J. reine angew. Math. **548** (2002), 149—166

Journal für die reine und angewandte Mathematik

Irreducible subfactors of $L(\mathbb{F}_\infty)$ of index $\lambda > 4$

By *Dimitri Shlyakhtenko* at Los Angeles and *Yoshimichi Ueda* at Fukuoka

---

Abstract. By utilizing an irreducible inclusion of type III_{q^2} factors coming from a free-product type action of the quantum group $\mathrm{SU}_q(2)$, we show that the free group factor $L(\mathbb{F}_\infty)$ possesses irreducible subfactors of arbitrary index > 4. Combined with earlier results of Radulescu, this shows that $L(\mathbb{F}_\infty)$ has irreducible subfactors with any index value in $\{4\cos^2(\pi/n)\colon n \geqq 3\} \cup [4, +\infty)$.

1. Introduction

Let M be a type II_1 factor, and let $N \subset M$ be a finite index subfactor. Recall that $N \subset M$ is called *irreducible*, if the relative commutant $N' \cap M$ is trivial.

In his fundamental paper [8], V. F. R. Jones introduced the notion of index $[M : N]$ of a subfactor $N \subset M$. Related to the index, he introduced two invariants of a factor M:

$$\mathscr{I}(M) = \{\lambda \in \mathbb{R}_+\colon \exists N \subset M \text{ subfactor, } [M : N] = \lambda\},$$

$$\mathscr{C}(M) = \{\lambda \in \mathbb{R}_+\colon \exists N \subset M \text{ irreducible subfactor, } [M : N] = \lambda\},$$

measuring the possible index values of subfactors of M.

In the same paper, Jones showed that for any factor M, the index of a subfactor cannot be arbitrary; in fact, $\mathscr{C}(M) \subset \mathscr{I}(M) \subset \mathscr{I} = \{4\cos^2(\pi/n)\colon n \geqq 3\} \cup [4, +\infty]$. Moreover, he produced subfactors of the hyperfinite II_1 factor R of arbitrary allowable index, thus showing that $\mathscr{I}(R) = \mathscr{I}$. However, the subfactors of R with index > 4 that he constructed are not irreducible; to this day, it is still not known whether the equality $\mathscr{C}(R) = \mathscr{I}$ holds; in other words, whether every index value can be realized by an irreducible subfactor of the hyperfinite factor. The one general positive result is due to Popa [12] (see also [13] and [14]), stating that given a number $\lambda \in \mathscr{I}$, there is a (non-hyperfinite) II_1 factor M, for which $\lambda \in \mathscr{C}(M)$.

Thanks to the theory of free probability, pioneered by Voiculescu in the early 80s (see e.g. [25]), there has been much progress at understanding free group factors $L(\mathbb{F}_n)$ (Voiculescu's introduction to [25] states that these are the "best" of the "bad" non-hyperfinite factors). Following this philosophy, Radulescu undertook a study of subfactors of free group factors. Using a particular version of Popa's construction [16] and by employing free probability and random matrix techniques, he managed to show that for an interpolated free group factor $L(\mathbb{F}_t)$, $\mathscr{C}\big(L(\mathbb{F}_t)\big) \cap [0,4] = \mathscr{I}\big(L(\mathbb{F}_t)\big) \cap [0,4] = \mathscr{I} \cap [0,4]$. Coupled with his earlier result [15] on the fundamental group of $L(\mathbb{F}_\infty)$ this gave that $L(\mathbb{F}_\infty)$ has (possibly non-irreducible) subfactors of all indices, i.e., $\mathscr{I}\big(L(\mathbb{F}_\infty)\big) = \mathscr{I}$. However, this still left the question about whether subfactors of $L(\mathbb{F}_\infty)$ of index > 4 can be chosen to be irreducible.

The main theorem of this paper is

Theorem. *For each* $\lambda > 4$, *there exists an irreducible subfactor of* $L(\mathbb{F}_\infty)$ *of index* λ.

Combined with earlier results of F. Radulescu, this gives:

Corollary. $\mathscr{C}\big(L(\mathbb{F}_\infty)\big) = \mathscr{I}\big(L(\mathbb{F}_\infty)\big) = \mathscr{I}$.

