Lifting the infinite tensor product representation of the quantum toroidal algebra to the trivalent intertwiner
(量子トロイダル代数の無限テンソル積表現の,
三価型絡作用素への持ち上げ)

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Abstract

We lift the infinite tensor product construction of representations of the quantum toroidal $\mathfrak{gl}_1$ algebra to the construction of the trivalent intertwiners. This enables us to obtain the intertwiner for the vertical MacMahon representation, *i.e.* the MacMahon intertwiner. To this end we introduce the definition of the trivalent intertwiner in a more general setting than existing one, and observe that the regularization procedures also work well at the operator level. As an application we can compute the R-matrix for the MacMahon representation from the commutator of two MacMahon intertwiners.

Contents

1 Introduction

Quantum toroidal $\mathfrak{gl}_1$ algebra

Since a discovery of the AGT correspondence [1], the quantum toroidal $\mathfrak{gl}_1$ algebra and its degenerate algebras have played important roles in mathematical physics. In [1] they claimed the correspondence between the generating function of the instanton counting for the four dimensional super Yang-Mills gauge theory and the correlation function of the two dimensional conformal field theory. The generating function of the instanton counting is known as the Nekrasov function [2], and it is expressed in a certain combinatorial way. On the other hand, the conformal correlation function is formulated by making use of the representation theory of the conformal algebra, and it is typically decomposed into more fundamental building blocks called conformal blocks [3]. From a standpoint of the conformal field theory, the original AGT correspondence is related to the conformal block associated with the Virasoro algebra, and its generalizations have been also developed, for example the $\mathcal{W}_N$-algebra[1] version [4] and the q-deformed version [5, 6] are well known. Since the quantum toroidal $\mathfrak{gl}_1$ algebra is obtained as a deformation of the $\mathcal{W}_{1+\infty}$ algebra [13], it is a universal $\mathcal{W}$-algebra of A-type in the sense that any (deformed) $\mathcal{W}_N$-algebras are realized as certain representations of the quantum toroidal $\mathfrak{gl}_1$ algebra [21, 15]. Therefore we can understand the above correspondences by constructing an action of the quantum toroidal $\mathfrak{gl}_1$ algebra on cohomologies (or its K-theory lift) of the instanton moduli spaces [7] (see also [8] as for the Yangian limit version).

Mathematically the quantum toroidal $\mathfrak{gl}_n$ algebra for $n \geq 3$ was introduced along the geometric Langrands duality [9] and an analogue of the Schur duality [10]. Then its vertex operator representation was considered in [11]. After a while [12] and [13] revealed fundamental properties of the quantum toroidal $\mathfrak{gl}_1$ algebra[2], then [14, 15, 16, 17] have worked out its representation theory exhaustively. In this thesis we only deal with the quantum toroidal $\mathfrak{gl}_{n=1}$ algebra. This algebra has two deformation parameters. It is convenient to arrange these parameters into three dependent parameters q_i $(i = 1, 2, 3)$; $q_1 q_2 q_3 = 1$. q_i enters the algebra symmetrically, however a choice of the representation may break this symmetry (see Section 2).

Tensor product representation

From the viewpoint of representation theory, the existence of the (formal) coproduct

[1] We mean the $\mathcal{W}$-algebra of A_{N-1}-type by $\mathcal{W}_N$-algebra. The $\mathcal{W}_2$-algebra is just the Virasoro algebra.

[2] Therefore the same algebra is also called Ding-Iohara-Miki algebra (DIM algebra for short).

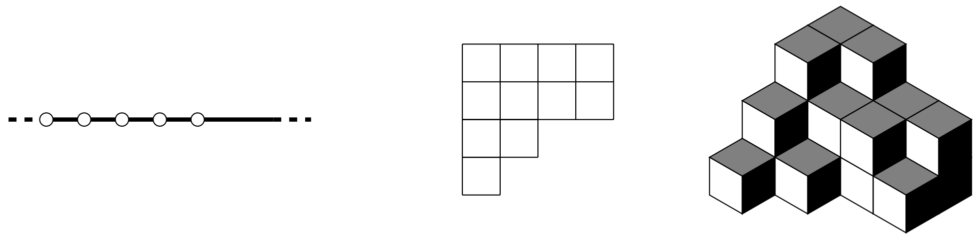


Figure 1: Each state of the vector, Fock and MacMahon representation corresponds to an integer number, partition and plane partition respectively, see Section 2.2 for details.

structure is quite important. In general, if we have a coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ for the algebra $\mathcal{A}$, then we can naturally take the tensor product representation as follows. Let (ρ_V, V), (ρ_W, W) be representations of $\mathcal{A}$. Then the tensor product representation $(\rho_{V\otimes W}, V \otimes W)$ is defined as

$$\rho_{V\otimes W}(a) := \rho_V \otimes \rho_W(\Delta(a)), \quad a \in \mathcal{A}. \tag{1.1}$$

In [14, 16] they have introduced three kinds of representations of the quantum toroidal $\mathfrak{gl}_1$ algebra: the vector, Fock and MacMahon representations. Roughly speaking these representation spaces are spanned by one, two and three dimensional Young diagrams respectively, see Figure 1. In their constructions the vector representation is the fundamental ingredient. One can construct the Fock representation by infinite tensor poduct of the vector representations. Furthermore, one can construct the MacMahon representation in a similar manner from the Fock representations. Note, however, that we need regularization procedures in order to obtain the stability conditions for validity of these infinite tensor products (see Section 2.2). These procedures are not only involved but also giving depth to the theory. For example, the process from the Fock representation to the MacMahon one allows the action to have an extra parameter, denoted by $K \in \mathbb{C}^\times$, which is out of the algebra itself, though the physical interpretation of this new parameter is not clear yet.

We would also like to mention that the MacMahon representation has a visually salient trait that the permutation of the parameters q_1, q_2, q_3 corresponds to the three dimensional transposition of coordinate axes. For example, see Figure 2. This symmetry shows the triality symmetry which is thought of the nature of $\mathcal{W}$-algebras [18, 19]. Therefore the MacMahon representation seems to be a natural setting to consider the relation with $\mathcal{W}$-algebras.

Intertwiner

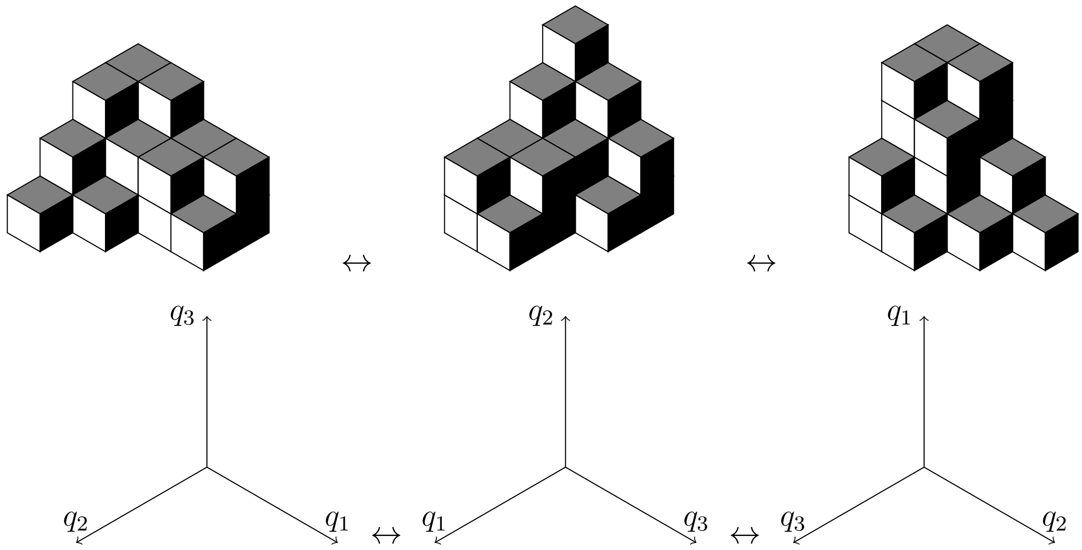


Figure 2: The three dimensional transposition corresponding to the cyclic permutation.

The intertwining operator is an operator $\Psi : V \to W$ between two representation spaces such that it commutes with the action of the algebra $\mathcal{A}$;

$$\rho_W(a)\Psi = \Psi\rho_V(a), \quad a \in \mathcal{A}. \tag{1.2}$$

Since we have the notion of the coproduct, we can ask the existance of the intertwiner between tensor product representations. For example, the existence of the R-matrix $\mathcal{R} : V \otimes W \to W \otimes V$, which is a solution to the Yang-Baxter equation, indicates the almost cocommutativity of the bialgebra $\mathcal{A}$.

Among other things it is important to understand the following trivalent intertwining operator of the quantum toroidal algebra $\mathcal{U}$ (we call it intertwiner for short) from both mathematical and physical points of view:

$$\Psi : \mathcal{V} \otimes \mathcal{H} \to \mathcal{H}', \quad \rho_{\mathcal{H}'}(a)\Psi = \Psi\ \rho_{\mathcal{V}} \otimes \rho_{\mathcal{H}}(\Delta(a)), \quad a \in \mathcal{U}, \tag{1.3}$$

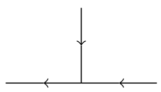
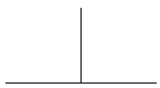
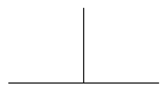
where $\mathcal{V}$ is a vertical representation and $\mathcal{H}, \mathcal{H}'$ are horizontal representations. The vertical representation is a representation in which the Cartan like part of the algebra acts diagonally such as the vector, Fock and MacMahon representations (see Section 2.2), while the horizontal representation is a vertex operator representation like [11] (see Section 2.3). The importance of such an intertwiner was recognized in [20] for the first time. Then, when the vertical representation is the Fock one, [22] eventually identified the intertwiner with the refined topological vertex [23, 24], which is a computational tool for the topological string theory. Because the Nekrasov function is reproduced by appropriately gluing the refined topological vertices, we can identify the Nekrasov function with the certain combination of the intertwiners. This realizes another aspect of the AGT correspondence, between the super Yang-Mills theory, conformal field theory and topological string theory, and we can understand that the algebraic structure of them is governed by the same quantum toroidal algebra. This is the reason why we concentrate on the theory of the intertwiner in this thesis.

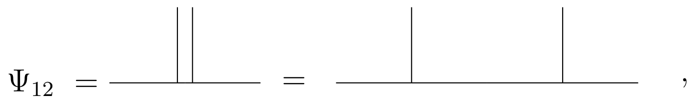
Many important objects in representation theory are naturally related to the intertwiner; for example, R-matrix [25], (q,t)-KZ equation [27] and qq-character [28] are investigated by making use of the intertwiner. Furthermore, by considering the intertwiner for the quantum toroidal $\mathfrak{gl}_n$ algebra ($n \in \mathbb{N}$) instead of the quantum toroidal $\mathfrak{gl}_1$ algebra, we can obtain a natural generalization of the refined topological vertex and identify the combination of these intertwiners with the function of instanton counting on the ALE space [29]. As another direction of a generalization, one might expect to replace the representation space of the intertwiner; then, this is the subject in this thesis. In what follows, we state the question, idea and result.

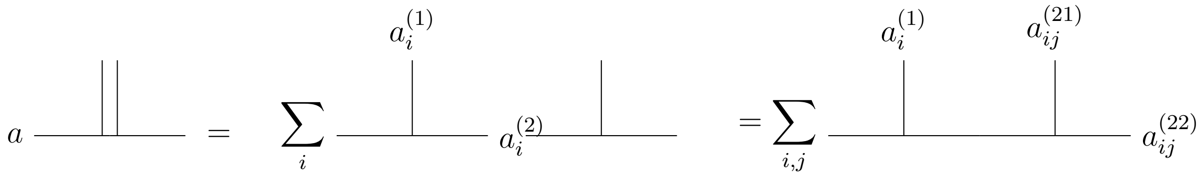
Systematic construction of intertwiner

Originally, in [22], they considered the trivalent intertwiner for a triple of the Fock representations and constructed it by making use of the vertex operator formalism (see Section 3.1 for details). Each state of three Fock representations was labeled by a partition (or a Young diagram)[14], and each partition corresponded to one of three labels of the refined topological vertex. However, we know that there are also a lot of representations other than the Fock representation. Then it is natural to consider whether there always exists an intertwiner for arbitrary representation or not. This thesis aims to answer this question partly. Now we would like to replace one of the Fock representations of the intertwiner [22] with another one and to discuss its properties, though we do not know the physically clear meaning of the resultant vertex.

Here we explain the strategy to construct a new intertwiner from given intertwiners in a general setting. To this end we identify the intertwiner with a trivalent vertex graphically in the following way:

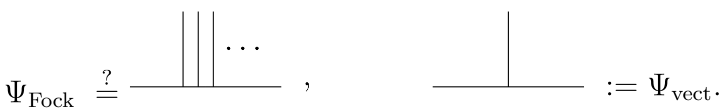
$\Psi =$  , a  $= \sum_i$ $a_i^{(1)}$  $a_i^{(2)}$, $\qquad a \in \mathcal{A}$, where we have used the notation $\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$ and we often omit arrow signs from diagrams. Note that we will associate the vertical and the horizontal representations to the vertical and the horizontal edges, respectively. If we have two intertwiners $\Psi_i : V_i \otimes H_i \to H_i'$ $(i = 1, 2)$ with $H_1 = H_2'$, then we can obtain the intertwiner for the tensor product $V_1 \otimes V_2$ by simply composing two intertwiners $\Psi_{12} = \Psi_1 \circ (\mathbf{id} \otimes \Psi_2) : V_1 \otimes V_2 \otimes H_2 \to H_1'$. The intertwining relation can be checked graphically as follows:





where we have used the notation $\Delta(a_i^{(2)}) = \sum_j a_{ij}^{(21)} \otimes a_{ij}^{(22)}$. For this construction, what we have to do is only to check the consistency condition for the composite operator to exist.

Since the Fock representation of the quantum toroidal $\mathfrak{gl}_1$ algebra can be constructed by infinite tensor product of the vector representations, one might expect that the intertwiner for the Fock representation can be obtained by infinitely many times composition of the intertwiners for the vector representation:



Actually taking the infinite tensor product requires some regularization procedure. Accordingly we have to take this effect into account and attach some modification operator to the composition of the intertwiners in order to stabilize the infinite composition. As concerns the intertwiner for the Fock representation, we obtain the following theorem (see Proposition 3.10 for details):

Theorem 1.1 ([30, Section 4.2.2])**.** *There exists a unique intertwiner for the Fock representation up to normalization if and only if a pair of horizontal representations satisfies the matching condition for the zero mode sector. Furthermore, for the suitable choice of horizontal representations, this intertwiner agrees with that of* [*22*].

It is the main part of this thesis to show how to construct the intertwiner in a systematic way along the above idea. Furthermore, by applying the same method, we can newly construct the intertwiner for the MacMahon representation [30], though we have to define the horizontal representation in a more abstract way compared to [22]. The following existence and uniqueness theorem of the intertwiner for the MacMahon representation is the main theorem of this thesis (see Proposition 3.13 for details):

Theorem 1.2 ([30, Section 4.3.2])**.** *There exists a unique intertwiner for the MacMahon representation up to normalization if and only if a pair of horizontal representations satisfies the matching condition for the zero mode sector.*

Here we sketch the explicit form of the intertwiner $\Xi(K;v)$ for the MacMahon representation (note that we express the intertwiner as an operator between a pair of horizontal representations $(\mathcal{H},\mathcal{H}')$, the Λ-component, see Definition 3.2):

$$\begin{aligned}\Xi_\Lambda(K;v) &= z_\Lambda \tilde{\Xi}_\Lambda(K;v) : \mathcal{H} \to \mathcal{H}', \\ \tilde{\Xi}_\Lambda(K;v) &= \mathcal{M}^{[n]}(K)\cdot \tilde{\Phi}^{[n]}_\Lambda(v)\circ \Gamma_n(K;v), \quad n > h(\Lambda),\end{aligned}$$

where $\tilde{\Phi}^{[n]}_\Lambda(v)$ denotes the finite composition of the intertwiners for the Fock representation, $\Gamma_n(K;v)$ denotes the above mentioned modification operator, $\mathcal{M}^{[n]}(K)$ denotes the coefficient which is required for the well-definedness of the operator $\tilde{\Xi}_\Lambda(K;v)$, and z_Λ is the contribution from the zero mode sector which is characteristic of the vertex operator formalism and there is no counterpart at the representation level.

In order to obtain an arbitrary web diagram by composition of the intertwiners, one needs the notion of the dual intertwiner [22]. We can also construct it in almost the same way as the original intertwiner by merely reversing the ordering of the composition. Hence we can obtain the counterparts of the above theorems (see Propositions 4.9 and 4.12). However, we should pay some attention to the dual basis. We know that the dual basis of the Fock representation corresponds to the dual Macdonald function, while we do not know what is the dual basis of the MacMahon representation. This thesis only suggests what should be the normalization factors, see Section 5.2; we do not have any insights into the relation with the theory of symmetric function.

Further direction

Strictly speaking we only deal with the *vacuum* MacMahon representation in this thesis. One can define the representation whose highest weight state has infinitely many boxes consistently, which is called MacMahon representaion *with boundary condition* [16]. This highly nontrivial representation can also be constructed as the infinite tensor product in somewhat peculiar way. Therefore we expect that the intertwiner for this representation can also be constructed in a spirit of this thesis.

Organization

In Section 2 we review the definition and the representations of the quantum toroidal $\mathfrak{gl}_1$ algebra. In Section 3 we make a detailed explanation of how to construct the intertwiners, which is the main part of this thesis. In Section 3.1 we state the definition of the intertwiner in somewhat more general settings than [22]. In Section 3.2 we construct the intertwiner for the vector representation by lifting the generating function of the vector representation to the vertex operator. In Section 3.3 we construct the intertwiner for the Fock representation as the composition of the vector intertwiners and the modification operator. In the same way, we construct the intertwiner for the MacMahon representation in Section 3.4. Section 4 is devoted to the construction of the dual intertwiner which is defined by exchanging the source and the target representation spaces in the definition of the original intertwiner. Section 5 is devoted to a few supplementary properties of the MacMahon intertwiner. In Section 5.1 we confirm the agreement of two ways to compute the R-matrix as an application of the MacMahon intertwiner. In Section 5.2 we discuss the relation between the normalization factor for the dual basis and the OPE factor arising from the certain arrangement of the intertwiners.

Most of parts of this thesis, particularly Sections 3 and 5.1, are based on [30], while Sections 4 and 5.2 are based on [31].

Notation

Geometric series: $\frac{a}{1-z} \in \mathbb{C}[[z]]$ denotes $\sum_{n\geq 0} az^n$.
Delta function: $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$.
$\mathfrak{q}$-integer: $[r] := \frac{\mathfrak{q}^r - \mathfrak{q}^{-r}}{\mathfrak{q}-\mathfrak{q}^{-1}}$ $(r \in \mathbb{Z})$.
Partition: $\lambda = (\lambda_i)_{i\geq 1}$ where $\lambda_i \geq \lambda_{i+1}$ for arbitrary $i \in \mathbb{Z}_{>0}$ and $\lambda_i = 0$ for sufficiently large i.

$\varnothing := (0)_{i\geq 1}$.

$(i,j) \in \lambda :\Longleftrightarrow \lambda_i \geq j$ $(i,j \in \mathbb{Z}_{>0})$.

$\ell(\lambda) := \min\{\ell \geq 0 | (\ell+1,1) \notin \lambda\}$.

$|\lambda| := \sum_{i=1}^{\ell(\lambda)} \lambda_i$.

$\lambda + 1_j := (\lambda'_i)_{i\geq 1}$ where $\lambda'_j = \lambda_j + 1$ and $\lambda'_i = \lambda_i$ except for $i = j$.

${}^t\lambda := (\lambda'_i)_{i\geq 1}$ where $\lambda'_i = |\{k \geq 1 | \lambda_k \geq i\}|$.

$a_\lambda(i,j) := \lambda_i - j,\ \ l_\lambda(i,j) := {}^t\lambda_j - i$.

Plane partition: $\Lambda = (\Lambda^{(k)})_{k\geq 1}$ where each $\Lambda^{(k)}$ is a partition, $\Lambda_i^{(k)} \geq \Lambda_i^{(k+1)}$ for arbitrary $i, k \in \mathbb{Z}_{>0}$ and $\Lambda^{(k)} = \varnothing$ for sufficiently large k.

$\boldsymbol{\varnothing} := (\varnothing)_{k\geq 1}$.

$(i,j,k) \in \Lambda :\Longleftrightarrow \Lambda_i^{(k)} \geq j \ (i,j,k \in \mathbb{Z}_{>0})$.

$h(\Lambda) := \min\{k \geq 0 | (1,1,k+1) \notin \Lambda\}$.

$|\Lambda| := \sum_{k=1}^{h(\Lambda)} |\Lambda^{(k)}| = \sum_{k=1}^{h(\Lambda)} \sum_{i=1}^{\ell(\Lambda^{(k)})} \Lambda_i^{(k)}$.

$\Lambda + 1_j^{(a)} := (\Lambda'^{(k)})_{k\geq 1}$ where $\Lambda'^{(a)} = \Lambda^{(a)} + 1_j$ and $\Lambda'^{(k)} = \Lambda^{(k)}$ except for $k = a$.

Coordinate: $x_{ij} := q_1^{j-1} q_2^{i-1}$, $x_{ijk} := q_1^{j-1} q_2^{i-1} q_3^{k-1}$.

q-Pochhammer symbol: $(z;q)_\infty := \prod_{k=0}^{\infty}(1-q^k z)$.

Theta function: $\theta_p(z) := (z;p)_\infty (p/z;p)_\infty = (1-z)\prod_{k=1}^{\infty}(1-p^k z)(1-p^k/z)$.

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2 Representation of quantum toroidal $\mathfrak{gl}_1$ algebra

2.1 Definition of quantum toroidal $\mathfrak{gl}_1$ algebra

The quantum toroidal $\mathfrak{gl}_1$ algebra, denoted by $\mathcal{U}$, has two deformation parameters $\mathfrak{q}, \mathfrak{d} \in \mathbb{C}$ and we follow the following convention:

$$q_1 = \mathfrak{d}\mathfrak{q}^{-1}, \quad q_2 = \mathfrak{d}^{-1}\mathfrak{q}^{-1}, \quad q_3 = \mathfrak{q}^2, \qquad q_1 q_2 q_3 = 1. \tag{2.1}$$

In this thesis we assume that q_i is generic, that is, $q_1^a q_2^b q_3^c = 1$ for a, b, $c \in \mathbb{Z}$ if and only if $a = b = c$. We introduce the structure function

$$g(z, w) = (z - q_1 w)(z - q_2 w)(z - q_3 w). \tag{2.2}$$

Definition 2.1. *Let $\mathcal{U}$ be a unital associative algebra over $\mathbb{C}$ generated by the currents*

$$E(z) = \sum_{k \in \mathbb{Z}} E_k z^{-k}, \quad F(z) = \sum_{k \in \mathbb{Z}} F_k z^{-k}, \quad K^{\pm}(z) = K^{\pm} + \sum_{r=1}^{\infty} K_r^{\pm} z^{\mp r}, \tag{2.3}$$

and the central element C. These generators satisfy the following defining relations:

$$K^{\pm}(z)K^{\pm}(w) = K^{\pm}(w)K^{\pm}(z), \tag{2.4}$$

$$\frac{g(C^{-1}z, w)}{g(Cz, w)}\ K^{-}(z)K^{+}(w) = \frac{g(w, C^{-1}z)}{g(w, Cz)}\ K^{+}(w)K^{-}(z), \tag{2.5}$$

$$g(z, w)\ K^{\pm}(C^{(1\mp1)/2}z)E(w) + g(w, z)\ E(w)K^{\pm}(C^{(1\mp1)/2}z) = 0, \tag{2.6}$$

$$g(w, z)\ K^{\pm}(C^{(1\pm1)/2}z)F(w) + g(z, w)\ F(w)K^{\pm}(C^{(1\pm1)/2}z) = 0, \tag{2.7}$$

$$[E(z), F(w)] = \tilde{g}\left\{\delta(C\frac{w}{z})\ K^{+}(z) - \delta(C\frac{z}{w})\ K^{-}(w)\right\}, \tag{2.8}$$

$$g(z, w)\ E(z)E(w) + g(w, z)\ E(w)E(z) = 0, \tag{2.9}$$

$$g(w, z)\ F(z)F(w) + g(z, w)\ F(w)F(z) = 0, \tag{2.10}$$

where $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ is the delta function.

Strictly speaking, $\mathcal{U}$ is defined by the above defining relations with the Serre relation [16, 13]. However, we adopt one without the Serre relation because we do not use it in this thesis and also it has been already known that the representations which we use in this thesis satisfy it [16]. Note that the coefficient $\tilde{g}$ in (2.8) only affects the relative normalization of currents $E(z), F(z)$ and $K^{\pm}(z)$. In this thesis we choose

$$\tilde{g} = \frac{(1 - q_1)(1 - q_2)}{1 - q_3^{-1}} \tag{2.11}$$

in order to make the coefficients for the right hand sides of (2.78)–(2.80) unit, while we need nontrivial multiplication factors in (2.22) and (2.23). In addition we impose $K^- = (K^+)^{-1}$, hence there are essentially two central elements C and K^-.

Definition 2.2. *Let $\gamma_{1,2} \in \mathbb{C}^\times$. A representation of $\mathcal{U}$ is called level (γ_1, γ_2) representation if $C = \gamma_1$ and $K^- = \gamma_2$.*

Following [25], we call $C = 1$ or $C = \mathfrak{q}$ representation *vertical* or *horizontal* representation respectively in this thesis. In particular we use the vector, Fock and MacMahon representations as the vertical representation (see Section 2.2), while the vertex operator representation as the horizontal representation (see Section 2.3). We need a coproduct structure of $\mathcal{U}$ in order to define the tensor product of two representations and we use the following one.

Proposition 2.3 ([12, Theorem 2.2])**.** *The following coproduct Δ defines a quasi-Hopf structure on $\mathcal{U}$:*

$$\Delta(K^+(z)) = K^+(z) \otimes K^+(C_1^{-1} z), \tag{2.12}$$

$$\Delta(K^-(z)) = K^-(C_2^{-1} z) \otimes K^-(z), \tag{2.13}$$

$$\Delta(E(z)) = E(z) \otimes 1 + K^-(C_1 z) \otimes E(C_1 z), \tag{2.14}$$

$$\Delta(F(z)) = F(C_2 z) \otimes K^+(C_2 z) + 1 \otimes F(z), \tag{2.15}$$

$$\Delta(C) = C \otimes C, \tag{2.16}$$

where $C_1 = C \otimes 1$ and $C_2 = 1 \otimes C$.

