On the density function for the value-distribution of automorphic L-functions

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Abstract

The Bohr-Jessen limit theorem is a probabilistic limit theorem on the value-distribution of the Riemann zeta-function in the critical strip. Moreover their limit measure can be written as an integral involving a certain density function. The existence of the limit measure is now known for a quite general class of zeta-functions, but the integral expression has been proved only for some special cases (such as Dedekind zeta-functions). In this paper we give an alternative proof of the existence of the limit measure for a general setting, and then prove the integral expression, with an explicitly constructed density function, for the case of automorphic L-functions attached to primitive forms with respect to congruence subgroups $\Gamma_0(N)$.

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1. Introduction

Let $s = \sigma + it$ be a complex variable, $\zeta(s)$ the Riemann zeta-function. Let R be a fixed rectangle in the complex plane $\mathbb{C}$, with the edges parallel to the axes. By μ_k we mean the k-dimensional usual Lebesgue measure. For $\sigma > 1/2$ and $T > 0$, define

$$V_\sigma(T, R; \zeta) = \mu_1\{t \in [-T, T] \mid \log \zeta(\sigma + it) \in R\}. \tag{1.1}$$

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(The rigorous definition of $\log\zeta(\sigma+it)$ will be given later, in Section 3.) In their classical paper [4], Bohr and Jessen proved the existence of the limit

$$W_\sigma(R;\zeta) = \lim_{T\to\infty}\frac{1}{2T}V_\sigma(T,R;\zeta). \tag{1.2}$$

This is now called the Bohr-Jessen limit theorem. Moreover they proved that this limit value can be written as

$$W_\sigma(R;\zeta) = \int_R \mathcal{M}_\sigma(z,\zeta)|dz|, \tag{1.3}$$

where $z = x+iy \in \mathbb{C}$, $|dz| = dxdy/2\pi$, and $\mathcal{M}_\sigma(z,\zeta)$ is a continuous non-negative, explicitly constructed function defined on $\mathbb{C}$, which we may call the density function for the value-distribution of $\zeta(s)$.

This work is a milestone in the value-distribution theory of $\zeta(s)$, and various alternative proofs and related results have been published; for example, Jessen and Wintner [9], Borchsenius and Jessen [5], Guo [6], and Ihara and the first author [7].

An important problem is to consider the generalization of the Bohr-Jessen theorem. The first author [13] proved that the formula (1.2) can be generalized to a fairly general class of zeta-functions with Euler products. However, (1.3) has not yet been generalized to such a general class. The reason is as follows.

The original proof of (1.2) and (1.3) by Bohr and Jessen depends on a geometric theory of certain "infinite sums" of convex curves, developed by themselves [3]. In later articles [9] and [5], the effect of the convexity of curves was embodied in a certain inequality due to Jessen and Wintner [9, Theorem 13]. Using this method, the Bohr-Jessen theory was generalized to Dirichlet L-functions (Joyner [10]) and Dedekind zeta-functions of Galois number fields (the first author [14]). These generalizations are possible because these zeta-functions have "convex" Euler products in the sense of [13, Section 5]. But this convexity cannot be expected for more general zeta-functions. For example, we cannot apply Jessen-Wintner theorem to each Euler factor of $\log L(f,s)$ or $(L'/L)(f,s)$, where $L(f,s)$ is the L-function associated with a primitive cuspform f. But Lebacque and Zykin [12] studied $\log L(f\otimes\chi,s)$ and $(L'/L)(f\otimes\chi,s)$ as modulas aspect in the way similar to [8] using Murty's result [20].

In [13], the first author developed a method of proving (1.2) without using any convexity, so succeeded in generalizing the theory. However, the method in [13] cannot give a generalization of (1.3).

So far, there is no proof of (1.3) or its analogues without using the convexity, or the Jessen-Wintner type of inequalities. For example, [7] gives a different argument of constructing the density functions for Dirichlet L-functions, but the argument in [7] also depends on the Jessen-Wintner inequality. Additionally, authors [18] studied the value-distribution of the difference between the $\log L(\mathrm{sym}_f^m, s)$ and $\log L(\mathrm{sym}_f^{m-2}, s)$ in level aspect, where $L(\mathrm{sym}_f^m, s)$ is the symmetric mth power L-function associated with a primitive cusp-form f. The argument in [18] is similar to one in [7] and it depends on the Jessen-Wintner inequality.

In [16] [17], the first author obtained certain quantitative results on the value-distribution of Dedekind zeta-functions of non-Galois fields and Hecke L-functions of ideal class characters, whose Euler products are not convex. But in these cases, they are "not so far" from the case of Dedekind zeta-functions of Galois fields. In fact, a simple generalization of the Jessen-Wintner inequality is proved ([17, Lemma 2]) and is essentially used in the proof.

Actually, analyzing the proof of [9, Theorems 12, 13] carefully, we can see that the convexity of curves is not essential. The indispensable tool is the inequality of the Jessen-Wintner type. (However the convex property is probably of independent interest; see Section 8.)

It is the purpose of the present paper to obtain an analogue of (1.3) in the case of automorphic L-functions. The main result (Theorem 2.1) will be stated in the next section. The key is Proposition 7.1, which is an analogue of the Jessen-Wintner inequality for the automorphic case. The novelty of this proposition will be discussed in Section 6.

Except for the proof of this inequality, the argument can be carried out in more general situation. In Section 3 we will introduce a general class of zeta-functions, and in Sections 4 to 6 we will generalize the method in [14] to that general class. Then in Section 7 we will prove the Jessen-Wintner inequality for the automorphic case to complete the proof of the main theorem.

2. Statement of the main result

Let f be a primitive form of weight κ and level N, that is a normalized Hecke-eigen new form of weight κ with respect to the congruence subgroup

$\Gamma_0(N)$, and write its Fourier expansion as

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(\kappa-1)/2} e^{2\pi i n z},$$

where the coefficients $\lambda_f(n)$ are real numbers with $\lambda_f(1) = 1$. Denote the associated L-function by

$$L_f(s) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s}.$$

This is absolutely convergent when $\sigma > 1$, and can be continued to the whole plan $\mathbb{C}$ as an entire function. We understand the rigorous meaning of $\log L_f(s)$ and of

$$V_\sigma(T, R; L_f) = \mu_1\{t \in [-T, T] \mid \log L_f(\sigma + it) \in R\}$$

in the sense explained in Section 3. The following is the main theorem of the present paper.

Theorem 2.1 (Main Theorem)**.** *For any $\sigma > 1/2$, the limit*

$$W_\sigma(R; L_f) = \lim_{T\to\infty} \frac{1}{2T} V_\sigma(T, R; L_f) \tag{2.1}$$

exists, and can be written as

$$W_\sigma(R; L_f) = \int_R \mathcal{M}_\sigma(z, L_f)|dz|, \tag{2.2}$$

where $\mathcal{M}_\sigma(z, L_f)$ is a continuous non-negative function (explicitly given by (6.4) *below) defined on $\mathbb{C}$.*

The above function $\mathcal{M}_\sigma(w, L_f)$ can be called the density function for the value-distribution of $L_f(s)$. The integral expression involving the density function is useful for quantitative studies; for example, in [14] [16] [17] we used such expressions to evaluate the speed of convergence of (3.4) below in the case of Dedekind zeta-functions and Hecke L-functions. Therefore we may expect that (2.2) can be used for quantitative investigation on the value-distribution of $L_f(s)$ (see also Remark 6.2).