In fact, more is true: given an allowable index value $\lambda \in \mathscr{I}$, there exists an irreducible subfactor $L(\mathbb{F}_\infty) \cong N \subset M \cong L(\mathbb{F}_\infty)$ of index λ.

We would like to point out that it is still not known whether every Popa's λ-lattice [13] (i.e., an abstract system of higher relative commutants of a subfactor) can be realized by a subfactor of some fixed universal II_1 factor M. The factor $M = L(\mathbb{F}_\infty)$ seems to be a likely candidate, although we don't know how to prove it[1)]; note, however, that Radulescu gave an affirmative answer to this question in the case that the λ-lattice is finite-depth. Also, by the results of this paper, all examples of irreducible subfactors whose λ-lattices come from the representation theory of $\mathrm{SU}_q(2)$ (see [2], [30], [27]) can be realized as $N \subset M$ with $N \cong M \cong L(\mathbb{F}_\infty)$.

We don't know whether our result holds for $L(\mathbb{F}_n)$, $n < \infty$.

The strategy to find irreducible subfactors of $L(\mathbb{F}_\infty)$ with index $\lambda > 4$ comes from the work of the second-named author [24] on quantum $\mathrm{SU}_q(2)$ actions on free products of von Neumann algebras, and analysis of resulting Wassermann-type inclusions of type III factors. As was pointed out in [24], Question 2, to prove the main theorem of the present paper, it is sufficient to identify a certain crossed product of a certain von Neumann algebra by $\mathrm{SU}_q(2)$, which we do in Section 2 by identifying this algebra as a free Araki-Woods factor [18]. For the convenience of the reader, we summarize the main necessary facts from [24] in Section 3.

Acknowledgement. The authors would like to express their sincere gratitude to MSRI and the organizers of the year-long program "Operator Algebras (2000–2001)" held at MSRI for inviting them to the program, where this joint work (although conceived some time ago) was carried out and completed.

1) **Note added in proof.** After this paper was written, S. Popa and the first-named author proved that, indeed, any λ-lattice can be realyzed by a subfactor of $L(\mathbb{F}_\infty)$.

D.S. would like to acknowledge support by a National Science Foundation Postdoctoral Fellowship. Y.U. would like to acknowledge support by the Japanese Ministry of Education, Science, Sports and Culture.

2. Amalgamated free products of free group factors with $B(H)$

In this section, we prove some technical results about amalgamated free products of free group factors with $B(H)$, identifying them as free Araki-Woods factors. The treatment given in this section is more general than strictly speaking necessary for our result, but allows for more succinct and abstract proofs. We encourage the interested reader to look through Appendix I, which can serve as an illustration of the proof given here.

We recall some notation from [20]. Let M be a von Neumann algebra and

$$\eta\colon M \to M \otimes B(\ell^2)$$

be a normal completely positive map, given as a matrix $\eta(m) = \big(\eta_{ij}(m)\big)_{ij}$. Let K be the M-Hilbert bimodule spanned by vectors ξ_i, and satisfying

$$\langle \xi_i, m\xi_j \rangle_M = \eta_{ij}(m), \quad \forall m \in M$$

(see [20], Lemma 2.2).

Let $\mathscr{T}$ be the universal C*-algebra generated by M and operators $L(\zeta)\colon \zeta \in K$ satisfying

$$L(\zeta)^* m L(\zeta') = \langle \zeta, m\zeta' \rangle, \quad \forall m \in M,\ \zeta, \zeta' \in K.$$

This is the Toeplitz extension of the Cuntz-Pimsner C*-algebra associated to the M-bimodule H (see [11]). The operators $L(\zeta)$ are called η-creation operators over M.

Set $L_i = L(\xi_i)$. Let E_M denote the canonical conditional expectation from $\mathscr{T}$ onto M (see [11], [20]). Choose a faithful normal state ϕ_0 on M and set $\phi = \phi_0 \circ E_M$. In the GNS representation associated to ϕ, consider the von Neumann algebra

$$\Phi(M, \eta) = W^*(M, L_i + L_i^*\colon i = 1, 2, \ldots).$$

It can be shown ([20]) that $\Phi(M, \eta)$ depends only on M and (η_{ij}).