Note that the right hand sides of (2.14) and (2.15) may not be inside $\mathcal{U} \otimes \mathcal{U}$ in general. However, they make sense on any tensor product representations which we consider in this thesis.

When one checks the defining relations of $\mathcal{U}$, the following formula is useful.

Proposition 2.4 ([14, Lemma 3.3])**.** *Let $f(u)$ be a rational function which is regular at $u = 0, \infty$ and has at most simple poles. Let $f^\pm(u)$ be the Taylor expansion of $f(u)$ in $u^{\mp 1}$. Then we have*

$$f^+(u) - f^-(u) = \sum_\alpha \delta(\alpha/u) \left(\operatorname*{Res}_{u=\alpha} f(u) \frac{du}{u} \right), \tag{2.17}$$

where the summation runs over all poles α of $f(u)$ and $\operatorname*{Res}_{u=\alpha}$ denotes the residue at the point $u = \alpha$.

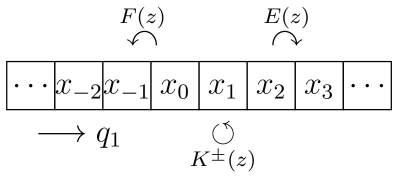


Figure 3: The vector $[v]_{i-1}$ corresponds to the position of x_i. It is convenient to assign the coordinate q_1^{i-1} to x_i, then the action of the vector representation can be written in terms of this coordinate. $E(z)$ shifts the vector to the right and $F(z)$ does to the left while $K^{\pm}(z)$ acts diagonally.

2.2 Vertical representation

In this section we summarize vertical representations which have the trivial first level $\gamma_1 = C = 1$. In particular we review the idea of the infinite tensor product construction worked out in [14, 16]. In these representations we can find a basis which simultaneously diagonalizes $K_r^{\pm}$ due to the commutation relation (2.5). In [14, 16] three kinds of vertical representations have been introduced: the vector, Fock and MacMahon representations. In these representations the basis on which $K_r^{\pm}$ acts diagonally is roughly speaking one, two and three dimensional Young diagrams respectively. Accordingly, we can define the Fock representation as an irreducible subrepresentation within the infinite tensor product of the vector representations [14]. In a similar manner, we can construct the MacMahon representation from the Fock representations [16]. The second level of the vector representation is $\gamma_2 = K^- = 1$. However, the regularization procedure required in the above procedure makes γ_2 nontrivial. Consequently, the Fock representation has a quantized level $\gamma_2 = \mathfrak{q}$, where $\mathfrak{q}$ is one of the parameters of $\mathcal{U}$. Moreover, quite interestingly, the MacMahon representation allows continuous level $\gamma_2 = K^{1/2} \in \mathbb{C}^{\times}$, where K is independent of the definition of $\mathcal{U}$. We can find a natural regularization for the Fock representation, while that for the MacMahon representation somehow looks ambiguous and leads to an arbitrary value $K^{1/2}$, which we can interpret formally as a limit of $\mathfrak{q}^N$ $(N \to \infty)$. In summary, the explicit actions of the vector, Fock and MacMahon representations are given by (2.20)–(2.23), (2.33)–(2.36) and (2.58)–(2.60), respectively.

2.2.1 Vector representation

Graphically, the vector representation is spanned by integral points of one dimensional line, see Figure 3.

We introduce the following generating function,

$$\tilde{\psi}(z) = \frac{(1-q_2^{-1}z)(1-q_3^{-1}z)}{(1-z)(1-q_1 z)} \in \mathbb{C}[[z]], \tag{2.18}$$

$$\tilde{\psi}(z)\tilde{\psi}(q_1^{-1}z)^{-1} = G(z) := -\frac{g(z,1)}{g(1,z)} = \frac{(1-q_1^{-1}z)(1-q_2^{-1}z)(1-q_3^{-1}z)}{(1-q_1 z)(1-q_2 z)(1-q_3 z)}. \tag{2.19}$$

One can confirm the following proposition by making use of Proposition 2.4.

Proposition 2.5 ([14, Proposition 3.1])**.** *Let $V(v)$ be the vector space over $\mathbb{C}$ generated by the basis $\{[v]_i\}_{i\in\mathbb{Z}}$, where $v \in \mathbb{C}$ is the spectral parameter. Then the following action on $V(v)$ defines a level $(1,1)$ representation of $\mathcal{U}$:*

$$K^+(z)[v]_i = \tilde{\psi}(q_1^i v/z)[v]_i, \tag{2.20}$$

$$K^-(z)[v]_i = \tilde{\psi}(q_1^{-i-1}z/v)[v]_i, \tag{2.21}$$

$$E(z)[v]_i = \mathcal{E}\cdot\delta(q_1^{i+1}v/z)[v]_{i+1}, \tag{2.22}$$

$$F(z)[v]_{i+1} = \mathcal{F}\cdot\delta(q_1^{i+1}v/z)[v]_i, \tag{2.23}$$

where the multiplication factors $\mathcal{E}$ and $\mathcal{F}$ are determined from the choice of $\tilde{g}$:

$$\mathcal{E}\cdot\mathcal{F} = \tilde{g}\frac{(1-q_2^{-1})(1-q_3^{-1})}{1-q_1} = (1-q_2)(1-q_2^{-1}). \tag{2.24}$$

This representation is called vector representation and denoted by $(\rho^{\mathrm{v}}, V(v))$.

In this thesis we choose

$$\mathcal{E} = 1-q_2, \quad \mathcal{F} = 1-q_2^{-1}, \tag{2.25}$$

in order to write down the intertwiner concisely, for example see (3.34) and (3.36). Note that we choose q_1 as the special direction, hence the symmetry of the parameters $q_{1,2,3}$, which exists at the level of algebra, is broken at the level of representation.

In general we can consider the tensor product of the vector representations by making use of the coproduct of Proposition 2.3. However, this procedure may become false if the spectral parameter takes the special value. As concerns this problem, we have the following sufficient condition for the well-definedness of the tensor product.

Proposition 2.6 ([14, Lemma 3.6])**.** *If $w \neq q_1^n v$ for any $n \in \mathbb{Z}$, then the coproduct of Proposition 2.3 define the tensor product representation on $V(v)\otimes V(w)$.*

Proof. Let us evaluate the action on $[v]_i \otimes [w]_j \in V(v) \otimes V(w)$ $(i, j \in \mathbb{Z})$. The parts in question are as follows:

$$K^-(z)[v]_i \otimes E(z)[w]_j = \mathcal{E} \cdot \tilde{\psi}(q_1^{-i-1} z/v) \delta(q_1^{j+1} w/z) [v]_i \otimes [w]_{j+1}, \tag{2.26}$$

$$\begin{cases} \text{pole}: & w = q_1^{i-j} v,\ q_1^{i-j-1} v, \\ \text{zero}: & w = q_1^{i-j} q_2 v,\ q_1^{i-j} q_3 v, \end{cases} \tag{2.27}$$

$$F(z)[v]_{i+1} \otimes K^+(z)[w]_j = \mathcal{F} \cdot \tilde{\psi}(q_1^j w/z) \delta(q_1^{i+1} v/z) [v]_i \otimes [w]_j, \tag{2.28}$$

$$\begin{cases} \text{pole}: & w = q_1^{i-j} v,\ q_1^{i-j+1} v, \\ \text{zero}: & w = q_1^{i-j+1} q_2 v,\ q_1^{i-j+1} q_3 v. \end{cases} \tag{2.29}$$

We have elucidated the poles and zeros by substituting the support of the delta functions for the function $\tilde{\psi}$. Therefore the assumption $w \neq q_1^n v$ is nothing but the condition in order for the action to avoid poles. $\square$

Actually the Fock representation $\mathcal{F}(v)$ which we will consider in the next subsection only has tensor components of type $V(v) \otimes V(q_2^n v)$, hence there is no problem at least for the *finite* tensor product $\bigotimes_{i=1}^N V(q_2^{i-1} v)$. On the other hand the tensor product of the Fock representations $\bigotimes_{k=1}^N \mathcal{F}(q_3^{k-1} v)$ cannot be defined even if N is finite due to the specialization of the spectral parameters. However, we can define the representation of $\mathcal{U}$ consistently by taking the certain subspace in it, which is the MacMahon representation [16].

2.2.2 Fock representation

Graphically, the Fock representation is spanned by (ordinary) Young diagrams, see Figure 4. Let $\mathcal{F}(v)$ be the subspace generated by the basis $\{|\lambda)\}_{\lambda:\text{partition}}$:

$$\bigotimes_{i=1}^{\infty} V(q_2^{i-1} v) \supset \mathcal{F}(v) \ni |\lambda) := \bigotimes_{i=1}^{\infty} [q_2^{i-1} v]_{\lambda_i - 1}. \tag{2.30}$$

In [14], they introduced the level $(1, \mathfrak{q})$ representation of $\mathcal{U}$ on $\mathcal{F}(v)$ by considering a certain limit of the tensor product of the vector representations. We introduce the coordinate $x_s = x_{s\lambda_s}$, $x_{ij} = q_1^{j-1} q_2^{i-1}$. It is also onvenient to decompose $\tilde{\psi}(z)$ into a ratio of more fundamental function $\psi(z)$ with shifted variable as follows:

$$\tilde{\psi}(z) = \psi(z) \psi(q_2^{-1} z)^{-1}, \quad \psi(z) = \mathfrak{q} \frac{1 - q_3^{-1} z}{1 - z} \in \mathbb{C}[[z]]. \tag{2.31}$$

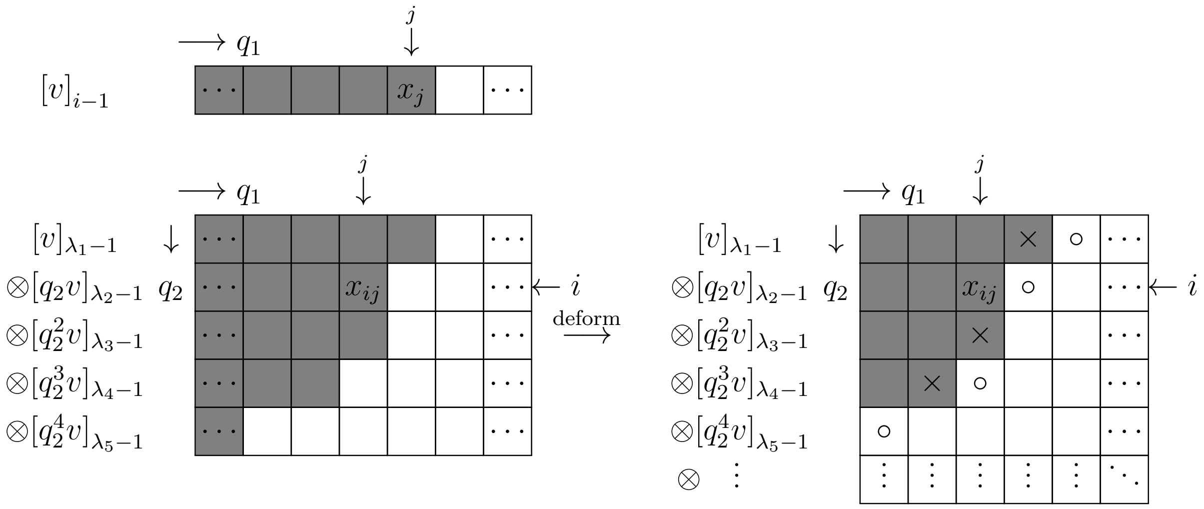


Figure 4: The basis vector of the vector representation looks a row of boxes. The tensor product of these vectors looks a perpendicular arrangement of these rows. The Young diagram results from taking the infinite tensor product and cutting the left half side. It is convenient to assign the coordinate $q_1^{j-1}q_2^{i-1}$ to the box x_{ij}. $E(z)$ adds the box to concave positions (denoted by $\circ$) and $F(z)$ removes the box from convex positions (denoted by $\times$) while $K^{\pm}(z)$ picks up the energy from each of boxes. The above example is the case of $\lambda = (4,3,3,2)$.

Note that, as rational functions, we have

$$\tilde{\psi}(u) = \tilde{\psi}(q_1^{-1}/u), \quad \psi(u) = \psi(q_3/u)^{-1}. \tag{2.32}$$

Proposition 2.7 ([14, Theorem 4.3, Corollary 4.4]). *The following action on $\mathcal{F}(v)$ defines a level $(1,\mathfrak{q})$ representation of $\mathcal{U}$:*

$$K^+(z)|\lambda\rangle = \prod_{s=1}^{\ell(\lambda)} \tilde{\psi}(x_s v/z)\ \beta^+_{\ell(\lambda)}(v/z)|\lambda\rangle = \prod_{s=1}^{\ell(\lambda)} \psi(x_s v/z) \prod_{s=1}^{\ell(\lambda)+1} \psi(q_2^{-1}x_s v/z)^{-1}|\lambda\rangle, \tag{2.33}$$

$$K^-(z)|\lambda\rangle = \prod_{s=1}^{\ell(\lambda)} \tilde{\psi}(q_1^{-1}x_s^{-1}z/v)\ \beta^-_{\ell(\lambda)}(z/v)|\lambda\rangle = \prod_{s=1}^{\ell(\lambda)} \psi(q_3 x_s^{-1}z/v)^{-1} \prod_{s=1}^{\ell(\lambda)+1} \psi(q_1^{-1}x_s^{-1}z/v)|\lambda\rangle, \tag{2.34}$$

$$\begin{aligned} E(z)|\lambda\rangle &= (1-q_2) \sum_{j=1}^{\ell(\lambda)+1} \prod_{s=1}^{j-1} \tilde{\psi}(q_1^{-1}x_s/x_j)\delta(q_1 x_j v/z)|\lambda+1_j\rangle \\ &= (1-q_2) \sum_{j=1}^{\ell(\lambda)+1} \prod_{s=1}^{j-1} \psi(q_1^{-1}x_s/x_j)\psi(q_3 x_s/x_j)^{-1}\delta(q_1 x_j v/z)|\lambda+1_j\rangle, \end{aligned} \tag{2.35}$$

$$
\begin{aligned}
F(z)|\lambda) &= (1-q_2^{-1})\sum_{j=1}^{\ell(\lambda)}\prod_{s=j+1}^{\ell(\lambda)}\tilde{\psi}(x_s/x_j)\delta(x_jv/z)\ \beta^{+}_{\ell(\lambda)}(1/x_j)|\lambda-1_j) \\
&= (1-q_2^{-1})\sum_{j=1}^{\ell(\lambda)}\prod_{s=j+1}^{\ell(\lambda)}\psi(x_s/x_j)\prod_{s=j+1}^{\ell(\lambda)+1}\psi(q_2^{-1}x_s/x_j)^{-1}\delta(x_jv/z)|\lambda-1_j),
\end{aligned} \tag{2.36}
$$

where

$$
\beta^{+}_{N}(v/z) := \mathfrak{q}^{-1}\frac{1-q_2^Nq_3v/z}{1-q_2^Nv/z}, \quad \beta^{-}_{N}(z/v) := \mathfrak{q}\frac{1-q_2^{-N}q_3^{-1}z/v}{1-q_2^{-N}z/v}. \tag{2.37}
$$

This representation is called Fock representation and denoted by $(\rho^{\mathrm{F}}, \mathcal{F}(v))$.

In [14] they proved that the actions (2.33)–(2.36) are indeed closed in the subspace $\mathcal{F}(v) \subset \bigotimes_{i=1}^{\infty} V(q_2^{i-1}v)$. This fact is assured by the q_2-shift of spectral parameters among adjacent vector representations in the tensor product. The Fock representation is the highest weight representation with $|\varnothing)$ being the highest weight state. The generating function of eigenvalues of the vacuum is

$$
\psi_{\varnothing}(v/z) := (\varnothing|K^{+}(z)|\varnothing) = \psi(q_3v/z)^{-1}, \tag{2.38}
$$

where we have defined the bra operator $(\lambda|$ by the condition $(\lambda|\mu) := \delta_{\lambda\mu}$. By making use of the relation (2.19), one can rewrite (2.33) in a symmetric way

$$
(\lambda|K^{+}(z)|\lambda) = \psi_{\varnothing}(v/z)\prod_{(i,j)\in\lambda}G(x_{ij}v/z). \tag{2.39}
$$

Note that the finite tensor product of the vector representations has the trivial level $(1,1)$ as well as the vector representation itself does. Now, however, the Fock representation has the nontrivial level $(1,\mathfrak{q})$ due to the regularization (2.51) we will choose.

At first following [14], we outline the idea to obtain the Fock representation $(\rho^{\mathrm{F}}, \mathcal{F}(v))$. One can construct a natural tensor product of the vector representations by making use of the coproduct (2.12)–(2.16) of $\mathcal{U}$. We would like to define an infinite tensor product of the vector representations and find an irreducible subrepresentation whose basis is spanned by partitions, that is, the Fock representation. Let us consider the following finite tensor product with q_2-shifted spectral parameters:

$$
\bigotimes_{i=1}^{N} V(q_2^{i-1}v) \ni |\lambda) := \bigotimes_{i=1}^{N}[q_2^{i-1}v]_{\lambda_i-1}, \quad \lambda = (\lambda_1,\dots,\lambda_N)\in\mathbb{Z}^N. \tag{2.40}
$$

The tensor product representation is defined by $\rho^{\rm v}_N(X(z)) = \rho^{\rm v}_{v_1}\otimes\cdots\otimes\rho^{\rm v}_{v_N}(\Delta^{N-1}(X(z)))$, where $\rho^{\rm v}_{v_i}$ denotes the vector representation with the spectral parameter $v_i = q_2^{i-1}v$. Since $C=1$ for vector representations, the coproduct (2.12)–(2.15) gives

$$\Delta^{N-1}(K^{\pm}(z)) = \underbrace{K^{\pm}(z)\otimes\cdots\otimes K^{\pm}(z)}_{N}, \tag{2.41}$$

$$\Delta^{N-1}(E(z)) = \sum_{k=1}^{N}\underbrace{K^{-}(z)\otimes\cdots K^{-}(z)}_{k-1}\otimes E(z)\otimes\underbrace{1\otimes\cdots\otimes 1}_{N-k}, \tag{2.42}$$

$$\Delta^{N-1}(F(z)) = \sum_{k=1}^{N}\underbrace{1\otimes\cdots\otimes 1}_{k-1}\otimes F(z)\otimes\underbrace{K^{+}(z)\otimes\cdots\otimes K^{+}(z)}_{N-k}. \tag{2.43}$$

One can naturally consider $\lambda\in\mathbb{Z}^N$ also as $\lambda\in\mathbb{Z}^{N+1}$ with $\lambda_{N+1}=0$. However, $(\rho^{\rm v}_N, \bigotimes_{i=1}^N V(q_2^{i-1}v))$ does not form an inductive system because actions of $\rho^{\rm v}_N$ and $\rho^{\rm v}_{N+1}$ on $|\lambda)$ ($\lambda\in\mathbb{Z}^N$) are different. For this reason one cannot take a limit $N\to\infty$ naively. In order to settle this problem, we *modify* the action of $(\rho^{\rm v}_N, \bigotimes_{i=1}^N V(q_2^{i-1}v))$ to some $(\bar\rho^{\rm v}_N, \bigotimes_{i=1}^N V(q_2^{i-1}v))$ so that it forms an inductive system of $\mathcal{U}$-modules. Then one can take the inductive limit $\bar\rho^{\rm v}_\infty = \lim_{N\to\infty}\bar\rho^{\rm v}_N$ to find the Fock representation $(\rho^{\rm F},\mathcal{F}(v))$ as an irreducible subrepresentation of $(\bar\rho^{\rm v}_\infty, \bigotimes_{i=1}^\infty V(q_2^{i-1}v))$.

In the following, instead of giving a full account of the proof for Proposition 2.7, we describe the idea of *modification* in detail and reveal the role of $\beta_N^{\pm}$ (2.37) in Proposition 2.7. Let us modify the action of $\rho^{\rm v}_N$ to $\bar\rho^{\rm v}_N$ so as to obtain the condition

$$\bar\rho^{\rm v}_N(X(z)) = \bar\rho^{\rm v}_{N+M}(X(z)) \text{ on } |\lambda),\ \lambda\in\mathbb{Z}^{N-1}, \quad \forall M\in\mathbb{Z}_{>0}, \tag{2.44}$$

for $X=K^{\pm},E$ and F. Then we can define the action of $\bar\rho^{\rm v}_\infty(X(z))$ on $|\lambda)$ ($\lambda\in\mathbb{Z}^{N-1}$) as $\bar\rho^{\rm v}_N(X(z))$. Therefore, we should search for a modified action which satisfies the condition (2.44). Actually, we have to modify only $\rho^{\rm v}_N(K^{\pm}(z))$ and $\rho^{\rm v}_N(F(z))$ keeping $\rho^{\rm v}_N(E(z))$ intact. To begin with, let us see that the action $\bar\rho^{\rm v}_N(E(z))$ can be the same as $\rho^{\rm v}_N(E(z))$. This action satisfies the condition (2.44) due to the vanishing property:

$$(K^-(z)\otimes E(z))([q_2^{N-1}v]_{-1}\otimes[q_2^N v]_{-1}) = \tilde\psi(q_2^{-N+1}z/v)\delta(q_2^N v/z)\ [q_2^{N-1}v]_{-1}\otimes[q_2^N v]_0 = 0. \tag{2.45}$$

Next, let us focus on the action of $K^{\pm}(z)$. This time the actions of $\rho^{\rm v}_N$ and $\rho^{\rm v}_{N+M}$ differ. Let $K^{\pm}_N(z) = \bar\rho^{\rm v}_N(K^{\pm}(z)) = \rho^{\rm v}_N(K^{\pm}(z))\cdot\beta^{\pm}_N = \Delta^{N-1}(K^{\pm}(z))\cdot\beta^{\pm}_N$, where $\beta^{\pm}_N = \beta^{\pm}_N((v/z)^{\pm1})$ is a *modification factor* which satisfies $\beta^{+}_N(u)=\beta^{-}_N(1/u)$ as a rational function. Since

$$K^+(z)[q_2^N v]_{-1} = \tilde\psi(q_1^{-1}q_2^N v/z)[q_2^N v]_{-1}, \tag{2.46}$$

one gets the recursion relation for the modification factor β_N^+:

$$1 = \frac{(\lambda|K_{N+1}^+(z)|\lambda)}{(\lambda|K_N^+(z)|\lambda)} = \frac{\beta_{N+1}^+}{\beta_N^+}\ \tilde{\psi}(q_1^{-1}q_2^N v/z). \tag{2.47}$$

This is equivalent to

$$\begin{aligned}\beta_N^+ &= \tilde{\psi}(q_1^{-1}q_2^{N-1}v/z)^{-1}\beta_{N-1}^+ = \psi(q_1^{-1}q_2^{N-1}v/z)^{-1}\psi(q_1^{-1}q_2^{N-2}v/z)\beta_{N-1}^+ \\ &= \psi(q_1^{-1}q_2^{N-1}v/z)^{-1}\psi(q_1^{-1}v/z)\beta_1^+,\end{aligned} \tag{2.48}$$

which determines $\beta_N^\pm$ up to an appropriate initial condition. Naturally, the initial condition should be related to the regularization problem of the vacuum of the Fock representation. To see this let us look at the unmodified representation ρ_N^{v} on the *" vacuum state* $|\varnothing)$*,"* $\varnothing = (0,\dots,0) \in \mathbb{Z}^N$,

$$\begin{aligned}(\varnothing|\Delta^{N-1}(K^+(z))|\varnothing) &= \prod_{k=1}^N \tilde{\psi}(q_1^{-1}q_2^{k-1}v/z) = \prod_{k=1}^N \psi(q_1^{-1}q_2^{k-1}v/z)\psi(q_2^{k-1}q_3 z/v)^{-1} \\ &= \frac{1-q_2^N v/z}{1-q_2^N q_3 v/z}\frac{1-q_3 v/z}{1-v/z}.\end{aligned} \tag{2.49}$$

This expression makes no sense at the limit of $N \to \infty$, but we can formally regularize it by specifying the ordering of infinite products:

$$\psi(q_3 v/z)^{-1}\prod_{k\geq 1}\left(\psi(q_1^{-1}q_2^{k-1}v/z)\psi(q_2^k q_3 v/z)^{-1}\right) = \psi(q_3 v/z)^{-1}. \tag{2.50}$$

Therefore, in the modified representation $\bar{\rho}_N^{\mathrm{v}}$, one expects the vacuum expectation value to be

$$\begin{aligned}(\varnothing|K_N^+(z)|\varnothing) &= \prod_{k=1}^N \tilde{\psi}(q_1^{-1}q_2^{k-1}v/z)\beta_N^+(v/z) \\ &= \psi(q_3 v/z)^{-1} = \mathfrak{q}^{-1}\frac{1-q_3 v/z}{1-v/z}, \quad \forall N \in \mathbb{Z}_{>0}.\end{aligned} \tag{2.51}$$

Thus, the problematic factor $\frac{1-q_2^N v/z}{1-q_2^N q_3 v/z}$ in (2.49) has been replaced by the factor $\mathfrak{q}^{-1}$ by the regularization. Now (2.51) leads to the initial condition

$$\beta_1^+(v/z) = \psi(q_1^{-1}v/z)^{-1}. \tag{2.52}$$

Hence, the above prescription for the regularization gives

$$\beta_N^+(v/z) = \psi(q_1^{-1}q_2^{N-1}v/z)^{-1} = \psi(q_2^{-N}z/v) = \beta_N^-(z/v). \tag{2.53}$$

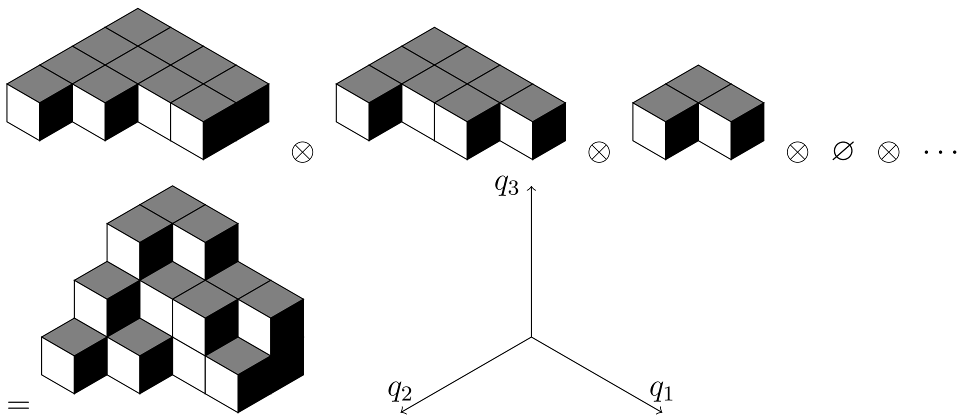


Figure 5: The three dimensional Young diagram (or plane partition) is a pile of ordinary (two dimensional) Young diagrams. Now the extra direction (dimension) corresponds to the parameter q_3, which measures the hight. The above example is the case of $\Lambda = ((4,4,2,1),(4,3,1),(2,1))$.

