W^*-BIMODULES AND THE DILATION THEORY RELATED TO PRODUCT SYSTEMS
(W^*-双加群とプロダクトシステムに関する伸張理論)

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Abstract

This thesis can be roughly divided into two contents.

Firstly, we will discuss an equivalence between left and right relative tensor products. The notion of relative tensor product have been introduced by Connes. For two W^*-bimodules, their relative tensor product is defined by the two ways. We call them the left and right relative tensor product. We shall show that the two bicategories $\mathcal{M}^{\leftthreetimes}$ and $\mathcal{M}^{\rightthreetimes}$ of W^*-bimodules with left and right relative tensor products as tensor functors, are monoidally equivalent. Each of $\mathcal{M}^{\leftthreetimes}$ and $\mathcal{M}^{\rightthreetimes}$ has an involutive structure, and they will be involutionally and monoidally equivalent. Such equivalence is basic when we consider relative tensor products.

Secondly, we will discuss the dilation theory for CP_0-semigroups. The existence of the minimal dilation of a given CP_0-semigroups was shown by Bhat-Skeide, Muhly-Solel and Arveson. We will provide a notion of relative product system, that is, a W^*-bimodule version of Arveson's product systems, which enables as to describe a relation between Bhat-Skeide's and Muhly-Solel's approaches. We will then construct the minimal dilation by relative product systems. We also discuss the construction for the discrete semigroup generated by a single normal UCP-map via discrete relative product systems. Product systems have been originally introduced to classify E_0-semigroups on type I factors. We will develop the classification theory of E_0-semigroups in terms of relative product systems with the help of the dilation theory.

Contents

Abstract 2
1. Introduction 3
Notation 9
2. Preliminaries 10
2.1. W^*-bimodules 10
2.2. W^*-bicategories 12
2.3. Hilbert bimodules 14
2.4. CP_0-semigroups, E_0-semigroups and Arveson's product systems 16
3. W^*-bicategories of W^*-bimodules 20
3.1. Monoidal equivalence 21
3.2. Unitary involutions and involutionally equivalence 22
4. Minimal dilations of CP_0-semigroups 26
4.1. Bhat-Skeide's and Muhly-Solel's constructions 26
4.2. Relative product systems from CP_0-semigroups 29
4.3. Construction of dilations by relative product systems 33
4.4. Relation between the two constructions 36
4.5. The discrete case 39
5. E_0-semigroups and relative product systems 41
5.1. Correspondence between CP_0-semigroups and units 41
5.2. Classification of E_0-semigroups by relative product systems 46
Acknowledgments 50
References 50

1. Introduction

One of von Neumann's achievements is the mathematical formulation of quantum mechanics. In a quantum system, observables are associated with self-adjoint operators on a Hilbert space $\mathcal{H}$ and physical states are described by unit vectors in $\mathcal{H}$. Within the framework of von Neumann, self-adjoint algebras of bounded operators on a Hilbert space are especially focused on. The set $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ has several topologies. A von Neumann algebra M, introduced by Murray and von Neumann in 1930's, is a self-adjoint unital closed subalgebra of $\mathcal{B}(\mathcal{H})$ with respect to the weak operator topology. By von Neumann's double commutant theorem, von Neumann algebras can be characterized by an algebraic property as follows: for a self-adjoint subalgebra M of $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if $M = M''$, where the commutant S' of a subset $S \subset \mathcal{B}(\mathcal{H})$ is the set of operators commuting with all operators in S. We call a von Neumann algebra a W^*-algebra when the representation is not especially referred to.

In the representation theory of von Neumann algebras, the Gelfand-Naimark-Segal construction guarantees that for every von Neumann algebra M and every faithful normal positive functional ϕ on M, there is a faithful normal representation π_ϕ on a Hilbert space $\mathcal{H}_\phi$ and $\phi^{\frac{1}{2}} \in \mathcal{H}_\phi$ such that $\mathcal{H}_\phi = \overline{\pi_\phi(M)\phi^{\frac{1}{2}}}$ and $\phi(x) = \langle \phi^{\frac{1}{2}}, \pi_\phi(x)\phi^{\frac{1}{2}} \rangle$

for all $x \in M$. Tomita and Takesaki have established the modular theory for von Neumann algebras which has made a substantial contribution to the noncommutative integration theory. More presicely, if we regard as $M \subset \mathcal{B}(\mathcal{H}_\phi)$, there are an anti-unitary operator J_ϕ called the modular conjugation and an unbounded self-adjoint operator Δ called the modular operator on $\mathcal{H}_\phi$ such that the commutant M' of M is $J_\phi M J_\phi$ and $\Delta^{it} M \Delta^{-it} = M$ for all $t \in \mathbb{R}$. An automorphism group $\{\sigma_t^\phi\}_{t\in\mathbb{R}}$ defined by $\sigma_t(x) = \Delta^{it} x \Delta^{-it}$ for each $t \in \mathbb{R}$ and $x \in M$ is called the modular automorphism group and it plays important roles in the structure theory of type III factors, the theory of noncommutative L^p-spaces, etc. The modular conjugation gives a right M-module structure on $\mathcal{H}_\phi$ which commutes with the left action of M, and hence $\mathcal{H}_\phi$ is an M-bimodule and referred to as a W^*-M-bimodule in this thesis. Also, there is a W^*-M-bimodule $L^2(M)$ called the standard space of M independent of the choice of ϕ such that $\overline{\phi^{\frac{1}{2}} M} = L^2(M) = \overline{M \phi^{\frac{1}{2}}}$ like the regular representation of a group. The standard space $L^2(\mathcal{B}(\mathcal{H}))$ is the Hilbert space $\mathcal{C}_2(\mathcal{H})$ of Hilbert-Schmidt operators on $\mathcal{H}$. Connes[9] has introduced the notion of relative tensor product for two W^*-M-bimodules. The standard space $L^2(M)$ is the unit object with respect to the relative tensor product $\otimes^M$, that is, we have isomorphisms $L^2(M) \otimes^M \mathcal{H} \cong \mathcal{H} \cong \mathcal{H} \otimes^M L^2(M)$ for any W^*-bimodule $\mathcal{H}$.

A dynamics on a quantum system is described by a semigroup $\{T_t\}_{t\geq 0}$ of normal unital completely positive (UCP-) maps T_t on a von Neumann algebra M for which t represents a parameter of time development. We often assume that the semigroup $\{T_t\}_{t\geq 0}$ is continuous with respect to the parameter in the σ-weak operator topology, and then we call it a CP_0-semigroup. If each T_t is a $*$-homomorphism, $\{T_t\}_{t\geq 0}$ is called an E_0-semigroup. By Stinespring's theorem, a single normal UCP-map T on $M \subset \mathcal{B}(\mathcal{H})$ can be dilated to a representation on an extended Hilbert space, that is, there exist a Hilbert space $\mathcal{K}$ and a representation π of M on $\mathcal{K}$ satisfying $T(x) = p\pi(x)|_{\mathcal{H}}$ for all $x \in M$, where p is the projection from $\mathcal{K}$ onto $\mathcal{H}$. Roughly speaking, the aim of dilation theory is to extend a UCP-map to a unital $*$-homomorphism. A dilation of a CP_0-semigroup $\{T_t\}_{t\geq 0}$ is a triple $(N, p, \{\theta_t\}_{t\geq 0})$ of a von Neumann algebra N, a projection $p \in N$ and an E_0-semigroup $\{\theta_t\}_{t\geq 0}$ such that $M = pNp$ and $T_t(x) = p\theta_t(x)p$ for all $t \geq 0$ and $x \in M$. A dilation $(N, p, \{\theta_t\}_{t\geq 0})$ is said to be minimal if N is generated by $\theta_{[0,\infty)}(M)$ and the central support of p in N is 1_N. Arveson has defined a hierarchy for dilations in [3], and a minimal dilation is the minimal one in the hierarchy (if it exists). Note that Stinespring's theorem can not be applied to CP_0-semigroups. Some researchers have shown an existence of minimal dilations gradually. In [6] and [7], Bhat has shown it in the case when M is a type I factor and in the case when M is a C^*-algebra, respectively. Bhat-Skeide[8] constructed minimal dilations by a method which is valid for both of von Neumann algebras and C^*-algebras. Also, we know Muhly-Solel's[17] and Arveson's[4] constructions, which differ from each other, of the minimal dilation of a CP_0-semigroup on a von Neumann algebra. Some direct relationships among the constructions have not been clarified yet.

The study of E_0-semigroups have been initiated by Powers[19]. Arveson[2] has provided the notion of product system. A product system $\{\mathcal{H}_t\}_{t>0}$ is a family of

Hilbert spaces $\mathcal{H}_t$ parameterized by nonnegative real numbers equipped with isomorphisms $\mathcal{H}_s \otimes \mathcal{H}_t \cong \mathcal{H}_{s+t}$ with the associativity. He also classified E_0-semigroups on type I factors by product systems up to cocycle conjugacy. The theory of product systems influenced the constructions of minimal dilations of CP_0-semigroups. The classification theory of E_0-semigroups has been developed in terms of Hilbert modules (that is, modules over a C^*-algebra A with A-valued inner products). A Bhat-Skeide's[8] product system is a family $\{E_t\}_{t\geq 0}$ of Hilbert bimodules over a C^*-algebra satisfying a similar property with Arveson's one with respect to tensor products of Hilbert bimodules. They classified E_0-semigroups on a (general) C^*-algebra by their product systems up to cocycle conjugacy. There have been no approaches to the classification theory of E_0-semigroups on a von Neumann algebra by the W^*-bimodule theory.

In this thesis, we will discuss the equivalence between left and right relative tensor products. Also, we will construct the minimal dilation of a CP_0-semigroup in terms of W^*-bimodules and describe a relation between Bhat-Skeide's and Muhly-Solel's constructions of minimal dilations. The classification theory of E_0-semigroups on a von Neumann algebra will be developed by the W^*-bimodules theory with the help of our construction of minimal dilations. Now, we shall give an outline of the contents in this thesis.

We shall prepare the notions of W^*-bimodule, W^*-bicategory, Hilbert modules, CP_0-semigroups and E_0-semigroups in Section 2 related to the later sections.

In Section 3, we will discuss the W^*-bicategory whose objects are W^*-algebras and the category associated with W^*-algebras M and N consists of W^*-M-N-bimodules with relative tensor products as tensor functors. The relation between the W^*-bicategory of W^*-bimodules and general W^*-categories has been clarified by Yamagami[37]. For W^*-M-N-bimodule $\mathcal{H}$ and W^*-N-P-bimodule $\mathcal{K}$, the relative tensor product $\mathcal{H} \otimes^M \mathcal{K}$ is defined by two ways. To distinguish them, we call them the left and the right relative tensor product and denote by $\mathcal{H} \leftthreetimes^M \mathcal{K}$ and $\mathcal{H} \rightthreetimes^M \mathcal{K}$, respectively. Of course, the two relative tensor products are isomorphic: $\mathcal{H} \leftthreetimes^M \mathcal{K} \cong \mathcal{H} \rightthreetimes^M \mathcal{K}$. However, just the existence of isomorphisms is not enough to show that left and right relative tensor products are equivalent in the sense of tensor categories. The following theorem is the main result in Subsection 3.1.

Theorem 1.0.1. *Let $\mathcal{M}^{\leftthreetimes}$ and $\mathcal{M}^{\rightthreetimes}$ be W^*-bicategories of W^*-bimodules with left and right relative tensor products as tensor functors, respectively. Then $\mathcal{M}^{\leftthreetimes}$ and $\mathcal{M}^{\rightthreetimes}$ are monoidally equivalent, that is, there exists a natural unitary isomorphism $\{m_{X,Y} : X \leftthreetimes^B Y \to X \rightthreetimes^B Y\}$, where X is a W^*-A-B-bimodule and Y is a W^*-B-C-bimodule, such that the following diagram commutes for all W^*-A-B-bimodule X, W^*-B-C-bimodule Y and W^*-C-D-bimodule Z.*

$$\begin{tikzcd}
(X \leftthreetimes Y) \leftthreetimes Z \arrow[r, "m_{X,Y} \leftthreetimes \mathrm{id}_Z"] \arrow[d, "a_{X,Y,Z}"] & (X \rightthreetimes Y) \leftthreetimes Z \arrow[r, "m_{X \rightthreetimes Y, Z}"] & (X \rightthreetimes Y) \rightthreetimes Z \arrow[d, "a'_{X,Y,Z}"] \\
X \leftthreetimes (Y \leftthreetimes Z) \arrow[r, "\mathrm{id}_X \leftthreetimes m_{Y,Z}"] & X \leftthreetimes (Y \rightthreetimes Z) \arrow[r, "m_{X, Y \rightthreetimes Z}"] & X \rightthreetimes (Y \rightthreetimes Z)
\end{tikzcd}$$

Also, the dual Hilbert space $\mathcal{H}^*$ of a W^*-M-N-bimodule $\mathcal{H}$ has the canonical W^*-N-M-bimodule structure: $y\xi^*x = (x^*\xi y^*)^*$ for each $x \in M, y \in N$ and $\xi \in \mathcal{H}$. We define involutive structures on $\mathcal{M}^{\leftthreetimes}$ and $\mathcal{M}^{\rightthreetimes}$ and show that they are involutionally and monoidally equivalent in Subsection 3.2 as follows:

Theorem 1.0.2. *We denote $\Box = \leftthreetimes$ or $\rightthreetimes$. There are natural unitary isomorphisms $\{c^{\Box}_{X,Y} : Y^*\Box^B X^* \to (X\Box^B Y)^*\}$, where X is a W^*-A-B-bimodule and Y is a W^*-B-C-bimodule, on $\mathcal{M}^{\Box}$. Then $\mathcal{M}^{\leftthreetimes}$ and $\mathcal{M}^{\rightthreetimes}$ are involutive W^*-bicategories and they are involutionally and monoidally equivalent with respect to $\{m_{X,Y}\}$, that is, the following diagram commutes for all W^*-A-B-bimodule X and W^*-B-C-bimodule Y.*

$$\begin{tikzcd}
Y^* \leftthreetimes X^* \arrow[rr, "c^{\leftthreetimes}_{X,Y}"] \arrow[d, "m_{Y^*,X^*}"] & & (X \leftthreetimes Y)^* \\
Y^* \rightthreetimes X^* \arrow[rr, "c_{X,Y}{}^{\rightthreetimes}"] & & (X \rightthreetimes Y)^* \arrow[u, "{}^t m_{X,Y}"']
\end{tikzcd}$$

The equivalence is basic when we consider relative tensor products. Section 3 is based on [24] which is a joint work with S. Yamagami.

In Section 4, we will provide a notion of relative product system, that is, a W^*-bimodule version of Arveson's or Bhat-Skeide's product systems, where tensor products taken into account are relative tensor products, and discuss dilations in terms of relative product systems. The notion gives a new approach to the dilation theory and the classification theory of E_0-semigroups in Section 5. Let T be a CP_0-semigroup T on a von Neumann algebra M. In Subsection 4.2, we will associate a relative product system $\tilde{\mathcal{H}}^{\otimes} = \{\tilde{\mathcal{H}}_t\}_{t\geq 0}$ with T as follows: for $t > 0$, let $\tilde{\mathcal{H}}_t$ be the inductive limit of $\{\tilde{\mathcal{H}}(\mathfrak{p}, t) \mid \mathfrak{p}$ is a partition of $[0, t]\}$ with respect to refinements of partitions, and $\tilde{\mathcal{H}}_0 = L^2(M)$. In Subsection 4.3, we will construct the minimal dilation of T. The inductive limit $\tilde{\mathcal{H}}$ of $\tilde{\mathcal{H}}^{\otimes}$ has a right W^*-module structure, and we can identify $\tilde{\mathcal{H}} \otimes^M \tilde{\mathcal{H}}_t$ with $\tilde{\mathcal{H}}$ for all $t \geq 0$. If we define a map θ_t on $\mathrm{End}(\tilde{\mathcal{H}}_M)$ by $\theta_t(a) = \mathrm{id}_{\tilde{\mathcal{H}}_t} \otimes a$ for each $a \in \mathrm{End}(\tilde{\mathcal{H}}_M)$, then we the following theorem.

Theorem 1.0.3. *There is a faithful representation π of M on $\tilde{\mathcal{H}}$ and $\theta = \{\theta_t\}_{t\geq 0}$ is an E_0-semigroup on $\mathrm{End}(\tilde{\mathcal{H}}_M)$. The triple $(\mathrm{End}(\tilde{\mathcal{H}}_M), \pi(1_M), \theta)$ is a dilation of T. Moreover, if we denote the von Neumann algebra generated by $\bigcup_{t\geq 0} \theta_t(M)$ by N, then the triple $(N, \pi(1_M), \theta|_N)$ is the minimal dilation of T.*

There are some advantages of the construction of the minimal dilation of a given CP_0-semigroup T on M in Section 4 as follows: we use the Hilbert (or von Neumann) module theory in Bhat-Skeide's and Muhly-Solel's constructions, however Theorem 1.0.3 makes it possible to construct the minimal dilation without the theory. Also, Muhly-Solel's construction depends on a representation of M and our construction enables to get the canonical representation of the dilated von Neumann algebra N independent of the choice of representations of M. We describe a relation between Bhat-Skeide's and Muhly-Solel's constructions. Suppose M acts on a (separable) Hilbert space $\mathcal{H}$. A common point of Bhat-Skeide's and Muhly-Solel's construction is to establish the product systems of von Neumann bimodules, and to dilate T to an E_0-semigroup on the inductive limits of the product systems. However, Bhat-Skeide's product system $\{E_t\}_{t\geq 0}$ consists of von Neumann M-bimodule and Muhly-Solel's one $\{E(t)\}_{t\geq 0}$ consists of von Neumann M'-bimodules. Now, we have the one-to-one correspondence between von Neumann bimodules and W^*-bimodules. The relative product system $\tilde{\mathcal{H}}^{\otimes} = \{\tilde{\mathcal{H}}_t\}_{t\geq 0}$ associated with T gives a relation between Bhat-Skeide's and Muhly-Solel's constructions of minimal dilations:

Theorem 1.0.4. *There is a one-to-one correspondence*

$$E_t \longleftrightarrow \tilde{\mathcal{H}}_t, \ E(t) \longleftrightarrow \mathcal{H}^* \otimes^M \tilde{\mathcal{H}}_t \otimes^M \mathcal{H}$$

between von Neumann bimodules and W^*-bimodules for each $t \geq 0$.

The relation is different from the one described by Skeide's commutant duality in [27]. Section 4 is based on [22] mainly. Bhat-Skeide's and Muhly-Solel's constructions can apply to a discrete CP_0-semigroup, that is, the semigroup $\{T^n\}_{n\in\mathbb{Z}_{\geq 0}}$ generated by a single normal UCP-map T. The construction (Theorem 1.0.3) and the relation (Theorem 1.0.4) in this case will be described by discrete relative product systems in Subsection 4.5 which is based on [21].

In Section 5, we will develop the classification theory of E_0-semigroups in terms of relative product systems. We fix a faithful normal state ϕ on a von Neumann algebra M. A unit of a relative product system $\mathcal{H}^{\otimes} = \{\mathcal{H}_t\}_{t\geq 0}$ is a family $\xi^{\otimes} = \{\xi(t)\}_{t\geq 0}$ of $\xi(t) \in \mathcal{H}_t$ satisfying $\xi(s) \otimes^M \xi(t) = \xi(s+t)$ for all $s, t \geq 0$. For a given pair $(\mathcal{H}^{\otimes}, \xi^{\otimes})$, the inductive limit $\mathcal{H}^{\xi}$ can be defined. We will give a one-to-one correspondence between CP_0-semigroups on M without the continuity called algebraic CP_0-semigroups and units of relative product systems. Then the continuity of the CP_0-semigroup associated with a unit is described by the one of the unit. These can be summarized as follows.

Theorem 1.0.5. *There is a one-to-one correspondence between algebraic CP_0-semigroups T and pairs $(\mathcal{H}^{\otimes}, \xi^{\otimes})$ of relative product systems $\mathcal{H}^{\otimes}$ and generating unital units $\xi^{\otimes}$ up to unit-preverving isomorphisms. The algebraic CP_0-semigroups T associated with a pair $(\mathcal{H}^{\otimes}, \xi^{\otimes})$ is a CP_0-semigroup if and only if we have*

$$\xi\phi^{-\frac{1}{2}}\xi(t) \to \xi \qquad (t \to +0)$$

for all ϕ-bounded vector $\xi \in \mathcal{H}^{\xi}$, where $\xi\phi^{-\frac{1}{2}}\xi(t)$ means the relative tensor product with respect to ϕ and we can identify $\mathcal{H}^{\xi} \otimes^M \mathcal{H}_t$ with $\mathcal{H}^{\xi}$ for all $s \geq 0$.

For a pair $(\mathcal{H}^{\otimes}, \xi^{\otimes})$ of a relative product system $\mathcal{H}^{\otimes} = \{\mathcal{H}_t\}_{t\geq 0}$ and a (continuous unital) unit $\xi^{\otimes} = \{\xi(t)\}_{t\geq 0}$, we can get an E_0-semigroup $\theta = \{\theta_t\}_{t\geq 0}$ on a von Neumann algebra $\mathrm{End}(\mathcal{H}^{\xi}_M)$ called the maximal dilation of $(\mathcal{H}^{\otimes}, \xi^{\otimes})$. The main result in Subsection 5.2 is as follows:

Theorem 1.0.6. *Let $\theta = \{\theta_t\}_{t\geq 0}$ be the maximal dilation of a pair $(\mathcal{H}^{\otimes}, \xi^{\otimes})$. There is a one-to-one correspondence between contractive adapted right cocycles $w = \{w_t\}_{t\geq 0}$ on $\mathrm{End}(\mathcal{H}_M)$ and contractive units $\eta^{\otimes} = \{\eta(t)\}_{t\geq 0}$ in $\mathcal{H}^{\otimes}$.*

As a corollary of the previous result, E_0-semigroups on a von Neumann algebra M will be classified up to cocycle equivalence by relative product systems. For E_0-semigroup θ on M, we can get the pair $(\tilde{\mathcal{H}}^{\theta\otimes}, \xi^{\theta\otimes})$ of the relative product system

$\tilde{\mathcal{H}}^{\theta\otimes}$ and the unit $\xi^{\theta\otimes}$ as CP_0-semigroups in Theorem 1.0.5. Then we have the following theorem which is an analogue of the classification of E_0-semigroups in [8, Theorem 7.9] and [30, Theorem 12.3].

Theorem 1.0.7. *Let $\theta = \{\theta_t\}_{t\geq 0}$ and $\theta' = \{\theta'_t\}_{t\geq 0}$ be E_0-semigroups on a von Neumann algebra M. Suppose $(\tilde{\mathcal{H}}^{\theta\otimes}, \xi^{\theta\otimes})$ and $(\tilde{\mathcal{H}}^{\theta\otimes'}, \xi^{\theta\otimes'})$ are the pairs associated with θ and θ', respectively. Then $\tilde{\mathcal{H}}^{\theta\otimes}$ and $\tilde{\mathcal{H}}^{\theta'\otimes}$ are isomorphic if and only if θ and θ' are cocycle equivalent by a strongly continuous unitary cocycle.*

Theorem 1.0.3 and the sequential discussions related to the classification by relative product systems are reflected by Bhat-Skeide's observations.

We list some problems in the future as follows:

(1) E_0-semigroups on type I factors are classified by (Arveson's) product systems up to cocycle conjugacy as follows: suppose θ and θ' are E_0-semigroups on $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$, respectively. Then θ and θ' are cocycle conjugate if and only if they have isomorphic product systems. Bhat-Skeide's classification of E_0-semigroups is generalization of Arveson's one. Is there a concrete relation between our classification of E_0-semigroups on type I factors and Arveson's one?
(2) Arveson[2] have introduced the index invariant of E_0-semigroups on type I factors. In particular, type I E_0-semigroups on type I factors are completely classified by indices. Also, Alevras[1] and Margetts-Srinivasan[15] have defined index invariants of E_0-semigroups on type II_1 factors. Is there an analogous theory for (general) von Neumann algebras in terms of relative product systems, with their observation?
(3) Let E be a Hilbert (or von Neumann) module and $\mathcal{B}^a(E)$ the algebra of adjointable right linear maps on E. Within the framework of the Hilbert module theory, E_0-semigroups on $\mathcal{B}^a(E)$ are classified by product systems of Hilbert (von Neumann) bimodules. We refer the reader to Skeide's monograph [30] for details. Can we refine the classification theory of E_0-semigroups in terms of relative product systems?
(4) Recall that there are the three approaches to construct minimal dilation by Bhat-Skeide, Muhly-Solel and Arveson. We will describe a relation between Bhat-Skeide's and Muhly-Solel's constructions in Subsection 4.4. However, some concrete relationships among Arveson's construction and them have not been clarified yet.

Notation

· $\mathbb{C}$: The set of all complex numbers.
· $\mathbb{R}$: The set of all real numbers.
· $\mathbb{R}_{\geq 0}$: The set of real numbers which are not less than 0.
· $\mathbb{Z}$: The set of all integers.
· $\mathbb{Z}_{\geq 0}$: The set of integers which are not less than 0.
· $\mathbb{N}$: The set of all natural numbers which are grater than 1.
· $\langle \cdot, \cdot \rangle$: Inner products on Hilbert spaces or Hilbert modules whose second terms are linear.
· $\mathcal{H}^*$: The dual Hilbert space of a Hilbert space $\mathcal{H}$.
· $l^2(I)$: The Hilbert space of square summable sequences indexed by a set I.
· $\mathrm{span}(S)$: The linear span of a subset S in a vector space.
· $\overline{\mathrm{span}}(S)$: The closure of $\mathrm{span}(S)$ with respect to a suitable topology.
· $\mathcal{B}(\mathcal{H})$: The set of all bounded linear operators on a Hilbert space $\mathcal{H}$.
· $\mathcal{C}_2(\mathcal{H})$: The Hilbert space of all Hilbert-Schmidt operators on a Hilbert space $\mathcal{H}$.
· $\xi\eta^*$: The operator on a Hilbert space $\mathcal{H}$ given by $\xi\eta^*(\zeta) = \langle \xi, \eta \rangle \xi$ for each $\zeta \in \mathcal{H}$, for a $\xi \in \mathcal{H}$ and an $\eta^* \in \mathcal{H}^*$.
· S' : The commutant of a subset S in $\mathcal{B}(\mathcal{H})$.
· M_* : The predual of a von Neumann algebra M.
· M_*^+ : The set of σ-weakly positive functionals in M_*.
· A° : The opposite algebra of an algebra A.
· $\|x\|$: The operator norm of an operator $x \in \mathcal{B}(\mathcal{H})$.
· id_A : The identity map on a set A.
· 1_M : The unit of a von Neumann algebra M.
· $[\phi]$: The support projection of a functional $\phi \in M_*^+$.

2. Preliminaries

In this section, we prepare some notions and notations for later sections. Subsection 2.1 is a preparation for W^*-bimodules related with all sections. Subsection 2.2 is for W^*-bicategories and we will discuss the W^*-bicategory of W^*-bimodules in Section 3. In Subsection 4.1, the minimal dilation is constructed in terms of Hilbert modules prepared in Subsection 2.3 by Bhat-Skeide's and Muhly-Solel's approaches. In Subsection 2.4, we recall the notions of CP_0-semigroups and E_0-semigroups related with Section 4 and 5

2.1. W^*-bimodules.

W^*-bimodules are Hilbert spaces on which von Neumann algebras act from the left and the right. More precisely, for von Neumann algebras N and M, a Hilbert space $\mathcal{H}$ with normal $*$-representations of N and the opposite von Neumann algebra M° of M is a W^*-N-M-bimodule if their representations commute. When $N = \mathbb{C}$ or $M = \mathbb{C}$, we call $\mathcal{H}$ a right W^*-M-module or a left W^*-N-module, respectively. We write a W^*-N-M-bimodule, a right W^*-M-module and a left W^*-N-module by ${}_N\mathcal{H}_M$, $\mathcal{H}_M$ and ${}_N\mathcal{H}$, respectively.

Let N be a von Neumann algebra, $\mathcal{H}_N$ and $\mathcal{K}_N$ be right W^*-N-modules, and ${}_N\mathcal{H}'$ and ${}_N\mathcal{K}'$ be left W^*-N-modules. $\mathrm{Hom}(\mathcal{H}_N, \mathcal{K}_N)$ and $\mathrm{Hom}({}_N\mathcal{H}', {}_N\mathcal{K}')$ are the sets of all right and left N-linear bounded maps, respectively. If $\mathcal{H} = \mathcal{K}$ and $\mathcal{H}' = \mathcal{K}'$, they are denoted by $\mathrm{End}(\mathcal{H}_N)$ and $\mathrm{End}({}_N\mathcal{H}')$, respectively. Suppose the space $\mathrm{Hom}({}_N\mathcal{H}', {}_N\mathcal{K}')^\circ$ is bijevtive to $\mathrm{Hom}({}_N\mathcal{H}', {}_N\mathcal{K}')$ as sets and each operator $f \in \mathrm{Hom}({}_N\mathcal{H}', {}_N\mathcal{K}')^\circ$ acts from the right like opposite algebras.

For a von Neumann algebra M and $\phi \in M_*^+$, the left (right) GNS-representation of M with respect to ϕ is defined as follows: we define a left (right) ideal $\mathcal{N}_\phi^l = \{x \in M \mid \phi(x^*x) = 0\}$ ($\mathcal{N}_\phi^r = \{x \in M \mid \phi(xx^*) = 0\}$) of M and denote as $x\phi^{\frac{1}{2}} = x + \mathcal{N}_\phi^l \in M/\mathcal{N}_\phi^l$ ($\phi^{\frac{1}{2}}x = x + \mathcal{N}_\phi^r \in M/\mathcal{N}_\phi^r$) for each $x \in M$. The left (right) GNS-space is the completion $\overline{M\phi^{\frac{1}{2}}}$ ($\overline{\phi^{\frac{1}{2}}M}$) with respect to an inner product defined by $\langle x\phi^{\frac{1}{2}}, y\phi^{\frac{1}{2}}\rangle = \phi(x^*y)$ ($\langle \phi^{\frac{1}{2}}x, \phi^{\frac{1}{2}}y\rangle = \phi(yx^*)$) for each $x, y \in M$. The left (right) GNS-representation is a $*$-homomorphism $\pi : M \to \mathcal{B}(\overline{M\phi^{\frac{1}{2}}})$ ($*$-anti-homomorphism $\rho : M \to \mathcal{B}(\overline{\phi^{\frac{1}{2}}M})$) defined by $\pi(a)x\phi^{\frac{1}{2}} = (ax)\phi^{\frac{1}{2}}$ ($\rho(a)\phi^{\frac{1}{2}}x = \phi^{\frac{1}{2}}(xa)$) for each $a, x \in M$.