Let $\mathbb{P}$ be the set of all prime numbers. Since f is a common Hecke eigen form, $L_f(s)$ has the Euler product

$$\begin{aligned} L_f(s) &= \prod_{\substack{p\in\mathbb{P}\\ p|N}} (1-\lambda_f(p)p^{-s})^{-1} \prod_{\substack{p\in\mathbb{P}\\ p\nmid N}} (1-\lambda_f(p)p^{-s}+p^{-2s})^{-1} \\ &= \prod_{\substack{p\in\mathbb{P}\\ p|N}} (1-\lambda_f(p)p^{-s})^{-1} \prod_{\substack{p\in\mathbb{P}\\ p\nmid N}} (1-\alpha_f(p)p^{-s})^{-1}(1-\beta_f(p)p^{-s})^{-1}, \end{aligned} \tag{2.3}$$

where $\alpha_f(p)+\beta_f(p)=\lambda_f(p)$, $\beta_f(p)=\overline{\alpha_f(p)}$, and

$$|\alpha_f(p)| = |\beta_f(p)| = 1. \tag{2.4}$$

Also we know

$$|\lambda_f(p)| \le 1 \qquad (\text{if } p|N) \tag{2.5}$$

(see [19, Theorem 4.6.17]).

It is known that, for any $\varepsilon>0$, there exists a set of prime $\mathbb{P}_f(\varepsilon)$ of positive density in $\mathbb{P}$, such that the inequality

$$|\lambda_f(p)| > \sqrt{2}-\varepsilon \tag{2.6}$$

holds for any $p\in\mathbb{P}_f(\varepsilon)$ (M. R. Murty [20, Corollary 2 of Theorem 4] in the full modular case, and M. R. Murty and V. K. Murty [21, Chapter 4, Theorem 8.6] for general $\Gamma_0(N)$ case). This fact is used essentially in the course of the proof.

3. The general formulation

A large part of the proof of our Theorem 2.1 can be carried out under a more general framework, that is, for general Euler products introduced in [13]. We begin with recalling the definition of those Euler products.

Let $\mathbb{N}$ be the set of all positive integers, and $g(n)\in\mathbb{N}$, $f(j,n)\in\mathbb{N}$ $(1\le j\le g(n))$ and $a_n^{(j)}\in\mathbb{C}$. Denote by p_n the n-th prime number. We assume

$$g(n) \le C_1 p_n^{\alpha}, \quad |a_n^{(j)}| \le p_n^{\beta} \tag{3.1}$$

with constants $C_1>0$ and $\alpha,\beta\ge 0$. Define

$$\varphi(s) = \prod_{n=1}^{\infty} A_n(p_n^{-s})^{-1}, \tag{3.2}$$

where $A_n(X)$ are polynomials in X given by

$$A_n(X) = \prod_{j=1}^{g(n)} (1 - a_n^{(j)} X^{f(j,n)}).$$

Then $\varphi(s)$ is convergent absolutely in the half-plane $\sigma > \alpha + \beta + 1$ by (3.1).

Definition. We denote by $\mathcal{M}$ the set of all φ satisfying the following three conditions.

(i) $\varphi(s)$ can be continued meromorphically to $\sigma \geq \sigma_0$, where $\alpha + \beta + 1/2 \leq \sigma_0 < \alpha + \beta + 1$, and all poles in this region are included in a compact subset of $\{s \mid \sigma > \sigma_0\}$,

(ii) $\varphi(\sigma + it) = O((|t| + 1)^C)$ for any $\sigma \geq \sigma_0$, with a constant $C > 0$,

(iii) It holds that

$$\int_{-T}^{T} |\varphi(\sigma_0 + it)|^2 dt = O(T). \tag{3.3}$$

Remark 3.1. Here we note that $L_f(s)$ defined in the preceding section belongs to $\mathcal{M}$. In fact, the Euler product is given by (2.3). The condition (3.1) is satisfied with $\alpha = \beta = 0$ by (2.4), (2.5). It is entire, so (i) is obvious. Since it satisfies a functional equation, (ii) follows by using the Phragmén-Lindelöf convexity principle. Lastly, (iii) follows (with any $\sigma_0 > 1/2$) by Potter's result [22].

Now let us define $\log \varphi(s)$. First, when $\sigma > \alpha + \beta + 1$, it is defined by the sum

$$\log \varphi(s) = -\sum_{n=1}^{\infty} \sum_{j=1}^{g(n)} \mathrm{Log}(1 - a_n^{(j)} p_n^{-f(j,n)s}),$$

where Log means the principal branch. Next, let

$$B(\rho) = \{\sigma + i\Im\rho \mid \sigma_0 \leq \sigma \leq \Re\rho\}$$

for any zero or pole ρ with $\Re\rho \geq \sigma_0$. We exclude all $B(\rho)$ from $\{s \mid \sigma \geq \sigma_0\}$, and denote the remaining set by$G(\varphi)$. Then, for any $s \in G(\varphi)$, we may define $\log \varphi(s)$ by the analytic continuation along the horizontal path from the right. Define

$$V_\sigma(T, R; \varphi) = \mu_1\{t \in [-T, T] \mid \sigma + it \in G(\varphi), \log \varphi(\sigma + it) \in R\}.$$

Then, as a generalization of (1.2), the first author [13] proved the following

Theorem 3.1 ([13]). *Let $\varphi \in \mathcal{M}$. For any $\sigma > \sigma_0$, the limit*

$$W_\sigma(R;\varphi) = \lim_{T\to\infty} \frac{1}{2T} V_\sigma(T,R;\varphi) \tag{3.4}$$

exists.

This theorem may be regarded as a result on weak convergence of probability measures, and Prokhorov's theorem in probability theory is used in the proof given in [13].

In [14], the first author presented an alternative argument of proving such a limit theorem, again without using any convexity. This argument is based on Lévy's convergence theorem. The method in [14] is more suitable to discuss the matter of density functions, so in the present paper we follow the method in [14].

In [14], only the case of Dedekind zeta-functions is discussed, but, as mentioned in [15], the idea in [14] can be applied to any $\varphi \in \mathcal{M}$. Such a generalization has, however, not yet been published, so we will give a sketch of the argument in the following Sections 4 and 5.

4. The method of Fourier transforms

Let $\sigma > \sigma_0$, and $N \in \mathbb{N}$. The starting point of the argument is to consider the finite truncation of $\varphi(s)$, that is

$$\varphi_N(s) = \prod_{n\le N} A_n(p_n^{-s})^{-1} = \prod_{n\le N}\prod_{j=1}^{g(n)} \left(1 - r_n^{(j)} p_n^{-if(j,n)t}\right)^{-1},$$

where $r_n^{(j)} = a_n^{(j)} p_n^{-f(j,n)\sigma}$. Then

$$\log \varphi_N(s) = -\sum_{n\le N}\sum_{j=1}^{g(n)} \log\left(1 - r_n^{(j)} e^{-itf(j,n)\log p_n}\right). \tag{4.1}$$

Note that

$$|r_n^{(j)}| \le |a_n^{(j)}| p_n^{-f(j,n)\sigma} \le p_n^{\beta-\sigma} \le p_n^{\beta-(\alpha+\beta+1/2)} \le p_n^{-1/2} \le 1/\sqrt{2}.$$

Let $\mathbb{Z}$ be the set of all integers, $\mathbb{R}$ the set of all real numbers, $\mathbb{T}^N = (\mathbb{R}/\mathbb{Z})^N$ be the N-dimensional unit torus, and define the mapping $S_N : \mathbb{T}^N \to \mathbb{C}$, attached to (4.1), by

$$S_N(\theta_1,\dots,\theta_N) = -\sum_{n\le N}\sum_{j=1}^{g(n)} \log\left(1 - r_n^{(j)} e^{2\pi i f(j,n)\theta_n}\right). \tag{4.2}$$

(Though S_N depends on σ and φ, we do not write explicitly in the notation, for brevity. Similar abbreviation is applied to the notation of λ_N, Λ, K_n below.) We write $z_n^{(j)}(\theta_n) = -\log(1 - r_n^{(j)} e^{2\pi i f(j,n)\theta_n})$ and $z_n(\theta_n) = \sum_{j=1}^{g(n)} z_n^{(j)}(\theta_n)$. Then

$$S_N(\theta_1,\dots,\theta_N) = \sum_{n\le N} z_n(\theta_n). \tag{4.3}$$

For any Borel subset $A \subset \mathbb{C}$, we define $W_{N,\sigma}(A;\varphi) = \mu_N(S_N^{-1}(A))$. Then $W_{N,\sigma}$ is a probability measure on $\mathbb{C}$.