As concerns the action of $F(z)$, we also need some modification factor. This factor should be precisely the same as $\beta_N^+(v/z)$ for the sake of (2.8). In fact, it satisfies the condition (2.44) due to the vanishing property,

$$\beta_{N+1}^+ F(z)[q_2^N v]_{-1} = \psi(q_1^{-1}q_2^N v/z)^{-1}\delta(q_1^{-1}q_2^N v/z)[q_2^N v]_{-2} = 0. \tag{2.54}$$

At last, we can find the invariant subspace $\mathcal{F}(v)$ which consists only of partitions by investigating the positions of zeros appearing in the action of the creation operator $E(z)$ and the annihilation operator $F(z)$. This irreducible subrepresentation generated by the empty Young diagram $\varnothing$ is nothing but what we have wanted.

2.2.3 MacMahon representation

Graphically, the MacMahon representation is spanned by three dimensional Young diagrams or plane partitions, see Figure 5.

Let $\mathcal{M}(v)$ be the subspace generated by the basis $\{|\Lambda)\}_{\Lambda:\text{plane partition}}$:

$$\bigotimes_{k=1}^{\infty}\mathcal{F}(q_3^{k-1}v) \supset \mathcal{M}(v) \ni |\Lambda) := \bigotimes_{k=1}^{\infty}|\Lambda^{(k)}). \tag{2.55}$$

A plane partition $\Lambda = (\Lambda^{(k)})_{k\geq 1}$ is a sequence of (ordinary) partitions which satisfy $\Lambda_i^{(k+1)} \leq \Lambda_i^{(k)}$ $(\forall i,k)$ and $\Lambda^{(n)} = \varnothing$ for sufficiently large $n \in \mathbb{Z}_{>0}$. We also use the following notations:

$$h(\Lambda) := \min\{k \geq 0 | (1,1,k+1) \notin \Lambda\}, \tag{2.56}$$

$$(i,j,k)\in\Lambda :\Longleftrightarrow \Lambda_i^{(k)}\geq j. \tag{2.57}$$

Let $|\Lambda^{(k)})\in\mathcal{F}(q_3^{k-1}v)$ unless otherwise mentioned. In [16], they introduced the level $(1,K^{1/2})$ representation of $\mathcal{U}$ on $\mathcal{M}(v)$ in a similar way to the construction of the Fock representation. We introduce the coordinate $x_{ijk}=q_1^{j-1}q_2^{i-1}q_3^{k-1}=q_1^{j}q_2^{i}q_3^{k}$.

Proposition 2.8 ([16, Section 3.1])**.** *The following action on $\mathcal{M}(v)$ defines a level $(1,K^{1/2})$ representations of $\mathcal{U}$ for $K^{1/2}\in\mathbb{C}^\times$:*

$$\begin{aligned}
K^{\pm}(z)|\Lambda) &= \prod_{k=1}^{h(\Lambda)}(\Lambda^{(k)}|K^{\pm}(z)|\Lambda^{(k)})|_{\mathcal{F}(q_3^{k-1}v)}\ \gamma^{\pm}_{h(\Lambda)}((v/z)^{\pm1})|\Lambda)\\
&= \boldsymbol{\psi}^{\pm}_{\varnothing}\left(K^{1/2};(v/z)^{\pm1}\right)\prod_{(i,j,k)\in\Lambda}G((x_{ijk}v/z)^{\pm1})^{\pm1}|\Lambda),
\end{aligned}\tag{2.58}$$

$$\begin{aligned}
E(z)|\Lambda) &= \sum_{k=1}^{h(\Lambda)+1}\sum_{i=1}^{\ell(\Lambda^{(k)})+1}\prod_{s=1}^{k-1}(\Lambda^{(s)}|K^{-}(q_1x_{ijk}v)|\Lambda^{(s)})|_{\mathcal{F}(q_3^{s-1}v)}\\
&\quad\times(\Lambda^{(k)}+1_i|E(z)|\Lambda^{(k)})|_{\mathcal{F}(q_3^{k-1}v)}|\Lambda+1_i^{(k)}),
\end{aligned}\tag{2.59}$$

$$\begin{aligned}
F(z)|\Lambda) &= \sum_{k=1}^{h(\Lambda)}\sum_{i=1}^{\ell(\Lambda^{(k)})}(\Lambda^{(k)}-1_i|F(z)|\Lambda^{(k)})|_{\mathcal{F}(q_3^{k-1}v)}\\
&\quad\times\prod_{s=k+1}^{h(\Lambda)}(\Lambda^{(s)}|K^{+}(x_{ijk}v)|\Lambda^{(s)})|_{\mathcal{F}(q_3^{s-1}v)}\ \gamma^{+}_{h(\Lambda)}(x_{ijk}^{-1})|\Lambda-1_i^{(k)}),
\end{aligned}\tag{2.60}$$

where

$$\begin{aligned}
\gamma_N^{+}(v/z)=\gamma_N^{+}(K;v/z) &:= \frac{K^{-1/2}(1-Kv/z)}{\mathfrak{q}^{-N}(1-q_3^{N}v/z)},\\
\gamma_N^{-}(z/v)=\gamma_N^{-}(K;z/v) &:= \frac{K^{1/2}(1-K^{-1}z/v)}{\mathfrak{q}^{N}(1-q_3^{-N}z/v)},
\end{aligned}\tag{2.61}$$

and we have introduced the generating function of eigenvalues of the vacuum

$$\boldsymbol{\psi}^{\pm}_{\varnothing}(K^{1/2};u)=K^{\mp1/2}\frac{1-K^{\pm1}u}{1-u}. \tag{2.62}$$

This representation is called MacMahon representation and denoted by $(\rho^{\mathrm{M}}(K),\mathcal{M}(v))$.

We note that $j=\Lambda_i^{(k)}$ is understood at the right hand sides of (2.59) and (2.60). The action of $K^{\pm}(z)$ has a manifestly symmetric expression with respect to the permutation of $q_{1,2,3}$ but that of $E(z)$ and $F(z)$ do not. The MacMahon representation is the highest weight representation with the empty plane partition $\boldsymbol{\varnothing}$ being the highest weight state. Note that, so far it makes sense, the N-times tensor product of the Fock representations has the level $(1, \mathfrak{q}^N)$. Now, however, the MacMahon representation has the level $(1, K^{1/2})$ with a continuous parameter K, which is outside the algebra $\mathcal{U}$, due to the regularization (2.67) we will choose. Hereafter we use the notation $\mathcal{M}(K; v)$ for $\mathcal{M}(v)$ in order to specify the K dependance. We also use the notation $\mathcal{M}_N(v)$ for the subspace in $\bigotimes_{k=1}^N \mathcal{F}(q_3^{k-1}v)$ which is spanned by the plane partition whose hight is at most N.

We can obtain the above formulas in almost the same way as the construction of the Fock representation, while the regularization procedure becomes somehow different. To see this, let us focus on the action of $K^\pm(z)$ here. Let $K_N^\pm(z) = \bar{\rho}_N^{\mathrm{F}}(K^\pm(z)) = \Delta^{N-1}(K^\pm(z)) \cdot \gamma_N^\pm$, where $\gamma_N^\pm = \gamma_N^\pm((v/z)^{\pm 1})$ is a *modification factor* which satisfies $\gamma_N^+(u) = \gamma_N^-(1/u)$ as a rational function. Since

$$(\varnothing|K^+(z)|\varnothing)|_{\mathcal{F}(q_3^N v)} = \psi(q_3^{N+1}v/z)^{-1}, \quad |\varnothing) \in \mathcal{F}(q_3^N v), \tag{2.63}$$

one gets the recursion relation for the modification factor γ_N^+:

$$1 = \frac{(\Lambda|K_{N+1}^+(z)|\Lambda)|_{\mathcal{M}_{N+1}(v)}}{(\Lambda|K_N^+(z)|\Lambda)|_{\mathcal{M}_N(v)}} = \frac{\gamma_{N+1}^+}{\gamma_N^+}\ \psi(q_3^{N+1}v/z)^{-1}, \tag{2.64}$$

which means

$$\gamma_N^+ = \psi(q_3^N v/z)\gamma_{N-1}^+ = \mathfrak{q}^{N-1}\frac{1-q_3 v/z}{1-q_3^N v/z}\gamma_1^+, \tag{2.65}$$

and determines $\gamma_N^\pm$ up to an appropriate initial condition. As concerns the initial condition, let us look at the unmodified representation ρ_N^{F} on the *"vacuum state* $|\boldsymbol{\varnothing})$*,"* $\boldsymbol{\varnothing} = (\varnothing, \ldots, \varnothing)$,

$$(\boldsymbol{\varnothing}|\Delta^{N-1}(K^+(z))|\boldsymbol{\varnothing})|_{\mathcal{M}_N(v)} = \prod_{k=1}^N \psi(q_3^k v/z)^{-1} = \mathfrak{q}^{-N}\prod_{k=1}^N \frac{1-q_3^k v/z}{1-q_3^{k-1}v/z} = \mathfrak{q}^{-N}\frac{1-q_3^N v/z}{1-v/z}. \tag{2.66}$$

This expression makes no sense at the limit of $N \to \infty$, but we cannot use the same strategy as in the previous subsection due to the presence of the coefficient $\mathfrak{q}^{-N}$. Here we have to regularize it in another way; we formally substitute $\mathfrak{q}^N$ by an arbitrary parameter $K^{1/2} \in \mathbb{C}^\times$. Then, in the modified representation $\bar{\rho}_N^{\mathrm{F}}$, one expects the vacuum expectation value to be

$$(\boldsymbol{\varnothing}|K_N^+(z)|\boldsymbol{\varnothing})|_{\mathcal{M}_N(v)} = \prod_{k=1}^N \psi(q_3^k v/z)^{-1}\ \gamma_N^+(v/z) = K^{-1/2}\frac{1-Kv/z}{1-v/z}, \tag{2.67}$$

which leads to the initial condition

$$\gamma_1^+(v/z) = \frac{K^{-1/2}(1 - Kv/z)}{\mathfrak{q}^{-1}(1 - q_3 v/z)}. \tag{2.68}$$

Hence our regularization gives

$$\gamma_N^+(v/z) = \frac{K^{-1/2}(1 - Kv/z)}{\mathfrak{q}^{-N}(1 - q_3^N v/z)} = \frac{K^{1/2}(1 - K^{-1}z/v)}{\mathfrak{q}^{N}(1 - q_3^{-N} z/v)} = \gamma_N^-(z/v). \tag{2.69}$$

2.3 Horizontal representation

In this section we summarize horizontal representations which have the unit first level $\gamma_1 = C = \mathfrak{q}$. We introduce the *Heisenberg mode* $H_{\pm r}$ ($r \in \mathbb{Z}_{>0}$) by

$$K^\pm(z) = K^\pm \exp\left(\pm(\mathfrak{q} - \mathfrak{q}^{-1}) \sum_{r=1}^{\infty} H_{\pm r} z^{\mp r} \right). \tag{2.70}$$

When $C = \mathfrak{q}$, the Heisenberg mode satisfies the following relation due to (2.5):

$$[H_r, H_s] = \delta_{r+s,0} \frac{[r]^2}{r} \mathfrak{q}^r (1 - q_1^r)(1 - q_2^r), \tag{2.71}$$

where $[r] := (\mathfrak{q}^r - \mathfrak{q}^{-r})/(\mathfrak{q} - \mathfrak{q}^{-1})$ is the $\mathfrak{q}$-integer. We define the *fundamental vertex operator* $V^\pm(z)$ as follows:

$$V^\pm(z) = \exp\left(\mp \sum_{r=1}^{\infty} \frac{H_{\pm r}}{[r]} z^{\mp r} \right), \tag{2.72}$$

then the fundamental OPE (operator product expansion) by the normal ordering is

$$V^+(z)V^-(w) = s(w/z) : V^+(z)V^-(w) :, \tag{2.73}$$

where the scattering factor $s(z)$ is

$$s(z) = \frac{(1 - \mathfrak{q}z)(1 - \mathfrak{q}^{-1}z)}{(1 - \mathfrak{d}z)(1 - \mathfrak{d}^{-1}z)} \in \mathbb{C}[[z]], \tag{2.74}$$

and satisfies the following formulas as a rational function:

$$s(u) = s(1/u), \quad G(u) = s(\mathfrak{q}^{-1}u)s(\mathfrak{q}u)^{-1}. \tag{2.75}$$

The normal ordering is the procedure to carry positive modes to the right, for example,

$$: H_r H_{-r} := H_{-r} H_r, \ : H_{-r} H_r := H_{-r} H_r, \ : H_r H_{-s} := H_r H_{-s} = H_{-s} H_r, \quad r, s \in \mathbb{Z}_{>0}, \ r \neq s. \tag{2.76}$$

The Fock space for the Heisenberg mode, the Heisenberg Fock space, is the vector space over $\mathbb{C}$ generated by the basis $|H_\lambda\rangle = H_{-\lambda_1}H_{-\lambda_2}\cdots|0\rangle$ where $\lambda = (\lambda_s)_{s\geq 1}$ is a partition and the vacuum state $|0\rangle$ is defined by the annihilation condition

$$H_r|0\rangle = 0, \quad r > 0. \tag{2.77}$$

One can confirm the following proposotion by making use of the fundamental OPE (2.73).

Proposition 2.9 ([20, Proposition A.6])**.** *The following vertex operators on the Heisenberg Fock space define a level* $(\mathfrak{q}, 1)$ *representation of* $\mathcal{U}$:

$$K^\pm(\mathfrak{q}^{1/2}z) \to \varphi^\pm(z) = V^\pm(\mathfrak{q}^{\pm 1}z)V^\pm(\mathfrak{q}^{\mp 1}z)^{-1}, \tag{2.78}$$

$$E(z) \to \eta(z) = V^-(\mathfrak{q}^{-\frac{1}{2}}z)V^+(\mathfrak{q}^{\frac{1}{2}}z), \tag{2.79}$$

$$F(z) \to \xi(z) = V^-(\mathfrak{q}^{\frac{1}{2}}z)^{-1}V^+(\mathfrak{q}^{-\frac{1}{2}}z)^{-1}. \tag{2.80}$$

The shift of the argument in $K^\pm(z)$ is conventional. Furthermore, for any $\gamma_2 \in \mathbb{C}^\times$, we can employ a more general level $(\mathfrak{q}, \gamma_2)$ representation with zero modes $\mathbf{e}(z)$, $\mathbf{f}(z)$ and $\mathbf{k}^\pm(z)$ which are operators acting on the Heisenberg Fock space and commuting with non-zero modes $H_{\pm r}$. We assume that the zero modes commute with one another,[3] namely the zero modes are $\mathbb{C}[[z^{\pm 1}]]$-valued.

Proposition 2.10 ([30, Section 2.3])**.** *When the zero modes satisfy the following conditions*:

$$\mathbf{k}^\pm(0) = \gamma_2^{\mp 1}, \tag{2.81}$$

$$\mathbf{e}(z)\mathbf{f}(\mathfrak{q}^{\mp 1}z) = \mathbf{k}^\pm(\mathfrak{q}^{\mp 1/2}z), \tag{2.82}$$

the following vertex operators on the Heisenberg Fock space define a level $(\mathfrak{q}, \gamma_2)$ *representation of* $\mathcal{U}$:

$$K^\pm(\mathfrak{q}^{1/2}z) \to \varphi^\pm(z)\ \mathbf{k}^\pm(z), \tag{2.83}$$

$$E(z) \to \eta(z)\ \mathbf{e}(z), \tag{2.84}$$

$$F(z) \to \xi(z)\ \mathbf{f}(z). \tag{2.85}$$

This representation is called horizontal representation and denoted by $\mathcal{H} = \mathcal{H}(\mathbf{k}^\pm(z), \mathbf{e}(z), \mathbf{f}(z))$.

[3] In the case of general quantum toroidal $\mathfrak{gl}_{n>1}$ algebra, the zero modes do not commute with one another.

Proof. To begin with, the second level is fixed by $K^-(0) = \mathbf{k}^-(0) = \gamma_2$ $(= (K^+(0))^{-1} = (\mathbf{k}^+(0))^{-1})$. Then it suffices to show that the above assignments (2.83)–(2.85) solve the defining relations (2.4)–(2.10). Since we have already known that the Heisenberg parts (2.78)–(2.80) themselves solve the defining relations, the nontrivial check is only for (2.8). We can check this from (2.82) by substituting the support of the delta function. $\square$

Note that $\mathbf{e}(z)$ can be uniquely determined from $\mathbf{f}(z)$ and $\mathbf{k}^\pm(z)$ as

$$\mathbf{e}(z) = \sqrt{(\mathbf{k}^+(\mathfrak{q}^{-1/2}z)\mathbf{k}^-(\mathfrak{q}^{1/2}z)) \,/\, (\mathbf{f}(\mathfrak{q}^{-1}z)\mathbf{f}(\mathfrak{q}z))}. \tag{2.86}$$

Furthermore, we make an ansatz that $\mathbf{k}^\pm(z)$ is independent of z:

$$\mathbf{k}^\pm(z) = \mathbf{k}^\pm(0) = \gamma_2^{\mp 1}, \tag{2.87}$$

so that we can lift the modification factors (2.37) and (2.69) of vertical representations to vertex operators uniquely later. As we will see, we need nontrivial zero modes for the existence of the intertwiner.

For example, the level $(\mathfrak{q}, \mathfrak{q}^N)$ representation used in [22] is defined by

$$\mathbf{k}^\pm(z) = \mathfrak{q}^{\mp N}, \qquad \mathbf{e}(z) = (\mathfrak{q}/z)^N u, \qquad \mathbf{f}(z) = (\mathfrak{q}/z)^{-N} u^{-1}, \tag{2.88}$$

where u is the spectral parameter of the representation. We can express this horizontal representation as $\mathcal{F}_u^{(\mathfrak{q},\mathfrak{q}^N)} := \mathcal{H}(\mathfrak{q}^{\mp N}, (\mathfrak{q}/z)^N u, (\mathfrak{q}/z)^{-N} u^{-1})$.

It is also useful to introduce the dual vertex operator $\tilde{V}^\pm(z)$ which satisfies

$$V^+(z)\tilde{V}^-(w) = (1 - w/z)^{-1} \; : V^+(z)\tilde{V}^-(w) :, \tag{2.89}$$

$$\tilde{V}^+(z)V^-(w) = (1 - w/z) \; : \tilde{V}^+(z)V^-(w) : . \tag{2.90}$$

It is expressed explicitly as

$$\tilde{V}^\pm(z) = \exp\left(\mp \sum_{r=1}^\infty \Lambda_{\pm r} z^{\mp r}\right), \qquad \Lambda_r := \frac{\mathfrak{q} - \mathfrak{q}^{-1}}{k_r} H_r, \tag{2.91}$$

where

$$k_r = \prod_{i=1}^3 (q_i^{r/2} - q_i^{-r/2}) = -\prod_{i=1}^3 (1 - q_i^r) = \sum_{i=1}^3 (q_i^r - q_i^{-r}), \tag{2.92}$$

and we have

$$[\Lambda_r, H_s] = \delta_{r+s,0} \frac{[r]}{r}. \tag{2.93}$$

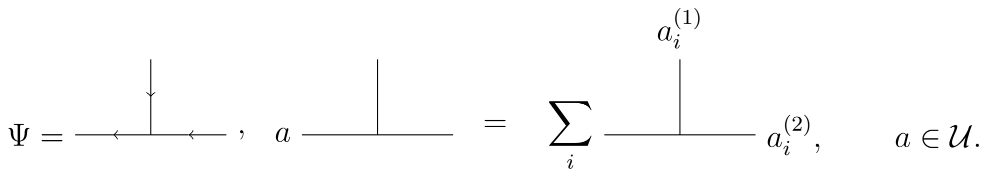


Figure 6: We use the notation $\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$.

3 Construction of intertwining operator

3.1 Definition of intertwiner

In [22] they introduced the trivalent intertwining operator of $\mathcal{U}$. It was defined for a triple of representations $(\mathcal{F}_v^{(1,\mathfrak{q})}, \mathcal{F}_u^{(\mathfrak{q},\mathfrak{q}^N)}, \mathcal{F}_{-uv}^{(\mathfrak{q},\mathfrak{q}^{N+1})})$, where $\mathcal{F}_v^{(1,\mathfrak{q})} = \mathcal{F}(v)$ was the vertical Fock representation and $(\mathcal{F}_u^{(\mathfrak{q},\mathfrak{q}^N)}, \mathcal{F}_{-uv}^{(\mathfrak{q},\mathfrak{q}^{N+1})})$ was a pair of the horizontal representations. In more general we define the trivalent intertwining operator as follows.

Definition 3.1. *Let $\mathcal{V}$ be a vertical representation and $(\mathcal{H}, \mathcal{H}')$ be a pair of horizontal representations. The trivalent intertwining operator is a linear operator $\Psi : \mathcal{V} \otimes \mathcal{H} \to \mathcal{H}'$ which satisfies the following intertwining condition*:

$$a\Psi = \Psi\Delta(a), \qquad \forall a \in \mathcal{U}. \tag{3.1}$$

Note that the coproduct of proposition 2.3 makes sense in the tensor product $\mathcal{V} \otimes \mathcal{H}$ because of the annihilation condition (2.77). Hereafter this trivalent intertwining operator is called *intertwiner* simply. The intertwiner Ψ whose vertical representation is the vector, Fock and MacMahon one is called and denoted as vector, Fock and MacMahon intertwiner $\mathbb{I}, \Phi$ and Ξ respectively. Graphically the intertwiner is represented as a trivalent vertex with a single vertical edge and two horizontal edges, and the intertwining condition (3.1) is represented as in Figure 6. Actually what we will do is to construct the following *component* of the intertwiner.

Definition 3.2. *Let $\{|\alpha)\}_\alpha$ be a basis of the vertical representation $\mathcal{V}$, which simultaneously diagonalizes the action of $K^{\pm}(z)$. The α-component of the intertwiner $\Psi : \mathcal{V} \otimes \mathcal{H} \to \mathcal{H}'$ is a linear operator $\Psi_\alpha : \mathcal{H} \to \mathcal{H}'$ defined by*

$$\Psi_\alpha(\bullet) = \Psi(|\alpha) \otimes \bullet), \quad \bullet \in \mathcal{H}. \tag{3.2}$$

We can use the following proposition as a definition of the component of the intertwiner.

Proposition 3.3 ([30, Section 4]). *Let Ψ_α be the α-component of the intertwiner $\Psi : \mathcal{V} \otimes \mathcal{H} \to \mathcal{H}'$. Then Ψ_α is characterized by the following intertwining relations*:

$$K^+(z)\Psi_\alpha = (\alpha|K^+(z)|\alpha)\ \Psi_\alpha K^+(z), \tag{3.3}$$

$$K^-(\mathfrak{q}z)\Psi_\alpha = (\alpha|K^-(z)|\alpha)\ \Psi_\alpha K^-(\mathfrak{q}z), \tag{3.4}$$

$$E(z)\Psi_\alpha = \sum_\beta (\beta|E(z)|\alpha)\ \Psi_\beta + (\alpha|K^-(z)|\alpha)\ \Psi_\alpha E(z), \tag{3.5}$$

$$F(z)\Psi_\alpha = \sum_\beta (\beta|F(\mathfrak{q}z)|\alpha)\ \Psi_\beta K^+(\mathfrak{q}z) + \Psi_\alpha F(z), \tag{3.6}$$

where operators on the right side of Ψ_α act on the source horizontal representation $\mathcal{H}$, while operators on the left side of Ψ_α act on the target one $\mathcal{H}'$. The matrix elements $(\beta|X|\alpha)$ are computed in the vertical representation $\mathcal{V}$.

Proof. It suffices to check the intertwining condition (3.1) for the generators of $\mathcal{U}$: $K^\pm(z)$, $E(z)$, $F(z)$ and C. (3.1) for the central element C is automatically satisfied by definition because $C = 1$ and $C = \mathfrak{q}$ for the vertical and the horizontal representations respectively. As concerns the other generators, we can obtain (3.3)–(3.6) by making (3.1) act on $|\alpha) \otimes \bullet \in \mathcal{V} \otimes \mathcal{H}$ and using the notation $1 = \sum_\beta |\beta)(\beta|$. $\square$

By evaluating the matrix elements in each of vertical representations, we can summarize definitions of the intertwiners as follows.