The standard space of M denoted by $L^2(M)$ is defined as a W^*-M-bimodule such that all left and right GNS-spaces are included and $[\phi]\overline{M\psi^{\frac{1}{2}}} = \overline{\phi^{\frac{1}{2}}M}[\psi]$ in $L^2(M)$ and $\overline{\phi^{\frac{1}{2}}M} = [\phi]L^2(M)$ for all $\phi, \psi \in M_*^+$. In particular, we have $\overline{\phi^{\frac{1}{2}}M} = L^2(M) = \overline{M\phi^{\frac{1}{2}}}$ if $\phi \in M_*^+$ is faithful. This observation will be helpful under the assumption which a von Neumann algebra has a faithful normal state in Section 4 and 5. Also $L^2(M)$ has an involutive structure $J : L^2(M) \to L^2(M)$ called the modular conjugation such that $J(x\xi y) = y^*J\xi x^*$ for all $x, y \in M$ and $\xi \in L^2(M)$. Note that for $\phi \in M_*^+$, we have $J(x\phi^{\frac{1}{2}}) = \phi^{\frac{1}{2}}x^*$ for all $x \in [\phi]M[\phi]$. We refer the reader to [32, Chapter IX], [38], [34] and [35] for details of the definition and properties of standard spaces included in the modular theory.

Example 2.1.1. *We consider the standard space of a type I factor. Let $\mathcal{H}$ be a separable Hilbert space and $M = \mathcal{B}(\mathcal{H})$. The one-to-one correspondence between* *positive normal functionals on M and positive trace-class operators on $\mathcal{H}$ makes it possible to identify $L^2(M)$ with $\mathcal{C}_2(\mathcal{H})$. Moreover, by the correspondence $\mathcal{H}\otimes\mathcal{H}^* \ni \xi\otimes\eta^* \mapsto \xi\eta^* \in \mathcal{B}(\mathcal{H})$, we have M-bimodule isomorphism $\mathcal{H}\otimes\mathcal{H}^* \cong \mathcal{C}_2(\mathcal{H})$. The standard representation of M on $\mathcal{H}\otimes\mathcal{H}^*$ is defined by $\pi(x)(\xi\otimes\eta^*) = (x\xi)\otimes\eta^*$ for each $x\in M$ and $\xi,\eta\in\mathcal{H}$. The modular conjugation J is given by $J(\xi\otimes\eta^*) = \eta\otimes\xi^*$ for each $\xi,\eta\in\mathcal{H}$. The vector $\Omega = \sum_{n=1}^\infty \frac{1}{n}\xi_n\otimes\xi_n^*$ is cyclic and separating for $\pi(M)$, where $\{\xi_n\}_{n=1}^\infty$ is an orthonormal basis for $\mathcal{H}$.*

For a W^*-M-N-bimodule ${}_M\mathcal{H}_N$, $\mathrm{Hom}({}_ML^2(M), {}_M\mathcal{H})^\circ$ and $\mathrm{Hom}(L^2(N)_N, \mathcal{H}_N)$ have the canonical M-N-bimodule structures. For $f_1, f_2 \in \mathrm{Hom}({}_ML^2(M), {}_M\mathcal{H})^\circ$ and $g_1, g_2 \in \mathrm{Hom}(L^2(N)_N, \mathcal{H}_N)$, we define $\xi(f_2f_1^*) = (\xi f_2)f_1^*$ and $(g_1^*g_2)\xi = g_1^*(g_2\eta)$ for each $\xi \in L^2(M)$ and $\eta \in L^2(N)$. Note that $f_2f_1^* \in \mathrm{End}({}_ML^2(M))^\circ$ and $g_1^*g_2 \in \mathrm{End}(L^2(N)_N)$ are elements in M and N acting from right and left, respectively.

Now, we introduce relative tensor products as follows:

Definition 2.1.2. *Let $\mathcal{H}_N$ and ${}_N\mathcal{K}$ be right and left W^*-N-modules, respectively. The left relative tensor product $\mathcal{H}\curlywedge^M\mathcal{K}$ of $\mathcal{H}$ and $\mathcal{K}$ is defined by the completion of the tensor product* $\mathrm{Hom}(L^2(N)_N, \mathcal{H}_N)\otimes_N \mathcal{K}$ *of Banach spaces with respect an M-valued inner product*

$$\langle f_1\otimes\eta_1, f_2\otimes\eta_2\rangle = \langle \eta_1, (f_1^*f_2)\eta_2\rangle$$

for all $f_1, f_2 \in \mathrm{Hom}(L^2(N)_N, \mathcal{H}_N)$, $\eta_1, \eta_2 \in \mathcal{K}$. We can also define the right relative tensor product $\mathcal{H}\curlyvee^N\mathcal{K}$ by the completion of $\mathcal{H}\otimes_N \mathrm{Hom}({}_NL^2(N), {}_N\mathcal{K})^\circ$ similarly.

For more details, see [5], [25] and [33]. When $\mathcal{H}$ is a W^*-M-N-bimodule and $\mathcal{K}$ is a W^*-N-P-bimodule, the left and right relative tensor products $\mathcal{H}\curlywedge^M\mathcal{K}$ and $\mathcal{H}\curlyvee^M\mathcal{K}$ have W^*-M-P-bimodules structures. In Section 3, we will give an isomorphism $\mathcal{H}\curlywedge^M\mathcal{K}\cong\mathcal{H}\curlyvee^M\mathcal{K}$ as W^*-bimodules by the canonical way, and show that the two W^*-bicategories of W^*-bimmodules with left and right tensor products as tensor functors are monoidally equivalent in the sense of Definition 2.2.6.

Let ${}_M\mathcal{H}_N, {}_M\mathcal{H}'_N, {}_N\mathcal{K}_P$ and ${}_N\mathcal{K}'_P$ be W^*-bimodules. For two bilinear maps $f : {}_M\mathcal{H}_N \to {}_M\mathcal{H}'_N$ and $g : {}_N\mathcal{K}_P \to {}_N\mathcal{K}'_P$, we can define M-P-linear maps $f\curlywedge g : {}_M\mathcal{H}\curlywedge\mathcal{K}_P \to {}_M\mathcal{H}'\curlywedge\mathcal{K}'_P$ and $f\curlyvee g : {}_M\mathcal{H}\curlyvee\mathcal{K}_P \to {}_M\mathcal{H}'\curlyvee\mathcal{K}'_P$ by

$$(f\curlywedge g)(x\otimes\eta) = (fx)\otimes(g\eta),\ (f\curlyvee g)(\xi\otimes y) = (f\xi)\otimes(gy)$$

for each $x \in \mathrm{Hom}(L^2(N)_N, \mathcal{H}_N), \eta\in\mathcal{K}, y\in\mathrm{Hom}({}_NL^2(N), {}_N\mathcal{K})^\circ$ and $\xi\in\mathcal{H}$, where gy is defined by $\zeta(gy) = g(\zeta y)$ for each $\zeta\in L^2(N)$.

Now, we introduce the alternative definition of (left) lelative tensor products in[9, Chapter 5, Appendix B] or [32, Chapter IX, Section 3]. Suppose $\mathcal{H}$ is a W^*-M-N-bimodule and $\mathcal{K}$ is a W^*-N-P-bimodule. For simplicity, assume that N has a faithful normal state ϕ. A vector $\xi\in\mathcal{H}$ is called a (left) ϕ-bounded vector if there is $c>0$ such that $\|\xi x\| \le c\|\phi^{\frac{1}{2}}x\|$ for all $x\in M$. We denote the set of all ϕ-bounded vectors in $\mathcal{H}$ by $\mathcal{D}(\mathcal{H};\phi)$. The (left) relative tensor product $\mathcal{H}\curlywedge_\phi^N\mathcal{K}$ with respect to ϕ is the completion $\overline{\mathcal{D}(\mathcal{H};\phi)\otimes_{\mathrm{alg}}\mathcal{K}}$ with respect to an inner product defined by

$$\langle \xi_1\phi^{-\frac{1}{2}}\eta_1, \xi_2\phi^{-\frac{1}{2}}\eta_2\rangle = \langle \eta_1, \pi_\phi(\xi_1)^*\pi_\phi(\xi_2)\eta_2\rangle,$$

for each $\xi_1, \xi_2 \in \mathcal{D}(\mathcal{H}; \phi)$ and $\eta_1, \eta_2 \in \mathcal{K}$, where $\pi_\phi(\xi) : L^2(N) \ni \phi^{\frac{1}{2}} x \to \xi x \in \mathcal{H}$ and we usually use a notation $\xi \phi^{-\frac{1}{2}} \eta$ rather than $\xi \otimes \eta$. By [32, Chapter IX, Lemma 3.3], we have an isomorphism $\mathcal{H} \leftthreetimes_\phi^N \mathcal{K} \cong \mathcal{H} \leftthreetimes \mathcal{K}$. For relative tensor product, we will use this definition in Section 4 and 5. Also, we can define the right relative tensor product $\mathcal{H} \rightthreetimes_\phi^N \mathcal{K}$ by right ϕ-bounded vectors. We already know the isomorphism $\mathcal{H} \leftthreetimes_\phi^N \mathcal{K} \cong \mathcal{H} \rightthreetimes_\phi^N \mathcal{K}$ and the associativity $(\mathcal{H} \leftthreetimes_\phi^N \mathcal{K}) \rightthreetimes_\psi^P \mathcal{L} \cong \mathcal{H} \leftthreetimes_\phi^N (\mathcal{K} \rightthreetimes_\psi^P \mathcal{L})$ in [32, Chapter IX, Theorem 3.20]. However, they are not enough to show the pentagonal identity in Definition 2.2.1.

2.2. W^***-bicategories.** For the general theory of categories and tensor categories, see [14] and [10]. For convenience, we recall the notion of naturality as the following. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A natural isomorphism from F to G is a family $\{t_C\}_{C \in \mathcal{C}}$ of isomorphisms $t_C : F(C) \to G(C)$ in $\mathcal{D}$ such that for all morphism $f : C \to C'$, the following diagram commutes.

$$\begin{tikzcd} F(C) \arrow[r, "t_C"] \arrow[d, "F(f)"] & G(C) \arrow[d, "G(f)"] \\ F(C') \arrow[r, "t_{C'}"] & G(C') \end{tikzcd}$$

Definition 2.2.1. *A bicategory $\mathcal{B}$ consists of objects $A, B, C, \cdots$ and linear categories ${}_A\mathcal{B}_B$ labeled by a pair (A, B) of objects A and B satisfying the following properties.*

(1) *There is a bifunctor (called a tensor functor) $\otimes : {}_A\mathcal{B}_B \times {}_B\mathcal{B}_C \to {}_A\mathcal{B}_C$ for each three objects A, B and C.*
(2) *There is an object I_A called a unit object in ${}_A\mathcal{B}_A$ for each abject A such that there are natural isomorphisms $\{l_X : I_A \otimes X \to X\}_{X \in {}_A\mathcal{B}_B}$ called a left unit isomorphism and $\{r_X : X \otimes I_B \to X\}_{X \in {}_A\mathcal{B}_B}$ called a right unit isomorphism for each objects A and B.*
(3) *There is a natural isomorphism $\{a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z) \mid X \in {}_A\mathcal{B}_B,\ Y \in {}_B\mathcal{B}_C,\ Z \in {}_C\mathcal{B}_D\}$ called an associativity isomorphism for each objects A, B, C and D.*
(4) *The isomorphisms l_X, r_X and $a_{X,Y,Z}$ satisfy the following commutativity.*

$$\begin{tikzcd} (X \otimes I) \otimes Y \arrow[rr] \arrow[dr] & & X \otimes (I \otimes Y) \arrow[dl] \\ & X \otimes Y & \end{tikzcd}$$

$$\begin{tikzcd} & (X \otimes Y) \otimes (Z \otimes W) \arrow[dr] & \\ ((X \otimes Y) \otimes Z) \otimes W \arrow[ur] \arrow[d] & & X \otimes (Y \otimes (Z \otimes W)) \\ (X \otimes (Y \otimes Z)) \otimes W \arrow[rr] & & X \otimes ((Y \otimes Z)) \otimes W) \arrow[u] \end{tikzcd}$$

The left commutativity is called the triangle identity and the right one is called the pentagonal identity.

For the naturalities of $\{l_X\}$, $\{r_x\}$ and $\{a_{X,Y,Z}\}$, related functors are not specified in the above definition. However it is clear, for example the left unit isomorphism

$\{l_X\}_{X \in {}_A\mathcal{B}_B}$ is a natural isomorphism from the functor ${}_A\mathcal{B}_B \ni X \to I_A \otimes X \in {}_A\mathcal{B}_B$ to the identity functor on ${}_A\mathcal{B}_B$.

The triangle identity implies the following lemma which will be used in Section 3.

Lemma 2.2.2. ([10, Proposition 2.2.4]) *For any objects X in ${}_A\mathcal{B}_B$ and Y in ${}_B\mathcal{B}_C$, the following diagrams are commutative.*

$$\begin{tikzcd}
(I_A \otimes X) \otimes Y \arrow[rr] \arrow[dr] & & I_A \otimes (X \otimes Y) \arrow[dl] \\
& X \otimes Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
(X \otimes Y) \otimes I_C \arrow[rr] \arrow[dr] & & X \otimes (Y \otimes I_C) \arrow[dl] \\
& X \otimes Y &
\end{tikzcd}$$

Remark 2.2.3. *Any bicategory satisfies the coherence theorem in the following sense. For $m, n \in \mathbb{N}$, if $X, Y \in {}_A\mathcal{B}_B$ are objects given by tensor products of composable m and n objects with any order of parentheses, and $f_1, f_2 : X \to Y$ are isomorphisms given by products of the left and right isomorphisms and the associativity isomorphisms and their inverses, then we have $f_1 = f_2$. The above lemma is a special case of the coherence.*

We shall give the formal definition of W^*-bicategories.

Definition 2.2.4. *A linear category $\mathcal{C}$ is called a C^*-category if there is an antilinear contravariant functor $* : \mathcal{C} \to \mathcal{C}$ preserving any objects, for every objects X and Y, the space* $\mathrm{Hom}(X, Y)$ *of morphisms from X to Y is a Banach space satisfying the following properties.*

(1) *For every $x \in \mathrm{Hom}(X, Y)$, x^*x is a positive element in* $\mathrm{Hom}(X, X)$ *and we have $\|x\|^2 = \|x^*x\|$.*
(2) *For every $x \in \mathrm{Hom}(Y, Z)$ and $y \in \mathrm{Hom}(X, Y)$, we have $\|xy\| \le \|x\|\|y\|$.*

We require functors between C^*-categories to be $*$-preserving and natural isomorphisms between such functors to consist of unitaries.

If each $\mathrm{Hom}(X, Y)$ *is the dual of a Banach space, $\mathcal{C}$ is called W^*-categories.*

A C^*-bicategory (W^*-bicategory) is a bicategories consisting of C^*-category (W^*-categories) such that the left and the right unit isomorphisms and the associativity isomorphism consist of unitaries.

Example 2.2.5. (1) *The category whose objects are Hilbert spaces and a morphism is a bounded operator, is a W^*-categories.*
(2) *Let M be a von Neumann algebra. The category whose objects are projections in M and a morphism from a projection p to a projection q is qxp for some $x \in M$, is a W^*-category.*

An involutive W^*-bicategory is a W^*-bicategory $\mathcal{B}$ equipped with a family of contravariant functors denoted by

$${}_A\mathcal{B}_B \ni X \mapsto X^* \in {}_B\mathcal{B}_A, \qquad \mathrm{Hom}(X, Y) \ni f \mapsto {}^t f \in \mathrm{Hom}(Y^*, X^*),$$

a natural unitary isomorphisms $\{c_{X,Y} : Y^* \otimes X^* \to (X \otimes Y)^* \mid X \in {}_A\mathcal{B}_B,\ Y \in {}_B\mathcal{B}_C\}$ and $\{d_X : X \to (X^*)^*\}_{X \in {}_A\mathcal{B}_B}$ satisfying ${}^t d_X = d_{X^*}^{-1}$ for all object $X \in {}_A\mathcal{B}_B$, ${}^t(f^*) =$

$({}^t f)^*$ for all $f \in \mathrm{Hom}(X, Y)$ and the following commutativities.

$$\begin{tikzcd}
((X \otimes Y) \otimes Z)^* \arrow[r] \arrow[d] & (X \otimes (Y \otimes Z))^* \arrow[d] \\
Z^* \otimes (X \otimes Y)^* \arrow[d] & (Y \otimes Z)^* \otimes X^* \arrow[d] \\
Z^* \otimes (Y^* \otimes X^*) \arrow[r] & (Z^* \otimes Y^*) \otimes X^*
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes Y \arrow[r] \arrow[d] & X^{**} \otimes Y^{**} \arrow[d] \\
(X \otimes Y)^{**} \arrow[r] & (Y^* \otimes X^*)^*
\end{tikzcd}$$

The left and right diagrams are called the hexagon identity and the square identity, respectively. The triple $(*, \{c_{X,Y}\}, \{d_X\})$ is called a unitary involution on $\mathcal{B}$. For involutions on tensor categories, see [36].

Finally, we shall provide the notion of monoidal equivalence between two (involutive) W^*-bicategories with same objects by the identify functor. The following definition is a special case of the general definition of (involutionally and) monoidal equivalences, however it is enough for the aims in Section 3.

Definition 2.2.6. *Two W^*-bicategories $\mathcal{B} = (\{A\}, \{{}_A\mathcal{B}_B\}, \{\otimes_{A,B,C}\}, \{a_{X,Y,Z}\})$ and $\mathcal{B}' = (\{A\}, \{{}_A\mathcal{B}_B\}, \{\otimes'_{A,B,C}\}, \{a'_{X,Y,X}\})$ are said to be monoidally equivalent if for each objects A, B and C, there is a natural unitary isomorphism $\{m_{X,Y} : X \otimes Y \to X \otimes' Y \mid X \in {}_A\mathcal{B}_B,\ Y \in {}_B\mathcal{B}_C\}$ such that the following diagram commutes for every composable objects X, Y and Z.*

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "m_{X,Y} \otimes \mathrm{id}_Z"] \arrow[d, "a_{X,Y,Z}"] & (X \otimes' Y) \otimes Z \arrow[r, "m_{X \otimes' Y, Z}"] & (X \otimes' Y) \otimes' Z \arrow[d, "a'_{X,Y,Z}"] \\
X \otimes (Y \otimes Z) \arrow[r, "\mathrm{id}_X \otimes m_{Y,Z}"] & X \otimes (Y \otimes' Z) \arrow[r, "m_{X, Y \otimes' Z}"] & X \otimes' (Y \otimes' Z)
\end{tikzcd}$$

In addition, suppose $\mathcal{B}$ and $\mathcal{B}'$ have unitary involutions $\{c_{X,Y}\}$ and $\{c'_{X,Y}\}$, respectively. If the following diagram commutes for all composable objects X and Y, then $\mathcal{B}$ and $\mathcal{B}'$ are said to be involutionally and monoidal equivalent with respect to $\{m_{X,Y}\}$.

$$\begin{tikzcd}
Y^* \otimes X^* \arrow[r, "c_{X,Y}"] \arrow[d, "m_{Y^*,X^*}"] & (X \otimes Y)^* \\
Y^* \otimes' X^* \arrow[r, "c'_{X,Y}"] & (X \otimes' Y)^* \arrow[u, "{}^t m_{X,Y}"']
\end{tikzcd}$$

2.3. **Hilbert bimodules.** We refer the reader to [12] for the general theory of Hilbert modules. For a von Neumann algebra M, a Hilbert M-module E is a right M-module with an M-valued inner product such that $\langle X, Yx \rangle = \langle X, Y \rangle x$ for all $X, Y \in E$ and $x \in M$, and E is complete with respect to a norm defined by $\|X\| = \sqrt{\|\langle X, X \rangle\|}$. For Hilbert M-module E and F, a bounded right module homomorphism $b : E \to F$ is said to be adjointable if there is a bounded right module homomorphism $b^* : F \to E$ satisfying $\langle X, bY \rangle = \langle b^* X, Y \rangle$ for all $Y \in E$ and $X \in F$. We denote the set of all adjointable bounded right module homomorphisms from E to F by $\mathcal{B}^a(E, F)$. We write a C^*-algebra $\mathcal{B}^a(E, E)$ by $\mathcal{B}^a(E)$. If there is a surjection $u \in \mathcal{B}^a(E, F)$ preserving the inner products (that is, unitary), E and F

are said to be isomorphic as Hilbert M-modules. For another von Neumann algebra N, when there is a $*$-homomorphism $j : N \to \mathcal{B}^a(E)$, E is called a Hilbert N-M-bimodule or a C^*-correspondence from N to M. We assume that the unit of N acts as a unit on E. (This means the non-degeneracy.)

Now, we discuss some tensor products related with Hilbert modules. In this process, we will define a von Neumann modules as a complete Hilbert module with respect to a suitable topology.

Definition 2.3.1. *For a Hilbert N-M-bimodule E and a Hilbert M-P-bimodule F, the tensor product $E \odot_M F$ of E and F over M is a Hilbert N-P-bimodule given by the completion of the algebraic tensor product $E \otimes_{\mathrm{alg}} F$ with respect to a P-valued sesquilinear form defined by*

$$\langle X \otimes Y, X' \otimes Y' \rangle = \langle Y, \langle X, X' \rangle Y' \rangle$$

for each $X, X' \in E$ and $Y, Y' \in F$. (Precisely, we take the completion of a quotient space $E \otimes_{\mathrm{alg}} F/\mathcal{N}$, where $\mathcal{N} = \{Z \in E \otimes_{\mathrm{alg}} F \mid \langle Z, Z \rangle = 0\}$. We often omit this argument when we define new tensor products.)

Note that the tensor product of Hilbert bimodules is associative.

Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and E be a Hilbert M-module. Then $\mathcal{H}$ and E are a Hilbert M-$\mathbb{C}$-bimodule and a Hilbert $\mathbb{C}$-M-bimodule, respectively, and hence we can define the tensor product $E \odot_M \mathcal{H}$ as Hilbert bimodules. We can regard E as a right M-submodule of $\mathcal{B}(\mathcal{H}, E \odot_M \mathcal{H})$ by an embedding $L_X : \mathcal{H} \ni \xi \mapsto X \otimes \xi \in E \odot_M \mathcal{H}$ for each $X \in E$. If $E \subset \mathcal{B}(\mathcal{H}, E \odot_M \mathcal{H})$ is closed with respect to the strong operator topology, E is called a von Neumann M-module. Suppose N is another von Neumann algebra. A von Neumann M-module E is called a von Neumann N-M-bimodule if E is a Hilbert N-M-bimodule, and a map $\rho : N \to \mathcal{B}(E \odot_M \mathcal{H})$ defined by

$$\rho(x)(\xi \otimes h) = x\xi \otimes h \qquad (x \in N\ \xi \in E,\ h \in \mathcal{H})$$

is normal.

Example 2.3.2. *A C^*-algebra A and a von Neumann algebra M are a Hilbert A-bimodule and a von Neumann M-bimodule, respectively by the canonical inner product $\langle x, y \rangle = x^* y$. For a right W^*-M-module, the space* $\mathrm{Hom}(L^2(M)_M, \mathcal{H}_M)$ *is a von Neumann M-module.*

For a von Neumann M-N-bimodule E and a von Neumann N-P-bimodule F, the strong closure denoted by $E \otimes_N F$ of $E \odot_N F$ via the above argument is a von Neumann M-P-bimodule.

Like von Neumann algebras, von Neumann modules are also characterized by self-duality, see [18] and [26].

We define the tensor product of Hilbert modules with respect to a UCP-map T whose inner products have information for T, like GNS-representation of C^*-algebra A with respect to a positive functional ϕ on A.

Definition 2.3.3. *Suppose M is a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and $T : M \to M$ is a normal unital completely positive map (UCP-map). We* *define a Hilbert M-M-bimodule $\mathcal{E}_T$ by the completion of algebraic tensor product $M \otimes_{\mathrm{alg}} M$ with respect to an M-valued sesquilinear form defined by*

$$\langle x \otimes y, x' \otimes y' \rangle_T = y^* T(x^* x') y'$$

for each $a, a', b, b' \in M$. The left and right actions are defined naturally. If we write the image of $1_M \otimes 1_M$ in $\mathcal{E}_T$ by Z_T, then $\mathrm{span}(M Z_T M)$ *is dense in $\mathcal{E}_T$ and $T(x) = \langle Z_T, x Z_T \rangle$ holds for all $x \in M$. We call the couple $(\mathcal{E}_T, Z_T)$ the GNS-representation with respect to T.*

Note that the strong closure $\overline{\mathcal{E}_T}$ in $\mathcal{B}(\mathcal{H}, \mathcal{E}_T \otimes_M \mathcal{H})$ is a von Neumann M-M-bimodule. The following tensor product will be helpful for the construction of minimal dilation by Muhly-Solel in Subsection 4.1 and the construction of relative product systems from CP_0-semigroups in Subsection 4.3 and Section 5.

Definition 2.3.4. *Suppose M is a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and $T : M \to M$ is a normal unital completely positive map (UCP-map) The tensor product $M \otimes_T \mathcal{H}$ of M and $\mathcal{H}$ with respect to T is defined as a Hilbert space given by the completion of the algebraic tensor product $M \otimes_{\mathrm{alg}} \mathcal{H}$ with respect to a sesquilinear form defined by*

$$\langle x \otimes \xi, y \otimes \eta \rangle = \langle \xi, T(x^* y) \eta \rangle$$

for each $x, y \in M$ and $\xi, \eta \in \mathcal{H}$. $M \otimes_T \mathcal{H}$ has a left W^*-M-module structure by $a(x \otimes \xi) = (ax) \otimes \xi$ for all $a, x \in M$ and $\xi \in \mathcal{H}$.

Finally, we see a relation between the above tensor products and the GNS-representations:

Proposition 2.3.5. *Let T be a normal UCP-map on a von Neumann algebra M and $\mathcal{H}$ a left W^*-M-module. Then we have $\mathcal{E}_T \otimes_M \mathcal{H} \cong M \otimes_T \mathcal{H}$ as left W^*-M-modules.*

Proof. We can define a left M-module map $u : \mathcal{E}_T \otimes_M \mathcal{H} \to M \otimes_T \mathcal{H}$ by $u((x \otimes y) \otimes \xi) = x \otimes y\xi$ for each $x, y \in M$ and $\xi \in \mathcal{H}$. Then we have

$$\begin{aligned}
\langle u((x_1 \otimes y_1) \otimes \xi_1), u((x_2 \otimes y_2) \otimes \xi_2) \rangle &= \langle x_1 \otimes y_1 \xi_1, x_2 \otimes y_2 \xi_2 \rangle \\
&= \langle y_1 \xi_1, T(x_1^* x_2) y_2 \xi_2 \rangle \\
&= \langle \xi_1, \langle x \otimes y_1, x_2 \otimes y_2 \rangle_T \xi_2 \rangle \\
&= \langle (x_1 \otimes y_1) \otimes \xi_1, (x_2 \otimes y_2) \otimes \xi_2 \rangle,
\end{aligned}$$

and hence it is a unitary. □

2.4. **CP_0-semigroups, E_0-semigroups and Arveson's product systems.** In this subsection, we provide the basic notion related with CP_0-semigroups and E_0-semigroups. Also, we simply explain Arveson's classification of E_0-semigroups on type I factors by his product systems.

We provide the formal definitions of CP_0-semigroups and E_0-semigroups as the following. A family $T = \{T_t\}_{t \geq 0}$ of normal UCP-maps T_t on a von Neumann algebra M is called a CP_0-semigroup if $T_0 = \mathrm{id}_M$, $T_s T_t = T_{s+t}$ for all $s, t \geq 0$, and for every $x \in M$ and $\phi \in M_*$, the function $\phi(T_t(x))$ on $[0, \infty)$ is continuous. If each T_t is a $*$-homomorphism, T is called an E_0-semigroup. A CP_0-semigroup (E_0-semigroup) without the continuity is called an algebraic CP_0-semigroup (algebraic E_0-semigroup, respectively).

Example 2.4.1. *Let $\{v_t\}_{t\geq 0}$ be a family of isometries v_t in a von Neumann algebra M such that $v_{s+t} = v_s v_t$ for all $s,t \geq 0$ and $v_0 = 1_M$. Suppose $\{v_t\}_{t\geq 0}$ is strongly continuous with respect to the parameter. If we define $T = \{T_t\}_{t\geq 0}$ by $T_t(x) = v_t^* x v_t$ for each $x \in M$ and $t \geq 0$, then T is a CP_0-semigroup. If each v_t is unitary, T is an E_0-semigroup.*

Example 2.4.2. *The CCR heat flow is a CP_0-semigroup T which has the noncommutative Laplacian Δ as generators. This will be immediately and concretely defined by the Weyl system. For more details, see* [4, Section 7].

Let $\mathcal{H} = L^2(\mathbb{R})$ and $M = \mathcal{B}(\mathcal{H})$. For $\mathbf{x} = (x,y) \in \mathbb{R}^2$, the concrete Weyl operator is $W_{\mathbf{x}} = \exp(\frac{xy}{2}i)U_x V_y$, where $\{U_x\}_{x\in\mathbb{R}}$ and $\{V_x\}_{x\in\mathbb{R}}$ are the unitary groups which have the position operator Q and the momentum operator P as generators, respectively, i.e.

$$(U_t f)(x) = e^{itx} f(x),\ (V_t f)(x) = f(x+t)$$

for $f \in L^2(\mathbb{R})$ and $t, x \in \mathbb{R}$. Then the family $\{W_{\mathbf{x}}\}_{\mathbf{x}\in\mathbb{R}^2}$ of the unitaries satisfies the Weyl relations

$$W_{\mathbf{x}_1} W_{\mathbf{x}_2} = \exp\left(\frac{i}{2}(x_2 y_2 - x_1 y_2)\right) W_{\mathbf{x}_1+\mathbf{x}_2} \tag{2.1}$$

for $\mathbf{x}_1 = (x_1, y_1), \mathbf{x}_2 = (x_2, y_2) \in \mathbb{R}^2$. The CCR heat flow is defined as the unique CP_0-semigroup $T = \{T_t\}_{t\geq 0}$ on M satisfying $T_t(W_{\mathbf{x}}) = \exp(-t\|x\|^2)W_{\mathbf{x}}$ for all $\mathbf{x} \in \mathbb{R}^2$ and $t \geq 0$. More precisely, we define T_t for $t \geq 0$ by a weak integral

$$T_t(x) = \int_{\mathbb{R}^2} W_{\frac{\mathbf{x}}{\sqrt{2}}} x W^*_{\frac{\mathbf{x}}{\sqrt{2}}} d\mu_t(\mathbf{x})$$

for each $x \in M$, where μ_t is the probability measure whose Fourier transformation is $u_t(\mathbf{x}) = \exp(-t\|\mathbf{x}\|^2)$.

According to Stinespring's dilation theorem, for a UCP-map T from a C^*-algebra A into $\mathcal{B}(\mathcal{H})$, there exist a Hilbert space $\mathcal{K}$, a unital representation of A on $\mathcal{K}$ and an isometry $v : \mathcal{H} \to \mathcal{K}$ such that $T(a) = v^*\pi(a)v$ for all $a \in A$. However, Stinespring's theorem does not apply to CP_0-semigroup. The notion of dilation of CP_0-semigroups are introduced as the following.

