $(\pi_\Omega, L^2(\mathbb{R}^2))$

of G_I as in (2.7) can be written of the form

$$C_\pi f(x,y) = e^{-y} f(x,y) \qquad (\ f \in L^2(\mathbb{R}^2)\ ). \tag{2.21}$$

We can relate this result to Pfaffian of $\mathfrak{g}_I$. For $\pi(e^{aX_1})f(x,y) = e^{2\pi i a e^y} f(x,y)$ we have

$$d\pi(X_1) f(x,y) = 2\pi i e^y f(x,y), \tag{2.22}$$

and since $\mathrm{Pf}(\mathfrak{g}_I) = Q_{\mathfrak{g}_I} = X_1^2$, then we obtain

$$d\pi(Q) = (d\pi(X_1))^2 = 4\pi^2 i^2 e^{2y} f(x,y). \tag{2.23}$$

Therefore, we obtain the following proposition

Proposition 2.5. *The Duflo-Moore operator C_{π_Ω} for representation π_Ω of G_I as in (2.7) is written in terms of the Pfaffian $Q_{\mathfrak{g}_I}$ as*

$$C_{\pi_\Omega} = 2\pi |d\pi(Q_{\mathfrak{g}_I})|^{-1/2}. \tag{2.24}$$

We also notice that G_I is a semi-direct product of $N := \exp\langle X_1, X_2, X_3\rangle$ which is isomorphic to Heisenberg Lie group, and a closed subgroup $H := \exp\langle X_4\rangle$ of $\mathrm{Aut}(N)$. It is known that the irreducible unitary representation σ_α of N corresponding to 2-dimensional coadjoint orbits through αX_1^* $(\alpha \in \mathbb{R}\setminus\{0\})$ can be characterized by formula $\sigma_\alpha(\exp aX_1) = e^{2\pi\alpha a}\mathrm{Id}$ for $a \in \mathbb{R}$. When $\alpha = \pm 1$, we define $\sigma_\pm$ to be a standard Schrödinger representation of N on $L^2(\mathbb{R})$ given by

$$\begin{aligned} \sigma_\pm(e^{aX_1}) f(x) &= e^{\pm 2\pi i a} f(x), \\ \sigma_\pm(e^{bX_2}) f(x) &= e^{\pm 2\pi i b x} f(x), \\ \sigma_\pm(e^{cX_3}) f(x) &= f(x+c), \end{aligned} \tag{2.25}$$

where $f \in L^2(\mathbb{R})$. We observe that $\sigma_\pm$ is equivalent to induced representation $\mathrm{Ind}_{\exp\langle X_1,X_2\rangle}^N \nu_{\pm X_1^*}$. Using the action of H on $\mathbb{R}\backslash\{0\}$ given by

$$h \cdot \alpha = e^{-q}\alpha \quad (h = \exp qX_4,\ \alpha \in \mathbb{R}\backslash\{0\}),$$

we define for general $\alpha \in \mathbb{R}\backslash\{0\}$ a representation $(\sigma_\alpha, L^2(\mathbb{R}))$ by

$$\sigma_\alpha(n) := \sigma_\varepsilon(h^{-1} \cdot n) \quad (n \in N),$$

where $h \in H$ and $\varepsilon = \pm 1$ are unique elements for which $\alpha = h \cdot \varepsilon$.

Remark 3. In [37, Kurniadi and Ishi], we realize the representation $\pi_\Omega = \mathrm{Ind}_{\exp \mathfrak{p}}^G \nu_\pm$ as a subrepresentation of the quasi-regular representation of G_I on $L^2(N)$. Then it was shown implicitly that $\pi_\Omega|_N = \int_{\pm\mathbb{R}_+}^{\oplus} \sigma_\alpha \, d\alpha$ for $\Omega = \Omega_\pm$.

2.2 The Second type $\mathfrak{g}_{II}(\tau)$

First of all, let us see the structure of coadjoint orbits of $G_{II}(\tau)$

Theorem 2.6. *Let $G_{II}(\tau)$ be an exponential solvable Lie group of the 4-dimensional real Frobenius Lie algebra $\mathfrak{g}_{II}(\tau), \tau \in \mathbb{R}$ with $\{X_1, X_2, X_3, X_4\}$ as basis. Then, the set $G_{II}(\tau)\backslash\mathfrak{g}_{II}(\tau)^*$ of all coadjoint orbits for the group $G_{II}(\tau)$ (i.e. $\Omega_{(\alpha,\beta,\gamma,\delta)} := \mathrm{Ad}^*(G_{II}(\tau))(\alpha X_1^* + \beta X_2^* + \gamma X_3^* + \delta X_4^*)$) consists of*

1. *the 0-dimensional coadjoint orbit with $\alpha = \beta = \gamma = 0$ given by*

$$\Omega_{(0,0,0,\delta)} = \{(0,0,0,\delta)\}, \tag{2.26}$$

   for any $\delta \in \mathbb{R}$.

2. *the 2-dimensional coadjoint orbits with $(\beta, \gamma) \neq (0,0)$:*

$$\Omega_{(0,\beta,\gamma,0)} = \{(0, f_\tau(q;\beta,\gamma), g_\tau(q;\beta,\gamma), t); t, q \in \mathbb{R}\}, \tag{2.27}$$

   where

$$\begin{pmatrix} f_\tau(q;\beta,\gamma) \\ g_\tau(q;\beta,\gamma) \end{pmatrix} = \exp q \begin{pmatrix} 0 & -1 \\ \tau & -1 \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \end{pmatrix}.$$

3. *the 4-dimensional coadjoint orbit of the form :*

$$\Omega_{(\pm 1,0,0,0)} = \{xX_1^* + yX_2^* + zX_3^* + tX_4^* \ ; \ \pm x > 0, y, z, t \in \mathbb{R}\}. \tag{2.28}$$

Proof. Let us compute coadjoint orbits of $G_{II}(\tau)$ with respect to the basis $\{X_1, X_2, X_3, X_4\}$ and $F = \alpha X_1^* + \beta X_2^* + \gamma X_3^* + \delta X_4^*$. We obtain

$$\begin{aligned}
&\mathrm{Ad}^*(e^{aX_1})F = \alpha X_1^* + \beta X_2^* + \gamma X_3^* + (\alpha a + \delta)X_4^*,\\
&\mathrm{Ad}^*(e^{bX_2})F = \alpha X_1^* + \beta X_2^* + (\alpha b + \gamma)X_3^* + (\frac{\alpha b^2}{2} + \gamma b + \delta)X_4^*,\\
&\mathrm{Ad}^*(e^{cX_3})F = \alpha X_1^* + (-\alpha c + \beta)X_2^* + \gamma X_3^* + (\frac{\tau\alpha c^2}{2} - \tau\beta c + \gamma c + \delta)X_4^*,\\
&\mathrm{Ad}^*(e^{qX_4})(\alpha X_1^*) = \alpha e^{-q} X_1^*.
\end{aligned} \tag{2.29}$$

We note that $\mathrm{Ad}^*(e^{qX_4})F$ is complicated in general, while the formula $\mathrm{Ad}^*(e^{qX_4})(\alpha X_1^*)$ is sufficient for our observation of coadjoint orbits. We observe that for $\alpha = \beta = \gamma = 0$, we have the 0-dimensional coadjoint orbits in the form $\Omega_{(0,0,0,\delta)} = \{(0,0,0,\delta)\}$ for $\delta \in \mathbb{R}$. Let $U = aX_1 + bX_2 + cX_3 + qX_4$, we have

$$\mathrm{ad}(U) = \begin{pmatrix} q & c & -b & -a \\ 0 & 0 & -\tau q & \tau c \\ 0 & q & q & -b-c \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.30}$$

By considering $\mathrm{ad}^*(U)$ and $\alpha = 0$, we obtain

$$\begin{pmatrix} y \\ z \\ t \end{pmatrix} = \mathrm{Ad}^*(e^U)(\beta X_2^* + \gamma X_3^* + \delta X_4^*) = (\exp \begin{pmatrix} 0 & -q & 0 \\ \tau q & -q & 0 \\ -\tau c & b+c & 0 \end{pmatrix}) \begin{pmatrix} \beta \\ \gamma \\ \delta \end{pmatrix}. \tag{2.31}$$

Moreover, we have

$$\begin{aligned}
\begin{pmatrix} y \\ z \end{pmatrix} &= (\exp \begin{pmatrix} 0 & -q \\ \tau q & -q \end{pmatrix}) \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \\
&= \begin{pmatrix} f_\tau(q;\beta,\gamma) \\ g_\tau(q;\beta,\gamma) \end{pmatrix}.
\end{aligned} \tag{2.32}$$

If $q = 0$, then we have

$$\begin{aligned}\begin{pmatrix} y \\ z \\ t \end{pmatrix} &= (\exp\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\tau c & b+c & 0 \end{pmatrix})\begin{pmatrix} \beta \\ \gamma \\ \delta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\tau c & b+c & 1 \end{pmatrix}\begin{pmatrix} \beta \\ \gamma \\ \delta \end{pmatrix}. \end{aligned} \tag{2.33}$$

If $(\beta, \gamma) \neq 0$, then $t = -\tau c\beta + (b+c)\gamma + \delta$ runs over $\mathbb{R}$ with (b, c) runs over $\mathbb{R}^2$. Thus we have 2-dimensional coadjoint orbits $\Omega_{(0,\beta,\gamma,0)} = \{(0, f_\tau(q; \beta, \gamma), g_\tau(q; \beta, \gamma), t) \ ; \ q, t \in \mathbb{R}\}$. Now if we consider $q = 0$, then

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \mathrm{Ad}^*(e^U)F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -c & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ a + 1/2(b^2 + bc + \tau c^2) & -\tau c & b+c & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}. \tag{2.34}$$

Furthermore, if $\alpha \neq 0$, then (y, z, t) will run over $\mathbb{R}^3$ with (a, b, c) runs over $\mathbb{R}^3$. In addition, if $(a, b, c) = (0, 0, 0)$

$$\mathrm{Ad}^*(e^U) = \begin{pmatrix} e^{-q} & & \\ & \exp\begin{pmatrix} 0 & -q \\ \tau q & -q \end{pmatrix} & \\ & & 1 \end{pmatrix}.$$

Therefore, we obtain 4-dimensional coadjoint orbits in the forms $\Omega_{(\pm 1,0,0,0)} = \{(x, y, z, t) \ ; \ \pm x > 0, y, z, t \in \mathbb{R}\}$. □

Remark 4. To compute $f_\tau(q; \beta, \gamma)$ and $g_\tau(q; \beta, \gamma)$ in (2.27), we explain by case of τ. First we consider the case $\tau < 1/4$. Eigenvalues of $\mathrm{ad}(X_4)$ are $\lambda_2 = \frac{1+\sqrt{1-4\tau}}{2}$ and $\lambda_3 = \frac{1-\sqrt{1-4\tau}}{2}$ corresponding to eigenvectors $v_2 = \begin{pmatrix} -\lambda_3 \\ 1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} -\lambda_2 \\ 1 \end{pmatrix}$ respectively. For simpler computations we put

$$Y_2 = -\lambda_3 X_2 + X_3 \text{ and } Y_3 = -\lambda_2 X_2 + X_3.$$

Thus, $\mathrm{ad}(X_4)Y_2 = \lambda_2 Y_2$ and $\mathrm{ad}(X_4)Y_3 = \lambda_3 Y_3$ and we get $e^{-qX_4}Y_2 = e^{-\lambda_2 q}Y_2$ and $e^{-qX_4}Y_3 = e^{-\lambda_3 q}Y_3$. Moreover, let us take

$$Y_2^* = \frac{1}{\lambda_2 - \lambda_3}(X_2^* + \lambda_3 X_3^*) \text{ and } Y_3^* = \frac{1}{\lambda_3 - \lambda_2}(X_2^* + \lambda_4 X_3^*)$$

such that $\langle Y_i, Y_j^* \rangle = \delta_{ij}$ where $i, j = 1, 2$. In these terms for $\beta \in \mathbb{R}$, we obtain 2 dimensional coadjoint orbits of the forms

$$\begin{aligned} \Omega_{\pm Y_2^* + \gamma Y_3^*} &= \{\pm e^{-\lambda_2 q} Y_2^* + \gamma e^{-\lambda_3 q} Y_3^* + t X_4^* \ ; \ q, t \in \mathbb{R}\} \\ &= \{\pm y Y_2^* + z Y_3^* + t X_4^* \ ; \ z = \gamma y^{\frac{\lambda_3}{\lambda_2}}, y > 0, t \in \mathbb{R}\}. \end{aligned} \tag{2.35}$$

For the second case $\tau = 1/4$, we take

$$\begin{aligned} Y_2 &= -X_2 + 2X_3, & Y_3 &= 2X_2, \\ Y_2^* &= \frac{1}{2}X_3^*, & Y_3^* &= \frac{1}{2}X_2^* + \frac{1}{4}X_3^*, \end{aligned}$$

where $\langle Y_i, Y_j^* \rangle = \delta_{ij}$. Using the equality

$$e^{\mathrm{ad}(-qX_4)} \begin{pmatrix} Y_2 & Y_3 \end{pmatrix} = \begin{pmatrix} Y_2 & Y_3 \end{pmatrix} \begin{pmatrix} e^{-q/2} & -qe^{-q/2} \\ 0 & e^{-q/2} \end{pmatrix}$$

and (2.33), we obtain

$$\begin{aligned} \Omega_{\beta Y_2^* + \gamma Y_3^*} = \{y Y_2^* + z Y_3^* + t X_4^* \ ; \ q, t \in \mathbb{R}\}, \ \text{where} \\ y = \beta e^{-q/2}, \\ z = -\beta q e^{-q/2} + \gamma e^{-q/2}. \end{aligned} \tag{2.36}$$

For the third case $\tau > 1/4$, we practice the argument similar to $\tau < 1/4$. Let us consider

$$Y_2 = -\lambda_3 X_2 + X_3, \qquad Y_3 = -\lambda_2 X_2 + X_3,$$

where $\lambda_2 = \frac{1}{2} + i\frac{\sqrt{4\tau - 1}}{2}$ and $\lambda_3 = \frac{1}{2} - i\frac{\sqrt{4\tau - 1}}{2}$. Then, we obtain $\mathrm{ad}(X_4)Y_2 = \lambda_2 Y_2$

and $\mathrm{ad}(X_4)Y_3 = \lambda_3 Y_3$. Let us take

$$Y_2^* = \frac{1}{\lambda_2 - \lambda_3}(X_2^* + \lambda_2 X_3^*) = \frac{X_3^*}{2} - \frac{i}{\sqrt{4\tau - 1}}(X_2^* + \frac{X_3^*}{2}),$$
$$Y_3^* = \frac{1}{\lambda_3 - \lambda_2}(X_2^* + \lambda_3 X_3^*) = \frac{X_3^*}{2} + \frac{i}{\sqrt{4\tau - 1}}(X_2^* + \frac{X_3^*}{2}),$$

where $\langle Y_i, Y_j^* \rangle = \delta_{ij}$ $(i, j = 1, 2)$. Then we put

$$Z_2^* = \frac{X_3^*}{2}, \quad Z_3^* = -\frac{1}{\sqrt{4\tau - 1}}(X_2^* + \frac{X_3^*}{2}),$$

where

$$\begin{pmatrix} Y_2^* & Y_3^* \end{pmatrix} = \begin{pmatrix} Z_2^* & Z_3^* \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

Indeed, we have

$$\mathrm{ad}(X_4) \begin{pmatrix} Z_2 & Z_3 \end{pmatrix} = \begin{pmatrix} Z_2 & Z_3 \end{pmatrix} \begin{pmatrix} 1/2 & \frac{\sqrt{4\tau-1}}{2} \\ -\frac{\sqrt{4\tau-1}}{2} & 1/2 \end{pmatrix},$$
$$e^{\mathrm{ad}(-qX_4)} \begin{pmatrix} Z_2 & Z_3 \end{pmatrix} = \begin{pmatrix} Z_2 & Z_3 \end{pmatrix} e^{-q/2} \begin{pmatrix} \cos(\frac{q}{2}\sqrt{4\tau - 1}) & -\sin(\frac{q}{2}\sqrt{4\tau - 1}) \\ \sin(\frac{q}{2}\sqrt{4\tau - 1}) & \cos(\frac{q}{2}\sqrt{4\tau - 1}) \end{pmatrix}.$$

Using (2.33) we obtain

$$\begin{aligned} \Omega_{\beta Z_2^* + \gamma Z_3^*} &= \{yZ_2^* + zZ_3^* + tX_4^* \ ; \ q, t \in \mathbb{R}\}, \text{ where} \\ y &= \beta e^{-q/2} \cos(\frac{q}{2}\sqrt{4\tau - 1})) + \gamma e^{-q/2} \sin(\frac{q}{2}\sqrt{4\tau - 1})), \\ z &= -\beta e^{-q/2} \sin(\frac{q}{2}\sqrt{4\tau - 1})) + \gamma e^{-q/2} \cos(\frac{q}{2}\sqrt{4\tau - 1})). \end{aligned} \tag{2.37}$$

We also give another way to compute coadjoint orbits when $\alpha \neq 0$. Let us now observe for $\alpha > 0$. Using (2.29) we consider for $F = \alpha X_1^* + \beta X_2^* + \gamma X_3^* + \delta X_4^*$

$$\mathrm{Ad}^*(e^{(\log \alpha) X_4} e^{-\frac{\delta''}{\alpha} X_1} e^{\frac{\beta}{\alpha} X_3} e^{-\frac{\gamma}{\alpha} X_2}) F = X_1^*, \tag{2.38}$$

where $\delta'' = \frac{\tau\alpha(\frac{\beta}{\alpha})^2}{2} - \tau\beta(\frac{\beta}{\alpha}) + \gamma(\frac{\beta}{\alpha}) + \delta$ and $q = \log \alpha$. We also use the same arguments for $\alpha < 0$. Therefore, for $\alpha \neq 0$ we obtain 2 types of 4-dimensional coadjoint orbits in the following patterns

$$\Omega_{(1,0,0,0)} = \{F \in \mathfrak{g}_{II}^*(\tau)\ ;\ \alpha > 0\}, \tag{2.39}$$

and

$$\Omega_{(-1,0,0,0)} = \{F \in \mathfrak{g}_{II}^*(\tau)\ ;\ \alpha < 0\}. \tag{2.40}$$

We can observe that only $\Omega_\pm := \Omega_{(\pm 1,0,0,0)}$ is an open orbit for $G_{II}(\tau)$. Furthermore, we shall construct irreducible unitary representation for $G_{II}(\tau)$ as follows.

Theorem 2.7. *The irreducible unitary representation of $G_{II}(\tau)$ on the space $L^2(\mathbb{R}^2, dxdy)$ corresponding to the open orbit $\Omega_\pm$ can be written as*

$$\begin{aligned}(\pi_{\Omega_\pm}(\exp qX_4)f)(x,y) &= f(x, y+q),\\ (\pi_{\Omega_\pm}(\exp Z)f)(x,y) &= \sigma_\pm(\exp(\mathrm{Ad}(\exp yX_4)Z))f(x,y),\end{aligned} \tag{2.41}$$

where $Z = aX_1 + bX_2 + cX_3$, and $\sigma_\pm$ acts on $f(\cdot, y) \in L^2(\mathbb{R})$ for each $y \in \mathbb{R}$.

Proof. Let $f = \pm X_1^*$ be element of $\mathfrak{g}_{II}^*(\tau)$ and $\mathfrak{p} = \langle X_1, X_2\rangle$ be a polarization of $\mathfrak{g}_{II}(\tau)$ at f satisfying Pukanszky condition. Identifying the coset space $\exp\mathfrak{p}\backslash G_{II}(\tau)$ with $\mathbb{R}^2$ by

$$\mathbb{R}^2 \ni (x,y) \mapsto \exp\mathfrak{p}\exp xX_3\exp yX_4 \in \exp\mathfrak{p}\backslash G_{II}(\tau),$$

we have a section

$$s : \exp\mathfrak{p}\backslash G_{II}(\tau) \simeq \mathbb{R}^2 \ni (x,y) \mapsto \exp xX_3 \exp yX_4 \in G_{II}(\tau),$$

then the master equation $s(\dot{x})g = h_s(\dot{x}, g)s(\dot{x}\cdot g)\quad (\dot{x} \in \exp\mathfrak{p}\backslash G_{II}(\tau),\ g \in G_{II}(\tau), h_s(\dot{x}, g) \in \exp\mathfrak{p})$ for the representation $\pi_{\Omega_\pm}$ of $G_{II}(\tau)$ becomes in our case

$$\exp(xX_3)\cdot\exp(yX_4)\cdot g = h'\cdot\exp(x'X_3)\cdot\exp(y'X_4). \tag{2.42}$$

Substituting $g = \exp qX_4$ to (2.42), we have

$$\exp(xX_3)\cdot\exp(yX_4)\cdot\exp(qX_4) = \exp(xX_3)\cdot\exp(y+q)X_4.$$

Therefore, the representation $\pi_{\Omega_\pm}$ with respect to the basis $\exp qX_4$ can be written as follows.

$$(\pi_{\Omega_\pm}(\exp qX_4)f)(x,y) = f(x, y+q) \qquad (\ f \in L^2(\mathbb{R}^2)\ ). \tag{2.43}$$

Furthermore, let $Z = aX_1 + bX_2 + cX_3$ be element of $\langle X_1, X_2, X_3\rangle$. We see that $N = \exp\langle X_1, X_2, X_3\rangle$ is the Heisenberg Lie group and it is well known that the master equation for the representation $\sigma_\pm := \mathrm{Ind}_{\exp\langle X_1,X_2\rangle}^{N} \nu_\pm$ of N can be written as

$$\exp(xX_3)\cdot n = h'\cdot \exp(x'X_3),$$

where $n \in N, x, x' \in \mathbb{R}, h' \in \exp\langle X_1, X_2\rangle$. Then the representation $\sigma_\pm$ takes the form

$$(\sigma_\pm(n)f)(x) = \nu_\pm(h')f(x') \qquad (f \in L^2(\mathbb{R}^2), \tag{2.44}$$

and it can be computed with respect to the basis $\{X_1, X_2, X_3\}$ as given in (2.25). On the other hand, the master equation for representation $\pi_{\Omega_\pm} := \mathrm{Ind}_{\exp\langle X_1,X_2\rangle}^{G} \nu_\pm$ of $G_{II}(\tau)$ is given in (2.42). Now, substituting $g = \exp Z$ to (2.42), we obtain

$$\begin{aligned} \exp(xX_3)\cdot\exp(yX_4)\cdot\exp Z &= \exp(xX_3)\cdot\exp(\mathrm{Ad}(e^{yX_4})Z)\cdot\exp(yX_4) \\ &= h'\cdot\exp(xX_3)\cdot\exp(yX_4). \end{aligned} \tag{2.45}$$

We see that for $n = \exp(\mathrm{Ad}(e^{yX_4})Z)$, the formula

$$\exp(xX_3)\cdot\exp(\mathrm{Ad}(e^{yX_4})Z) = h'\cdot\exp(x'X_3),$$

is the master equation for N. Therefore, we obtain the irreducible unitary representation for $G_{II}(\tau)$ on $L^2(\mathbb{R}^2)$ of the form

$$(\pi_{\Omega_\pm}(\exp Z)f)(x,y) = \sigma_\pm(\exp(\mathrm{Ad}(\exp yX_4)Z)f(x,y) \qquad (\ f\in L^2(\mathbb{R}^2)\ ), \tag{2.46}$$

as required. □

Next task is to consider the Duflo-Moore operator for the representation $\pi := \pi_{\Omega_\pm}$ of $G_{II}(\tau)$ given by (2.41) corresponding to the open coadjoint orbit$\Omega_{(\pm1,0,0,0)}$

by computing directly as follows. For $\phi \in C_c(G_{II}(\tau))$, we have

$$\int_{G_{II}(\tau)} \phi(g)\, dg = \int_{\mathbb{R}^4} \phi(e^{aX_1}e^{bX_2}e^{cX_3}e^{qX_4})\, \frac{dadbdcdq}{e^{2q}}.$$

We compute for $f_1, f_2 \in L^2(\mathbb{R}^2)$ the integral

$$\int_{G_{II}(\tau)} |(f_1|\pi(g)f_2)|^2\, dg. \tag{2.47}$$

Now we put $g = e^{aX_1}g'$ where $g' = e^{bX_2}e^{cX_3}e^{qX_4}$. We obtain

$$\begin{aligned}(f_1|\pi(g)f_2) &= \int_{\mathbb{R}^2} f_1(x,y)\overline{e^{\pm 2\pi i a e^y}\pi(g')f_2(x,y)}\, dxdy \\ &= \int_{\mathbb{R}} e^{\mp 2\pi i a e^y}\{\int_{\mathbb{R}} f_1(x,y)\overline{\pi(g')f_2(x,y)}\, dx\}\, dy \\ &= \int_0^{\infty} e^{\mp 2\pi i a\eta}\{\int_{\mathbb{R}} f_1(x,\log\eta)\overline{\pi(g')f_2(x,\log\eta)}\, dx\}\, \frac{d\eta}{\eta} \\ &\qquad (\eta = e^y,\ dy = \frac{d\eta}{\eta}).\end{aligned} \tag{2.48}$$

Using Plancherel formula, we have

$$\begin{aligned}&\int_{\mathbb{R}} |(f_1|\pi(e^{aX_1})\pi(g')f_2)|^2\, da \\ &\qquad = \int_0^{\infty} \left|\int_{\mathbb{R}} f_1(x,\log\eta)\overline{\pi(g')f_2(x,\log\eta)}\, dx\right|^2 \frac{d\eta}{\eta^2} \\ &\qquad = \int_0^{\infty} \left|\int_{\mathbb{R}} f_1(x,\log\eta)\overline{\pi(e^{bX_2}e^{cX_3})\pi(e^{qX_4})f_2(x,\log\eta)}\, dx\right|^2 \frac{d\eta}{\eta^2}.\end{aligned} \tag{2.49}$$

Before continuing computations, let us see the useful lemma below, which will be used in our next computations.