Proposition 3.4 ([30, Section 4.1.1]). *Let $\{[v]_{n-1}\}_{n\in\mathbb{Z}}$ be the basis of the vertical vector representation $V(v)$. The n-component of the vector intertwiner defined by*

$$\mathbb{I}_n(v)(\bullet) = \mathbb{I}([v]_{n-1} \otimes \bullet) \colon \mathcal{H} \to \mathcal{H}', \quad \bullet \in \mathcal{H}, \tag{3.7}$$

is characterized by the following intertwining relations:

$$K^+(z)\mathbb{I}_n(v) = \tilde{\psi}(q_1^{n-1}v/z)\ \mathbb{I}_n(v)K^+(z), \tag{3.8}$$

$$K^-(\mathfrak{q}z)\mathbb{I}_n(v) = \tilde{\psi}(q_1^{-n}z/v)\ \mathbb{I}_n(v)K^-(\mathfrak{q}z), \tag{3.9}$$

$$E(z)\mathbb{I}_n(v) = (1-q_2)\delta(q_1^n v/z)\ \mathbb{I}_{n+1}(v) + \tilde{\psi}(q_1^{-n}z/v)\ \mathbb{I}_n(v)E(z), \tag{3.10}$$

$$F(z)\mathbb{I}_n(v) = (1-q_2^{-1})\delta(\mathfrak{q}^{-1}q_1^{n-1}v/z)\ \mathbb{I}_{n-1}(v)K^+(\mathfrak{q}z) + \mathbb{I}_n(v)F(z). \tag{3.11}$$

Proof. It follows from Propositions 2.5 and 3.3. $\square$

Proposition 3.5 ([22, Lemma 3.2], [30, Section 4.2.1]). *Let $\{|\lambda)\}_{\lambda:\mathit{partition}}$ be the basis of the vertical Fock representation $\mathcal{F}(v)$. The λ-component of the Fock intertwiner defined by*

$$\Phi_\lambda(v)(\bullet) = \Phi(|\lambda) \otimes \bullet) \colon \mathcal{H} \to \mathcal{H}', \quad \bullet \in \mathcal{H}, \tag{3.12}$$

is characterized by the following intertwining relations:

$$K^{+}(z)\Phi_{\lambda}(v) = (\lambda|K^{+}(z)|\lambda)\ \Phi_{\lambda}(v)K^{+}(z), \tag{3.13}$$

$$K^{-}(\mathfrak{q}z)\Phi_{\lambda}(v) = (\lambda|K^{-}(z)|\lambda)\ \Phi_{\lambda}(v)K^{-}(\mathfrak{q}z), \tag{3.14}$$

$$E(z)\Phi_{\lambda}(v) = \sum_{k=1}^{\ell(\lambda)+1} (\lambda+1_k|E(z)|\lambda)\ \Phi_{\lambda+1_k}(v) + (\lambda|K^{-}(z)|\lambda)\ \Phi_{\lambda}(v)E(z), \tag{3.15}$$

$$F(z)\Phi_{\lambda}(v) = \sum_{k=1}^{\ell(\lambda)} (\lambda-1_k|F(\mathfrak{q}z)|\lambda)\ \Phi_{\lambda-1_k}(v)K^{+}(\mathfrak{q}z) + \Phi_{\lambda}(v)F(z), \tag{3.16}$$

where matrix elements can be read from (2.33)–(2.36)*.*

Proof. It follows from Propositions 2.7 and 3.3. □

Proposition 3.6 ([30, Section 4.3.1])**.** *Let* $\{|\Lambda)\}_{\Lambda:\textit{plane partition}}$ *be the basis of the vertical MacMahon representation* $\mathcal{M}(K;v)$*. The* Λ*-component of the MacMahon intertwiner defined by*

$$\Xi_{\Lambda}(K;v)(\bullet) = \Xi(|\Lambda)\otimes\bullet)\colon \mathcal{H}\to\mathcal{H}', \quad \bullet\in\mathcal{H}, \tag{3.17}$$

is characterized by the following intertwining relations:

$$K^{+}(z)\Xi_{\Lambda}(v) = (\Lambda|K^{+}(z)|\Lambda)\ \Xi_{\Lambda}(v)K^{+}(z), \tag{3.18}$$

$$K^{-}(\mathfrak{q}z)\Xi_{\Lambda}(v) = (\Lambda|K^{-}(z)|\Lambda)\ \Xi_{\Lambda}(v)K^{-}(\mathfrak{q}z), \tag{3.19}$$

$$E(z)\Xi_{\Lambda}(v) = \sum_{k=1}^{h(\Lambda)+1}\sum_{i=1}^{\ell(\Lambda^{(k)})+1} (\Lambda+1_i^{(k)}|E(z)|\Lambda)\ \Xi_{\Lambda+1_i^{(k)}}(v) + (\Lambda|K^{-}(z)|\Lambda)\ \Xi_{\Lambda}(v)E(z), \tag{3.20}$$

$$F(z)\Xi_{\Lambda}(v) = \sum_{k=1}^{h(\Lambda)}\sum_{i=1}^{\ell(\Lambda^{(k)})} (\Lambda-1_i^{(k)}|F(\mathfrak{q}z)|\Lambda)\ \Xi_{\Lambda-1_i^{(k)}}(v)K^{+}(\mathfrak{q}z) + \Xi_{\Lambda}(v)F(z), \tag{3.21}$$

where the matrix elements can be read from (2.58)–(2.60)*.*

Proof. It follows from Propositions 2.8 and 3.3. □

Note that, as shown in [22], the intertwiner cannot exist for an arbitrary pair of horizontal representations, and the existance requires some relative condition between them, which can be seen as *the charge conservation law.* In the following, we will construct the above component of intertwiner in the vertex operator formalism and specify the admissible pair of horizontal representations. Note that we can typically specify just the relative conditions between $\mathcal{H}$ and $\mathcal{H}'$, but there seemingly remains some freedom for each of them.

We use the notation $\mathbf{x}(z), \mathbf{x}'(z)$ and $\mathbf{x}''(z)$ for the zero modes of the horizontal representation $\mathcal{H}, \mathcal{H}'$ and $\mathcal{H}''$ respectively, where $\mathbf{x} = \mathbf{e}, \mathbf{f}, \mathbf{k}^{\pm}$. We also write the second level γ_2 of $\mathcal{H}, \mathcal{H}'$ and $\mathcal{H}''$ as γ, γ' and γ'' respectively unless otherwise mentioned.

3.2 Construction of vector intertwiner

Proposition 3.7 ([30, Section 4.1.2]). *There exists a unique vector intertwiner $\mathbb{I}(v) : V(v) \otimes \mathcal{H} \to \mathcal{H}'$ up to normalization if and only if a pair of horizontal representations $(\mathcal{H}, \mathcal{H}')$ satisfies $\gamma' = \gamma$ on the second level and $\mathbf{e}'(z) = q_2^{-1}\mathbf{e}(z)$, $\mathbf{f}'(z) = q_2\mathbf{f}(z)$ for the zero mode sector. The n-component is written explicitly as*

$$\mathbb{I}_n(v) = z_n \tilde{\mathbb{I}}_n(v), \quad \tilde{\mathbb{I}}_n(v) = \tilde{\mathbb{I}}_0(q_1^n v), \quad n \in \mathbb{Z}, \tag{3.22}$$

$$\tilde{\mathbb{I}}_0(v) = \exp\left(-\sum_{r=1}^{\infty} \frac{H_{-r}}{[r]} \frac{(\mathfrak{q}^{-1/2}v)^r}{1-q_1^r}\right) \exp\left(-\sum_{r=1}^{\infty} \frac{H_r}{[r]} \frac{(\mathfrak{q}^{1/2}q_1^{-1}v)^{-r}}{1-q_1^r}\right), \tag{3.23}$$

where $z_n = z_n(v)$ is a stack of zero modes:

$$z_0 = 1, \quad z_n = q_2^{-n} \prod_{j=1}^{n} \mathbf{e}(q_1^{j-1}v) \quad (n > 0), \quad z_n = q_2^{-n} \prod_{j=n}^{-1} \mathbf{e}(q_1^{j}v)^{-1} \quad (n < 0). \tag{3.24}$$

If we choose the horizontal representation as $\mathcal{H} = \mathcal{F}_u^{(\mathfrak{q},\mathfrak{q}^N)}$ (2.88) like [22], then $\mathcal{H}' = \mathcal{F}_{q_2^{-1}u}^{(\mathfrak{q},\mathfrak{q}^N)}$ and the zero mode stack $z_n = z_n(N; u|v)$ is

$$z_n = (u/q_2)^n \prod_{j=1}^{n} \left(\frac{\mathfrak{q}}{q_1^{j-1}v}\right)^N \quad (n > 0), \qquad z_n = (u/q_2)^n \prod_{j=n}^{-1} \left(\frac{\mathfrak{q}}{q_1^{j-1}v}\right)^N \quad (n < 0). \tag{3.25}$$

Before proving Proposition 3.7, we summarize properties of the Heisenberg part of the zero component $\tilde{\mathbb{I}}_0$.

Proposition 3.8 ([30, Section 4.1.2]). *The Heisenberg part of the zero component* (3.23) *satisfies the following OPE relations*:

$$\varphi^+(\mathfrak{q}^{-1/2}z)\tilde{\mathbb{I}}_0(v) = \tilde{\psi}(q_1^{-1}v/z)\, \tilde{\mathbb{I}}_0(v)\varphi^+(\mathfrak{q}^{-1/2}z), \tag{3.26}$$

$$\tilde{\mathbb{I}}_0(v)\varphi^-(\mathfrak{q}^{1/2}z) = \tilde{\psi}(z/v)^{-1}\, \varphi^-(\mathfrak{q}^{1/2}z)\tilde{\mathbb{I}}_0(v), \tag{3.27}$$

$$\xi(z)\tilde{\mathbb{I}}_0(v) = \frac{1-\mathfrak{q}v/z}{1-\mathfrak{q}q_2v/z} : \xi(z)\tilde{\mathbb{I}}_0(v) :, \quad \tilde{\mathbb{I}}_0(v)\xi(z) = \frac{1-\mathfrak{q}q_3^{-1}z/v}{1-\mathfrak{q}q_1z/v} : \tilde{\mathbb{I}}_0(v)\xi(z) :, \tag{3.28}$$

$$\eta(z)\tilde{\mathbb{I}}_0(v) = \frac{1-q_2v/z}{1-v/z} : \eta(z)\tilde{\mathbb{I}}_0(v) :, \quad \tilde{\mathbb{I}}_0(v)\eta(z) = \frac{1-q_1z/v}{1-q_3^{-1}z/v} : \tilde{\mathbb{I}}_0(v)\eta(z) : . \tag{3.29}$$

Proof. By definitions (2.71), (2.72) and (3.23), we have

$$V^+(z)\tilde{\mathbb{I}}_0(v) = \frac{1-\mathfrak{q}^{1/2}q_2v/z}{1-\mathfrak{q}^{1/2}v/z} : V^+(z)\tilde{\mathbb{I}}_0(v) :, \quad \tilde{\mathbb{I}}_0(v)V^-(z) = \frac{1-\mathfrak{q}^{1/2}q_1z/v}{1-\mathfrak{q}^{1/2}q_3^{-1}z/v} : \tilde{\mathbb{I}}_0(v)V^-(z) : . \tag{3.30}$$

Since $\varphi^\pm(z), \eta(z)$ and $\xi(z)$ are written in terms of the fundamental vertex operator $V^\pm(z)$, we can check (3.26)–(3.29) from (3.30) immediately. □

Incidentally one can understand the relation (3.26) as follows. (The same is true of the relation (3.27).) By making use of the relation

$$\varphi^+(\mathfrak{q}^{-1/2}z)\eta(w) = G(w/z) : \varphi^+(\mathfrak{q}^{-1/2}z)\eta(w) :, \quad G(u) = \tilde{\psi}(u)\tilde{\psi}(q_1^{-1}u)^{-1}, \tag{3.31}$$

one notices that

$$\varphi^+(\mathfrak{q}^{-1/2}z) : \prod_{j=1}^{N} \eta(q_1^{j-1}v)^{-1} := \tilde{\psi}(q_1^{N-1}u)^{-1}\tilde{\psi}(q_1^{-1}u) : \varphi^+(\mathfrak{q}^{-1/2}z)\prod_{j=1}^{N} \eta(q_1^{j-1}v)^{-1} : . \tag{3.32}$$

Therefore the equation (3.26) is a formal limit $N \to \infty$ of the equation (3.32). In other words, we can say that $\tilde{\mathbb{I}}_0(v)$ is just a regularized expression of the infinite product

$$: \prod_{j=1}^{\infty} \eta(q_1^{j-1}v)^{-1} : \tag{3.33}$$

through the geometric series of q_1.

Proof of Proposition 3.7. To begin with, (3.26) and (3.27) imply that, in order to satisfy the intertwining relation (3.8) and (3.9), the Heisenberg part of the vector intertwiner must be proportional to $\tilde{\mathbb{I}}_n(v)$ and the second levels of intertwined horizontal representations must coincide; $\gamma' = \gamma$. Note that we have assumed the constant condition (2.87) for the zero mode of $K^\pm(z)$. In the next we should check whether the already fixed $\tilde{\mathbb{I}}_n(v)$ solves the intertwining relations (3.10) and (3.11) up to contributions from zero modes. Let us check (3.11) for the zero-component. In order to obtain (3.11) from (3.28) by making use of Proposition 2.4, one finds that the additional factor q_2 is necessary for $F(z)$ on the left compared to $F(z)$ on the right, namely,

$$q_2\xi(z)\mathbf{f}(z)\mathbb{I}_0(v)-\mathbb{I}_0(v)\xi(z)\mathbf{f}(z)=(1-q_2^{-1})\delta\left(\frac{q_1^{-1}v}{\mathfrak{q}z}\right)q_2\mathbf{e}(q_1^{-1}v)^{-1}\tilde{\mathbb{I}}_{-1}(v)\varphi^+(\mathfrak{q}^{1/2}z)\gamma^{-1}, \tag{3.34}$$

where we have used the relation (2.82) for the source horizontal representation $\mathcal{H}$ and

$$\delta\left(\frac{q_1^{-1}v}{\mathfrak{q}z}\right)\xi(z)=\delta\left(\frac{q_1^{-1}v}{\mathfrak{q}z}\right)\eta(q_1^{-1}v)^{-1}\varphi^+(\mathfrak{q}^{1/2}z). \tag{3.35}$$

At the same time the required additional factor for $E(z)$ is automatically determined due to the relation (2.82) for the target horizontal representation $\mathcal{H}'$. In summary we have obtained the constraint $\mathbf{e}'(z)=q_2^{-1}\mathbf{e}(z)$, $\mathbf{f}'(z)=q_2\mathbf{f}(z)$ for the intertwined horizontal representations, and the zero mode part of the vector intertwiner z_n (3.24) have been also determined now. At the last, we should confirm that the operator $\mathbb{I}_n(v)$ (3.22) we have obtained solves the relation (3.10) consistently. This can be checked from (3.29) as follows:

$$q_2^{-1}\eta(z)\mathbf{e}(z)\mathbb{I}_0(v)-\tilde{\psi}(z/v)\mathbb{I}_0(v)\eta(z)\mathbf{e}(z)=(1-q_2)\delta(v/z)q_2^{-1}\mathbf{e}(v)\tilde{\mathbb{I}}_1(v). \tag{3.36}$$

Now we have proven Proposition 3.7. □

Essentially the same vector intertwiner was given in [20], though the expression for the vector representation there might look different from ours. We can see that the *generating function* of the intertwiner in [20, (A.5)] corresponds to the *zero component* $\mathbb{I}_0(v)$ of the intertwiner in this thesis by appropriate redefinition of the parameters.

In what follows we would like to consider the composition of the vector intertwiners whose spectral parameters take relatively special values. Such a composition is not always allowed, and we should check that we can take the normal ordered product safely. Note that this argument corresponds to Proposition 2.6 at the representation level. The following condition for well-definedness results from OPE computations.

Proposition 3.9. *Let $v,w\in\mathbb{C}^\times$ be the spectral parameters of the vector representations. By making use of the coefficient $\tilde{\mathcal{A}}(w/v)$:*

$$\tilde{\mathcal{A}}(w/v)\tilde{\mathbb{I}}_0(v)\circ\tilde{\mathbb{I}}_0(w)=:\tilde{\mathbb{I}}_0(v)\circ\tilde{\mathbb{I}}_0(w):, \tag{3.37}$$

$$\tilde{\mathcal{A}}(w/v)=\exp\left(\sum_{r\geq1}\frac{1}{r}\frac{1-q_2^r}{1-q_1^r}(q_1w/v)^r\right), \tag{3.38}$$

we define the OPE factor $\mathcal{A}_{mn}(w/v)$ *as follows*:

$$\tilde{\mathcal{A}}(w/v)\tilde{\mathbb{I}}_m(v)\circ\tilde{\mathbb{I}}_n(w)=\mathcal{A}_{mn}(w/v)^{-1}:\tilde{\mathbb{I}}_m(v)\circ\tilde{\mathbb{I}}_n(w):. \tag{3.39}$$

Then we have

$$\mathcal{A}_{mn}(w/v)=\begin{cases}\prod_{i=1}^{m-n}\frac{1-q_1^{n-m+i}q_2w/v}{1-q_1^{n-m+i}w/v} & (m>n),\\ 1 & (m=n),\\ \prod_{i=1}^{n-m}\frac{1-q_1^iw/v}{1-q_1^iq_2w/v} & (m<n).\end{cases} \tag{3.40}$$

In particular if $m>n$ *and* $w\neq q_1^{i-1}q_2^{-1}v$ $(1\leq i\leq m-n)$ *or* $m<n$ *and* $w\neq q_1^{-i}v$ $(1\leq i\leq n-m)$*, then the composite operator* $\tilde{\mathcal{A}}(w/v)\tilde{\mathbb{I}}_m(v)\circ\tilde{\mathbb{I}}_n(w)$ *is well-defined.*

Proof. Let us decompose $\tilde{\mathbb{I}}_n(v)$ into the negative and the positive modes:

$$\tilde{\mathbb{I}}_n(v)=\tilde{\mathbb{I}}_n^-(v)\tilde{\mathbb{I}}_n^+(v), \tag{3.41}$$

$$\tilde{\mathbb{I}}_n^-(v)=\exp\left(-\sum_{r=1}^{\infty}\frac{H_{-r}}{[r]}\frac{(\mathfrak{q}^{-1/2}q_1^nv)^r}{1-q_1^r}\right),\quad \tilde{\mathbb{I}}_n^+(v)=\exp\left(-\sum_{r=1}^{\infty}\frac{H_r}{[r]}\frac{(\mathfrak{q}^{1/2}q_1^{n-1}v)^{-r}}{1-q_1^r}\right). \tag{3.42}$$

Then we have

$$\tilde{\mathbb{I}}_m^+(v)\tilde{\mathbb{I}}_n^-(v)=\exp\left(\sum_{r\geq1}\frac{1}{r}\frac{1-q_2^r}{1-q_1^r}(q_1^{n-m+1}w/v)^r\right):\tilde{\mathbb{I}}_m^+(v)\tilde{\mathbb{I}}_n^-(v):. \tag{3.43}$$

Therefore

$$\mathcal{A}_{mn}(w/v)=\exp\left(\sum_{r\geq1}\frac{1}{r}\frac{1-q_2^r}{1-q_1^r}(1-q_1^{r(n-m)})(q_1w/v)^r\right). \tag{3.44}$$

One can obtain (3.40) taking into account the equations

$$\frac{1-q^N}{1-q}=\sum_{i=1}^{N}q^i\quad(N>0),\qquad \frac{1-q^N}{1-q}=-\sum_{i=N}^{-1}q^i\quad(N<0). \tag{3.45}$$

□

Note that, when we construct the Fock and MacMahon intertwiners, the above well-definedness conditions are kept.

3.3 Construction of Fock intertwiner

We construct the Fock *intertwiner* in a parallel way to the construction of the Fock *representation* (Section 2.2.2) in contrast to the rather direct way of [22]. In our approach, taking the tensor product of the vertical representations is realized merely by the composition of the corresponding vector intertwiners. The essential work that we have to do is to find the *modification operator* $B_n(v) : \mathcal{H} \to \mathcal{H}''$ corresponding to the *modification factor* $\beta_n^{\pm}$ (2.37), where $\mathcal{H}$ is the same horizontal representation as the source of the Fock intertwiner while $\mathcal{H}''$ is some intermediate horizontal representation whose zero mode sector we should specify.

Proposition 3.10 ([30, Section 4.2.2])**.** *There exists a unique Fock intertwiner $\Phi(v) : \mathcal{F}(v) \otimes \mathcal{H} \to \mathcal{H}'$ up to normalization if and only if a pair of horizontal representations $(\mathcal{H}, \mathcal{H}')$ satisfies $\gamma' = \mathfrak{q}\gamma$ on the second level and $\mathbf{e}'(z) = (-\mathfrak{q}v/z)\mathbf{e}(z)$, $\mathbf{f}'(z) = (-\mathfrak{q}v/z)^{-1}\mathbf{f}(z)$ for the zero mode sector. The λ-component is written as*

$$\Phi_\lambda(v) = z_\lambda \tilde{\Phi}_\lambda(v) : \mathcal{H} \to \mathcal{H}', \tag{3.46}$$

$$\tilde{\Phi}_\lambda(v) = \mathcal{G}^{[n]} \cdot \tilde{\mathbb{I}}^{[n]}_\lambda(v) \circ B_n(v), \quad n > \ell(\lambda), \tag{3.47}$$

where $\tilde{\mathbb{I}}^{[n]}_\lambda(v) = \tilde{\mathbb{I}}_{\lambda_1}(v) \circ \cdots \circ \tilde{\mathbb{I}}_{\lambda_n}(q_2^{n-1}v)$ and the coefficient $\mathcal{G}^{[n]}$ is defined by the normal ordering[4]

$$\tilde{\mathbb{I}}^{[n]}_\varnothing(v) B_n(v) = (\mathcal{G}^{[n]})^{-1} : \tilde{\mathbb{I}}^{[n]}_\varnothing(v) B_n(v) : . \tag{3.48}$$

The modification operator $B_n(v)$ is defined by making use of the dual vertex operator (2.91) *as*

$$B_n(v) = \tilde{V}^-(\mathfrak{q}^{1/2} q_2^n v) \tilde{V}^+(\mathfrak{q}^{3/2} q_2^n v)^{-1}, \tag{3.49}$$

and $z_\lambda = z_\lambda(v)$ is a stack of zero modes:

$$z_\lambda(v) = \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} \left(-\mathfrak{q} q_2^{i-1} x_{ij}^{-1}\right) \mathbf{e}(x_{ij} v) = q_2^{n(\lambda)} (-\mathfrak{q})^{|\lambda|} \prod_{(i,j)\in\lambda} x_{ij}^{-1} \mathbf{e}(x_{ij} v). \tag{3.50}$$

Before proving Proposition 3.10, we summarize properties of the modification operator $B_n(v)$.

[4] Let $\lambda_n = 0$ for $n > \ell(\lambda)$.

Proposition 3.11 ([30, Section 4.2.2]). *When a pair of horizontal representatins* $(\mathcal{H}, \mathcal{H}'')$ *satisfies*

$$\gamma'' = \mathfrak{q}\gamma, \quad \mathbf{e}''(z) = (-\mathfrak{q}q_2^n v/z)\mathbf{e}(z), \quad \mathbf{f}''(z) = (-\mathfrak{q}q_2^n v/z)^{-1}\mathbf{f}(z), \tag{3.51}$$

the modification operator $B_n(v)\colon \mathcal{H} \to \mathcal{H}''$ *satisfies the following relations*:

$$K^+(z)B_n(v) = \beta_n^+(v/z)\ B_n(v)K^+(z), \tag{3.52}$$

$$B_n(v)K^-(\mathfrak{q}z) = \beta_n^-(z/v)^{-1}\ K^-(\mathfrak{q}z)B_n(v), \tag{3.53}$$

$$F(z)B_n(v) - B_n(v)F(z) = 0, \tag{3.54}$$

$$E(z)B_n(v) - \beta_n^-(z/v)B_n(v)E(z) = -\mathfrak{q}\delta(q_2^n v/z) : \eta(z)B_n(v) : . \tag{3.55}$$

Proof. By making use of the OPE relations (2.89) and (2.90), one can check the following relations for $B_n(v)$:

$$\varphi^+(\mathfrak{q}^{-1/2}z)B_n(v) = \mathfrak{q}\beta_n^+(v/z)\ B_n(v)\varphi^+(\mathfrak{q}^{-1/2}z), \tag{3.56}$$

$$B_n(v)\varphi^-(\mathfrak{q}^{1/2}z) = \mathfrak{q}\beta_n^-(z/v)^{-1}\ \varphi^-(\mathfrak{q}^{1/2}z)B_n(v), \tag{3.57}$$

$$\xi(z)B_n(v) = (1 - \mathfrak{q}q_2^n v/z) : \xi(z)B_n(v) :, \quad B_n(v)\xi(z) = (1 - \mathfrak{q}^{-1}q_2^{-n}z/v) : \xi(z)B_n(v) :, \tag{3.58}$$

$$\eta(z)B_n(v) = (1 - q_2^n v/z)^{-1} : \eta(z)B_n(v) :, \quad B_n(v)\eta(z) = (1 - q_2^{-n}q_3^{-1}z/v)^{-1} : \eta(z)B_n(v) : . \tag{3.59}$$

Taking into account the relative level shift $\gamma'' = \mathfrak{q}\gamma$, one obtains (3.52) and (3.53) from (3.56) and (3.57) respectively. (3.58) and (3.59) lead to the following relations

$$\left(\frac{\mathfrak{q}}{z}(-q_2^n v)\right)^{-1} \xi(z)B_n(v) - B_n(v)\xi(z) = 0, \tag{3.60}$$

$$\frac{\mathfrak{q}}{z}(-q_2^n v)\eta(z)B_n(v) - \beta_n^-(z/v)B_n(v)\eta(z) = -\mathfrak{q}\delta(q_2^n v/z) : \eta(z)B_n(v) :, \tag{3.61}$$

where the contribution of the delta function in (3.61) is due to Proposition 2.4. Then taking into account the relative shift of the zero modes $\mathbf{e}''(z) = (-\mathfrak{q}q_2^n v/z)\mathbf{e}(z)$, $\mathbf{f}''(z) = (-\mathfrak{q}q_2^n v/z)^{-1}\mathbf{f}(z)$, one obtains (3.54) and (3.55). $\square$

It is convenient to identify the modification operator $B_n(z)$ with a single edge (with a coefficient) so that we can see Proposition 3.11 graphically as in Figure 7. Incidentally, along the same argument as lifting the function $\tilde{\psi}(q_1^{n-1})$ to the operator $\tilde{\mathbb{I}}_n(v)$ before, one can understand that $B_n(v)$ is just a regularized expression of the formal infinite product

$$: \prod_{i,j=1}^{\infty} \eta(q_1^{j-1}q_2^{n+i-1}v)^{-1} : . \tag{3.62}$$

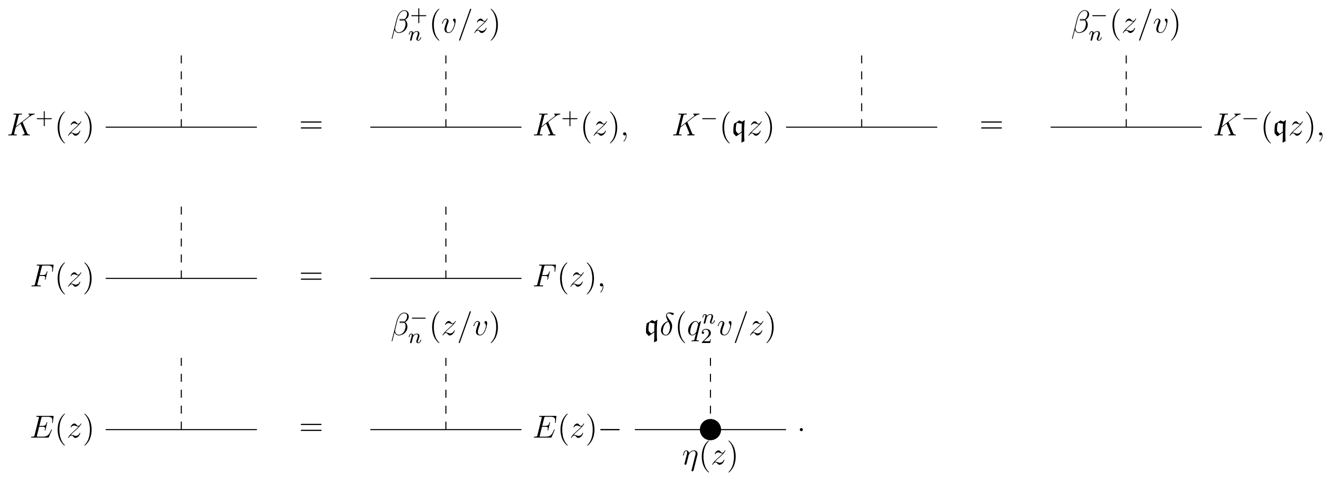


Figure 7: The solid line denotes $B_n(z)$, the dashed line denotes the multiplication of residing function (a blank denotes 1) and the symbol $\bullet$ denotes the normal ordered product operator such as $:\eta(z)B_n(z):$.