Let $K^\eta \subset K$ be the $\mathbb{R}$-linear subspace of the Hilbert M-bimodule K, given as the norm closure of the set of vectors

$$\sum_{\text{finite}} m_j \xi_j n_j^* + n_j \xi_j m_j^*, \quad m_j, n_j \in M.$$

Then ([20], Proposition 2.19) $\Phi(M, \eta)$ depends only on the inclusion $K^\eta \subset K$. In fact

$$\Phi(M, \eta) = W^*\big(M, L(\zeta) + L(\zeta)^*\colon \zeta \in K^\eta \subset K\big) \tag{2.1}$$

(in the GNS representation associated to ϕ).

In the particular case that $M = \mathbb{C}$, the matrix $\eta_{ij}(1)$ defines a scalar-valued inner product on K, and $K^\eta \subset K$ ends up a particular real subspace. In this case, under the assumption that the state $E_M = E_\mathbb{C}$ is faithful, $\Phi(M, \eta)$ is nothing other than the free Araki-Woods factor $\Gamma(K^\eta \subset K)''$ in the notation of Remark 2.6 of [18] (see also [20], Example 3.5).

A very particular example of such a matrix μ is

$$\mu_\lambda = \begin{pmatrix} 1 & -i\dfrac{\lambda - 1}{\lambda + 1} \\ i\dfrac{\lambda - 1}{\lambda + 1} & 1 \end{pmatrix}, \quad 0 < \lambda < 1;$$

in this case $\Phi(\mathbb{C}, \mu_\lambda)$ is of type III_λ and is the unique free Araki-Woods factor T_λ of this type.

Theorem 2.1. *Let H be a finite or infinite dimensional Hilbert space. Let*

$$(\eta_{ij})\colon B(H) \to B(H) \otimes B(\ell^2)$$

be any normal completely positive map. Assume that $E_{B(H)}\colon \Phi\big(B(H), \eta\big) \to B(H)$ is faithful. Then the von Neumann algebra $\Phi\big(B(H), \eta\big)$ is stably isomorphic to the von Neumann algebra $\Phi(\mathbb{C}, \mu)$ for some (completely) positive map $\mu\colon \mathbb{C} \to B(\ell^2)$; moreover, $E_\mathbb{C}\colon \Phi(\mathbb{C}, \mu) \to \mathbb{C}$ is faithful.

Proof. By equation (2.1),

$$N = \Phi\big(B(H), \eta\big) = W^*\big(B(H), L(\zeta) + L(\zeta)^*\colon \zeta \in K^\eta \subset K\big),$$

where K and K^η are as described above. Let e_{ij}, $0 \leqq i, j < \infty$ be a family of matrix units generating $B(H)$; thus

$$e_{ij} e_{j'i'} = \delta_{jj'} e_{ii'}, \quad e_{ij}^* = e_{ji}.$$

Consider the algebra

$$P = e_{00} N e_{00}.$$

It is clear that $P \otimes B(\ell^2) \cong N \otimes B(\ell^2)$; thus it is sufficient to prove that P is isomorphic to $\Phi(\mathbb{C}, \mu)$ for some μ. Note that P is generated by the family

$$e_{0i}\big(L(\zeta) + L(\zeta)^*\big) e_{j0}, \quad 0 \leqq i \leqq j, \ \zeta \in K^\eta;$$

hence

$$P = W^*\big(L(e_{0i}\zeta e_{j0}) + L^*(e_{0j}\zeta e_{i0})\colon \zeta \in K^\eta\big).$$