Definition 2.4.3. *Let $T = \{T_t\}_{t\geq 0}$ be a CP_0-semigroup on a von Neumann algebra M. A dilation of T consists of a von Neumann algebra N, a projection $p \in N$ and an E_0-semigroup $\{\theta_t\}_{t\geq 0}$ on N such that $M = pNp$ and $T_t(x) = p\theta_t(x)p$ for all $x \in M$ and $t \geq 0$. In addition, if N is generated by $\theta_{[0,\infty)}(M)$ and the central support of p in N is 1_N, the dilation is said to be minimal.*

Note that a minimal dilation of a CP_0-semigroup is unique (if it exists). The existence of minimal dilations is proved by Bhat-Skeide and Muhly-solel which we recall in Subsection 4.1. In Subsection 4.4, a relation between the two constructions will be clarified. Arveson also constructed the minimal dilation by other approach in [4] (or [3]).

The notion of cocycle for E_0-semigroups is introduced as the following, and the notion is useful for the classification of E_0-semigroups.

Definition 2.4.4. *Let $\theta = \{\theta_t\}_{t\geq 0}$ be an E_0-semigroup on a von Neumann algebra M. A family $w = \{w_t\}_{t\geq 0} \subset M$ is called a right cocycle for θ if $w_{s+t} = \theta_t(w_s)w_t$ for all $s, t \geq 0$. If each w_t is unitary (contractive), then w is called a right unitary (contractive, respectively) cocycle.*

Two E_0-semigroups $\theta = \{\theta_t\}_{t\geq 0}$ and $\theta' = \{\theta'_t\}_{t\geq 0}$ on M are said to be cocycle equivalent if there is a strongly continuous right unitary cocycle $w = \{w_t\}_{t\geq 0}$ such that $\theta'_t(x) = w_t^*\theta_t(x)w_t$ for all $t \geq 0$ and $x \in M$. Then θ' is called a cocycle perturbation of θ with respect to w.

Definition 2.4.5. *Let θ and θ' be E_0-semigroups on von Neumann algebras M and N, respectively. If there is a $*$-isomorphism $\alpha : M \to N$ such that $\theta'_t \circ \alpha = \alpha \circ \theta_t$ for all $t \geq 0$, then θ and θ' are said to be conjugate. If θ' is conjugate to a cocycle perturbation of θ, then θ and θ' are said to be cocycle conjugate.*

Example 2.4.6. *We give an example of two E_0-semigroups which are conjugate in* [16, Example 3.11]. *Let $M = \mathcal{B}(\mathcal{H})$ with a separable Hilbert space $\mathcal{H}$. The standard space $L^2(M)$ of M is $\mathcal{H} \otimes \mathcal{H}^*$. We use the notations in* Example 2.1.1. *Let θ be an E_0-semigroup on M and $\tilde{\theta}$ the conjugate E_0-semigroup $\pi \circ \theta \circ \pi^{-1}$ on $\pi(M)$. For $x \in \mathcal{B}(\mathcal{H})$, let $\overline{x}$ be the operator on $\mathcal{H}^*$ defined by $\overline{x}(\xi^*) = (x\xi)^*$, and $\overline{\theta}$ the E_0-semigroup defined on $\mathcal{B}(\mathcal{H}^*)$ by $\overline{\theta}_t(\overline{x}) = \overline{\theta_t(x)}$ for each $t \geq 0$ and $x \in M$. Margetts-Srinivasan*[15] *introduced the dual E_0-semigroup $\tilde{\theta}^J$ on $\pi(M)' \subset \mathcal{B}(\mathcal{H}\otimes\mathcal{H}^*)$ by*

$$\tilde{\theta}^J_t(x') = J\tilde{\theta}_t(Jx'J)J$$

for each $t \geq 0$ and $x' \in \pi(M)'$. Then $\tilde{\theta}^J$ is conjugate to $\overline{\theta}$ via an $*$-isomorphism: $\mathcal{B}(\mathcal{H}^*) \ni \overline{x} \mapsto 1_M \otimes \overline{x} \in \pi(M)'$.

Arveson[2] classified E_0-semigroups on type I factors up to cocycle congugacy by his product systems. We introduce a product system in the sense of Arveson as the follows:

Definition 2.4.7. *Let $\mathcal{H}^{\otimes} = \{\mathcal{H}_t\}_{t>0}$ be a family of separable Hilbert spaces $\mathcal{H}_t$. If there is a unitary $U_{s,t} : \mathcal{H}_s \otimes \mathcal{H}_t \to \mathcal{H}_{s+t}$ for each $s, t > 0$ satisfying the associativity*

$$U_{r,s+t}(\mathrm{id}_{\mathcal{H}_r} \otimes U_{s,t}) = U_{r+s,t}(U_{r,s} \otimes \mathrm{id}_{\mathcal{H}_t})$$

for all $r, s, t \geq 0$. Then the pair $(\mathcal{H}^{\otimes}, \{U_{s,t}\}_{s,t>0})$ is called a product system.

Note that we require a product system of a measurable structure in the original definition by Arveson. For more details, see [4] or [13]. We call such a product system a continuous product system. We will introduce the notion of product system of Hilbert bimodules (Bhat-Skeide's product system) and the one of relative product system in Section 4, however we will not consider their measurable structures. Indeed, the construction of minimal dilations and the classification of E_0-semigroups do not require measurable structures of Bhat-Skeide's product systems and relative ones.

For E_0-semigroup $\theta = \{\theta_t\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{H})$ with a separable Hilbert space $\mathcal{H}$, Arveson associated with the (continuous) product system $\mathcal{E}^{\theta\otimes} = \{\mathcal{E}_t^\theta\}_{t>0}$ by

$$\mathcal{E}_t^\theta = \{x \in \mathcal{B}(\mathcal{H}) \mid \theta_t(a)x = xa \ (\forall a \in \mathcal{B}(\mathcal{H}))\}$$

equipped with the inner product defined by $\langle x, y\rangle = x^*y \in \mathcal{B}(\mathcal{H})' = \mathbb{C}1_{\mathcal{B}(\mathcal{H})}$ for each $x, y \in \mathcal{E}_t^\theta$. He showed that two E_0-semigroup θ on $\mathcal{B}(\mathcal{H})$ and θ' on $\mathcal{B}(\mathcal{K})$ are cocycle conjugate if and only if the associated product systems $\mathcal{E}^{\theta\otimes}$ and $\mathcal{E}^{\theta'\otimes}$ are isomorphic ([4, Theorem 2.4.10]). Now, units of product systems are defined as the following.

Definition 2.4.8. *A unit of a (continuous) product system $(\mathcal{H}^\otimes, \{U_{s,t}\}_{s,t>0})$ is a (measurable) family $\{\xi(t)\}_{t>0}$ of $\xi(t) \in \mathcal{H}_t$ for each $t > 0$ satisfying*

$$\xi_{s+t} = U_{s,t}(\xi(s) \otimes \xi(t))$$

for all $s, t > 0$. We denote the set of all units of $\mathcal{H}^\otimes$ by $\mathcal{U}(\mathcal{H}^\otimes)$.

The notion of unit gives the index invariant of E_0-semigroups on type I factors (see [4, Section 2.5]). Let θ be an E_0-semigroup on $\mathcal{B}(\mathcal{H})$. We can define a Hilbert space structure on $\mathcal{H}(\mathcal{E}^\theta)$ of all functions $f : \mathcal{U}(\mathcal{E}^{\theta\otimes}) \to \mathbb{C}$ that are finitely nonzero and sum to zero. The index of θ is defined by $\mathrm{ind}(\theta) = \dim \mathcal{H}(\mathcal{E}^\theta)$ if $\mathcal{U}(\mathcal{E}^{\theta\otimes}) \neq \emptyset$ and $\mathrm{ind}(\theta) = 2^{\aleph_0}$ if $\mathcal{U}(\mathcal{E}^{\theta\otimes}) = \emptyset$. The index is a numerical invariant for cocycle conjugacy: if θ and θ' are cocycle conjugate, then we have $\mathrm{ind}(\theta) = \mathrm{ind}(\theta')$.

3. W^*-bicategories of W^*-bimodules

In this section, we discuss the results in [24]. We will show that two W^*-bicategories $\mathcal{M}^{\leftthreetimes}$ and $\mathcal{M}^{\rightthreetimes}$ of W^*-bimodules with left and right relative tensor product as tensor functors, respectively, are monoidally equivalent in Subsection 3.1. In Subsection 3.2, we will define unitary involutions on $\mathcal{M}^{\leftthreetimes}$ and $\mathcal{M}^{\rightthreetimes}$, and show that they are also involutionally and monoidally equivalent. We will use notations $A, B, C \cdots$ for W^*-algebras and $X, Y, Z, \cdots$ for W^*-bimodules in accordance with Subsection 2.2.

Before we construct $\mathcal{M}^{\leftthreetimes}$ and $\mathcal{M}^{\rightthreetimes}$, we recall matrix extensions in [34] and Dixmier's structure theorem on homomorphisms between von Neumann algebras. Let A be a W^*-algebra, and I and J index sets. A matrix extension of A is defined by

$$M_{I,J}(A) = \mathrm{Hom}(l^2(J) \otimes L^2(A)_A, l^2(I) \otimes L^2(A)_A).$$

In particular, $M_I(A) = M_{I,I}(A)$ is a von Neumann algebra acting on $l^2(I) \otimes L^2(A)$. $M_{I,J}(A)$ can be identified with a subspace of $A^{I\times J} = \{(x_{i,j}) \mid i \in I,\ j \in J,\ x_{i,j} \in A\}$, and it has a dense subspace of $(x_{i,j})$ such that $x_{i,j} = 0$ except for finite numbers of $i \in I$ and $j \in J$ in the strong operator topology. For a W^*-bimodule ${}_AX_B$, we define a matrix extension of X by

$${}^IX^J = \left\{ (\xi_{i,j}) \mid i \in I,\ j \in J,\ \xi_{i,j} \in X,\ \sum_{i\in I, j\in J} \|\xi_{i,j}\|^2 < \infty \right\}$$

which is a W^*-$M_I(A)$-$M_J(B)$-bimodule. If I or J is any singleton, we denote ${}^IX^J$ by X^J or ${}^IX$, respectively.

Dixmier's theorem asserts the following projective module realization (see [31]). For W^*-A-B-bimodule X, there exist an index set I and a projection $p \in M_I(A)$ such that $X \cong L^2(A)^I p$ as left W^*-modules, and there is a normal homomorphism $B \to \mathrm{End}({}_AL^2(A)^I p)^\circ \cong pM_I(A)p$. It is also similar as right W^*-modules.

We shall define two W^*-bicategories $\mathcal{M}^{\leftthreetimes}$ and $\mathcal{M}^{\rightthreetimes}$. Of course, their objects are W^*-algebras, and for W^*-algebra A and B, both of W^*-categories ${}_A\mathcal{M}_B^{\leftthreetimes}$ and ${}_A\mathcal{M}_B^{\rightthreetimes}$ is the W^*-categories of W^*-A-B-bimodules. Suppose ${}_A\mathcal{M}_B^{\leftthreetimes}$ and ${}_A\mathcal{M}_B^{\rightthreetimes}$ have tensor functor $\leftthreetimes$ and $\rightthreetimes$, respectively. For W^*-A-B-bimodule ${}_AX_B$, unit isomorphisms $l_X^{\leftthreetimes}, r_X^{\leftthreetimes}$ in ${}_A\mathcal{M}_B^{\leftthreetimes}$ and $l_X^{\rightthreetimes}, r_X^{\rightthreetimes}$ in ${}_A\mathcal{M}_B^{\rightthreetimes}$ are defined by

$$l_X^{\leftthreetimes}(a \otimes \xi) = a\xi,\ r_X^{\leftthreetimes}(x \otimes \beta) = y\beta,\ l_X^{\rightthreetimes}(\alpha \otimes x) = \alpha x,\ r_X^{\rightthreetimes}(\xi \otimes b) = \xi b.$$

for each $a \in A$, $b \in B$, $\alpha \in L^2(A)$, $\beta \in L^2(B)$, $x \in \mathrm{Hom}({}_AL^2(A), {}_AX)^\circ$, $y \in \mathrm{Hom}(L^2(B)_B, X_B)$ and $\xi \in X$. Let ${}_AX_B, {}_BY_C$ and ${}_CZ_D$ be W^*-bimodules. Note that the following projective module realizations imply that

$$\begin{aligned} &\mathrm{Hom}(L^2(B)_B, X_B) \otimes_B \mathrm{Hom}(L^2(C)_C, Y_C) \otimes_C Z \subset X \leftthreetimes (Y \leftthreetimes Z), \\ &X \otimes_B \mathrm{Hom}({}_BL^2(B), {}_BY)^\circ \otimes_C \mathrm{Hom}({}_CL^2(C), {}_CZ)^\circ \subset X \rightthreetimes (Y \rightthreetimes Z) \end{aligned}$$

are dense. Hence, we can define associativity isomorphisms $a_{X,Y,Z}^{\leftthreetimes}$ in ${}_A\mathcal{M}_B^{\leftthreetimes}$ and $a_{X,Y,Z}^{\rightthreetimes}$ in ${}_A\mathcal{M}_B^{\rightthreetimes}$ by

$$a_{X,Y,Z}^{\leftthreetimes}((x \otimes y) \otimes \zeta) = x \otimes (y \otimes \zeta),\ a_{X,Y,Z}^{\rightthreetimes}((\xi \otimes z) \otimes w) = \xi \otimes (z \otimes w)$$

for each $x \in \mathrm{Hom}(L^2(B)_B, X_B)$, $y \in \mathrm{Hom}(L^2(C)_C, Y_C)$, $\zeta \in Z$, $\xi \in X$, $z \in \mathrm{Hom}({}_B L^2(B), {}_B Y)^\circ$ and $w \in \mathrm{Hom}({}_C L^2(C), {}_C Z)^\circ$.

We can check that the unit isomorphism $\{l^{\leftthreetimes}_X\}, \{r^{\leftthreetimes}_X\}$ and the associativity isomorphism $\{a^{\leftthreetimes}_{X,Y,Z}\}$ in $\mathcal{M}^{\leftthreetimes}$ satisfy the triangle and the pentagonal identity by the definitions, and similarly for $\mathcal{M}^{\rightthreetimes}$.

3.1. **Monoidal equivalence.** In this subsection, we shall construct a natural unitary isomorphism $\{m_{X,Y} : X \leftthreetimes Y \to X \rightthreetimes Y\}$ giving a monoidal equivalence in the sense of Definition 2.2.6 by the explicit and canonical way.

Let ${}_A X_B$ and ${}_B Y_C$ be W^*-bimodules. Suppose $u : X \to p^I L^2(B)$ and $v : Y \to L^2(B)^J q$ are B-linear unitaries for some index sets I and J and projections $p \in M_I(B)$ and $q \in M_J(B)$. Note that $({}^I L^2(B)) \leftthreetimes (L^2(B)^J)$ and $({}^I L^2(B)) \rightthreetimes (L^2(B)^J)$ can be identified with ${}^I(L^2(B) \leftthreetimes L^2(B))^J$ and ${}^I(L^2(B) \rightthreetimes L^2(B))^J$, respectively. Thus, the unit isomorphisms $l^{\leftthreetimes} = r^{\leftthreetimes} : L^2(B) \leftthreetimes L^2(B) \to L^2(B)$ and $l^{\rightthreetimes} = r^{\rightthreetimes} : L^2(B) \rightthreetimes L^2(B) \to L^2(B)$ induce $M_I(B)$-$M_J(B)$-linear unitaries

$$m_\Box : ({}^I L^2(B)) \Box (L^2(B)^J) \cong {}^I(L^2(B) \Box L^2(B))^J \to {}^I L^2(B)^J,$$

where $\Box$ is $\leftthreetimes$ or $\rightthreetimes$. We define an $M_I(B)$-$M_J(B)$-linear unitary ${}^I m^J = (m_{\rightthreetimes})^* m_{\leftthreetimes}$: $({}^I L^2(B)) \leftthreetimes (L^2(B)^J) \to ({}^I L^2(B)) \rightthreetimes (L^2(B)^J)$.

Definition 3.1.1. *We define a unitary $m_{X,Y} : X \leftthreetimes Y \to X \rightthreetimes Y$ by the following diagram.*

$$\begin{tikzcd}
X \leftthreetimes Y \arrow[r, "u \leftthreetimes v"] \arrow[d, "m_{X,Y}"] & (p^I L^2(B)) \leftthreetimes (L^2(B)^J q) = p(({}^I L^2(B)) \leftthreetimes (L^2(B)^J))q \arrow[d, "{}^I m^J"] \\
X \rightthreetimes Y \arrow[r, "u \rightthreetimes v"] & ((p^I L^2(B)) \rightthreetimes (L^2(B)^J q) = p(({}^I L^2(B)) \rightthreetimes (L^2(B)^J))q
\end{tikzcd}$$

Since ${}^I m^J$ is $M_I(B)$-$M_J(B)$-linear and we have the isomorphisms

$$\mathrm{End}(p^I L^2(B)_B) \cong pM_I(B)p, \ \mathrm{End}({}_B L^2(B)^J q)^\circ \cong qM_J(B)q,$$

$m_{X,Y}$ is A-C-linear. The naturalities of $\{m_{X,Y}\}$ is then reduced to those of the left and the right unit isomorphisms.

Theorem 3.1.2. *The W^*-bicategories $\mathcal{M}^{\leftthreetimes}$ and $\mathcal{M}^{\rightthreetimes}$ are monoidally equivalent.*

Proof. For a W^*-A-B-bimodule X, by the definitions of $m_{L^2(A),X}$ and $m_{X,L^2(A)}$, the following diagrams commute.

$$\begin{tikzcd}
L^2(A) \leftthreetimes X \arrow[rr, "m_{L^2(A),X}"] \arrow[dr, "l^{\leftthreetimes}_X"'] & & L^2(A) \rightthreetimes X, \arrow[dl, "l^{\rightthreetimes}_X"] \\
& X &
\end{tikzcd}
\quad
\begin{tikzcd}
X \leftthreetimes L^2(B) \arrow[rr, "m_{X,L^2(B)}"] \arrow[dr, "r^{\leftthreetimes}_X"'] & & X \rightthreetimes L^2(B). \arrow[dl, "r^{\rightthreetimes}_X"] \\
& X &
\end{tikzcd} \tag{3.1}$$

Let ${}_A X_B$, ${}_B Y_C$ and ${}_C Z_D$ be W^*-bimodules. We shall check that the hexagonal diagram in Definition 2.2.6 commutes. It is enough to show it in the case when

$X = L^2(B)$ and $Z = L^2(C)$ by projective module realizations. For simplicity, we put $I_B = L^2(B)$ and $I_C = L^2(C)$. We consider the following diagram.

$$\begin{tikzcd}
(I_B \leftthreetimes Y) \leftthreetimes I_C \arrow[rr] \arrow[dr] \arrow[dddd] \arrow[dddr, phantom, "\textcircled{1}"] & & (I_B \rightthreetimes Y) \leftthreetimes I_C \arrow[rr] \arrow[dr] \arrow[dl, phantom, "\textcircled{2}"] \arrow[drr, phantom, "\textcircled{3}"] & & (I_B \rightthreetimes Y) \rightthreetimes I_C \arrow[dl] \arrow[dddd] \arrow[dddl, phantom, "\textcircled{1}"] \\
& I_B \leftthreetimes Y \arrow[rr] \arrow[dr] \arrow[dr, phantom, "\textcircled{3}", bend left] \arrow[ddr, phantom, "\textcircled{2}"] & & I_B \rightthreetimes Y \arrow[dl] \arrow[ddl, phantom, "\textcircled{2}"] & \\
& & Y & & \\
& Y \leftthreetimes I_C \arrow[ur] \arrow[rr] \arrow[ur, phantom, "\textcircled{3}"] \arrow[dr, phantom, "\textcircled{2}"] & & Y \rightthreetimes I_C \arrow[ul] \arrow[d, phantom, "\textcircled{3}"] & \\
I_B \leftthreetimes (Y \leftthreetimes I_C) \arrow[rr] \arrow[uuur] \arrow[ur] & & I_B \leftthreetimes (Y \rightthreetimes I_C) \arrow[rr] \arrow[ur] & & I_B \rightthreetimes (Y \rightthreetimes I_C), \arrow[ul] \arrow[uuul]
\end{tikzcd}$$

The diagrams around ① commute by the triangle identity of the unit isomorphisms of Lemma 2.2.2, ② commute by the naturalities of the unit isomorphisms and ③ commute by (3.1). □

3.2. **Unitary involutions and involutionally equivalence.** In this subsection, we define unitary involutions on $\mathcal{M}^{\leftthreetimes}$ and $\mathcal{M}^{\rightthreetimes}$, and show that they are involutionally and monoidally equivalent with respect to the natural unitary isomorphism $\{m_{X,Y}\}_{X,Y}$ which was constructed in Subsection 3.1.

For a W^*-A-B-bimodule X, the dual Hilbert space X^* has the canonical W^*-B-A-bimodule structure: $b\xi^*a = (a^*\xi b^*)^*$ for each $a \in A$, $b \in B$ and $\xi \in X$, where the notation ξ^* means the linear functional $\xi^* : X \ni \eta \mapsto \langle \xi, \eta \rangle$. We can take the transpose ${}^t f : Y^* \to X^*$ for each A-B-linear map f from ${}_AX_B$ to ${}_AY_B$, more precisely ${}^t f$ is defined by $({}^t f\eta^*)(\xi) = \langle \eta, f\xi \rangle$ for each $\xi \in X$ and $\eta \in Y$. The operation gives a contravariant functor ${}_A\mathcal{M}_B \to {}_B\mathcal{M}_A$. It is well known that $(X^*)^*$ can be identified with X by the canonical isomorphism $d_X : X \ni \xi \mapsto (\xi^*)^* \in (X^*)^*$ satisfying ${}^t d_X = d_{X^*}^{-1}$. Note that the family $\{d_X\}_X$ is natural.

Now, we shall define natural unitary isomorphisms $\{c^{\square}_{X,Y} : Y^* \square X^* \to (X \square Y)^*\}_{X,Y}$, where $\square$ is $\leftthreetimes$ or $\rightthreetimes$. Fix W^*-bimodules ${}_AX_B$ and ${}_BY_C$. For each $x \in \mathrm{Hom}(L^2(B)_B, X_B)$, we define a conjugation $\overline{x} \in \mathrm{Hom}({}_BL^2(B), {}_BX^*)$ of x by $\overline{x}(\beta) = (x(J\beta))^*$ for each $\beta \in L^2(B)$, where J is the modular conjugation. We denote $\overline{x}$ regarded as in $\mathrm{Hom}({}_BL^2(B), {}_BX^*)^\circ$ by $\Psi_X(x)$. Then $\Psi_X : \mathrm{Hom}(L^2(B)_B, X_B) \to \mathrm{Hom}({}_BL^2(B), {}_BX^*)^\circ$ is an isometric isomorphism satisfying

$$\Psi_X(axb) = b^*\Psi_X(x)a^*, \; x^*x' = \Psi_X(x')\Psi_X(x)^* \tag{3.2}$$

for all $x, x' \in \mathrm{Hom}(L^2(B)_B, X_B)$, $a \in A$ and $b \in B$.

Proposition 3.2.1. *We define $c_{X,Y} : Y^* \rightthreetimes X^* \ni \eta \otimes \Psi(x) \mapsto (x \otimes \eta)^* \in (X \leftthreetimes Y)^*$. Then $\{c_{X,Y}\}_{X,Y}$ is a unitary satisfying the following commutativity for all*

W^*-bimodules ${}_AX_B, {}_BY_C$ and ${}_CZ_D$.

$$\begin{tikzcd}
(Z^* \leftthreetimes Y^*) \leftthreetimes X^* \arrow[r, "c_{Y,Z} \leftthreetimes \mathrm{id}_{X^*}"] \arrow[d, "a^{\leftthreetimes}_{Z^*,Y^*,X^*}"] & (Y \rightthreetimes Z)^* \leftthreetimes X^* \arrow[r, "c_{X,Y \rightthreetimes Z}"] & (X \rightthreetimes (Y \rightthreetimes Z))^* \arrow[d, "{}^t a^{\rightthreetimes}_{X,Y,Z}"] \\
Z^* \leftthreetimes (Y^* \leftthreetimes X^*) \arrow[r, "\mathrm{id}_{Z^*} \leftthreetimes c_{X,Y}"] & Z^* \leftthreetimes (X \rightthreetimes Y)^* \arrow[r, "c_{X \rightthreetimes Y, Z}"] & ((X \rightthreetimes Y) \rightthreetimes Z)^*
\end{tikzcd}$$

Proof. The property (3.2) implies an equality $\|\eta^* \otimes \Psi_X(x)\| = \|x \otimes \eta\|$. Also, the morphism $c_{X,Y}$ has a dense range of vectors of the form $\eta^* \otimes \Psi_X(x)$, and hence $c_{X,Y}$ is a unitary.

For every $x \in \mathrm{Hom}(L^2(B)_B, X_B)$ and $y \in \mathrm{Hom}(L^2(C)_C, X_C)$, we have

$$\Psi_{X \rightthreetimes Y}(x \otimes y) = c_{X,Y}(\Psi_Y(y) \otimes \Psi_X(x)),$$

because we have

$$\begin{aligned}
&\gamma(c_{X,Y}(\Psi_Y(y) \otimes \Psi_X(x))) = c_{X,Y}(\gamma\Psi_Y(y) \otimes \Psi_X(x)) = c_{X,Y}((y(J\gamma))^* \otimes \Psi_X(x)) \\
&= ((x \otimes y)(J\gamma))^* = \gamma\Psi_{X \rightthreetimes Y}(x \otimes y)
\end{aligned}$$

for all $\gamma \in L^2(C)$. Thus, we have the following commutative diagram.

$$\begin{tikzcd}
(\zeta^* \otimes \Psi_Y(y)) \otimes \Psi_X(x) \arrow[r, "c_{Y,Z} \leftthreetimes \mathrm{id}_{X^*}"] \arrow[d, "a^{\leftthreetimes}_{Z^*,Y^*,X^*}"] & (y \otimes \zeta)^* \otimes \Psi_X(x) \arrow[r, "c_{X,Y \rightthreetimes Z}"] & (x \otimes (y \otimes \zeta))^* \arrow[d, "{}^t a^{\rightthreetimes}_{X,Y,Z}"] \\
\zeta^* \otimes (\Psi_Y(y) \otimes \Psi_X(x)) \arrow[r, "\mathrm{id}_{Z^*} \leftthreetimes c_{X,Y}"] & \zeta^* \otimes \Psi_{\rightthreetimes Y}(x \otimes y) \arrow[r, "c_{X \rightthreetimes Y, Z}"] & ((x \otimes y) \otimes \zeta)^*
\end{tikzcd}$$

for all $\zeta^* \in Z^*,\ x \in \mathrm{Hom}(L^2(B)_B, X_B)$ and $y \in \mathrm{Hom}(L^2(C)_C, X_C)$. By the surjectivity of Ψ, the proof is completed. $\square$

Definition 3.2.2. *We define natural unitary isomorphisms $\{c^{\leftthreetimes}_{X,Y}\}_{X,Y}$ and $\{c^{\rightthreetimes}_{X,Y}\}_{X,Y}$ by*

$$\begin{aligned}
c^{\leftthreetimes}_{X,Y} &= {}^t m^{-1}_{X,Y} c_{X,Y} : Y^* \leftthreetimes X^* \to (X \leftthreetimes Y)^*, \\
c^{\rightthreetimes}_{X,Y} &= c_{X,Y} m_{Y^*,X^*} : Y^* \rightthreetimes X^* \to (X \rightthreetimes Y)^*
\end{aligned}$$

for each W^*-bimodules ${}_AX_B$ and ${}_BY_C$.

Theorem 3.2.3. *The triples $(*, \{c^{\rightthreetimes}_{X,Y}\}_{X,Y}, \{d_X\}_X)$ and $(*, \{c^{\leftthreetimes}_{X,Y}\}_{X,Y}, \{d_X\}_X)$ are involutions on $\mathcal{M}^{\rightthreetimes}$ and $\mathcal{M}^{\leftthreetimes}$, respectively, and they are involutionally and monoidally equivalent with respect to $\{m_{X,Y}\}_{X,Y}$.*

Proof. We shall consider $\{c^{\leftthreetimes}_{X,Y}\}_{X,Y}$. By Proposition 3.2.1, it satisfies the hexagon identity. We shall show the square identity. Let ${}_AX_B$ and ${}_BY_C$ be W^*-bimodules and $u : p^I L^2(B) \to X$ a projective module realization as right B-modules. By the canonical isomorphism $L^2(B)^* \cong L^2(B)$, we can get a left B-linear unitary $v : X^* \to L^2(B)^I p$. Note that for $\eta^* \in Y^*,\ b_i \in B,\ \beta \in L^2(B)$ and $y \in \mathrm{Hom}({}_BL^2(B), {}_BY)^\circ$, we can define unitaries by

$$\begin{aligned}
&Y^* \leftthreetimes L^2(B)^I p \ni \eta^* \otimes (b_i^*)p \mapsto (p(b_i) \otimes \eta)^* \in (p^I L^2(B) \rightthreetimes Y)^*, \\
&(p^I L^2(B) \rightthreetimes Y)^* \ni (p(b_i) \otimes \eta)^* \mapsto (p(\beta_i \otimes y))^* \in (p^I (L^2(B) \leftthreetimes Y))^*
\end{aligned}$$

under the relation $p(b_i\eta) = p(\beta_i y)$, where (b_i) and (b_i^*) are a column and a row vector, respectively. Also assume that other notations $(\,\cdot_i)$ will be used as row or column vectors as products of matrices and vectors are compatible. Then by the naturalities of $c_{X,Y}$ and $m_{X,Y}$, the following diagram commutes.

$$\begin{tikzcd}
Y^* \leftthreetimes X^* \arrow[r, "c_{X,Y}"] \arrow[d, "\mathrm{id}_{Y^*}\leftthreetimes v"] & (X \rightthreetimes Y)^* \arrow[d, "{}^t(u\rightthreetimes \mathrm{id}_Y)"] & (X \leftthreetimes Y)^* \arrow[l, "{}^t m_{X,Y}"'] \arrow[d] \\
Y^* \leftthreetimes L^2(B)^I p \arrow[r] & (p^I L^2(B) \rightthreetimes Y)^* \arrow[r] & (p({}^I L^2(B) \leftthreetimes Y))^*
\end{tikzcd}$$

Then $c_{X,Y}$ is specified as the following diagram with the bottom arrow given by $\eta^* \otimes (b_i^*)p = \left(\sum_i \Psi_{Y^*}^{-1}(y)(J\beta_i) \otimes \delta_i\right) p \mapsto (p(\beta_i) \otimes y)$, where we identify $(Y^*)^*$ with Y via d_Y and $\delta_i = (0, \cdots, 0, 1_B, 0 \cdots)$ is the canonical row vector whose i-th component is 1_B.

$$\begin{tikzcd}
Y^* \leftthreetimes X^* \arrow[r, "c^{\leftthreetimes}_{X,Y}"] \arrow[d, "\mathrm{id}_{Y^*}\leftthreetimes v"] & (X \leftthreetimes Y)^* \arrow[d, "{}^t(u\leftthreetimes \mathrm{id}_Y)"] \\
Y^* \leftthreetimes L^2(B)^I p \arrow[r] & (p^I L^2(B) \leftthreetimes Y)^*
\end{tikzcd} \tag{3.3}$$

Similarly, we have also the following commutativity for W^*-bimodules ${}_B Z_A$ and ${}_C W_B$ with a projective module realization $Z \cong L^2(B)^J q$.

$$\begin{tikzcd}
Z^* \leftthreetimes W^* \arrow[r] \arrow[d] & (W \leftthreetimes Z)^* \arrow[d] \\
q^J L^2(B) \leftthreetimes W^* \arrow[r] & (W \leftthreetimes L^2(B)^I q)^*
\end{tikzcd} \tag{3.4}$$

Here, the bottom arrow is given by

(3.5)

$$q^J L^2(B) \leftthreetimes W^* \ni q(\beta_j) \otimes \Psi_W(w) \mapsto \left(\left(\sum_j w(J\beta_j) \otimes \delta_j\right) q\right)^* \in (W \leftthreetimes L^2(B)^I q)^*$$

for $\beta_j \in L^2(B)$ and $w \in \mathrm{Hom}(L^2(B)_B, W_B)$.