(3.56) and (3.57) mean that the operator $B_n(v)$ exactly corresponds to the modification factor $\beta_n^\pm$ up to the factor $\mathfrak{q}$, and this discrepancy leads to the relative level shift.

Note that we need the coefficient $\mathcal{G}^{[n]}$ in order for $\tilde{\Phi}_\lambda(v)$ to exist independently of sufficiently large $n > \ell(\lambda)$, this requirement is peculiar to the operator formalism and is absent at the level of construction of the Fock representation; functions do commute but operators do not.

Proposition 3.12 ([30, Section 4.2.2])**.** *We define* $\mathcal{G}^{[n]}$, $\tilde{\mathbb{I}}_\lambda^{[n]}(v)$ *and* $B_n(v)$ *as Proposition* 3.10*, then the operator* $\tilde{\Phi}_\lambda(v) := \mathcal{G}^{[n]} \cdot \tilde{\mathbb{I}}_\lambda^{[n]}(v) \circ B_n(v)$ *is independent of* $n > \ell(\lambda)$.

Proof. As for $\lambda = \varnothing$, by definition of $\mathcal{G}^{[n]}$, we have

$$\begin{aligned}\tilde{\Phi}_\varnothing(v) = B_0(v) := \exp&\left(-\sum_{r=1}^{\infty} \frac{H_{-r}}{[r]}(\mathfrak{q}^{-1/2}v)^r \frac{1}{(1-q_1^r)(1-q_2^r)}\right)\\ &\times \exp\left(\sum_{r=1}^{\infty} \frac{H_r}{[r]}(\mathfrak{q}^{1/2}v)^{-r} \frac{q_3^{-r}}{(1-q_1^r)(1-q_2^r)}\right). \end{aligned} \tag{3.63}$$

As for general λ, if we introduce the factor $\mathcal{G}_\lambda$ by

$$\tilde{\mathbb{I}}_\lambda^{[n]}(v)B_n(v) = (\mathcal{G}_\lambda)^{-1}(\mathcal{G}^{[n]})^{-1} : \tilde{\mathbb{I}}_\lambda^{[n]}(v)B_n(v) :, \tag{3.64}$$

then we can show that $\mathcal{G}_\lambda$ is independent of $n > \ell(\lambda)$ in fact (see Section 5.2.1). Hence we have

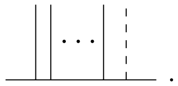
$$\mathcal{G}_\lambda \cdot \tilde{\Phi}_\lambda(v) =: \tilde{\mathbb{I}}^{[n]}_\lambda(v) B_n(v) :=: B_0(v) \prod_{(i,j\in\lambda)} \eta(x_{ij}v) :$$

$$= \exp\left(\sum_{r=1}^{\infty} \frac{H_{-r}}{[r]} (\mathfrak{q}^{-1/2}v)^r \left(\sum_{(i,j)\in\lambda} x_{ij}^r - \frac{1}{(1-q_1^r)(1-q_2^r)} \right)\right)$$
$$\times \exp\left(-\sum_{r=1}^{\infty} \frac{H_{r}}{[r]} (\mathfrak{q}^{1/2}v)^{-r} \left(\sum_{(i,j)\in\lambda} x_{ij}^{-r} - \frac{q_3^{-r}}{(1-q_1^r)(1-q_2^r)} \right)\right), \tag{3.65}$$

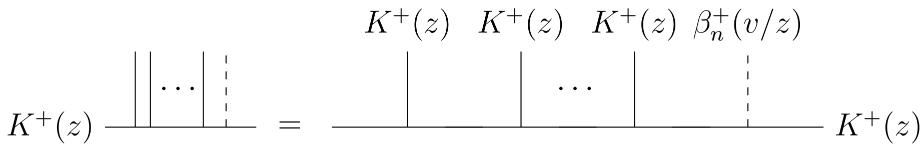
where $x_{ij} = q_1^{j-1} q_2^{i-1}$. □

The following diagrammatic expression for the composite operator assists us in confirming the intertwining relation:

$$\Phi(v) = \mathcal{G}^{[n]} \cdot \mathbb{I}(v) \circ \cdots \circ \mathbb{I}(q_2^{n-1}v) \circ B_n(v) = \tag{3.66}$$



Note that we can obtain the λ-component $\Phi_\lambda(v)$ by evaluating (3.66) on the state $|\lambda) = [v]_{\lambda_1 - 1} \otimes \cdots \otimes [q_2^{\ell-1}v]_{\lambda_\ell - 1} \in \mathcal{F}(v)$. For example, *if* $\gamma'' = \mathfrak{q}\gamma$, the Fock intertwining relation with $K^+(z)$ can be seen easily by applying (3.52) and the intertwining relation (3.1) for $\mathbb{I}(q_2^{i-1}v)$ repeatedly:



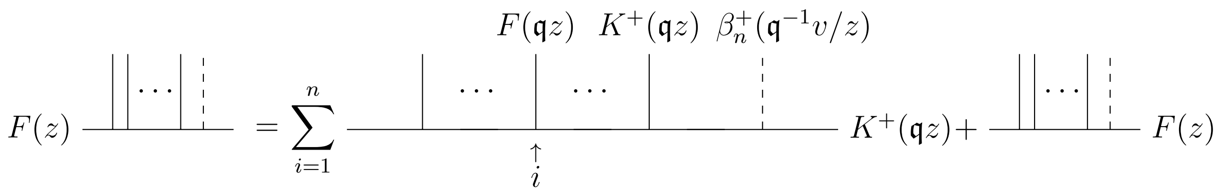
(3.67)

Evaluating this on $|\lambda)$ yields just (3.13). The relation with $K^-(z)$ can be also checked in the same way.

Proof of Proposition 3.10. To begin with, (3.26), (3.27), (3.52) and (3.53) imply that, in order to satisfy the intertwining relation (3.13) and (3.14), the Heisenberg part of the Fock intertwiner must be proportional to $\tilde{\Phi}_\lambda(v)$ (3.47), and the relative second level shift of intertwined horizontal representations must satisfy $\gamma' = \mathfrak{q}\gamma$. Now let us introduce the intermediate horizontal representation $\mathcal{H}''$ and let $B_n(v) : \mathcal{H} \to \mathcal{H}''$, $\mathbb{I}^{[n]}_\lambda(v) : \mathcal{H}'' \to \mathcal{H}'$, where $\mathbb{I}^{[n]}_\lambda(v) = \mathbb{I}_{\lambda_1}(v) \circ \cdots \circ \mathbb{I}_{\lambda_n}(q_2^{n-1}v)$ and we fixed $n > \ell(\lambda)$. Then the composition $\mathbb{I}^{[n]}_\lambda(v) \circ B_n(v)$ makes sense. Note that the constraint for a pair of horizontal representations $(\mathcal{H}'', \mathcal{H}')$ is already fixed in order for $\mathbb{I}^{[n]}_\lambda(v)$ to make sense, namely $\mathbf{e}'(z) = q_2^{-n}\mathbf{e}''(z)$

and $\mathbf{f}'(z) = q_2^n \mathbf{f}''(z)$. Therefore it suffices to check that the constraint for $(\mathcal{H}, \mathcal{H}'')$ is (3.51) and that (3.47) solves the remaining Fock intertwining relation (3.15) and (3.16) up to normalization between different λ-components; this discrepancy fixes the zero mode contribution z_λ (3.50).

Since the relation (3.52) has been already fixed by the condition $\gamma' = \gamma'' = \mathfrak{q}\gamma$, $F(z)$ must commute with $B_n(v)$, namely we need the relation (3.54). Therefore we can see that the constraint $\mathbf{f}''(z) = (-\mathfrak{q} q_2^n v/z)^{-1}\mathbf{f}(z)$ is necessary from (3.60). Indeed if (3.54) is satisfied, we have



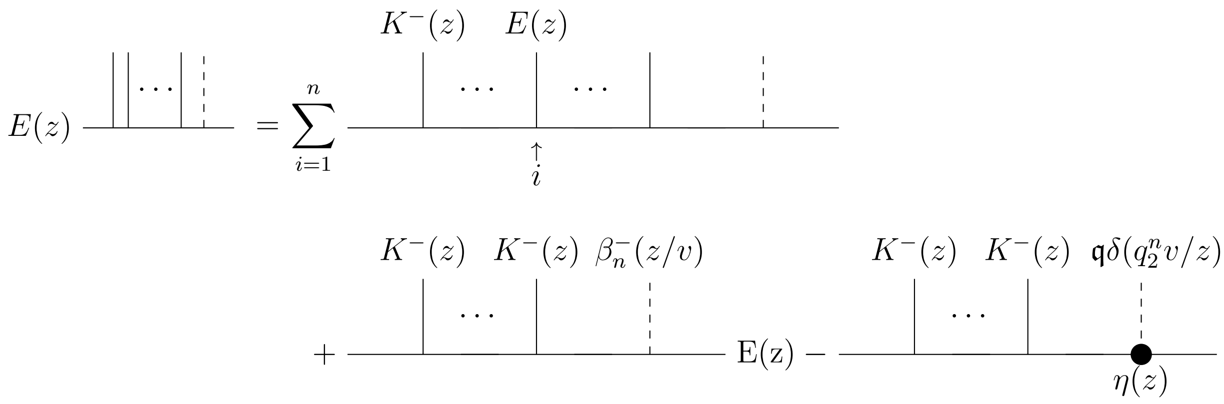
(3.68)

and evaluating this on $|\lambda)$ yields

$$
\begin{aligned}
&F(z)\Phi_\lambda(v) - \Phi_\lambda(v)F(z) \\
&= z_\lambda \mathcal{G}^{[n]} \cdot \sum_{k=1}^{n} \prod_{s=k+1}^{n} \tilde{\psi}(q_1^{\lambda_s-1} q_2^{s-1} v/z)\ \beta_n^+(v/z) \mathbb{I}_{\lambda_1}(v) \cdots \mathbb{I}_{\lambda_{k-1}}(q_2^{k-2}v) \\
&\qquad \times (1-q_2^{-1})\delta(\mathfrak{q}^{-1}x_k v/z)(-\mathfrak{q}q_2^{k-1}v/z)^{-1}\mathbb{I}_{\lambda_k-1}(q_2^{k-1}v)\ \mathbb{I}_{\lambda_{k+1}}(q_2^{k}v) \cdots \mathbb{I}_{\lambda_n}(q_2^{n-1}v) B_n(v) K^+(\mathfrak{q}z) \\
&= \sum_{k=1}^{n} (\lambda_k - 1|F(\mathfrak{q}z)|\lambda)\Phi_{\lambda-1_k}(v)K^+(\mathfrak{q}z). \qquad (3.69)
\end{aligned}
$$

This relation is nothing but (3.16). At the same time, the zero mode contribution from each box in λ, namely $z_{\lambda+1_k}/z_\lambda$, can be read. At the last let us check that (3.15) is satisfied. Note that the constraint (3.51) is already imposed due to (2.86), hence we have

(3.55). Therefore we have



(3.70)

and evaluating this on $|\lambda)$ yields

$$
\begin{aligned}
&E(z)\Phi_\lambda(v) - (\lambda|K^-(z)|\lambda)\ \Phi_\lambda(v)E(z)\\
&= z_\lambda \mathcal{G}^{[n]} \cdot \sum_{k=1}^{n}\prod_{s=1}^{k-1}\tilde{\psi}(q_1^{-\lambda_s}q_2^{1-s}z/v)\mathbb{I}_{\lambda_1}(v)\cdots\mathbb{I}_{\lambda_{k-1}}(q_2^{k-2}v)\\
&\qquad\times(1-q_2)\delta(q_1x_kv/z)(-\mathfrak{q}q_2^{k-1}v/z)\mathbb{I}_{\lambda_k+1}(q_2^{k-1}v)\mathbb{I}_{\lambda_{k+1}}(q_2^{k}v)\cdots\mathbb{I}_{\lambda_n}(q_2^{n-1}v)B_n(v)\\
&\quad - z_\lambda\mathcal{G}^{[n]}\cdot\prod_{s=1}^{n}\tilde{\psi}(q_1^{-\lambda_s}q_2^{1-s}z/v)\mathfrak{q}\delta(q_2^nv/z)\mathbb{I}_\lambda^{[n]}(v) :\eta(z)B_n(v):\\
&= \sum_{k=1}^{n}(\lambda_k+1|E(z)|\lambda)\Phi_{\lambda+1_k}(v).
\end{aligned}
\tag{3.71}
$$

This relation is nothing but (3.15). Note that, in the above equation, the coefficient of $:\eta(z)B_n(v):$ vanishes due to the factor $\tilde{\psi}(q_2^{-n+1}z/v)\delta(q_2^nv/z)$. □

One can confirm that the operator $\Phi_\lambda(v)$ agrees with the one of [22], and the factor $\mathcal{G}_\lambda$ plays the same role as c_λ in [22]. An explicit form of $\mathcal{G}_\lambda$ is

$$
\mathcal{G}_\lambda = \prod_{\square\in\lambda}\left(1-q_1^{-a_\lambda(\square)}q_2^{l_\lambda(\square)+1}\right) = q_1^{-n(\lambda')}q_2^{n(\lambda)+|\lambda|}c_\lambda,
\tag{3.72}
$$

$$
c_\lambda = \prod_{\square\in\lambda}(1-q_1^{a_\lambda(\square)}q_2^{-l_\lambda(\square)-1}).
\tag{3.73}
$$

If we choose the horizontal representation as $\mathcal{H}=\mathcal{F}_u^{(\mathfrak{q},\mathfrak{q}^N)}$, then $\mathcal{H}'=\mathcal{F}_{-vu}^{(\mathfrak{q},\mathfrak{q}^{N+1})}$ and the zero mode stack $z_\lambda = z_\lambda(N;u|v)$ is

$$
z_\lambda(N;u|v) = \prod_{i=1}^{\ell(\lambda)} z_{\lambda_i}(N+1;-q_2^ivu|q_2^{i-1}v) = \prod_{i=1}^{\ell(\lambda)}(-q_2^{i-1}vu)^{\lambda_i}\prod_{j=1}^{\lambda_i}\left(\frac{\mathfrak{q}}{q_1^{j-1}q_2^{i-1}v}\right)^{N+1}
$$

$$= q_2^{n(\lambda)}(-vu)^{|\lambda|} \prod_{(i,j)\in\lambda} \left(\mathfrak{q} x_{ij}^{-1} v^{-1}\right)^{N+1}, \tag{3.74}$$

where $n(\lambda) = \sum_{j=1}^{\ell(\lambda)} (j-1)\lambda_j$.

3.4 Construction of MacMahon intertwiner

We construct the MacMahon intertwiner in the same way as the construction of the Fock intertwiner in previous section. Though the regularization procedure for the construction of the MacMahon representation was different from that of the Fock representation, the strategy for the construction of the MacMahon intertwiner is the same as that of the Fock intertwiner; we have already known the modification factor to consider, namely $\gamma_n^{\pm}$ (2.69). Hence the essential work that we have to do is to find the modification operator $\Gamma_n(K;v) : \mathcal{H} \to \mathcal{H}''$ corresponding to it in this case too. We introduce the theta function

$$\theta_p(z) := (z;p)_\infty (p/z;p)_\infty = (1-z)\prod_{k=1}^{\infty}(1-p^k z)(1-p^k/z), \quad p \in \mathbb{C},\ |p|<1, \tag{3.75}$$

$$(z;q)_\infty := \prod_{k=0}^{\infty}(1-q^k z). \tag{3.76}$$

Hereafter we assume $|q_3| < 1$.

Proposition 3.13 ([30, Section 4.3.2])**.** *There exists a unique MacMahon intertwiner $\Xi(K;v) : \mathcal{M}(K;v) \otimes \mathcal{H} \to \mathcal{H}'$ up to normalization if and only if a pair of horizontal representations $(\mathcal{H}, \mathcal{H}')$ satisfies $\gamma' = K^{1/2}\gamma$ on the second level and*

$$\mathbf{e}'(z) = K^{1/2}\frac{\theta_{q_3}(v/z)}{\theta_{q_3}(Kv/z)}\mathbf{e}(z), \qquad \mathbf{f}'(z) = \frac{\theta_{q_3}(\mathfrak{q}Kv/z)}{\theta_{q_3}(\mathfrak{q}v/z)}\mathbf{f}(z) \tag{3.77}$$

for the zero mode sector. The Λ-component is written as

$$\Xi_\Lambda(K;v) = z_\Lambda \tilde{\Xi}_\Lambda(K;v) : \mathcal{H} \to \mathcal{H}', \tag{3.78}$$

$$\tilde{\Xi}_\Lambda(K;v) = \mathcal{M}^{[n]}(K)\cdot \tilde{\Phi}_\Lambda^{[n]}(v) \circ \Gamma_n(K;v), \quad n > h(\Lambda), \tag{3.79}$$

where $\tilde{\Phi}_\Lambda^{[n]}(v) = \tilde{\Phi}_{\Lambda^{(1)}}(v) \circ \cdots \circ \tilde{\Phi}_{\Lambda^{(n)}}(q_3^{n-1}v)$ and the coefficient $\mathcal{M}^{[n]}(K)$ is defined by the normal oerdering[5]

$$\tilde{\Phi}_\varnothing^{[n]}(v)\Gamma_n(K;v) = \left(\mathcal{M}^{[n]}(K)\right)^{-1} : \tilde{\Phi}_\varnothing^{[n]}(v)\Gamma_n(K;v) : . \tag{3.80}$$

[5] Let $\Lambda^{(n)} = \varnothing$ for $n > h(\Lambda)$.

The modification operator $\Gamma_n(K;v)$ is defined as

$$\Gamma_n(K;v) = \exp\left(\sum_{r=1}^{\infty} \frac{H_{-r}}{[r]} \frac{q_3^{nr} - K^r}{k_r} (\mathfrak{q}^{-1/2}v)^r\right) \exp\left(\sum_{r=1}^{\infty} \frac{H_r}{[r]} \frac{q_3^{-nr} - K^{-r}}{k_r} (\mathfrak{q}^{1/2}v)^{-r}\right), \tag{3.81}$$

and $z_\Lambda = z_\Lambda(K;v)$ is a stack of zero modes:

$$z_\Lambda(K;v) = \prod_{(i,j,k)\in\Lambda} \frac{K^{1/2}}{\mathfrak{q}^{k-1}} \frac{\theta_{q_3}(q_3^{k-1}/x_{ijk})}{\theta_{q_3}(K/x_{ijk})} \mathbf{e}(x_{ijk}v). \tag{3.82}$$

Before proving Proposition 3.13, we summarize properties of the modification operator $\Gamma_n(K;v)$. Along the same argument as before, one can guess that $\Gamma_n(K;v)$ will be the limit $N \to \infty$ of

$$:\prod_{i,j=1}^{\infty} \prod_{k=n+1}^{N} \eta(q_1^{j-1}q_2^{i-1}q_3^{k-1}v)^{-1} :=: \prod_{i,j,k=1}^{\infty} \eta(q_1^{j-1}q_2^{i-1}q_3^{n+k-1}v)^{-1}\eta(q_1^{j-1}q_2^{i-1}q_3^{k-1}q_3^{N}v) : . \tag{3.83}$$

Here recall that we formally substitute $q_3^N \to K$ in the regularization procedure for the construction of the MacMahon representation. Then one can understand that $\Gamma_n(K;v)$ is just a regularized expression of the formal infinite product

$$:\prod_{i,j,k=1}^{\infty} \eta(q_1^{j-1}q_2^{i-1}q_3^{n+k-1}v)^{-1}\eta(q_1^{j-1}q_2^{i-1}q_3^{k-1}Kv) : . \tag{3.84}$$

Proposition 3.14 ([30, Section 4.3.2])**.** *When a pair of horizontal representatins $(\mathcal{H}, \mathcal{H}'')$ satisfies*

$$\gamma'' = \mathfrak{q}^{-n}K^{1/2}\gamma, \qquad \mathbf{e}''(z) = \frac{K^{1/2}}{\mathfrak{q}^n} \frac{\theta_{q_3}(q_3^n v/z)}{\theta_{q_3}(Kv/z)} \mathbf{e}(z), \qquad \mathbf{f}''(z) = \frac{\theta_{q_3}(\mathfrak{q}Kv/z)}{\theta_{q_3}(\mathfrak{q}q_3^n v/z)} \mathbf{f}(z), \tag{3.85}$$

the modification operator $\Gamma_n(K;v) : \mathcal{H} \to \mathcal{H}''$ satisfies the following relations:

$$K^+(z)\Gamma_n(K;v) = \gamma_n^+(v/z)\ \Gamma_n(K;v)K^+(z), \tag{3.86}$$

$$\Gamma_n(K;v)K^-(\mathfrak{q}z) = \gamma_n^-(z/v)^{-1}\ K^-(\mathfrak{q}z)\Gamma_n(K;v), \tag{3.87}$$

$$F(z)\Gamma_n(K;v) - \Gamma_n(K;v)F(z) = 0, \tag{3.88}$$

$$E(z)\Gamma_n(K;v) - \gamma_n^-(z/v)\ \Gamma_n(K;v)E(z) = 0. \tag{3.89}$$

Proof. By making use of the OPE relations

$$V^+(z)\Gamma_n(K;v) = \exp\left(\sum_{r=1}^{\infty}\frac{1}{r}\frac{q_3^{nr}-K^r}{1-q_3^r}(\mathfrak{q}^{1/2}v/z)^r\right) : V^+(z)\Gamma_n(K;v) :, \tag{3.90}$$

$$\Gamma_n(K;v)V^-(z) = \exp\left(-\sum_{r=1}^{\infty}\frac{1}{r}\frac{q_3^{-nr}-K^{-r}}{1-q_3^r}(\mathfrak{q}^{3/2}v/z)^{-r}\right) : \Gamma_n(K;v)V^-(z) :, \tag{3.91}$$

one can check the following relations for $\Gamma_n(K;v)$:

$$\varphi^+(\mathfrak{q}^{-1/2}z)\Gamma_n(K;v) = \frac{K^{1/2}}{\mathfrak{q}^n}\gamma_n^+(v/z)\ \Gamma_n(K;v)\varphi^+(\mathfrak{q}^{-1/2}z), \tag{3.92}$$

$$\Gamma_n(K;v)\varphi^-(\mathfrak{q}^{1/2}z) = \frac{K^{1/2}}{\mathfrak{q}^n}\gamma_n^-(z/v)^{-1}\ \varphi^-(\mathfrak{q}^{1/2}z)\Gamma_n(K;v), \tag{3.93}$$

$$\xi(z)\Gamma_n(K;v) = \frac{(\mathfrak{q}q_3^n v/z;q_3)_\infty}{(K\mathfrak{q}v/z;q_3)_\infty} : \xi(z)\Gamma_n(K;v) :, \quad \Gamma_n(K;v)\xi(z) = \frac{(K^{-1}\mathfrak{q}z/v;q_3)_\infty}{(\mathfrak{q}q_3^{-n}z/v;q_3)_\infty} : \xi(z)\Gamma_n(K;v) :, \tag{3.94}$$

$$\eta(z)\Gamma_n(K;v) = \frac{(Kv/z;q_3)_\infty}{(q_3^n v/z;q_3)_\infty} : \eta(z)\Gamma_n(K;v) :, \quad \Gamma_n(K;v)\eta(z) = \frac{(q_3^{-n}z/v;q_3)_\infty}{(K^{-1}z/v;q_3)_\infty} : \eta(z)\Gamma_n(K;v) : . \tag{3.95}$$

Taking into account the relative level shift, one obtains (3.86) and (3.87) from (3.92) and (3.93) respectively. (3.94) and (3.95) lead to the following relations

$$\frac{\theta_{q_3}(\mathfrak{q}Kv/z)}{\theta_{q_3}(\mathfrak{q}q_3^n v/z)}\ \xi(z)\Gamma_n(K;v) - \Gamma_n(K;v)\xi(z) = 0, \tag{3.96}$$

$$\frac{K^{1/2}}{\mathfrak{q}^n}\frac{\theta_{q_3}(q_3^n v/z)}{\theta_{q_3}(Kv/z)}\ \eta(z)\Gamma_n(K;v) - \gamma_n^-(z/v)\ \Gamma_n(K;v)\eta(z) = 0. \tag{3.97}$$

Then taking into account the relative shift of the zero modes, one obtains (3.88) and (3.89). □

Note that (3.89) vanishes by itself, while the corresponding equation (3.55) does only after multiplying with the factor $\tilde{\psi}(q_2^{-n+1}z/v)$. (3.92) and (3.93) mean that the operator $\Gamma_n(K;v)$ exactly corresponds to the modification factor $\gamma_n^\pm$ up to the monomial factor.

Note that we need the coefficient $\mathcal{M}^{[n]}(K)$ in order for $\tilde{\Xi}_\Lambda(K;v)$ to exist independently of sufficiently large $n > h(\Lambda)$.