Consider the linear space

$$V_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\big(\{e_{0i}\zeta e_{i0} : \zeta \in K^\eta, i \geqq 0\} \cup \{e_{0i}\zeta e_{j0} + e_{0j}\zeta e_{i0} : \zeta \in K^\eta, 0 \leqq i < j\} \cup \{\sqrt{-1}(e_{0i}\zeta e_{j0} - e_{0j}\zeta e_{i0}) : \zeta \in K^\eta, 0 \leqq i < j\}\big) \subset K^\eta,$$

as a subspace of the complex linear space

$$V = \mathrm{span}\{e_{0i}\zeta e_{j0} : \zeta \in K\} \subset K.$$

Since for $i < j$,

$$\begin{aligned}\big(L(e_{0i}\zeta e_{j0}) + L^*(e_{0j}\zeta e_{i0})\big) = \frac{1}{2}L(e_{0i}\zeta e_{j0} + e_{0j}\zeta e_{i0}) + L(e_{0i}\zeta e_{j0} + e_{0j}\zeta e_{i0})^* \\ -\frac{1}{2}\sqrt{-1}\big(L\big(\sqrt{-1}(e_{0i}\zeta e_{j0} - e_{0j}\zeta e_{i0})\big) \\ + L\big(\sqrt{-1}(e_{0i}\zeta e_{j0} - e_{0j}\zeta e_{i0})\big)^*\big),\end{aligned}$$

it follows that

$$P = W^*\big(L(\zeta) + L(\zeta)^* : \zeta \in V_{\mathbb{R}}\big).$$

Since for any $\zeta, \zeta' \in V$, $e_{00}\zeta e_{00} = \zeta$ and $e_{00}\zeta' e_{00} = \zeta'$, we get that

$$\langle \zeta, \zeta' \rangle_{B(H)} = \langle e_{00}\zeta e_{00}, e_{00}\zeta' e_{00} \rangle = e_{00}\langle \zeta, \zeta' \rangle e_{00} \in \mathbb{C}e_{00}.$$

It follows that the restriction of the inner product on K to V is scalar-valued; we denote the restriction by $\langle \cdot, \cdot \rangle_V$. Hence we get an inclusion of a real Hilbert space into a complex Hilbert space

$$V_{\mathbb{R}} \subset V.$$

In particular, for any $\zeta, \zeta' \in V$, we have

$$L(\zeta)^* L(\zeta') = \langle \zeta, \zeta' \rangle_V. \tag{2.2}$$

Denote by θ the state $P \ni m \mapsto e_{00}E_{B(H)}(m)e_{00}$. Since $E_{B(H)}$ is faithful by assumption, and $\theta(e_{00}me_{00}) = E_{B(H)}(e_{00}me_{00})$, we get that θ is a faithful state. Furthermore,

$$\theta\big(L(\zeta_1)\cdots L(\zeta_k)L(\zeta_{k+1})^*\cdots L(\zeta_r)^*\big) = 0 \tag{2.3}$$

if $r \neq 0$; note that this, together with the relations (2.2) determines θ on P.

Choose now a basis ζ_i for $V_{\mathbb{R}}$ as a real Hilbert space, and let

$$\mu_{ij} = \langle \zeta_i, \zeta_j \rangle_V.$$

Then $\Phi(\mathbb{C}, \mu)$ is generated in the GNS representation associated to $E_{\mathbb{C}}$ by $l_i + l_i^*$, subject to the relations

$$l_i^* l_j = \mu_{ij} = \langle \zeta_i, \zeta_j \rangle_V.$$

Moreover, $\psi = E_{\mathbb{C}}$ is determined by

$$\psi(l_{i_1} l_{i_2} \cdots l_{i_k} l_{i_{k+1}}^* \cdots l_{i_r}^*) = 0, \quad r \neq 0.$$

Comparing these with (2.2) and (2.3), and using the fact that θ is faithful, we obtain that the map $L(\zeta_i) + L(\zeta_i)^* \mapsto l_i + l_i^*$ extends to an isomorphism

$$P \cong W^*(l_i + l_i^* : i = 1, 2, \ldots) = \Phi(\mathbb{C}, \mu),$$

which conjugates θ and $\psi = E_{\mathbb{C}}$. It also follows that $E_{\mathbb{C}}$ is faithful. □

We can now deduce the main technical result needed for further computations; for convenience, we write $L(\mathbb{F}_1) = L(\mathbb{Z})$.