Lemma 2.8 (see [19])**.** *Let $\sigma_{\pm}$ be the representation of $N = \exp\langle X_1, X_2, X_3\rangle$ on $L^2(\mathbb{R})$ as in (2.25). For $\phi_1, \phi_1 \in L^2(\mathbb{R})$, we have*

$$\int_{\mathbb{R}^2} |(\phi_1|\sigma_{\pm}(e^{bX_2}e^{cX_3})\phi_2)_{L^2(\mathbb{R})}|^2\, dbdc = ||\phi_1||^2_{L^2(\mathbb{R})}||\phi_2||^2_{L^2(\mathbb{R})}. \tag{2.50}$$

Proof. Let ϕ_1, ϕ_2 be elements of $L^2(\mathbb{R})$. We compute the inner product

$$(\phi_1|\sigma_\pm(e^{bX_2}e^{cX_3})\phi_2) = \int_{\mathbb{R}} \phi_1(x)\overline{e^{\pm 2\pi ibx}\phi_2(x+c)}\,dx$$
$$= \int_{\mathbb{R}} e^{\mp 2\pi ibx}\phi_1(x)\overline{\phi_2(x+c)}\,dx, \tag{2.51}$$

and by Plancherel formula we get

$$\begin{aligned}\int_{\mathbb{R}^2} |(\phi_1|\sigma_\pm(e^{bX_2}e^{cX_3})\phi_2)|^2\,db\,dc &= \int_{\mathbb{R}} |\phi_1(x)|^2 \{\int_{\mathbb{R}} |\phi_2(x+c)|^2\,dc\}\,dx \\ &= \int_{\mathbb{R}} |\phi_1(x)|^2\,dx \int_{\mathbb{R}} |\phi_2(x+c)|^2\,dc \\ &= \int_{\mathbb{R}} |\phi_1(x)|^2\,dx \int_{\mathbb{R}} |\phi_2(c')|^2\,dc' \quad (c' = x+c) \\ &= ||\phi_1||^2_{L^2(\mathbb{R})}||\phi_2||^2_{L^2(\mathbb{R})}.\end{aligned} \tag{2.52}$$

□

Furthermore, since X_2 and X_3 commute with their commutator, that is,

$$[X_2,[X_2,X_3]] = [X_3,[X_2,X_3]] = 0,$$

by the Baker-Campbell-Hausdorff formula we have

$$\begin{aligned}e^{bX_2}e^{cX_3} &= e^{bX_2+cX_3+bc[X_2,X_3]/2} \\ &= e^{bX_2+cX_3-bcX_1/2} \\ &= e^{-bcX_1/2}e^{bX_2+cX_3}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}|(\phi_1|\sigma_\pm(e^{bX_2}e^{cX_3})\phi_2)|^2 &= |e^{\pm 2\pi i(bc/2)}(\phi_1|\sigma_\pm(e^{bX_2+cX_3})\phi_2)|^2 \\ &= |(\phi_1|\sigma_\pm(e^{bX_2+cX_3})\phi_2)|^2.\end{aligned}$$

Thus we obtain

Corollary 2.9.

$$\int_{\mathbb{R}^2} |(\phi_1|\sigma_\pm(e^{bX_2+cX_3}\phi_2))_{L^2(\mathbb{R})}|^2\, dbdc = ||\phi_1||^2_{L^2(\mathbb{R})}||\phi_2||^2_{L^2(\mathbb{R})}. \tag{2.53}$$

Using (2.41), (2.49), and Corollary 2.9, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}}|(f_1|\pi(e^{aX_1})\pi(e^{bX_2}e^{cX_3})\pi(e^{qX_4})f_2)|^2\, dadbdc \\
&\quad= \int_0^\infty \int_{\mathbb{R}^2} |(f_1(\cdot,\log\eta)|\pi(e^{bX_2+cX_3})\pi(e^{qX_4})f_2(\cdot,\log\eta))|^2\, dbdc\, \frac{d\eta}{\eta^2} \\
&\quad= \int_0^\infty \int_{\mathbb{R}^2} |(f_1(\cdot,\log\eta)|\sigma_\pm(\exp(\mathrm{Ad}(e^{\log\eta X_4})(bX_2+cX_3)))\pi(e^{qX_4})f_2(\cdot,\log\eta))|^2\, dbdc\, \frac{d\eta}{\eta^2} \\
&\quad= \int_0^\infty \int_{\mathbb{R}^2} |(f_1(\cdot,\log\eta)|\sigma_\pm(e^{b'X_2+c'X_3})\pi(e^{qX_4})f_2(\cdot,\log\eta))|^2\, \frac{db'\,dc'}{\eta}\, \frac{d\eta}{\eta^2} \\
&\qquad (b'X_2+c'X_3 := \mathrm{Ad}(e^{\log\eta X_4})(bX_2+cX_3), \quad db'dc' = \eta\, dbdc) \\
&\quad= \int_0^\infty ||(f_1(\cdot,\log\eta)||^2_{L^2(\mathbb{R})}||\pi(e^{qX_4})f_2(\cdot,\log\eta))||^2_{L^2(\mathbb{R})}\, \frac{d\eta}{\eta^3}.
\end{aligned} \tag{2.54}$$

Therefore, we get

$$\begin{aligned}
&\int_{G_{II}(\tau)}|(f_1|\pi(g)f_2)|^2\, dg = \int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}} |(f_1|\pi(e^{aX_1})\pi(e^{bX_2+cX_3})\pi(e^{qX_4})f_2)|^2\, \frac{dadbdcdq}{e^{2q}} \\
&\quad= \int_0^\infty ||f_1(\cdot,\log\eta)||^2_{L^2(\mathbb{R})}\{\int_{\mathbb{R}} ||\pi(e^{qX_4})f_2(\cdot,\log\eta)||^2_{L^2(\mathbb{R})}\, \frac{dq}{e^{2q}}\}\, \frac{d\eta}{\eta^3} \\
&\quad= \int_0^\infty ||f_1(\cdot,\log\eta)||^2_{L^2(\mathbb{R})}\{\int_{\mathbb{R}} ||f_2(\cdot,\log\eta+q)||^2_{L^2(\mathbb{R})}\, \frac{dq}{e^{2q}}\}\, \frac{d\eta}{\eta^3} \\
&\quad= \int_0^\infty ||f_1(\cdot,\log\eta)||^2_{L^2(\mathbb{R})}\{\int_{\mathbb{R}} ||f_2(\cdot,q')||^2_{L^2(\mathbb{R})}\, \eta^2\, \frac{dq}{e^{2q'}}\}\, \frac{d\eta}{\eta^3} \\
&\qquad (q' = \log\eta + q) \\
&\quad= \int_0^\infty ||f_1(\cdot,\log\eta)||^2_{L^2(\mathbb{R})}\frac{d\eta}{\eta} \cdot \int_{\mathbb{R}} ||f_2(\cdot,q')||^2_{L^2(\mathbb{R})}\, \frac{dq}{e^{2q'}} \\
&\quad= \int_{\mathbb{R}} ||f_1(\cdot,s)||^2_{L^2(\mathbb{R})}\, ds \cdot \int_{\mathbb{R}} ||f_2(\cdot,q')||^2_{L^2(\mathbb{R})}\, \frac{dq}{e^{2q'}} \\
&\qquad (s = \log\eta, ds = \frac{d\eta}{\eta}).
\end{aligned} \tag{2.55}$$

Moreover, we have

$$\int_{G_{II}(\tau)} |(f_1|\pi(g)f_2)|^2 \, dg = ||f_1||^2_{L^2(\mathbb{R})} \cdot \int_{\mathbb{R}} ||f_2(\cdot, q')||^2_{L^2(\mathbb{R})} \frac{dq}{e^{2q'}}$$
$$= ||f_1||^2_{L^2(\mathbb{R})} \cdot \int_{\mathbb{R}} ||e^{-q'} f_2(\cdot, q')||^2_{L^2(\mathbb{R})} \, dq. \tag{2.56}$$

Based on computations above, we have proved the following theorem.

Theorem 2.10. *The Duflo-Moore operator for the representation $(\pi, L^2(\mathbb{R}^2))$ of $G_{II}(\tau)$ can be written of the form*

$$C_\pi f(x, y) = e^{-y} f(x, y) \qquad (\ f \in L^2(\mathbb{R}^2)\ ). \tag{2.57}$$

Furthermore, in notion of Pfaffian of $\mathfrak{g}_{II}(\tau)$, we obtain

Proposition 2.11. *The Duflo-Moore operator C_π for the representation π of $G_{II}(\tau)$ which related to the Pfaffian of $\mathfrak{g}_{II}(\tau)$ is given by*

$$C_\pi = 2\pi |i^2 d\pi(Q_{\mathfrak{g}_{II}(\tau)})|^{-1/2}, \tag{2.58}$$

where $Q_{\mathfrak{g}_{II}(\tau)} := \mathrm{Pf}(\mathfrak{g}_{II}(\tau)) = X_1^2$.

2.3 The Third type $\mathfrak{g}_{III}(\pm 1)$

The Lie algebra $\mathfrak{g}_{III}(\varepsilon)$ is the semi-direct-direct product of a commutative ideal $\mathfrak{n} := \langle X_1, X_2 \rangle$ and a commutative subalgebra $\mathfrak{h} := \langle X_3, X_4 \rangle$. The matrix expressions of $\mathrm{ad}(aX_1 + bX_2)$ and $\mathrm{ad}(cX_3 + qX_4)$ with respect to the basis $\{X_i\}_{i=1}^4$ are written as

$$\begin{pmatrix} 0 & & -a & -b \\ & 0 & -b & \varepsilon a \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} c & q & & \\ -\varepsilon q & c & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

respectively. The Lie group $G_{III}(\varepsilon) \subset \mathrm{GL}(\mathfrak{g}_{III}(\varepsilon))$ corresponding to $\mathrm{ad}(\mathfrak{g}_{III}(\varepsilon))$ can be written as the semi-direct product $N \rtimes H$ with

$$N = \left\{ \begin{pmatrix} 1 & & -a & -b \\ & 1 & -b & \varepsilon a \\ & & 1 & \\ & & & 1 \end{pmatrix} \; ; \; a, b \in \mathbb{R} \right\},$$

and when $\varepsilon = -1$,

$$H = \left\{ \begin{pmatrix} e^c \cosh q & e^c \sinh q & & \\ e^c \sinh q & e^c \cosh q & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \; ; \; c, q \in \mathbb{R} \right\},$$

and when $\varepsilon = 1$,

$$H = \left\{ \begin{pmatrix} e^c \cos q & e^c \sin q & & \\ -e^c \sin q & e^c \cos q & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \; ; \; c, q \in \mathbb{R} \right\}.$$

Furthermore, we can see that the matrix expression for $\mathrm{Ad}^*(\exp(aX_1 + bX_2))$ with respect to the basis $\{X_i^*\}_{i=1}^4$ is

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ a & b & 1 & \\ b & -\varepsilon a & & 1 \end{pmatrix}.$$

Now for $f = \alpha X_1^* + \beta X_2^* + \gamma X_3^* + \delta X_4^* \in \mathfrak{g}_{III}^*(\varepsilon)$, the N-orbit $\mathrm{Ad}^*(N)f$ equals

$$\{\alpha X_1^* + \beta X_2^* + (\gamma + \alpha a + \beta b)X_3^* + (\delta - \varepsilon\beta a + \alpha b)X_4^* \; ; \; a, b \in \mathbb{R}\}.$$

We can observe that $\mathrm{Ad}^*(N)f$ is 2-dimensional if and only if the linear map

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \beta \\ -\varepsilon\beta & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

is non-singular, that is, $\det \begin{pmatrix} \alpha & \beta \\ -\varepsilon\beta & \alpha \end{pmatrix} = \alpha^2 + \varepsilon\beta^2 \neq 0$. In this case, we have

$$\mathrm{Ad}^*(N)f = \alpha X_1^* + \beta X_2^* + \langle X_3^*, X_4^* \rangle.$$

Moreover, these orbits are mapped each other by the H-action. When $\varepsilon = -1$, the set $\{\alpha X_1^* + \beta X_2^* \ ; \ \alpha^2 - \beta^2 \neq 0\}$ is decomposed into four orbits

$$\mathrm{Ad}^*(H)(\pm X_1^*) = \{\alpha X_1^* + \beta X_2^* \ ; \ \alpha^2 - \beta^2 > 0, \, \pm(\alpha + \beta) > 0\},$$

and

$$\mathrm{Ad}^*(H)(\pm X_2^*) = \{\alpha X_1^* + \beta X_2^* \ ; \ \alpha^2 - \beta^2 < 0, \, \pm(\alpha - \beta) < 0\}.$$

Therefore, there exist four open coadjoint orbits $\Omega_{\pm X_i^*}$ $(i = 1, 2)$ in $\mathfrak{g}_{III}^*(-1)$.

Next task is to consider the irreducible unitary representation $\pi := \pi_{\Omega_{\pm X_1^*}}$ of $G_{III}(-1)$ corresponding to the open coadjoint orbit $\Omega_{\pm X_1^*}$. To do so, let $f := \pm X_1^* \in \Omega_{\pm X_1^*}$ and $\mathfrak{p} = \mathfrak{n} = \langle X_1, X_2 \rangle$ be a polarization of $\mathfrak{g}_{III}(-1)$ at the point f. The 1-dimensional representation of $N = \exp \mathfrak{p}$ can be written as

$$\nu_{\pm X_1^*}(\exp(aX_1 + bX_2)) = e^{\pm 2\pi i a},$$

where $aX_1 + bX_2 \in \mathfrak{p}$. Identifying the coset space $N \backslash G_{III}(-1)$ with $\mathbb{R}^2$ by

$$\mathbb{R}^2 \ni (x, y) \mapsto N \exp xX_3 \exp yX_4 \in N \backslash G_{III}(-1),$$

we have a section

$$s : N \backslash G_{III}(-1) \simeq \mathbb{R}^2 \ni (x, y) \mapsto \exp xX_3 \exp yX_4 \in G_{III}(-1),$$

then the master equation

$$s(\dot{x})g = n_s(\dot{x}, g)s(\dot{x} \cdot g), \quad (\dot{x} \in N \backslash G_{III}(-1), \ g \in G_{III}(-1), \ n_s(\dot{x}, g) \in N)$$

for the representation $\pi_{\Omega_{\pm X_1^*}}$ of $G_{III}(-1)$ becomes in our case of the form

$$\begin{pmatrix} e^x\cosh y & e^x\sinh y & & \\ e^x\sinh y & e^x\cosh y & & \\ & & 1 & \\ & & & 1 \end{pmatrix}(nh) = \begin{pmatrix} 1 & & -a_1 & -b_1 \\ & 1 & -b_1 & -a_1 \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} e^{x'}\cosh y' & e^{x'}\sinh y' & & \\ e^{x'}\sinh y' & e^{x'}\cosh y' & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

where $nh \in G_{III}(-1) = N \rtimes H$ is of the form

$$\begin{pmatrix} 1 & & -a & -b \\ & 1 & -b & -a \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} e^c\cosh q & e^c\sinh q & & \\ e^c\sinh q & e^c\cosh q & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Solving the master equation above, we have $a_1 = ae^x\cosh y + be^x\sinh y, b_1 = ae^x\sinh y + be^x\cosh y, x' = x + c$, and $y' = y + q$. Then, the representation π of $G_{III}(-1)$ can be induced from the representation $\nu_{\pm X_1^*}$ of N. Namely, $\pi = \mathrm{Ind}_N^G \nu_{p_0}$. Therefore, we obtain the following theorem.

Theorem 2.12. *The representation π of $G_{III}(-1)$ on $L^2(\mathbb{R}^2, dxdy)$ corresponding to the open coadjoint orbits $\Omega_{\pm X_1^*}$ can be written as*

$$\pi_{\Omega_{\pm X_1^*}}(g(a,b,c,q)f(x,y) = e^{\pm 2\pi i(ae^x\cosh y + be^x\sinh y)} f(x+c, y+q) \quad (f \in L^2(\mathbb{R}^2)). \tag{2.59}$$

We also compute the representation π of $G_{III}(-1)$ with respect to basis $\{X_1, X_2, X_3, X_4\}$ by computing each master equation as follows.

- the master equation with respect to e^{aX_1} is of the form

$$\begin{aligned} \exp xX_3\, \exp yX_4\, \exp aX_1 &= \exp xX_3\, \exp(e^{\mathrm{ad}yX_4}aX_1)\, \exp yX_4 \\ &= \exp xX_3\, \exp((a\cosh y)X_1 + (a\sinh y)X_2)\, \exp yX_4 \\ &= \exp(e^{\mathrm{ad}xX_3}((a\cosh y)X_1 + (a\sinh y)X_2))\, \exp xX_3\, \exp yX_4 \\ &= \exp((ae^x\cosh y)X_1 + (ae^x\sinh y)X_2)\, \exp xX_3\, \exp yX_4. \end{aligned}$$

- the master equation with respect to e^{bX_2} is of the form

$$\begin{aligned}\exp xX_3\ \exp yX_4\ \exp bX_2 &= \exp xX_3\ \exp(e^{\mathrm{ad}\, yX_4} bX_2)\ \exp yX_4 \\ &= \exp((be^x \sinh y)X_1 + (be^x \cosh y)X_2)\ \exp xX_3\ \exp yX_4.\end{aligned}$$

- the master equation with respect to e^{cX_3} is of the form

$$\exp xX_3\ \exp yX_4\ \exp cX_3 = \exp(x+c)X_3\ \exp yX_4.$$

- the master equation with respect to e^{qX_4} is of the form

$$\exp xX_3\ \exp yX_4\ \exp qX_4 = \exp xX_3\ \exp(y+q)X_4.$$

Therefore, we have the formulas below.

$$\begin{aligned}(\pi_{\Omega_{\pm X_1^*}}(\exp aX_1)f)(x,y) &= e^{\pm 2\pi i a e^x \cosh y} f(x,y), \\ (\pi_{\Omega_{\pm X_1^*}}(\exp bX_2)f)(x,y) &= e^{\pm 2\pi i b e^x \sinh y} f(x,y), \\ (\pi_{\Omega_{\pm X_1^*}}(\exp cX_3)f)(x,y) &= f(x+c,y), \\ (\pi_{\Omega_{\pm X_1^*}}(\exp qX_4)f)(x,y) &= f(x,y+q) \qquad (f \in L^2(\mathbb{R}^2)).\end{aligned} \tag{2.60}$$

Similarly we can apply exactly the same argument as the one for construction of representation π of $G_{III}(-1)$ corresponding to the open coadjoint orbit $\Omega_{\pm X_1^*}$ to construction of representation π of $G_{III}(-1)$ corresponding to the open coadjoint orbit $\Omega_{\pm X_2^*}$.

Now we shall compute the Duflo-Moore operator of representation $\pi := \pi_{\Omega_{\pm X_1^*}}$ of $G_{III}(-1)$ given by (2.60) corresponding to the open coadjoint orbits $\Omega_{\pm X_1^*}$ directly. For $\phi \in C_c(G_{III}(-1))$, we have

$$\int_{G_{III}(-1)} \phi(g)\, dg = \int_{\mathbb{R}^4} \phi(e^{aX_1} e^{bX_2} e^{cX_3} e^{qX_4})\, \frac{dadbdcdq}{e^{2c}}.$$

We compute for $f_1, f_2 \in L^2(\mathbb{R}^2)$ the integral

$$\int_{G_{III}(-1)} |(f_1|\pi(g)f_2)|^2\, dg. \tag{2.61}$$

Now we put $g = e^{aX_1}e^{bX_2}g'$ where $g' = e^{cX_3}e^{qX_4}$. We obtain

$$\begin{aligned}
(f_1|\pi(g)f_2) &= \int_{\mathbb{R}^2} f_1(x,y)\overline{e^{\pm 2\pi i e^x(a\cosh y + b\sinh y)}\pi(g')f_2(x,y)}\; dxdy \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{\mp 2\pi i(ae^x\cosh y + be^x\sinh y)} f_1(x,y)\overline{\pi(g')f_2(x,y)}\; dx\,dy \\
&= \int_{\left\{\begin{array}{l}\zeta_1-\zeta_2>0\\ \zeta_1+\zeta_2>0\end{array}\right\}} e^{\mp 2\pi i(a\zeta_1 + b\zeta_2)} f_1(x,y)\overline{\pi(g')f_2(x,y)}\; \frac{d\zeta_1\,d\zeta_2}{\zeta_1^2-\zeta_2^2} \\
&\qquad (\zeta_1 = e^x\cosh y, \quad \zeta_2 = e^x\sinh y, \quad dxdy = \frac{1}{\zeta_1^2-\zeta_2^2}\; d\zeta_1\,d\zeta_2). \qquad (2.62)
\end{aligned}$$

Using Plancherel formula and (2.62), then we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} |(f_1|\pi(e^{aX_1})\pi(e^{bX_2})\pi(g')f_2)|^2\, da\,db \\
&\quad = \int_{\left\{\begin{array}{l}\zeta_1-\zeta_2>0\\ \zeta_1+\zeta_2>0\end{array}\right\}} |f_1(x,y)\pi(g')f_2(x,y)|^2\; \frac{1}{(\zeta_1^2-\zeta_2^2)^2}\; d\zeta_1\,d\zeta_2 \\
&\quad = \int_{\mathbb{R}^2} |f_1(x,y)\pi(g')f_2(x,y)|^2\; \frac{1}{e^{4x}}\; e^{2x}\,dx\,dy \\
&\quad = \int_{\mathbb{R}^2} |f_1(x,y)|^2\,|\pi(g')f_2(x,y)|^2\; e^{-2x}\,dx\,dy. \qquad (2.63)
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
\pi(g')f_2(x,y) &= \pi(e^{cX_3})\pi(e^{qX_4})f_2(x,y) \\
&= f_2(x+c, y+q). \qquad (2.64)
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\pi(e^{cX_3})\pi(e^{qX_4})f_2(x,y)|^2\, \frac{dcdq}{e^{2c}} \\
&\qquad = \int_{\mathbb{R}^2} |f_2(x+c, y+q)|^2\, \frac{dc}{e^{2c}}\; dq \\
&\qquad = \int_{\mathbb{R}^2} |f_2(c', q')|^2\; e^{2x}\; \frac{dc'}{e^{2c'}}\; dq' \\
&\qquad\qquad (c' = x+c,\; q' = y+q). \qquad (2.65)
\end{aligned}$$

Combining (2.63) and (2.65), then the formula (2.61) becomes

$$\begin{aligned}\int_{G_{III}(-1)} |(f_1|\pi(g)f_2)|^2\,dg &= \int_{\mathbb{R}^4} |(f_1|\pi(e^{aX_1})\pi(e^{bX_2})\pi(e^{cX_3})\pi(e^{qX_4})f_2|^2\,\frac{dadbdcdq}{e^{2c}} \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |f_1(x,y)|^2\ |f_2(c',q')|^2\ e^{2x}\,\frac{dc'dq'}{e^{2c'}}\,e^{-2x}\,dxdy \\ &= \int_{\mathbb{R}^2} |f_1(x,y)|^2\,dxdy\ \cdot \int_{\mathbb{R}^2} |e^{-c'}f_2(c',q')|^2\,dc'dq' \\ &= ||f_1||_{L^2(\mathbb{R})}\ \cdot \int_{\mathbb{R}^2} |e^{-c'}f_2(c',q')|^2\,dc'dq'.\end{aligned} \tag{2.66}$$

Therefore, we obtain the following theorem.