Proposition 3.15 ([30, Section 4.3.2]). *We define $\mathcal{M}^{[n]}(K)$, $\tilde{\Phi}^{[n]}_\Lambda(v)$ and $\Gamma_n(K;v)$ as Proposition* 3.13*, then the operator $\tilde{\Xi}_\Lambda(K;v) := \mathcal{M}^{[n]}(K)\cdot\tilde{\Phi}^{[n]}_\Lambda(v)\circ\Gamma_n(K;v)$ is independent of $n > h(\Lambda)$.*

Proof. As for $\Lambda = \varnothing$, by definition of $\mathcal{M}^{[n]}(K)$, we have

$$\begin{aligned}\tilde{\Xi}_\varnothing(K;v) = \Gamma_0(K;v) := \exp&\left(\sum_{r=1}^\infty \frac{H_{-r}}{[r]}\frac{1-K^r}{k_r}(\mathfrak{q}^{-1/2}v)^r\right)\\ &\times\exp\left(\sum_{r=1}^\infty \frac{H_r}{[r]}\frac{1-K^{-r}}{k_r}(\mathfrak{q}^{1/2}v)^{-r}\right).\end{aligned}\tag{3.98}$$

As for general Λ, if we introduce the factor $\mathcal{C}_\Lambda$ by

$$\tilde{\Phi}^{[n]}_\Lambda(v)\Gamma_n(K;v) = (\mathcal{C}_\lambda)^{-1}(\mathcal{M}^{[n]}(K))^{-1} :\tilde{\Phi}^{[n]}_\Lambda(v)\Gamma_n(K;v):,\tag{3.99}$$

then we can show that $\mathcal{C}_\lambda$ is independent of $n > h(\Lambda)$ in fact (see Section 5.2.2). Hence we have

$$\begin{aligned}\mathcal{C}_\Lambda\cdot\tilde{\Xi}_\Lambda(K;v) &=: \tilde{\Phi}^{[n]}_\Lambda(v)\Gamma_n(K;v) :=: \Gamma_0(K;v)\prod_{(i,j,k\in\Lambda)}\eta(x_{ijk}v):\\ &= \exp\left(\sum_{r=1}^\infty \frac{H_{-r}}{[r]}(\mathfrak{q}^{-1/2}v)^r\left(\sum_{(i,j,k)\in\Lambda} x_{ijk}^r + \frac{1-K^r}{k_r}\right)\right)\\ &\quad\times\exp\left(-\sum_{r=1}^\infty \frac{H_r}{[r]}(\mathfrak{q}^{1/2}v)^{-r}\left(\sum_{(i,j,k)\in\Lambda} x_{ijk}^{-r} - \frac{1-K^{-r}}{k_r}\right)\right).\end{aligned}\tag{3.100}$$

where $x_{ijk} = q_1^{j-1}q_2^{i-1}q_3^{k-1}$. □

Proof of Proposition 3.13. We can prove it in the same way as Proposition 3.10, in particular the MacMahon intertwining relation almost automatically follows from the Fock intertwining relation and Proposition 3.14. Here let us check the constraint for the zero mode sector:

$$\begin{aligned}\mathbf{f}'(z) &= q_3^{\frac{n}{2}(n-1)}(-\mathfrak{q}v/z)^{-n}\mathbf{f}''(z)\\ &= q_3^{\frac{n}{2}(n-1)}(-\mathfrak{q}v/z)^{-n}\frac{\theta_{q_3}(\mathfrak{q}Kv/z)}{\theta_{q_3}(\mathfrak{q}q_3^n v/z)}\mathbf{f}(z) = \frac{\theta_{q_3}(\mathfrak{q}Kv/z)}{\theta_{q_3}(\mathfrak{q}v/z)}\mathbf{f}(z),\end{aligned}\tag{3.101}$$

$$\begin{aligned}\mathbf{e}'(z) &= q_3^{-\frac{n}{2}(n-1)}(-\mathfrak{q}v/z)^{n}\mathbf{e}''(z)\\ &= q_3^{-\frac{n}{2}(n-1)}(-\mathfrak{q}v/z)^{n}\frac{K^{1/2}}{\mathfrak{q}^n}\frac{\theta_{q_3}(q_3^n v/z)}{\theta_{q_3}(Kv/z)}\mathbf{e}(z) = K^{1/2}\frac{\theta_{q_3}(v/z)}{\theta_{q_3}(Kv/z)}\mathbf{e}(z),\end{aligned}\tag{3.102}$$

where we have used the formula for the theta function

$$\theta_p(zp^n) = (-z)^{-n} p^{-\frac{n}{2}(n-1)} \theta_p(z), \qquad n \in \mathbb{Z}. \tag{3.103}$$

□

4 Construction of dual intertwining operator

4.1 Definition of dual intertwiner

We define the dual intertwiner by exchanging the source and the target representation spaces of the original intertwiner as follows.

Definition 4.1. *Let $\mathcal{V}$ be a vertical representation and $(\mathcal{H}, \mathcal{H}')$ be a pair of horizontal representations. The dual intertwiner is a linear operator $\Psi^* : \mathcal{H}' \to \mathcal{H} \otimes \mathcal{V}$ which satisfies the following dual intertwining condition*:

$$\Delta(a)\Psi^* = \Psi^* a, \qquad \forall a \in \mathcal{U}. \tag{4.1}$$

Note that the ordering of the tensor product of $\mathcal{V}$ and $\mathcal{H}$ is also exchanged compared with Definition 3.1. Graphically the dual intertwiner is represented by reversing the directions of arrows in Figure 6, and the dual intertwining condition (4.1) is represented as in Figure 8.

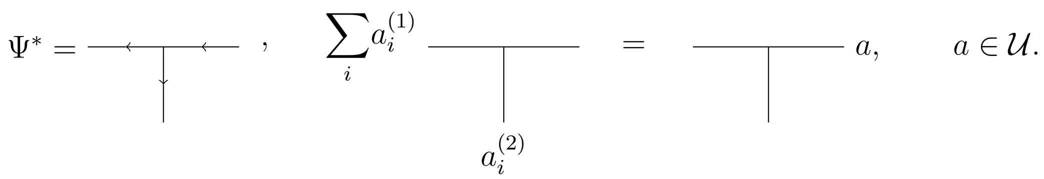


Figure 8: We use the notation $\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$.

Definition 4.2. *Let $\{|\alpha)\}_\alpha$ be a basis of the vertical representation $\mathcal{V}$, which simultaneously diagonalizes the action of $K^\pm(z)$. The α-component of the dual intertwiner $\Psi^* : \mathcal{H}' \to \mathcal{H} \otimes \mathcal{V}$ is a linear operator $\Psi^*_\alpha : \mathcal{H}' \to \mathcal{H}$ defined by*

$$\Psi^*(\bullet) = \sum_\alpha \Psi^*_\alpha(\bullet) \otimes |\alpha), \quad \bullet \in \mathcal{H}'. \tag{4.2}$$

We can use the following proposition as a definition of the component of the dual intertwiner.

Proposition 4.3 ([31]). *Let Ψ^*_α be the α-component of the dual intertwiner $\Psi^* : \mathcal{H}' \to \mathcal{H} \otimes \mathcal{V}$. Then Ψ^*_α is characterized by the following dual intertwining relations*:

$$\Psi^*_\alpha K^+(\mathfrak{q}z) = (\alpha|K^+(z)|\alpha)\ K^+(\mathfrak{q}z)\Psi^*_\alpha, \tag{4.3}$$
$$\Psi^*_\alpha K^-(z) = (\alpha|K^-(z)|\alpha)\ K^-(z)\Psi^*_\alpha, \tag{4.4}$$
$$\Psi^*_\alpha E(z) = K^-(\mathfrak{q}z)\sum_\beta(\alpha|E(\mathfrak{q}z)|\beta)\ \Psi^*_\beta +\ E(z)\Psi^*_\alpha, \tag{4.5}$$
$$\Psi^*_\alpha F(z) = \sum_\beta(\alpha|F(z)|\beta)\ \Psi^*_\beta + (\alpha|K^+(z)|\alpha)\ F(z)\Psi^*_\alpha, \tag{4.6}$$

where operators on the right side of Ψ^*_α act on the source horizontal representation $\mathcal{H}'$, while operators on the left side of Ψ^*_α act on the target one $\mathcal{H}$. The matrix elements $(\beta|X|\alpha)$ are computed in the vertical representation $\mathcal{V}$.

Proof. We can obtain (4.3)–(4.6) by making (4.1) act on $\bullet \in \mathcal{H}$ and using the notation $1 = \sum_\alpha |\alpha)(\alpha|$. □

By evaluating the matrix elements in each of vertical representations, we can summarize definitions of the components of the dual intertwiners as follows.

Proposition 4.4 ([31]). *Let $\{[v]_{n-1}\}_{n\in\mathbb{Z}}$ be the basis of the vertical vector representation $V(v)$. The n-component of the dual vector intertwiner defined by*

$$\mathbb{I}^*(v)(\bullet) = \sum_{n\in\mathbb{Z}}(\mathbb{I}^*_n(v)(\bullet)) \otimes [v]_{n-1}, \quad \bullet \in \mathcal{H}', \tag{4.7}$$

is characterized by the following dual intertwining relations:

$$\mathbb{I}^*_n(v)K^+(\mathfrak{q}z) = \tilde{\psi}(q_1^{n-1}v/z)\ K^+(\mathfrak{q}z)\mathbb{I}^*_n(v), \tag{4.8}$$
$$\mathbb{I}^*_n(v)K^-(z) = \tilde{\psi}(q_1^{-n}z/v)\ K^-(z)\mathbb{I}^*_n(v), \tag{4.9}$$
$$\mathbb{I}^*_n(v)E(z) = (1-q_2)\delta(\mathfrak{q}^{-1}q_1^{n-1}v/z)K^-(\mathfrak{q}z)\ \mathbb{I}^*_{n-1}(v) +\ E(z)\mathbb{I}^*_n(v), \tag{4.10}$$
$$\mathbb{I}^*_n(v)F(z) = (1-q_2^{-1})\delta(q_1^n v/z)\ \mathbb{I}^*_{n+1}(v) + \tilde{\psi}(q_1^{n-1}v/z)\ F(z)\mathbb{I}^*_n(v). \tag{4.11}$$

Proof. It follows from Propositions 2.5 and 4.3. □

Proposition 4.5 ([22, Lemma 3.5], [31]). *Let* $\{|\lambda)) := c_\lambda/c'_\lambda|\lambda)\}_{\lambda:partition}$ *be the dual basis of the vertical Fock representation* $\mathcal{F}(v)$. *(See* (5.43) *and* (5.44) *for the definition of* c'_λ *and* c_λ.*) The* λ*-component of the dual Fock intertwiner defined by*

$$\Phi^*(v)(\bullet) = \sum_\lambda (\Phi^*_\lambda(v)(\bullet)) \otimes |\lambda)), \quad \bullet \in \mathcal{H}', \tag{4.12}$$

is characterized by the following dual intertwining relations:

$$\Phi^*_\lambda(v)K^+(\mathfrak{q}z) = (\lambda|K^+(z)|\lambda)\ K^+(\mathfrak{q}z)\Phi^*_\lambda(v), \tag{4.13}$$

$$\Phi^*_\lambda(v)K^-(z) = (\lambda|K^-(z)|\lambda)\ K^-(z)\Phi^*_\lambda(v), \tag{4.14}$$

$$\Phi^*_\lambda(v)E(z) = K^-(\mathfrak{q}z)\sum_{k=1}^{\ell(\lambda)}((\lambda|E(\mathfrak{q}z)|\lambda-1_k))\ \Phi^*_{\lambda-1_k}(v) + \ E(z)\Phi^*_\lambda(v), \tag{4.15}$$

$$\Phi^*_\lambda(v)F(z) = \sum_{k=1}^{\ell(\lambda)+1}((\lambda|F(z)|\lambda+1_k))\ \Phi^*_{\lambda+1_k}(v) + (\lambda|K^+(z)|\lambda)\ F(z)\Phi^*_\lambda(v). \tag{4.16}$$

Proof. It follows from Propositions 2.7 and 4.3. $\square$

Proposition 4.6 ([31]). *Let* $\{|\Lambda)) := \mathcal{C}_\Lambda/\mathcal{C}'_\Lambda|\Lambda)\}_{\Lambda:plane\ partition}$ *be the dual basis of the vertical MacMahon representation* $\mathcal{M}(K;v)$. *(See section* 5.2.2 *for the definition of* $\mathcal{C}_\Lambda$ *and* $\mathcal{C}'_\Lambda$.*) The* Λ*-component of the dual MacMahon intertwiner defined by*

$$\Xi^*(K;v)(\bullet) = \sum_\Lambda (\Xi^*_\Lambda(K;v)(\bullet)) \otimes |\Lambda)), \quad \bullet \in \mathcal{H}', \tag{4.17}$$

is characterized by the following dual intertwining relations:

$$\Xi^*_\Lambda(v)K^+(\mathfrak{q}z) = (\Lambda|K^+(z)|\Lambda)\ K^+(\mathfrak{q}z)\Xi^*_\Lambda(v), \tag{4.18}$$

$$\Xi^*_\Lambda(v)K^-(z) = (\Lambda|K^-(z)|\Lambda)\ K^-(z)\Xi^*_\Lambda(v), \tag{4.19}$$

$$\Xi^*_\Lambda(v)E(z) = K^-(\mathfrak{q}z)\sum_{k=1}^{h(\Lambda)}\sum_{i=1}^{\ell(\Lambda^{(k)})}((\Lambda|E(\mathfrak{q}z)|\Lambda-1_i^{(k)}))\ \Xi^*_{\Lambda-1_i^{(k)}}(v) + \ E(z)\Xi^*_\Lambda(v), \tag{4.20}$$

$$\Xi^*_\Lambda(v)F(z) = \sum_{k=1}^{h(\Lambda)+1}\sum_{i=1}^{\ell(\Lambda^{(k)})+1}((\Lambda|F(z)|\Lambda+1_i^{(k)}))\ \Xi^*_{\Lambda+1_i^{(k)}}(v) + (\Lambda|K^+(z)|\Lambda)\ F(z)\Xi^*_\Lambda(v). \tag{4.21}$$

Proof. It follows from Propositions 2.8 and 4.3. $\square$

Note that, as concerns the Fock representation, $|\lambda)$ is identified with the Macdonald symmetric function, while $|\lambda))$ is identified with the dual basis of it with respect to the

Macdonald inner product [32], see [22]. However, as concerns the MacMahon representation, we do not have the corresponding theory of the orthogonal symmetric function. Therefore we assume the existence of such a system of orthogonal function in Proposition 4.6, and the terminology "the dual basis" for $\{|\Lambda))\}$ is merely an imitation of that for $\{|\lambda))\}$, see Section 5.2 for details. Note also that we can rewrite the dual matrix elements $((\alpha|X|\beta))$ in terms of ordinary matrix elements $(\alpha|X|\beta)$, see Section 5.2.

Roughly speaking, we can construct the dual intertwiner by exchanging the roles of $E(z)$ and $F(z)$ compared with the original intertwiner.

4.2 Construction of dual vector intertwiner

Proposition 4.7 ([31])**.** *There exists a unique dual vector intertwiner $\mathbb{I}^*(v) : \mathcal{H}' \to \mathcal{H} \otimes V(v)$ up to normalization if and only if a pair of horizontal representations $(\mathcal{H}, \mathcal{H}')$ satisfies $\gamma' = \gamma$ on the second level and $\mathbf{e}'(z) = q_2^{-1}\mathbf{e}(z)$, $\mathbf{f}'(z) = q_2\mathbf{f}(z)$ for the zero mode sector. The n-component is written explicitly as*

$$\mathbb{I}^*_n(v) = z^*_n \tilde{\mathbb{I}}^*_n(v), \quad \tilde{\mathbb{I}}^*_n(v) = \tilde{\mathbb{I}}^*_0(q_1^n v), \quad n \in \mathbb{Z}, \tag{4.22}$$

$$\tilde{\mathbb{I}}^*_0(v) = \exp\left(\sum_{r=1}^{\infty} \frac{H_{-r}}{[r]} \frac{(\mathfrak{q}^{1/2}v)^r}{1-q_1^r}\right) \exp\left(\sum_{r=1}^{\infty} \frac{H_r}{[r]} \frac{(\mathfrak{q}^{-1/2}q_1^{-1}v)^{-r}}{1-q_1^r}\right), \tag{4.23}$$

where $z^*_n = z^*_n(v)$ is a stack of zero modes:

$$z^*_0 = 1, \quad z^*_n = q_2^n \prod_{j=1}^{n} \mathbf{f}(q_1^{j-1}v) \quad (n > 0), \quad z^*_n = q_2^n \prod_{j=n}^{-1} \mathbf{f}(q_1^j v)^{-1} \quad (n < 0). \tag{4.24}$$

As concerns the vector case, the proof is the same as the original vector intertwiner. The relations corresponding to proposition 3.8 are as follows.

Proposition 4.8 ([31])**.** *The Heisenberg part of the zero component* (4.23) *satisfies the following OPE relations:*

$$\varphi^+(\mathfrak{q}^{1/2}z)\mathbb{I}^*_0(v) = \tilde{\psi}(q_1^{-1}v/z)^{-1}\ \mathbb{I}^*_0(v)\varphi^+(\mathfrak{q}^{1/2}z), \tag{4.25}$$

$$\mathbb{I}^*_0(v)\varphi^-(\mathfrak{q}^{-1/2}z) = \tilde{\psi}(z/v)\ \varphi^-(\mathfrak{q}^{-1/2}z)\mathbb{I}^*_0(v), \tag{4.26}$$

$$\eta(z)\mathbb{I}^*_0(v) = \frac{1-\mathfrak{q}v/z}{1-\mathfrak{q}q_2v/z} :\eta(z)\mathbb{I}^*_0(v):, \quad \mathbb{I}^*_0(v)\eta(z) = \frac{1-\mathfrak{q}q_3^{-1}z/v}{1-\mathfrak{q}q_1z/v} :\mathbb{I}^*_0(v)\eta(z):, \tag{4.27}$$

$$\xi(z)\mathbb{I}^*_0(v) = \frac{1-q_1^{-1}v/z}{1-q_3v/z} :\xi(z)\mathbb{I}^*_0(v):, \quad \mathbb{I}^*_0(v)\xi(z) = \frac{1-q_2^{-1}z/v}{1-z/v} :\mathbb{I}^*_0(v)\xi(z): . \tag{4.28}$$

Proof. By definitions (2.71), (2.72) and (4.23), we have

$$V^+(z)\mathbb{I}_0^*(v) = \frac{1-\mathfrak{q}^{3/2}v/z}{1-\mathfrak{q}^{3/2}q_2v/z} :V^+(z)\mathbb{I}_0^*(v):, \quad \mathbb{I}_0^*(v)V^-(z) = \frac{1-\mathfrak{q}^{3/2}q_3^{-1}z/v}{1-\mathfrak{q}^{3/2}q_1z/v} :\mathbb{I}_0^*(v)V^-(z):, \tag{4.29}$$

then (4.25)–(4.28) follow from (4.29) immediately. □

One can understand that $\tilde{\mathbb{I}}_0^*(v)$ is just a regularized expression of the infinite product

$$:\prod_{j=1}^{\infty}\xi(q_1^{j-1}v)^{-1}: \tag{4.30}$$

through the geometric series of q_1.

Proof of Proposition 4.7. We can prove it in the same way as Proposition 3.7 by making use of proposition 4.8. □

4.3 Construction of dual Fock intertwiner

Proposition 4.9 ([31])**.** *There exists a unique dual Fock intertwiner $\Phi^*(v) : \mathcal{H}' \to \mathcal{H}\otimes\mathcal{F}(v)$ up to normalization if and only if a pair of horizontal representations $(\mathcal{H},\mathcal{H}')$ satisfies $\gamma' = \mathfrak{q}\gamma$ on the second level and $\mathbf{e}'(z) = (-\mathfrak{q}v/z)\mathbf{e}(z)$, $\mathbf{f}'(z) = (-\mathfrak{q}v/z)^{-1}\mathbf{f}(z)$ for the zero mode sector. The λ-component is written as*

$$\Phi^*_\lambda(v) = z^*_\lambda\tilde{\Phi}^*_\lambda(v) : \mathcal{H}' \to \mathcal{H}, \tag{4.31}$$

$$\tilde{\Phi}^*_\lambda(v) = \mathcal{G}^{[n]*}\cdot B_n^*(v)\circ\tilde{\mathbb{I}}_\lambda^{[n]*}(v), \quad n > \ell(\lambda), \tag{4.32}$$

where $\tilde{\mathbb{I}}_\lambda^{[n]*}(v) = \tilde{\mathbb{I}}^*_{\lambda_n}(q_2^{n-1}v)\circ\cdots\circ\tilde{\mathbb{I}}^*_{\lambda_1}(v)$ and the coefficient $\mathcal{G}^{[n]*}$ is defined by the normal ordering

$$B_n^*(v)\tilde{\mathbb{I}}_\varnothing^{[n]*}(v) = \left(\mathcal{G}^{[n]*}\right)^{-1} : B_n^*(v)\tilde{\mathbb{I}}_\varnothing^{[n]*}(v): . \tag{4.33}$$

The modification operator $B_n^*(v)$ is defined by making use of the dual vertex operator (2.91) *as*

$$B_n^*(v) = \tilde{V}^-(\mathfrak{q}^{3/2}q_2^nv)^{-1}\tilde{V}^+(\mathfrak{q}^{1/2}q_2^nv), \tag{4.34}$$

and $z^*_\lambda = z^*_\lambda(v)$ is a stack of zero modes:

$$z^*_\lambda(v) = \mathfrak{q}^{-|\lambda|}\prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i}\left(-\mathfrak{q}q_2^{i-1}x_{i,j}^{-1}\right)^{-1}\mathbf{f}(x_{i,j}v) = q_2^{-n(\lambda)}(-q_3)^{-|\lambda|}\prod_{(i,j)\in\lambda}x_{i,j}\mathbf{f}(x_{i,j}v). \tag{4.35}$$

The difference from the original Fock intertwiner is that the ordering of the operator composition here is opposite to there. Furthermore the factor $\mathfrak{q}^{-|\lambda|}$ in z_λ^* is not from the zero modes, but due to the change of basis $|\lambda) \to |\lambda))$.

The relations corresponding to proposition 3.11 are as follows:

Proposition 4.10 ([31])**.** *When a pair of horizontal representatins $(\mathcal{H}, \mathcal{H}'')$ satisfies*

$$\gamma'' = \mathfrak{q}\gamma, \quad \mathbf{e}''(z) = (-\mathfrak{q}q_2^n v/z)\mathbf{e}(z), \quad \mathbf{f}''(z) = (-\mathfrak{q}q_2^n v/z)^{-1}\mathbf{f}(z), \tag{4.36}$$

the modification operator $B_n^*(v)\colon \mathcal{H}'' \to \mathcal{H}$ satisfies the following relations:

$$B_n^*(v)K^+(\mathfrak{q}z) = \beta_n^+(v/z)\ K^+(\mathfrak{q}z)B_n^*(v), \tag{4.37}$$

$$K^-(z)B_n^*(v) = \beta_n^-(z/v)^{-1}\ B_n^*(v)K^-(z), \tag{4.38}$$

$$B_n^*(v)E(z) - E(z)B_n^*(v) = 0, \tag{4.39}$$

$$B_n^*(v)F(z) - \beta_n^+(v/z)F(z)B_n^*(v) = -\mathfrak{q}^{-1}\delta(q_2^n v/z) : F(z)B_n^*(v) : . \tag{4.40}$$

Proof. By making use of the OPE relations (2.89) and (2.90), one can check the following relations for $B_n^*(v)$:

$$\varphi^+(\mathfrak{q}^{1/2}z)B_n^*(v) = \mathfrak{q}^{-1}\beta_n^+(v/z)^{-1}\ B_n^*(v)\varphi^+(\mathfrak{q}^{1/2}z), \tag{4.41}$$

$$B_n^*(v)\varphi^-(\mathfrak{q}^{-1/2}z) = \mathfrak{q}^{-1}\beta_n^-(z/v)\ \varphi^-(\mathfrak{q}^{-1/2}z)B_n^*(v), \tag{4.42}$$

$$\eta(z)B_n^*(v) = (1 - \mathfrak{q}q_2^n v/z) : \eta(z)B_n^*(v) :, \quad B_n^*(v)\eta(z) = (1 - \mathfrak{q}^{-1}q_2^{-n}z/v) : \eta(z)B_n^*(v) :, \tag{4.43}$$

$$\xi(z)B_n^*(v) = (1 - q_2^n q_3 v/z)^{-1} : \xi(z)B_n^*(v) :, \quad B_n^*(v)\xi(z) = (1 - q_2^{-n}z/v)^{-1} : \xi(z)B_n^*(v) :, \tag{4.44}$$

$$(-\mathfrak{q}q_2^n v/z)\ B_n^*(v)\eta(z) - \eta(z)B_n^*(v) = 0, \tag{4.45}$$

$$(-\mathfrak{q}q_2^n v/z)^{-1}\ B_n^*(v)\xi(z) - \beta_n^+(v/z)\xi(z)B_n^*(v) = -\mathfrak{q}^{-1}\delta(q_2^n v/z) : B_n^*(v)\xi(z) : . \tag{4.46}$$

Taking (4.36) into account, (4.37)–(4.40) follow from the above relations. □

One can understand that $B_n^*(v)$ is just a regularized expression of the formal infinite product

$$: \prod_{i,j=1}^{\infty} \xi(q_1^{j-1}q_2^{n+i-1}v)^{-1} : . \tag{4.47}$$

Proposition 4.11 ([31])**.** *We define $\mathcal{G}^{[n]*}$, $\tilde{\mathbb{I}}_\lambda^{[n]*}(v)$ and $B_n^*(v)$ as Proposition* 4.9*, then the operator $\tilde{\Phi}_\lambda^*(v) := \mathcal{G}^{[n]*} \cdot B_n^*(v) \circ \tilde{\mathbb{I}}_\lambda^{[n]*}(v)$ is independent of $n > \ell(\lambda)$.*

Proof. As for $\lambda = \varnothing$, by definition of $\mathcal{G}^{[n]*}$, we have

$$\tilde{\Phi}^*_{\varnothing}(v) = B^*_0(v) := \exp\left(\sum_{r=1}^{\infty} \frac{H_{-r}}{[r]}(\mathfrak{q}^{1/2}v)^r \frac{1}{(1-q_1^r)(1-q_2^r)}\right) \\ \times \exp\left(-\sum_{r=1}^{\infty} \frac{H_r}{[r]}(\mathfrak{q}^{-1/2}v)^{-r} \frac{q_3^{-r}}{(1-q_1^r)(1-q_2^r)}\right). \tag{4.48}$$