Put $Z = X^*$ and $W = Y^*$. For $\beta_i \in L^2(B)$ and $y \in \mathrm{Hom}({}_B L^2(B), {}_B Y)$, by (3.5), we have

$$\begin{tikzcd}
p(\beta_j) \otimes y \arrow[r, "\mathrm{id}\leftthreetimes d_Y"] \arrow[d, "d"] & p(\beta_j) \otimes d_Y y \arrow[d, "c^{\leftthreetimes}"] \\
(p(\beta_j) \otimes y)^{**} \arrow[r, "{}^t c^{\leftthreetimes}"] & ((\sum_i \Psi_{Y^*}^{-1}(d_Y y)(J\beta_i) \otimes \delta_i)p)^*,
\end{tikzcd}$$

and hence the following commutative diagram.

$$\begin{tikzcd}
p^I L^2(B) \leftthreetimes Y \arrow[r, "\mathrm{id}\leftthreetimes d_Y"] \arrow[d, "d"] & p^I L^2(B) \leftthreetimes Y^{**} \arrow[d, "c^{\leftthreetimes}"] \\
(p^I L^2(B) \leftthreetimes Y)^{**} \arrow[r, "{}^t c^{\leftthreetimes}"] & (Y^* \leftthreetimes L^2(B)^I p)^*
\end{tikzcd} \tag{3.6}$$

Here, note that the isomorphisms $(p^I L^2(B))^{**} \cong p^I L^2(B)$ and $(p^I L^2(B))^* \cong L^2(B)^I p$. Now, we consider the following diagram.

$$\begin{tikzcd}
X \leftthreetimes Y \arrow[rrr] \arrow[ddd] \arrow[dr] & & & X^{**} \leftthreetimes Y^{**} \arrow[ddd] \\
& p^I L^2(B) \leftthreetimes Y \arrow[r] \arrow[d] & p^I L^2(B) \leftthreetimes Y^{**} \arrow[ur] \arrow[d] & \\
& (p^I L^2(B) \leftthreetimes Y)^* \arrow[r] & (Y^* \leftthreetimes L^2(B)^I p)^* & \\
(X \leftthreetimes Y)^{**} \arrow[rrr] \arrow[ur] & & & (Y^* \leftthreetimes X^*)^* \arrow[ul]
\end{tikzcd}$$

The left diagram commutes by the naturality of d, the top one commutes by the definition, the bottom one commutes by (3.3), the right one commutes by (3.4) and the central one commutes by (3.6).

We have shown that $\mathcal{M}^{\leftthreetimes}$ is an involutive W^*-bicategory with a unitary involution $(*, \{c^{\leftthreetimes}_{X,Y}\}_{X,Y}, \{d_X\}_X)$. By similar arguments, it can be proved for the case of $\mathcal{M}^{\rightthreetimes}$.

By the definition of $\{c^{\rightthreetimes}_{X,Y}\}$ and $\{c^{\leftthreetimes}_{X,Y}\}$, the square diagram in Definition 2.2.6 commutes clearly. $\square$

4. Minimal dilations of CP_0-semigroups

In this section, we discuss constructions of minimal dilations of CP_0-semigroups. The minimal dilation of a given CP_0-semigroup T in the sense of Definition 2.4.3 is constructed by some approaches. We will introduce Bhat-Skeide's and Muhly-Solel's constructions in Subsection 4.1. In Subsection 4.2, we will provide the notion of relative product system, that is, a W^*-bimodule version of Arveson's and Bhat-Skeide's product systems, and establish a relative product system from a CP_0-semigroup. The notion naturally arise from a relation between the two construction of minimal dilations, which will be provided in Subsection 4.4. In the discrete case, that is, T is the semigroup generated by a normal UCP-map, we have a similar relation, which will be discussed in Subsection 4.5. There is also Arveson's approach ([3] or [4]) which is different from the two ways. Obviously the three constructions are mutually related but no explicit connection has been established yet. In Subsection 4.3, we will construct minimal dilations in terms of relative product systems. We associate a given CP_0-semigroup with the relative product system and take the inductive limit of the relative product system, which is inspired by Bhat-Skeide's construction. Subsection 4.2, 4.3 and 4.4 are based on [22].

In Section 4 and 5, we assume that a von Neumann algebra M on which CP_0-semigroups act has a faithful normal state

4.1. **Bhat-Skeide's and Muhly-Solel's constructions.** Let M be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$ and $T = \{T_t\}_{t\geq 0}$ a CP_0-semigroup. In this subsection, we review Bhat-Skeide's[8] and Muhly-Solel's[17] constructions of the minimal dilation of T.

Bhat-Skeide's construction

First, we prepare notations related with partitions. We fix $t > 0$. Let $\mathfrak{P}_t$ be the set of all finite tuples $\mathfrak{p} = (t_1, \cdots, t_n)$ with $t_i > 0$ such that $\sum_{i=1}^n t_i = t$. Now, for $\mathfrak{p} = (t_1, \cdots, t_n), \mathfrak{q} = (s_1, \cdots, s_m) \in \mathfrak{P}_t$, we define the joint tuple by $\mathfrak{p} \vee \mathfrak{q} = (t_1, \cdots, t_n, s_1, \cdots, s_m)$ and write $\mathfrak{p} \succ \mathfrak{q}$ if for each $i = 1, \cdots, m$ there is $\mathfrak{q}_i \in \mathfrak{P}_{s_i}$ such that $\mathfrak{p} = \mathfrak{q}_1 \vee \cdots \vee \mathfrak{q}_m$. Let $\mathfrak{P}_0$ be the singleton of the empty tuple $()$ satisfying $\mathfrak{p} \vee () = () \vee \mathfrak{p} = \mathfrak{p}$. Note that when we consider partitions of an interval $[0, t]$, treating $\mathfrak{P}_t$ or the set $\mathfrak{P}'_t$ of all finite tuples $(t_1, \cdots, t_n)$ such that $t = t_n > t_{n-1} > \cdots > t_1 > 0$ is equivalent because $\mathfrak{P}_t$ and $\mathfrak{P}'_t$ are order isomorphic via a map $\mathfrak{o} : \mathfrak{P}_t \to \mathfrak{P}'_t$ defined by

$$\mathfrak{o}(t_1, t_2, \cdots, t_n) = \left(\sum_{i=1}^{1} t_i, \sum_{i=1}^{2} t_i, \cdots, \sum_{i=1}^{n} t_i \right).$$

Now, we introduce the notion of product system and units in the sense of Bhat-Skeide. These are Hilbert (or von Neumann) bimodule versions of Arveson's product systems and units in Definition 2.4.7 and Definition 2.4.8.

Definition 4.1.1. *Let A be a C^*-algebra and $E^{\otimes} = \{E_t\}_{t\geq 0}$ a family of Hilbert A-bimodules E_t. Suppose $E_0 = A$ and there is a A-bilinear unitary $u_{s,t} : E_s \odot E_t \to E_{s+t}$*

for each $s, t \geq 0$ with the associativity

$$u_{r,s+t}(\mathrm{id}_{E_r} \odot u_{s,t}) = u_{r+s,t}(u_{r,s} \odot \mathrm{id}_{E_t})$$

for all $r, s, t \geq 0$. Then the pair $(E^{\otimes}, \{u_{s,t}\}_{s,t\geq 0})$ is called a product system of Hilbert bimodules over A. We can define a product system of von Neumann bimodules similarly. Then $\odot$ is replaced by $\otimes$.

Definition 4.1.2. *A unit $Z^{\otimes} = \{Z^t\}_{t\geq 0}$ of a product system $(E^{\otimes}, \{u_{s,t}\}_{s,t\geq 0})$ of Hilbert (or von Neumann) bimodules over A is a family of $Z^t \in E_t$ satisfying $Z^0 = 1_A$ and $u_{s,t}(Z^s \odot Z^t) = Z^{s+t}$ for all $s, t \geq 0$. A unit $Z^{\otimes}$ is unital if $\langle Z^t, Z^t\rangle = 1_A$ for all $t \geq 0$.*

For each GNS-representation $(\mathcal{E}_{T_t}, Z_{T_t})$ with respect to T_t, we denote the strong closure as von Neumann M-M-bimodules by the same notation $\mathcal{E}_{T_t}$, and $(\mathcal{E}_{T_t}, Z_{T_t})$ is denoted by $(\mathcal{E}_t, Z_t)$ simply. We fix $t \geq 0$ and $\mathfrak{p} \in \mathfrak{P}_t$. We define a von Neumann M-M-bimodule $E_{\mathfrak{p},t}$ by the tensor product $\mathcal{E}_{t_1} \otimes_M \cdots \otimes_M \mathcal{E}_{t_n}$ if $t > 0$, and $E_{()} = \mathcal{E}_0$ if $t = 0$. For $\mathfrak{p} = \mathfrak{q}_1 \vee \cdots \vee \mathfrak{q}_m \succ \mathfrak{q}$ with $\mathfrak{p} = (t_1, \cdots, t_n), \mathfrak{q} = (s_1, \cdots, s_m) \in \mathfrak{P}_t$ and $\mathfrak{q}_i = (s_{i,1}, \cdots, s_{i,k(i)}) \in \mathfrak{P}_{s_i}$, we define an M-bilinear isometry $\beta_{\mathfrak{p},\mathfrak{q}} : E_{\mathfrak{q},t} \to E_{\mathfrak{p},t}$ by

$$\beta_{\mathfrak{p},\mathfrak{q}} = \beta_{\mathfrak{q}_1(s_1)} \otimes \cdots \otimes \beta_{\mathfrak{q}_m(s_m)},$$

where each $\beta_{\mathfrak{q}_i(s_i)} : \mathcal{E}_{s_i} \to E_{\mathfrak{q}_i,t}$ is an M-bilinear isometry defined by the map: $Z_{s_i} \mapsto Z_{s_{i,1}} \otimes \cdots \otimes Z_{s_{i,k(i)}}$. Then the pair $(\{E_{\mathfrak{p},t}\}_{\mathfrak{p}\in\mathfrak{P}_t}, \{\beta_{\mathfrak{p},\mathfrak{q}}\}_{\mathfrak{p}\succ\mathfrak{q}})$ becomes an inductive system with respect to the partial order $\succ$, and the inductive limit E_t of the inductive system is a von Neumann M-bimodule. For $t \geq 0$, put $Z^t = \iota_{(t),t} Z_t$, where $\iota_{\mathfrak{p},t} : E_{\mathfrak{p},t} \to E_t$ is the canonical embedding for each $\mathfrak{p} \in \mathfrak{P}_t$. For $s, t \geq 0$, we define an M-bilinear unitary $u_{t,s}$ is defined by

$$u_{t,s}(\iota_{\mathfrak{p},t} X_{\mathfrak{p}} \otimes \iota_{\mathfrak{q},t} Y_{\mathfrak{q}}) = \iota_{\mathfrak{p}\vee\mathfrak{q},t}(X_{\mathfrak{p}} \otimes Y_{\mathfrak{q}})$$

for each $X_{\mathfrak{p}} \in E_{\mathfrak{p},t}$ and $Y_{\mathfrak{q}} \in E_{\mathfrak{q},s}$.

Theorem 4.1.3. ([8, Theorem 4.8]) *The family $(\{E_t\}_{t\geq 0}, \{u_{s,t}\}_{s,t\geq 0})$ is a product system of von Neumann M-bimodules with the unital unit $\{Z^t\}_{t\geq 0}$.*

Note that the unit $\{Z^t\}_{t\geq 0}$ satisfies $T_t(x) = \langle Z^t, xZ^t\rangle$ for all $t \geq 0$ and $x \in M$.

For $t \geq s \geq 0$, we define an isometry

$$\gamma_{t,s} : E_s \ni X \mapsto Z^{t-s} \otimes X \in E_{t-s} \otimes_M E_s \cong E_t.$$

Then $(\{E_t\}_{t\geq 0}, \{\gamma_{t,s}\}_{t\geq s})$ is an inductive system of von Neumann M-modules. The inductive limit E of $(\{E_t\}_{t\geq 0}, \{\gamma_{t,s}\}_{t\geq s})$ is a von Neumann M-module and satisfies $E \otimes_M E_t \cong E$ as von Neumann M-modules for all $t \geq 0$. Let k_0 be the canonical embedding from $E_0 = M$ into E and $Z = k_0 1_M$. If we define

$$\begin{aligned}
j_0(x)(X) &= Zx\langle Z, X\rangle \qquad (X \in E, x \in M),\\
\alpha_t(a) &= a \otimes \mathrm{id}_{E_t} \qquad (a \in \mathcal{B}^a(E)),\\
j_t(x) &= \alpha_t(j_0(x)) \qquad (x \in M),
\end{aligned}$$

$\{\alpha_t\}_{t\geq 0}$ is a semigroup of endomorphism on the von Neumann algebra $\mathcal{B}^a(E)$. We identify M with $j_0(M)$ in $\mathcal{B}^a(E)$. Let N be a von Neumann algebra generated by $\bigcup_{t\geq 0} j_t(M)$, $p = j_0(1_M)$ and $\theta_t = \alpha_t^0|_N$ for each ≥ 0. Then we obtained the minimal dilation of T by [8, Theorem 5.8].

Muhly-Solel's construction

For $t \geq 0$ and $\mathfrak{p} = (0 = t_0, t_1, \cdots, t_{n-1}, t_n = t) \in \mathfrak{P}'_t$, we define a Hilbert space

$$\mathcal{H}_{\mathfrak{p},t} = M \otimes_{t_1-t_0} (M \otimes_{t_2-t_1} (\cdots (M \otimes_{t_n-t_{n-1}} \mathcal{H}) \cdots), \tag{4.1}$$

where we denote the tensor product $M \otimes_{T_s} \mathcal{K}$ in Definition 2.3.4 by $M \otimes_s \mathcal{K}$ for a left W^*-M-module $\mathcal{K}$ and $s \geq 0$ simply. Then $\mathcal{H}_{\mathfrak{p},t}$ has a left W^*-M-module structure given by

$$x(a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes \xi) = (xa_1) \otimes a_2 \otimes \cdots \otimes a_n \otimes \xi$$

for each $x, a_1, a_2, \cdots, a_n \in M$ and $\xi \in \mathcal{H}$, and also $\mathrm{Hom}({}_M\mathcal{H}, {}_M\mathcal{H}_{\mathfrak{p},t})$ has an M'-bimodule structure given by

$$(Xx')\xi = X(x'\xi),\ (x'X)\xi = (1_M \otimes \cdots \otimes 1_M \otimes x')X\xi$$

for each $X \in \mathrm{Hom}({}_M\mathcal{H}, {}_M\mathcal{H}_{\mathfrak{p},t})$, $x' \in M'$ and $\xi \in \mathcal{H}$. Note that if we write $F_s = \mathrm{Hom}({}_M\mathcal{H}, {}_M M \otimes_s \mathcal{H})$, we have an isomorphism

$$\mathrm{Hom}({}_M\mathcal{H}, {}_M\mathcal{H}_{\mathfrak{p},t}) \cong F_{t_n-t_{n-1}} \otimes_{M'} F_{t_{n-1}-t_{n-2}} \otimes_{M'} \cdots \otimes_{M'} F_{t_2-t_1} \otimes_{M'} F_{t_1-t_0}$$

as von Neumann M'-bimodules. For $k = 1, \cdots, n$ and $t_k < \tau < t_{k+1}$, put $\mathfrak{p}_{k,\tau} = (t_0, t_1, \cdots, t_k, \tau, t_{k+1}, \cdots, t_{n-1}, t_n) \in \mathfrak{P}'_t$. We get a left M-linear isometry $v_0 : \mathcal{H}_{\mathfrak{p},t} \to \mathcal{H}_{\mathfrak{p}_{k,\tau},t}$ by

$$v_0(a_1 \otimes \cdots a_n \otimes \xi) = a_1 \otimes \cdots \otimes a_k \otimes 1_M \otimes a_{k+1} \otimes \cdots \otimes a_n \otimes \xi$$

for each $a_1, \cdots, a_n \in M$ and $\xi \in \mathcal{H}$, and an M'-bilinear isometry

$$v : \mathrm{Hom}({}_M\mathcal{H}, {}_M\mathcal{H}_{\mathfrak{p},t}) \to \mathrm{Hom}({}_M\mathcal{H}, {}_M\mathcal{H}_{\mathfrak{p}_{k,\tau},t})$$

by $v(X) = v_0X$ for each $X \in \mathrm{Hom}({}_M\mathcal{H}, {}_M\mathcal{H}_{\mathfrak{p},t})$. For a refinement pair $\mathfrak{p} \succ \mathfrak{q}$, we can define a left M-linear isometry $v_{0,\mathfrak{p},\mathfrak{q}} : \mathcal{H}_{\mathfrak{q},t} \to \mathcal{H}_{\mathfrak{p},t}$ and an M'-bilinear isometry

$$v_{\mathfrak{p},\mathfrak{q}} : \mathrm{Hom}({}_M\mathcal{H}, {}_M\mathcal{H}_{\mathfrak{q},t}) \to \mathrm{Hom}({}_M\mathcal{H}, {}_M\mathcal{H}_{\mathfrak{p},t})$$

by a repetition of the above construction. Then $\{\mathcal{H}_{\mathfrak{p},t}\}_{\mathfrak{p}\in\mathfrak{P}'_t}$ and $\{\mathrm{Hom}({}_M\mathcal{H}, {}_M\mathcal{H}_{\mathfrak{p},t})\}_{\mathfrak{p}\in\mathfrak{P}'_t}$ become inductive systems with respect to the sets of isometries $\{v_{0,\mathfrak{p},\mathfrak{q}}\}_{\mathfrak{p}\succ\mathfrak{q}}$ and $\{v_{\mathfrak{p},\mathfrak{q}}\}_{\mathfrak{p}\succ\mathfrak{q}}$, and their inductive limits are denoted by $\mathcal{H}_t$ and $E(t)$, respectively. We also denote the canonical embedding from $\mathcal{H}_{\mathfrak{p},t}$ into $\mathcal{H}_t$ by $v_{0,\mathfrak{p},\infty}$. Note that the space $\mathrm{Hom}({}_M\mathcal{H}, {}_M\mathcal{H}_t)$ has a natural M'-M'-bimodule structure and is isomorphic to $E(t)$ as von Neumann M'-bimodules ([17, Lemma 3.1]). The family $\{E(t)\}_{t\geq 0}$ is a product system of von Neumann M'-bimodules.

We shall construct a fully coisometric completely contractive covariant representation $\{\Phi_t\}_{t\geq 0}$ of $\{E(t)\}_{t\geq 0}$ on $\mathcal{H}$ and dilate $\{\Phi_t\}_{t\geq 0}$ to the minimal isometric dilation $(\{V_t\}_{t\geq 0}, u_0, \mathcal{K})$. Here, a contractive covariant representation $\{\Phi_t\}_{t\geq 0}$ is a set of completely contractive continuous linear maps $\Phi_t : E(t) \to \mathcal{B}(\mathcal{H})$ with respect to the σ-topology ([5]) on $E(t)$ and the σ-weak topology on $\mathcal{B}(\mathcal{H})$ such that $\Phi_0 : M' \to \mathcal{B}(\mathcal{H})$ is a normal $*$-homomorphism and we have $\Phi_s \otimes \Phi_t = \Phi_{s+t}$ for all $s, t \geq 0$. Let $E(t) \otimes_{\Phi_0} \mathcal{H}$ be the completion of $E(t) \otimes_{\mathrm{alg}} \mathcal{H}$ with respect to the inner product defined by $\langle X \otimes \xi, Y \otimes \eta \rangle = \langle \xi, \Phi_0(\langle X, Y \rangle)\eta \rangle$ for each $X, Y \in E(t)$ and $\xi, \eta \in \mathcal{H}$. $\{\Phi_t\}_{t\geq 0}$ is said to be fully coisometric if $\tilde{\Phi}_t\tilde{\Phi}_t^* = \mathrm{id}_{\mathcal{H}}$ for all $t \geq 0$, where $\tilde{\Phi}_t : E(t) \otimes_{\Phi_0} \mathcal{H} \ni X \otimes \xi \mapsto \Phi_t(X)\xi \in \mathcal{H}$. A dilation $(\{V_t\}_{t\geq 0}, u_0, \mathcal{K})$ in the sense of [17, Section 3] consists of a Hilbert space $\mathcal{K}$, an isometry $u_0 : \mathcal{H} \to \mathcal{K}$ and a fully coisometric completely contractive covariant representation $\{V_t\}_{t\geq 0}$ of $\{E(t)\}_{t\geq 0}$ on $\mathcal{K}$ satisfying that $\Phi_t(X) = u_0^* V_t(X) u_0$ and $V_t(X)^*$ leaves $u_0(\mathcal{H})$ invariant for all $X \in E(t)$ and $t \geq 0$. If $V_t(X)^* V_t(Y) = V_0(\langle X, Y\rangle)$ for all $X, Y \in E(t)$ and $t \geq 0$, and the smallest subspace $\mathcal{K}_0$ of $\mathcal{K}$ containing $u_0(\mathcal{H})$ and $V_t(X)\mathcal{K}_0 = \mathcal{K}_0$ for all $X \in E(t)$ and $t \geq 0$, is $\mathcal{K}$, then $(\{V_t\}_{t\geq 0}, u_0, \mathcal{K})$ is called the minimal isometric dilation of $\{\Phi_t\}_{t\geq 0}$.

For $t \geq 0$ and $\mathfrak{p} \in \mathfrak{P}'_t$, we define $\iota_{\mathfrak{p}} : \mathcal{H} \ni \xi \mapsto 1_M \otimes \cdots 1_M \otimes \xi \in \mathcal{H}_{\mathfrak{p},t}$. Then we have

$$\iota_{\mathfrak{p}}^*(a_1 \otimes \cdots \otimes a_n \xi) = (T_{t_n - t_{n-1}}(T_{t_{n-1}-t_{n-2}}(\cdots (T_{t_1}(a_1)a_2)\cdots)a_{n-1})a_n)\xi$$

for all $a_1, \cdots, a_n \in M$ and $\xi \in \mathcal{H}$. Put $v_{\mathfrak{p}}^* = \iota_{\mathfrak{q}}$ when $\mathfrak{p} \succ \mathfrak{q}$. By the universality of inductive limits there is a unique map $\iota_t^* : \mathcal{H}_t \to \mathcal{H}$ such that $\iota_t^* v_{0,\mathfrak{p},\infty} = \iota_{\mathfrak{p}}^*$. We define $\{\Phi_t\}_{t\geq 0}$ by $\Phi_t(X) = \iota_t^* X$ for each $X \in \mathrm{Hom}({}_M\mathcal{H}, {}_M\mathcal{H}_t) \cong E(t)$. Then $\{\Phi_t\}_{t\geq 0}$ is a fully coisometric completely contractive covariant representation $\{\Phi_t\}_{t\geq 0}$ of $\{E(t)\}_{t\geq 0}$ on $\mathcal{H}$ and satisfies $T_t(x) = \tilde{\Phi}_t(\mathrm{id}_{E(t)} \otimes x)\tilde{\Phi}_t^*$ for all $x \in M$ and $t \geq 0$ ([17, Theorem 3.9]).

For $0 \leq t < s$, we denote the isomorphism from $E(t) \otimes_{M'} E(s-t)$ to $E(s)$ by $U_{t,s}$ and define a left M'-linear isometry

$$u_{t,s} = (U_{t,s} \otimes \mathrm{id}_{\mathcal{H}})(\mathrm{id}_{E(t)} \otimes \tilde{\Phi}_{s-t}^*) : E(t) \otimes_{\Phi_0} \mathcal{H} \to E(s) \otimes_{\Phi_0} \mathcal{H}.$$

Then the pair $(\{E(t) \otimes_{\Phi_0} \mathcal{H}\}_{t\geq 0}, \{u_{t,s}\}_{t<s})$ is an inductive system. Let $\mathcal{K}_\infty$ be the inductive limit of $(\{E(t) \otimes_{\Phi_0} \mathcal{H}\}_{t\geq 0}, \{u_{t,s}\}_{t<s})$ and $u_t : E(t) \otimes_{\Phi_0} \mathcal{H} \to \mathcal{K}_\infty$ the canonical embeddings. We can get a well-defined map $V_t : E(t) \to \mathcal{B}(\mathcal{K}_\infty)$ by $V_t(X)u_s(Y \otimes \xi) = u_{t+s}(U_{t,t+s}(X \otimes Y) \otimes \xi)$ for each $s, t \geq 0$, $X \in E(t)$, $Y \in E(s)$ and $\xi \in \mathcal{H}$. If we define a subspace $\mathcal{K}$ of $\mathcal{K}_\infty$ as the smallest subspace containing $u_0(\mathcal{H})$, then the triplet $(\{V_t\}_{t\geq 0}, u_0, \mathcal{K})$ is the minimal isometric dilation of $\{\Phi_t\}_{t\geq 0}$. Moreover, if we put $N = V_0(M')' \subset \mathcal{B}(\mathcal{K})$ and $\theta_t(x) = \tilde{V}_t(\mathrm{id}_{E(t)} \otimes x)\tilde{V}_t^*$ for each $x \in N$ and $t \geq 0$, we get the minimal dilation of $\{T_t\}_{t\geq 0}$ ([17, Theorem 3.11]).

4.2. **Relative product systems from CP_0-semigroups.** In this section, we introduce a notion of relative product system and construct a relative product system from a given CP_0-semigroup. The following definition of relative product systems is inspired by the definitions of Arveson's and Bhat-Skeide's product systems in Definition 2.4.7 and Definition 4.1.1. Also, units of relative product systems will be defined in Definition 5.1.1.

Definition 4.2.1. *Let M be a von Neumann algebra and $\mathcal{H}^\otimes = \{\mathcal{H}_t\}_{t\geq 0}$ a family of W^*-M-M-bimodules with $\mathcal{H}_0 = L^2(M)$. If there exist bimodule unitaries $U_{s,t} : \mathcal{H}_s \otimes^M \mathcal{H}_t \to \mathcal{H}_{s+t}$ with the associativity*

$$U_{r,s+t}(\mathrm{id}_{\mathcal{H}_r} \otimes^M U_{s,t}) = U_{r+s,t}(U_{r,s} \otimes^M \mathrm{id}_{\mathcal{H}_t}) \tag{4.2}$$

for all $r, s, t \geq 0$ such that $U_{s,0}$ and $U_{0,t}$ are the canonical identifications, then the pair $(\mathcal{H}^\otimes, \{U_{s,t}\}_{s,t\geq 0})$ is called a relative product system over M, where the notation $\otimes^M$ denotes the relative tensor product over M

Remark 4.2.2. *Precisely speaking, associativity* (4.2) *means that the following diagram commutes.*

$$\begin{tikzcd}
 & \mathcal{H}_{r+s+t} & \\
\mathcal{H}_{r+s} \otimes^M \mathcal{H}_t \arrow[ur, "U_{r+s,t}"] & & \mathcal{H}_r \otimes^M \mathcal{H}_{s+t} \arrow[ul, "U_{r,s+t}"'] \\
(\mathcal{H}_r \otimes^M \mathcal{H}_s) \otimes^M \mathcal{H}_t \arrow[u, "U_{r,s}\otimes \mathrm{id}_{\mathcal{H}_t}"] \arrow[rr, "a"] & & \mathcal{H}_r \otimes^M (\mathcal{H}_s \otimes^M \mathcal{H}_t) \arrow[u, "\mathrm{id}_{\mathcal{H}_r}\otimes U_{s,t}"']
\end{tikzcd}$$

Here, the morphism a is the associativity isomorphism discussed in Section 3*. By* Theorem 3.1.2*, we can choose either the left or the right relative tensor product. We will construct a relative product system from a given CP_0-semigroup by left relative tensor products.*

Now, we provide a formula related to the relative tensor products and normal UCP-maps for the convenience in later arguments.

Proposition 4.2.3. *Let M be a von Neumann algebra and T a normal UCP-map on M and ϕ a faithful normal state on M. For $x, y \in M$, $x \otimes y\phi^{\frac{1}{2}}$ is a ϕ-bounded vector in $M \otimes_T L^2(M)$. For $x_1, x_2, y_1, y_2 \in M$, we have*

$$\pi_\phi(x_1 \otimes y_1\phi^{\frac{1}{2}})^*\pi_\phi(x_2 \otimes y_2\phi^{\frac{1}{2}}) = y_1^*T(x_1^*x_2)y_2 \in M.$$

Proof. For $x', y', z \in M$, we can compute as

$$\begin{aligned}
&\langle \pi_\phi(x_1 \otimes y_1\phi^{\frac{1}{2}})\phi^{\frac{1}{2}}z, x' \otimes y'\phi^{\frac{1}{2}}z'\rangle = \langle x_1 \otimes y_1\phi^{\frac{1}{2}}z, x' \otimes y'\phi^{\frac{1}{2}}z'\rangle \\
&= \langle y_1\phi^{\frac{1}{2}}z, T(x_1^*x')y'\phi^{\frac{1}{2}}z'\rangle = \langle \phi^{\frac{1}{2}}z, y_1^*T(x_1^*x')y'\phi^{\frac{1}{2}}z'\rangle,
\end{aligned}$$

and hence $\pi_\phi(x_1 \otimes y_1\phi^{\frac{1}{2}})^*(x' \otimes y'\phi^{\frac{1}{2}}z') = y_1^*T(x_1^*x')y'\phi^{\frac{1}{2}}z'$. Thus, we have

$$\pi_\phi(x_1 \otimes y_1\phi^{\frac{1}{2}})^*\pi_\phi(x_2 \otimes y_2\phi^{\frac{1}{2}})\phi^{\frac{1}{2}}z = y_1^*T(x_1^*x_2)y_2\phi^{\frac{1}{2}}z.$$

□

Now, we fix a CP_0-semigroup $T = \{T_t\}_{t\geq 0}$ on a von Neumann algebra M with a faithful normal state ϕ. We shall construct a relative product system from T.