Let $R \subset \mathbb{C}$ be any rectangle with the edges parallel to the axes. The idea of considering the inverse image $S_N^{-1}(R) \subset \mathbb{T}^N$ goes back to Bohr's work (Bohr and Courant [2], Bohr [1], and Bohr and Jessen [4]). Also let E be any strip, parallel to the real or imaginary axis. We have the following two facts, whose proofs are exactly the same as the proofs of [14, Lemma 1].

Fact 1. The sets $S_N^{-1}(R)$, $S_N^{-1}(E)$ are Jordan measurable.

Fact 2. For any $\varepsilon > 0$, there exists a positive number η such that, for any strip E whose width is not larger than η, it holds that $W_{N,\sigma}(E;\varphi) < \varepsilon$.

Now define

$$V_{N,\sigma}(T,R;\varphi) = \mu_1\{t \in [-T,T] \mid \log\varphi_N(\sigma+it) \in R\}.$$

We see that $\log\varphi_N(\sigma+it) \in R$ if and only if

$$\left(\left\{-\frac{t}{2\pi}\log p_1\right\},\dots,\left\{-\frac{t}{2\pi}\log p_N\right\}\right) \in S_N^{-1}(R)$$

(where $\{x\}$ means the fractional part of x). Since $\log p_1,\dots,\log p_N$ are linearly independent over the rational number field$\mathbb{Q}$, in view of Fact 1, we can apply the Kronecker-Weyl theorem to obtain

Proposition 4.1. *For any $N \in \mathbb{N}$, we have*

$$W_{N,\sigma}(R;\varphi) = \lim_{T\to\infty} \frac{1}{2T} V_{N,\sigma}(T,R;\varphi). \tag{4.4}$$

This is the "finite truncation" version of Theorem 3.1. Therefore, the remaining task to arrive at Theorem 3.1 is to discuss the limit $N \to \infty$. For this purpose, we consider the Fourier transform

$$\Lambda_N(w) = \int_{\mathbb{C}} e^{i\langle z,w\rangle} dW_{N,\sigma}(z;\varphi),$$

where $\langle z, w\rangle = \Re z \Re w + \Im z \Im w$. Our next aim is to show the following

Proposition 4.2. *As $N \to \infty$, $\Lambda_N(w)$ converges to a certain function $\Lambda(w)$, uniformly in$\{w \in \mathbb{C} \mid |w| \le a\}$ for any $a > 0$.*

Proof. The proof is quite similar to the argument in [14, Section 3]. It is easy to see that

$$\Lambda_N(w) = \int_{\mathbb{T}^N} e^{i\langle S_N(\theta_1,\ldots,\theta_N),w\rangle} d\mu_N(\theta_1, \ldots, \theta_N),$$

so in view of (4.3) we can write

$$\Lambda_N(w) = \prod_{n\le N} K_n(w) \tag{4.5}$$

with

$$K_n(w) = \int_0^1 e^{i\langle z_n(\theta_n),w\rangle} d\theta_n.$$

Noting $|z_n^{(j)}(\theta_n)| \ll |r_n^{(j)}| \le p_n^{\beta-\sigma}$ and (3.1), we have

$$|z_n(\theta_n)|^2 = \left|\sum_{j=1}^{g(n)} z_n^{(j)}(\theta_n)\right|^2 \ll p_n^{2(\alpha+\beta-\sigma)}.$$

Therefore, analogously to [14, (3.2)], we obtain

$$|K_n(w) - 1| \ll |w|^2 p_n^{2(\alpha+\beta-\sigma)}, \tag{4.6}$$

which implies

$$|\Lambda_{n+1}(w) - \Lambda_n(w)| = |\Lambda_n(w)| \cdot |K_{n+1}(w) - 1| \ll |w|^2 p_{n+1}^{2(\alpha+\beta-\sigma)}. \tag{4.7}$$

Therefore, for $M > N$,

$$|\Lambda_M(w) - \Lambda_N(w)| \leq \sum_{n=N}^{M-1} |\Lambda_{n+1}(w) - \Lambda_n(w)|$$
$$\ll |w|^2 \sum_{n=N}^{M-1} p_{n+1}^{2(\alpha+\beta-\sigma)} \leq |w|^2 \sum_{n=N}^{\infty} p_{n+1}^{2(\alpha+\beta-\sigma)}. \tag{4.8}$$

Since $\sigma > \sigma_0 \geq \alpha + \beta + 1/2$, the last sum tends to 0 as $N \to \infty$, uniformly in the region $|w| \leq a$. This implies the assertion of the proposition. $\square$

From Proposition 4.2, in view of Lévy's convergence theorem, we immediately obtain

Corollary 4.1. *There exists a regular probability measure $W_\sigma(\cdot\,; \varphi)$, to which $W_{N,\sigma}(\cdot\,; \varphi)$ converges weakly as $N \to \infty$, and*

$$\Lambda(w) = \int_{\mathbb{C}} e^{i\langle z,w\rangle} dW_\sigma(z; \varphi). \tag{4.9}$$

Moreover, taking the limit $M \to \infty$ on (4.8), we obtain

$$|\Lambda(w) - \Lambda_N(w)| \ll |w|^2 \sum_{n=N}^{\infty} p_{n+1}^{2(\alpha+\beta-\sigma)}. \tag{4.10}$$

5. Proof of Theorem 3.1

In this section we show how to prove Theorem 3.1 in the framework of our present method. The argument is very similar to that given in [14, Sections 3 and 4], so we omit some details.

First, using Fact 2 in Section 4, we can show (analogously to the argument in the last part of [14, Section 3]) that R is a continuity set with respect to W_σ, and hence

$$W_\sigma(R; \varphi) = \lim_{N \to \infty} W_{N,\sigma}(R; \varphi). \tag{5.1}$$

Now, following the method in [14, Section 4], we prove Theorem 3.1. Put

$$R_N(s; \varphi) = \log \varphi(s) - \log \varphi_N(s), \quad f_N(s; \varphi) = \frac{\varphi(s)}{\varphi_N(s)} - 1.$$

When $\sigma > \alpha + \beta + 1$, since

$$R_N(s;\varphi) \ll \sum_{n>N} \sum_{j=1}^{g(n)} |a_n^{(j)}| p_n^{-f(j,n)\sigma} \ll \sum_{n>N} p_n^{\alpha+\beta-\sigma} \tag{5.2}$$

which tends to 0 as $N \to \infty$, the assertion of the theorem directly follows from Proposition 4.1 and (5.1).