Theorem 2.13. *The Duflo Moore operator for the representation* π *of* $G_{III}(-1)$ *given in* (2.60) *can be written of the form*

$$C_\pi f(x,y) = e^{-x} f(x,y) \qquad (\ f \in L^2(\mathbb{R}^2)\ ). \tag{2.67}$$

Now from (2.60), we have

$$\begin{aligned} d\pi(X_1)f(x,y) &= \pm 2\pi i e^x \cosh y\ f(x,y), \\ d\pi(X_2)f(x,y) &= \pm 2\pi i e^x \sinh y\ f(x,y). \end{aligned} \tag{2.68}$$

Furthermore, we have

$$d\pi(X_1)^2 f(x,y) - d\pi(X_2)^2 f(x,y) = 4\pi^2 i^2 e^{2x} f(x,y).$$

Therefore, the Duflo-Moore operator for the representation π of $G_{III}(-1)$ as in (2.60) related to the Pfaffian of $\mathfrak{g}_{III}(-1)$ can be written as

Proposition 2.14. *The Duflo-Moore operator for the representation* π *of* $G_{III}(-1)$ *as in* (2.60) *related to the Pfaffian of* $\mathfrak{g}_{III}(-1)$ *is the form*

$$C_\pi = 2\pi |d\pi(Q_{\mathfrak{g}_{III}(-1)})|^{-1/2}, \tag{2.69}$$

where $Q_{\mathfrak{g}_{III}(-1)} := \mathrm{Pf}(\mathfrak{g}_{III}(-1)) = -X_1^2 + X_2^2$.

Secondly, when $\varepsilon = 1$, the set $\{\alpha X_1^* + \beta X_2^*\ ;\ \alpha^2 + \beta^2 \neq 0\}$ is an H-orbit

$\mathrm{Ad}^*(H)X_1^*$. In conclusion, we have just one open coadjoint orbit of the form $\Omega_{X_1^*}$ in $\mathfrak{g}^*_{III}(1)$. Now let us construct the irreducible unitary representation $\pi := \pi_{\Omega_{X_1^*}}$ of $G_{III}(1)$ corresponding to the open coadjoint orbit $\Omega_{X_1^*}$. Let $f := X_1^*$ be element of $\Omega_{X_1^*}$ and $\mathfrak{p} := \mathbb{R}X_1 \oplus \mathbb{R}X_2 \subset \mathfrak{g}_{III}(1)$ be a polarization at the point f. The 1-dimensional representation of N can be written as $\nu_{X_1^*}(\exp(aX_1 + bX_2)) = e^{2\pi i a}$, where $aX_1 + bX_2 \in \mathfrak{p}$. Moreover, we choose a section $s : N|G \to G$ which identified by $\mathbb{R}\times[0, 2\pi) \to G$. Our task is to solve the master equation $s(x)g = h_s(x, g)s(x\cdot g)$ with respect to the basis $\{X_1, X_2, X_3, X_4\}$ as follows.

- the master equation with respect to e^{aX_1} is of the form

$$\begin{aligned}
\exp xX_3\ \exp yX_4\ \exp aX_1 &= \exp xX_3\ \exp(e^{\mathrm{ad} yX_4} aX_1)\ \exp yX_4 \\
&= \exp xX_3\ \exp((a\cos y)X_1 - (a\sin y)X_2)\ \exp yX_4 \\
&= \exp(e^{\mathrm{ad} xX_3}((a\cos y)X_1 - (a\sin y)X_2))\ \exp xX_3\ \exp yX_4 \\
&= \exp((ae^x\cos y)X_1 - (ae^x\sin y)X_2)\ \exp xX_3\ \exp yX_4.
\end{aligned}$$

- the master equation with respect to e^{bX_2} is of the form

$$\begin{aligned}
\exp xX_3\ \exp yX_4\ \exp bX_2 &= \exp xX_3\ \exp(e^{\mathrm{ad} yX_4} bX_2)\ \exp yX_4 \\
&= \exp((be^x\sin y)X_1 + (be^x\cos y)X_2)\ \exp xX_3\ \exp yX_4.
\end{aligned}$$

- the master equation with respect to e^{cX_3} is of the form

$$\exp xX_3\ \exp yX_4\ \exp cX_3 = \exp(x + c)X_3\ \exp yX_4.$$

- the master equation with respect to e^{qX_4} is of the form

$$\exp xX_3\ \exp yX_4\ \exp qX_4 = \exp xX_3\ \exp(y + q)X_4.$$

Therefore, the irreducible unitary representation $\pi_{\Omega_{X_1^*}}$ of $G_{III}(1)$ on $L^2(\mathbb{R}\times[0, 2\pi))$ corresponding to the open coadjoint orbit $\Omega_{X_1^*}$ with respect to the basis $\{X_1, \ldots, X_4\}$

can be written as

$$\begin{aligned}
(\pi_{\Omega_{X_1^*}}(\exp aX_1)f)(x,y) &= e^{2\pi i a e^x \cos y} f(x,y),\\
(\pi_{\Omega_{X_1^*}}(\exp bX_2)f)(x,y) &= e^{2\pi i b e^x \sin y} f(x,y),\\
(\pi_{\Omega_{X_1^*}}(\exp cX_3)f)(x,y) &= f(x+c,y),\\
(\pi_{\Omega_{X_1^*}}(\exp qX_4)f)(x,y) &= f(x,y+q) \qquad (f \in L^2(\mathbb{R}_+ \times [0,2\pi))).
\end{aligned} \tag{2.70}$$

Furthermore, the Duflo-Moore operator for the representation π of $G_{III}(1)$ in (2.70) can be computed in the following way. For $\phi \in C_c(G_{III}(1))$, we have

$$\int_{G_{III}(1)} \phi(g)\, dg = \int_{\mathbb{R}^4} \phi(e^{aX_1}e^{bX_2}e^{cX_3}e^{qX_4})\, \frac{dadbdcdq}{e^{2c}}.$$

We compute for $f_1, f_2 \in L^2(\mathbb{R}^2)$ the integral

$$\int_{G_{III}(1)} |(f_1|\pi(g)f_2)|^2\, dg. \tag{2.71}$$

Now we put $g = e^{aX_1}e^{bX_2}g'$ where $g' = e^{cX_3}e^{qX_4}$. We obtain

$$\begin{aligned}
(f_1|\pi(g)f_2) &= \int_{[0,2\pi)}\int_{\mathbb{R}} f_1(x,y)\overline{e^{2\pi i e^x(a\cos y + b\sin y)}\pi(g')f_2(x,y)}\; dxdy\\
&= \int_{[0,2\pi)}\int_{\mathbb{R}} e^{2\pi i(ae^x\cos y + be^x \sin y)} f_1(x,y)\overline{\pi(g')f_2(x,y)}\; dx\,dy\\
&= \int_{\mathbb{R}^2} e^{2\pi i(a\zeta_1 + b\zeta_2)} f_1(x,y)\overline{\pi(g')f_2(x,y)}\; \frac{d\zeta_1\, d\zeta_2}{\zeta_1^2+\zeta_2^2}\\
&\quad (\zeta_1 = e^x\cos y, \quad \zeta_2 = e^x \sin y, \quad dxdy = \frac{1}{\zeta_1^2+\zeta_2^2}\; d\zeta_1\, d\zeta_2).
\end{aligned} \tag{2.72}$$

Using Plancherel formula and (2.72), then we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} |(f_1|\pi(e^{aX_1})\pi(e^{bX_2})\pi(g')f_2)|^2 \, da \, db \\
&\quad = \int_{\mathbb{R}^2} |f_1(x,y)\pi(g')f_2(x,y)|^2 \, \frac{1}{(\zeta_1^2+\zeta_2^2)^2} \, d\zeta_1 \, d\zeta_2 \\
&\quad = \int_{[0,2\pi)} \int_{\mathbb{R}} |f_1(x,y)\pi(g')f_2(x,y)|^2 \, \frac{1}{e^{4x}} \, e^{2x} \, dx \, dy \\
&\quad = \int_{[0,2\pi)} \int_{\mathbb{R}} |f_1(x,y)|^2 \, |\pi(g')f_2(x,y)|^2 \, e^{-2x} \, dx \, dy. \qquad (2.73)
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
\pi(g')f_2(x,y) &= \pi(e^{cX_3})\pi(e^{qX_4})f_2(x,y) \\
&= f_2(x+c, y+q). \qquad (2.74)
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\pi(e^{cX_3})\pi(e^{qX_4})f_2(x,y)|^2 \, \frac{dcdq}{e^{2c}} \\
&\qquad = \int_{\mathbb{R}^2} |f_2(x+c, y+q)|^2 \, \frac{dc}{e^{2c}} \, dq \\
&\qquad = \int_{\mathbb{R}^2} |f_2(c', q')|^2 \, e^{2x} \, \frac{dc'}{e^{2c'}} \, dq' \\
&\qquad \quad (c' = x+c, \; q' = y+q). \qquad (2.75)
\end{aligned}$$

Combining (2.73) and (2.75), then the formula (2.71) becomes

$$\begin{aligned}
\int_{G_{III}(1)} |(f_1|\pi(g)f_2)|^2 \, dg &= \int_{\mathbb{R}^4} |(f_1|\pi(e^{aX_1})\pi(e^{bX_2})\pi(e^{cX_3})\pi(e^{qX_4})f_2|^2 \, \frac{dadbdcdq}{e^{2c}} \\
&= \int_{[0,2\pi)} \int_{\mathbb{R}} \int_{\mathbb{R}^2} |f_1(x,y)|^2 \, |f_2(c',q')|^2 \, e^{2x} \, \frac{dc'dq'}{e^{2c'}} \, e^{-2x} \, dxdy \\
&= \int_{[0,2\pi)} \int_{\mathbb{R}} |f_1(x,y)|^2 \, dxdy \cdot \int_{\mathbb{R}^2} |e^{-c'} f_2(c',q')|^2 \, dc'dq' \\
&= ||f_1||_{L^2(\mathbb{R})} \cdot \int_{\mathbb{R}^2} |e^{-c'} f_2(c',q')|^2 \, dc'dq'.
\end{aligned}$$

(2.76)

Therefore, we obtain the following theorem.

Theorem 2.15. *The Duflo Moore operator for the representation* π *of* $G_{III}(1)$ *given in* (2.70) *can be written of the form*

$$C_\pi f(x,y) = e^{-x} f(x,y) \qquad (\ f \in L^2(\mathbb{R}\times[0,2\pi))\ ). \tag{2.77}$$

Now from (2.70), we have

$$\begin{aligned} d\pi(X_1)f(x,y) &= 2\pi i e^x \cos y\ f(x,y), \\ d\pi(X_2)f(x,y) &= 2\pi i e^x \sin y\ f(x,y). \end{aligned} \tag{2.78}$$

From equations above, then we have

$$d\pi(X_1)^2 f(x,y) + d\pi(X_2)^2 f(x,y) = 4\pi^2 i^2 e^{2x} f(x,y).$$

Thus, the Duflo-Moore operator for the representation π of $G_{III}(1)$ related to the Pfaffian of $\mathfrak{g}_{III}(1)$ can be written as

Proposition 2.16. *The Duflo-Moore operator for the representation* π *of* $G_{III}(1)$ *given in* (2.70) *related to the Pfaffian of* $\mathfrak{g}_{III}(1)$ *can be written as*

$$C_\pi = 2\pi |d\pi(Q_{\mathfrak{g}_{III}(1)})|^{-1/2}, \tag{2.79}$$

where $Q_{\mathfrak{g}_{III}(1)} := \mathrm{Pf}(\mathfrak{g}_{III}(1)) = -X_1^2 - X_2^2$.

We also compute the Duflo-Moore operator for the representation of $G_{III}(1)$ given in (2.70) by a different way as in Proposition 4.2.

Chapter 3

General Results for $V \rtimes H$

Let $\mathfrak{g}$ be a semi-direct sum of $V \cong \mathbb{R}^n$ and $\mathfrak{h} \subset \mathfrak{gl}(V)$, denoted by $\mathfrak{g} := V \rtimes \mathfrak{h}$ and its Lie group be denoted by $G := V \rtimes H$, where H is the connected subgroup of $\mathrm{GL}(V)$ correponding to $\mathfrak{h}$. In this chapter we give general results for G, especially conditions for coadjoint orbits of G to be open in $\mathfrak{g}^*$. If this happens, namely the coadjoint orbit of G at $\xi_0 \in \mathfrak{g}^*$, denoted by Ω_{ξ_0}, is open in $\mathfrak{g}^*$, then we have that the Lie algebra $\mathfrak{g} = V \rtimes \mathfrak{h}$ is Frobenius. Furthermore, when the representation of G is square-integrable, then we compute its Duflo-Moore operator, particularly for $\mathfrak{h}_{p_0} = \{0\}$ and the other for $\mathfrak{h}_{p_0} \neq \{0\}$.

3.1 Conditions for coadjoint orbits of $G = V \rtimes H$ to be open in $\mathfrak{g}^*$

Let G be a semi-direct product of V which is isomorphic to an n-dimensional real vector space $\mathbb{R}^n$ and a connected subgroup H of $\mathrm{GL}(V)$, and let $\mathfrak{g} := V \rtimes \mathfrak{h}$ be its Lie algebra with $\mathfrak{g}^* := V^* \oplus \mathfrak{h}^*$ as its dual space. For $p \in V^*$ and $A \in \mathrm{End}(V)$, we define a linear functional A^*p by

$$\langle A^*p, v\rangle = \langle p, Av\rangle \qquad (v \in V). \tag{3.1}$$

For $u, v \in V, A \in \mathfrak{h}$, and $a \in H$, let $X(v, A)$ be an element of $\mathfrak{g}$, and let $g(u, a)$ be an element of G. We also write $g(a) = g(0, a)$ and $g(u) = g(u, I)$. We decribe adjoint actions of G on $\mathfrak{g}$ by

$$\mathrm{Ad}(g(a))X(v,A) = X(av, \mathrm{Ad}(a)A),$$

and

$$\mathrm{Ad}(g(u))X(v,A) = X(v - Au, A).$$

Furthermore, we have formulas for coadjoint actions of G and $\mathfrak{g}$ on $\mathfrak{g}^*$ respectively as follows :

$$\begin{aligned}
\mathrm{Ad}^*(g(a))\xi(p,\alpha) &= \xi((a^{-1})^* p, \mathrm{Ad}^*(a)\alpha), \\
\mathrm{Ad}^*(g(u))\xi(p,\alpha) &= \xi(p, \alpha + u.p), \\
\mathrm{ad}^*(X(v,A))\xi(p,\alpha) &= \xi(-A^* p, \mathrm{ad}^*(A)\alpha + v.p),
\end{aligned} \tag{3.2}$$

where $u.p \in \mathfrak{h}^*$ is defined by $\langle u.p, A\rangle := \langle p, Au\rangle \quad (A \in \mathfrak{h})$. Let $\xi_0 = \xi(p_0, \alpha_0)$ be an element of $\mathfrak{g}^*$ with $p_0 \in V^*$ and $\alpha_0 \in \mathfrak{h}^*$, we consider conditions for coadjoint orbit $\Omega_{\xi_0} := \mathrm{Ad}^*(G)\xi_0 \subset \mathfrak{g}^*$ to be open in $\mathfrak{g}^*$ as follows.

Lemma 3.1. *The coadjoint orbit Ω_{ξ_0} is open in $\mathfrak{g}^*$ if and only if the map*

$$f : \mathfrak{g} \to \mathfrak{g}^* \tag{3.3}$$

defined by

$$f : \mathfrak{g} \ni X(v,A) \mapsto \mathrm{ad}^*(X(v,A))\xi_0 = \xi(-A^* p_0, \mathrm{ad}^*(A)\alpha_0 + v.p_0) \in \mathfrak{g}^* \tag{3.4}$$

is bijective.

Proof. Assume that the coadjoint orbit Ω_{ξ_0} is open in $\mathfrak{g}^*$. This means that f is surjective. Since $\dim \mathfrak{g} = \dim \mathfrak{g}^*$, surjectivity of f is equivalent to bijectivity. □

Let $\mathfrak{h}_{p_0}$ be the stabilizer of $\mathfrak{h}$ at $p_0 \in V^*$. In addition, let $\mathfrak{l}$ be a subspace of $\mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{h}_{p_0}$. Now let ϖ be a canonical projection of $\mathfrak{g}^* = V^* \oplus \mathfrak{h}^*$ onto V^*.

Lemma 3.2. *The coadjoint orbit $\varpi(\Omega_{\xi_0})$ is open in V^* if and only if*

$$V \ni v \longmapsto v \cdot p_0 \in \mathfrak{l}^*$$

is bijective.

Proof. First of all, let assume that $\varpi(\Omega_{\xi_0})$ is open in V^*. We observe that

$$\begin{aligned}\varpi(\Omega_{\xi_0}) &= \varpi(\{\mathrm{Ad}^*(g)\xi_0 \ ; \ g \in G\}) \\ &= \varpi(\{\mathrm{Ad}^*(g(v,h))\xi_0 \ ; \ v \in V, \ h \in H\}) \\ &= \{(h^{-1})^* p_0 \ ; \ h \in H\} \simeq H/H_{p_0}.\end{aligned}$$

Since $\varpi(\Omega_{\xi_0}) \simeq H/H_{p_0}$ and $\varpi(\Omega_{\xi_0})$ is open in V^*, then $\dim V^* = \dim \mathfrak{h} - \dim \mathfrak{h}_{p_0}$. On the other hand, $\dim \mathfrak{h} = \dim \mathfrak{l} + \dim \mathfrak{h}_{p_0}$. Therefore, we obtain $\dim \mathfrak{l} = \dim V^*$. Furthermore, let B an element of $\mathfrak{l}$ for which $-B^* p_0 = 0$, then $B \in \mathfrak{h}_{p_0}$. We get $B \in \mathfrak{l} \cap \mathfrak{h}_{p_0} = \{0\}$. Therefore, $B = 0$. We obtain that the map

$$\mathfrak{l} \ni B \longmapsto -B^* p_0 \in V^*$$

is always injective, so that bijective since $\dim \mathfrak{l} = \dim V^*$. Let us take $v \in V$ for which $v \cdot p_0 = 0$, then for all $B \in \mathfrak{l}$ we have

$$\langle v \cdot p_0, B \rangle = \langle B^* p_0, v \rangle = 0.$$

Thus, $v = 0$. This means the map $V \ni v \longmapsto v \cdot p_0 \in \mathfrak{l}^*$ is injective, so that it is bijective since $\dim V = \dim \mathfrak{l}^*$.
Now we assume that the map $V \ni v \longmapsto v \cdot p_0 \in \mathfrak{l}^*$ is bijective. We have $\dim V = \dim \mathfrak{l}^*$. Let us take $B \in \mathfrak{l}$ for which $B^* p_0 = 0$, then for all $v \in V$ we have

$$\langle B^* p_0, v \rangle = \langle v \cdot p_0, B \rangle = 0.$$

Therefore, $B = 0$. This means, $\mathfrak{l} \ni B \longmapsto -B^* p_0 \in V^*$ is injective, so that bijective. Thus, $\varpi(\Omega_{\xi_0})$ is open. □

Theorem 3.3. Ω_{ξ_0} *is open if and only if the following two conditions are satisfied :*

1. $\varpi(\Omega_{\xi_0})$ *is open in* V^*.
2. $\mathfrak{h}_{p_0} = 0$, *or the coadjoint orbit* $\mathrm{Ad}^*(H_{p_0})(\alpha_0|_{\mathfrak{h}_{p_0}})$ *in* $\mathfrak{h}_{p_0}^*$ *through* $\alpha_0|_{\mathfrak{h}_{p_0}} \in \mathfrak{h}_{p_0}^*$ *is open.*

Proof. In the previous construction, $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{h}_{p_0}$, so that we have $\mathfrak{g} = V \oplus \mathfrak{l} \oplus \mathfrak{h}_{p_0}$ and $\mathfrak{g}^* = V^* \oplus \mathfrak{l}^* \oplus \mathfrak{h}_{p_0}^*$. We note that $\mathfrak{l}^*$ is naturally identified with $(\mathfrak{h}_{p_0}^{\perp})$. For $(p, \beta, \gamma) \in V^* \oplus \mathfrak{l}^* \oplus \mathfrak{h}_{p_0}^*$, we write $\zeta(p, \beta, \gamma)$ for the corresponding element of $\mathfrak{g}^*$. Namely,

$$\langle \zeta(p, \beta, \gamma), X(v, B + C) \rangle := \langle p, v \rangle + \langle \beta, B \rangle + \langle \gamma, C \rangle \,,$$

where $B \in \mathfrak{l}$ and $C \in \mathfrak{h}_{p_0}$. In other words, $\xi(p, \alpha) = \zeta(p, \alpha|_{\mathfrak{l}}, \alpha|_{\mathfrak{h}_{p_0}})$ for $p \in V^*$ and $\alpha \in \mathfrak{h}^*$. Then, the map $f_0 : \mathfrak{g} \to \mathfrak{g}^*$ in (3.4) is described as

$$\begin{aligned} X(v, 0) &\longmapsto \zeta(0, v.p_0, 0), \\ X(0, B) &\longmapsto \zeta(-B^* p_0, \mathrm{ad}^*(B)(\alpha_0|_{\mathfrak{l}}), \mathrm{ad}^*(B)(\alpha_0|_{\mathfrak{h}_{p_0}}) \ (B \in \mathfrak{l}), \\ X(0, C) &\longmapsto \zeta(0, \mathrm{ad}^*(C)(\alpha_0|_{\mathfrak{l}}), \mathrm{ad}^*(C)(\alpha_0|_{\mathfrak{h}_{p_0}}) \ (C \in \mathfrak{h}_{p_0}). \end{aligned} \tag{3.5}$$

We shall prove "if" part. Assume $\varpi(\Omega_{\xi_0})$ is open and $\mathrm{Ad}^*(H_{p_0})(\alpha_0|_{\mathfrak{h}_{p_0}}) \subset \mathfrak{h}_{p_0}$ is open. As is seen before $\dim \mathfrak{l} = \dim V^*$. Therefore, the map $\mathfrak{l} \ni B \mapsto -B^* p_0 \in V^*$ is bijective. For a given $\zeta(p_1, \beta_1, \gamma_1) \in \mathfrak{g}^*$, we can find $X(v_1, B_1 + C_1)$ for which

$$\mathrm{ad}^*(X(v_1, B_1 + C_1))\xi_0 = \zeta(p_1, \beta_1, \gamma_1).$$

In fact, we have by (3.5)

$$p_1 = -B_1^* p_0, \tag{3.6}$$

$$\beta_1 = v_1.p_0 + \mathrm{ad}^*(B_1 + C_1)(\alpha_0|_{\mathfrak{l}}), \tag{3.7}$$

$$\gamma_1 = \mathrm{ad}^*(B_1 + C_1)(\alpha_0|_{\mathfrak{h}_{p_0}}). \tag{3.8}$$

First, by (3.6) we find $B_1 \in \mathfrak{l}$. Then by (3.8) and openness of $\mathrm{Ad}^*(H_{p_0})(\alpha|_{\mathfrak{h}_{p_0}})$, we can find $C_1 \in \mathfrak{h}_{p_0}$. Finally, we can find $v_1 \in V$ by (3.7) and Lemma 3.2. Thus, we prove "if part' for the case $\mathfrak{h}_{p_0} \neq \{0\}$, and the case $\mathfrak{h}_{p_0} = \{0\}$ is proved similarly. Now we shall prove the "only if" part. If $\varpi(\Omega_{\xi_0})$ is not open in V^*, then there exists $p_1 \in V^*$ such that $A^* p_0 \neq p_1$ for any $A \in \mathfrak{h}$. This implies that $\mathrm{ad}^*(X)\xi_0 \neq \zeta(p_1, *, *)$ by (3.6) for any $X \in \mathfrak{g}$. Therefore, Ω_{ξ_0} is not open in $\mathfrak{g}^*$.