Theorem 2.2. *Let H be a separable Hilbert space (finite or infinite-dimensional). Let $B \subset B(H)$ be a von Neumann subalgebra, and assume that E: $B(H) \to B$ is a normal faithful conditional expectation. For $m = 1, 2, \ldots$ or $+\infty$, consider the reduced amalgamated free product*

$$N = \big(L(\mathbb{F}_m) \otimes B, \tau \otimes \mathrm{id}\big) *_B \big(B(H), E\big).$$

Then N is stably isomorphic to a free Araki-Woods factor $\Gamma(\mathscr{H}_{\mathbb{R}}, U_t)''$.

In particular, if N is of type III_λ, $0 < \lambda < 1$, then N is isomorphic to the unique type III_λ free Araki-Woods factor T_λ.

Proof. Let η: $B(H) \to B(H) \otimes B(\ell^2)$ be given by

$$\eta(T) = \mathrm{diag}\big(E(T), E(T), \ldots, E(T), 0, 0, \ldots\big)$$

(m copies of $E(T)$). Since $\eta_{ij} = \delta_{ij} E$, the canonical conditional expectation

$$E_{B(H)}\colon \Phi\big(B(H), \eta\big) \to B(H)$$

is known to be faithful (see e.g. Appendix II). By [20], Example 3.3(c),

$$\Phi\big(B(H), \eta\big) \cong \big(B(H), E\big) *_B \big(\Phi(B, \eta|_B), E_B\big).$$

Moreover, by [20], Example 3.3(b),

$$\big(\Phi(B, \eta|_B), E_B\big) \cong \big(\Phi(\mathbb{C}, \eta|_{\mathbb{C}1_B}) \otimes B, E_{\mathbb{C}} \otimes \mathrm{id}\big) \cong \big(L(\mathbb{F}_m) \otimes B, \tau \otimes \mathrm{id}\big).$$

Hence

$$\Phi\big(B(H), \eta\big) \cong \big(L(\mathbb{F}_m) \otimes B, \tau \otimes \mathrm{id}\big) *_B \big(B(H), E\big) = N.$$

Applying Theorem 2.1, we get that

$$N \otimes B(\ell^2) \cong \Phi(\mathbb{C}, \mu) \otimes B(\ell^2)$$

for some μ. It follows that $\Phi(\mathbb{C}, \mu) \cong \Gamma(\mathscr{H}_{\mathbb{R}}, U_t)''$ for some real Hilbert space $\mathscr{H}_{\mathbb{R}}$ and one-parameter group U_t (in fact, $\mathscr{H}_{\mathbb{R}}$ is the real subspace $V_{\mathbb{R}}$ constructed in the proof of Theorem 2.1).

If N is of type III_λ, $0 < \lambda < 1$, it follows from the uniqueness of the type III_λ free Araki-Woods factor T_λ [18] that $N \cong T_\lambda$. $\square$

The following theorem can be deduced from the results of Radulescu [17] and Dykema [6], as was done in [18]. We have since found a somewhat shorter argument, which we give below:

Theorem 2.3. *Let T_λ be the (unique) free Araki-Woods factor of type III_λ. Let ϕ be any normal faithful state, so that the modular group σ_t^ϕ is periodic of period $-2\pi/\log\lambda$. Then the centralizer T_λ^ϕ is isomorphic to $L(\mathbb{F}_\infty)$. Also, the core of T_λ is isomorphic to $L(\mathbb{F}_\infty) \otimes B(H)$.*

Proof. Clearly, only the statement about the centralizer needs to be proved, since the core is isomorphic to the centralizer, tensor $B(H)$, as soon as the centralizer is a factor (see [4]). Also, we can restrict ourselves to dealing with a particular choice of the state ϕ, satisfying the hypothesis of this theorem; indeed, by [4], all centralizers of such states are stably isomorphic. Since by Radulescu's results [15], the fundamental group of $L(\mathbb{F}_\infty)$ is all of $(0, +\infty)$, a II_1 factor M is stably isomorphic to $L(\mathbb{F}_\infty)$ iff it is actually isomorphic to $L(\mathbb{F}_\infty)$.