For $t > 0$ and $\mathfrak{p} = (t_1, \cdots, t_n) \in \mathfrak{P}_t$, we denote the W^*-M-M-bimodule

$$(M \otimes_{T_{t_1}} L^2(M)) \otimes^M \cdots \otimes^M (M \otimes_{T_{t_n}} L^2(M))$$

by $\tilde{\mathcal{H}}(\mathfrak{p}, t)$ and $M \otimes_{T_t} L^2(M)$ by $M \otimes_t L^2(M)$. We shall define an inductive system structure on the set $\{\tilde{\mathcal{H}}(\mathfrak{p}, t)\}_{\mathfrak{p}\in\mathfrak{P}_t}$. Suppose $\mathfrak{p} \succ \mathfrak{q}$ with $\mathfrak{p} = (t_1, \cdots, t_n)$, $\mathfrak{q} = (s_1, \cdots, s_m) \in \mathfrak{P}_t$ and $\mathfrak{p} = \mathfrak{q}(s_1) \vee \cdots \vee \mathfrak{q}(s_m)$ with $\mathfrak{q}(s_i) = (s_{i,1}, \cdots, s_{i,k(i)}) \in \mathfrak{P}_{s_i}$. We define an M-bilinear map $\alpha_{\mathfrak{q}(s_i)} : M \otimes_{s_i} L^2(M) \to \tilde{\mathcal{H}}(\mathfrak{q}(s_i), s_i)$ by

$$\begin{aligned}
&\alpha_{\mathfrak{q}(s_i)}(x \otimes_{s_i} y\phi^{\frac{1}{2}}) \\
&= (x \otimes_{s_{i,1}} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(1_M \otimes_{s_{i,2}} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}(1_M \otimes_{s_{i,k(i)-1}} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(1_M \otimes_{s_{i,k(i)}} y\phi^{\frac{1}{2}})
\end{aligned}$$

for each $x, y \in M$ (we can check that $\alpha_{\mathfrak{p}(s_i)}$ is an isometry by Proposition 4.2.3), and an isometry

$$\alpha_{\mathfrak{p},\mathfrak{q}} = \alpha_{\mathfrak{q}(s_1)} \otimes^M \cdots \otimes^M \alpha_{\mathfrak{q}(s_m)} : \tilde{\mathcal{H}}(\mathfrak{q}, t) \to \tilde{\mathcal{H}}(\mathfrak{p}, t). \tag{4.3}$$

Then the pair $(\{\tilde{\mathcal{H}}(\mathfrak{p}, t)\}_{\mathfrak{p}\in\mathfrak{P}_t}, \{\alpha_{\mathfrak{p},\mathfrak{q}}\}_{\mathfrak{p}\succ\mathfrak{q}})$ is an inductive system of W^*-M-M-bimodules. Let $\tilde{\mathcal{H}}_t$ be the inductive limit and $\kappa_{\mathfrak{p},t} : \tilde{\mathcal{H}}(\mathfrak{p}, t) \to \tilde{\mathcal{H}}_t$ the canonical embedding. Put $\tilde{\mathcal{H}}_0 = L^2(M)$.

The following theorem is an analogue of Theorem 4.1.3 and the proof is essentially the same.

Theorem 4.2.4. *The family $\tilde{\mathcal{H}}^{\otimes} = \{\tilde{\mathcal{H}}_t\}_{t\geq 0}$ is a relative product system over M.*

Proof. For $s, t > 0$, we define a map $U_{s,t} : \tilde{\mathcal{H}}_s \otimes^M \tilde{\mathcal{H}}_t \to \tilde{\mathcal{H}}_{s+t}$ by

$$U_{s,t}((\kappa_{\mathfrak{q},s}\xi_{\mathfrak{q}})\phi^{-\frac{1}{2}}(\kappa_{\mathfrak{p},t}\eta_{\mathfrak{q}})) = \kappa_{\mathfrak{q}\vee\mathfrak{p},s+t}(\xi_{\mathfrak{q}}\phi^{-\frac{1}{2}}\eta_{\mathfrak{p}}).$$

for each $\mathfrak{q} = (s_1, \cdots, s_m) \in \mathfrak{P}_s, \mathfrak{p} = (t_1, \cdots, t_n) \in \mathfrak{P}_t$, $\xi_{\mathfrak{q}} \in \mathcal{D}(\tilde{\mathcal{H}}(\mathfrak{q}, s); \phi)$ and $\eta_{\mathfrak{p}} \in \tilde{\mathcal{H}}(\mathfrak{p}, t)$ Here, note that $\kappa_{\mathfrak{q},s}\xi_{\mathfrak{q}}$ is ϕ-bounded. We shall show that $U_{s,t}$ is an isometry, i.e. the equation

$$\langle(\kappa_{\mathfrak{q},s}\xi_{\mathfrak{q}})\phi^{-\frac{1}{2}}(\kappa_{\mathfrak{p},t}\eta_{\mathfrak{p}}), (\kappa_{\mathfrak{q}',s}\xi'_{\mathfrak{q}'})\phi^{-\frac{1}{2}}(\kappa_{\mathfrak{p}',t}\eta'_{\mathfrak{p}'})\rangle = \langle\kappa_{\mathfrak{q}\vee\mathfrak{p},s+t}(\xi_{\mathfrak{q}}\phi^{-\frac{1}{2}}\eta_{\mathfrak{p}}), \kappa_{\mathfrak{q}'\vee\mathfrak{p}',s+t}(\xi'_{\mathfrak{q}'}\phi^{-\frac{1}{2}}\eta'_{\mathfrak{p}'})\rangle$$

holds for all $\mathfrak{q}, \mathfrak{q}' \in \mathfrak{P}_s$, $\mathfrak{p}, \mathfrak{p}' \in \mathfrak{P}_t$, $\xi_{\mathfrak{q}} \in \mathcal{D}(\tilde{\mathcal{H}}(\mathfrak{q}, s); \phi), \xi'_{\mathfrak{q}'} \in \mathcal{D}(\tilde{\mathcal{H}}(\mathfrak{q}', s); \phi), \eta_{\mathfrak{p}} \in \tilde{\mathcal{H}}(\mathfrak{p}, t)$ and $\eta'_{\mathfrak{p}'} \in \tilde{\mathcal{H}}(\mathfrak{p}', t)$. If $\mathfrak{q} = \mathfrak{q}'$ and $\mathfrak{p} = \mathfrak{p}'$, we have

$$\begin{aligned}
&\langle(\kappa_{\mathfrak{q},s}\xi_{\mathfrak{q}})\phi^{-\frac{1}{2}}(\kappa_{\mathfrak{p},t}\eta_{\mathfrak{p}}), (\kappa_{\mathfrak{q},s}\xi'_{\mathfrak{q}})\phi^{-\frac{1}{2}}(\kappa_{\mathfrak{p},t}\eta'_{\mathfrak{p}})\rangle = \langle\kappa_{\mathfrak{p},t}\eta_{\mathfrak{p}}, \pi_\phi(\kappa_{\mathfrak{q},s}\xi_{\mathfrak{q}})^*\pi_\phi(\kappa_{\mathfrak{q},s}\xi'_{\mathfrak{q}})\kappa_{\mathfrak{p},t}\eta'_{\mathfrak{p}}\rangle \\
&= \langle\kappa_{\mathfrak{q},t}\eta_{\mathfrak{p}}, \kappa_{\mathfrak{p},t}(\pi_\phi(\kappa_{\mathfrak{q},s}\xi_{\mathfrak{q}})^*\pi_\phi(\kappa_{\mathfrak{q},s}\xi'_{\mathfrak{q}}))\eta')\rangle = \langle\eta_{\mathfrak{p}}, \pi_\phi(\kappa_{\mathfrak{q},s}\xi_{\mathfrak{q}})^*\pi_\phi(\kappa_{\mathfrak{q},s}\xi'_{\mathfrak{q}})\eta'_{\mathfrak{p}}\rangle \\
&= \langle\eta_{\mathfrak{p}}, \pi_\phi(\xi_{\mathfrak{q}})^*\pi_\phi(\xi'_{\mathfrak{q}})\eta'_{\mathfrak{p}})\rangle = \langle\kappa_{\mathfrak{q}\vee\mathfrak{p},s+t}(\xi_{\mathfrak{q}}\phi^{-\frac{1}{2}}\eta_{\mathfrak{p}}), \kappa_{\mathfrak{q}\vee\mathfrak{p},s+t}(\xi'_{\mathfrak{q}}\phi^{-\frac{1}{2}}\eta'_{\mathfrak{p}})\rangle.
\end{aligned}$$

In general case, since $\kappa_{\mathfrak{s},s} = \kappa_{\mathfrak{s}',s}\alpha_{\mathfrak{s}',\mathfrak{s}}$ for all $\mathfrak{s}, \mathfrak{s}' \in \mathfrak{P}_s$ with $\mathfrak{s}' \succ \mathfrak{s}$, if we take $\hat{\mathfrak{q}} \in \mathfrak{P}_s, \hat{\mathfrak{p}} \in \mathfrak{P}_t$, such that $\hat{\mathfrak{q}} \succ \mathfrak{q}, \mathfrak{q}'$ and $\hat{\mathfrak{p}} \succ \mathfrak{p}, \mathfrak{p}'$, then

$$\begin{aligned}
&\langle(\kappa_{\mathfrak{q},s}\xi_{\mathfrak{q}})\phi^{-\frac{1}{2}}(\kappa_{\mathfrak{p},t}\eta_{\mathfrak{p}}), (\kappa_{\mathfrak{q}',s}\xi'_{\mathfrak{q}'})\phi^{-\frac{1}{2}}(\kappa_{\mathfrak{p}',t}\eta'_{\mathfrak{p}'})\rangle \\
&= \langle(\kappa_{\hat{\mathfrak{q}},s}\alpha_{\hat{\mathfrak{q}},\mathfrak{q}}\xi_{\mathfrak{q}})\phi^{-\frac{1}{2}}(\kappa_{\hat{\mathfrak{p}},t}\alpha_{\hat{\mathfrak{p}},\mathfrak{p}}\eta_{\mathfrak{p}}), (\kappa_{\hat{\mathfrak{q}},s}\alpha_{\hat{\mathfrak{q}},\mathfrak{q}'}\xi'_{\mathfrak{q}'})\phi^{-\frac{1}{2}}(\kappa_{\hat{\mathfrak{p}},t}\alpha_{\hat{\mathfrak{p}},\mathfrak{p}'}\eta'_{\mathfrak{p}'})\rangle \\
&= \langle\kappa_{\hat{\mathfrak{q}}\vee\hat{\mathfrak{p}},s+t}((\alpha_{\hat{\mathfrak{q}},\mathfrak{q}}\xi_{\mathfrak{q}})\phi^{-\frac{1}{2}}(\alpha_{\hat{\mathfrak{p}},\mathfrak{p}}\eta_{\mathfrak{p}})), \kappa_{\hat{\mathfrak{q}}\vee\hat{\mathfrak{p}},s+t}((\alpha_{\hat{\mathfrak{q}},\mathfrak{q}'}\xi'_{\mathfrak{q}'})\phi^{-\frac{1}{2}}(\alpha_{\hat{\mathfrak{p}},\mathfrak{p}'}\eta'_{\mathfrak{p}'}))\rangle \\
&= \langle(\alpha_{\hat{\mathfrak{q}},\mathfrak{q}}\xi_{\mathfrak{q}})\phi^{-\frac{1}{2}}(\alpha_{\hat{\mathfrak{p}},\mathfrak{p}}\eta_{\mathfrak{p}}), (\alpha_{\hat{\mathfrak{q}},\mathfrak{q}'}\xi'_{\mathfrak{q}'})\phi^{-\frac{1}{2}}(\alpha_{\hat{\mathfrak{p}},\mathfrak{p}'}\eta'_{\mathfrak{p}'})\rangle \\
&= \langle(\alpha_{\hat{\mathfrak{q}},\mathfrak{q}} \otimes^M \alpha_{\hat{\mathfrak{p}},\mathfrak{p}})(\xi_{\mathfrak{q}}\phi^{-\frac{1}{2}}\eta_{\mathfrak{q}}), (\alpha_{\hat{\mathfrak{q}},\mathfrak{q}'} \otimes^M \alpha_{\hat{\mathfrak{p}},\mathfrak{p}'})(\xi'_{\mathfrak{q}'}\phi^{-\frac{1}{2}}\eta'_{\mathfrak{q}'})\rangle \\
&= \langle\alpha_{\hat{\mathfrak{q}}\vee\hat{\mathfrak{p}},\mathfrak{q}\vee\mathfrak{p}}(\xi_{\mathfrak{q}}\phi^{-\frac{1}{2}}\eta_{\mathfrak{q}}), \alpha_{\hat{\mathfrak{q}}\vee\hat{\mathfrak{p}},\mathfrak{q}'\vee\mathfrak{p}'}(\xi'_{\mathfrak{q}'}\phi^{-\frac{1}{2}}\eta'_{\mathfrak{q}'})\rangle \\
&= \langle\kappa_{\hat{\mathfrak{q}}\vee\hat{\mathfrak{p}},s+t}\alpha_{\hat{\mathfrak{q}}\vee\hat{\mathfrak{p}},\mathfrak{q}\vee\mathfrak{p}}(\xi_{\mathfrak{q}}\phi^{-\frac{1}{2}}\eta_{\mathfrak{q}}), \kappa_{\hat{\mathfrak{q}}\vee\hat{\mathfrak{p}},s+t}\alpha_{\hat{\mathfrak{q}}\vee\hat{\mathfrak{p}},\mathfrak{q}'\vee\mathfrak{p}'}(\xi'_{\mathfrak{q}'}\phi^{-\frac{1}{2}}\eta'_{\mathfrak{q}'})\rangle \\
&= \langle\kappa_{\mathfrak{q}\vee\mathfrak{p},s+t}(\xi_{\mathfrak{q}}\phi^{-\frac{1}{2}}\eta_{\mathfrak{p}}), \kappa_{\mathfrak{q}'\vee\mathfrak{p}',s+t}(\xi'_{\mathfrak{q}'}\phi^{-\frac{1}{2}}\eta'_{\mathfrak{p}'})\rangle.
\end{aligned}$$

In particular, we conclude that $U_{s,t}$ is well-defined and can be extended to an isometry from $\tilde{\mathcal{H}}_s \otimes^M \tilde{\mathcal{H}}_t$ to $\tilde{\mathcal{H}}_{s+t}$, and also denote the isometry by $U_{s,t}$ again. The surjectivity and the two-sides linearity of $U_{s,t}$ are obvious.

The isomorphisms $U_{0,t}$ and $U_{s,0}$ are defined as the canonical maps giving the isomorphisms $L^2(M) \otimes^M \tilde{\mathcal{H}}_s$ and $\tilde{\mathcal{H}}_t \otimes^M L^2(M)$, respectively.

To show (4.2), it is enough to check it for a vector $(\kappa_{\mathfrak{p},r}\xi_{\mathfrak{p}})\phi^{-\frac{1}{2}}(\kappa_{\mathfrak{q},s}\eta_{\mathfrak{q}})\phi^{-\frac{1}{2}}\zeta$ with the forms

$$\begin{cases} \xi_{\mathfrak{p}} = (x_1 \otimes_{r_1} y_1\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}(x_m \otimes_{r_m} y_m\phi^{\frac{1}{2}}), \\ \eta_{\mathfrak{q}} = (z_1 \otimes_{s_1} w_1\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}(z_n \otimes_{s_n} w_n\phi^{\frac{1}{2}}) \end{cases}$$

for some $x_i, y_i, z_j, w_j \in M$ and $\zeta_t \in \tilde{\mathcal{H}}_t$. □

Example 4.2.5. *Let M be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$ and $\{v_t\}_{t\geq 0}$ a family of isometries in M satisfying $v_s v_t = v_{s+t}$ for each $s, t \geq 0$. We define a CP_0-semigroup $T = \{T_t\}_{t\geq 0}$ by $T_t(x) = v_t^* x v_t$ for each $x \in M$* (Example 2.4.1). *For each $t \geq 0$, we can identify $M \otimes_t L^2(M)$ with $L^2(M)$ by a bilinear unitary $U^t(x \otimes_t y\phi^{\frac{1}{2}}) = xv_t y\phi^{\frac{1}{2}}$ for $x, y \in M$. For $t \geq 0$ and $\mathfrak{p} \in \mathfrak{P}_t$, the unitaries induce a bilinear unitary $U^{\mathfrak{p}} : \tilde{\mathcal{H}}(\mathfrak{p}, t) \to L^2(M)$ such that $U^{\mathfrak{p}}\alpha_{\mathfrak{p},\mathfrak{q}} = U^{\mathfrak{q}}$ for all $\mathfrak{p} \succ \mathfrak{q}$.*

Example 4.2.6. *We consider the CP_0-semigroup generated by a family of stochastic matrices. Let $M = \mathbb{C} \oplus \mathbb{C}$ be a von Neumann algebra regarded as a von Neumann subalgebra of $M_2(\mathbb{C})$. Then $L^2(M) = \mathbb{C} \oplus \mathbb{C}$. Let $T = \{T_t\}_{t\geq 0}$ be the CP_0-semigroup on M associated with stochastic matrices $\left\{\begin{pmatrix} e^{-t} & 1-e^{-t} \\ 0 & 1 \end{pmatrix}\right\}_{t\geq 0}$, that is, each T_t is defined by*

$$T_t(a \oplus b) = (e^{-t}a + (1 - e^{-t})b) \oplus b$$

for each $a, b \in \mathbb{C}$. By using the normalized canonical trace on $M_2(\mathbb{C})$, for $t \geq 0$ and $\mathfrak{p} = (t_1, \cdots, t_n) \in \mathfrak{P}_t$, it turns out that $\tilde{\mathcal{H}}(\mathfrak{p}, t)$ is $\mathbb{C}^n \oplus \mathbb{C}$ on which M acts as

$$(a \oplus b)(x_1 \oplus \cdots x_n \oplus y \oplus z) = ax_1 \oplus bx_2 \oplus \cdots \oplus bx_n \oplus by \oplus bz$$
$$(x_1 \oplus \cdots x_n \oplus y \oplus z)(a \oplus b) = ax_1 \oplus \cdots \oplus ax_n \oplus ay \oplus bz$$

for $a, b, x_1, \cdots, x_n, y, z \in \mathbb{C}$. Thus, $\tilde{\mathcal{H}}(\mathfrak{p}, t)$ depends on only the number n of the partition $\mathfrak{p}$.

Example 4.2.7. *We consider the W^*-$\mathcal{B}(\mathcal{H})$-bimodule $\tilde{\mathcal{H}}(\mathfrak{p}, t)$ associated with the CCR heat flow in* Example 2.4.2 *for $t \geq 0$ and $\mathfrak{p} \in \mathfrak{P}_t$.*

Let $\mathcal{H} = L^2(\mathbb{R})$ and $M = \mathcal{B}(\mathcal{H})$. Then the standard space $L^2(\mathcal{B}(\mathcal{H}))$ of M is isomorphic to $\mathcal{H} \otimes \mathcal{H}^* \cong \mathcal{C}_2(\mathcal{H})$. For $t \geq 0$, $x, x' \in M$ and $\xi \otimes \eta^*, \xi' \otimes \eta'^* \in \mathcal{H} \otimes \mathcal{H}^*$, the inner product on $M \otimes_t L^2(M)$ is given by

$$\langle x \otimes (\xi \otimes \eta^*), x' \otimes (\xi' \otimes \eta'^*)\rangle = \langle \eta', \eta\rangle \int_{\mathbb{R}^2} \langle xW^*_{\frac{\mathbf{x}}{\sqrt{2}}}\xi, x'W^*_{\frac{\mathbf{x}}{\sqrt{2}}}\xi'\rangle d\mu_t(\mathbf{x}).$$

Let $\mathfrak{p} = (t_1, \cdots, t_n) \in \mathfrak{P}_t$. Fix a faithful normal state ϕ on M and suppose $\rho \in \mathcal{C}_1(\mathcal{H})$ is associated with ϕ by $\phi(x) = \mathrm{tr}(\rho x)$ for all $x \in M$. In terms of $\mathcal{C}_2(\mathcal{H})$, the inner product on $\tilde{\mathcal{H}}(\mathfrak{p}, t)$ is

$$\begin{aligned} &\langle (x_1 \otimes_{t_1} a_1\rho^{\frac{1}{2}})\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}(x_n \otimes_{t_n} a_n\rho^{\frac{1}{2}}), (y_1 \otimes_{t_1} b_1\rho^{\frac{1}{2}})\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}(y_n \otimes_{t_n} b_n\rho^{\frac{1}{2}})\rangle \\ &= \int_{\mathbb{R}^{2n}} \langle x_1W^*_{\frac{\mathbf{x}_1}{\sqrt{2}}}a_1 \cdots x_nW^*_{\frac{\mathbf{x}_n}{\sqrt{2}}}a_n\rho^{\frac{1}{2}}, y_1W^*_{\frac{\mathbf{x}_1}{\sqrt{2}}}b_1 \cdots y_nW^*_{\frac{\mathbf{x}_n}{\sqrt{2}}}b_n\rho^{\frac{1}{2}}\rangle d\mu_{t_1}(\mathbf{x}_1) \cdots d\mu_{t_n}(\mathbf{x}_n) \end{aligned}$$

for each $x_1, \cdots, x_n, y_1, \cdots, y_n, a_1, \cdots, a_n, b_1, \ldots, b_n \in M$. *The properties* (2.1) *and* $\mu_s * \mu_t = \mu_{s+t}$ *ensure the fact that* $\alpha_{\mathfrak{p},\mathfrak{q}}$ *defined by* (4.3) *is isometry for* $\mathfrak{p}, \mathfrak{q} \in \mathfrak{P}_t$. *For example, for* $s, t \geq 0$ *and* $x, y, a, b \in M$, *we have*

$$\begin{aligned}
&\langle \alpha_{(s,t),(s+t)}(x \otimes_{s+t} a\rho^{\frac{1}{2}}), \alpha_{(s,t),(s+t)}(y \otimes_{s+t} b\rho^{\frac{1}{2}})\rangle \\
&= \langle (x \otimes_s \rho^{\frac{1}{2}})\phi^{-\frac{1}{2}}(1_M \otimes_t a\rho^{\frac{1}{2}}), (y \otimes_s \rho^{\frac{1}{2}})\phi^{-\frac{1}{2}}(1_M \otimes_t b\rho^{\frac{1}{2}})\rangle \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \langle xW^*_{\frac{\mathbf{x}}{\sqrt{2}}}W^*_{\frac{\mathbf{y}}{\sqrt{2}}}a\rho^{\frac{1}{2}}, yW^*_{\frac{\mathbf{x}}{\sqrt{2}}}W^*_{\frac{\mathbf{y}}{\sqrt{2}}}b\rho^{\frac{1}{2}}\rangle d\mu_t(\mathbf{x})d\mu_s(\mathbf{y}) \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \langle xW^*_{\frac{\mathbf{x}+\mathbf{y}}{\sqrt{2}}}a\rho^{\frac{1}{2}}, yW^*_{\frac{\mathbf{x}+\mathbf{y}}{\sqrt{2}}}b\rho^{\frac{1}{2}}\rangle d\mu_t(\mathbf{x})d\mu_s(\mathbf{y}) \\
&= \int_{\mathbb{R}^2} \langle xW^*_{\frac{\mathbf{z}}{\sqrt{2}}}a\rho^{\frac{1}{2}}, yW^*_{\frac{\mathbf{z}}{\sqrt{2}}}b\rho^{\frac{1}{2}}\rangle d(\mu_t * \mu_s)(\mathbf{z}) \\
&= \int_{\mathbb{R}^2} \langle xW^*_{\frac{\mathbf{z}}{\sqrt{2}}}a\rho^{\frac{1}{2}}, yW^*_{\frac{\mathbf{z}}{\sqrt{2}}}b\rho^{\frac{1}{2}}\rangle d\mu_{s+t}(\mathbf{z}) = \langle (x \otimes_{s+t} a\rho^{\frac{1}{2}}), (y \otimes_{s+t} b\rho^{\frac{1}{2}})\rangle.
\end{aligned}$$

We have considered the three examples of CP_0-semigroups, however it seems to be difficult to concretely realize their relative product systems and minimal dilations which will be constructed by the method in the next subsection. This also is so for Bhat-Skeide's and Muhly-Solel's constructions. Remark that in Example 4.2.5 we identified $\tilde{\mathcal{H}}(\mathfrak{p}, t)$ with the standard space $L^2(M)$, however the relative product system associated with T is not always isomorphic to the trivial system $\{L^2(M)\}_{t\geq 0}$ in the sense of Definition 5.1.5.

4.3. **Construction of dilations by relative product systems.** We shall construct the minimal dilation (N, p, θ) of T. The von Neumann algebra N acts on the inductive limit of the relative product system $\tilde{\mathcal{H}}^{\otimes}$ associated with T. We shall introduce an inductive system structure on $\tilde{\mathcal{H}}^{\otimes}$. For $s \leq t$, we define a right M-linear isometry $\tilde{\alpha}_{t,s} : \tilde{\mathcal{H}}_s \to \tilde{\mathcal{H}}_t$ by

$$\tilde{\alpha}_{t,s}(\xi) = U_{t-s,s}(\kappa_{(t-s),t-s}(1_M \otimes_{t-s} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}\xi)$$

for each $\xi \in \tilde{\mathcal{H}}_r$. Note that for $s, t \geq 0$, we have

$$\begin{aligned}
&U_{s,t}(\kappa_{(s),s}(1_M \otimes_s \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}\kappa_{(t),t}(1_M \otimes_t \phi^{\frac{1}{2}})) \\
&= \kappa_{(s,t),s+t}((1_M \otimes_s \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(1_M \otimes_t \phi^{\frac{1}{2}})) = \kappa_{(s,t),s+t}\alpha_{(s,t),(s+t)}(1_M \otimes_{s+t} \phi^{\frac{1}{2}}) \\
&= \kappa_{(s+t),s+t}(1_M \otimes_{s+t} \phi^{\frac{1}{2}}).
\end{aligned} \tag{4.4}$$

The following lemma shows that the couple $(\tilde{\mathcal{H}}^{\otimes}, \{\tilde{\alpha}_{t,s}\}_{s\leq t})$ is an inductive system.

Lemma 4.3.1. *For* $r \leq s \leq t$, *we have* $\tilde{\alpha}_{t,r} = \tilde{\alpha}_{t,s} \circ \tilde{\alpha}_{s,r}$.

Proof. For all $\xi \in \tilde{\mathcal{H}}_r$, we can calculate as

$$\begin{aligned}
&\tilde{\alpha}_{t,s} \circ \tilde{\alpha}_{s,r}(\xi) \\
&= U_{t-s,s}(\kappa_{(t-s),t-s}(1_M \otimes_{t-s} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} U_{s-r,r}(\kappa_{(s-r),s-r}(1_M \otimes_{s-r} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}\xi)) \\
&= U_{t-s,s}(\mathrm{id}_{\tilde{\mathcal{H}}_{t-s}} \otimes^M U_{s-r,r})(\kappa_{(t-s),t-s}(1_M \otimes_{t-s} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \kappa_{(s-r),s-r}(1_M \otimes_{s-r} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}\xi) \\
&= U_{t-r,r}(U_{t-s,s-r} \otimes^M \mathrm{id}_{\tilde{\mathcal{H}}_r})(\kappa_{(t-s),t-s}(1_M \otimes_{t-s} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \kappa_{(s-r),s-r}(1_M \otimes_{s-r} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}\xi) \\
&= U_{t-r,r}(U_{t-s,s-r}(\kappa_{(t-s),t-s}(1_M \otimes_{t-s} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \kappa_{(s-r),s-r}(1_M \otimes_{s-r} \phi^{\frac{1}{2}}))\phi^{-\frac{1}{2}}\xi) \\
&= U_{t-r,r}(\kappa_{(t-r),t-r}(1_M \otimes_{t-r} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}\xi) = \tilde{\alpha}_{t,r}(\xi)
\end{aligned}$$

by the associativity and (4.4). □

Remark 4.3.2. *If we denote $\kappa_{(t),t}(1_M \otimes_t \phi^{\frac{1}{2}})$ by $\tilde{\xi}(t)$ for each $t > 0$ and $\tilde{\xi}(0) = \kappa_{(0),0}\phi^{\frac{1}{2}}$, then the family $\tilde{\xi}^{\otimes} = \{\tilde{\xi}(t)\}_{t\geq 0}$ is a unital unit in the sense of* Definition 4.4 *by* (4.4). *Clearly, we have $\phi(T_t(x)) = \langle \tilde{\xi}(t), x\tilde{\xi}(t)\rangle$ for all $t \geq 0$ and $x \in M$. The unit $\{\tilde{\xi}(t)\}_{t\geq 0}$ is comparable with the unit $\{Z^t\}_{t\geq 0}$ in Bhat-Skeide's construction* (*see* Remark 5.1.3).

We denote the inductive limit of the inductive system $(\tilde{\mathcal{H}}^{\otimes}, \{\tilde{\alpha}_{t,s}\}_{s\leq t})$ by $\tilde{\mathcal{H}}$ and the canonical embedding from $\tilde{\mathcal{H}}_t$ into $\tilde{\mathcal{H}}$ by κ_t. Note that $\tilde{\mathcal{H}}$ is a right W^*-M-module.

Theorem 4.3.3. *For $t \geq 0$, we can define a right M-linear unitary $U_t : \tilde{\mathcal{H}} \otimes^M \tilde{\mathcal{H}}_t \to \tilde{\mathcal{H}}$ by*

$$U_t(\xi\phi^{-\frac{1}{2}}\eta_t) = \kappa_{s+t}U_{s,t}(\xi_s\phi^{-\frac{1}{2}}\eta_t)$$

for $\xi = \kappa_s\xi_s \in \tilde{\mathcal{H}}$ and $\eta_t \in \tilde{\mathcal{H}}_t$, where $\xi_s \in \tilde{\mathcal{H}}_s$ is a ϕ-bounded vector.