Assume that $\varpi(\Omega_{\xi_0})$ is open. If $\mathfrak{h}_{p_0} \neq \{0\}$ and $\mathrm{Ad}^*(H_{p_0})(\alpha_0|_{\mathfrak{h}_{p_0}})$ is not open in $\mathfrak{h}_{p_0}^*$, then there exists $\gamma_1 \in \mathfrak{h}_{p_0}^*$ such that $\mathrm{ad}^*(C)(\alpha|_{\mathfrak{h}_{p_0}}) \neq \gamma_1$ for any $C \in \mathfrak{h}_{p_0}$. This implies that $\mathrm{ad}^*(X)\xi_0 \neq \zeta(0, 0, \gamma_1)$ for any $X \in \mathfrak{g}$. In fact, if $\mathrm{ad}^*(X)\xi_0 = \zeta(0, 0, \gamma_1)$ with $X = X(v_1, B_1 + C_1)$, then $B_1^* p_0 = 0$ means $B_1 = 0$, because as is already seen that openness of $\varpi(\Omega_{\xi_0})$ implies $B_1 \mapsto -B_1^* p_0$ is injective. On the other hand, (3.8) means $\gamma_1 = \mathrm{ad}^*(C_1)(\alpha_0|_{\mathfrak{h}_{p_0}})$. This is a contradiction, therefore, Ω_{ξ_0} is not open in $\mathfrak{g}^*$ either. We have shown that if Ω_{ξ_0} is open in $\mathfrak{g}^*$ then $\varpi(\Omega_{\xi_0})$ is open in V^* and $\mathrm{Ad}^*(H_{p_0})(\alpha_0|_{\mathfrak{h}_{p_0}})$ is open in $\mathfrak{h}_{p_0}^*$ or $\mathfrak{h}_{p_0} = \{0\}$. □

As a corollary of Theorem 3.3, we obtain the following result.

Corollary 3.4. *The Lie algebra $\mathfrak{g} = V \rtimes \mathfrak{h}$ is a Frobenius Lie algebra if and only if there exists $p_0 \in V^*$ such that $\mathfrak{h}.p_0 = V^*$ and the stabilizer $\mathfrak{h}_{p_0} \subset \mathfrak{h}$ is a Frobenius Lie algebra.*

We shall see later that Corollary 3.4 will be applied to similitude and connected affine Lie group in Chapter IV.

3.2 Duflo-Moore Operators for $V \rtimes H$ with trivial stabilizer case of H

As in the previous section, let G be a semi-direct product of $V \cong \mathbb{R}^n$ and a Lie subgroup H of $\mathrm{GL}(V)$. We take $\xi_0 := \xi(p_0, \alpha_0)$ in $\mathfrak{g}^*$ and assume that Ω_{ξ_0} is open as described in Theorem 3.3. Furthermore, we assume $\mathfrak{h}_{p_0} = \{0\}$. The case $\mathfrak{h}_{p_0} \neq \{0\}$, it shall be discussed in the Section 3.3.

Lemma 3.5.

$$\Omega_{\xi_0} = \Omega_{\xi(p_0,0)}. \tag{3.9}$$

Proof. Since $\mathfrak{h}_{p_0} = \{0\}$, we have $\mathfrak{h}_{p_0}^{\perp} = \mathfrak{h}^*$ and the bijection

$$V \ni u \mapsto u \cdot p_0 \in \mathfrak{h}^*$$

as in the proof of Lemma 3.2. Let $u_0 \in V$ for which $u_0 \cdot p_0 = \alpha_0$. Then,

$$\xi_0 = \xi(p_0, \alpha_0) = \mathrm{Ad}^*(g(u_0))\xi(p_0, 0)$$

by (3.2). Therefore, $\Omega_{\xi_0} = \Omega_{\xi(p_0,0)}$. □

We define an action of H on V^* by

$$h \cdot p := (h^{-1})^* p \qquad (h \in H, p \in V^*).$$

Then $\mathcal{O}_{p_0} := \varpi(\Omega_{p_0}) \subset V^*$ is the H-orbit $\{h \cdot p_0; h \in H\}$. By Theorem 3.3, $\mathcal{O}_{p_0}$ is open in V^*. By Lemma 3.5, we assume $\xi_0 = \xi(p_0, 0)$ in what follows. We also note that $\Omega_{\xi_0} = \mathcal{O}_{p_0} \oplus V^*$.

Furthermore, since the stabilizer $H_{p_0} = \{1\}$, we have a bijection $H \ni h \mapsto h.p_0 \in \mathcal{O}_{p_0}$. In addition, for $p \in \mathcal{O}_{p_0}$, we denote by h_p a unique element of H for which $h_p \cdot p_0 = p$.

Lemma 3.6. *Define a linear form*

$$\mu : C_c(H) \to \mathbb{R}$$

by

$$\mu(\psi) := \int_{\mathcal{O}_{p_0}} \psi(h_p)\, |\det h_p|\, dp \quad (\psi \in C_c(H)). \tag{3.10}$$

Then, μ is left invariant.

Proof. For each $a \in H$, we define left translation $L(a)\psi(h) := \psi(a^{-1}h)$. Then we have,

$$\begin{aligned}
\mu(L(a)\psi) &= \int_{\mathcal{O}_{p_0}} \psi(a^{-1}h_p)|\det h_p|\, dp \\
&= \int_{\mathcal{O}_{p_0}} \psi(h_{a^{-1}\cdot p})|\det h_p|\, dp \\
&= \int_{\mathcal{O}_{p_0}} \psi(h_{p'})|\det h_{a\cdot p'}|\, |\det a|^{-1}\, dp' \quad (p' = a^{-1} \cdot p) \\
&= \int_{\mathcal{O}_{p_0}} \psi(h_{p'})|\det a \cdot h_{p'}|\, |\det a|^{-1}\, dp' \\
&= \int_{\mathcal{O}_{p_0}} \psi(h_{p'})|\det a| \cdot |\det h_{p'}|\, |\det a|^{-1}\, dp' \\
&= \int_{\mathcal{O}_{p_0}} \psi(h_{p'})|\det h_{p'}|\, dp' = \mu(\psi).
\end{aligned}$$

Therefore, μ is left invariant. □

By a uniqueness of Haar measure and Lemma 3.6, there exists a $C_0 > 0$ such that

$$\int_H \psi(h)\, dh = C_0\mu(\psi) = C_0 \int_{\mathcal{O}_{p_0}} \psi(h_p)|\det h_p|\, dp. \tag{3.11}$$

If we define $D(p) := C_0|\det h_p| \quad (p \in \mathcal{O}_{p_0})$, then (3.11) is rewritten as

$$\int_H \psi(h)\, dh = \int_{\mathcal{O}_{p_0}} \psi(p)\, D(p)\, dp \quad (\psi \in C_c(H)). \tag{3.12}$$

Let us consider a unitary representation of G corresponding to $\Omega_{\xi_0} \subset \mathfrak{g}^*$. In this case, $V \subset \mathfrak{g}$ is a polarization at ξ_0. Let π be the induced representation $\mathrm{Ind}_V^G \nu_{p_0}$ of G. By Mackey Theory, π is irreducible and we realize π on $L^2(H, dh)$ by

$$\pi(g(a))f(h) = f(a^{-1}h), \tag{3.13}$$

$$\pi(g(v))f(h) = \nu_{p_0}(h^{-1}v)f(h) \ \ (f \in L^2(H, dh),\ a, h \in H,\ v \in V). \tag{3.14}$$

Since $\nu_{p_0}(h^{-1}v) = e^{2\pi i \langle p_0, h^{-1}v \rangle} = e^{2\pi i \langle h \cdot p_0, v\rangle}$, we rewrite (3.14) as

$$\pi(g(v))f(h) = e^{2\pi i \langle h\cdot p_0, v\rangle} f(h). \tag{3.15}$$

Now we shall compute the Duflo-Moore operator C_π for the representation π. To do that, let f_1 and f_2 be elements in $L^2(H)$. We shall consider the integral

$$\int_H \int_V |(f_1|\pi(g(v))\pi(g(a))f_2)_{L^2(H)}|^2\, dv\, |\det a|^{-1}\, da. \tag{3.16}$$

We obtain by (3.15)

$$\begin{aligned} (f_1|\pi(g(v))\pi(g(a))f_2)_{L^2(H)} &= \int_H f_1(h)\overline{\pi(g(v))\pi(g(a))f_2(h)}\, d\mu_l(h) \\ &= \int_H e^{-2\pi i \langle h.p_0, v\rangle} f_1(h)\overline{\pi(g(a))f_2(h)}\, d\mu_l(h). \end{aligned} \tag{3.17}$$

Using (3.12), the last term equals

$$\int_{\mathcal{O}_{p_0}} e^{-2\pi i \langle p, v\rangle} f_1(h_p)\overline{\pi(g(a))f_2(h_p)}\, D(p)\, dp. \tag{3.18}$$

By the Plancherel formula, we have

$$\int_V |(f_1|\pi(g(v))\pi(g(a))f_2)_{L^2(H)}|^2\, dv = \int_{\mathcal{O}_{p_0}} |f_1(h_p)|^2 |\pi(g(a))f_2(h_p)|^2\, D(p)^2\, dp. \tag{3.19}$$

On the other hand, we have

$$\int_H |\pi(g(a))f_2(h_p)|^2 \, |\det a|^{-1} \, da = \int_H |f_2(a^{-1} \cdot h_p)|^2 \, |\det a|^{-1} \, da$$
$$= |\det h_p|^{-1} \int_H |f_2(h')|^2 \, |\det h'| \, \Delta_H(h')^{-1} \, dh' \quad (h' = a^{-1} \cdot h_p). \tag{3.20}$$

By (3.19) and Fubini Theorem, we obtain

$$\int_H \int_V |(f_1|\pi(g(v))\pi(g(a))f_2)_{L^2(H)}|^2 \, dv \, |\det a|^{-1} \, da$$
$$= \int_H \left\{ \int_{\mathcal{O}_{p_0}} |f_1(h_p)|^2 \, |\pi(g(a))f_2(h_p)|^2 D(p)^2 \, dp \right\} |\det a|^{-1} \, da$$
$$= \int_{\mathcal{O}_{p_0}} |f_1(h_p)|^2 \left\{ \int_H |f_2(a^{-1}h_p)|^2 \, |\det a|^{-1} \, da \right\} D(p)^2 \, dp. \tag{3.21}$$

Then we apply (3.20) and $D(p) = C_0 |\det h_p|$, we see that (3.21) equals

$$\int_{\mathcal{O}_{p_0}} |f_1(h_p)|^2 C_0 |\det h_p| \, dp \left\{ C_0 \int_H |f_2(h')|^2 \, |\det h'| \Delta_H(h')^{-1} \, dh' \right\}. \tag{3.22}$$

Using (3.11), we see that (3.22) equals

$$||f_1||^2_{L^2(H)} \cdot \left\{ C_0 \int_H |f_2(h')|^2 \, |\det h'| \Delta_H(h')^{-1} \, dh' \right\}. \tag{3.23}$$

Substituting $\Delta_G(h')^{-1} = |\det h'| \Delta_H(h')^{-1}$, the result equals

$$||f_1||^2_{L^2(H)} \cdot \left\{ C_0 \int_H |f_2(h')|^2 \, \Delta_G(h')^{-1} \, dh' \right\}. \tag{3.24}$$

Therefore, from (3.24) we obtain the Duflo-Moore operator for the representation π of G as follows.

Theorem 3.7. *We assume that* $\mathfrak{h}_{p_0} = \{0\}$. *The Duflo-Moore operator* $C_\pi : L^2(H) \to L^2(H)$ *of the representation* $\pi := \mathrm{Ind}_V^G \nu_{p_0}$ *is described as*

$$C_\pi f_2(h) = C_0^{1/2} \, \Delta_G(h)^{-1/2} f_2(h) \quad (f_2 \in L^2(H)), \tag{3.25}$$

where C_0 is defined in (3.11).

3.3 General Formula of Duflo-Moore Operator for $V \rtimes H$

As in the previous section, let G be a semi-direct product of $V \cong \mathbb{R}^n$ and a Lie subgroup H of $\mathrm{GL}(V)$. We take $\xi_0 := \xi(p_0, \alpha_0)$ in $\mathfrak{g}^*$ and assume that Ω_{ξ_0} is open as described in Theorem 3.3. Furthermore, we assume $\mathfrak{h}_{p_0} \neq \{0\}$.

Lemma 3.8. *For $h_0 \in H_{p_0}$, we have*

$$\Delta_G(h_0) = \Delta_{H_{p_0}}(h_0). \tag{3.26}$$

Proof. We shall consider the coadjoint orbit $\Omega_{p_0} := \mathrm{Ad}^*(G)p_0 \subset \mathfrak{g}^*$ and the isotropy representation

$$\tau : H_{p_0} \to \mathrm{GL}(T_{p_0}\Omega_{p_0}).$$

We have

$$\begin{aligned} T_{p_0}\Omega_{p_0} &= \{\mathrm{ad}^*(X(v,A))p_0; v \in V, A \in \mathfrak{h}\} \\ &= \{\xi(-A^*p_0, v.p_0); v \in V, A \in \mathfrak{h}\}, \end{aligned} \tag{3.27}$$

and we see that $v.p_0 \in (\mathfrak{h}_{p_0})^\perp$ by Lemma 3.2. Thus,

$$T_{p_0}\Omega_{p_0} = V^* \oplus (\mathfrak{h}_{p_0})^\perp. \tag{3.28}$$

For $h_0 \in H_{p_0}$, the linear map

$$\tau(h_0) : T_{p_0}\Omega_{p_0} \to T_{p_0}\Omega_{p_0} \tag{3.29}$$

is symplectic, so that $\det \tau(h_0) = 1$. On the other hand by (3.28)

$$\begin{aligned} \det \tau(h_0) &= \det(\tau(h_0)|_{V^*}) \cdot \det(\tau(h_0)|_{\mathfrak{h}_{p_0}^\perp}) \\ &= (\det h_0)^{-1} \cdot \det \mathrm{Ad}_{\mathfrak{h}}(h_0)^{-1} / \det \mathrm{Ad}_{\mathfrak{h}_{p_0}}(h_0)^{-1} \\ &= (\det h_0)^{-1} \cdot \Delta_H(h_0)/\Delta_{H_{p_0}}(h_0). \end{aligned} \tag{3.30}$$

Therefore,

$$\det h_0 = \Delta_H(h_0)/\Delta_{H_{p_0}}(h_0).$$

On the other hand,

$$\det h_0 = \Delta_H(h_0)/\Delta_G(h_0),$$

which implies that $\Delta_G(h_0) = \Delta_{H_{p_0}}(h_0)$. □

We also obtain

Theorem 3.9. *Let $\mathfrak{m}_0 \subset \mathfrak{h}_{p_0}$ be a polarization at $\alpha_0|_{\mathfrak{h}_{p_0}}$ satisfying Pukanszky condition and let π_0 be the induced representation $\mathrm{Ind}_{\exp \mathfrak{m}_0}^{H_{p_0}} \nu_{\alpha_0}$ of H_{p_0}.*

1. *$\mathfrak{p}_0 := V \rtimes \mathfrak{m}_0 \subset \mathfrak{g}$ is a polarization at $\xi_0 = \xi(p_0, \alpha_0)$ satisfying Pukanszky condition.*

2. *$\pi := \mathrm{Ind}_{\exp \mathfrak{p}_0}^{G} \nu_{\xi_0}$ is irreducible if π_0 is irreducible.*

3. *π is square-integrable if π_0 is square-integrable.*

Proof. For the first part, let us observe the value of linear functional ξ_0 on $[\mathfrak{p}_0, \mathfrak{p}_0]$. Since we have

$$\langle \xi_0, [v + A, v' + A'] \rangle = p_0(Av' - A'v) + \alpha_0([A, A']) = 0 \quad (v, v' \in V, A, A' \in \mathfrak{m}_0),$$

we see that $\langle \xi_0, [\mathfrak{p}_0, \mathfrak{p}_0] \rangle = 0$. Moreover, since $\mathfrak{m}_0$ is a polarization at $\alpha_0|_{\mathfrak{h}_{p_0}}$, we have $\dim \mathfrak{m}_0 = \frac{1}{2} \dim \mathfrak{h}_{p_0} = \frac{1}{2}(\dim \mathfrak{h} - n)$. Therefore we obtain

$$\dim \mathfrak{p} = n + \dim \mathfrak{m}_0 = \frac{1}{2}(n + \dim \mathfrak{h}) = \frac{1}{2} \dim \mathfrak{g}.$$

Now we shall prove that $\mathfrak{p}_0$ satisfies Pukanszky condition, namely, for each $\xi_0' \in \xi_0 + \mathfrak{p}_0^{\perp}$, there exist $g(v, h) \in G$ so that $\mathrm{Ad}^*(g(v, h))\xi_0 = \xi_0'$. We recall the notation in the proof of Theorem 3.3 so that $\xi_0 = \zeta(p_0, \beta_0, \gamma_0)$ and $\xi_0' = \zeta(p_0, \beta_0', \gamma_0')$ where $\beta_0, \beta_0' \in \mathfrak{l}^*$ and $\gamma_0, \gamma_0' \in \mathfrak{h}_0^*$. By the assumption that $\mathfrak{m}_0$ satisfies Pukansky condition, we can take $h \in H_{p_0}$ for which $\gamma_0' = \mathrm{Ad}^*(h)\gamma_0$. Let $\xi_0'' := \mathrm{Ad}^*(h)\xi_0$. For $C \in \mathfrak{h}_{p_0}$, we have

$$\begin{aligned}\langle \xi_0'', C\rangle &= \langle \mathrm{Ad}^*(h)\xi_0, C\rangle \\ &= \langle \xi_0, \mathrm{Ad}(h^{-1})C\rangle \\ &= \langle \zeta(p_0, \beta_0, \gamma_0), \mathrm{Ad}(h^{-1})C\rangle \\ &= \langle \gamma_0, \mathrm{Ad}(h^{-1})C\rangle \\ &= \langle \mathrm{Ad}^*(h)\gamma_0, C\rangle \\ &= \langle \gamma_0', C\rangle.\end{aligned}$$

Therefore, we have $\xi_0'' = \zeta(p_0, \beta_0'', \gamma_0')$. On the other hand, by Lemma 3.2, we can take $v \in V$ for which $\beta_0' - \beta_0'' = v \cdot p_0$. Then we have

$$\begin{aligned}\mathrm{Ad}^*(g(v,h))\xi_0 &= \mathrm{Ad}^*(g(v))\mathrm{Ad}^*(g(h)\xi_0 \\ &= \mathrm{Ad}^*(g(v))\xi_0'' \\ &= \mathrm{Ad}^*(g(v))\zeta(p_0, \beta_0'', \gamma_0') \\ &= \zeta(p_0, \beta_0'' + v \cdot p_0, \gamma_0') \\ &= \zeta(p_0, \beta_0', \gamma_0') = \xi_0'.\end{aligned}$$

Therefore, $\mathfrak{p}_0$ satisfies Pukanszky condition.

For the second part, the proof of the statement can be found in [4] and [23] in the context of Mackey Machine. The last part about square-integrability of $\pi := \mathrm{Ind}_{\exp \mathfrak{p}_0}^{G} \nu_{\xi_0}$, we can find the detailed proof in [2]. □

Let us assume that π_0 is irreducible and square integrable in what follows. We shall describe the Duflo-Moore operator of square-integrable representation π of G. To do that, we realize induced representations in the Blattner model in ([8], [23]) as follows.