In the case $\sigma_0 < \sigma \le \alpha+\beta+1$, naturally we have to discuss more carefully. Let $\delta > 0$, and define

$$K_N^\delta(T;\varphi) = \left\{ t \in [-T,T] \;\middle|\; \begin{array}{l} \sigma + it \in G(\varphi), \\ |\log\varphi(\sigma+it) - \log\varphi_N(\sigma+it)| \ge \delta \end{array} \right\},$$

and $k_N^\delta(T;\varphi) = \mu_1(K_N^\delta(T;\varphi))$. We will prove that $k_N^\delta(T;\varphi)$ is negligible, that is, for any $\varepsilon > 0$ we can choose $N_0 = N_0(\delta,\varepsilon)$ for which

$$\limsup_{T\to\infty} T^{-1} k_N^\delta(T;\varphi) \le \varepsilon \tag{5.3}$$

holds for any $N \ge N_0$.

Let $\alpha_0 = \sigma - \varepsilon$, $\alpha_1 = \sigma - 2\varepsilon$. We choose ε so small that $\sigma_0 < \alpha_1 < \alpha_0 < \sigma$. For any $t_0 \in [-T,T]$, put

$$H(t_0) = \{s \mid \sigma > \alpha_0, t_0 - 1/2 < t < t_0 + 1/2\},$$

and define $\psi_N^\delta(t_0;\varphi) = 0$ if $H(t_0) \subset G(\varphi)$ and$|R_N(s;\varphi)| < \delta$ for any $s \in H(t_0)$, and $\psi_N^\delta(t_0;\varphi) = 1$ otherwise. Then clearly

$$k_N^\delta(T;\varphi) \le \int_{-T}^{T} \psi_N^\delta(t_0;\varphi) dt_0. \tag{5.4}$$

Using (5.2) we can find $\beta_0 = \alpha + \beta + 1 + C\delta^{-1}$ (with an absolute positive constant C) for which $|R_N(s;\varphi)| < \delta$ holds for any s satisfying $\sigma \ge \beta_0$. Let $Q(t_0) = H(t_0) \cap \{s \mid \sigma < \beta_0\}$.

Lemma 5.1. *If $|f_N(s;\varphi)| < \delta/2$ for any $s \in Q(t_0)$, then $\psi_N^\delta(t_0;\varphi) = 0$.*

This is a generalization of [14, Lemma 2], which further goes back to Bohr [1, Hilfssatz 5]. Bohr's proof in [1] can be applied without change to the above general case, so we omit the proof.

Let $\beta_1 = 2\beta_0$, and let $P(t_0)$ be the rectangle given by $\alpha_1 \leq \sigma \leq \beta_1$, $t_0 - 1 \leq t \leq t_0 + 1$. Put

$$F_N(t_0;\varphi) = \iint_{P(t_0)} |f_N(s;\varphi)|^2 d\sigma dt.$$

(This can be defined only when $P(t_0)$ does not include a pole of $\varphi(s)$.) We use Lemma 5.1 and [14, Lemma 3] (which is [1, Hilfssatz 4]) to see that if

$$F_N(t_0;\varphi) < \pi\left(\varepsilon/2\right)^2\left(\delta/2\right)^2$$

then $\psi_N^\delta(t_0;\varphi) = 0$. Therefore

$$\frac{1}{2T}\int_{-T}^{T} \psi_N^\delta(t_0;\varphi)dt_0 \leq b + \frac{\mu_1(\mathcal{S})}{2T}, \tag{5.5}$$

where $\mathcal{S}$ is the set of all $t \in [-T,T]$ for which we can find a pole s' of$\varphi(s)$ satisfying $|t - \Im s'| \leq 2$, and

$$b = \frac{1}{2T}\mu_1\left(\left\{t_0 \in [-T,T]\setminus\mathcal{S} \;\middle|\; F_N(t_0;\varphi) \geq \pi(\varepsilon/2)^2(\delta/2)^2\right\}\right).$$

From the definition of b we obtain

$$\begin{aligned}&\pi(\varepsilon/2)^2(\delta/2)^2 b\\ \leq&\frac{1}{2T}\int_{t_0\in[-T,T]\setminus\mathcal{S}} F_N(t_0;\varphi)dt_0 = \frac{1}{2T}\int_{\alpha_1}^{\beta_1}\int_{-T-1}^{T+1}|f_N(s;\varphi)|^2\int^{\#} dt_0 dt d\sigma,\end{aligned}$$

where the innermost integral (with the # symbol) is on $t_0 \in [-T,T]\setminus\mathcal{S}$, $t-1 \leq t_0 \leq t+1$. This innermost integral is trivially ≤ 2, and is equal to 0 if there exists a pole s' of $\varphi(s)$ such that $|t - \Im s'| \leq 1$ (because then all $t_0 \in [t-1,t+1]$ belongs to $\mathcal{S}$). Therefore

$$\pi(\varepsilon/2)^2(\delta/2)^2 b \leq \frac{1}{T}\int_{\alpha_1}^{\beta_1}\int_{J(T+1)}|f_N(s;\varphi)|^2 dt d\sigma, \tag{5.6}$$

where

$$J(T) = \{t \in [-T,T] \mid |t - \Im s'| > 1 \text{ for any pole } s' \text{ of } \varphi(s)\}.$$

From (5.4), (5.5) and (5.6) we now obtain

$$\frac{1}{2T}k_N^{\delta}(T;\varphi) \leq \frac{1}{\pi(\varepsilon/2)^2(\delta/2)^2T}\int_{\alpha_1}^{\beta_1}\int_{J(T+1)}|f_N(s;\varphi)|^2 dtd\sigma + \frac{\mu_1(\mathcal{S})}{2T}. \quad (5.7)$$

On the double integral on the right-hand side, as an analogue of [14, Lemma 4], we can show the following lemma.

Lemma 5.2. *For any $\eta > 0$, There exists $N_0 = N_0(\eta)$, such that*

$$\frac{1}{T}\int_{\alpha_1}^{\beta_1}\int_{J(T+1)}|f_N(s;\varphi)|^2 dtd\sigma < \eta \quad (5.8)$$

for any $N \geq N_0$ and any $T \geq T_0$ with some $T_0 = T_0(N)$.

Proof. Write the Dirichlet series expansion of $\varphi(s)$ in the region $\sigma > \alpha+\beta+1$ as

$$\varphi(s) = \sum_{k=1}^{\infty} c_k k^{-s}.$$

Then the Dirichlet series expansion of $f_N(s)$ is

$$f_N(s;\varphi) = \sum_k{}' c_k k^{-s},$$

where the symbol $\sum'$ means that the summation is restricted to $k > 1$ which is co-prime with $p_1p_2\cdots p_N$. In [11, Appendix] it has been shown that, for any $\varepsilon > 0$, we can choose a sufficiently large $N = N(\varepsilon)$ such that

$$c_k = O(k^{\alpha+\beta+\varepsilon}) \quad (5.9)$$

for all k co-prime with $p_1p_2\cdots p_N$.