Proof. We shall show that U_t is an isometry. For $s \geq 0$, $\xi_s, \xi'_s \in \tilde{\mathcal{H}}_s$ and $\eta_t, \eta'_t \in \tilde{\mathcal{H}}_t$, we have

$$\begin{aligned}
&\langle U_t((\kappa_s\xi_s)\phi^{-\frac{1}{2}}\eta_t), U_t((\kappa_s\xi'_s)\phi^{-\frac{1}{2}}\eta'_t)\rangle = \langle \xi_s\phi^{-\frac{1}{2}}\eta_t, \xi'_s\phi^{-\frac{1}{2}}\eta'_t\rangle \\
&= \langle \eta_t, \pi_\phi(\xi_s)^*\pi_\phi(\xi'_s)\eta'_t\rangle = \langle \eta_t, \pi_\phi(\kappa_s\xi_s)^*\pi_\phi(\kappa_s\xi'_s)\eta'_t\rangle = \langle (\kappa_s\xi_s)\phi^{-\frac{1}{2}}\eta_t, (\kappa_s\xi'_s)\phi^{-\frac{1}{2}}\eta'_t\rangle.
\end{aligned}$$

This implies that for $s \geq 0$, $\xi_s \in \tilde{\mathcal{H}}_s$, $\zeta_r \in \tilde{\mathcal{H}}_r$ and $\eta_t, \eta'_t \in \tilde{\mathcal{H}}_t$, in the general case, we have

$$\begin{aligned}
&\langle U_t((\kappa_s\xi_s)\phi^{-\frac{1}{2}}\eta_t), U_t((\kappa_r\zeta_r)\phi^{-\frac{1}{2}}\eta'_t)\rangle \\
&= \langle U_t((\kappa_{s+t}\beta_{s+r.s}\xi_s)\phi^{-\frac{1}{2}}\eta_t), U_t((\kappa_{s+r}\beta_{s+r,r}\zeta_r)\phi^{-\frac{1}{2}}\eta'_t)\rangle \\
&= \langle (\kappa_{s+t}\beta_{s+r.s}\xi_s)\phi^{-\frac{1}{2}}\eta_t, (\kappa_{s+r}\beta_{s+r,r}\zeta_r)\phi^{-\frac{1}{2}}\eta'_t\rangle = \langle (\kappa_s\xi_s)\phi^{-\frac{1}{2}}\eta_t, (\kappa_r\zeta_r)\phi^{-\frac{1}{2}}\eta'_t\rangle.
\end{aligned}$$

We shall check that U_t is surjective. In the case when $s \leq t$, for $\eta = \kappa_s\eta_s \in \tilde{\mathcal{H}}$, we can conclude that the image of $(\kappa_0\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}\kappa_t^*\kappa_s\eta_s$ by U_t is η. In the case when $s > t$, let $\mathcal{D}$ be a subspace of $\tilde{\mathcal{H}}_{s-t} \otimes^M \tilde{\mathcal{H}}_t$ spanned by vectors $\eta_{s-t}\phi^{-\frac{1}{2}}\eta_t$ for all ϕ-bounded vectors $\eta_{s-t} \in \tilde{\mathcal{H}}_{s-t}$ and $\eta_t \in \tilde{\mathcal{H}}_t$. For $\eta = \kappa_s\eta_s \in \tilde{\mathcal{H}}$, η_s can be approximated by vectors $U_{s-t,t}\zeta$ for some $\zeta \in \mathcal{D}$ and we have $U_t(\kappa_{s-t} \otimes \mathrm{id}_{\tilde{\mathcal{H}}_t}) = \kappa_s U_{s-t,t}$ on $\mathcal{D}$. □

Now, the von Neumann algebra M can be represented faithfully on $\tilde{\mathcal{H}}$ by $\pi(x)\xi = \kappa_0(x(\kappa_0^*\xi))$ for each $x \in M$ and $\xi \in \tilde{\mathcal{H}}$. Note that $\pi(M) \subset \mathrm{End}(\tilde{\mathcal{H}}_M)$. For $t \geq 0$, we define a map $\theta_t : \mathrm{End}(\tilde{\mathcal{H}}_M) \to \mathrm{End}(\tilde{\mathcal{H}}_M)$ by

$$\theta_t(a) = U_t(a \otimes^M \mathrm{id}_{\tilde{\mathcal{H}}_t})U_t^*$$

for each $a \in \mathrm{End}(\tilde{\mathcal{H}}_M)$. Since $U_{s+t}(\mathrm{id}_{\tilde{\mathcal{H}}} \otimes U_{s,t}) = U_s(U_t \otimes \mathrm{id}_{\tilde{\mathcal{H}}_s})$ for all $s, t \geq 0$, $\theta = \{\theta_t\}_{t\geq 0}$ is a semigroup. Note that $\pi(1_M)\mathrm{End}(\tilde{\mathcal{H}}_M)\pi(1_M) = \pi(M)$ since we have $\pi(1_M)a\kappa_0 = \kappa_0\pi_\phi(\kappa_0\phi^{\frac{1}{2}})\pi_\phi(a\kappa_0\phi^{\frac{1}{2}})$ for $a \in \mathrm{End}(\tilde{\mathcal{H}}_M)$.

It will be proved that the semigroup $\theta = \{\theta_t\}_{t\geq 0}$ is a dilation in Theorem 4.3.5. The following proposition will ensure that θ is an E_0-semigroup.

Proposition 4.3.4. *We have*

$$U_t(\xi\phi^{-\frac{1}{2}}\tilde{\xi}(t)) \to \xi \qquad (t \to 0) \tag{4.5}$$

for all $\xi \in \mathcal{D}(\tilde{\mathcal{H}};\phi)$, where recall that $\tilde{\xi}(t) = \kappa_{(t),t}(1_M \otimes_t \phi^{\frac{1}{2}})$.

Proof. Suppose that ξ has a form

$$\kappa_s\kappa_{\mathfrak{q},s}((x_1 \otimes_{s_1} y_1\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}(x_m \otimes_{s_m} y_m\phi^{\frac{1}{2}})) \tag{4.6}$$

for some $s \geq 0$, $\mathfrak{q} = (s_1, \cdots, s_m) \in \mathfrak{P}_s$, $x_1, \cdots, x_m, y_1, \cdots, y_m \in M$, and $t < \min\{s_1, \cdots, s_m\}$. Let $\mathfrak{p}' = (t, s_1 - t, t, s_2 - t, t, \cdots, s_m - t, t) \succ (t) \vee \mathfrak{p}, \mathfrak{p} \vee (t)$. Then ξ coincides with

$$\begin{aligned}
&\kappa_{s+t}\tilde{\alpha}_{s+t,s}((x_1 \otimes_{s_1} y_1\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}(x_m \otimes_{s_m} y_m\phi^{\frac{1}{2}}))\\
&= \kappa_{s+t}\kappa_{(t)\vee\mathfrak{p},s+t}((1_M \otimes_t \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(x_1 \otimes_{s_1} y_1\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}(x_m \otimes_{s_m} y_m\phi^{\frac{1}{2}}))\\
&= \kappa_{s+t}\kappa_{\mathfrak{p}',s+t}\alpha_{\mathfrak{p}',(t)\vee\mathfrak{p}}((1_M \otimes_t \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(x_1 \otimes_{s_1} y_1\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}(x_m \otimes_{s_m} y_m\phi^{\frac{1}{2}}))\\
&= \kappa_{s+t}\kappa_{\mathfrak{p}',s+t}((1_M \otimes_t \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}((x_1 \otimes_{s_1-t} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(1_M \otimes_t y_1\phi^{\frac{1}{2}}))\phi^{-\frac{1}{2}}\\
&\qquad\qquad \cdots \phi^{-\frac{1}{2}}((x_m \otimes_{s_m-t} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(1_M \otimes_t y_m\phi^{\frac{1}{2}}))).
\end{aligned}$$

On the other hand, $U_t(\xi\phi^{-\frac{1}{2}}\tilde{\xi}(t))$ is

$$\begin{aligned}
&\kappa_{s+t}\kappa_{\mathfrak{q}\vee(t),s+t}((x_1 \otimes_{s_1} y_1\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}(x_m \otimes_{s_m} y_m\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(1_M \otimes_t \phi^{\frac{1}{2}}))\\
&= \kappa_{s+t}\kappa_{\mathfrak{p}',s+t}\alpha_{\mathfrak{p}',\mathfrak{p}\vee(t)}((x_1 \otimes_{s_1} y_1\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}(x_m \otimes_{s_m} y_m\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(1_M \otimes_t \phi^{\frac{1}{2}}))\\
&= \kappa_{s+t}\kappa_{\mathfrak{p}',s+t}(((x_1 \otimes_t \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(1_M \otimes_{s_1-t} y_1\phi^{\frac{1}{2}}))\phi^{-\frac{1}{2}}\\
&\qquad \cdots \phi^{-\frac{1}{2}}((x_m \otimes_t \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(1_M \otimes_{s_m-t} y_m\phi^{\frac{1}{2}}))\phi^{-\frac{1}{2}}(1_M \otimes_t \phi^{\frac{1}{2}})).
\end{aligned}$$

By calculations of inner products and [30, A.2 Lemma], when t tend to 0, we conclude that $U_t(\xi\phi^{-\frac{1}{2}}\tilde{\xi}(t))$ converges to ξ. □

Theorem 4.3.5. *The triple $(\mathrm{End}(\tilde{\mathcal{H}}_M), \pi(1_M), \theta)$ is a dilation of T. Moreover, if we denote by N the von Neumann algebra generated by $\bigcup_{t\geq 0}\theta_t(\pi(M))$, the triplet $(N, \pi(1_M), \theta|_N)$ is the minimal dilation of T.*

Proof. We shall show that $\pi(T_t(x)) = p\theta_t(\pi(x))p$ for all $t \geq 0$ and $x \in M$. For all $y, z \in M$, we have

$$\begin{aligned}
&\langle \phi^{\frac{1}{2}} y, \kappa_0^* \theta_t(\pi(x))\kappa_0 \phi^{\frac{1}{2}} z\rangle\rangle = \langle U_t^* \kappa_0 \phi^{\frac{1}{2}} y, (\pi(x) \otimes^M \mathrm{id}_{\tilde{\mathcal{H}}_t}) U_t^* \kappa_0 \phi^{\frac{1}{2}} z)\rangle \\
&= \langle (\kappa_0 \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \kappa_t^* \kappa_0 \phi^{\frac{1}{2}} y, (\pi(x) \otimes^M \mathrm{id}_{\tilde{\mathcal{H}}_t})(\kappa_0 \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \kappa_t^* \kappa_0 \phi^{\frac{1}{2}} z\rangle \\
&= \langle (\kappa_0 \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \kappa_t^* \kappa_0 \phi^{\frac{1}{2}} y, (\kappa_0 x \kappa_0^* \kappa_0 \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \kappa_t^* \kappa_0 \phi^{\frac{1}{2}} z\rangle \\
&= \langle (\kappa_0 \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \kappa_t^* \kappa_0 \phi^{\frac{1}{2}} y, \kappa_0(x \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \kappa_t^* \kappa_0 \phi^{\frac{1}{2}} z\rangle \\
&= \langle (\kappa_0 \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \tilde{\alpha}_{t,0} \phi^{\frac{1}{2}} y, \kappa_0(x \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \tilde{\alpha}_{t,0} \phi^{\frac{1}{2}} z\rangle \\
&= \langle (\kappa_0 \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \tilde{\alpha}_{t,0} \phi^{\frac{1}{2}} y, \kappa_0(x \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \tilde{\alpha}_{t,0} \phi^{\frac{1}{2}} z\rangle \\
&= \langle (\kappa_0 \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \kappa_{(t),t}(1_M \otimes_t \phi^{\frac{1}{2}} y), \kappa_0(x \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \kappa_{(t),t}(1_M \otimes_t \phi^{\frac{1}{2}} z)\rangle \\
&= \langle 1_M \otimes_t \phi^{\frac{1}{2}} y, x \otimes_t \phi^{\frac{1}{2}} z\rangle = \langle \phi^{\frac{1}{2}} y, T_t(x)\phi^{\frac{1}{2}} z\rangle,
\end{aligned}$$

where the fifth equality is implied by the following formula

$$\kappa_t^* \kappa_s = \begin{cases} \tilde{\alpha}_{t,s} & (t \geq s), \\ \tilde{\alpha}_{s,t}^* & (t < s). \end{cases}$$

Thus we have $T_t(x) = \kappa_0^* \theta_t(\pi(x))\kappa_0$.

Now, we discuss the continuity of the semigroup θ. For each $a \in \mathrm{End}(\tilde{\mathcal{H}}_M)$ and each ϕ-bounded vector $\xi \in \tilde{\mathcal{H}}$, by Proposition 4.3.4,

$$\begin{aligned}
\theta_t(a)\xi - a\xi &= \theta_t(a)\xi - U_t(a\xi\phi^{-\frac{1}{2}}\tilde{\xi}(t)) + U_t(a\xi\phi^{-\frac{1}{2}}\tilde{\xi}(t)) - a\xi \\
&= \theta_t(a)(\xi - U_t(\xi\phi^{-\frac{1}{2}}\tilde{\xi}(t))) + U_t(a\xi\phi^{-\frac{1}{2}}\tilde{\xi}(t)) - a\xi \to 0
\end{aligned}$$

when t tend to 0. Thus, the map $t \mapsto \theta_t(a)$ is σ-weakly continuous for each $a \in \mathrm{End}(\tilde{\mathcal{H}}_M)$, i.e. θ is an E_0-semigroup on $\mathrm{End}(\tilde{\mathcal{H}}_M)$.

For $t \geq 0$, $\mathfrak{p} = (t_1, \cdots, t_n) \in \mathfrak{P}_t$, $x_1, \cdots, x_n, y \in M$, we can check that

$$\begin{aligned}
&\theta_t(\pi(x_1))\theta_{t-t_1}(\pi(x_2)) \cdots \theta_{t_{n-1}+t_n}(\pi(x_{n-1}))\theta_{t_n}(\pi(x_n))\kappa_0 \phi^{\frac{1}{2}} y \\
&= \kappa_t \kappa_{\mathfrak{p},t}((x_1 \otimes_{t_1} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(x_2 \otimes_{t_2} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}(x_{n-1} \otimes_{t_{n-1}} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(x_n \otimes_{t_n} \phi^{\frac{1}{2}} y)).
\end{aligned}$$

Hence, we have $\overline{\mathrm{span}}(N\pi(1_M)\tilde{\mathcal{H}}) \supset \overline{\mathrm{span}}(N\kappa_0 L^2(M)) = \tilde{\mathcal{H}}$. Since the central support $c(\pi(1_M))$ of $\pi(1_M)$ in N is the projection onto $\overline{\mathrm{span}} N\pi(1_M)\tilde{\mathcal{H}}$, we have $c(\pi(1_M)) = 1_N$. We conclude that the triplet $(N, \pi(1_M), \theta)$ is the minimal dilation of T. $\square$

4.4. **Relation between the two constructions.** In this section, we provide a relation between Bhat-Skeide's and Muhly-Solel's constructions of the minimal dilation of a given CP_0-semigroup T acting on a von Neumann algebra M by the relative product system associated with T. A common point of the two ways is to establish the product systems of von Neumann bimodules by the inductive limits with respect to refinements of partitions, and to dilate T to an E_0-semigroup on the inductive limits of the product systems (see Subsection 4.1). However, Bhat-Skeide's product system $\{E_t\}_{t\geq 0}$ consists of von Neumann M-bimodule and Muhly-Solel's one $\{E(t)\}_{t\geq 0}$ consists of von Neumann M'-bimodules. A concrete aim in this section is to find the relation between E_t and $E(t)$ for each $t \geq 0$.

First, we provide some general results related to tensor product with respect to finite numbers of normal UCP-maps and relative tensor product.

Definition 4.4.1. *Let $T_1, T_2, \cdots, T_n$ be normal UCP-maps on a von Neumann algebra M. We define a Hilbert space*

$$\mathcal{H}(T_1, \cdots, T_n) = M \otimes_{T_1} (M \otimes_{T_2} (\cdots (M \otimes_{T_n} L^2(M)) \cdots),$$

and a W^*-M-M-bimodule structure by

$$x(a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes \xi) = (xa_1) \otimes a_2 \otimes \cdots \otimes a_n \otimes \xi,$$
$$(a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes \xi)x = a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes (\xi x)$$

for each $x, a_1, a_2, \cdots, a_n \in M$ and $\xi \in L^2(M)$.

Proposition 4.4.2. *For normal UCP-maps T and S on a von Neumann algebra M and a W^*-M-M-bimodule $\mathcal{H}$, we have an isomorphisms*

$$\mathcal{H}(T) \otimes^M \mathcal{H}(S) \cong \mathcal{H}(T, S), \ \mathcal{H}(T) \otimes^M \mathcal{H} \cong M \otimes_T \mathcal{H}$$

as W^*-bimodules.

Proof. If we define maps by

$$\mathcal{H}(T) \otimes^M \mathcal{H}(S) \ni (x \otimes_T y\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(z \otimes_S \phi^{\frac{1}{2}}w) \mapsto x \otimes_T ((yz) \otimes_S (\phi^{\frac{1}{2}}w)) \in \mathcal{H}(T, S),$$
$$\mathcal{H}(T) \otimes^M \mathcal{H} \ni (x \otimes_T y\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}\xi \mapsto x \otimes_T y\xi \in M \otimes_T \mathcal{H}$$

for each $x, y, z, w \in M$ and $\xi \in \mathcal{H}$, then they give isomorphisms $\mathcal{H}(T) \otimes^M \mathcal{H}(S) \cong \mathcal{H}(T, S)$ and $\mathcal{H}(T) \otimes^M \mathcal{H} \cong M \otimes_T \mathcal{H}$ as W^*-bimodules, respectively. □

Corollary 4.4.3. *Let $T_1, \cdots, T_2$ be normal UCP-maps on a von Neumann algebra M and $\mathcal{H}$ a W^*-M-M-bimodule. We have an isomorphisms*

$$\mathcal{H}(T_1) \otimes^M \cdots \otimes^M \mathcal{H}(T_n) \cong \mathcal{H}(T_1, \cdots, T_n),$$
$$\mathcal{H}(T_1) \otimes^M \cdots \otimes^M \mathcal{H}(T_n) \otimes^M \mathcal{H} \cong M \otimes_{T_1} (M \otimes_{T_2} (\cdots (M \otimes_{T_n} \mathcal{H}) \cdots)$$

as W^*-bimodules.

Proof. For example we consider the case of $n = 3$. (In general case, we can prove by an induction.) By Proposition 4.4.2, we have W^*-bimodule isomorphisms

$$\begin{aligned}
\mathcal{H}(T_1) \otimes^M \mathcal{H}(T_2) \otimes^M \mathcal{H}(T_3) &\cong \mathcal{H}(T_1) \otimes^M \mathcal{H}(T_2, T_3) \\
&\cong M \otimes_{T_1} \mathcal{H}(T_2, T_3) = \mathcal{H}(T_1, T_2, T_3), \\
\mathcal{H}(T_1) \otimes^M \mathcal{H}(T_2) \otimes^M \mathcal{H}(T_3) \otimes^M \mathcal{H} &\cong \mathcal{H}(T_1) \otimes^M \mathcal{H}(T_2) \otimes^M (M \otimes_{T_3} \mathcal{H}) \\
&\cong \mathcal{H}(T_1) \otimes^M (M \otimes_{T_2} (M \otimes_{T_1} \mathcal{H})) \\
&\cong M \otimes_{T_1} (M \otimes_{T_2} (M \otimes_{T_1} \mathcal{H})).
\end{aligned}$$

□

We fix a CP_0-semigroup $T = \{T_t\}_{t \geq 0}$ on a von Neumann algebra M which acts on a separable Hilbert space $\mathcal{H}$, and we will use the notation in Subsection 4.1 and 4.2.

For each $t \geq 0$ and $\mathfrak{p} = (t_1, \cdots, t_n) \in \mathfrak{P}_t$, the left W^*-M-module $\mathcal{H}_{\mathfrak{p},t}$ is defined as (4.1). By the second assertion in Corollary 4.4.3, we have the following isomorphism as von Neumann M'-modules.

$$\begin{aligned}
&\mathrm{Hom}({}_M\mathcal{H}, {}_M\mathcal{H}_{\mathfrak{p},t}) \cong \mathrm{Hom}({}_M\mathcal{H}, {}_M\tilde{\mathcal{H}}(\mathfrak{p},t) \otimes^M \mathcal{H}) \\
&\cong \mathrm{Hom}({}_{(M')^\circ}\mathcal{H}^* \otimes^M \mathcal{H}, {}_{(M')^\circ}\mathcal{H}^* \otimes^M \tilde{\mathcal{H}}(\mathfrak{p},t) \otimes^M \mathcal{H}) \\
&\cong \mathrm{Hom}(\mathcal{H}^* \otimes^M \mathcal{H}_{M'}, \mathcal{H}^* \otimes^M \tilde{\mathcal{H}}(\mathfrak{p},t) \otimes^M \mathcal{H}_{M'}) \\
&\cong \mathrm{Hom}(L^2(M')_{M'}, \mathcal{H}^* \otimes^M \tilde{\mathcal{H}}(\mathfrak{p},t) \otimes^M \mathcal{H}_{M'}).
\end{aligned} \tag{4.7}$$

Now, for a von Neumann algebra N, we recall the correspondence between von Neumann N-bimodules and W^*-N-bimodules by the relations

$$F \cong \mathrm{Hom}(L^2(N)_N, F \otimes_N L^2(N)_N), \ \mathcal{K} \cong \mathrm{Hom}(L^2(N)_N, \mathcal{K}_N) \otimes_N L^2(N).$$

up to isomorphism. We refer the reader to [27, Section 2] for details. The correspondence implies that the tensor category of von Neumann M-M-bimodules and the tensor category of W^*-M-M-bimodules are tensor equivalent.

Hence, by (4.7), the von Neumann M'-bimodule $\mathrm{Hom}({}_M\mathcal{H}, {}_M\mathcal{H}_{\mathfrak{p},t})$ corresponds to $\mathcal{H}^* \otimes^M \tilde{\mathcal{H}}(\mathfrak{p},t) \otimes^M \mathcal{H}$.

Theorem 4.4.4. *For normal UCP-maps $T_1, \cdots, T_n$ on a von Neumann algebra M, we have a W^*-$(M')^\circ$-$(M')^\circ$-bimodules isomorphism*

$$\begin{aligned}
&\mathcal{H}^* \otimes^M \mathcal{H}(T_1) \otimes^M \cdots \otimes^M \mathcal{H}(T_n) \otimes^M \mathcal{H} \\
&\cong (\mathcal{H}^* \otimes^M \mathcal{H}(T_1) \otimes^M \mathcal{H}) \otimes^{(M')^\circ} \cdots \otimes^{(M')^\circ} (\mathcal{H}^* \otimes^M \mathcal{H}(T_n) \otimes^M \mathcal{H}).
\end{aligned}$$

Thus, the set $\{\mathcal{H}^* \otimes^M \tilde{\mathcal{H}}_t \otimes^M \mathcal{H}\}_{t \geq 0}$ is a relative product system of W^*-$(M')^\circ$-bimodules.

On the other hand, by using Proposition 2.3.5 repeatedly, we have $E_{\mathfrak{p},t} \otimes_M L^2(M) \cong \tilde{\mathcal{H}}(\mathfrak{p},t)$ as W^*-M-bimodule. Thus, the above correspondence implies that $E_{\mathfrak{p},t}$ corresponds to $\tilde{\mathcal{H}}(\mathfrak{p},t)$. Note that we also have the direct identification via a unitary $v_{\mathfrak{p},t} : E_{\mathfrak{p},t} \otimes_M L^2(M) \to \tilde{\mathcal{H}}(\mathfrak{p},t)$ defined by

$$\begin{aligned}
&v_{\mathfrak{p},t}((x_1 \otimes y_1) \otimes \cdots \otimes (x_n \otimes y_n) \otimes \phi^{\frac{1}{2}} z) \\
&= (x_1 \otimes y_1 \phi^{\frac{1}{2}}) \phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}} (x_{n-1} \otimes y_{n-1} \phi^{\frac{1}{2}}) \phi^{-\frac{1}{2}} (x_n \otimes y_n \phi^{\frac{1}{2}} z)
\end{aligned}$$

for each $x_i, y_i, z \in M$ and a faithful normal state ϕ on M. Now, we have such correspondences between the inductive limits $E(t)$ and the left W^*-M'-modules $\mathcal{H}^* \otimes^M \tilde{\mathcal{H}}_t \otimes^M \mathcal{H}$, and between the inductive limits E_t and the left W^*-M-modules $\tilde{\mathcal{H}}_t$ as the following theorem.

Theorem 4.4.5. *Fix $t \geq 0$. We have an isomorphism $E_t \otimes_M L^2(M) \cong \tilde{\mathcal{H}}_t$ as W^*-M-bimodule by an M-bilinear unitary $V_t : E_t \otimes_M L^2(M) \to \tilde{\mathcal{H}}_t$ by*

$$V_t(\iota_{\mathfrak{p},t} X \otimes \phi^{\frac{1}{2}} z) = \kappa_{\mathfrak{p},t} v_{\mathfrak{p},t}(X \otimes \phi^{\frac{1}{2}} z)$$

for each $\mathfrak{p} \in \mathfrak{P}_t$, $X \in E_{\mathfrak{p},t}$, where recall that $\iota_{\mathfrak{p},t} : E_{\mathfrak{p},t} \to E_t$ and $\kappa_{\mathfrak{p},t} : \tilde{\mathcal{H}}(\mathfrak{p},t) \to \tilde{\mathcal{H}}_t$ are the canonical embeddings. We have also $E(t) \otimes_{M'} L^2(M') \cong \mathcal{H}^* \otimes^M \tilde{\mathcal{H}}_t \otimes^M \mathcal{H}$ as W^*-M'-bimodules.

Proof. It is clear that U_t is isometry, and hence U_t is well-defined. The bilinearity and the surjectivity is obvious. We also have the following isomorphisms

$$\begin{aligned}\mathcal{H}^* \otimes^M \tilde{\mathcal{H}}_t \otimes^M \mathcal{H} &\cong \varinjlim_{\mathfrak{p}} (\mathcal{H}^* \otimes^M \tilde{\mathcal{H}}(\mathfrak{p}, t) \otimes^M \mathcal{H}) \\ &\cong \varinjlim_{\mathfrak{p}} (\mathrm{Hom}({}_M\mathcal{H}, {}_M\mathcal{H}_{\mathfrak{p},t}) \otimes_{M'} L^2(M')) \\ &\cong (\varinjlim_{\mathfrak{p}} (\mathrm{Hom}({}_M\mathcal{H}, {}_M\mathcal{H}_{\mathfrak{p},t})) \otimes_{M'} L^2(M') \\ &= E(t) \otimes_{M'} L^2(M').\end{aligned}$$

as W^*-M'-bimodules, where each $\varinjlim_{\mathfrak{p}}$ means the inductive limits of the inductive system which is defined canonically with respect to refinements of partitions. □

We conclude that there is a one-to-one correspondence

$$E_t \longleftrightarrow \tilde{\mathcal{H}}_t, \ E(t) \longleftrightarrow \mathcal{H}^* \otimes^M \tilde{\mathcal{H}}_t \otimes^M \mathcal{H}$$

between von Neumann bimodules and W^*-bimodules for each $t \geq 0$.

Remark 4.4.6. *Skeide have provided the notion of commutant duality for* (*concrete*) *von Neumann bimodules in* [27] *and* [28] (*or see* [29, Section 6]). *For a von Neumann M-M-bimodule E, a von Neumann M'-M'-bimodule E' called the commutatnt of E is defined as the intertwiners space with respect to the left action of E. We have the duality $(E')' = E$ and it is known that the product system in Muhly-Solel's construction is the commutant of the one in Bhat-Skeide's construction.*

4.5. **The discrete case.** Let T be a normal UCP-map on a von Neumann algebra M. For $n \in \mathbb{N}$, we define a W^*-M-bimodule

$$\tilde{\mathcal{H}}_n = \underbrace{(M \otimes_T L^2(M)) \otimes^M \cdots \otimes^M (M \otimes_T L^2(M))}_{n \text{ times}}.$$

Of course, the set $\tilde{\mathcal{H}}^{d\otimes} = \{\tilde{\mathcal{H}}_n\}_{n \in \mathbb{Z}_{\geq 0}}$ satisfies $\tilde{\mathcal{H}}_m \otimes^M \tilde{\mathcal{H}}_n \cong \tilde{\mathcal{H}}_{m+n}$ for all $n, m \in \mathbb{Z}_{\geq 0}$ with the associativity, and hence it can be regarded as a discrete relative product system. By a similar way of Subsection 4.3, $\tilde{\mathcal{H}}^{d\otimes}$ has a inductive system structure, and if we denote the inductive limit by $\tilde{\mathcal{H}}^d$ with the canonical embeddings $\kappa_n^d : \tilde{\mathcal{H}}_n \to \tilde{\mathcal{H}}^d$, then for each $n \in \mathbb{Z}_{\geq 0}$, we have an isomorphism $\tilde{\mathcal{H}}^d \otimes^M \tilde{\mathcal{H}}_n \cong \tilde{\mathcal{H}}^d$ as right W^*-M-modules by a right M-linear unitary U_n^d defined by

$$U_n^d(\kappa_m^d(\xi_m)\phi^{-\frac{1}{2}}\eta_n) = \kappa_{m+n}^d(\xi_m\phi^{-\frac{1}{2}}\eta_n)$$

for each $m \in \mathbb{Z}_{\geq 0}$, $\xi \in \mathcal{D}(\tilde{\mathcal{H}}_m; \phi)$ and $\eta_n \in \tilde{\mathcal{H}}_n$. We have $U_m(U_n \otimes \mathrm{id}_{\tilde{\mathcal{H}}_n}) = U_{m+n}$ for all $n \in \mathbb{Z}_{\geq 0}$. In this case, we also have a faithful representation π^d of M on $\tilde{\mathcal{H}}^d$. Thus, a $*$-homomorphism θ on $\mathrm{End}(\tilde{\mathcal{H}}_M^d)$ defined by $\theta(a) = U_1^*(a \otimes \mathrm{id}_{\tilde{\mathcal{H}}_n})U_1$ gives a dilation of $\{T^n\}_{n \in \mathbb{Z}_{\geq 0}}$, that is, $\{\theta^n\}_{n \in \mathbb{Z}_{\geq 0}}$ is a semigroup, $\pi(M) = \pi^d(1_M)\mathrm{End}(\tilde{\mathcal{H}}_M^d)\pi^d(1_M)$ and $\pi^d(T^n(x)) = \pi^d(1_M)\theta^n(\pi(x))\pi^d(1_M)$ for all $x \in M$.

Now, suppose M acts in a separable Hilbert space $\mathcal{H}$. Bhat-Skeide's and Muhly-Solel's constructions of minimal dilations can apply to $\{T^n\}_{n \in \mathbb{Z}_{\geq 0}}$. The (discrete) product systems $\{E_n\}_{n\in\mathbb{Z}_{\geq 0}}$ and $\{E(n)\}_{n\in\mathbb{Z}_{\geq 0}}$ associated with $\{T^n\}_{n\in\mathbb{Z}_{\geq 0}}$ which appear in Bhat-Skeide's and Muhly-Solel's constructions, are defined by

$$E_n = \underbrace{\mathcal{E}_T \otimes_M \cdots \otimes_M \mathcal{E}_T}_{n \text{ times}},\ E_0 = M$$

$$E(n) = \mathrm{Hom}({}_M\mathcal{H}, {}_M \underbrace{M \otimes_T (M \otimes_T (\cdots (M \otimes_T \mathcal{H}) \cdots))}_{n \text{ times}}),\ E(0) = M'$$

for each $n \in \mathbb{Z}_{\geq 0}$, respectively. Thus, we have similar correspondences

$$E_n \longleftrightarrow \tilde{\mathcal{H}}_n,\ E(n) \longleftrightarrow \mathcal{H}^* \otimes^M \tilde{\mathcal{H}}_n \otimes^M \mathcal{H}.$$

This subsection is based on [21].