The representation space $\mathcal{H}_\pi$ of π is given by

$$\mathcal{H}_\pi = L^2(G/\exp\mathfrak{p}_0, \nu_{\xi_0})$$
$$= \left\{ \phi : G \to \mathbb{C}\ ;\ \begin{array}{r} \phi(g\exp X) = \Delta_{G,\exp\mathfrak{p}_0}(\exp X)^{-1/2}\nu_{\xi_0}(\exp X)^{-1}\phi(g) \\ (\ g \in G,\ \ X \in \mathfrak{p}_0\ ), \\ ||\phi||^2_{\mathcal{H}_\pi} = \oint_{G/\exp\mathfrak{p}_0} |\phi(g)|^2\, d\mu(\dot{g}) < \infty. \end{array} \right\}. \tag{3.31}$$

Since $G/\exp\mathfrak{p}_0 \simeq H/\exp\mathfrak{m}_0$, the space $\mathcal{H}_\pi$ is identified with $L^2(H/\exp\mathfrak{m}_0, \nu_{\alpha_0})$ through the restriction map

$$\mathcal{H}_\pi \ni \phi \longmapsto \phi|_H \in L^2(H/\exp\mathfrak{m}_0, \nu_{\alpha_0}). \tag{3.32}$$

Then π is realized on $L^2(H/\exp\mathfrak{m}_0, \nu_{\alpha_0})$ by

$$\begin{aligned} &\pi(g(a))\phi(h) = \phi(a^{-1}h), \\ &\pi(g(v))\phi(h) = \nu_{p_0}(h^{-1}\cdot v)\phi(h) \quad (\ \phi \in L^2(H/\exp\mathfrak{m}_0, \nu_{\alpha_0}),\ \ a, h \in H,\ \ v \in V\ ), \end{aligned} \tag{3.33}$$

and the latter formula is rewritten as

$$\pi(g(v))\phi(h) = e^{2\pi i\langle h\cdot p_0, v\rangle}\phi(h). \tag{3.34}$$

Because of induction by stage, we have

$$\begin{aligned} \mathrm{Ind}^H_{\exp\mathfrak{m}_0}\nu_{\alpha_0} &\simeq \mathrm{Ind}^H_{H_{p_0}}(\mathrm{Ind}^{H_{p_0}}_{\exp\mathfrak{m}_0}\nu_{\alpha_0}) \\ &\simeq \mathrm{Ind}^H_{H_{p_0}}\pi_0. \end{aligned}$$

This equivalence is realized by the Hilbert space isomorphism

$$L^2(H/\exp\mathfrak{m}_0, \nu_{\alpha_0}) \ni \phi \longmapsto \tilde{\phi} \in L^2(H/H_{p_0}, \mathcal{H}_{\pi_0})$$

given by

$$\tilde{\phi}(l)(h) := \phi(lh)\Delta_{H,H_{p_0}}(h)^{1/2} \quad (\ l \in H,\ \ h \in H_{p_0}). \tag{3.35}$$

Indeed, for almost all $l \in H$, we see that $\tilde{\phi}(l) \in \mathcal{H}_{\pi_0}$ because

$$\begin{aligned}
||\phi||^2 &= \oint_{H/\exp \mathfrak{m}_0} |\phi(h)|^2 d\mu_0(\dot{h}) \\
&= \oint_{H/H_{p_0}} \{ \oint_{H_{p_0}/\exp \mathfrak{m}_0} |\phi(lh_0)|^2 \Delta_{H,H_{p_0}}(h_0) d\mu(\dot{h_0}) \} d\mu_0(\dot{l}) \\
&= \oint_{H/H_{p_0}} ||\tilde{\phi}(l)||^2_{\mathcal{H}_{\pi_0}} d\mu_0(\dot{l}). \qquad (3.36)
\end{aligned}$$

Theorem 3.10. *The representation* $\pi = \mathrm{Ind}_{\exp \mathfrak{p}_\circ}^G \nu_{\xi_0}$ *is isomorphic to* $\mathrm{Ind}_{G_{p_0}}^G (\nu_{p_0} \otimes \pi_0)$*, where* $G_{p_0} := V \rtimes H_{p_0}$ *and* $\nu_{p_0} \otimes \pi_0$ *is defined by*

$$\nu_{p_0} \otimes \pi_0(u, h) := \nu_{p_0}(u)\pi_0(h) \quad ((u, h) \in G_{p_0}).$$

Proof. Let $\pi' := \mathrm{Ind}_{G_{p_0}}^G (\nu_{p_0} \otimes \pi_0)$. The representation space of π' is $\mathcal{H}_{\pi'} = L^2(G/G_{p_0}, \mathcal{H}_{\pi_0})$ which is isomorphic to $L^2(H/H_{p_0}, \mathcal{H}_{\pi_0})$ via

$$L^2(G/G_{p_0}, \mathcal{H}_{\pi_0}) \ni \psi \longmapsto \psi|_H \in L^2(H/H_{p_0}, \mathcal{H}_{\pi_0}).$$

Then π' is realized on $L^2(H/H_{p_0}, \mathcal{H}_{\pi_0})$ by

$$\begin{aligned}
&\pi'(g(a))\psi(h) = \psi(a^{-1}h), \\
&\pi'(g(v))\psi(h) = \nu_{p_0}(h^{-1}v)\psi(h) \;\; (\, a, h \in H, \;\; v \in V, \;\; \psi \in L^2(H/\exp \mathfrak{m}_0, \nu_{\alpha_0}) \,).
\end{aligned} \qquad (3.37)$$

Comparing (3.33) and (3.37), we see that the isomorphism

$$L^2(H/\exp \mathfrak{m}_0, \nu_{\alpha_0}) \ni \phi \longmapsto \psi = \tilde{\phi} \in L^2(H/H_{p_0}, \mathcal{H}_{\pi_0})$$

defined in (3.35) gives an intertwining operator from π to π'. □

Based on Theorem 3.10, we describe the Duflo-Moore operator C_π for the representation π by using Duflo-Moore operator C_{π_0} for the representation π_0. Now take a section $L = \{l_p \; ; \; p \in \mathcal{O}_{p_0}\} \subset H$ of H/H_{p_0} such that $L \ni l_p \mapsto p = l_p.p_0 \in \mathcal{O}_{p_0}$ is bijective.

Lemma 3.11. *Define a linear form*

$$\mu : \mathcal{E}(H/H_{p_0}) \to \mathbb{R}$$

by

$$\mu(f) = \int_{\mathcal{O}_{p_0}} f(l_p)|\det l_p|\, dp \quad (f \in \mathcal{E}(H/H_{p_0})). \tag{3.38}$$

Then

1. *μ does not depend on the choice of L.*
2. *μ is left invariant.*
3. *There exists $C_0 > 0$ such that*

$$\oint_{H/H_{p_0}} f(l)\, d\mu_0(\dot{l}) = C_0 \mu(f). \tag{3.39}$$

Proof. 1. Let $L' := \{l'_p\}$ be another section of $\mathcal{O}_{p_0}$. We observe that for each $p \in \mathcal{O}_{p_0}$, we can find a unique $a_p \in H_{p_0}$ such that $l'_p = l_p a_p$. Moreover, for $f \in \mathcal{E}(H/H_{p_0})$ we have $f(la) = \Delta_{H,H_{p_0}}(a)^{-1} f(l)$, so that

$$\begin{aligned}\int_{\mathcal{O}_{p_0}} f(l'_p)|\det l'_p|\, dp &= \int_{\mathcal{O}_{p_0}} f(l_p a_p)|\det l_p a_p|\, dp \\ &= \int_{\mathcal{O}_{p_0}} f(l_p)\Delta_{H,H_{p_0}}(a_p)^{-1}\, |\det l_p|\, |\det a_p|\, dp.\end{aligned}$$

By Lemma 3.8, we have $\det a_p = \Delta_{H,H_{p_0}}(a_p)$. Therefore, the last term equals

$$\int_{\mathcal{O}_{p_0}} f(l_p)|\det l_p|\, dp$$

as required.

2. Let $l'_p = a^{-1} \cdot l_{a\cdot p}$ which is a section of $\mathcal{O}_{p_0}$. We have

$$\begin{aligned}\mu(L(a)f) &= \int_{\mathcal{O}_{p_0}} f(a^{-1}l_p)\,|\det l_p|\,dp \\ &= \int_{\mathcal{O}_{p_0}} f(a^{-1}l_{a\cdot p'})\,|\det l_{a\cdot p'}|\,|\det a|^{-1}\,dp' \quad (p = a\cdot p') \\ &= \int_{\mathcal{O}_{p_0}} f(l'_{p'})\,|\det l'_{p'}|\,dp'.\end{aligned}$$

By the first statement, the last term equals $\mu(f)$.
3. Since μ is left invariant, the statement follows from a uniqueness of the linear functional μ_0 in Proposition 1.11. □

We define $D_L(p) := C_0|\det l_p|$. Then by Lemma 3.11(3), we have

$$\oint_{H/H_{p_0}} f(l)\,d\mu_0(\dot{l}) = \int_{\mathcal{O}_{p_0}} f(l_p)\,D_L(p)\,dp \quad (f \in \mathcal{E}(H/H_{p_0})). \tag{3.40}$$

Now for $\phi_1, \phi_2 \in \mathcal{H}_\pi$, we shall evaluate the integral

$$\int_G |(\phi_1|\pi(g)\phi_2)_{\mathcal{H}_\pi}|^2\,dg = \int_H\int_V |(\phi_1|\pi(g(v))\pi(g(h))\phi_2)_{\mathcal{H}_\pi}|^2\,|\det h|^{-1}dh\,dv. \tag{3.41}$$

We have for $g = g(v)g(h) \in G$

$$\begin{aligned}(\phi_1|\pi(g)\phi_2)_{\mathcal{H}_\pi} &= \int_{H/H_{p_0}} (\tilde{\phi}_1(l)|\widetilde{\pi(g)\phi_2}(l))_{\mathcal{H}_{\pi_0}}\,d\mu(\dot{l}) \\ &= \int_{\mathcal{O}_{p_0}} (\tilde{\phi}_1(l_p)|\widetilde{\pi(g)\phi_2}(l_p))_{\mathcal{H}_{\pi_0}}\,D_L(p)\,dp \\ &= \int_{\mathcal{O}_{p_0}} e^{-2\pi i\langle p,v\rangle}(\tilde{\phi}_1(l_p)|\widetilde{\pi(g(h))\phi_2}(l_p))_{\mathcal{H}_{\pi_0}}\,D_L(p)\,dp\end{aligned}$$

by (3.36), (3.40), and (3.34). Furthermore, using the Plancherel formula, we have

$$\int_V |(\phi_1|\pi(g(v))\pi(g(h))\phi_2)_{\mathcal{H}_\pi}|^2\,dv = \int_{\mathcal{O}_{p_0}} |(\tilde{\phi}_1(l_p)|\widetilde{\pi(g(h))\phi_2}(l_p))_{\mathcal{H}_{\pi_0}}|^2\,D_L(p)^2\,dp. \tag{3.42}$$

Next task is to consider for each $p \in \mathcal{O}_{p_0}$ the following integral

$$\begin{aligned}
&\int_H |(\tilde{\phi}_1(l_p)|\pi(\widetilde{g(h)})\phi_2(l_p))_{\mathcal{H}_{\pi_0}}|^2 \,|\det h|^{-1}\, dh \\
&\quad = \int_H |(\tilde{\phi}_1(l_p)|\pi(\widetilde{g(h^{-1})})\phi_2(l_p))_{\mathcal{H}_{\pi_0}}|^2 \,|\det h|\Delta_H(h)^{-1}\, dh \quad (h \to h^{-1}) \\
&\quad = \int_H |(\tilde{\phi}_1(l_p)|\pi(\widetilde{g(h^{-1})})\phi_2(l_p))_{\mathcal{H}_{\pi_0}}|^2 \,\Delta_G(h)^{-1}\, dh. \qquad (3.43)
\end{aligned}$$

By Proposition 1.11, we have

$$\int_H f(h)\, dh = \oint_{H/H_{p_0}} \{\int_{H_{p_0}} f(lh_0)\Delta_{H,H_{p_0}}(h_0)\, dh_0\}\, d\mu_0(\dot{l}) \quad (f \in C_c(H)).$$

Indeed, the formula is valid for a continuous function f on H if the integrals in both sides converges. Thus, for $a \in H$ we have

$$\begin{aligned}
\Delta_H(a)\int_H f(h)\, dh &= \int_H f(aha^{-1})\, dh \\
&= \oint_{H/H_{p_0}} \{\int_{H_{p_0}} f(alh_0a^{-1})\Delta_{H,H_{p_0}}(h_0)\, dh_0\}\, d\mu_0(\dot{l}) \\
&= \oint_{H/H_{p_0}} \{\int_{H_{p_0}} f(ala^{-1}.ah_0a^{-1})\Delta_{H,H_{p_0}}(h_0)\, dh_0\}\, d\mu_0(\dot{l}).
\end{aligned}$$

Using the identification

$$H/H_{p_0} \ni \dot{l} \mapsto \dot{l}' \in H/aH_{p_0}a^{-1} = H/H_{a.p_0} \qquad (l' := ala^{-1}),$$

we define a linear functional μ_a on $\mathcal{E}(H/H_{a.p_0})$ in such a way that

$$\int_{H/H_{a.p_0}} \phi(l')\, d\mu_a(\dot{l}') = \int_{H/H_{p_0}} \phi(ala^{-1})\, d\mu_0(\dot{l}). \qquad (3.44)$$

Based on the observation above, we obtain

Lemma 3.12. *For each $a \in H$, we have*

$$\int_H f(h)\, dh = \frac{1}{\Delta_H(a)} \oint_{H/H_{a.p_0}} \{\int_{H_{p_0}} f(l'ah_0a^{-1})\Delta_{H,H_{p_0}}(h_0)\, dh_0\}\, d\mu_a(\dot{l}'). \qquad (3.45)$$

Substituting $a = l_p$ and

$$f(h) = |(\tilde{\phi}_1(l_p)|\pi(\widetilde{g(h^{-1})})\phi_2(l_p))_{\mathcal{H}_{\pi_0}}|^2\,\Delta_G(h)^{-1}$$

to Lemma 3.12, we see that (3.43) equals

$$\begin{aligned}
&\frac{1}{\Delta_H(l_p)}\oint_{H/H_p}\Big\{\int_{H_{p_0}}|(\tilde{\phi}_1(l_p)|[\pi(g(l_ph_0^{-1}l_p^{-1}))\pi(g(l'))^{-1}\phi_2]^{\sim}(l_p))_{\mathcal{H}_{\pi_0}}|^2\\
&\qquad\qquad\times\Delta_G(l'l_ph_0l_p^{-1})^{-1}\Delta_{H,H_{p_0}}(h_0)\,dh_0\Big\}\,d\mu_{l_p}(\dot{l}')\\
&=\frac{1}{\Delta_H(l_p)}\oint_{H/H_p}\Big\{\int_{H_{p_0}}|(\tilde{\phi}_1(l_p)|[\pi(g(l_ph_0^{-1}l_p^{-1}))\pi(g(l'))^{-1}\phi_2]^{\sim}(l_p))_{\mathcal{H}_{\pi_0}}|^2\\
&\qquad\qquad\times\Delta_G(l')^{-1}\Delta_G(h_0)^{-1}\Delta_{H,H_{p_0}}(h_0)\,dh_0\Big\}\,d\mu_{l_p}(\dot{l}').
\end{aligned}\tag{3.46}$$

Lemma 3.13. *For each $\phi \in \mathcal{H}_\pi, h_0 \in H_{p_0}$ and $a \in H$ we have*

$$[\pi(g(ah_0a^{-1}))\phi]^{\sim}(a) = \Delta_{H,H_{p_0}}(h_0)^{1/2}\pi_0(h_0)[\tilde{\phi}(a)] \tag{3.47}$$

as element of $\mathcal{H}_{\pi_0}$.

Proof. For $h' \in H_{p_0}$, we observe

$$\begin{aligned}
[\pi(g(ah_0a^{-1}))\phi]^{\sim}(a)(h') &= \pi(g(ah_0a^{-1}))\phi(ah')\Delta_{H,H_{p_0}}(h')^{1/2}\\
&= \phi(ah_0^{-1}a^{-1}ah')\Delta_{H,H_{p_0}}(h')^{1/2}\\
&= \phi(ah_0^{-1}h')\Delta_{H,H_{p_0}}(h_0^{-1}h')^{1/2}\Delta_{H,H_{p_0}}(h_0)^{1/2}.
\end{aligned}$$

Furthermore, the last term equals

$$\Delta_{H,H_{p_0}}(h_0)^{1/2}\tilde{\phi}(a)(h_0^{-1}h') = \Delta_{H,H_{p_0}}(h_0)^{1/2}\pi_0(h_0)[\tilde{\phi}(a)](h'),$$

which completes the proof. □

Substituting $\phi = \pi(g(l'))^{-1}\phi_2$ and $a = l_p$ to Lemma 3.13, we have

$$\begin{aligned}
&|(\tilde{\phi}_1(l_p)|[\pi(g(l_ph_0^{-1}l_p^{-1}))\pi(g(l'))^{-1}\phi_2)]^{\sim}(l_p))_{\mathcal{H}_{\pi_0}}|^2\\
&\qquad= |(\tilde{\phi}_1(l_p)|\pi_0(h_0^{-1})[\pi(\widetilde{g(l'))^{-1}}\phi_2(l_p)])_{\mathcal{H}_{\pi_0}}|^2\Delta_{H,H_{p_0}}(h_0)^{-1},
\end{aligned}$$

so that (3.46) becomes

$$\frac{1}{\Delta_H(l_p)}\oint_{H/H_p}\{\Delta_G(l')^{-1}\int_{H_{p_0}}|(\tilde{\phi}_1(l_p)|\pi_0(h_0^{-1})[\widetilde{\pi(g(l'))^{-1}}\phi_2(l_p)])_{\mathcal{H}_{\pi_0}}|^2\Delta_G(h_0)^{-1}\,dh_0\}\;d\mu_{l_p}(\dot{l}'). \tag{3.48}$$

By Lemma 3.8, (3.48) equals

$$\begin{aligned}&\frac{1}{\Delta_H(l_p)}\oint_{H/H_p}\{\Delta_G(l')^{-1}\int_{H_{p_0}}|(\tilde{\phi}_1(l_p)|\pi_0(h_0^{-1})[\widetilde{\pi(g(l'))^{-1}}\phi_2(l_p)])_{\mathcal{H}_{\pi_0}}|^2\Delta_{H_{p_0}}(h_0)^{-1}\,dh_0\}\;d\mu_{l_p}(\dot{l}')\\ &\quad(h_0^{-1}\mapsto h_0)\\ &\quad=\frac{1}{\Delta_H(l_p)}\oint_{H/H_{p_0}}\{\Delta_G(l')^{-1}\int_{H_{p_0}}|(\tilde{\phi}_1(l_p)|\pi_0(h_0)[\widetilde{\pi(g(l'))^{-1}}\phi_2(l_p)])_{\mathcal{H}_{\pi_0}}|^2\,dh_0\}\;d\mu_{l_p}(\dot{l}').\end{aligned} \tag{3.49}$$

Now by square-integrability of π_0, we have

$$\int_{H_{p_0}}|(\tilde{\phi}_1(l_p)|\pi_0(h_0)[\widetilde{\pi(g(l'))^{-1}}\phi_2(l_p)])_{\mathcal{H}_{\pi_0}}|^2\,dh_0=||\tilde{\phi}_1(l_p)||^2||C_{\pi_0}[\widetilde{\pi(g(l'))^{-1}}\phi_2(l_p)]||^2_{\mathcal{H}_{\pi_0}}.$$

Thus, (3.49) is equal to

$$\frac{1}{\Delta_H(l_p)}\oint_{H/H_p}\Delta_G(l')^{-1}\,||\tilde{\phi}_1(l_p)||^2||C_{\pi_0}[\widetilde{\pi(g(l'))^{-1}}\phi_2(l_p)]||^2_{\mathcal{H}_{\pi_0}}\;d\mu_{l_p}(\dot{l}'). \tag{3.50}$$

We note that

$$\begin{aligned}\widetilde{\pi(g(l'))^{-1}}\phi_2(l_p)(h')&=\pi(g(l'))^{-1}\phi_2(l_ph')\Delta_{H,H_{p_0}}(h')^{1/2}\\ &=\phi_2(l'l_ph')\Delta_{H,H_{p_0}}(h')^{1/2}\\ &=\tilde{\phi}_2(l'l_p)(h').\end{aligned}$$

Therefore, (3.50) is equal to

$$
\begin{aligned}
&\frac{1}{\Delta_H(l_p)}\oint_{H/H_p} \Delta_G(l')^{-1}\,||\tilde{\phi}_1(l_p)||^2_{\mathcal{H}_{\pi_0}}||C_{\pi_0}\tilde{\phi}_2(l'l_p)||^2_{\mathcal{H}_{\pi_0}}\;d\mu_{l_p}(\dot{l}')\\
&\qquad=\frac{1}{\Delta_H(l_p)}\oint_{H/H_{p_0}} \Delta_G(l_p l l_p^{-1})^{-1}\,||\tilde{\phi}_1(l_p)||^2||C_{\pi_0}\widetilde{\phi}_2(l_p l l_p^{-1}l_p)||^2_{\mathcal{H}_{\pi_0}}\;d\mu_0(\dot{l})\\
&\qquad=||\tilde{\phi}_1(l_p)||^2_{\mathcal{H}_{\pi_0}}\cdot\frac{1}{\Delta_H(l_p)}\oint_{H/H_{p_0}} \Delta_G(l)^{-1}||C_{\pi_0}\widetilde{\phi}_2(l_p l)||^2_{\mathcal{H}_{\pi_0}}\;d\mu_0(\dot{l})\\
&\qquad=||\tilde{\phi}_1(l_p)||^2_{\mathcal{H}_{\pi_0}}\cdot\frac{1}{\Delta_H(l_p)}\oint_{H/H_{p_0}} \Delta_G(l_p^{-1}l')^{-1}||C_{\pi_0}\widetilde{\phi}_2(l')||^2_{\mathcal{H}_{\pi_0}}\;d\mu_0(\dot{l}')\quad(l'=l_p l)\\
&\qquad=||\tilde{\phi}_1(l_p)||^2_{\mathcal{H}_{\pi_0}}\cdot\frac{\Delta_G(l_p)}{\Delta_H(l_p)}\oint_{H/H_{p_0}} ||\Delta_G(l)^{-1/2}\cdot C_{\pi_0}\widetilde{\phi}_2(l')||^2_{\mathcal{H}_{\pi_0}}\;d\mu_0(\dot{l}'),
\end{aligned}
\tag{3.51}
$$

where we use (3.44) for the first equality. Therefore, using (3.42) and (3.43), the formula (3.41) equals

$$
\begin{aligned}
\int_G |(\phi_1|\pi(g)\phi_2)_{\mathcal{H}_\pi}|^2\,dg = &\int_{\mathcal{O}_{p_0}} ||\tilde{\phi}_1(l_p)||^2_{\mathcal{H}_{\pi_0}}\frac{\Delta_G(l_p)}{\Delta_H(l_p)}\,D_L(p)^2\,dp\\
&\times\oint_{H/H_{p_0}} ||\Delta_G(l')^{-1/2}\cdot C_{\pi_0}\tilde{\phi}_2(l')||^2_{\mathcal{H}_{\pi_0}}\,d\mu_0(\dot{l}').
\end{aligned}
\tag{3.52}
$$

Using Lemma 3.11, the formula (3.52) equals

$$
\begin{aligned}
\int_G |(\phi|\pi(g)\phi_2)|^2\,dg = &\int_{\mathcal{O}_{p_0}} ||\tilde{\phi}_1(l_p)||^2_{\mathcal{H}_{\pi_0}}\cdot C_0|\det l_p|\,dp\\
&\times\oint_{H/H_{p_0}} ||C_0^{1/2}\Delta_G(l')^{-1/2}\cdot C_{\pi_0}\tilde{\phi}_2(l')||^2_{\mathcal{H}_{\pi_0}}\,d\mu_0(\dot{l}')\\
&= ||\phi_1||^2_{\mathcal{H}_\pi}\cdot\oint_{H/H_{p_0}} ||C_0^{1/2}\Delta_G(l')^{-1/2}\cdot C_{\pi_0}\tilde{\phi}_2(l')||^2_{\mathcal{H}_{\pi_0}}\,d\mu_0(\dot{l}').
\end{aligned}
\tag{3.53}
$$

Therefore, by (3.53) we obtain

Theorem 3.14. *The the Duflo-Moore operator of* $(\pi,\mathcal{H}_\pi)$ *can be described as*

$$
\widetilde{C_\pi\phi}(l) = C_0^{1/2}\Delta_G^{-1/2}(l)C_{\pi_0}\tilde{\phi}(l)\quad(\text{ a.a. } l\in H,\ \tilde{\phi}(l)\in\mathcal{H}_{\pi_0}\,). \tag{3.54}
$$

Chapter 4

Application for similitude and affine Lie group cases

The goal of this Chapter is to apply previous results to the Lie algebras of similitude Lie group $\mathrm{Sim}(n) := \mathbb{R}^n \rtimes (\mathbb{R}_+ \times \mathrm{SO}(n))$ and the connected affine Lie group $\mathrm{Aff}^+(n) := \mathbb{R}^n \rtimes \mathrm{GL}_n^+(\mathbb{R})$. We shall describe the condition for the coadjoint orbits of $\mathrm{Sim}(n)$ and $\mathrm{Aff}^+(n)$ to be open in each dual space respectively. Particularly, when a representation of $\mathrm{Aff}^+(n)$ is square-integrable, we compute its Duflo-Moore operator. We shall also describe the Pfaffian of $\mathfrak{aff}(n) = \mathrm{Lie}(\mathrm{Aff}^+(n))$.