By (3.3) and the convexity principle we have

$$\int_{J(T)}|\varphi(\sigma+it)|^2 dt = O(T) \quad (5.10)$$

for any $\sigma \geq \sigma_0$. On the other hand, using (4.1) we have

$$\varphi_N(\sigma+it)^{-1} \leq \exp\left(C\sum_{n\leq N}\sum_{j=1}^{g(n)}|a_n^{(j)}|p_n^{-f(j,n)\sigma}\right) \leq \exp\left(C'N^{\alpha+\beta+1-\sigma}\right)$$

(where C, C' are positive constants). Combining this estimate with (5.10) we obtain

$$\frac{1}{T}\int_{J(T)} |f_N(\sigma+it;\varphi)|^2 dt \ll \exp\left(2C'N^{\alpha+\beta+1-\sigma}\right),$$

which is $O(1)$ with respect to T. Therefore by Carlson's mean value theorem (see [23, Section 9.51])

$$\lim_{T\to\infty}\frac{1}{T}\int_{J(T)} |f_N(\sigma+it;\varphi)|^2 dt = {\sum_k}' c_k^2 k^{-2\sigma}, \tag{5.11}$$

uniformly in σ. Using (5.9), we can estimate the right-hand side of (5.11) as

$$\ll \sum_{k\geq p_{N+1}} k^{2(\alpha+\beta+\varepsilon-\sigma)} \ll N^{1+2(\alpha+\beta+\varepsilon-\sigma)},$$

whose exponent is negative for $\sigma > \sigma_0$ (if ε is sufficiently small). This immediately implies the assertion of the lemma. $\square$

Now, applying Lemma 5.2 with $\eta = \pi\delta^2\varepsilon^3/16$ to (5.7), we arrive at (5.3). The assertion of the theorem in the case $\sigma_0 < \sigma \leq \alpha+\beta+1$ then follows by the same argument as in the last part of [14, Section 4].

6. The density function

In this section σ is any real number larger than σ_0. We discuss when it is possible to show that $W_\sigma(\cdot; L_f)$ is absolutely continuous. Then by measure theory we can write

$$W_\sigma(R;\varphi) = \int_R \mathcal{M}_\sigma(z,\varphi)|dz| \tag{6.1}$$

with the Radon-Nikodým density function $\mathcal{M}_\sigma(z;\varphi)$.

For this purpose, we aim to show

$$\Lambda_N(w) = O(|w|^{-(2+\eta)}) \qquad (|w|\to\infty) \tag{6.2}$$

uniformly in N, with some $\eta > 0$.

If (6.2) is valid, then

$$\int_{\mathbb{C}} |\Lambda_N(w)||dw| < \infty.$$

Therefore $W_{N,\sigma}$ is absolutely continuous, and the Radon-Nikodým density function $\mathcal{M}_{N,\sigma}(z;\varphi)$ is given by

$$\mathcal{M}_{N,\sigma}(z;\varphi) = \int_{\mathbb{C}} e^{-i\langle z,w\rangle}\Lambda_N(w)|dw| \tag{6.3}$$

and is continuous (see [9, p.53], [5, p.105]). Moreover, the above uniformity in N implies that the same estimate as (6.2) is valid for the limit function $\Lambda(w)$. Therefore W_σ is also absolutely continuous, hence (6.1) is valid with the continuous density function given by

$$\mathcal{M}_{\sigma}(z;\varphi) = \int_{\mathbb{C}} e^{-i\langle z,w\rangle}\Lambda(w)|dw|. \tag{6.4}$$

The following proposition reduces the problem to the evaluation of $K_n(w)$:

Proposition 6.1. *If there are at least five n's, say $n_1,\dots,n_5$, for which $K_n(w) = O_n(|w|^{-1/2})$ holds as $|w|\to\infty$, then* (6.2) *is valid for any $N \geq \max\{n_1,\dots,n_5\}$, and so* (6.1) *and* (6.4) *are also valid.*

Remark 6.1. The proof of (6.2) in the above proposition is simple: just apply $K_n(w) = O_n(|w|^{-1/2})$ (for $n_1,\dots,n_5$) and the trivial estimate $|K_n(w)| \leq 1$ to the product formula (4.5). The result is (6.2) with $\eta = 1/2$, uniform in N.

Remark 6.2. The existence of the density function is useful for quantitative studies. For instance, if there are at least ten n's with $K_n(w) = O(|w|^{-1/2})$, then $\Lambda_N(w) = O(|w|^{-5})$ for large N. This fact with (4.6), (4.10) leads the estimate

$$|W_\sigma(R;L_f) - W_{N,\sigma}(R;L_f)| = O(\mu_2(R)N^{1+2(\alpha+\beta-\sigma)}(\log N)^{2(\alpha+\beta-\sigma)}) \tag{6.5}$$

for $\sigma > \sigma_0$, as an analogue of [14, (6.4)].

In [14], when $\varphi = \zeta_K$ (the Dedekind zeta-function of a Galois number field K), the key estimate (6.2) was proved by using [9, Theorem 13]. In this case, ζ_K has the Euler product of the form (3.2) with $f(1,n) = \cdots = f(g(n),n)$ ($= f(n)$, say, the inertia degree) and $a_n^{(j)} = 1$ (and hence$r_n^{(1)} = \cdots = r_n^{(g(n))} = p_n^{-f(n)\sigma}$ ($= r_n$, say)). Therefore

$$z_n(\theta_n) = -g(n)\log(1 - r_n e^{2\pi i f(n)\theta_n}),$$

which describes a curve when θ_n moves from 0 to 1. This curve is convex, so the original Jessen-Wintner inequality ([9, Theorem 13]) can be directly applied. In this case we encounter only one type of curve, that is, the curve $-\log(1-\xi)$ ($\xi \in \mathbb{C}$, $|\xi| = r_n$).

When K is non-Galois, $f(1,n), \ldots, f(g(n),n)$ are not necessarily the same as each other, so

$$z_n(\theta_n) = -\sum_{j=1}^{g(n)} \log(1 - r_n^{(j)} e^{2\pi i f(j,n)\theta_n}).$$

However, still in this case, the number of relevant types of curves

$$-\sum_{j=1}^{g(n)} \log(1 - \xi^{f(j,n)}) \qquad (\xi \in \mathbb{C},\ |\xi| = p_n^{-\sigma})$$

is finite, because there are only finitely many patterns of the decomposition of prime numbers into prime ideals in K. Because of this finiteness, we can use [17, Lemma 2] (which is a simple generalization of [9, Theorem 13]) to show (6.2) in this case. The case of Hecke L-functions of ideal class characters can be treated in a similar way.

However in the automorphic case, we encounter infinitely many types of curves, because in this case $z_n(\theta_n)$ describes a curve

$$-\log(1 - \alpha_f(p_n)\xi) - \log(1 - \beta_f(p_n)\xi) \qquad (\xi \in \mathbb{C},\ |\xi| = p_n^{-\sigma}), \tag{6.6}$$

which depends on $\alpha_f(p_n), \beta_f(p_n)$. Therefore we have to prove a new type of Jessen-Wintner inequality, suitable for the automorphic case. This will be done in the next section.

7. An analogue of the Jessen-Wintner inequality for automorphic *L*-functions

Now we restrict ourselves to the case of automorphic L-functions. Except for the (finitely many) prime factors of N, the Euler factor of $L_f(s)$ is of the form

$$(1 - \alpha_f(p_n)p_n^{-s})^{-1}(1 - \beta_f(p_n)p_n^{-s})^{-1},$$

so $z_n(\theta_n) = A_n(p_n^{-\sigma} e^{2\pi i \theta_n})$ with

$$A_n(X) = -\log(1 - \alpha_f(p_n)X) - \log(1 - \beta_f(p_n)X).$$

When θ_n moves from 0 to 1, the points $z_n(\theta_n)$ describes a curve (6.6) on the complex plane, which we denote by Γ_n.