As for general λ, if we introduce the factor $\mathcal{G}^*_\lambda$ by

$$B^*_n(v)\tilde{\mathbb{I}}^{[n]*}_\lambda(v) = (\mathcal{G}^*_\lambda)^{-1}(\mathcal{G}^{[n]*})^{-1} : B^*_n(v)\tilde{\mathbb{I}}^{[n]*}_\lambda(v) :, \tag{4.49}$$

then we can show that $\mathcal{G}^*_\lambda$ is independent of $n > \ell(\lambda)$ in fact (see section 5.2.1). Hence we have

$$\mathcal{G}^*_\lambda \cdot \tilde{\Phi}^*_\lambda(v) =: B^*_n(v)\tilde{\mathbb{I}}^{[n]*}_\lambda(v) :=: B^*_0(v) \prod_{(i,j\in\lambda)} \xi(x_{ij}v) : \\ = \exp\left(-\sum_{r=1}^{\infty} \frac{H_{-r}}{[r]}(\mathfrak{q}^{1/2}v)^r \left(\sum_{(i,j)\in\lambda} x_{ij}^r - \frac{1}{(1-q_1^r)(1-q_2^r)}\right)\right) \\ \times \exp\left(\sum_{r=1}^{\infty} \frac{H_r}{[r]}(\mathfrak{q}^{-1/2}v)^{-r} \left(\sum_{(i,j)\in\lambda} x_{ij}^{-r} - \frac{q_3^{-r}}{(1-q_1^r)(1-q_2^r)}\right)\right), \tag{4.50}$$

where $x_{ij} = q_1^{j-1}q_2^{i-1}$. □

Proof of Proposition 4.9. We can prove it in the same way as Proposition 3.10 by making use of Proposition 4.10 and the dual vector intertwining relation. □

4.4 Construction of dual MacMahon intertwiner

Proposition 4.12 ([31])**.** *There exists a unique dual MacMahon intertwiner $\Xi^*(K;v) : \mathcal{H}' \to \mathcal{H} \otimes \mathcal{M}(K;v)$ up to normalization if and only if a pair of horizontal representations $(\mathcal{H}, \mathcal{H}')$ satisfies $\gamma' = K^{1/2}\gamma$ on the second level and*

$$\mathbf{e}'(z) = \frac{\theta_{q_3}(\mathfrak{q}v/z)}{\theta_{q_3}(\mathfrak{q}Kv/z)}\mathbf{e}(z), \qquad \mathbf{f}'(z) = K^{-1/2}\frac{\theta_{q_3}(Kv/z)}{\theta_{q_3}(v/z)}\mathbf{f}(z) \tag{4.51}$$

for the zero mode sector. The Λ-component is written as

$$\Xi^*_\Lambda(K;v) = z^*_\Lambda \tilde{\Xi}^*_\Lambda(K;v) \colon \mathcal{H}' \to \mathcal{H}, \tag{4.52}$$

$$\tilde{\Xi}^*_\Lambda(K;v) = \mathcal{M}^{[n]*}(K) \cdot \Gamma^*_n(K;v) \circ \tilde{\Phi}^{[n]*}_\Lambda(v), \quad n > h(\Lambda), \tag{4.53}$$

where $\tilde{\Phi}^{[n]*}_{\Lambda}(v) = \tilde{\Phi}^{*}_{\Lambda^{(n)}}(q_3^{n-1}v) \circ \cdots \circ \tilde{\Phi}^{*}_{\Lambda^{(1)}}(v)$ *and the coefficient* $\mathcal{M}^{[n]*}(K)$ *is defined by the normal ordering*

$$\Gamma^*_n(K;v)\tilde{\Phi}^{[n]*}_{\varnothing}(v) = \left(\mathcal{M}^{[n]*}(K)\right)^{-1} : \Gamma^*_n(K;v)\tilde{\Phi}^{[n]*}_{\varnothing}(v) : . \tag{4.54}$$

The modification operator $\Gamma^*_n(K;v)$ *is defined as*

$$\Gamma^*_n(K;v) = \exp\left(-\sum_{r=1}^{\infty}\frac{H_{-r}}{[r]}\frac{q_3^{nr}-K^r}{k_r}(\mathfrak{q}^{1/2}v)^r\right)\exp\left(-\sum_{r=1}^{\infty}\frac{H_r}{[r]}\frac{q_3^{-nr}-K^{-r}}{k_r}(\mathfrak{q}^{-1/2}v)^{-r}\right) \tag{4.55}$$

and $z^*_\Lambda = z^*_\Lambda(K;v)$ *is a stack of zero modes*:

$$z^*_\Lambda(K;v) = K^{-|\Lambda|/2}\prod_{(i,j,k)\in\Lambda}\left(\frac{K^{1/2}}{\mathfrak{q}^{k-1}}\frac{\theta_{q_3}(q_3^{k-1}/x_{ijk})}{\theta_{q_3}(K/x_{ijk})}\right)^{-1}\mathbf{f}(x_{ijk}v). \tag{4.56}$$

Note that the ordering of the operator composition here is opposite to that for the ordinary MacMahon intertwiner. One can understand that $\Gamma^*_n(K;v)$ is just a regularized expression of the formal infinite product

$$:\prod_{i,j,k=1}^{\infty}\xi(q_1^{j-1}q_2^{i-1}q_3^{n+k-1}v)^{-1}\xi(q_1^{j-1}q_2^{i-1}q_3^{k-1}Kv) : . \tag{4.57}$$

The relations corresponding to proposition 3.14 are as follows:

Proposition 4.13 ([31]). *When a pair of horizontal representatins* $(\mathcal{H},\mathcal{H}'')$ *satisfies*

$$\gamma'' = \mathfrak{q}^{-n}K^{1/2}\gamma, \qquad \mathbf{e}''(z) = \frac{\theta_{q_3}(\mathfrak{q}q_3^n v/z)}{\theta_{q_3}(\mathfrak{q}Kv/z)}\mathbf{e}(z), \qquad \mathbf{f}''(z) = \frac{K^{-1/2}}{\mathfrak{q}^{-n}}\frac{\theta_{q_3}(Kv/z)}{\theta_{q_3}(q_3^n v/z)}\mathbf{f}(z). \tag{4.58}$$

the modification operator $\Gamma^*_n(K;v):\mathcal{H}''\to\mathcal{H}$ *satisfies the following relations*:

$$\Gamma^*_n(K;v)K^+(\mathfrak{q}z) = \gamma_n^+(v/z)K^+(\mathfrak{q}z)\Gamma^*_n(K;v), \tag{4.59}$$

$$K^-(z)\Gamma^*_n(K;v) = \gamma_n^-(z/v)^{-1}\Gamma^*_n(K;v)K^-(z), \tag{4.60}$$

$$\Gamma^*_n(K;v)E(z) - E(z)\Gamma^*_n(K;v) = 0, \tag{4.61}$$

$$\Gamma^*_n(K;v)F(z) - \gamma_n^+(v/z)\ F(z)\Gamma^*_n(K;v) = 0. \tag{4.62}$$

Proof. By making use of the OPE relations,

$$V^+(z)\Gamma^*_n(K;v) = \exp\left(-\sum_{r=1}^{\infty}\frac{1}{r}\frac{q_3^{nr}-K^r}{1-q_3^r}(\mathfrak{q}^{3/2}v/z)^r\right) : V^+(z)\Gamma^*_n(K;v) :, \tag{4.63}$$

$$\Gamma_n^*(K;v)V^-(z) = \exp\left(\sum_{r=1}^{\infty}\frac{1}{r}\frac{q_3^{-nr}-K^{-r}}{1-q_3^r}(\mathfrak{q}^{1/2}v/z)^{-r}\right) : \Gamma_n^*(K;v)V^-(z) :, \tag{4.64}$$

one can check the following relations for $\Gamma_n^*(K;v)$:

$$\varphi^+(\mathfrak{q}^{1/2}z)\Gamma_n^*(K;v) = \frac{\mathfrak{q}^n}{K^{1/2}}\gamma_n^+(v/z)^{-1}\ \Gamma_n^*(K;v)\varphi^+(\mathfrak{q}^{1/2}z), \tag{4.65}$$

$$\Gamma_n^*(K;v)\varphi^-(\mathfrak{q}^{-1/2}z) = \frac{\mathfrak{q}^n}{K^{1/2}}\gamma_n^-(z/v)\ \varphi^-(\mathfrak{q}^{-1/2}z)\Gamma_n^*(K;v), \tag{4.66}$$

$$\eta(z)\Gamma_n^*(K;v) = \frac{(\mathfrak{q}q_3^n v/z;q_3)_\infty}{(\mathfrak{q}Kv/z;q_3)_\infty} : \eta(z)\Gamma_n^*(K;v) :, \quad \Gamma_n^*(K;v)\eta(z) = \frac{(\mathfrak{q}K^{-1}z/v;q_3)_\infty}{(\mathfrak{q}q_3^{-n}z/v;q_3)_\infty} : \eta(z)\Gamma_n^*(K;v) :, \tag{4.67}$$

$$\xi(z)\Gamma_n^*(K;v) = \frac{(q_3Kv/z;q_3)_\infty}{(q_3^{n+1}v/z;q_3)_\infty} : \xi(z)\Gamma_n^*(K;v) :, \quad \Gamma_n^*(K;v)\xi(z) = \frac{(q_3^{-n+1}z/v;q_3)_\infty}{(q_3K^{-1}z/v;q_3)_\infty} : \xi(z)\Gamma_n^*(K;v) : . \tag{4.68}$$

Taking (4.58) into account, (4.59)–(4.62) follow from the above relations. □

Proposition 4.14 ([31])**.** *We define $\mathcal{M}^{[n]*}(K)$, $\tilde{\Phi}_\Lambda^{[n]*}(v)$ and $\Gamma_n^*(K;v)$ as Proposition 4.12, then the operator $\tilde{\Xi}_\Lambda^*(K;v) := \mathcal{M}^{[n]*}(K)\cdot\Gamma_n^*(K;v)\circ\tilde{\Phi}_\Lambda^{[n]*}(v)$ is independent of $n > h(\Lambda)$.*

Proof. As for $\Lambda = \varnothing$, by definition of $\mathcal{M}^{[n]*}(K)$, we have

$$\begin{aligned}\tilde{\Xi}_\varnothing^*(K;v) = \Gamma_0^*(K;v) := \exp&\left(-\sum_{r=1}^{\infty}\frac{H_{-r}}{[r]}\frac{1-K^r}{k_r}(\mathfrak{q}^{1/2}v)^r\right)\\ &\times\exp\left(-\sum_{r=1}^{\infty}\frac{H_r}{[r]}\frac{1-K^{-r}}{k_r}(\mathfrak{q}^{-1/2}v)^{-r}\right).\end{aligned} \tag{4.69}$$

As for general Λ, if we introduce the factor $\mathcal{C}_\Lambda^*$ by

$$\Gamma_n^*(K;v)\tilde{\Phi}_\Lambda^{[n]*}(v) = (\mathcal{C}_\Lambda^*)^{-1}(\mathcal{M}^{[n]*}(K))^{-1} : \Gamma_n^*(K;v)\tilde{\Phi}_\Lambda^{[n]*}(v) :, \tag{4.70}$$

then we can show that $\mathcal{C}_\Lambda^*$ is independent of $n > h(\Lambda)$ in a similar manner to Section 5.2.2. Hence we have

$$\begin{aligned}\mathcal{C}_\Lambda^*\cdot\tilde{\Xi}_\Lambda^*(K;v) &=: \Gamma_n^*(K;v)\tilde{\Phi}_\Lambda^{[n]*}(v) :=: \Gamma_0^*(K;v)\prod_{(i,j,k\in\lambda)}\xi(x_{ijk}v) :\\ &= \exp\left(-\sum_{r=1}^{\infty}\frac{H_{-r}}{[r]}(\mathfrak{q}^{1/2}v)^r\left(\sum_{(i,j,k)\in\Lambda}x_{ijk}^r + \frac{1-K^r}{k_r}\right)\right)\end{aligned}$$

$$\times \exp\left(\sum_{r=1}^{\infty}\frac{H_r}{[r]}(\mathfrak{q}^{-1/2}v)^{-r}\left(\sum_{(i,j,k)\in\Lambda}x_{ijk}^{-r}-\frac{1-K^{-r}}{k_r}\right)\right), \tag{4.71}$$

where $x_{ijk}=q_1^{j-1}q_2^{i-1}q_3^{k-1}$. □

Proof of Proposition 4.12. We can prove it in the same way as Proposition 3.13 by making use of Proposition 4.13 and the dual Fock intertwining relation. □

5 Properties of MacMahon intertwiner

5.1 MacMahon R-matrix

In this section we will check that two ways to compute the R-matrix lead to the same results; one way is to use the formula of the universal R-matrix on two vertical representations [17] and the other way is to read the commutator of two intertwiners. We can explicitly check this agreement for the MacMahon case now [30] (the agreement for the Fock case was checked in [26]).

5.1.1 R-matrix for vertical MacMahon representation

As one way to compute the R-matrix, let us use the formula of the universal R-matrix in [17].

Definition 5.1. *We define the diagonal part of the universal R-matrix by*

$$\mathcal{R}_0 := (K^+\otimes 1)^{1\otimes d_1}(1\otimes K^+)^{d_1\otimes 1}\exp\left(-\sum_{r=1}^{\infty} rk_r(h_{-r}\otimes h_r)\right), \tag{5.1}$$

where h_r is the mode of $K^{\pm}(z)$ introduced by

$$K^{\pm}(z)=K^{\pm}\exp\left(\sum_{r=1}^{\infty}k_r h_{\pm r}z^{\mp r}\right), \tag{5.2}$$

and d_1 is the first grading operator of $\mathcal{U}$ defined by

$$[d_1,E(z)]=E(z),\quad [d_1,F(z)]=-F(z),\quad [d_1,K^{\pm}(z)]=0. \tag{5.3}$$

We would like to evaluate $\mathcal{R}_0$ on the tensor product of two MacMahon representations $\mathcal{M}(K_1;v_1)\otimes\mathcal{M}(K_2;v_2)$.

Proposition 5.2 ([30, Section 3]). *h_r ($r \neq 0, r \in \mathbb{Z}$) and $K^{\pm}$ act on the state $|\Lambda_i) \in \mathcal{M}(K_i; v_i)$ ($i = 1, 2$) as follows:*

$$h_r|\Lambda_i) = \frac{v_i^r}{r}\left[\frac{1-K_i^r}{k_r} + \sum_{\square\in\Lambda_i} x_\square^r\right]|\Lambda_i), \quad x_\square = q_1^j q_2^i q_3^k \quad \textit{for} \quad \square = (i,j,k) \in \Lambda_i, \tag{5.4}$$

$$K^{\pm}|\Lambda_i) = K_i^{\mp 1/2}|\Lambda_i). \tag{5.5}$$

Proof. Since we can rewrite the ingredients of the action for $K^{\pm}(z)$ (2.58), which are (2.62) and (2.19), as follows:

$$\boldsymbol{\psi}_{\boldsymbol{\varnothing}}^{\pm}(K^{1/2}; u) = K^{\mp 1/2}\exp\left(\sum_{r=1}^{\infty}\frac{u^r}{r}(1-K^{\pm r})\right), \qquad G(u) = \exp\left(\sum_{r=1}^{\infty}\frac{u^r}{r}k_r\right), \tag{5.6}$$

we can evaluate the action by definition (5.2). □

Also note that the grading operator is represented as $d_1|\Lambda) = |\Lambda||\Lambda)$. Therefore we can evaluate $\mathcal{R}_0$ as follows.

Proposition 5.3 ([30, Section 3]). *Let $|\Lambda)\otimes|\Pi) \in \mathcal{M}(K_1; v_1)\otimes\mathcal{M}(K_2; v_2)$. The matrix element of $\mathcal{R}_0$ is defined and evaluated as*

$$R_{\Lambda\Pi}^{K_1,K_2}(v_2/v_1) := \frac{(\Lambda|\otimes(\Pi|\,\mathcal{R}_0\,|\Lambda)\otimes|\Pi)}{(\varnothing|\otimes(\varnothing|\,\mathcal{R}_0\,|\varnothing)\otimes|\varnothing)} \tag{5.7}$$

$$= K_1^{-|\Pi|/2}K_2^{-|\Lambda|/2}\prod_{\square\in\Lambda}\prod_{\blacksquare\in\Pi}G(x_\blacksquare v_2/x_\square v_1)\times\prod_{\square\in\Lambda}\frac{1-(K_2v_2)/(x_\square v_1)}{1-v_2/(x_\square v_1)}\prod_{\blacksquare\in\Pi}\frac{1-(x_\blacksquare v_2)/v_1}{1-(x_\blacksquare v_2)/(K_1v_1)}. \tag{5.8}$$

Proof. As for $\Lambda = \Pi = \boldsymbol{\varnothing}$, by making use of Proposition 5.2, we have

$$(\varnothing|\otimes(\varnothing|\,\mathcal{R}_0\,|\varnothing)\otimes|\varnothing) = \exp\left(-\sum_{r=1}^{\infty}\frac{(v_2/v_1)^r}{r}\frac{(1-K_1^{-r})(1-K_2^r)}{k_r}\right). \tag{5.9}$$

As for general Λ and Π, again by making use of Proposition 5.2, we have

$$\begin{aligned}&(\Lambda|\otimes(\Pi|\,\mathcal{R}_0\,|\Lambda)\otimes|\Pi)\\ &= K_1^{-|\Pi|/2}K_2^{-|\Lambda|/2}\exp\left(\sum_{r=1}^{\infty}\frac{(v_2/v_1)^r}{r}\left[k_r\sum_{\square\in\Lambda}x_\square^{-r}\sum_{\blacksquare\in\Pi}x_\blacksquare^r\right.\right.\\ &\qquad\left.\left.+(1-K_2^r)\sum_{\square\in\Lambda}x_\square^{-r}-(1-K_1^{-r})\sum_{\blacksquare\in\Pi}x_\blacksquare^r-\frac{(1-K_1^{-r})(1-K_2^r)}{k_r}\right]\right).\end{aligned} \tag{5.10}$$

By introducing the generating functions

$$\prod_{\square\in\Lambda}\prod_{\blacksquare\in\Pi} G(x_\blacksquare v_2/x_\square v_1) = \exp\left(\sum_{r=1}^{\infty}\frac{(v_2/v_1)^r}{r}\left[k_r \sum_{\square\in\Lambda} x_\square^{-r}\sum_{\blacksquare\in\Pi} x_\blacksquare^{r}\right]\right), \tag{5.11}$$

$$\prod_{\square\in\Lambda}\frac{1-(K_2v_2)/(x_\square v_1)}{1-v_2/(x_\square v_1)} = \exp\left(\sum_{r=1}^{\infty}\frac{(v_2/v_1)^r}{r}\left[(1-K_2^r)\sum_{\square\in\Lambda} x_\square^{-r}\right]\right), \tag{5.12}$$

$$\prod_{\blacksquare\in\Pi}\frac{1-(x_\blacksquare v_2)/v_1}{1-(x_\blacksquare v_2)/(K_1v_1)} = \exp\left(\sum_{r=1}^{\infty}\frac{(v_2/v_1)^r}{r}\left[-(1-K_1^{-r})\sum_{\blacksquare\in\Pi} x_\blacksquare^{r}\right]\right), \tag{5.13}$$

we obtain the expression (5.8). □

5.1.2 R-matrix as the commutator of MacMahon intertwiners

As the other way, let us compute the commutator of two MacMahon intertwiners and check that the arising R-factor is the same as the R-matrix previously computed in Proposition 5.3.

Definition 5.4. *We define the matrix element of the R-factor $\mathcal{R}$ by*

$$\Xi_\Pi(K_2;v_2)\Xi_\Lambda(K_1;v_1) = \Upsilon^{(+)}(K_1,K_2;v_1/v_2)\mathcal{R}_{\Lambda\Pi}^{K_1,K_2}(v_2/v_1)^{-1}\Xi_\Lambda(K_1;v_1)\Xi_\Pi(K_2;v_2), \tag{5.14}$$

where $\Upsilon^{(+)}$ denotes the vacuum contribution, explicitly,

$$\Upsilon^{(+)}(K_1,K_2;v_1/v_2) = \exp\left(\sum_{r=1}^{\infty}\frac{1}{r}\frac{(1-K_1^r)(1-K_2^{-r})}{k_r(1-q_3^r)}\left[\left(\frac{K_1v_1}{K_2v_2}\right)^{-r} - \left(\frac{v_1}{v_2}\right)^{r}\right]\right). \tag{5.15}$$

Proposition 5.5 ([30, Section 3])**.**

$$\mathcal{R}_{\Lambda\Pi}^{K_1,K_2}(v_2/v_1) = R_{\Lambda\Pi}^{K_1,K_2}(v_2/v_1). \tag{5.16}$$

Proof. To begin with, let us check the commutator of vacuum components. We can write the vacuum component of the MacMahon intertwiner as follows:

$$\Xi_\varnothing(K;v) = \Gamma_0(K;v) := \Gamma_0^-(K;v)\Gamma_0^+(K;v), \tag{5.17}$$

$$\Gamma_0^-(K;v) := \exp\left(\sum_{r=1}^{\infty}\frac{H_{-r}}{[r]}\frac{1-K^r}{k_r}(\mathfrak{q}^{-1/2}v)^r\right), \tag{5.18}$$

$$\Gamma_0^+(K;v) := \exp\left(\sum_{r=1}^{\infty}\frac{H_{r}}{[r]}\frac{1-K^{-r}}{k_r}(\mathfrak{q}^{1/2}v)^{-r}\right). \tag{5.19}$$

From the commutation relation of the Heisenberg mode (2.71), we obtain

$$\Gamma_0^+(K_2;v_2)\Gamma_0^-(K_1;v_1) = \exp\left(-\sum_{r=1}^{\infty}\frac{(v_1/v_2)^r}{r}\frac{(1-K_1^r)(1-K_2^{-r})}{k_r(1-q_3^r)}\right)\Gamma_0^-(K_1;v_1)\Gamma_0^+(K_2;v_2), \tag{5.20}$$

$$\Gamma_0^-(K_2;v_2)\Gamma_0^+(K_1;v_1) = \exp\left(\sum_{r=1}^{\infty}\frac{(v_1/v_2)^{-r}}{r}\frac{(1-K_1^{-r})(1-K_2^{r})}{k_r(1-q_3^r)}\right)\Gamma_0^+(K_1;v_1)\Gamma_0^-(K_2;v_2), \tag{5.21}$$

hence we have

$$\Xi_{\varnothing}(K_2;v_2)\Xi_{\varnothing}(K_1;v_1) = \Upsilon^{(+)}(K_1,K_2;v_1/v_2)\Xi_{\varnothing}(K_1;v_1)\Xi_{\varnothing}(K_2;v_2). \tag{5.22}$$

In general, taking into account the shift of the zero mode contribution (3.82), we obtain the following relations:

$$\begin{aligned}
&\Gamma_0(K_2;v_2)\Xi_\Lambda(K_1;v_1)\\
&= \Upsilon^{(+)}(K_1,K_2;v_1/v_2)\prod_{\square\in\Lambda}\gamma_0^-(K_2;x_\square v_1/v_2)^{-1}\ \Xi_\Lambda(K_1;v_1)\Gamma_0(K_2;v_2) &(5.23)\\
&= \Upsilon^{(+)}(K_1,K_2;v_1/v_2)\prod_{\square\in\Lambda}K_2^{-1/2}\frac{1-(x_\square v_1)/v_2}{1-(x_\square v_1)/(K_2v_2)}\ \Xi_\Lambda(K_1;v_1)\Gamma_0(K_2;v_2), &(5.24)\\
&\Xi_\Pi(K_2;v_2)\Gamma_0(K_1;v_1)\\
&= \Upsilon^{(+)}(K_1,K_2;v_1/v_2)\prod_{\blacksquare\in\Pi}\gamma_0^-(K_1;x_\blacksquare v_2/v_1)\ \Gamma_0(K_1;v_1)\Xi_\Pi(K_2;v_2) &(5.25)\\
&= \Upsilon^{(+)}(K_1,K_2;v_1/v_2)\prod_{\blacksquare\in\Pi}K_1^{1/2}\frac{1-(x_\blacksquare v_2)/(K_1v_1)}{1-(x_\blacksquare v_2)/v_1}\ \Gamma_0(K_1;v_1)\Xi_\Pi(K_2;v_2), &(5.26)
\end{aligned}$$

where

$$\gamma_0^-(K;z) = K^{1/2}\frac{1-z/K}{1-z} \tag{5.27}$$

is (2.69) for $N=0$. By combining the above relations with

$$\eta(x_\blacksquare v_2)\eta(x_\square v_1) = G((x_\square v_1)/(x_\blacksquare v_2))\ \eta(x_\square v_1)\eta(x_\blacksquare v_2), \tag{5.28}$$

we can conclude that

$$\begin{aligned}
&\Xi_\Pi(K_2;v_2)\Xi_\Lambda(K_1;v_1)\\
&= \Upsilon^{(+)}(K_1,K_2;v_1/v_2)K_1^{|\Pi|/2}K_2^{|\Lambda|/2}\prod_{\square\in\Lambda}\prod_{\blacksquare\in\Pi}G((x_\blacksquare v_2)/(x_\square v_1))^{-1}
\end{aligned}$$

$$\times \prod_{\square\in\Lambda} \frac{1 - v_2/(x_\square v_1)}{1-(K_2 v_2)/(x_\square v_1)} \prod_{\blacksquare\in\Pi} \frac{1-(x_\blacksquare v_2)/(K_1 v_1)}{1-(x_\blacksquare v_2)/v_1}\, \Xi_\Lambda(K_1; v_1)\Xi_\Pi(K_2; v_2). \tag{5.29}$$

Hence Proposition 5.5 has been proven. □

5.2 Normalization factor

When we consider the dual intertwiners, we need the notion of the dual basis for the vertical representation (except for the vector case). As concerns the (dual) Fock intertwiner, some calculations tell us that the certain OPE factors agree with the normalization factors c_λ and c'_λ of the Macdonald function, see (5.43) and (5.44). Therefore, one can understand that this change of basis of the Fock representation corresponds to reversing the ordering of the operator composition. This fact causes the exchange between roles of $E(z)$ and $F(z)$, see (5.48) and (5.49). Hence one can confirm the dual Fock intertwining relation in Section 4.3.