4.1 The Lie algebra $\mathbb{R}^n \rtimes (\mathbb{R} \oplus \mathfrak{so}(n))$

As an application of Theorem 3.3 and Corollary 3.4, we see that

Theorem 4.1. *The Lie algebra $\mathfrak{g} := \mathbb{R}^n \rtimes (\mathbb{R} \oplus \mathfrak{so}(n))$ of the similitude Lie group $\mathrm{Sim}(n)$ is not a Frobenius Lie algebra for $n \geq 3$.*

Proof. Let H be the group $\mathbb{R}_+ \times \mathrm{SO}(n)$ acting on $\mathbb{R}^n$ by

$$h \cdot x = rAx \qquad (x \in \mathbb{R}^n,\ h := (r, A) \in H).$$

Then H acts on $(\mathbb{R}^n)^*$ identified with the space of row vectors by

$$h \cdot p = r^{-1}pA^{-1} \qquad (p \in (\mathbb{R}^n)^*,\ h := (r, A) \in H).$$

Let $\xi_0(p_0, \alpha_0) \in \mathfrak{g}^* \cong (\mathbb{R}^n)^* \oplus (\mathbb{R}^* \oplus \mathfrak{so}(n)^*)$ and we choose $p_0 = (0, 0, \ldots, 1) \in (\mathbb{R}^n)^*$. Then

$$
\begin{aligned}
H_{p_0} &= \{h \in H \ ; \ h \cdot p_0 = p_0\} \\
&= \{(r, A) \in H \ ; \ r^{-1} p_0 A^{-1} = p_0\} \\
&\cong \left\{ (1, \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}) \ ; \ M \in \mathrm{SO}(n-1) \right\} \cong \mathrm{SO}(n-1).
\end{aligned} \tag{4.1}
$$

Now we consider the orbits of H at p_0 as follows.

$$
\begin{aligned}
H \cdot p_0 &= \{h \cdot p_0 \ ; \ h \in H\} \\
&= \{r^{-1} p_0 A^{-1} \ ; \ A \in \mathrm{SO}(n), \ r > 0\} \\
&= \{r p_0 A \ ; \ A \in \mathrm{SO}(n), \ r > 0\}.
\end{aligned}
$$

This set is equal to $(\mathbb{R}^n)^* \setminus \{0\}$. To see this, let $p \in (\mathbb{R}^n)^* \setminus \{0\}$. Put $r := ||p|| > 0$ and $q := \frac{1}{r} p \in (\mathbb{R}^n)^* \setminus \{0\}$. Then $||q|| = 1$ so that we can take $A \in \mathrm{SO}(n)$ for which $q = p_0 A$. Thus, $p = rq = r p_0 A \in H \cdot p_0$ as required. Moreover, we note for $p_1 = h_1 \cdot p_0, \ (h_1 \in H)$, we have $H_{p_1} = h_1 H_{p_0} h_1^{-1}$. Therefore, we get the stabilizer of H at p as

$$
H_p \simeq \begin{cases} H = \mathbb{R}_+ \times \mathrm{SO}(n) & (p = 0), \\ \mathrm{SO}(n-1) & (p \in H \cdot p_0). \end{cases} \tag{4.2}
$$

We can see from Theorem 3.3 and Corollary 3.4 that

1. If $n = 2$, then $\mathfrak{so}(1) = \{0\}$ and H_{p_0} is trivial, so that $\mathfrak{g} := \mathbb{R}^2 \rtimes (\mathbb{R}_+ \times \mathfrak{so}(2))$ is a Frobenius Lie algebra as proved in [12], but

2. If $n \geq 3$, then $\mathfrak{so}(n-1) \neq \{0\}$ which is unimodular. Thus, $\mathfrak{so}(n-1)$ nor $\mathbb{R}_+ \oplus \mathfrak{so}(n-1)$ is not a Frobenius Lie algebra. Therefore, $\mathfrak{g} := \mathbb{R}^n \rtimes (\mathbb{R}_+ \times \mathfrak{so}(n))$ is not a Frobenius Lie algebra either.

□

Remark 5. Although the Lie algebra of $\mathrm{Sim}(n)$ $(n \geq 3)$ is not Frobenius, $\mathrm{Sim}(n)$ has square-integrable representations as is found in [1, p.308] and [20].

Let us consider the similitude group $\mathrm{Sim}(n)$, particularly for the case $n = 2$, that is, $\mathrm{Sim}(2) := V \rtimes H$ where $V \cong \mathbb{R}^2$ and $H \cong \mathbb{R}_+ \times \mathrm{SO}(2)$. The multiplication in Sim(2) is given as follow.

$$g(v,r,A)g(v',r',A') = g(v+rAv',rr',AA') \quad (v,v' \in \mathbb{R}^2,\ r,r' \in \mathbb{R}_+,\ A,A' \in \mathrm{SO}(2)).$$

Recalling that the Lie group $G_{III}(1)$ corresponds to $\mathfrak{g}_{III}(1)$ and noting that

$$\mathrm{ad}(X_4)(X_2\ X_1) = (X_2\ X_1)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, we have an isomorphism

$$G_{III}(1) \ni e^{aX_1}e^{bX_2}e^{cX_3}e^{qX_4} \longmapsto (\begin{pmatrix} b \\ a \end{pmatrix}, (e^c, \begin{pmatrix} \cos q & -\sin q \\ \sin q & \cos q \end{pmatrix})) \in \mathrm{Sim}(2).$$

We recall the representation $\pi_{\Omega_{X_1^*}}$ of $G_{III}(1)$ on $L^2(\mathbb{R}_+ \times [0,2\pi))$ as in (2.70) as follows.

$$\begin{aligned}
(\pi_{\Omega_{X_1^*}}(\exp aX_1)f)(x,y) &= e^{2\pi i a e^x \cos y} f(x,y), \\
(\pi_{\Omega_{X_1^*}}(\exp bX_2)f)(x,y) &= e^{2\pi i b e^x \sin y} f(x,y), \\
(\pi_{\Omega_{X_1^*}}(\exp cX_3)f)(x,y) &= f(x+c,y), \\
(\pi_{\Omega_{X_1^*}}(\exp qX_4)f)(x,y) &= f(x,y+q) \qquad (f \in L^2(\mathbb{R}_+ \times [0,2\pi))).
\end{aligned}$$

On the other hand, we obtain the representation π of Sim(2) on $L^2(H,dh)$ from (3.13) as follows.

$$\begin{aligned}
(\pi(\exp aX_1)F)(h) &= e^{2\pi i a e^x \cos y} F(h), \\
(\pi(\exp bX_2)F)(h) &= e^{2\pi i b e^x \sin y} F(h) \quad (h \in H,\ \ F \in L^2(H,dh)). \qquad (4.3)
\end{aligned}$$

Moreover, from (3.15) we also obtain the representation π of Sim(2) on $L^2(H,dh)$

as below

$$
\begin{aligned}
(\pi(\exp cX_3)F)(e^{-x}, \begin{pmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{pmatrix}) &= F(e^{-x-c}, \begin{pmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{pmatrix}), \\
(\pi(\exp qX_4)F)(e^{-x}, \begin{pmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{pmatrix}) &= F(e^{-c}, \begin{pmatrix} \cos q & \sin q \\ -\sin q & \cos q \end{pmatrix} \begin{pmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{pmatrix}) \\
&= F(e^{-c}, \begin{pmatrix} \cos(y+q) & \sin(y+q) \\ -\sin(y+q) & \cos(y+q) \end{pmatrix}),
\end{aligned} \tag{4.4}
$$

where $F \in L^2(H, dh)$. In this case, $f(x, y) = F(e^{-x}, \begin{pmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{pmatrix})$. Furthermore, we shall prove that

$$
\phi : L^2(H, dh) \to L^2(\mathbb{R} \times [0, 2\pi)) \tag{4.5}
$$

is an intertwining operator, namely, $\phi \circ \pi(g)F = \pi_{\Omega_{X_1^*}}(g) \circ \phi(F)$. To do so, let us observe that

- for $g = e^{aX_1}e^{bX_2}$

$$
\begin{aligned}
&\phi \circ \pi(e^{aX_1}e^{bX_2})F(x, y) \\
&\quad = \exp\{2\pi i \left\langle p_0,\ e^x \begin{pmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} \right\rangle\} F(e^{-x}, \begin{pmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{pmatrix}) \\
&\quad = \exp\{2\pi i \left\langle p_0,\ e^x \begin{pmatrix} b\cos y - a\sin y \\ b\sin y + a\cos y \end{pmatrix} \right\rangle\} F(e^{-x}, \begin{pmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{pmatrix}) \\
&\quad = \exp\{2\pi i e^x (a\cos y + b\sin y)\} F(e^{-x}, \begin{pmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{pmatrix}) \quad (\ p_0 = (0, 1)\ ).
\end{aligned}
$$

- for $g = e^{cX_3}e^{qX_4}$

$$
\begin{aligned}
\phi \circ \pi(e^{cX_3}e^{qX_4})F(x, y) &= F(e^{-x-c}, \begin{pmatrix} \cos q & \sin q \\ -\sin q & \cos q \end{pmatrix} \begin{pmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{pmatrix}) \\
&= F(e^{-x-c}, \begin{pmatrix} \cos(y+q) & \sin(y+q) \\ -\sin(y+q) & \cos(y+q) \end{pmatrix}).
\end{aligned}
$$

Therefore, $\phi \circ \pi(g)F = \pi_{\Omega_{X_1^*}}(g) \circ \phi(F)$ as required.

Now we shall compute the Duflo-Moore operator C_π as follows. We know that $\Delta_{G_{III}(1)}(g) = r^{-2}$ if $g = g(v, r, A)$ and C_0 defined in (3.11) equals 1. To prove the latter statement, that is $C_0 = 1$, let us consider $\psi \in C_c(H), p_0 = (0, 1), p = (x, y) \in (\mathbb{R}^2)^*$ and $h_p = rA = r \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in H$ for which $h_p \cdot p_0 = p$. Solving the equation, we get $p = (x, y) = (r^{-1}\sin\theta, r^{-1}\cos\theta)$. Thus, using (3.11) we obtain C_0 as follows :

$$C_0 = \frac{\int_H \psi(h)\, dh}{\int_{\mathcal{O}_{p_0}} \psi(h_p)\, |\det h_p|\, dp} = \frac{\int_H \psi(h)\, dh}{\int_{\mathbb{R}^2\setminus\{(0,0)\}} \psi(h_p)\, |\det h_p|\, dp},$$

but since

$$\begin{aligned} \int_{\mathbb{R}^2\setminus\{(0,0)\}} \psi(h_p)\, |\det h_p|\, dp &= \int_{\mathbb{R}^2\setminus\{(0,0)\}} \psi(rA)\, r^2\, \frac{dr d\theta}{r^3} \\ &= \int_{\mathbb{R}^2\setminus\{(0,0)\}} \psi(rA)\, \frac{dr d\theta}{r} \\ &= \int_H \psi(h)\, dh, \end{aligned}$$

then we obtain $C_0 = 1$. Therefore, due to the general result for $V \rtimes H$ in Chapter III in Theorem 3.7, for trivial stabilizer of H, we get

Proposition 4.2. *The Duflo-Moore operator for the representation* $(\pi, L^2(H, dh)$ *of* $\mathrm{Sim}(2)$ *given by* (4.3) *and* (4.4) *is described by*

$$C_\pi F(h) = rF(h) \qquad (h = h(r, A) \in H,\ F \in L^2(H, dh)). \tag{4.6}$$

4.2 Duflo-Moore Operator for $\mathrm{Aff}^+(1)$

In this sub-section we recall the Duflo-Moore operator for 2-dimensional affine Lie group. Although it is well known as in [21] and [28], we shall give the detailed computation in order to compare the general result in Theorem 3.7 for the Duflo-Moore operator of representation π of $V \rtimes H$ in $\mathfrak{h}_{p_0} = \{0\}$ case. Note that in [21] and [28] the Duflo-Moore operator for the representation $\mathrm{Aff}^+(1)$ is given by using

Fourier transform whereas our formula is more direct. Let us denote

$$G=\left\{g(x,a)=\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\ ;\ a>0, x\in\mathbb{R}\right\}.$$

Then we have $G = V \rtimes H$, where $V = \left\{v(x)=\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\ ;\ x\in\mathbb{R}\right\} \simeq \mathbb{R}$ and $H = \left\{h(a)=\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\ ;\ a>0\right\} \simeq \mathbb{R}_+$ and $g(x,a) = v(x)h(a)$. Moreover, the Lie algebra of G is of the form

$$\mathfrak{g}=\langle X_1, X_2\rangle=\left\{\begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}\ ;\ x_1,x_2\in\mathbb{R}\right\}.$$

It is well known that the open coadjoint orbit for G is of the form $\Omega_\pm = \{xX_1^* + yX_2^*\ ;\ \pm y > 0\}$. Now let $f = \pm X_2^*$ be an element of $\Omega_\pm$ and let $\mathfrak{p} = \mathbb{R}X_2$ be a polarization in $\mathfrak{g}$ at $\pm X_2^*$ satisfying Pukanszky condition, namely, $\pm X_2^* + \mathfrak{p}^\perp \subset \Omega_\pm$. We also have $V = \exp\mathfrak{p}$. Furthermore, Let $\pi_\pm := \mathrm{Ind}_V^G \nu_{\pm X_2^*}$ be an irreducible unitary representations of G induced from the representation

$$\nu_{\pm X_2^*} : V \ni v(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto e^{\pm 2\pi i x} \in \mathbb{C}.$$

acts on the space

$$\mathcal{H}_\pm := \left\{\phi : G\to\mathbb{C}\ ;\ \begin{array}{l} \phi(gv) = \nu_{\pm X_2^*}(v)^{-1}\phi(g)\ (g\in G,\ v\in V) \\ \oint_{G/V} |\phi(g)|^2\, d\dot{g} < \infty \end{array}\right\},$$

where

$$\oint_{G/V} |\phi(g)|^2\, d\dot{g} = \int_H |\phi(h)|^2\, dh = \int_0^\infty \left|\phi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right|^2 \frac{da}{a}.$$

Particularly, by restrictions of ϕ to H with $\mathcal{H}_\pm \ni \phi \mapsto \phi|_H =: f \in L^2(H, dh)$, we obtain

$$\pi_{\pm}(h_0)f(h) = f(h_0^{-1}h),$$
$$\pi_{\pm}(v_0)f(h) = e^{\pm 2\pi i x_0 a^{-1}} f(h) \quad (\, h_0, h = h(a) \in H, v_0 = v(x_0) \in V\,), \tag{4.7}$$

where $f(h) = \phi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$.

Now we shall compute the integral

$$\int_0^{\infty} \int_{-\infty}^{\infty} |(f_1|\pi_{\pm}(v(x))\pi_{\pm}(h(a))f_2)_{L^2(H,dh)}|^2 \, dx \, \frac{da}{|a|^2}. \tag{4.8}$$

Let us consider

$$\begin{aligned}(f_1|\pi_{\pm}(v(x))\pi_{\pm}(h(a))f_2)_{L^2(H,dh)} &= \int_0^{\infty} f_1(h(b))\overline{\pi_{\pm}(v(x))\pi_{\pm}(h(a))f_2(h(b))} \, \frac{db}{b} \\ &= \int_0^{\infty} f_1(h(b))\overline{e^{\pm 2\pi i x b^{-1}}\pi_{\pm}(h(a))f_2(h(b))} \, \frac{db}{b} \\ &= \int_0^{\infty} e^{\mp 2\pi i x \xi} f_1(h(\xi^{-1}))\overline{\pi_{\pm}(h(a))f_2(h(\xi^{-1}))} \, \frac{d\xi}{\xi} \quad (\xi = b^{-1}).\end{aligned} \tag{4.9}$$

By Plancherel formula, we have

$$\begin{aligned}\int_{-\infty}^{\infty} |(f_1|\pi_{\pm}(v(x))\pi_{\pm}(h(a))f_2)_{L^2(H,dh)}|^2 \, dx &= \int_0^{\infty} |f_1(h(\xi^{-1}))\overline{\pi_{\pm}(h(a))f_2(h(\xi^{-1}))}|^2 \, \frac{d\xi}{\xi^2} \\ &= \int_0^{\infty} |f_1(h(\eta))\overline{\pi_{\pm}(h(a))f_2(h(\eta))}|^2 \, d\eta \quad (\eta = \xi^{-1}).\end{aligned} \tag{4.10}$$

Therefore, (4.8) equals

$$\begin{aligned}
&\int_0^\infty \int_{-\infty}^\infty |(f_1|\pi_\pm(v(x))\pi_\pm(h(a))f_2)_{L^2(H,dh)}|^2 \ dx \, \frac{da}{a^2} \\
&\qquad = \int_0^\infty |f_1(h(\eta))|^2 \left\{ \int_0^\infty \overline{|f_2(h(a^{-1})h(\eta))|^2} \, \frac{da}{a^2} \right\} d\eta \\
&\qquad = \int_0^\infty |f_1(h(\eta))|^2 \left\{ \int_0^\infty \overline{|f_2(h(a^{-1}\eta))|^2} \, \frac{da}{a^2} \right\} d\eta \\
&\qquad = \int_0^\infty |f_1(h(\eta))|^2 \left\{ \int_0^\infty |f_2(h(a'))|^2 \, \frac{da'}{\eta} \right\} d\eta \quad (a' = a^{-1}\eta) \\
&\qquad = \int_0^\infty |f_1(h(\eta))|^2 \, \frac{d\eta}{\eta} \left\{ \int_0^\infty |(a')^{1/2} f_2(h(a'))|^2 \, \frac{da'}{|a'|} \right\}.
\end{aligned} \tag{4.11}$$

Thus, we have

Proposition 4.3. *The Duflo-Moore operator $C_{\pi_\pm}$ for the representation $(\pi_\pm, L^2(H))$ of $\mathrm{Aff}^+(1)$ as in (4.7) can be written as*

$$C_{\pi_\pm} f(h) = a^{1/2} f(h) \qquad (f \in L^2(H, dh), \ \ h = h(a) \in H, \ \ a > 0). \tag{4.12}$$

Remark 6. It is well known that $\Delta_G(g(x,a)) = a^{-1}$, and applying Theorem 3.7, the Duflo-Moore operator of representation $\pi_\pm$ of $\mathrm{Aff}(1)$ is nothing but (4.12).

On the other hand, for $h := h(a) \in H$, we have

$$d\pi(X_2)f(h) = \pm 2\pi i a^{-1} f(h), \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}.$$

Therefore, the formula in (4.12) corresponding to Pfaffian equals

$$\begin{aligned}
C_{\pi_\pm} &= \sqrt{2\pi} \ |i d\pi(X_2)|^{-1/2}, \\
&= \sqrt{2\pi} \ |i d\pi(Q_{\mathfrak{g}})|^{-1/2},
\end{aligned} \tag{4.13}$$

where $Q_{\mathfrak{g}} := X_2$ is the Pfaffian of $\mathfrak{g}$. Thus, we obtain

Proposition 4.4. *The Duflo-Moore operator $C_{\pi_\pm}$ for the representation $(\pi_\pm, L^2(H))$*

of $\mathrm{Aff}^+(1)$ *corresponding to Pfaffian of* $\mathfrak{g} := \mathfrak{aff}(1)$ *is written as*

$$C_{\pi_\pm} = \sqrt{2\pi}\ |id\pi(Q_\mathfrak{g})|^{-1/2}, \tag{4.14}$$

where $Q_\mathfrak{g} := X_2$ *is the Pfaffian of* $\mathfrak{g}$.

4.3 $\mathfrak{aff}(n)$ as the Frobenius Lie algebra

In this sub-section, we shall prove that the real affine Lie algebra $\mathfrak{g} := \mathfrak{aff}(n)$ of $G = \mathrm{Aff}^+(n)$ is Frobenius. Although it is mentioned in ([48], p.497), we give a direct proof and give all open coadjoint orbits. We shall recall notations in Section 3.1 and rephrase some formulas in concrete form. Now, let $X(v, A), X(v', A')$ be elements of $\mathfrak{g}$ and $g(u, a), g(u', a')$ be elements of G. The Lie bracket of $\mathfrak{g}$ and the multiplication in G are given as follows.

$$\begin{aligned} [X(v, A), X(v', A')] &= X(Av' - A'v, [A, A']), \\ g(u, a)g(u', a') &= g(u + au', aa'). \end{aligned} \tag{4.15}$$

We write $g(u) := g(u, I)$ and $g(a) := g(0, a)$. We obtain adjoint actions of G on $\mathfrak{g}$ as follow

$$\begin{aligned} \mathrm{Ad}(g(a))X(v, A) &= g(a)X(v, A)g(a^{-1}) = X(av, aAa^{-1}), \\ \mathrm{Ad}(g(u))X(v, A) &= g(u)X(v, A)g(-u) = X(v - Au, A). \end{aligned} \tag{4.16}$$

Therefore, we have

$$\mathrm{Ad}(g(u, a))X(v, A) = X(av - aAa^{-1}u, aAa^{-1}). \tag{4.17}$$

We shall regard $p \in (\mathbb{R}^n)^*$ as a row vector and we identify $(\mathfrak{gl}_n(\mathbb{R}))^*$ with $\mathrm{Mat}_n(\mathbb{R})$ by

$$\langle X, \alpha \rangle = \mathrm{tr}\,(\alpha X) \qquad (\alpha \in \mathrm{Mat}_n(\mathbb{R}), X \in \mathfrak{gl}_n(\mathbb{R})).$$

Then the coadjoint actions of G and $\mathfrak{g}$ on $\mathfrak{g}^*$ are given respectively by

$$\mathrm{Ad}^*(g(u, a))\xi(p, \alpha) = \xi(pa^{-1}, a\alpha a^{-1} + upa^{-1}), \tag{4.18}$$

$$\mathrm{ad}^*(X(v, A))\xi(p, \alpha) = \xi(-pA, [A, \alpha] + vp), \tag{4.19}$$

where $g(u,a) \in G$, $X(v,A) \in \mathfrak{g}$, and $\xi(p,\alpha) \in \mathfrak{g}^*$. Let $\xi_0 := \xi(p_0,\alpha_0)$ be an element of $\mathfrak{g}^*$, and the coadjoint orbit $\mathrm{Ad}^*(G)\xi_0$ of G at point ξ_0 be denoted by Ω_{ξ_0}. By definition, the coadjoint orbit Ω_{ξ_0} is open in $\mathfrak{g}^*$ if the dimension of Ω_{ξ_0} is equal to the dimension of G. Using Lemma 3.1 we obtain that the coadjoint orbit Ω_{ξ_0} is open in $\mathfrak{g}^*$ if and only if the map $f : \mathfrak{g} \to \mathfrak{g}^*$ defined by

$$f(X(v,A)) = \mathrm{ad}^*(X(v,A))\xi_0 = \xi(-p_0A, [A,\alpha_0] + vp_0) \in \mathfrak{g}^* \tag{4.20}$$

is bijective. We obtain the proposition below.