Let $x_n(\theta_n) = \Re z_n(\theta_n)$ and $y_n(\theta_n) = \Im z_n(\theta_n)$. Writing $w = |w|e^{i\tau}$ ($\tau \in [0, 2\pi)$) we have $w = |w|\cos\tau + i|w|\sin\tau$. Then

$$\langle z_n(\theta_n), w\rangle = |w| g_{\tau,n}(\theta_n), \tag{7.1}$$

where

$$g_{\tau,n}(\theta_n) = x_n(\theta_n)\cos\tau + y_n(\theta_n)\sin\tau.$$

Therefore

$$K_n(w) = \int_0^1 e^{i|w|g_{\tau,n}(\theta_n)} d\theta_n. \tag{7.2}$$

Lemma 7.1. *Let $n \in \mathbb{N}$ such that $p_n \nmid N$. For any fixed τ, the function $g_{\tau,n}(\theta_n)$ (as a function in θ_n) is a C^∞-class function. Moreover, if $p_n \in \mathbb{P}_f(\varepsilon)$ and n is sufficiently large, then $g''_{\tau,n}(\theta_n)$ has exactly two zeros on the interval $[0, 1)$.*

Proof. Hereafter, for brevity, we write $p_n = p$, $p_n^{-\sigma} = q$, $2\pi\theta_n = \theta$, $z_n(\theta_n) = z(\theta)$, $g_{\tau,n}(\theta_n) = g_\tau(\theta)$, $x_n(\theta_n) = x(\theta)$, and $y_n(\theta_n) = y(\theta)$. Since the Taylor expansion of $A_n(x)$ is given by

$$A_n(x) = \sum_{j=1}^{\infty} a_j x^j \qquad \text{with} \qquad a_j = \frac{1}{j}(\alpha_f(p)^j + \beta_f(p)^j),$$

we have

$$z(\theta) = \sum_{j=1}^{\infty} a_j q^j e^{ij\theta}.$$

Therefore, putting $b_j = \Re a_j$ and $c_j = \Im a_j$, we have

$$x(\theta) = \sum_{j=1}^{\infty} q^j u_j(\theta), \quad y(\theta) = \sum_{j=1}^{\infty} q^j v_j(\theta),$$

where

$$u_j(\theta) = b_j\cos(j\theta) - c_j\sin(j\theta), \quad v_j(\theta) = b_j\sin(j\theta) + c_j\cos(j\theta).$$

Differentiate these series termwise with respect to θ; for example

$$x'(\theta) = -\sum_{j=1}^{\infty} jq^j v_j(\theta), \quad y'(\theta) = \sum_{j=1}^{\infty} jq^j u_j(\theta)$$

and so on. From (2.4) we have $|a_j| \le 2/j$, so

$$|b_j| \le 2/j, \quad |c_j| \le 2/j. \tag{7.3}$$

Noting these estimates and $q < 1$, we see that these differentiated series are convergent absolutely. Therefore $x(\theta)$, $y(\theta)$ are belonging to the C^∞-class, and so is $g_\tau(\theta)$. In particular the above termwise differentiation is valid, and we have

$$\begin{aligned} g_\tau'(\theta) &= -\sum_{j=1}^{\infty} j q^j v_j(\theta) \cos\tau + \sum_{j=1}^{\infty} j q^j u_j(\theta) \sin\tau \\ &= -q v_1(\theta)\cos\tau + q u_1(\theta)\sin\tau + E_1(q;\theta,\tau), \end{aligned} \tag{7.4}$$

where $E_1(q;\theta,\tau)$ denotes the sum corresponding to $j \ge 2$, and

$$\begin{aligned} |E_1(q;\theta,\tau)| &\le 2\sum_{j\ge 2} j q^j (|b_j| + |c_j|) \\ &\le 2\sum_{j\ge 2} j q^j \left(\frac{2}{j} + \frac{2}{j}\right) = 8\sum_{j\ge 2} q^j = \frac{8q^2}{1-q}. \end{aligned} \tag{7.5}$$

Since $q = p_n^{-\sigma} \le 2^{-1/2} = 1/\sqrt{2}$, we find that $E_1(q;\theta,\tau) = O(q^2)$ as $q \to 0$ (that is, $n \to \infty$), where the implied constant is absolute. Therefore from (7.4) we have

$$g_\tau'(\theta) = -q b_1 \sin(\theta - \tau) - q c_1 \cos(\theta - \tau) + O(q^2).$$

Write $\gamma_1 = \arg a_1$. Then $b_1 = |a_1|\cos\gamma_1$, $c_1 = |a_1|\sin\gamma_1$, and so

$$\begin{aligned} g_\tau'(\theta) &= -q|a_1|(\cos\gamma_1 \sin(\theta-\tau) + \sin\gamma_1\cos(\theta-\tau)) + O(q^2) \\ &= -q(|a_1|\sin(\gamma_1 + \theta - \tau) + O(q)). \end{aligned} \tag{7.6}$$

Similarly, one more differentiation gives

$$\begin{aligned} g_\tau''(\theta) &= -\sum_{j=1}^{\infty} j^2 q^j u_j(\theta)\cos\tau - \sum_{j=1}^{\infty} j^2 q^j v_j(\theta)\sin\tau \\ &= -q|a_1|\cos(\gamma_1 + \theta - \tau) + E_2(q;\theta,\tau), \end{aligned} \tag{7.7}$$

where $E_2(q;\theta,\tau)$, the sum corresponding to $j \geq 2$, satisfies

$$\begin{aligned}|E_2(q;\theta,\tau)| &\leq 2\sum_{j\geq 2} j^2 q^j (|b_j| + |c_j|)\\ &\leq 2\sum_{j\geq 2} j^2 q^j \left(\frac{2}{j} + \frac{2}{j}\right) = 8\sum_{j\geq 2} jq^j = \frac{8q^2(2-q)}{(1-q)^2}.\end{aligned} \tag{7.8}$$

(The proof of the last equality: Put $J = \sum_{j\geq 2} jq^j$, and observe that $J = \sum_{j\geq 1}(j+1)q^{j+1} = q\sum_{j\geq 1} jq^j + \sum_{j\geq 1} q^{j+1} = q^2 + qJ + q^2/(1-q)$.) Therefore $E_2(q;\theta,\tau) = O(q^2)$ with an absolute implied constant (by using again $q \leq 1/\sqrt{2}$), and hence

$$g''_\tau(\theta) = -q(|a_1|\cos(\gamma_1 + \theta - \tau) + O(q)). \tag{7.9}$$

Furthermore

$$g'''_\tau(\theta) = q|a_1|\sin(\gamma_1 + \theta - \tau) + E_3(q;\theta,\tau) \tag{7.10}$$

with

$$\begin{aligned}|E_3(q;\theta,\tau)| &\leq 2\sum_{j\geq 2} j^3 q^j (|b_j| + |c_j|)\\ \leq 8\sum_{j\geq 2} j^2 q^j &= 8q^2\left(\frac{3}{1-q} + \frac{1}{1-q^2} + \frac{2q(2-q)}{(1-q)^3}\right) = O(q^2)\end{aligned} \tag{7.11}$$

with an absolute implied constant. (The evaluation of $\sum_{j\geq 2} j^2 q^j$ can be done similarly to the last equality of (7.8).)

Now we assume that $p_n \in \mathbb{P}_f(\varepsilon)$, where ε is a small positive number. Recall $a_1 = \alpha_f(p) + \beta_f(p) = \lambda_f(p)$. Therefore from (2.6) we have $|a_1| > \sqrt{2} - \varepsilon$. On the other hand, the term $O(q)$ can be arbitrarily small when n is sufficiently large. Therefore from (7.9) we find that, for sufficiently large n, if $\theta = \theta_0$ is a solution of $g''_\tau(\theta) = 0$, then $\cos(\gamma_1 + \theta_0 - \tau)$ is to be close to 0. That is, writing $\theta = \theta_1^c, \theta_2^c$ be two solutions of $\cos(\gamma_1 + \theta - \tau) = 0$ in the interval $0 \leq \theta < 2\pi$, we see that θ_0 is close to θ_1^c or θ_2^c.