5. E_0-SEMIGROUPS AND RELATIVE PRODUCT SYSTEMS

In this section, we develop the classification theory of E_0-semigroup in terms of relative product systems. In Subsection 5.1, we will find a one-to-one correspondence between algebraic CP_0-semigroups and unital units of relative product systems and consider the continuities of CP_0-semigroups by the ones of units. In Subsection 5.2, we will give a one-to-one correspondence between cocycles and units, and the correspondence will make the classification of E_0-semigroups possible. The correspondences and the classification are analogous to Bhat-Skeide's observation in [8], however they considered algebraic CP_0-semigroups and algebraic E_0-semigroups on C^*-algebras. One of issues in the future is to find some examples related with the classification other than Example 5.2.3 and 5.2.4.

5.1. **Correspondence between CP_0-semigroups and units.** We shall define units of relative product systems similarly with the definitions of units of Arveson's and Bhat-Skeide's product systems in Definition 2.4.8 and Definition 4.1.2, respectively. A unit in the sense of Bhat-Skeide is a family of vectors in Hilbert modules, however our unit is a family of vectors in Hilbert spaces like a unit in the sense of Arveson.

Definition 5.1.1. *Let M be a von Neumann algebra with a faithful normal state ϕ and $(\mathcal{H}^\otimes = \{\mathcal{H}_t\}_{t\geq 0}, \{U_{s,t}\}_{s,t\geq 0})$ a relative product system over M. A family $\xi^\otimes = \{\xi(t)\}_{t\geq 0}$ of $\xi(t) \in \mathcal{D}(\mathcal{H}_t;\phi)$ is called a unit of $\mathcal{H}^\otimes$ with respect to ϕ if $\xi(0) = \phi^{\frac{1}{2}}$ and $U_{s,t}(\xi(s)\phi^{-\frac{1}{2}}\xi(t)) = \xi(s+t)$ for all $s,t \geq 0$. If a unit $\xi^\otimes = \{\xi(t)\}_{t\geq 0}$ satisfies $\pi_\phi(\xi(t))^*\pi_\phi(\xi(t)) = 1_M$ ($\|\pi_\phi(\xi(t))^*\pi_\phi(\xi(t))\| \leq 1$), it is said to be unital (contractive).*

Now, we fix a faithful normal state ϕ on a von Neumann algebra M. When we say a unit merely, suppose that it is a unit with respect to ϕ, unless otherwise specified.

Let $\xi^\otimes = \{\xi(t)\}_{t\geq 0}$ be a unital unit of a relative product system $\mathcal{H}^\otimes = \{\mathcal{H}_t\}_{t\geq 0}$ with respect to ϕ. We define a unital linear map $T_t^{\xi^\otimes}$ on M by

$$T_t^{\xi^\otimes}(x) = \pi_\phi(\xi(t))^*\pi_\phi(x\xi(t)) \in M \tag{5.1}$$

for $t \geq 0$ and $x \in M$.

Lemma 5.1.2. *The family $T^{\xi^\otimes} = \{T_t^{\xi^\otimes}\}_{t\geq 0}$ is an algebraic CP_0-semigroup.*

Proof. By the definition, it is clear that each $T_t^{\xi^\otimes}$ is normal completely positive map.

For $s,t \geq 0$ and $x,y,z \in M$, we can compute as

$$\begin{aligned}
&\langle T_s^{\xi^\otimes}(T_t^{\xi^\otimes}(x))\phi^{\frac{1}{2}}y, \phi^{\frac{1}{2}}z\rangle = \langle \pi_\phi(\xi(s))^*\pi_\phi(\pi_\phi(\xi(t))^*\pi(x\xi(t))\xi(s))\phi^{\frac{1}{2}}y, \phi^{\frac{1}{2}}z\rangle \\
&= \langle \pi_\phi(\xi(t))^*\pi(x\xi(t))\xi(s)y, \xi(s)z\rangle = \langle x\xi(t)\phi^{-\frac{1}{2}}\xi(s)y, \xi(t)\phi^{-\frac{1}{2}}\xi(s)z\rangle \\
&= \langle xU_{s,t}(\xi(t)\phi^{-\frac{1}{2}}\xi(s))y, U_{s,t}(\xi(t)\phi^{-\frac{1}{2}}\xi(s)z)\rangle = \langle x\xi(t+s)y, \xi(t+s)z\rangle \\
&= \langle \pi_\phi(\xi(s+t))^*\pi_\phi(x\xi(s+t))\phi^{\frac{1}{2}}y, \phi^{\frac{1}{2}}z\rangle = \langle T_{s+t}^{\xi^\otimes}(x)\phi^{\frac{1}{2}}y, \phi^{\frac{1}{2}}z\rangle,
\end{aligned}$$

and hence $T_s^{\xi^\otimes} \circ T_t^{\xi^\otimes} = T_{s+t}^{\xi^\otimes}$. □

Remark 5.1.3. *In Bhat-Skeide's*[8] *observation, for a unit $Z^{\otimes} = \{Z^t\}_{t\geq 0}$ of a product system $E^{\otimes} = \{E_t\}_{t\geq 0}$ of Hilbert bimodule over a C^*-algebra A, they defined an algebraic CP_0-semigroup $T^{Z^{\otimes}}$ on A by $T_t^{Z^{\otimes}}(x) = \langle Z^t, xZ^t\rangle$ for $t \geq 0$ and $x \in A$.*

Let $\xi^{\otimes} = \{\xi(t)\}_{t\geq 0}$ be a unital unit of a relative product system $(\mathcal{H}^{\otimes}, \{U_{s,t}\}_{s,t})$. Similarly for Subsection 4.3, we can define an inductive system structure on $\mathcal{H}^{\otimes}$ and have an isomorphism $\varinjlim_s \mathcal{H}_s \otimes^M \mathcal{H}_t \cong \varinjlim_s \mathcal{H}_s$ for each $t \geq 0$ as follows. For $0 \leq s \leq t$, we define a right M-linear isometry $\beta_{t,s} : \mathcal{H}_s \to \mathcal{H}_t$ by

$$\beta_{t,s}(\xi) = U_{t-s,s}(\xi(t-s)\phi^{-\frac{1}{2}}\xi)$$

for each $\xi \in \mathcal{H}_s$. Note that $\beta_{t,s} \circ \beta_{s,r} = \beta_{t,r}$ for $0 \leq r \leq s \leq t$ by the proof of Lemma 4.3.1. Let $\mathcal{H}$ be the inductive limit of the inductive system $(\mathcal{H}^{\otimes}, \{\beta_{t,s}\}_{s\leq t})$ and $\kappa_t : \mathcal{H}_t \to \mathcal{H}$ the canonical embedding for each $t \geq 0$. The right W^*-M-module $\mathcal{H}$ is called an inductive limit of the pair $(\mathcal{H}^{\otimes}, \xi^{\otimes})$.

Fix $t \geq 0$. By a similar way of Theorem 4.3.3, if we define

$$U_t((\kappa_s\xi_s)\phi^{-\frac{1}{2}}\eta_t) = \kappa_{s+t}U_{s,t}(\xi_s\phi^{-\frac{1}{2}}\eta_t).$$

for $s \geq 0$, $\xi_s \in \mathcal{D}(\mathcal{H}_t; \phi)$ and $\eta_t \in \mathcal{H}_t$, then U_t can be extended to a unitary from $\mathcal{H} \otimes^M \mathcal{H}_t$ onto $\mathcal{H}$.

We can describe the continuity for the algebraic CP_0-semigroup $T^{\xi^{\otimes}}$ as the one for the unit $\xi^{\otimes}$ as the following theorem.

Theorem 5.1.4. *The semigroup $T^{\xi^{\otimes}} = \{T_t^{\xi^{\otimes}}\}_{t\geq 0}$ associated with a unital unit $\xi^{\otimes} = \{\xi(t)\}_{t\geq 0}$, is a CP_0-semigroup if and only if*

$$U_t(\kappa_0(x\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}\xi(t)) = \kappa_t(x\xi(t)) \to \kappa_0(x\phi^{\frac{1}{2}}) \qquad (t \to +0) \tag{5.2}$$

holds for each $x \in M$.

Proof. Suppose (5.2) for all $x \in M$. For $t \geq 0$ and $x, y, z \in M$, we have

$$\begin{aligned}\langle T_t^{\xi^{\otimes}}(x)\phi^{\frac{1}{2}}y, \phi^{\frac{1}{2}}z\rangle &= \langle \pi_\phi(\xi(t))^*\pi_\phi(x\xi(t))\phi^{\frac{1}{2}}y, \phi^{\frac{1}{2}}z\rangle = \langle x\xi(t)y, \xi(t)z\rangle \\ &= \langle U_t(\kappa_0(x\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}\xi(t)y), U_t(\kappa_0(\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}\xi(t)z)\rangle.\end{aligned}$$

Thus, when $t \to +0$, the inner product $\langle T_t^{\xi^{\otimes}}(x)\phi^{\frac{1}{2}}y, \phi^{\frac{1}{2}}z\rangle$ tends to

$$\langle \kappa_0(x\phi^{\frac{1}{2}})y, \kappa_0(\phi^{\frac{1}{2}})z\rangle = \langle x\phi^{\frac{1}{2}}y, \phi^{\frac{1}{2}}z\rangle.$$

We conclude that for every $x \in M$, $T_t^{\xi^{\otimes}}(x) \to x$ weakly when $t \to +0$, and hence $T^{\xi^{\otimes}}$ is a CP_0-semigroup by the boundedness of $\{\|T_t^{\xi^{\otimes}}(x)\|\}_{t\geq 0}$.

Conversely, we assume that $T^{\xi^{\otimes}}$ is a CP_0-semigroup. We can compute as

$$\begin{aligned}\langle x\xi(t), x\xi(t)\rangle &= \langle xU_{t,0}(\xi(t)\phi^{-\frac{1}{2}}\phi^{\frac{1}{2}}), xU_{t,0}(\xi(t)\phi^{-\frac{1}{2}}\phi^{\frac{1}{2}})\rangle \\ &= \langle \xi(t)\phi^{-\frac{1}{2}}\phi^{\frac{1}{2}}, x^*x\xi(t)\phi^{-\frac{1}{2}}\phi^{\frac{1}{2}}\rangle \\ &= \langle \phi^{\frac{1}{2}}, \pi_\phi(\xi(t))^*\pi_\phi(x^*x\xi(t))\phi^{\frac{1}{2}}\rangle = \langle \phi^{\frac{1}{2}}, T_t^{\xi^{\otimes}}(x^*x)\phi^{\frac{1}{2}}\rangle,\end{aligned}$$

$$\begin{aligned}
\langle \kappa_t(x\xi(t)), \kappa_0(x\phi^{\frac{1}{2}})\rangle &= \langle \kappa_t(x\xi(t)), \kappa_t\beta_{t,0}\kappa_0(x\phi^{\frac{1}{2}})\rangle \\
&= \langle x\xi(t), U_{t,0}(\xi(t)\phi^{-\frac{1}{2}}\kappa_0(x\phi^{\frac{1}{2}}))\rangle \\
&= \langle U_{t,0}(x\xi(t)\phi^{-\frac{1}{2}}\phi^{\frac{1}{2}}), U_{t,0}(\xi(t)\phi^{-\frac{1}{2}}\kappa_0(x\phi^{\frac{1}{2}}))\rangle \\
&= \langle \phi^{\frac{1}{2}}, \pi_\phi(\xi(t))^*\pi_\phi(x^*\xi(t))x\phi^{\frac{1}{2}}\rangle = \langle \phi^{\frac{1}{2}}, T_t^{\xi^\otimes}(x^*)x\phi^{\frac{1}{2}}\rangle.
\end{aligned}$$

Thus, when $t \to +0$, we have $\|\kappa_t(x\xi(t)) - \kappa_0(x\phi^{\frac{1}{2}})\|^2 \to 0$. □

We have constructed the algebraic CP_0-semigroup T^ξ from a given unital unit $\xi^\otimes$ of a relative product system $\mathcal{H}^\otimes$. Conversely, in Subsection 4.3, we constructed the relative product system $\tilde{\mathcal{H}}^\otimes = \{\tilde{\mathcal{H}}_t\}_{t\geq 0}$ from a CP_0-semigroup $T = \{T_t\}_{t\geq 0}$ on M and the unital unit $\tilde{\xi}^\otimes = \{\tilde{\xi}(t)\}_{t\geq 0}$ satisfying the condition (5.2) for all $x \in M$ in Remark 4.3.2. Note that we can apply the method to algebraic CP_0-semigroups, and then the unital unit $\tilde{\xi}^\otimes = \{\tilde{\xi}(t)\}_{t\geq 0}$ does not necessarily satisfy (5.2) for all $x \in M$. The main aim in this subsection is to show that the correspondence

$$(\mathcal{H}^\otimes, \xi) \mapsto T^{\xi^\otimes},\ T \mapsto (\tilde{\mathcal{H}}, \tilde{\xi}^\otimes)$$

is one-to-one up to isomorphisms. As the corollary of the result, we will show that a unit with (5.2) for all $x \in M$ satisfies $U_t(\xi\phi^{-\frac{1}{2}}\xi(t)) \to \xi$ $(t \to +0)$ for all $\xi \in \mathcal{D}(\mathcal{H};\phi)$ in Corollary 5.1.8.

For an algebraic CP_0-semigroup $T = \{T_t\}_{t\geq 0}$, it turns out that the algebraic CP_0-semigroup $T^{\tilde{\xi}^\otimes} = \{T_t^{\tilde{\xi}^\otimes}\}_{t\geq 0}$ associated with $\tilde{\xi}^\otimes$ as (5.1) coincides to $T = \{T_t\}_{t\geq 0}$. To show the converse, we introduce the natural notion of isomorphism between relative product systems and a generating property for units as the following.

Definition 5.1.5. *Let $(\mathcal{H}^\otimes, \{U_{s,t}\}_{s,t\geq 0})$ and $(\mathcal{K}^\otimes, \{V_{s,t}\}_{s,t\geq 0})$ be relative product systems over M. An isomorphism is a family $u^\otimes = \{u_t\}_{t\geq 0}$ of M-bilinear unitaries $u_t : \mathcal{H}_t \to \mathcal{K}_t$ satisfying*

$$V_{s,t}((u_s\xi_s)\phi^{-\frac{1}{2}}(u_t\eta_t)) = u_{s+t}U_{s,t}(\xi_s\phi^{-\frac{1}{2}}\eta_t) \tag{5.3}$$

for all $s,t \geq 0, \xi_s \in \mathcal{D}(\mathcal{H}_s;\phi)$ and $\eta_t \in \mathcal{H}_t$. Then $\mathcal{H}^\otimes$ is said to be isomorphic to $\mathcal{K}^\otimes$ and we denote as $\mathcal{H}^\otimes \cong \mathcal{K}^\otimes$.

We introduce the notion of generating for units. Let $\mathcal{H}^\otimes = \{\mathcal{H}_t\}_{t\geq 0}$ be a relative product system over M with M-bilinear unitaries $\{U_{s,t}\}_{s,t\geq 0}$. For $t \geq 0$ and $\mathfrak{p} = (t_1, \cdots, t_n) \in \mathfrak{P}_t$, we denote the M-bilinear unitary

$$U(\mathfrak{p}) = U_{t_1,t_1'}(\mathrm{id}_{t_1} \otimes U_{t_2,t_2'})(\mathrm{id}_{t_1,t_2} \otimes U_{t_3,t_3'}) \cdots (\mathrm{id}_{t_1,\cdots,t_{n-2}} \otimes U_{t_{n-1},t_{n-1}'})$$

from $\mathcal{H}_{t_1} \otimes^M \cdots \otimes^M \mathcal{H}_{t_n}$ onto $\mathcal{H}_t$, where $t_i' = t_{i+1} + \cdots + t_n$ and $\mathrm{id}_{t_1,\cdots,t_i} = \mathrm{id}_{t_1} \otimes \cdots \otimes \mathrm{id}_{t_i}$. A unital unit $\xi^\otimes = \{\xi(t)\}_{t\geq 0}$ is said to be generating when the set

$$\{U(\mathfrak{p})(x_1\xi(t_1)\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}x_{n-1}\xi(t_{n-1})\phi^{-\frac{1}{2}}x_n\xi(t_n)y) \mid \mathfrak{p} \in \mathfrak{P}_t,\ x_1, \cdots, x_n, y \in M\}$$

is dense in $\mathcal{H}_t$ for all $t \geq 0$.

Note that the unital unit $\tilde{\xi}$ associated with an algebraic CP_0-semigroup T is generating.

Proposition 5.1.6. *Let $(\mathcal{H}^\otimes, \xi^\otimes)$ be a pair of a relative product system $\mathcal{H}^\otimes = \{\mathcal{H}_t\}_{t\geq 0}$ and a unital unit $\xi^\otimes = \{\xi(t)\}_{t\geq 0}$, and $u = \{u_t\}_{t\geq 0}$ an isomorphism from $\mathcal{H}^\otimes = \{\mathcal{H}_t\}_{t\geq 0}$ to a relative product system $\mathcal{K}^\otimes = \{\mathcal{K}_t\}_{t\geq 0}$. We denote $\eta(t) = u_t\xi(t)$ for each $t \geq 0$ and $\eta^\otimes = \{\eta(t)\}_{t\geq 0}$.*

(1) *$\eta^\otimes = \{\eta(t)\}_{t\geq 0}$ is a unital unit.*
(2) *If $\xi^\otimes$ is generating, $\eta^\otimes = \{\eta(t)\}_{t\geq 0}$ is so.*
(3) *Let $\mathcal{H}$ and $\mathcal{K}$ be the inductive limits of $(\mathcal{H}^\otimes, \xi^\otimes)$ and $(\mathcal{K}^\otimes, \eta^\otimes)$, respectively. If $\xi^\otimes$ satisfies* (5.2) *for all $x \in M$, so is $\eta^\otimes$.*

Proof. (1) and (2) are clear by the definitions.

We shall show (3). We denote unitaries associated with the relative product systems $\mathcal{H}^\otimes$ and $\mathcal{K}^\otimes$ by $U_{s,t}$ and $V_{s,t}$, respectively. Suppose $\beta_{t,s}$ and $\gamma_{t,s}$ give the inductive system structures on $\mathcal{H}^\otimes$ and $\mathcal{K}^\otimes$. For each $x \in M$, we have

$$\begin{aligned} u_t\beta_{t,0}(x\phi^{\frac{1}{2}}) &= u_tU_{t,0}(\xi(t)\phi^{-\frac{1}{2}}x\phi^{\frac{1}{2}}) = V_{t,0}(u_t\xi(t)\phi^{-\frac{1}{2}}u_0x\phi^{\frac{1}{2}}) \\ &= V_{t,0}(\eta(t)\phi^{-\frac{1}{2}}x\phi^{\frac{1}{2}}) = \gamma_{t,0}(x\phi^{\frac{1}{2}}). \end{aligned}$$

Thus, $\|\lambda_t(x\eta(t)) - \lambda_0(x\phi^{\frac{1}{2}})\| = \|\kappa_t(x\xi(t)) - \kappa_0(x\phi^{\frac{1}{2}})\|$, where $\kappa_t : \mathcal{H}_t \to \mathcal{H}$ and $\lambda_t : \mathcal{K}_t \to \mathcal{K}$ are the canonical embeddings. □

Theorem 5.1.7. *Let $(\mathcal{H}^\otimes, \{U_{s,t}\}_{s,t\geq 0})$ be a relative product system over M with a generating unital unit $\xi^\otimes = \{\xi(t)\}_{t\geq 0}$, $T^{\xi^\otimes} = \{T^{\xi^\otimes}_t\}_{t\geq 0}$ the algebraic CP_0-semigroup associated with $\xi^\otimes$ and $(\tilde{\mathcal{H}}^\otimes, \{\tilde{U}_{s,t}\}_{s,t\geq 0})$ the relative product system associated with $T^{\xi^\otimes}$ with the unit $\tilde{\xi}^\otimes$ defined in* Remark 4.3.2. *Then, there is an isomorphism from $\tilde{\mathcal{H}}^\otimes$ onto $\mathcal{H}^\otimes$ preserving the unit.*

Proof. For $t \geq 0$ and a partition $\mathfrak{p} = (t_1, \cdots, t_n) \in \mathfrak{P}_t$, we define a map $u_t : \tilde{\mathcal{H}}_t \to \mathcal{H}_t$ by

$$\begin{aligned} &u_t(\tilde{\kappa}_{\mathfrak{p},t}((x_1 \otimes_{t_1} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}(x_{n-1} \otimes_{t_{n-1}} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(x_n \otimes_{t_n} \phi^{\frac{1}{2}}y))) \\ &= U(\mathfrak{p})(x_1\xi(t_1)\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}x_{n-1}\xi(t_{n-1})\phi^{-\frac{1}{2}}x_n\xi(t_n)y) \end{aligned}$$

for each $x_1, \cdots, x_n, y \in M$, where $\tilde{\kappa}_{\mathfrak{p},t} : \tilde{\mathcal{H}}(\mathfrak{p}, t) \to \tilde{\mathcal{H}}_t$ is the canonical embedding. It is proved that u_t is an isometry by the similar way of the proof of Theorem 4.3.3. Since $\xi^\otimes$ is generating, u_t can be extended as unitary from $\tilde{\mathcal{H}}_t$ onto $\mathcal{H}_t$.

We must show that $U_{s,t}((u_s\xi_s)\phi^{-\frac{1}{2}}(u_t\eta_t)) = u_{s+t}\tilde{U}_{s,t}(\xi_s\phi^{-\frac{1}{2}}\eta_t)$ for all $\xi_s \in \mathcal{D}(\tilde{\mathcal{H}}_s; \phi)$ and all $\eta_t \in \tilde{\mathcal{H}}_t$. It enough to show it for

$$\begin{aligned} \xi_s &= \tilde{\kappa}_{\mathfrak{q},s}((x_1 \otimes_{s_1} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}(x_{m-1} \otimes_{s_{m-1}} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(x_m \otimes_{s_m} \phi^{\frac{1}{2}})), \\ \eta_t &= \tilde{\kappa}_{\mathfrak{p},t}((z_1 \otimes_{t_1} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}(z_{n-1} \otimes_{t_{n-1}} \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(z_n \otimes_{t_n} \phi^{\frac{1}{2}}w)), \end{aligned}$$

where $\mathfrak{q} = (s_1, \cdots, s_m) \in \mathfrak{P}_s$, $\mathfrak{p} = (t_1, \cdots, t_n) \in \mathfrak{P}_t$ and $x_1, \cdots, x_m, z_1, \cdots, z_n, w \in M$. If we put

$$\begin{aligned} \zeta_1 &= x_1\xi(s_1)\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}x_m\xi(s_m), \\ \zeta_2 &= z_1\xi(t_1)\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}}z_{n-1}\xi(t_{n-1})\phi^{-\frac{1}{2}}z_n\xi(t_n)w, \end{aligned}$$

then we have

$$\begin{aligned}
u_{s+t}\tilde{U}_{s,t}(\xi_s\phi^{-\frac{1}{2}}\eta_t) &= U_{s_1,s_1'+t}(\mathrm{id}_{s_1}\otimes U_{s_2,s_2'+t})\cdots(\mathrm{id}_{s_1,\cdots,s_{m-1}}\otimes U_{s_m,t})(\mathrm{id}_{s_1,\cdots,s_m}\otimes U_{t_1,t_1'}) \\
&\qquad \cdots(\mathrm{id}_{s_1,\cdots,s_m}\otimes \mathrm{id}_{t_1,\cdots,t_{n-2}}\otimes U_{t_{n-1},t_{n-1}'})(\zeta_1\phi^{-\frac{1}{2}}\zeta_2) \\
&= U_{s_1,s_1'+t}(\mathrm{id}_{s_1}\otimes U_{s_2,s_2'+t})(\mathrm{id}_{s_1,s_2}\otimes U_{s_3,s_3'+t}) \\
&\qquad \cdots(\mathrm{id}_{s_1,\cdots,s_{m-1}}\otimes U_{s_m,t})(\zeta_1\phi^{-\frac{1}{2}}U(\mathfrak{p})\zeta_2). \qquad (5.4)
\end{aligned}$$

Also, we have

$$\begin{aligned}
&U_{s,t}((u_s\xi_s)\phi^{-\frac{1}{2}}(u_t\eta_t)) = U_{s,t}(U(\mathfrak{q})(\zeta_1)\phi^{-\frac{1}{2}}U(\mathfrak{p})(\zeta_2)) \\
&= U_{s,t}(U_{s_1,s_1'}\otimes \mathrm{id}_t)(\mathrm{id}_{s_1}\otimes U_{s_2,s_2'}\otimes \mathrm{id}_t)(\mathrm{id}_{s_1,s_2}\otimes U_{s_3,s_3'}\otimes \mathrm{id}_t) \\
&\qquad \cdots(\mathrm{id}_{s_1,\cdots,s_{n-2}}\otimes U_{s_{n-1},s_{n-1}'}\otimes \mathrm{id}_t)(\zeta_1\phi^{-\frac{1}{2}}U(\mathfrak{p})(\zeta_2)).
\end{aligned}$$

By the associativity of $\{U_{s,t}\}_{s,t\geq 0}$, we have

$$\begin{aligned}
&U_{s,t}((u_s\xi_s)\phi^{-\frac{1}{2}}(u_t\eta_t)) = U_{s_1,s_1'+t}(\mathrm{id}_{s_1}\otimes U_{s_1',t})(\mathrm{id}_{s_1}\otimes U_{s_2,s_2'}\otimes \mathrm{id}_t)(\mathrm{id}_{s_1,s_2}\otimes U_{s_3,s_3'}\otimes \mathrm{id}_t) \\
&\qquad \cdots(\mathrm{id}_{s_1,\cdots,s_{n-2}}\otimes U_{s_{n-1},s_{n-1}'}\otimes \mathrm{id}_t)(\zeta_1\phi^{-\frac{1}{2}}U(\mathfrak{p})(\zeta_2)), \\
&= U_{s_1,s_1'+t}(\mathrm{id}_{s_1}\otimes U_{s_1',t}(U_{s_2,s_2'}\otimes \mathrm{id}_t))(\mathrm{id}_{s_1,s_2}\otimes U_{s_3,s_3'}\otimes \mathrm{id}_t) \\
&\qquad \cdots(\mathrm{id}_{s_1,\cdots,s_{n-2}}\otimes U_{s_{n-1},s_{n-1}'}\otimes \mathrm{id}_t)(\zeta_1\phi^{-\frac{1}{2}}U(\mathfrak{p})(\zeta_2)) \\
&= U_{s_1,s_1'+t}(\mathrm{id}_{s_1}\otimes U_{s_2,s_2'+t}(\mathrm{id}_{s_2}\otimes U_{s_2',t}))(\mathrm{id}_{s_1,s_2}\otimes U_{s_3,s_3'}\otimes \mathrm{id}_t) \\
&\qquad \cdots(\mathrm{id}_{s_1,\cdots,s_{n-2}}\otimes U_{s_{n-1},s_{n-1}'}\otimes \mathrm{id}_t)(\zeta_1\phi^{-\frac{1}{2}}U(\mathfrak{p})(\zeta_2)) \\
&= U_{s_1,s_1'+t}(\mathrm{id}_{s_1}\otimes U_{s_2,s_2'+t})(\mathrm{id}_{s_1,s_2}\otimes U_{s_2',t}(U_{s_3,s_3'}\otimes \mathrm{id}_t))(\mathrm{id}_{s_1,s_2,s_3}\otimes U_{s_4,s_4'}\otimes \mathrm{id}_t) \\
&\qquad \cdots(\mathrm{id}_{s_1,\cdots,s_{n-2}}\otimes U_{s_{n-1},s_{n-1}'}\otimes \mathrm{id}_t)(\zeta_1\phi^{-\frac{1}{2}}U(\mathfrak{p})(\zeta_2)).
\end{aligned}$$

By repeating the above calculations and (5.4), we have

$$\begin{aligned}
&U_{s,t}((u_s\xi_s)\phi^{-\frac{1}{2}}(u_t\eta_t)) \\
&= U_{s_1,s_1'+t}(\mathrm{id}_{s_1}\otimes U_{s_2,s_2'+t})(\mathrm{id}_{s_1,s_2}\otimes U_{s_3,s_3'+t})\cdots(\mathrm{id}_{s_1,\cdots,s_{m-1}}\otimes U_{s_m,t})(\zeta_1\phi^{-\frac{1}{2}}U(\mathfrak{p})\zeta_2) \\
&= u_{s+t}\tilde{U}_{s,t}(\xi_s\phi^{-\frac{1}{2}}\eta_t).
\end{aligned}$$

We conclude that $\{u_t\}_{t\geq 0}$ gives an isomorphism. □

We conclude that the correspondence between algebraic CP_0-semigroups T and pairs $(\mathcal{H}^{\otimes}, \xi^{\otimes})$ of relative product systems $\mathcal{H}^{\otimes}$ and generating unital units $\xi^{\otimes}$, is one-to-one. By the correspondence, we shall show that a unit with the condition (5.2) for all $x \in M$ has a stronger condition automatically. For this, a few preparations are required as the following.

Let $\mathcal{H}^{\otimes} = \{\mathcal{H}_t\}_{t\geq 0}$ and $\mathcal{K}^{\otimes} = \{\mathcal{K}_t\}_{t\geq 0}$ be relative product systems on a von Neumann algebra M with unital units $\xi^{\otimes} = \{\xi(t)\}_{t\geq 0}$ and $\eta^{\otimes} = \{\eta(t)\}_{t\geq 0}$, respectively. Suppose $u^{\otimes} = \{u_t\}_{t\geq 0}$ is an isomorphism from $\mathcal{H}^{\otimes}$ onto $\mathcal{K}^{\otimes}$. Then, we can define the canonical right M-linear unitary u from the inductive limit $\mathcal{H}$ of $\mathcal{H}^{\otimes}$ onto the one $\mathcal{K}$ of $\mathcal{K}^{\otimes}$ by

$$u(\kappa_t^{\mathcal{H}}(\xi_t)) = \kappa_t^{\mathcal{K}}u_t(\xi_t)$$

for each $t \geq 0$ and $\xi_t \in \mathcal{H}_t$, where $\kappa_t^{\mathcal{H}}$ means the canonical embedding from $\mathcal{H}_t$ into $\mathcal{H}$ and $\kappa_t^{\mathcal{K}}$ is similar.