Proposition 4.5. *For $\xi_0 = \xi(p_0,\alpha_0)$ with $p_0 := (1,1,\ldots,1)$ and $\alpha_0 := \mathrm{diag}\,\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$, $\alpha_i \neq \alpha_j$, the coadjoint orbit Ω_{ξ_0} is open.*

Proof. Using Lemma 3.1, we will show that the map (4.20) is bijective. We introduce sets $D := \{\mathrm{diag}\,\{d_1,d_2,\ldots,d_n\}\ ;\ d_i \in \mathbb{R}\}$ and $D^\perp := \{A \in \mathrm{Mat}_n(\mathbb{R})\ ;\ A_{ii} = 0,\ i = 1,2,\ldots,n\}$. Let us denote sets

$$\begin{aligned}\mathfrak{h}_1 &= \{X(v,0)\ ;\ v \in \mathbb{R}^n\},\\ \mathfrak{h}_2 &= \{X(0,B)\ ;\ B \in D\},\\ \mathfrak{h}_3 &= \{X(0,C)\ ;\ C \in D^\perp\},\end{aligned} \tag{4.21}$$

then the affine Lie algebra $\mathfrak{g}$ can be written as a direct sum of the form $\mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$. Furthermore, the map $f : \mathfrak{g} \to \mathfrak{g}^*$ defined in (4.20) can be described as

$$\begin{aligned} f_1 &: \mathbb{R}^n \ni v \longmapsto vp_0 \in \mathrm{Mat}_n(\mathbb{R}),\\ f_2 &: D \ni B \longmapsto p_0B = (B_{11}, B_{22}, \ldots, B_{nn}) \in (\mathbb{R}^n)^*,\\ f_3 &: D^\perp \ni C \longmapsto [C,\alpha_0] \in \mathrm{Mat}_n(\mathbb{R}).\end{aligned} \tag{4.22}$$

Since $vp_0 = \begin{pmatrix} v_1 & v_1 & \ldots & v_1 \\ v_2 & v_2 & \ldots & v_2 \\ \vdots & \vdots & \ddots & \vdots \\ v_n & v_n & \ldots & v_n \end{pmatrix}$ and $[C,\alpha_0] \in D^\perp$, we can see that $\mathrm{Im}\, f_1 \cap \mathrm{Im}\, f_3 =$

$\{0\}$. Furthermore, we also have

$$\begin{aligned}\dim(\operatorname{Im} f_1 + \operatorname{Im} f_3) &= \dim \operatorname{Im} f_1 + \dim \operatorname{Im} f_3 \\ &= \dim \mathbb{R}^n + \dim D^{\perp} \\ &= n + (n^2 - n) = n^2 \\ &= \dim \operatorname{Mat}_n(\mathbb{R}).\end{aligned}$$

Therefore, $\operatorname{Mat}_n(\mathbb{R}) = \operatorname{Im} f_1 \oplus \operatorname{Im} f_3$. This means that for each $\alpha \in \operatorname{Mat}_n(\mathbb{R})$, there exist $v \in \mathbb{R}^n$ and $C \in D^{\perp}$ uniquely such that $\alpha = [C, \alpha_0] + vp_0$. Therefore, $f_1 + f_3 : \mathbb{R}^n \oplus D^{\perp} \to \operatorname{Mat}_n(\mathbb{R})$ is surjective, so that $f_1 + f_3$ is bijective. On the other hand, f_2 is bijective. Therefore, f is bijective. □

Thus we obtain

Theorem 4.6. *The affine Lie algebra $\mathfrak{aff}(n)$ is Frobenius.*

Let $\xi_1 := \xi(p_1, \alpha_1)$ be an element of $\mathfrak{g}^*$ with $p_1 \in (\mathbb{R}^n)^*$ and $\alpha_1 \in \mathfrak{gl}_n(\mathbb{R})^*$. We shall give other criteria for coadjoint orbit Ω_{ξ_1} to be open in $\mathfrak{g}^*$. First, let us assume that Ω_{ξ_1} is open. Then the map (4.20) is surjective, so that

$$\begin{aligned}\operatorname{Mat}_n(\mathbb{R}) &= \{[A, \alpha_1] + vp_1 \ ; \ A \in \operatorname{Mat}_n(\mathbb{R}), v \in \mathbb{R}^n\} \\ &= \operatorname{Im} \operatorname{ad} \alpha_1 + \{vp_1 \ ; \ v \in \mathbb{R}^n\}.\end{aligned} \tag{4.23}$$

We observe

$$\operatorname{Im} \operatorname{ad}(\alpha_1) \cong \operatorname{Mat}_n(\mathbb{R}) / \operatorname{Ker} \operatorname{ad}(\alpha_1) \cong \operatorname{Mat}_n(\mathbb{R}) / \operatorname{Cent}(\alpha_1), \tag{4.24}$$

where $\operatorname{Cent}(\alpha_1)$ denotes the centralizer of α_1 in $\operatorname{Mat}_n(\mathbb{R})$ so that $\dim \operatorname{Im} \operatorname{ad}(\alpha_1) = n^2 - \dim \operatorname{Cent}(\alpha_1)$. Let

$$\phi_1 : \mathbb{R}^n \ni v \mapsto vp_1 \in \operatorname{Mat}_n(\mathbb{R}),$$

then we have $\operatorname{Mat}_n(\mathbb{R}) = \operatorname{Im} \operatorname{ad}(\alpha_1) + \operatorname{Im}(\phi_1)$ by (4.23). Therefore,

$$n^2 \leqq n^2 - \dim \operatorname{Cent}(\alpha_1) + n - \dim \operatorname{Ker}(\phi_1), \tag{4.25}$$

that is

$$\dim \operatorname{Ker}(\phi_1) \leqq n - \dim \operatorname{Cent}(\alpha_1).$$

Since $\dim \mathrm{Cent}\,(\alpha_1) \geqq n$ in general, the possibility is only that $\dim \mathrm{Cent}\,(\alpha_1) = n$ and $\dim \mathrm{Ker}\,(\alpha_1) = 0$ and the equality (4.25) holds. In particular

$$\mathrm{Mat}_n(\mathbb{R}) = \mathrm{Im}\,\mathrm{ad}\,(\alpha_1) \oplus \mathrm{Im}\,(\phi_1). \tag{4.26}$$

Therefore, we proved the lemma below.

Lemma 4.7. *If* Ω_{ξ_1} *is open then* $\dim \mathrm{Cent}\,(\alpha_1) = n$.

Again from (4.26), we have $\mathrm{Im}\,\mathrm{ad}\,(\alpha_1) \cap \{vp_1 \;;\; v \in \mathbb{R}^n\} = \{0\}$. In other words, we have

Lemma 4.8. *If* $\mathrm{Im}\,\mathrm{ad}\,\alpha_1 \cap \{vp_1 \;;\; v \in \mathbb{R}^n\} \neq \{0\}$*, then* Ω_{ξ_1} *is not open in* $\mathfrak{g}^*$.

If the coadjoint orbit Ω_{ξ_1} is open in $\mathfrak{g}^*$, f is injective by Lemma 3.1, so that the map

$$F : \mathrm{Cent}\,(\alpha_1) \ni A \longmapsto -p_1 A \in (\mathbb{R}^n)^*$$

is injective. On the other hand, if $\dim \mathrm{Cent}\,(\alpha_1) = n$ and $\mathrm{Ker}\, F = \{0\}$, then F is surjective. Based on the explanation above, we obtain necessary and sufficient conditions for Ω_{ξ_1} to be open in $\mathfrak{g}^*$ as follows.

Proposition 4.9. *The orbit* Ω_{ξ_1} *is open in* $\mathfrak{g}^*$ *if and only if the following three conditions are satisfied*

1. $\dim \mathrm{Cent}\,(\alpha_1) = n$.
2. $\mathrm{Im}\,\mathrm{ad}\,(\alpha_1) \cap \{vp_1 \;;\; v \in \mathbb{R}^n\} = \{0\}$.
3. *If* $A \in \mathrm{Cent}\,(\alpha_1) \setminus \{0\}$*, then* $-p_1 A \neq 0$.

To bring it down to earth, we shall give some examples as follow. If we choose $\xi_1 = \xi(p_1, \alpha_1)$ where $p_1 = (1, 0)$ and $\alpha_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then the coadjoint orbit Ω_{ξ_1} is not open in $\mathfrak{aff}(2)^*$ since $\mathrm{Im}\,\mathrm{ad}\,\alpha_1 \cap \{vp_1 \;;\; v \in \mathbb{R}^n\} \neq \{0\}$. On the other hand, if we choose $\xi_0 = \xi(p_0, \alpha_0)$ where $p_0 = (0, 1)$ and $\alpha_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, then the coadjoint orbit Ω_{ξ_0} is open in $\mathfrak{aff}(2)^*$, and this implies that $\mathfrak{aff}(2)$ is Frobenius Lie algebra.

Now we shall prove the openness of some coadjoint orbits of $\mathrm{Aff}^+(n)$. For simplicity, let $G^n := \mathrm{Aff}^+(n) = \mathbb{R}^n \rtimes H^n$ with $H^n := \mathrm{GL}_n^+(\mathbb{R})$, and $\mathfrak{g}^n := \mathfrak{aff}(n) = \mathbb{R}^n \rtimes \mathfrak{h}^n$ with $\mathfrak{h}^n := \mathfrak{gl}_n(\mathbb{R})$. Let $\xi_1^\pm = (\pm 1, 0)$ be an element of $(\mathfrak{g}^1)^*$ and $\xi_n^\pm := \xi(p_n, \alpha_n^\pm)$ be an element of $(\mathfrak{g}^n)^*$ for $n \geqq 2$ with

$$p_n = (0, 0, \ldots, 1), \qquad \alpha_n^\pm = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 0 \\ \pm 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix}. \tag{4.27}$$

Although we can check the coadjoint orbit $\Omega_{\xi_n^\pm}$ of G^n through $\xi_n^\pm = \xi(p_n, \alpha_n^\pm)$ is open in $(\mathfrak{g}^n)^*$ by applying Proposition 4.9, we shall show the openness $\Omega_{\xi_n^\pm}$ in other way by induction.

For $n = 1$, $\Omega_{\xi_1^\pm} = \Omega_{\pm X_2^*}$ through $\xi_1^\pm = (\pm 1, 0) = \pm X_2^*$ is open as is shown in Section 4.2. Assume that $n \geqq 2$ and $\Omega_{\xi_{n-1}^\pm}$ is open in $(\mathfrak{g}^{n-1})^*$. We observe that the stabilizer $H_{p_n}^n$ is equal to

$$\left\{ \begin{pmatrix} M & v \\ 0 & 1 \end{pmatrix} \ ; \ M \in \mathrm{GL}_{n-1}^+(\mathbb{R}),\ v \in \mathbb{R}^{n-1} \right\} \simeq G^{n-1} = \mathrm{Aff}^+(n-1). \tag{4.28}$$

Let $\iota_n : \mathfrak{g}^{n-1} \hookrightarrow \mathfrak{h}_{p_n}^n$ be the corresponding Lie algebra isomorphism defined by

$$\iota_n(X(v, A)) = \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \ \ (A \in \mathfrak{gl}_{n-1}(\mathbb{R}),\ v \in \mathbb{R}^{n-1}), \tag{4.29}$$

then $\xi_{n-1}^\pm = \alpha_n^\pm \circ \iota_n$. Since $\mathrm{Ad}^*(G^{n-1})\xi_{n-1}^\pm$ is open in $(\mathfrak{g}^{n-1})^*$ by induction hypothesis, $\mathrm{Ad}^*(H_{p_n}^n)(\alpha_n^\pm|_{\mathfrak{h}_{p_n}^n})$ is open in $(\mathfrak{h}_{p_n}^n)^*$. On the other hand,

$$\begin{aligned} \varpi(\Omega_{\xi_n^\pm}) &= \{-p_n A \ ; \ A \in H^n\} \\ &= \{-(a_{n1}, a_{n2}, \ldots, a_{nn}) \ ; \ A = (a_{ij}) \in H^n\} \\ &= (\mathbb{R}^n)^* \setminus \{(0, 0, \ldots, 0)\} \end{aligned}$$

is open in $(\mathbb{R}^n)^*$. Therefore, $\Omega_{\xi_n^\pm}$ is open in $(\mathfrak{g}^n)^*$ by Theorem 3.3. We shall show later that $\Omega_{\xi_n^+} \neq \Omega_{\xi_n^-}$ so that we have exactly two open coadjoint orbits in $(\mathfrak{g}^n)^*$.

4.4 The Duflo-Moore operator for $\mathrm{Aff}^+(n)$

We have already proved that $\Omega_{\xi_n^\pm}$ through $\xi_n^\pm := \xi(p_n, \alpha_n^\pm)$ with p_n and α_n in (4.27) is open in $(\mathfrak{g}^n)^*$. We shall observe representations $\pi_{\Omega_{\xi_n^\pm}}$ of G^n corresponding to the open coadjoint orbit $\Omega_{\xi_n^\pm}$ as follows. We have already described the representation $\pi_{\Omega_{\xi_n^\pm}}$ of G^n for $n = 1$ case in Section 4.2. Let $\mathfrak{p}_n = \mathbb{R}^n \rtimes \mathfrak{m}_n$ be defined inductively by $\mathfrak{m}_n := \iota_n(\mathfrak{p}_{n-1})$ starting from $\mathfrak{p}_1 = \mathbb{R}X_2 \subset \mathfrak{g}^1 = \mathfrak{aff}(1)$ with $\iota_n : \mathfrak{g}^{n-1} \hookrightarrow \mathfrak{h}^n_{p_n}$ be the corresponding Lie algebra isomorphism as is defined in (4.29). In this case, $\mathfrak{m}_n$ is a polarization of $\mathfrak{h}^n_{p_n}$ satisfying Pukanszky condition and $\xi^\pm_{n-1} = \alpha_n^\pm \circ \iota_n$. As is already seen that the stabilizer $H^n_{p_n}$ is of the form (4.28). Assume that $\mathfrak{p}_{n-1}$ is a polarization of $\mathfrak{g}^{n-1}$ satisfying Pukanszky condition and representations $\pi_{\Omega_{\xi^\pm_{n-1}}} = \mathrm{Ind}^{G^{n-1}}_{\exp \mathfrak{p}_{n-1}} \nu_{\xi^\pm_{n-1}}$ are irreducible and square-integrable. We shall show that $\mathfrak{p}_n$ is a polarization of $\mathfrak{g}^n$ satisfying Pukanszky condition. Since $\mathfrak{p}_{n-1} = \{X(v, A) \ ; \ v \in R^{n-1}, \ A \in \mathfrak{gl}_{n-1}(\mathbb{R})\}$ with

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \end{pmatrix}, \text{ and } A = \begin{pmatrix} 0 & x_{12} & x_{13} & \dots & x_{1,n-2} & x_{1,n-1} \\ 0 & 0 & x_{23} & \dots & x_{2,n-2} & x_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & x_{n-2,n-1} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

is a polarization of $\mathfrak{g}^{n-1}$ at $\xi^\pm_{n-1}$ satisfying Pukanszky condition by induction hypothesis, $\mathfrak{m}_n$ defined inductively by

$$\mathfrak{m}_n := \iota_n(\mathfrak{p}_{n-1}) = \left\langle \begin{pmatrix} 0 & x_{12} & x_{13} & \dots & x_{1,n-1} & v_1 \\ 0 & 0 & x_{23} & \dots & x_{2,n-1} & v_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & v_{n-1} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \right\rangle$$

is also a polarization of $\mathfrak{h}^n_{p_n}$ at $\alpha_n|_{\mathfrak{h}^n_{p_n}}$ satisfying Pukanszky condition. Then by Theorem 3.9 (1), $\mathfrak{p}_n := \mathbb{R}^n \rtimes \mathfrak{m}_n \subset \mathfrak{g}^n$ is a polarization at $\xi_n^\pm$ satisfying Pukanszky condition. Furthermore, through the identification of $H^n_{p_n}$ and $G^{n-1} = \mathrm{Aff}^+(n -$

1), the representations $\pi_0 = \mathrm{Ind}_{\exp \mathfrak{m}_n}^{H^n_{p_n}} \nu_{\alpha_n^\pm}|_{\mathfrak{h}^n_{p_n}}$ are exactly $\pi_{\Omega_{\xi^\pm_{n-1}}} = \mathrm{Ind}_{\exp \mathfrak{p}_{n-1}}^{G^{n-1}} \nu_{\xi^\pm_{n-1}}$ which are irreducible and square-integrable by induction hypothesis. Therefore, by Theorem (3.9) (2) and (3), the representation $\pi_{\Omega_{\xi^\pm_n}} = \mathrm{Ind}_{\exp \mathfrak{p}_n}^{G^n} \nu_{\xi^\pm_n}$ of G^n is irreducible and square-integrable. Moreover, using Theorem 3.10 we have $\pi_{\Omega_{\xi^\pm_n}} = \mathrm{Ind}_{\mathbb{R}^n \rtimes G^{n-1}}^{G^n} (\nu_{p_n} \otimes \pi_{\Omega_{\xi^\pm_{n-1}}})$. Therefore, we have already proved the Theorem below.

Theorem 4.10. *Let $\xi^\pm_n = (p_n, \alpha^\pm_n)$ be an element of $(\mathfrak{g}^n)^*$ as in (4.27).*

1. *$\mathfrak{p}_n$ is a polarization of $\mathfrak{g}^n$ at $\xi^\pm_n$ satisfying Pukanszky condition.*
2. *$\pi_{\Omega_{\xi^\pm_n}} = \mathrm{Ind}_{\exp \mathfrak{p}_n}^{G^n} \nu_{\xi^\pm_n}$ is irreducible and square-integrable.*
3. *Under the identification $H^n_{p_n} \simeq G^{n-1} = \mathrm{Aff}^+(n-1)$, the representation $\pi_{\Omega_{\xi^\pm_n}}$ is isomorphic to $\mathrm{Ind}_{\mathbb{R}^n \rtimes G^{n-1}}^{G_n} (\nu_{p_n} \otimes \pi_{\Omega_{\xi^\pm_{n-1}}})$.*

In our discussion above, the coadjoint orbit $\Omega_{\xi^\pm_n}$ is open and indeed, it satisfies Theorem 3.3. Furthermore, we also have known that stabilizer $\mathfrak{h}^n_{p_n} \neq \{0\}$ for $n \geq 2$ and $\pi_{\Omega_{\xi^\pm_{n-1}}}$ is the square-integrable representation of $H^n_{p_n} \simeq G^{n-1} = \mathrm{Aff}^+(n-1)$, then using Theorem 3.14 we can describe the Duflo-Moore operator $C_{\pi_{\Omega_{\xi^\pm_n}}}$ for the representation $(\pi_{\Omega_{\xi^\pm_n}}, L^2(H^n/H^n_{p_n})$ of G^n as follows.

Proposition 4.11. *The Duflo-Moore operator of $(\pi_{\Omega_{\xi^\pm_n}}, L^2(H^n/H^n_{p_n})$ of G^n ($n \geq 2$) can be described as*

$$\widetilde{C_{\pi_{\Omega_{\xi^\pm_n}}}} \phi(a) = C_0^{1/2} |\det a|^{1/2} C_{\pi_{\Omega_{\xi^\pm_{n-1}}}} \tilde{\phi}(a) \tag{4.30}$$

for almost all $a \in H^n$.

4.5 Pfaffian of $\mathfrak{aff}(n)$

In the end of this sub-section, we shall observe a general formula for the Pfaffian of the $N := n(n+1)$-dimensional affine Lie algebra $\mathfrak{g}^n := \mathfrak{aff}(n) = \mathbb{R}^n \rtimes \mathfrak{gl}_n(\mathbb{R})$ as follows. First we realize $\mathfrak{g}^n$ as the subalgebra of $\mathfrak{gl}_{n+1}(\mathbb{R})$ via

$$\iota_{n+1} : \mathfrak{g}^n \ni X(v, A) \longmapsto \iota(X(v, A)) := \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \in \mathfrak{h}^{n+1}_{p_{n+1}} \subset \mathfrak{gl}_{n+1}(\mathbb{R}).$$

Let $\xi := \xi(p,\alpha)$ be an element of $(\mathfrak{g}^n)^*$ with

$$p = (\beta_1, \beta_2, \dots, \beta_n), \ \alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{pmatrix}. \tag{4.31}$$

Furthermore, we denote the Pfaffian of $\mathfrak{g}^n$ by $Q_{\mathfrak{g}^n}(\xi) := \mathrm{Pf}\ \xi([X_i, X_j])_{i,j=1}^N$ with a basis $\{X_i\}_{i=1}^N$ of $\mathfrak{g}^n$ taken in a specific way explained below.

Let us consider the Pfaffian $Q_{\mathfrak{g}^n}(\xi)$ for the case $n = 1$. We take $\{E_{11}, E_{12}\}$ as a basis of $\mathfrak{g}^1$ realized by $\iota_2(\mathfrak{g}^1) = \mathfrak{h}^2_{p_2} \subset \mathfrak{gl}_2(\mathbb{R})$. Since $[E_{11}, E_{12}] = E_{12}$, we have for $\xi = (p,\alpha) \in (\mathfrak{g}^1)^*$

$$M_{\mathfrak{g}^1} = \begin{pmatrix} 0 & E_{12} \\ -E_{12} & 0 \end{pmatrix} \text{ and } M_{\mathfrak{g}^1}(\xi_1) = \begin{pmatrix} 0 & p \\ -p & 0 \end{pmatrix}.$$

Therefore, we get $Q_{\mathfrak{g}^1}(\xi) = p$.

Now let us consider the case $n = 2$. Let $\xi = \xi(p,\alpha)$ be an element of $(\mathfrak{g}^2)^*$ with

$$p = (\beta_1, \beta_2), \ \alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}.$$

We take $\{E_{11}, E_{12}, E_{21}, E_{22}, E_{13}, E_{23}\}$ as a basis for $\mathfrak{g}^2$ realized by $\iota_3(\mathfrak{g}^2) = \mathfrak{h}^3_{p_3} \subset \mathfrak{gl}_3(\mathbb{R})$. We obtain the matrix $M_{\mathfrak{g}^2}(\xi)$

$$M_{\mathfrak{g}^2}(\xi) = \begin{pmatrix} M_{11}(\xi)_{2\times2} & M_{12}(\xi)_{2\times2} & M_{13}(\xi)_{2\times2} \\ M_{21}(\xi)_{2\times2} & M_{22}(\xi)_{2\times2} & M_{23}(\xi)_{2\times2} \\ M_{31}(\xi)_{2\times2} & M_{32}(\xi)_{2\times2} & M_{33}(\xi)_{2\times2} \end{pmatrix} \tag{4.32}$$

with

$$M_{11}(\xi) = \begin{pmatrix} 0 & \alpha_{21} \\ -\alpha_{21} & 0 \end{pmatrix}, \ M_{31}(\xi) = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_1 \end{pmatrix}, \ M_{32}(\xi) = \begin{pmatrix} \beta_2 & 0 \\ 0 & \beta_2 \end{pmatrix}$$
$$M_{33}(\xi) = O, \quad M_{13}(\xi) = -\,{}^tM_{31}(\xi), \quad M_{23}(\xi) = -\,{}^tM_{32}(\xi).$$

Note that $M_{11}(\xi)$ can be identified with $M_{\mathfrak{g}^1}(\xi)$ via the map $\iota_2 : \mathfrak{g}^1 \to \mathfrak{h}^2_{p_2} \subset \mathfrak{g}^2$.