Now consider $g'''_\tau(\theta)$. From (7.10) and (7.11) we have

$$g'''_\tau(\theta) = q(|a_1|\sin(\gamma_1 + \theta - \tau) + O(q)).$$

Since

$$|\sin(\gamma_1 + \theta_1^c - \tau)| = |\sin(\gamma_1 + \theta_2^c - \tau)| = 1,$$

we see that $g_\tau'''(\theta) \neq 0$ around $\theta = \theta_j^c$ $(j = 1, 2)$, if $p_n \in \mathbb{P}_f(\varepsilon)$ and n is sufficiently large. This implies that $g_\tau''(\theta)$ is monotone around $\theta = \theta_j^c$. Therefore we conclude that there is at most one solution $\theta = \theta_0$ of $g_\tau''(\theta) = 0$ around θ_j^c.

Moreover, from (7.9) we see that $g_\tau''(\theta)$ is negative around the value of θ satisfying $\cos(\gamma_1 + \theta - \tau) = 1$, and is positive around the value of θ satisfying $\cos(\gamma_1 + \theta - \tau) = -1$. Therefore $g_\tau''(\theta)$ changes its sign twice in the interval $0 \leq \theta < 1$, so that the above solution θ_0 indeed exists both around θ_1^c and around θ_2^c. We denote those solutions by θ_1'' and θ_2'', respectively. That is, $g_\tau''(\theta) = 0$ has exactly two solutions in the interval $0 \leq \theta < 2\pi$. □

Remark 7.1. By the same reasoning as above, we can show that, if $p_n \in \mathbb{P}_f(\varepsilon)$ and n is sufficiently large, $g_\tau'(\theta) = 0$ also has exactly two solutions θ_1' and θ_2' in the interval $0 \leq \theta < 2\pi$. In fact, there exists two solutions $\theta = \theta_1^s, \theta_2^s$ of $\sin(\gamma_1 + \theta - \tau) = 0$ in the interval $0 \leq \theta < 2\pi$, and θ_j' is close to θ_j^s $(j = 1, 2)$. (We can further show that, for any $l \in \mathbb{N}$, there exist exactly two solutions of $g_\tau^{(l)}(\theta) = 0$.)

Now we can prove an analogue of the Jessen-Wintner inequality for automorphic L-functions. In the rest of this section, we follow the argument in the proof of [9, Theorem 12]. We use the notation defined in the proof of Lemma 7.1 and in Remark 7.1. The integral (7.2) can be rewritten as

$$K_n(w) = \frac{1}{2\pi} \int_0^{2\pi} e^{i|w|g_\tau(\theta)} d\theta. \tag{7.12}$$

Proposition 7.1. *If $p_n \in \mathbb{P}_f(\varepsilon)$ and n is sufficiently large, we have*

$$K_n(w) = O\left(\frac{1}{q^{1/2}|w|^{1/2}} + \frac{1}{q|w|}\right),$$

with the implied constant depending only on ε.

Proof. When θ moves between θ_i^s and θ_j^c $(1 \leq i, j \leq 2) \pmod{2\pi}$, the values of $\sin(\gamma_1 + \theta - \tau)$ and $\cos(\gamma_1 + \theta - \tau)$ varies continuously and monotonically, and there exists a unique value $\theta = \theta_{ij}$ between θ_i^s and θ_j^c at which

$$|\sin(\gamma_1 + \theta_{ij} - \tau)| = |\cos(\gamma_1 + \theta_{ij} - \tau)| = 1/\sqrt{2}$$

holds. We split the interval $0 \leq \theta < 2\pi \pmod{2\pi}$ into four subintervals at the values θ_{ij} $(1 \leq i, j \leq 2)$. Then on two of those subintervals (which we denote by I_A and I_B) the inequality$|\sin(\gamma_1+\theta-\tau)| \geq 1/\sqrt{2}$ holds, while on the other two subintervals (which we denote by I_C and I_D) the inequality $|\cos(\gamma_1+\theta-\tau)| \geq 1/\sqrt{2}$ holds.

Since $p_n \in \mathbb{P}_f(\varepsilon)$ and n is sufficiently large, we can again use the facts $|a_1| > \sqrt{2}-\varepsilon$ and the term $O(q)$ is small. Therefore from (7.6) we find

$$|g'_\tau(\theta)| \geq q((\sqrt{2}-\varepsilon)(1/\sqrt{2})-\varepsilon) \geq q(1-2\varepsilon) \tag{7.13}$$

for $\theta \in I_A \cup I_B$. Similarly from (7.9) we find that, for sufficiently large n,

$$|g''_\tau(\theta)| \geq q(1-2\varepsilon) \tag{7.14}$$

for $\theta \in I_C \cup I_D$.

The number θ_1^c is included in I_A or I_B, say I_A. Then $\theta_2^c \in I_B$. Therefore also $\theta_1'' \in I_A$ and $\theta_2'' \in I_B$. We split I_A into two subintervals at $\theta = \theta_1''$. Then in the both of those subintervals, $g'_\tau(\theta)$ is monotone. Therefore, applying the first derivative test (Titchmarsh [24, Lemma 4.2]) with (7.13) to those subintervals we have

$$\left|\int_{I_A} e^{i|w|g_\tau(\theta)}d\theta\right| \leq 2\cdot\frac{4}{\min\{|w||g'_\tau(\theta)|\}} \leq \frac{8}{q|w|(1-2\varepsilon)},$$

and the same inequality holds for the integral on I_B.

As for the integrals on the intervals I_C and I_D, we use the second derivative test ([24, Lemma 4.4]). The monotonicity is not required for the second derivative test, so we need not divide I_C into subintervals. Using (7.14), we have

$$\left|\int_{I_C} e^{i|w|g_\tau(\theta)}d\theta\right| \leq \frac{8}{\sqrt{|w|q(1-2\varepsilon)}},$$

and the same for I_D. Collecting these inequalities, we obtain the assertion of the proposition. □

Proposition 7.1 implies that

$$K_n(w) = O_{n,\varepsilon}(|w|^{-1/2}) \qquad (|w|\to\infty) \tag{7.15}$$

if $p_n \in \mathbb{P}_f(\varepsilon)$ and n is sufficiently large. The set $\mathbb{P}_f(\varepsilon)$ is of positive density, especially it includes infinitely many elements (so surely includes five elements). Therefore we can obviously apply Proposition 6.1 to $\varphi(s) = L_f(s)$, and the proof of Theorem 2.1 is now complete.

8. The convexity

In our proof of Theorem 2.1, the convexity of relevant curves plays no role. However the geometric property of the curve Γ_n is of independent interest. We conclude this paper with the following

Proposition 8.1. *If $p_n \in \mathbb{P}_f(\varepsilon)$ for a small positive number ε and n is sufficiently large, the curve Γ_n is a closed convex curve.*

Remark 8.1. Using [9, Theorem 13] we have that each curve Γ_n is convex if $|\xi|$ is sufficiently small. But their theorem does not give any explicit bound of $|\xi|$ (which may depend on n), so we cannot deduce the above proposition from their theorem.