As concerns the (dual) MacMahon intertwiner, we do not know the counterpart of the Macdonald function. Therefore we do not have an insight into the notion of the "dual basis" for the MacMahon representation, however we can define it by imitating the Fock case. Namely we define the normalization factors $\mathcal{C}_\Lambda$ and $\mathcal{C}'_\Lambda$ of the counterpart of the Macdonald function as the certain OPE factors, see (5.54) and (5.55). Though we cannot evaluate these factors in closed form, we can confirm that the "dual basis" $|\Lambda)) = \mathcal{C}_\Lambda/\mathcal{C}'_\Lambda|\Lambda)$ provides us appropriate formula in order to check the dual MacMahon intertwining relations in Section 4.4, see (5.62) and (5.63).

We expect that these factors are related to some theory of orthogonal symmetric function, however we only make a few calculations which we need in Section 4.

5.2.1 Normalization factor for Fock intertwiner

We introduce the *normalization factors* $\mathcal{G}_\lambda, \mathcal{G}'_\lambda, \mathcal{G}^*_\lambda$ and $\mathcal{G}^{*\prime}_\lambda$ as follows.

Definition 5.6. *We define the normalization factors $\mathcal{G}_\lambda, \mathcal{G}'_\lambda, \mathcal{G}^*_\lambda$ and $\mathcal{G}^{*\prime}_\lambda$ by*

$$\tilde{\mathbb{I}}^{[n]}_\lambda(v)B_n(v) = \mathcal{G}_\lambda^{-1}\left(\mathcal{G}^{[n]}\right)^{-1} : \tilde{\mathbb{I}}^{[n]}_\lambda(v)B_n(v) :, \qquad B_n(v)\tilde{\mathbb{I}}^{[n]\prime}_\lambda(v) = \mathcal{G}'^{-1}_\lambda\left(\mathcal{G}^{[n]\prime}\right)^{-1} : B_n(v)\tilde{\mathbb{I}}^{[n]\prime}_\lambda(v) :, \tag{5.30}$$

$$B^*_n(v)\tilde{\mathbb{I}}^{[n]*}_\lambda(v) = \mathcal{G}^{*-1}_\lambda\left(\mathcal{G}^{[n]*}\right)^{-1} : B^*_n(v)\tilde{\mathbb{I}}^{[n]*}_\lambda(v) :, \quad \tilde{\mathbb{I}}^{[n]*\prime}_\lambda(v)B^*_n(v) = \mathcal{G}^{*\prime -1}_\lambda\left(\mathcal{G}^{[n]*\prime}\right)^{-1} : \tilde{\mathbb{I}}^{[n]*\prime}_\lambda(v)B^*_n(v) :, \tag{5.31}$$

where $n > \ell(\lambda)$ and the coefficients $\mathcal{G}^{[n]}, \mathcal{G}^{[n]\prime}, \mathcal{G}^{[n]*}$ and $\mathcal{G}^{[n]*\prime}$ are defined by the normal ordering

$$\tilde{\mathbb{I}}_{\varnothing}^{[n]}(v)B_n(v) = (\mathcal{G}^{[n]})^{-1} : \tilde{\mathbb{I}}_{\varnothing}^{[n]}(v)B_n(v) :, \quad B_n(v)\tilde{\mathbb{I}}_{\varnothing}^{[n]\prime}(v) = (\mathcal{G}^{[n]\prime})^{-1} : B_n(v)\tilde{\mathbb{I}}_{\varnothing}^{[n]\prime}(v) :, \tag{5.32}$$

$$B_n^*(v)\tilde{\mathbb{I}}_{\varnothing}^{[n]*}(v) = (\mathcal{G}^{[n]*})^{-1} : B_n^*(v)\tilde{\mathbb{I}}_{\varnothing}^{[n]*}(v) :, \quad \tilde{\mathbb{I}}_{\varnothing}^{[n]*\prime}(v)B_n^*(v) = (\mathcal{G}^{[n]*\prime})^{-1} : \tilde{\mathbb{I}}_{\varnothing}^{[n]*\prime}(v)B_n^*(v) :, \tag{5.33}$$

and

$$\tilde{\mathbb{I}}_{\lambda}^{[n]}(v) := \tilde{\mathbb{I}}_{\lambda_1}(v) \circ \cdots \circ \tilde{\mathbb{I}}_{\lambda_n}(q_2^{n-1}v), \quad \tilde{\mathbb{I}}_{\lambda}^{[n]\prime}(v) := \tilde{\mathbb{I}}_{\lambda_n}(q_2^{n-1}v) \circ \cdots \circ \tilde{\mathbb{I}}_{\lambda_1}(v), \tag{5.34}$$

$$\tilde{\mathbb{I}}_{\lambda}^{[n]*}(v) := \tilde{\mathbb{I}}_{\lambda_n}^*(q_2^{n-1}v) \circ \cdots \circ \tilde{\mathbb{I}}_{\lambda_1}^*(v), \quad \tilde{\mathbb{I}}_{\lambda}^{[n]*\prime}(v) := \tilde{\mathbb{I}}_{\lambda_1}^*(v) \circ \cdots \circ \tilde{\mathbb{I}}_{\lambda_n}^*(q_2^{n-1}v). \tag{5.35}$$

By making use of OPE relations, we obtain the following recursion relations.

Proposition 5.7 ([31])**.** *The normalization factors $\mathcal{G}_\lambda, \mathcal{G}'_\lambda, \mathcal{G}^*_\lambda$ and $\mathcal{G}^{*\prime}_\lambda$ satisfy the following recursion relations for Young diagrams*:

$$\frac{\mathcal{G}_\lambda}{\mathcal{G}_{\lambda+1_j}} = \prod_{s=1}^{j-1} \frac{1-q_1x_j/x_s}{1-q_3^{-1}x_j/x_s} \prod_{s=j+1}^{\ell(\lambda)+1} \frac{1-q_2x_s/x_j}{1-x_s/x_j} \frac{1}{1-q_2x_{\ell(\lambda)+1}/x_j}, \tag{5.36}$$

$$\frac{\mathcal{G}'_\lambda}{\mathcal{G}'_{\lambda+1_j}} = \prod_{s=1}^{j-1} \frac{1-q_2x_s/x_j}{1-x_s/x_j} \prod_{s=j+1}^{\ell(\lambda)+1} \frac{1-q_1x_j/x_s}{1-q_3^{-1}x_j/x_s} \frac{1}{1-q_1x_j/x_{\ell(\lambda)+1}}, \tag{5.37}$$

$$\frac{\mathcal{G}^*_\lambda}{\mathcal{G}^*_{\lambda+1_j}} = \prod_{s=1}^{j-1} \frac{1-q_1^{-1}x_s/x_j}{1-q_3x_s/x_j} \prod_{s=j+1}^{\ell(\lambda)+1} \frac{1-q_2^{-1}x_j/x_s}{1-x_j/x_s} \frac{1}{1-q_2^{-1}x_j/x_{\ell(\lambda)+1}}, \tag{5.38}$$

$$\frac{\mathcal{G}^{*\prime}_\lambda}{\mathcal{G}^{*\prime}_{\lambda+1_j}} = \prod_{s=1}^{j-1} \frac{1-q_2^{-1}x_j/x_s}{1-x_j/x_s} \prod_{s=j+1}^{\ell(\lambda)+1} \frac{1-q_1^{-1}x_s/x_j}{1-q_3x_s/x_j} \frac{1}{1-q_1^{-1}x_{\ell(\lambda)+1}/x_j}. \tag{5.39}$$

In particular $\mathcal{G}_\lambda, \mathcal{G}'_\lambda, \mathcal{G}^*_\lambda$ and $\mathcal{G}^{*\prime}_\lambda$ are defined independently of $n > \ell(\lambda)$.

Proof. Since we can prove the others in the same way, let us show (5.36). It suffices to calculate the normal ordering products of $\tilde{\mathbb{I}}_{\lambda_s}(q_3^{s-1}v)\eta(q_1x_jv)$, $\eta(q_1x_jv)\tilde{\mathbb{I}}_{\lambda_s}(q_3^{s-1}v)$ and $\eta(q_1x_jv)B_n(v)$ because what we want to know is just the ratio between $\tilde{\mathbb{I}}_{\lambda+1_j}^{[n]}(v)B_n(v)$ and $\tilde{\mathbb{I}}_{\lambda}^{[n]}(v)B_n(v)$. As concerns these product, we have

$$\tilde{\mathbb{I}}_{\lambda_s}(q_3^{s-1}v)\eta(q_1x_jv) = \frac{1-q_1x_j/x_s}{1-q_3^{-1}x_j/x_s} : \tilde{\mathbb{I}}_{\lambda_s}(q_3^{s-1}v)\eta(q_1x_jv) :, \tag{5.40}$$

$$\eta(q_1x_jv)\tilde{\mathbb{I}}_{\lambda_s}(q_3^{s-1}v) = \frac{1-q_2x_s/x_j}{1-x_s/x_j} : \eta(q_1x_jv)\tilde{\mathbb{I}}_{\lambda_s}(q_3^{s-1}v) :, \tag{5.41}$$

$$\eta(q_1 x_j v) B_n(v) = \frac{1}{1 - x_{n+1}/x_j} : \eta(q_1 x_j v) B_n(v) : . \tag{5.42}$$

Hence these three contributions sum into (5.36). ☐

Proposition 5.8 ([31])**.** *The recursion relations of Proposition* 5.7 *are solved by the followings*:

$$\mathcal{G}_\lambda = c_\lambda(q_1^{-1}, q_2^{-1}), \qquad \mathcal{G}'_\lambda = c'_\lambda(q_1, q_2), \tag{5.43}$$

$$\mathcal{G}^*_\lambda = c_\lambda(q_1, q_2), \qquad \mathcal{G}^{*'}_\lambda = c'_\lambda(q_1^{-1}, q_2^{-1}), \tag{5.44}$$

where

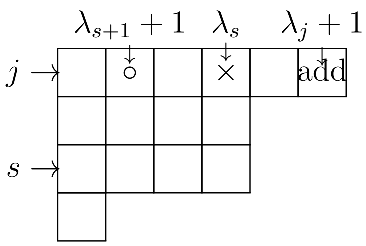
$$c_\lambda(q_1, q_2) := \prod_{\square\in\lambda} \left(1 - q_1^{a_\lambda(\square)} q_2^{-l_\lambda(\square)-1}\right), \quad c'_\lambda(q_1, q_2) := \prod_{\square\in\lambda} \left(1 - q_1^{a_\lambda(\square)+1} q_2^{-l_\lambda(\square)}\right). \tag{5.45}$$

Note that c_λ and c'_λ is just the normalization factors for the Macdonald function [32].

Proof. Note that the normalization factors are unit for $\lambda = \varnothing$ by definition. Since we can prove the others in the same way, let us show the proposition about $\mathcal{G}_\lambda$. Let us focus on the contributions related to the additional box $(j, \lambda_j + 1) \in \lambda + 1_j$. Firstly let us look at $\square = (s, \lambda_j + 1)$ for $s = 1, \dots, j-1$; the arm and the leg length are $a_\lambda(\square) = \lambda_s - (\lambda_j + 1)$ and $l_\lambda(\square) = (j-1) - s$, respectively. Then we can see that these parts contribute to $\mathcal{G}_\lambda/\mathcal{G}_{\lambda+1_j}$ as

$$\prod_{s=1}^{j-1} \frac{1 - q_1^{-\lambda_s+\lambda_j+1} q_2^{j-s}}{1 - q q_1^{-\lambda_s+\lambda_j+1} q_2^{j-s+1}} = \prod_{s=1}^{j-1} \frac{1 - q_1 x_j/x_s}{1 - q_3^{-1} x_j/x_s}. \tag{5.46}$$

Secondly let us look at the s-th row which satisfies $\lambda_s \neq \lambda_{s+1}$ for $s = j, \dots, \ell(\lambda)$; the arm and the leg length for the disappearing part are $a_\lambda(\times) = \lambda_j - \lambda_s$ and $l_\lambda(\times) = s - j$, respectively, while those for the appearing part are $a_{\lambda+1_j}(\circ) = (\lambda_j + 1) - (\lambda_{s+1} + 1)$ and $l_{\lambda+1_j}(\circ) = s - j$, see the diagram below. Therefore we can see that these second parts contribute as



$$\prod_{s=j}^{\ell(\lambda)+1} \frac{1 - q_1^{-\lambda_j+\lambda_s} q_2^{s-j+1}}{1 - q_1^{-\lambda_j+\lambda_{s+1}} q_2^{s-j+1}} = \prod_{s=j}^{\ell(\lambda)+1} \frac{1 - q_2 x_s/x_j}{1 - x_{s+1}/x_j}, \tag{5.47}$$

note that the s-th row which satisfies $\lambda_s = \lambda_{s+1}$ cancels in the above product. Thirdly the additoinal box itself contributes as $(1 - q_2)^{-1}$. Hence these three contributions sum into (5.36). ☐

Proposition 5.9 ([31]). *Nontrivial matrix elements for the dual basis $|\lambda)) := c_\lambda/c'_\lambda|\lambda)$ in the Fock representation $\mathcal{F}(v)$ are evaluated as follows*:

$$((\lambda|E(z)|\lambda-1_j)) = -\mathfrak{q}^{-1}q_2(\lambda-1_j|F(z)|\lambda), \tag{5.48}$$
$$((\lambda|F(z)|\lambda+1_j)) = -\mathfrak{q}q_2^{-1}(\lambda+1_j|E(z)|\lambda). \tag{5.49}$$

Proof. From Proposition 5.7, we have the following equations:

$$\frac{c_{\lambda-1_j}}{c_\lambda}\frac{c'_\lambda}{c'_{\lambda-1_j}} = \mathfrak{q}^{-1}\prod_{s=1}^{j-1}\tilde{\psi}(x_s/x_j)^{-1}\prod_{s=j+1}^{\ell(\lambda)+1}\tilde{\psi}(x_s/x_j)\ \beta^+_{\ell(\lambda)+1}(1/x_j), \tag{5.50}$$
$$\frac{c_{\lambda+1_j}}{c_\lambda}\frac{c'_\lambda}{c'_{\lambda+1_j}} = \mathfrak{q}\prod_{s=1}^{j-1}\tilde{\psi}(q_1^{-1}x_s/x_j)\prod_{s=j+1}^{\ell(\lambda)+1}\tilde{\psi}(q_1^{-1}x_s/x_j)^{-1}\ \beta^+_{\ell(\lambda)+1}(q_1^{-1}/x_j)^{-1}. \tag{5.51}$$

Hence we have

$$\begin{aligned}((\lambda|E(z)|\lambda-1_j)) &= \frac{c'_\lambda}{c_\lambda}\frac{c_{\lambda-1_j}}{c'_{\lambda-1_j}}(\lambda|E(z)|\lambda-1_j)\\ &= \mathfrak{q}^{-1}(1-q_2)\prod_{s=j+1}^{\ell(\lambda)+1}\tilde{\psi}(x_s/x_j)\ \beta^+_{\ell(\lambda)+1}(1/x_j)\ \delta(x_jv/z)\\ &= -\mathfrak{q}^{-1}q_2(\lambda-1_j|F(z)|\lambda), \qquad (5.52)\\ ((\lambda|F(z)|\lambda+1_j)) &= \frac{c'_\lambda}{c_\lambda}\frac{c_{\lambda+1_j}}{c'_{\lambda+1_j}}(\lambda|F(z)|\lambda+1_j)\\ &= \mathfrak{q}(1-q_2^{-1})\prod_{s=1}^{j-1}\tilde{\psi}(q_1^{-1}x_s/x_j)\ \delta(q_1x_jv/z)\\ &= -\mathfrak{q}q_2^{-1}(\lambda+1_j|E(z)|\lambda). \qquad (5.53)\end{aligned}$$

□

Note that the same formulas as above were also computed in [22, (6.1)].

5.2.2 Normalization factor for MacMahon intertwiner

We introduce the *normalization factors* $\mathcal{C}_\Lambda$ and $\mathcal{C}'_\Lambda$ as follows.

Definition 5.10. *We define the normalization factors $\mathcal{C}_\Lambda$ and $\mathcal{C}'_\Lambda$ by*

$$\Gamma_n(K;v)\tilde{\Phi}^{[n]\prime}_\Lambda(v) = {\mathcal{C}'_\Lambda}^{-1}\left(\mathcal{M}^{[n]\prime}(K)\right)^{-1} : \Gamma_n(K;v)\tilde{\Phi}^{[n]\prime}_\Lambda(v) :, \tag{5.54}$$
$$\Gamma^*_n(K;v)\tilde{\Phi}^{[n]*}_\Lambda(v) = {\mathcal{C}_\Lambda}^{-1}\left(\mathcal{M}^{[n]*}(K)\right)^{-1} : \Gamma^*_n(K;v)\tilde{\Phi}^{[n]*}_\Lambda(v) :, \tag{5.55}$$

where $n > h(\Lambda)$ and the coefficients $\mathcal{M}^{[n]\prime}(K)$ and $\mathcal{M}^{[n]*}(K)$ are defined by the normal ordering

$$\Gamma_n(K;v)\tilde{\Phi}^{[n]\prime}_{\varnothing}(v) = \left(\mathcal{M}^{[n]\prime}(K)\right)^{-1} :\Gamma_n(K;v)\tilde{\Phi}^{[n]\prime}_{\varnothing}(v):, \tag{5.56}$$

$$\Gamma^*_n(K;v)\tilde{\Phi}^{[n]*}_{\varnothing}(v) = \left(\mathcal{M}^{[n]*}(K)\right)^{-1} :\Gamma^*_n(K;v)\tilde{\Phi}^{[n]*}_{\varnothing}(v):, \tag{5.57}$$

and

$$\tilde{\Phi}^{[n]\prime}_{\Lambda}(v) := \tilde{\Phi}'_{\Lambda^{(n)}}(q_3^{n-1}v)\circ\cdots\circ\tilde{\Phi}'_{\Lambda^{(1)}}(v), \quad \tilde{\Phi}^{[n]*}_{\Lambda}(v) := \tilde{\Phi}^*_{\Lambda^{(n)}}(q_3^{n-1}v)\circ\cdots\circ\tilde{\Phi}^*_{\Lambda^{(1)}}(v), \tag{5.58}$$

$$\tilde{\Phi}'_{\lambda}(v) := B_m(v)\tilde{\mathbb{I}}^{[m]\prime}_{\lambda}(v), \quad m > \ell(\lambda). \tag{5.59}$$

By making use of OPE relations, we obtain the following recursion relations.

Proposition 5.11 ([31])**.** *The normalization factors $\mathcal{C}_\Lambda$ and $\mathcal{C}'_\Lambda$ satisfy the following recursion relations for plane partitions:*

$$\begin{aligned}\frac{\mathcal{C}'_\Lambda}{\mathcal{C}'_{\Lambda+1^{(k)}_j}} =& \prod_{i=1}^{k-1}\left[\prod_{s=1}^{\ell(\Lambda^{(i)})+1}\frac{1-q_2x_s/x}{1-x_s/x}\frac{1}{1-q_2x_{\ell(\Lambda^{(i)})+1}/x}\right]\\ &\times\prod_{s=1}^{j-1}\frac{1-q_2x_s/x}{1-x_s/x}\prod_{s=j+1}^{\ell(\Lambda^{(k)})+1}\frac{1-q_1x/x_s}{1-q_3^{-1}x/x_s}\frac{1}{1-q_1x/x_{\ell(\Lambda^{(k)})+1}}\\ &\times\prod_{i=k+1}^{h(\Lambda)+1}\left[\prod_{s=1}^{\ell(\Lambda^{(i)})+1}\frac{1-q_1x/x_s}{1-q_3^{-1}x/x_s}\frac{1}{1-q_1x/x_{\ell(\Lambda^{(i)})+1}}\right]\times\frac{(q_1q_3^{-h(\Lambda)-1}x;q_3)_\infty}{(q_1K^{-1}x;q_3)_\infty},\end{aligned} \tag{5.60}$$

$$\begin{aligned}\frac{\mathcal{C}_\Lambda}{\mathcal{C}_{\Lambda+1^{(k)}_j}} =& \prod_{i=1}^{k-1}\left[\prod_{s=1}^{\ell(\Lambda^{(i)})+1}\frac{1-q_1^{-1}x_s/x}{1-q_3x_s/x}\frac{1}{1-q_1^{-1}x_{\ell(\Lambda^{(i)})+1}/x}\right]\\ &\times\prod_{s=1}^{j-1}\frac{1-q_1^{-1}x_s/x}{1-q_3x_s/x}\prod_{s=j+1}^{\ell(\Lambda^{(k)})+1}\frac{1-q_2^{-1}x/x_s}{1-x/x_s}\frac{1}{1-q_2^{-1}x/x_{\ell(\Lambda^{(k)})+1}}\\ &\times\prod_{i=k+1}^{h(\Lambda)+1}\left[\prod_{s=1}^{\ell(\Lambda^{(i)})+1}\frac{1-q_2^{-1}x/x_s}{1-x/x_s}\frac{1}{1-q_2^{-1}x/x_{\ell(\Lambda^{(i)})+1}}\right]\times\frac{(q_1q_3^{-h(\Lambda)}x;q_3)_\infty}{(q_1q_3K^{-1}x;q_3)_\infty},\end{aligned} \tag{5.61}$$

where $x = q_1^{\Lambda^{(k)}_j-1}q_2^{j-1}q_3^{k-1}$ and $x_s = q_1^{\Lambda^{(i)}_s-1}q_2^{s-1}q_3^{i-1}$ for each $1 \le i \le h(\Lambda)+1$. In particular $\mathcal{C}_\Lambda$ and $\mathcal{C}'_\Lambda$ are defined independently of $n > h(\Lambda)$.

Proof. We can check in the same way as Proposition 5.7. □

Though we do not know the closed forms of $\mathcal{C}_\Lambda$ and $\mathcal{C}'_\Lambda$, we can calculate the matrix elements for the dual basis.

Proposition 5.12 ([31])**.** *Nontrivial matrix elements for the dual basis* $|\Lambda)) = \mathcal{C}_\Lambda/\mathcal{C}'_\Lambda|\Lambda)$ *in the MacMahon representation* $\mathcal{M}(K;v)$ *are evaluated as follows*:

$$((\Lambda|E(z)|\Lambda - 1_j^{(k)})) = -K^{-1/2}q_2(\Lambda - 1_j^{(k)}|F(z)|\Lambda), \tag{5.62}$$

$$((\Lambda|F(z)|\Lambda + 1_j^{(k)})) = -K^{1/2}q_2^{-1}(\Lambda + 1_j^{(k)}|E(z)|\Lambda). \tag{5.63}$$

Proof. From Proposition 5.11, we have the following equations:

$$\begin{aligned}\frac{\mathcal{C}_{\Lambda-1_j^{(k)}}}{\mathcal{C}_\Lambda}\frac{\mathcal{C}'_\Lambda}{\mathcal{C}'_{\Lambda-1_j^{(k)}}} = K^{-1/2}&\prod_{i=1}^{k-1}\left[\prod_{s=1}^{\ell(\Lambda^{(i)})+1}\tilde{\psi}(x_s/x)^{-1}\beta^-_{\ell(\Lambda^{(i)})+1}(q_3^{-i+1}x)^{-1}\right]\\ &\times\prod_{s=1}^{j-1}\tilde{\psi}(x_s/x)^{-1}\prod_{s=j+1}^{\ell(\Lambda^{(k)})+1}\tilde{\psi}(x_s/x)\beta^+_{\ell(\Lambda^{(k)})+1}(q_3^{k-1}/x)\\ &\times\prod_{i=k+1}^{h(\Lambda)+1}\left[\prod_{s=1}^{\ell(\Lambda^{(i)})+1}\tilde{\psi}(x_s/x)\beta^+_{\ell(\Lambda^{(i)})+1}(q_3^{i-1}/x)\right]\gamma^+_{h(\Lambda)+1}(1/x),\end{aligned} \tag{5.64}$$

$$\begin{aligned}\frac{\mathcal{C}_{\Lambda+1_j^{(k)}}}{\mathcal{C}_\Lambda}\frac{\mathcal{C}'_\Lambda}{\mathcal{C}'_{\Lambda+1_j^{(k)}}} = K^{1/2}&\prod_{i=1}^{k-1}\left[\prod_{s=1}^{\ell(\Lambda^{(i)})+1}\tilde{\psi}(q_1^{-1}x_s/x)\beta^-_{\ell(\Lambda^{(i)})+1}(q_1q_3^{-i+1}x)\right]\\ &\times\prod_{s=1}^{j-1}\tilde{\psi}(q_1^{-1}x_s/x)\prod_{s=j+1}^{\ell(\Lambda^{(k)})+1}\tilde{\psi}(q_1^{-1}x_s/x)^{-1}\beta^+_{\ell(\Lambda^{(k)})+1}(q_1^{-1}q_3^{k-1}/x)^{-1}\\ &\times\prod_{i=k+1}^{h(\Lambda)+1}\left[\prod_{s=1}^{\ell(\Lambda^{(i)})+1}\tilde{\psi}(q_1^{-1}x_s/x)^{-1}\beta^+_{\ell(\Lambda^{(i)})+1}(q_1^{-1}q_3^{i-1}/x)^{-1}\right]\gamma^+_{h(\Lambda)+1}(q_1^{-1}/x)^{-1},\end{aligned} \tag{5.65}$$

Hence we have

$$((\Lambda|E(z)|\Lambda - 1_j^{(k)})) = \frac{\mathcal{C}'_\Lambda}{\mathcal{C}_\Lambda}\frac{\mathcal{C}_{\Lambda-1_j^{(k)}}}{\mathcal{C}'_{\Lambda-1_j^{(k)}}}(\Lambda|E(z)|\Lambda - 1_j^{(k)}) = -K^{-1/2}q_2(\Lambda - 1_j^{(k)}|F(z)|\Lambda), \tag{5.66}$$

$$((\Lambda|F(z)|\Lambda + 1_j^{(k)})) = \frac{\mathcal{C}'_\Lambda}{\mathcal{C}_\Lambda}\frac{\mathcal{C}_{\Lambda+1_j^{(k)}}}{\mathcal{C}'_{\Lambda+1_j^{(k)}}}(\Lambda|F(z)|\Lambda + 1_j^{(k)}) = -K^{1/2}q_2^{-1}(\Lambda + 1_j^{(k)}|E(z)|\Lambda). \tag{5.67}$$

$\square$

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