Moreover, if $\beta_1 = 0$, then we have $M_{13}(\xi) = M_{31}(\xi) = O$ and $M_{23}(\xi) = -{}^tM_{32}(\xi) = -\beta_2 I_2$. In this case, the form (4.32) becomes

$$M_{\mathfrak{g}^3}(\xi) = \begin{pmatrix} M_{11}(\xi)_{2\times 2} & M_{12}(\xi)_{2\times 2} & O_{2\times 2} \\ M_{21}(\xi)_{2\times 2} & M_{22}(\xi)_{2\times 2} & -\beta_2 I_{2\times 2} \\ O_{2\times 2} & \beta_2 I_{2\times 2} & O_{2\times 2} \end{pmatrix}, \tag{4.33}$$

so that $Q_{\mathfrak{g}^2}(\xi) = \beta_2^2 \operatorname{Pf} M_{11}(\xi) = \beta_2^2\alpha_{21}$. For general $\xi = \xi(p, \alpha)$ with $\beta_2 \neq 0$, take $a = \begin{pmatrix} 1 & 0 \\ \beta_1/\beta_2 & 1 \end{pmatrix} \in H^2$ and $\xi' := \mathrm{Ad}^*(g(a))\xi$. Then $\beta_1' = 0$ because

$$\begin{aligned} \xi' &= \mathrm{Ad}^*(g(a))\xi \\ &= ((0, \beta_2), \begin{pmatrix} \alpha_{11} - \frac{\beta_1}{\beta_2}\alpha_{12} & \alpha_{11} \\ \alpha_{21} - \frac{\beta_1}{\beta_2}\alpha_{22} + \frac{\beta_1}{\beta_2}\alpha_{11} - (\frac{\beta_1}{\beta_2})^2\alpha_{12} & \alpha_{22} + \frac{\beta_1}{\beta_2}\alpha_{11} \end{pmatrix}), \end{aligned}$$

so that $Q_{\mathfrak{g}^2}(\xi') = \beta_2^2(\alpha_{21} - \frac{\beta_1}{\beta_2}\alpha_{22} + \frac{\beta_1}{\beta_2}\alpha_{11} - (\frac{\beta_1}{\beta_2})^2\alpha_{12})$. On the other hand, since $\det \mathrm{Ad}(g(a)) = 1$, we have $Q_{\mathfrak{g}^2}(\xi) = Q_{\mathfrak{g}^2}(\xi')$ by Proposition 1.7. Therefore, we get

$$Q_{\mathfrak{g}^2}(\xi) = \beta_2^2\alpha_{21} - \beta_1^2\alpha_{12} + \beta_1\beta_2(\alpha_{11} - \alpha_{22}).$$

Since the both sides are polynomial functions, the formula above is valid also for the case $\beta_2 = 0$. We define a rational map

$$\Phi : (\mathfrak{g}^2)^* \to (\mathfrak{g}^1)^*$$

given by

$$\Phi(\xi) = \xi' \circ \iota_2 = (\alpha_{11} - \frac{\beta_1}{\beta_2}\alpha_{12},\ \alpha_{21} - \frac{\beta_1}{\beta_2}\alpha_{22} + \frac{\beta_1}{\beta_2}\alpha_{11} - (\frac{\beta_1}{\beta_2})^2\alpha_{12}).$$

Then, the arguments above are summarized as an equality

$$Q_{\mathfrak{g}^2}(\xi) = \beta_2^2 Q_{\mathfrak{g}^1}(\Phi(\xi)). \tag{4.34}$$

Let us consider the case $n = 3$. We take a basis

$$\{E_{11}, E_{12}, E_{21}, E_{22}, E_{13}, E_{23}, E_{31}, E_{32}, E_{33}, E_{14}, E_{24}, E_{34}\}$$

of $\mathfrak{g}^3$. We take $\xi = \xi(p, \alpha)$ with

$$p = (\beta_1, \beta_2, \beta_3), \text{ and } \alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}.$$

The matrix $M_{\mathfrak{g}^3}(\xi)$ can be written as

$$M_{\mathfrak{g}^3}(\xi) = \begin{pmatrix} M_{11}(\xi)_{6\times 6} & M_{12}(\xi)_{6\times 3} & M_{13}(\xi)_{6\times 3} \\ M_{21}(\xi)_{3\times 6} & M_{22}(\xi)_{3\times 3} & M_{23}(\xi)_{3\times 3} \\ M_{31}(\xi)_{3\times 6} & M_{32}(\xi)_{3\times 3} & M_{33}(\xi)_{3\times 3} \end{pmatrix}, \tag{4.35}$$

where

$$M_{11}(\xi) = \begin{pmatrix} 0 & -\alpha_{12} & \alpha_{21} & 0 & -\alpha_{13} & 0 \\ \alpha_{12} & 0 & \alpha_{22} - \alpha_{11} & -\alpha_{12} & 0 & -\alpha_{13} \\ -\alpha_{21} & \alpha_{11} - \alpha_{22} & 0 & \alpha_{21} & -\alpha_{23} & 0 \\ 0 & \alpha_{12} & -\alpha_{21} & 0 & 0 & -\alpha_{23} \\ \alpha_{13} & 0 & \alpha_{23} & 0 & 0 & 0 \\ 0 & \alpha_{13} & 0 & \alpha_{23} & 0 & 0 \end{pmatrix},$$

$$M_{31}(\xi) = \begin{pmatrix} \beta_1 & 0 & \beta_2 & 0 & 0 & 0 \\ 0 & \beta_1 & 0 & \beta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_1 & \beta_2 \end{pmatrix}, \quad M_{32}(\xi) = \begin{pmatrix} \beta_3 & 0 & 0 \\ 0 & \beta_3 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}$$

$$M_{33}(\xi) = O, \quad M_{13}(\xi) = -\,{}^t M_{31}(\xi), \quad M_{23}(\xi) = -\,{}^t M_{32}(\xi).$$

Note that $M_{11}(\xi)$ can be identified with $M_{\mathfrak{g}^2}(\xi)$ via the map $\iota_3 : \mathfrak{g}^2 \to \mathfrak{h}^3_{p_3} \subset \mathfrak{g}^3$. Moreover, if $\beta_1 = \beta_2 = 0$, then we have $M_{13}(\xi) = M_{31}(\xi) = O$ and $M_{23}(\xi) = -\,{}^t M_{32}(\xi) = -\beta_3 I_3$. In this case, the form (4.35) becomes

$$M_{\mathfrak{g}^3}(\xi) = \begin{pmatrix} M_{11}(\xi)_{6\times 6} & M_{12}(\xi)_{6\times 3} & O_{3\times 3} \\ M_{21}(\xi)_{3\times 6} & M_{22}(\xi)_{3\times 3} & -\beta_3 I_{3\times 3} \\ O_{3\times 6} & \beta_3 I_{3\times 3} & O_{3\times 3} \end{pmatrix}. \tag{4.36}$$

We apply exactly the same arguments to the Pfaffian $Q_{\mathfrak{g}^3}(\xi)$ as the one for the Pfaffian $Q_{\mathfrak{g}^2}(\xi)$. We obtain that $Q_{\mathfrak{g}^3}(\xi) = \beta_3^3\, Q_{\mathfrak{g}_2}(\Phi_3(\xi))$ with $\Phi : (\mathfrak{g}^3)^* \ni \xi \longmapsto$

$\Phi_3(\xi) = \mathrm{Ad}^*(g(a))\xi \circ \iota_3 \in (\mathfrak{g}^2)^*$ and $a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta_1/\beta_3 & \beta_2/\beta_3 & 1 \end{pmatrix}$. Repeating the computations in these ways, we obtain

Proposition 4.12. *Let $\xi := \xi(p, \alpha)$ be an element of $(\mathfrak{g}^n)^*$ as in (4.31) and Φ_n be a map given by*

$$\Phi_n : (\mathfrak{g}^n)^* \ni \xi \longmapsto \Phi_n(\xi) = \mathrm{Ad}^*(g(a))\xi \circ \iota_n \in (\mathfrak{g}^{n-1})^*$$

with

$$a = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \beta_1/\beta_n & \beta_2/\beta_n & \beta_3/\beta_n & \dots & \beta_{n-1}/\beta_n & 1 \end{pmatrix}.$$

Then the Pfaffian of $\mathfrak{g}^n = \mathfrak{aff}(n)$ is of the form

$$Q_{\mathfrak{g}^n}(\xi) = \beta_n^n \, Q_{\mathfrak{g}^{n-1}}(\Phi_n(\xi)). \tag{4.37}$$

Proposition 4.13. *Let $\xi := \xi(p, \alpha)$ with $p \in (\mathbb{R}^n)^*$ and $\alpha \in \mathrm{Mat}_n(\mathbb{R})$ such that Ω_ξ is open in $(\mathfrak{g}^n)^*$, then $\Omega_\xi = \Omega_{\xi_n^+}$ or $\Omega_\xi = \Omega_{\xi_n^-}$.*

Proof. We apply induction on n as follows. It is true for $n = 1$. Now let us assume that for $n = k - 1$ the statement is true, that is, if Ω_ξ is open in $(\mathfrak{g}^{k-1})^*$, then $\Omega_\xi = \Omega_{\xi_{k-1}^+}$ or $\Omega_\xi = \Omega_{\xi_{k-1}^-}$. Let $\xi = \xi(p, \alpha)$ be an element of $\mathfrak{aff}(k)^*$ such that Ω_ξ is open in $(\mathfrak{g}^k)^*$. Then $p \neq 0$. Take $a \in \mathrm{GL}_k^+(\mathbb{R})$ such that $pa^{-1} = p_k = (0, \dots, 0, 1)$ then by (4.18) we have

$$\xi' = \mathrm{Ad}^*(g(a))\xi = \xi(p_k, \alpha') \in (\mathfrak{g}^k)^*, \text{ with } \alpha' = \begin{pmatrix} \alpha'_{11} & \alpha'_{12} & \dots & \alpha'_{1,k-1} & \alpha'_{1k} \\ \alpha'_{21} & \alpha'_{22} & \dots & \alpha'_{2,k-1} & \alpha'_{2k} \\ \vdots & \vdots & \ddots & \vdots & \\ \alpha'_{k1} & \alpha'_{k2} & \dots & \alpha'_{k,k-1} & \alpha'_{k,k} \end{pmatrix}.$$

Let $v = \begin{pmatrix} -\alpha'_{1,k} \\ -\alpha'_{2,k} \\ \vdots \\ -\alpha'_{k-1,k} \\ -\alpha'_{k,k} \end{pmatrix}$, we obtain by (4.18) that $\xi'' = \mathrm{Ad}^*(g(v))\xi' = \xi(p_k, \alpha' + vp_k) \in (\mathfrak{g}^k)^*$ with

$$\alpha' + vp_k = \begin{pmatrix} \alpha'_{11} & \alpha'_{12} & \dots & \alpha'_{1,k-1} & 0 \\ \alpha'_{21} & \alpha'_{22} & \dots & \alpha'_{2,k-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha'_{k1} & \alpha'_{k2} & \dots & \alpha'_{k,k-1} & 0 \end{pmatrix}.$$

We get that $(\mathfrak{g}^k)^* \supset \Omega_\xi = \Omega_{\xi'} = \Omega_{\xi''}$. By Theorem 3.3, since $\Omega_{\xi''}$ is open in $(\mathfrak{g}^k)^*$, then $\mathrm{Ad}^*(H^k_{p_k})(\xi''|_{\mathfrak{h}^k_{p_k}})$ is open in $(\mathfrak{h}^k_{p_k})^*$ with $\xi''|_{\mathfrak{h}^k_{p_k}} = \xi(p', \alpha'')$ is identified with $\xi' \circ \iota_k \in (\mathfrak{g}^{k-1})^*$ given by

$$p' = (\alpha'_{k1}, \alpha'_{k2}, \dots, \alpha'_{k,k-1}) \text{ and } \alpha'' = \begin{pmatrix} \alpha'_{11} & \alpha'_{12} & \dots & \alpha'_{1,k-1} \\ \alpha'_{21} & \alpha'_{22} & \dots & \alpha'_{2,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha'_{k-1,1} & \alpha'_{k-1,2} & \dots & \alpha'_{k-1,k-1} \end{pmatrix}.$$

On the other hand, $G^{k-1} \simeq H^k_{p_k}$ so that $\Omega_{\xi'' \circ \iota_k} := \mathrm{Ad}^*(G^{k-1})(\xi'' \circ \iota_k)$ is open in $(\mathfrak{g}^{k-1})^*$. By induction hypothesis, there exists $h \in G^{k-1}$ such that $\xi'' \circ \iota_k = \mathrm{Ad}^*(h)\xi^\pm_{k-1}$ and if we regard $h \in H^k_{p_k}$ then $\xi'' = \mathrm{Ad}^*(h)\xi^\pm_k$. Moreover, we obtain

$$\xi = \mathrm{Ad}^*(g(a)^{-1})\mathrm{Ad}^*(g(v)^{-1})\mathrm{Ad}^*(h)\xi^\pm_k.$$

Therefore, $\Omega_\xi = \Omega_{\xi^+_k}$ or $\Omega_\xi = \Omega_{\xi^-_k}$ as required. □

Proposition 4.14. *Let $\xi^\pm_n = \xi(p_n, \alpha^\pm_n)$ as in (4.27). We have*

1. $Q_{\mathfrak{g}^n}(\xi^\pm_n) = \pm 1$.
2. $\Omega_{\xi^+_n} \neq \Omega_{\xi^-_n}$.

Proof. Taking $\xi^\pm_n = \xi(p_n, \alpha^\pm_n)$ as in (4.27) and applying to Proposition 4.12, we obtain $Q_{\mathfrak{g}^n}(\xi^\pm_n) = \pm 1$. Suppose that there exists $g \in \mathrm{Aff}^+(n)$ such that $\mathrm{Ad}^*(g)\xi^+_n =$

ξ_n^-. Then $Q_{\mathfrak{g}^n}(\xi_n^-) = \det \mathrm{Ad}(g)\, Q_{\mathfrak{g}^n}(\xi_n^+)$ by Proposition 1.7. By connectedness of $\mathrm{Aff}^+(n)$, we have $\det \mathrm{Ad}(g) > 0$. Therefore, $Q_{\mathfrak{g}^n}(\xi_n^-) > 0$. But, this contradicts to the first assertion. □

In conclusion, using Proposition 4.13 and Proposition 4.14 above, we obtain that $\mathrm{Aff}^+(n)$ has exactly two open coadjoint orbits $\Omega_{\xi_n^+}$ and $\Omega_{\xi_n^-}$.

Bibliography

[1] Ali, S. T., Antoine, J.-P., and Gazeau, J.-P., *Coherent states, wavelets and their generalizations*, Springer-Verlag, New York, 2000.

[2] Aniello, P., Cassinelli, G., de Vito, E., and Levrero, A., *Square-integrability of induced representations of semidirect products*, Rev. Math. Phys. **10** (1998), 301–313.

[3] Auslander, L., and Kostant, B., *Polarization and unitary representations of solvable Lie groups*, Invent. Math. **14** (1971), 255–354.

[4] Barut, A. O., and Rączka, R., *Theory of group representations and applications*, Polish Scientific Publishers, Warszawa, Poland, 1980.

[5] Bernat, P., Conze, N., Duflo, M., Lévi-Nahas, M., Rais, M., Renouard, P., and Vergne, M., *Représentations des groupes de Lie résolubles*, Dunod, Paris, 1972.

[6] Berndt, R., *Representations of linear groups. An introduction based on examples from physics and number theory*, Vieweg, Wiesbaden, 2007.

[7] Bernier, D., and Taylor, K. F., *Wavelets from square-integrable representations*, SIAM J. Math. Anal. **27** (1996), 594–608.

[8] Blattner, R. J., *On induced representations*, Amer. J. Math. **83** (1961), 79–98.

[9] Blattner, R. J., *A theorem on induced representations*, Proc. Amer. Math. Soc. **13** (1962), 881–884.

[10] Carey, A. L., *Square-integrable representations of non-unimodular groups*, Bull. Austral. Math. Soc. **15** (1976), 1–12.

[11] Corwin, L. J., and Greenleaf, F. P., *Representations of nilpotent Lie groups and their applications. Part I. Basic theory and examples*, Cambridge University Press, Cambridge, 1990.

[12] Csikós, B., and Verhóczki, L., *Classification of Frobenius Lie algebras of dimension* ≤ 6, Publ. Math. Debrecen. **70** (2007), 427–451.

[13] Diatta, A., and Manga, B., *On properties of principal elements of Frobenius Lie algebras*, J. Lie Theory **24** (2014), 849–864.

[14] Dixmier, J., *Enveloping algebras*, Graduate Studies in Mathematics, vol. **11**. American Mathematical Society, Providence, RI, 1996.

[15] Duflo, M., and Moore, C. C., *On the regular representation of a nonunimodular locally compact group*, J. Funct. Anal. **21** (1976), 209–243.

[16] Duflo, M. and Raïs, M., *Sur l'analyse harmonique sur les groupes de Lie résolubles*, Ann. Sci. École Norm. Sup. **9** (1976), 107–144.

[17] Duflo, M., *On the Plancherel formula for almost algebraic real Lie groups. Lie groups representations, III (College Park, Md., 1982/1983)*, Lecture Notes in Math. **1077**, Springer, Berlin, 1984, 101–165.

[18] Élashvili, A. G., *Frobenius Lie algebras*, *Funkts. Anal. i Prilozh.* **16** (1982), 94–95.

[19] Folland, G. B., *A course in abstract harmonic analysis*, Second edition. Textbooks in Mathematics, CRC Press, Boca Raton, FL, 2016.

[20] Führ, H., *Wavelet frames and admissibility in higher dimensions*, J. Math. Phys. **37** (1996), 6353–6366

[21] Führ, H., *Abstract harmonic analysis of continuous wavelet transforms*, *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 2005.

[22] Führ, H., and Mayer, M., *Continuous wavelet transforms from semidirect products:cyclic representations and Plancherel measure*, J. Fourier Anal. Appl. **8** (2002), 375–397.

[23] Fujiwara, H., and Ludwig, J., *Harmonic analysis on exponential solvable Lie groups*, Springer Monographs in Mathematics, Springer, Tokyo, 2015.

[24] Fulton, W., and Harris, J., *Representation theory. A first course*, Springer-Verlag, Berlin-Hiedelberg-New York, 1988.

[25] Gelfand, I. M., and Neumark, M. A., *Unitary representations of the group of linear transformations of the straight line*, C. R. (Doklady) Acad. Sci URSS (N.S.), **55** (1947), 576–570.

[26] Gerstenhaber, M., and Giaquinto, A., *The principal element of a Frobenius Lie algebra*, Lett. Math. Phys. **88** (2009), 333–341.

[27] Grossmann, A., Morlet, J., and Paul, T., *Transforms associated to square integrable group representations.I. General results*, J. Math. Phys. **26** (1985), 2473–2479.

[28] Grossmann, A., Morlet, J., and Paul, T., *Transforms associated to square integrable group representations.II. Examples*, Ann. Inst. H. Poincaré Phys. Théor. **45** (1986), 293–309.

[29] He, J., and Liu, H., *Admissible wavelets associated with the affine automorphism group of the Siegel upper half-plane*, J. Math. Anal. Appl. **208** (1997), 58–70.

[30] Ishi, H., *On symplectic representations of normal j-algebras and their application to Xu's realizations of Siegel domains*, J. Differential Geom. Appl. **24** (2006), 588–612.

[31] Ishi, H., *Wavelet transforms for semidirect product groups with not necessarily commutative normal subgroups*, J. Fourier Anal. Appl. **12** (2006), 37–52.

[32] Ishi, H., *Continuous wavelet transforms and non-commutative Fourier analysis*, RIMS Kôkyûroku Bessatsu **B20** (2010), 173–185.

[33] Kirillov, A. A., *Unitary representations of nilpotent Lie groups*, Uspekhi Mat. Nauk **17** (1962), 57–110.

[34] Kirillov, A. A., *Elements of the theory of representations,*"Nauka", Moscow, 1972, 1978; English transl., Springer-Verlag, Berlin-Hiedelberg-New York, 1976.

[35] Kirillov, A. A., *Merits and demerits of the orbit method*, Bull. Amer. Math. Soc. (N.S.) **36** (1999), 433–488.

[36] Kirillov, A. A., *Lectures on the orbit method*, Graduate Studies in Mathematics, Vol.**64**, American Mathematical Society, Providence, RI, 2004.

[37] Kurniadi, E., and Ishi, H., *Harmonic analysis for 4-dimensional real Frobenius Lie algebras* , Springer Proceeding in Mathematics & Statistics, accepted, (2018).

[38] Lee, J. M., *Introduction to smooth manifolds*, Graduate Texts in Mathematics, Springer-Verlag, New York, 2003.

[39] Leptin, H., and Ludwig, J., *Unitary representation theory of exponential Lie groups*, De Gruyter Expositions in Mathematics, Walter de Gruyter & Co., Berlin, 1994.

[40] Lipsman, R. L., *An Orbital perspective on square-integrable representations*, Indiana Univ. Math. J. **34** (1985), 393–403.

[41] Lipsman, R. L., *Harmonic induction on Lie groups*, J. Reine Angew. Math. **344** (1983), 120–148.

[42] Lipsman, R. L., and Wolf, J. A., *Canonical semi-invariants and the Plancherel formula for parabolic groups*, Trans. Amer. Math. Soc. **269** (1982), 111–131.

[43] Mackey, G. W., *Imprimitivity for representations of locally compact groups.I*, Proc. Nat. Acad. Sci. U. S. A. **35** (1949), 537–545.

[44] Mackey, G. W., *Induced representations of locally compact groups.I*, Ann. of Math. **55** (1952), 101–139.

[45] Mackey, G. W., *Induced representations of locally compact groups.II.The Frobenius reciprocity theorem*, Ann. of Math. **58** (1953), 193–221.

[46] Mykytyuk, I. V., *Structure of the coadjoint orbits of Lie algebras*, J. Lie Theory **22** (2012), 251–268.

[47] Ooms, A. I., *On the field of the quotients of the universal enveloping algebra of a Lie algebra*, Ph.D thesis, Yale University, 1972.

[48] Ooms, A. I., *On Lie algebras having a primitive universal enveloping algebra*, J. Algebra **32** (1974), 488–500.

[49] Ooms, A. I., *On Lie algebras with primitive envelopes, supplements*, Proc. Amer. Math. Soc. **58** (1976), 67–72.

[50] Ooms, A. I., *On Frobenius Lie algebras*, Comm. Algebra **8** (1980), 13–52.

[51] Ooms, A. I., *Computing invariants and semi-invariants by means of Frobenius Lie algebras*, J. Algebra **321** (2009), 1293–1312.

[52] Panyushev, D. I., *The index of a Lie algebra, the centralizer of a nilpotent element, and the normalizer of the centralizer*, Math. Proc. Cambridge Philos. Soc. **134** (2003), 41–59.

[53] Phillips, J., *A note on square-integrable representations*, J. Functional Analysis **20** (1975), 83–92.

[54] Zeitlin, A. M., *On the unitary representations of the affine $ax+b$-group, $\hat{\mathfrak{sl}}(2,\mathbb{R})$ and their relatives*, *Lie algebras, Lie superalgebras, vertex algebras and related topics*, Proc. Sympos. Pure Math., **92**, Amer. Math. Soc., Providence, RI, 2016, 325–355.