Proof of Proposition 8.1. Assume $p_n \in \mathbb{P}_f(\varepsilon)$ and n is large. Then

$$u_1(\theta)^2 + v_1(\theta)^2 = b_1^2 + c_1^2 = |a_1|^2 = |\alpha_f(p) + \beta_f(p)|^2 > (\sqrt{2} - \varepsilon)^2$$

by (2.6). Therefore at least one of $|u_1(\theta)|^2$ and $|v_1(\theta)|^2$ is larger than $(\sqrt{2} - \varepsilon)^2/2$, that is, at least one of $|u_1(\theta)|$ and $|v_1(\theta)|$ is larger than $(\sqrt{2}-\varepsilon)/\sqrt{2} > 1 - \varepsilon$. Let

$$\Theta(u_1, n) = \{\theta \in [0, 2\pi) \mid |u_1(\theta)| > 1 - \varepsilon\},$$
$$\Theta(v_1, n) = \{\theta \in [0, 2\pi) \mid |v_1(\theta)| > 1 - \varepsilon\}.$$

Then $\Theta(u_1, n) \cup \Theta(v_1, n) = [0, 2\pi)$.

First consider the case when $\theta \in \Theta(v_1, n)$. The curve Γ_n consists of the points $z(\theta) = x(\theta) + iy(\theta)$. We identify $\mathbb{C}$ with the $\mathbb{R}^2$-space $\{(x, y) \mid x, y \in \mathbb{R}\}$, and identify $z(\theta)$ with $(x(\theta), y(\theta))$. We study the behavior of the tangent line of the planar curve Γ_n at $z(\theta)$, when θ varies. By $\Xi(\theta)$ we denote the tangent of the angle of inclination of the tangent line $z(\theta)$. Then

$$\Xi(\theta) = \frac{y'(\theta)}{x'(\theta)} = -\left(\sum_{j=1}^{\infty} jq^j u_j(\theta)\right) \Bigg/ \left(\sum_{j=1}^{\infty} jq^j v_j(\theta)\right). \tag{8.1}$$

It is to be noted that the denominator is $qv_1(\theta) + O(q^2)$, so this is non-zero for sufficiently small q (that is, sufficiently large n), because now we assume$\theta \in \Theta(v_1, n)$.

We evaluate $\Xi'(\theta)$. First, by differentiation we have

$$\Xi'(\theta) = X_1(\theta) + X_2(\theta) + X_3(\theta) + X_4(\theta), \tag{8.2}$$

say, where

$$X_1(\theta) = qv_1(\theta) \Bigg/ \left(\sum_{j=1}^{\infty} jq^j v_j(\theta) \right),$$

$$X_2(\theta) = \left(\sum_{j=2}^{\infty} j^2 q^j v_j(\theta) \right) \Bigg/ \left(\sum_{j=1}^{\infty} jq^j v_j(\theta) \right),$$

$$X_3(\theta) = (qu_1(\theta))^2 \Bigg/ \left(\sum_{j=1}^{\infty} jq^j v_j(\theta) \right)^2,$$

and

$$X_4(\theta) = \left(\sum_{\substack{j,k\in\mathbb{N} \\ j+k\geq 3}} jk^2 q^{j+k} u_j(\theta) u_k(\theta) \right) \Bigg/ \left(\sum_{j=1}^{\infty} jq^j v_j(\theta) \right)^2.$$

We write

$$\sum_{j=1}^{\infty} jq^j v_j(\theta) = qv_1(\theta)(1 + Y(\theta)), \tag{8.3}$$

where

$$Y(\theta) = \sum_{j=2}^{\infty} jq^{j-1} \frac{v_j(\theta)}{v_1(\theta)}.$$

Since $|v_1(\theta)| > 1 - \varepsilon$, using (7.3) we have

$$|Y(\theta)| \leq \frac{4}{1-\varepsilon} \sum_{j=2}^{\infty} q^{j-1} = \frac{4q}{(1-\varepsilon)(1-q)} = O(q)$$

(noting q is small). Therefore

$$\begin{aligned} \left(\sum_{j=1}^{\infty} jq^j v_j(\theta) \right)^{-1} &= \frac{1}{qv_1(\theta)} \left(1 - \frac{Y(\theta)}{1+Y(\theta)} \right) \\ =& \frac{1}{qv_1(\theta)} + O\left(\frac{1}{q(1-\varepsilon)} \frac{|Y(\theta)|}{1-|Y(\theta)|} \right) = \frac{1}{qv_1(\theta)} + O(1). \end{aligned} \tag{8.4}$$

This implies

$$X_1(\theta) = 1 + O(q). \tag{8.5}$$

The numerator of $X_2(\theta)$ can be evaluated, as in (7.8), by $O(q^2)$. Therefore with (8.4) (whose right-hand side is $O(q^{-1})$) we have

$$X_2(\theta) = O(q^2 \cdot q^{-1}) = O(q). \tag{8.6}$$

As for $X_3(\theta)$, again using $|v_1(\theta)| > 1 - \varepsilon$ and (8.4) we obtain

$$X_3(\theta) = \frac{u_1(\theta)^2}{v_1(\theta)^2}\left(1 - \frac{Y(\theta)}{1+Y(\theta)}\right)^2 = \frac{u_1(\theta)^2}{v_1(\theta)^2} + O(q). \tag{8.7}$$

Lastly, we have

$$X_4(\theta) \ll \sum_{\substack{j,k\in\mathbb{N}\\ j+k\geq 3}} kq^{j+k} \cdot q^{-2} \ll q, \tag{8.8}$$

because

$$\begin{aligned}\sum_{\substack{j,k\in\mathbb{N}\\ j+k\geq 3}} kq^{j+k} &= \sum_{j\geq 1} q^j \sum_{k\geq \max\{1,3-j\}} kq^k = q\sum_{k\geq 2} kq^k + \sum_{j\geq 2} q^j \sum_{k\geq 1} kq^k \\ &= qJ + (q+J)\sum_{j\geq 2} q^j = O(q^3)\end{aligned}$$

(where J was defined just after (7.8)). Collecting (8.2), (8.5), (8.6), (8.7) and (8.8), we obtain

$$\Xi'(\theta) = 1 + \frac{u_1(\theta)^2}{v_1(\theta)^2} + O(q). \tag{8.9}$$

Note that all the implied constants in the above formulas are absolute. When n is large, $O(q)$ becomes small, so (8.9) implies that $\Xi'(\theta) > 0$. That is, if $p_n \in \mathbb{P}_f(\varepsilon)$, n is sufficiently large, and $\theta \in \Theta(v_1, n)$, then $\Xi(\theta)$ is monotonically increasing.

In the case when $\theta \in \Theta(u_1, n)$, we change the roles of the axes. That is, now we identify $z(\theta) \in \mathbb{C}$ with $(-y(\theta), x(\theta)) \in \mathbb{R}^2$. Instead of $\Xi(\theta)$, we consider $\Xi^*(\theta) = x'(\theta)/y'(\theta)$. (The denominator $y'(\theta)$ is non-zero for large n because $\theta \in \Theta(u_1, n)$.) Then $-\Xi^*(\theta)$ is the tangent of the angle of inclination of the tangent line, under this new choice of the axes. We can proceed similarly, and obtain, analogously to (8.9),

$$(-\Xi^*(\theta))' = 1 + \frac{v_1(\theta)^2}{u_1(\theta)^2} + O(q), \tag{8.10}$$

hence $-\Xi^*(\theta)$ is monotonically increasing when $\theta \in \Theta(u_1, n)$. Therefore the tangent of the angle of inclination is always increasing, which implies that the curve Γ_n is convex. $\square$

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