Corollary 5.1.8. *Let $(\mathcal{H}^{\otimes}, \xi^{\otimes})$ be a pair of a relative product system $\mathcal{H}^{\otimes} = \{\mathcal{H}_t\}_{t\geq 0}$ and a generating unital unit $\xi^{\otimes} = \{\xi(t)\}_{t\geq 0}$, and $\mathcal{H}$ the inductive limit of $(\mathcal{H}^{\otimes}, \xi^{\otimes})$. When $\xi^{\otimes}$ satisfies the condition (5.2) for all $x \in M$, we have*

$$U_t(\xi\phi^{-\frac{1}{2}}\xi(t)) \to \xi \qquad (t \to +0) \tag{5.5}$$

for all $\xi \in \mathcal{D}(\mathcal{H};\phi)$. Then the unit $\xi^{\otimes}$ is said to be continuous.

Proof. We use the notations in Theorem 5.1.7 and its proof. By Proposition 4.3.4, the unit $\tilde{\xi}^{\otimes} = \{\tilde{\xi}(t) = \tilde{\kappa}_{(t),t}(1_M \otimes_t \phi^{\frac{1}{2}})\}_{t\geq 0}$ satisfies

$$U_t(\xi\phi^{-\frac{1}{2}}\tilde{\xi}(t)) \to \xi \qquad (t \to +0) \tag{5.6}$$

for all $\xi \in \mathcal{D}(\tilde{\mathcal{H}};\phi)$. Suppose $\tilde{\mathcal{H}}$ is the inductive limit of $\tilde{\mathcal{H}}^{\otimes}$ and $\tilde{\kappa}_t : \tilde{\mathcal{H}}_t \to \tilde{\mathcal{H}}$ is the canonical embedding for each $t \geq 0$. Let u be the unitary from $\tilde{\mathcal{H}}$ onto $\mathcal{H}$ induced from $\{u_t\}_{t\geq 0}$ as the above arguments, and $\tilde{U}_t$ the unitary giving the right W^*-M-module isomorphism $\tilde{\mathcal{H}} \otimes^M \tilde{\mathcal{H}}_t \cong \tilde{\mathcal{H}}$ for each $t \geq 0$. For each $\tilde{\xi}_s \in \tilde{\mathcal{H}}_s$, we have

$$\begin{aligned} u\tilde{U}_t(\tilde{\kappa}_s\tilde{\xi}_s\phi^{-\frac{1}{2}}\tilde{\xi}(t)) &= u\tilde{\kappa}_{s+t}\tilde{U}_{s,t}(\tilde{\xi}_s\phi^{-\frac{1}{2}}\tilde{\xi}(t)) = \kappa_{s+t}u_{s+t}\tilde{U}_{s,t}(\tilde{\xi}_s\phi^{-\frac{1}{2}}\tilde{\xi}(t)) \\ &= \kappa_{s+t}U_{s,t}((u_s\tilde{\xi}_s)\phi^{-\frac{1}{2}}u_t\tilde{\xi}(t)) = U_t((\kappa_s u_s\tilde{\xi}_s)\phi^{-\frac{1}{2}}u_t\tilde{\xi}(t)) \\ &= U_t((u\tilde{\kappa}_s\tilde{\xi}_s)\phi^{-\frac{1}{2}}u_t\tilde{\xi}(t)), \end{aligned}$$

where the third equality is implied from (5.3). Thus, by (5.6) for all $\xi \in \mathcal{D}(\mathcal{H};\phi)$, we have

$$\begin{aligned} \|U_t(\xi\phi^{-\frac{1}{2}}\xi(t)) - \xi\| &= \|U_t(uu^*\xi\phi^{-\frac{1}{2}}u_t u_t^*\xi(t)) - \xi\| \\ &= \|uU_t(u^*\xi\phi^{-\frac{1}{2}}u_t^*\xi(t)) - \xi\| \\ &= \|U_t(u^*\xi\phi^{-\frac{1}{2}}u_t^*\xi(t)) - u^*\xi\| \to 0 \end{aligned}$$

when $t \to +0$. ∎

5.2. **Classification of E_0-semigroups by relative product systems.** In this subsection, we will show that there is a one-to-one correspondence between contractive adapted right cocycles and units of relative product systems. By the correspondence, E_0-semigroups will be classified by relative product systems up to the unitary cocycle equivalence.

Let $(\mathcal{H}^{\otimes}, \xi^{\otimes})$ be a pair of a relative product system $\mathcal{H}^{\otimes}$ on a von Neumann algebra M and a continuous unital unit $\xi^{\otimes}$, and $\mathcal{H}$ the inductive limit of $(\mathcal{H}^{\otimes}, \xi^{\otimes})$. We can construct an E_0-semigroup $\theta = \{\theta_t\}_{t\geq 0}$ on $\mathrm{End}(\mathcal{H}_M)$ by

$$\theta_t(a) = U_t(a \otimes^M \mathrm{id}_t)U_t^* \tag{5.7}$$

for each $a \in \mathrm{End}(\mathcal{H}_M)$, where U_t gives the isomorphism $\mathcal{H} \otimes^M \mathcal{H}_t \cong \mathcal{H}$. (The continuity of θ is implied from the one of the unit $\xi^{\otimes}$ and the proof of Theorem 4.3.5.) The E_0-semigroup θ is called the maximal dilation of $(\mathcal{H}^{\otimes}, \xi^{\otimes})$. A right cocycle $w = \{w_t\}_{t\geq 0}$ for θ is called adapted if $\kappa_t\kappa_t^* w_t \kappa_t\kappa_t^* = w_t$ for all $t \geq 0$, where κ_t is the canonical embedding from $\mathcal{H}_t$ into $\mathcal{H}$.

Theorem 5.2.1. *Let $\theta = \{\theta_t\}_{t\geq 0}$ be the E_0-semigroup associated with a pair $(\mathcal{H}^{\otimes}, \xi^{\otimes})$ of a relative product system $\mathcal{H}^{\otimes} = \{\mathcal{H}_t\}_{t\geq 0}$ and a continuous unital unit $\xi^{\otimes} = \{\xi(t)\}_{t\geq 0}$ by* (5.7). *There is a one-to-one correspondence between contractive adapted right cocycles $w = \{w_t\}_{t\geq 0}$ on $\mathrm{End}(\mathcal{H}_M)$ and contractive units $\eta^{\otimes} = \{\eta(t)\}_{t\geq 0}$ in $\mathcal{H}^{\otimes}$ by relations $\eta(t) = \kappa_t^* w_t \kappa_0 \phi^{\frac{1}{2}}$ and $w_t = \pi_\phi(\kappa_t \eta(t))\pi_\phi(\kappa_0\phi^{\frac{1}{2}})^*$ for all $t \geq 0$.*

Proof. Let $\{U_{s,t}\}_{s,t\geq 0}$ be a family giving the relative product system structure of $\mathcal{H}^{\otimes}$ and $\mathcal{H}$ the inductive limit of $(\mathcal{H}^{\otimes}, \xi^{\otimes})$.

Let $w = \{w_t\}_{t\geq 0}$ be a contractive adapted right cocycle for θ. Note that each $\eta(t) = \kappa_t^* w_t \kappa_0 \phi^{\frac{1}{2}}$ is ϕ-bounded. Moreover, for each $t \geq 0$, we have

$$\pi_\phi(\eta(t))^*\pi_\phi(\eta(t))\phi^{\frac{1}{2}}x = \kappa_0^* w_t^* w_t \kappa_0 \phi^{\frac{1}{2}} x \tag{5.8}$$

for each $x \in M$, and hence the contractivity of w_t implies that $\|\pi_\phi(\eta(t))^*\pi_\phi(\eta(t))\| \leq 1$. We shall show that $\eta^{\otimes}$ is a unit. For $s, t \geq 0$, $\kappa_{s+t} = U_t(\kappa_s \otimes \mathrm{id}_t)U_{s,t}^*$ implies the following calculations.

$$\begin{aligned}
\eta(s+t) &= \kappa_{s+t}^* w_{s+t}\kappa_0\phi^{\frac{1}{2}} = \kappa_{s+t}^*\theta_t(w_s)w_t\kappa_0\phi^{\frac{1}{2}} = \kappa_{s+t}^* U_t(w_s \otimes \mathrm{id}_t)U_t^* w_t \kappa_0 \phi^{\frac{1}{2}} \\
&= \kappa_{s+t}^* U_t(w_s \otimes \mathrm{id}_t)((\kappa_0\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(\kappa_t^* w_t \kappa_0 \phi^{\frac{1}{2}})) \\
&= \kappa_{s+t}^* U_t((w_s\kappa_0\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(\kappa_t^* w_t \kappa_0\phi^{\frac{1}{2}})) \\
&= U_{s,t}(\kappa_s^* \otimes \mathrm{id}_t)U_t^* U_t((w_s\kappa_0\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(\kappa_t^* w_t\kappa_0\phi^{\frac{1}{2}})) \\
&= U_{s,t}(\kappa_s^* w_s \kappa_0 \phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(\kappa_t^* w_t \kappa_0 \phi^{\frac{1}{2}})) = U_{s,t}(\eta(s)\phi^{-\frac{1}{2}}\eta(t)).
\end{aligned}$$

Conversely, let $\eta^{\otimes} = \{\eta(t)\}_{t\geq 0}$ be a contractive unit of $\mathcal{H}^{\otimes}$, and for each $t \geq 0$, $w_t = \pi_\phi(\kappa_t\eta(t))\pi_\phi(\kappa_0\phi^{\frac{1}{2}})^* \in \mathrm{End}(\mathcal{H}_M)$. For all $\xi \in \mathcal{H}$, the equation

$$w_t\xi = \kappa_t U_{t,0}(\eta(t)\phi^{-\frac{1}{2}}\kappa_0^*\xi) \tag{5.9}$$

is implied from the approximation of $\kappa_0^*\xi$ by vectors as the form of $\phi^{\frac{1}{2}}x$. In particular,

$$w_t(\kappa_0\phi^{\frac{1}{2}}x) = \kappa_t\eta(t)x \tag{5.10}$$

for all $t \geq 0$, $x \in M$, and hence $w_t = 0$ on the orthogonal complement of the closed subspace $\kappa_0\kappa_0^*\mathcal{H}$. Thus, computations

$$\begin{aligned}
\theta_t(w_s)w_t(\kappa_0\phi^{\frac{1}{2}}x) &= U_t(w_s \otimes \mathrm{id}_t)U_t^*\pi_\phi(\kappa_t\eta(t))\pi_\phi(\kappa_0\phi^{\frac{1}{2}})^*\kappa_0(\phi^{\frac{1}{2}}x) \\
&= U_t(w_s \otimes \mathrm{id}_t)U_t^*\kappa_t(\eta(t)x) = U_t(w_s \otimes \mathrm{id}_t)((\kappa_0\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}\eta(t)) \\
&= U_t((w_s\kappa_0\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}(\eta_t x)) = U_t((\kappa_s\eta(s))\phi^{-\frac{1}{2}}(\eta(t)x)) \\
&= \kappa_{s+t}U_{s,t}(\eta(s)\phi^{-\frac{1}{2}}(\eta(t)x)) = w_{s+t}(\kappa_0\phi^{\frac{1}{2}}x)
\end{aligned}$$

for every $x \in M$, implies that w is a right cocycle. We shall show that w is adapted. For all $t \geq 0$ and all $\xi \in \mathcal{H}$, by (5.9), we have

$$\begin{aligned}
\kappa_t\kappa_t^* w_t\xi &= \kappa_t\kappa_t^* w_t\kappa_0\kappa_0^*\xi = \kappa_t\kappa_t^*\pi_\phi(\kappa_t\eta(t))\pi_\phi(\kappa_0\phi^{\frac{1}{2}})^*\kappa_0\kappa_0^*\xi \\
&= \kappa_t\kappa_t^*\kappa_t U_{t,0}(\eta(t)\phi^{-\frac{1}{2}}(\kappa_0^*\xi)) = w_t\kappa_0\kappa_0^*\xi = w_t\xi.
\end{aligned}$$

By (5.9) again and the fact that the family $\{\kappa_t\kappa_t^*\}_{t\geq 0}$ is increasing, we have also

$$\begin{aligned} w_t\kappa_t\kappa_t^*\xi &= \pi_\phi(\kappa_t\eta(t))\kappa_0^*\kappa_t\kappa_t^*\xi = \kappa_t U_{t,0}(\eta(t)\phi^{-\frac{1}{2}}\kappa_0^*\kappa_t\kappa_t^*\xi) \\ &= \kappa_t U_{t,0}(\eta(t)\phi^{-\frac{1}{2}}\kappa_0^*\xi) = w_t\xi. \end{aligned}$$

We conclude that $\kappa_t\kappa_t^* w_t\kappa_t\kappa_t^* = w_t$, that is, the adaptedness.

We can check that the correspondence between contractive adapted right cocycles and contractive units is one-to-one by (5.9). □

Note that by (5.8) and (5.9), the unit associated with an adapted unitary right cocycle is unital, and the contractive adapted right cocycle associated with a unital unit preserves inner products on $\kappa_0\kappa_0^*\mathcal{H}$.

At the beginning of this subsection, we constructed the E_0-semigroup from a pair of a relative product system and a continuous unital unit by (5.7).

Conversely, for E_0-semigroup $\theta = \{\theta_t\}_{t\geq 0}$ on a von Neumann algebra M, we can get the relative product system $\tilde{\mathcal{H}}^{\theta\otimes} = \{\tilde{\mathcal{H}}_t^\theta\}_{t\geq 0}$ and the continuous (generating) unital unit $\tilde{\xi}^{\theta\otimes} = \{\tilde{\xi}^\theta(t)\}_{t\geq 0}$ by the way in Subsection 4.3 as CP_0-semigroups. Suppose $\mathcal{H}_t^\theta$ is $L^2(M)$ as sets and a left and a right actions of M are defined by $x\xi y = \theta_t(x)\xi y$ for each $x, y \in M$ and $\xi \in \mathcal{H}_t^\theta$, and put $\xi^\theta(t) = \phi^{\frac{1}{2}}$ for each $t \geq 0$. Then, $\mathcal{H}^{\theta\otimes} = \{\mathcal{H}_t^\theta\}_{t\geq 0}$ and $\xi^{\theta\otimes} = \{\xi^\theta(t)\}_{t\geq 0}$ canonically become a relative product system and continuous (generating) unital unit, respectively. It turns out that there is an isomorphism $u^\theta = \{u_t^\theta\}_{t\geq 0}$ from $\tilde{\mathcal{H}}^{\theta\otimes}$ onto $\mathcal{H}^{\theta\otimes}$ preserving the units. A family $\{f_t^\theta\}_{t\geq 0}$ of right M-linear unitaries $f_t^\theta : \mathcal{H}_t^\theta \ni \xi \mapsto \xi \in L^2(M)$ induces the right M-linear unitary $f^\theta : \mathcal{H}^\theta \to L^2(M)$, where $\mathcal{H}^\theta$ is the inductive limit of $(\mathcal{H}^{\theta\otimes}, \xi^{\theta\otimes})$. Note that the all canonical embeddings $\kappa_t^\theta : \mathcal{H}_t^\theta \to \mathcal{H}^\theta$ are unitaries and coincide. The E_0-semigroup θ is given by $(\mathcal{H}^{\theta\otimes}, \xi^{\theta\otimes})$ in the sense of (5.1), and the E_0-semigroup $\{(f^\theta)^*\theta_t(f^\theta \cdot (f^\theta)^*)f^\theta\}_{t\geq 0}$ coincides with the maximal dilation $\tilde{\theta}$ of $(\mathcal{H}^{\theta\otimes}, \xi^{\theta\otimes})$ on $\mathrm{End}(\mathcal{H}_M^\theta)$.

If $w = \{w_t\}_{t\geq 0} \subset M$ is a unitary right cocycle for θ and put $\theta_t'(\cdot) = w_t^*\theta_t(\cdot)w_t$ for each $t \geq 0$, then $u_t : \mathcal{H}_t^{\theta'} \ni x\phi^{\frac{1}{2}} \mapsto w_t x\phi^{\frac{1}{2}} \in \mathcal{H}_t^\theta$ gives an isomorphism $\mathcal{H}^{\theta'\otimes} \cong \mathcal{H}^{\theta\otimes}$. Thus, we have $\tilde{\mathcal{H}}^{\theta'\otimes} \cong \mathcal{H}^{\theta'\otimes} \cong \mathcal{H}^{\theta\otimes} \cong \tilde{\mathcal{H}}^{\theta\otimes}$. Conversely, we have the following theorem.

Theorem 5.2.2. *Let $\theta = \{\theta_t\}_{t\geq 0}$ and $\theta' = \{\theta_t'\}_{t\geq 0}$ be E_0-semigroups on a von Neumann algebra M. Suppose $(\tilde{\mathcal{H}}^{\theta\otimes}, \tilde{\xi}^{\theta\otimes})$ and $(\tilde{\mathcal{H}}^{\theta'\otimes}, \tilde{\xi}^{\theta'\otimes})$ are the pairs of the relative product systems and the continuous unital units associated with θ and θ', respectively. If $\tilde{\mathcal{H}}^{\theta\otimes}$ and $\tilde{\mathcal{H}}^{\theta'\otimes}$ are isomorphic, then θ and θ' are cocycle equivalent.*

Proof. We will use the above notations for θ and θ'.

Suppose a family $u = \{u_t\}_{t\geq 0}$ of M-bilinear unitaries gives an isomorphism from $\tilde{\mathcal{H}}^{\theta'\otimes}$ onto $\tilde{\mathcal{H}}^{\theta\otimes}$. Put $\eta(t) = u_t^\theta u_t\tilde{\xi}^{\theta'}(t) \in \mathcal{H}_t^\theta$ for each $t \geq 0$ and $\eta^\otimes = \{\eta(t)\}_{t\geq 0}$. Then, we have

$$\begin{aligned} \theta_t'(x) &= \pi_\phi(\tilde{\xi}^{\theta'}(t))^*\pi_\phi(x\tilde{\xi}^{\theta'}(t)) = \pi_\phi(u_t^\theta u_t\tilde{\xi}^{\theta'}(t))^*\pi_\phi(xu_t^\theta u_t\tilde{\xi}^{\theta'}(t)) \\ &= \pi_\phi(\eta(t))^*\pi_\phi(x\eta(t)) \end{aligned} \tag{5.11}$$

for all $x \in M$, that is, θ' is given by $(\mathcal{H}^{\theta\otimes}, \eta^{\otimes})$ in the sense of (5.1). By Proposition 5.1.6, the unit $\eta^{\otimes}$ is continuous, generating and unital. We denote the right cocycle for $\tilde{\theta}$ associated with $\eta^{\otimes}$ by $w^0 = \{w_t^0 = \pi_\phi(\kappa_t^\theta \eta(t))\pi_\phi(\kappa_0^\theta \phi^{\frac{1}{2}})^*\}_{t\geq 0}$ as Theorem 5.2.1.

By (5.9), each w_t^0 is isometry. For all $x, y \in M$, we have

$$\langle \eta(t)x, \eta(t)y\rangle = \langle \kappa_t^\theta w_t^0 \kappa_0^\theta \phi^{\frac{1}{2}} x, \kappa_t^\theta w_t^0 \kappa_0^\theta \phi^{\frac{1}{2}} y\rangle = \langle \phi^{\frac{1}{2}} x, \phi^{\frac{1}{2}} y\rangle,$$

and hence $L^2(M) \ni \phi^{\frac{1}{2}} x \mapsto \eta(t)x \in \overline{\mathrm{span}}\{\eta(t)x \mid x \in M\}$ is unitary. This implies that $\overline{\mathrm{span}}\{\eta(t)x \mid x \in M\} = L^2(M)$. Thus, by (5.10), each w_t^0 is surjective. Now, we shall show that w^0 is strongly continuous. For $s \geq 0$, by the continuity of $\eta^{\otimes}$, we can check that $\kappa_t^\theta \eta(t) \to \kappa_s^\theta \eta(s)$ when $t \to s$. On the other hand, for $\xi \in \mathcal{H}^\theta$ and $t \geq s$, we have

$$\begin{aligned}
&\langle w_t\xi, w_s\xi\rangle = \langle \kappa_t^\theta U_{t,0}^\theta(\eta(t)\phi^{-\frac{1}{2}}(\kappa_0^\theta)^*\xi), \kappa_s^\theta U_{s,0}^\theta(\eta(s)\phi^{-\frac{1}{2}}(\kappa_0^\theta)^*\xi)\rangle \\
&= \langle U_{t,0}^\theta(\eta(t)\phi^{-\frac{1}{2}}(\kappa_0^\theta)^*\xi), \beta_{t,s}^\theta U_{s,0}^\theta(\eta(s)\phi^{-\frac{1}{2}}(\kappa_0^\theta)^*\xi)\rangle \\
&= \langle U_{t,0}^\theta(\eta(t)\phi^{-\frac{1}{2}}(\kappa_0^\theta)^*\xi), U_{t-s,s}^\theta(\mathrm{id}_{t-s} \otimes U_{s,0}^\theta)(\xi(t-s)\phi^{-\frac{1}{2}}\eta(s)\phi^{-\frac{1}{2}}(\kappa_0^\theta)^*\xi)\rangle \\
&= \langle U_{t,0}^\theta(\eta(t)\phi^{-\frac{1}{2}}(\kappa_0^\theta)^*\xi), U_{t,0}^\theta(U_{t-s,s}^\theta \otimes \mathrm{id}_0)(\xi(t-s)\phi^{-\frac{1}{2}}\eta(s)\phi^{-\frac{1}{2}}(\kappa_0^\theta)^*\xi)\rangle \\
&= \langle \eta(t)\phi^{-\frac{1}{2}}(\kappa_0^\theta)^*\xi, U_{t-s,s}^\theta(\xi(t-s)\phi^{-\frac{1}{2}}\eta(s))\phi^{-\frac{1}{2}}(\kappa_0^\theta)^*\xi\rangle \\
&= \langle \eta(t)\phi^{-\frac{1}{2}}(\kappa_0^\theta)^*\xi, \beta_{t,s}^\theta \eta(s)\phi^{-\frac{1}{2}}(\kappa_0^\theta)^*\xi\rangle = \langle \kappa_t^\theta\eta(t)\phi^{-\frac{1}{2}}(\kappa_0^\theta)^*\xi, \kappa_s^\theta\eta(s)\phi^{-\frac{1}{2}}(\kappa_0^\theta)^*\xi\rangle \\
(5.12)&= \langle (\kappa_0^\theta)^*\xi, \pi_\phi(\kappa_t^\theta\eta(t))^*\pi_\phi(\kappa_s^\theta\eta(s))(\kappa_0^\theta)^*\xi\rangle.
\end{aligned}$$

Since $\pi_\phi(\kappa_t^\theta\eta(t))^*\pi_\phi(\kappa_s^\theta\eta(s)) \to 1_M$ weakly when $t \to s$ or $s \to t$, (5.12) tends to $\langle \xi, \xi\rangle$ when $t \to s+0$, and by the symmetry, $\langle w_t\xi, w_s\xi\rangle$ also tends to $\langle \xi, \xi\rangle$ when $t \to s-0$. We conclude that $w_t\xi \to w_s\xi$ when $t \to s$.

Put $w_t = f^\theta w_t^0 (f^\theta)^* \in M$. Then $w = \{w_t\}_{t\geq 0}$ is a strongly continuous right cocycle. For all $t \geq 0$ and $x, y, z \in M$, since

$$w_t\phi^{\frac{1}{2}}x = f^\theta \kappa_t^\theta U_{t,0}^\theta(\eta(t)\phi^{-\frac{1}{2}}(\kappa_0^\theta)^*(f^\theta)^*\phi^{\frac{1}{2}}x) = f_t^\theta \eta(t)x = \eta(t)x,$$

and θ' is given by $(\mathcal{H}^{\theta\otimes}, \eta^{\otimes})$ as (5.11), we have $\theta_t'(x) = w_t^*\theta_t(x)w_t$ by equations

$$\begin{aligned}
&\langle w_t^*\theta_t(x)w_t\phi^{\frac{1}{2}}y, \phi^{\frac{1}{2}}z\rangle = \langle \theta_t(x)f_t^\theta\eta(t)y, f_t^\theta\eta(t)z\rangle = \langle \theta_t(x)\eta(t)y, \eta(t)z\rangle \\
&= \langle \pi_\phi(\eta(t))^*\pi_\phi(x\cdot\eta(t))\phi^{\frac{1}{2}}y, \phi^{\frac{1}{2}}z\rangle = \langle \theta_t'(x)\phi^{\frac{1}{2}}y, \phi^{\frac{1}{2}}z\rangle.
\end{aligned}$$

□

Example 5.2.3. *The relative product system associated with the E_0-semigroup given by a semigroup in* Example 2.4.1 *of unitaries, is isomorphic to the trivial relative product system $\{L^2(M)\}_{t\geq 0}$.*

Example 5.2.4. *Let θ be an E_0-semigroup on a von Neumann algebra M and $u \in M$ a unitary. The inner conjugate of θ by u is an E_0-semigroup θ' defined by $\theta_t'(x) = u\theta_t(u^*xu)u^*$ for each $t \geq 0$ and $x \in M$. Then θ and θ' have isomorphic relative product systems and they are cocycle equivalent by a right unitary cocycle w defined by $w_t = \theta_t(u)u^*$ for each $t \geq 0$.*

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References

[1] A. Alevras, One parameter semigroups of endomorphisms of factors of type II_1, J. Operator Theory, 51 (2004), 161-179.
[2] W. Arveson, Continuous analogues of Fock space, Mem. Amer. Math. Soc. 80 (1989), no. 409, iv+66 pp.
[3] W. Arveson, Generators of non-commutative dynamics, Ergodic Theory Dynam. Systems 22 (2002), no. 4, 1017-1030.
[4] W. Arveson, Noncommutative dynamics and E-semigroups, Springer Monographs in Mathematics, Springer-Verlag, New York (2003), x+434 pp.
[5] M. Baillet, Y. Denizeau, J. F. Havet, Indice d'une espérance conditionnelle, Compositio Math. 66 (1988), no. 2, 199-236.
[6] B. V. R. Bhat, An index theory for quantum dynamical semigroups, Trans. Amer. Math. Soc. 348 (1996), no. 2, 561-583.
[7] B. V. R. Bhat, Minimal dilations of quantum dynamical semigroups to semigroups of endomorphisms of C^*-algebras, J. Ramanujan Math. Soc. 14 (1999), no. 2, 109-124.
[8] B. V. R. Bhat, M. Skeide, Tensor product systems of Hilbert modules and dilations of completely positive semigroups, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 3 (2000), no. 4, 519-575.
[9] A. Connes, Noncommutative Geometry. Academic Press (1994).
[10] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, Tensor categories, Mathematical Surveys and Monographs 205. American Mathematical Society, Providence, RI (2015), xvi+343 pp.
[11] P. Ghez, R. Lima, J. E. Roberts, W^*-categories, Pacific J. Math., 120 (1985), 79-109.
[12] E. C. Lance, Hilbert C^*-modules, A toolkit for operator algebraists. London Mathematical Society Lecture Note Series 210, Cambridge University Press, Cambridge (1995), x+130 pp.
[13] V. Liebscher, Random sets and invariants for (type II) continuous tensor product systems of Hilbert spaces. Mem. Amer. Math. Soc. 199 (2009), no. 930, xiv+101 pp.
[14] S. MacLane, Categories for the working mathematician. Secound edition. Springer-Verlag (1998).
[15] O. T. Margetts, R. Srinivasan, Invariants for E_0-semigroups on II_1 factors. Comm. Math. Phys. 323 (2013), no. 3, 1155-1184.
[16] O. T. Margetts, R. Srinivasan, Non-cocycle-conjugate E_0-semigroups on factors, Publ. Res. Inst. Math. Sci. 53 (2017), no. 2, 299336.
[17] P. S. Muhly, B. Solel, Quantum Markov processes (correspondences and dilations), Internat. J. Math. 13 (2002), no. 8, 863-906.
[18] W. L. Paschke, Inner product modules over B^*-algebras. Trans. Amer. Math. Soc. 182 (1973), 443-468.
[19] R. T. Powers, An index theory for semigroups of $*$-endomorphisms of $\mathfrak{B}(\mathfrak{H})$, Can. Jour. Math. 40 (1988), 86-114.
[20] J. L. Sauvageot, Sur le produit tensoriel relatif d'espaces de Hilbert, J. Operator Theory 9 (1983), no. 2, 237-252.
[21] Y. Sawada, A remark on the minimal dilation of the semigroup generated by a normal UCP-map, to appear in Bull. Belg. Math. Soc. Simon Stevin.

[22] Y. Sawada, Minimal dilations of CP_0-semigroups and product systems of W*-bimodules, in preparation.
[23] Y. Sawada, E_0-semigroups and relative product systems, in preparation.
[24] Y. Sawada, S. Yamagami, Notes on the bicategory of W^*-bimodules, to appear in J. Math. Soc. Japan.
[25] D. Sherman, Relative tensor products for modules over von Neumann algebras, Function spaces, 275-291, Contemp. Math. 328, Amer. Math. Soc. Providence, (2003).
[26] M. Skeide, Generalised matrix C^*-algebras and representations of Hilbert modules, Math. Proc. R. Ir. Acad. 100A (2000), no. 1, 11-38.
[27] M. Skeide, Commutants of von Neumann modules, representations of $\mathcal{B}^a(E)$ and other topics related to product systems of Hilbert modules, Advances in quantum dynamics, Contemp. Math., 335, Amer. Math. Soc. (2003), 253-262.
[28] M. Skeide, Commutants of von Neumann correspondences and duality of Eilenberg-Watts theorems by Rieffel and by Blecher, Quantum probability, Banach Center Publ. 73, Polish Acad. Sci. Inst. Math., Warsaw, (2006), 391-408.
[29] M. Skeide, Product systems; a survey with commutants in view, Quantum stochastics and information, World Sci. Publ., Hackensack, NJ, (2008), 47-86.
[30] M. Skeide, Classification of E_0-semigroups by product systems, Mem. Amer. Math. Soc. 240 (2016), no.1137, vi+126 pp.
[31] M. Takesaki, Theory of operator algebras. I, Encyclopaedia of Mathematical Sciences, 124. Operator Algebras and Non-commutative Geometry, 5. Springer-Verlag, Berlin (2002), xx+415 pp.
[32] M. Takesaki, Theory of operator algebras. II, Encyclopaedia of Mathematical Sciences, 125. Operator Algebras and Non-commutative Geometry, 6. Springer-Verlag, Berlin (2003), xxii+518 pp.
[33] A. Thom, A remark about the Connes fusion tensor product. Theory Appl. Categ. 25 (2011), No. 2, 38-50.
[34] S. Yamagami, Algebraic aspects in modular theory, Publ. Res. Inst. Math. Sci. 28 (1992), no. 6, 1075-1106.
[35] S. Yamagami, Modular theory for bimodules, J. Funct. Anal. 125 (1994), 327-357.
[36] S. Yamagami, Frobenius duality in C^*-tensor categories, J. Operator Theory 52 (2004), 3-20.
[37] S. Yamagami, Notes on operator categories, J. Math. Soc. Japan 59 (2007), 541-555.
[38] S. Yamagami, Operator Algebras and Their Representations, lecture notes (2014).