Recall [18] that T_λ can be described as the free product

$$L^\infty[0,1] * \big(B(H), \theta\big)$$

where θ is a normal faithful state on $B(H)$ given by

$$\theta(T) = \mathrm{Tr}(DT),$$

where

$$D = \mathrm{diag}\big(1-\lambda, \lambda(1-\lambda), \dots\big).$$

Consider now the completely-positive map η given by $\eta(T) = \theta(T)$, and consider $\Phi\big(B(H), \eta\big) = W^*\big(B(H), L + L^*\big)$, where $L^* T L = \eta(T) = \theta(T)$ for all $T \in B(H)$. By [20], Example 3.3(a), $\Phi\big(B(H), \eta\big) \cong \big(B(H), \theta\big) * (L^\infty[0,1],\ \text{Lebesgue measure}) \cong (T_\lambda, \phi_\lambda)$.

Note that $B(H)$ is generated, as a von Neumann algebra, by a partial isometry v, so that $v^* v = 1$ and $vv^* = \mathrm{diag}(0, 1, 1, \dots)$. Hence T_λ is generated by $L + L^*$ and v. The modular group of ϕ_λ acts by fixing $L + L^*$ and sending v to $\lambda^{it} v$. The centralizer of the free product state is then isomorphic to the von Neumann algebra generated by all elements of the form

$$v^k (L + L^*)(v^*)^k, \quad (v^*)^l (L + L^*) v^l, \quad k, l = 0, 1, \dots$$

as well as the projections $p_k = v^k (v^*)^k$.

Let for $k \geqq 0$

$$L_k = v^k L (v^*)^k$$

and for $-l < 0$,

$$L_{-l} = (v^*)^l L v^l.$$

Denote by Ψ the faithful normal conditional expectation

$$\Psi(m) = \int_0^{-2\pi/\log\lambda} \sigma_t^{\phi_\lambda}(m)\,dt, \quad \Psi\colon T_\lambda \to T_\lambda^{\phi_\lambda}.$$

It follows that any $m \in T_\lambda$ can be written as the sum (convergent in $L^2(T_\lambda, \phi_\lambda)$)

$$m = \sum_{k \geqq 0} v^k m_k + \sum_{k<0} m_k (v^*)^k, \quad m_k = \Psi\big((v^*)^{-k} m\big),\ k \geqq 0,\ m_k = \Psi(m v^{-k}),\ k < 0,$$

with $m_k \in T_\lambda^{\psi_\lambda}$. The centralizer is linearly spanned by words of the form

$$v^{k_1}(v^*)^{l_1} X v^{k_2}(v^*)^{l_2} X \cdots v^{k_p}(v^*)^{l_p}, \quad \sum k_j - \sum l_j = 0,$$

where $X = L + L^*$. Using the notation

$$X_k = L_k + L_k^*, \quad k \in \mathbb{Z}$$

and the relation $v^* v = 1$, $v^k (v^*)^k = p_k \in D$, we get that the centralizer is generated by

$$D \quad \text{and} \quad \{X_k\colon k \in \mathbb{Z}\}.$$

Note that for all diagonal elements $d \in D = W^*(p_k\colon k \geqq 0)$,

$$L_k^*\, d L_{k'} = v^k \delta_{kk'} \theta\big((v^*)^k\, d v^k\big)(v^*)^k,$$

with the convention that $v^{-1} = v^*$.

By [22] and [20], Example 2.6, L_k are free with amalgamation over D.

Let $M_1 = W^*(D, L_k + L_k^*, k \leqq 0)$. Since when $k \leqq 0$,

$$L_k^*\, d L_k = C_k \theta(d),$$

for some nonzero constants C_k, we get by [19], Theorem 2.3 that L_k is $*$-free from D, and

$$M_1 = (D, \theta) * \Big(\mathop{*}_{k \leqq 0} W^*(L_k + L_k^*) \Big).$$

Since $L_k + L_k^*$ is semicircular, we get

$$M_1 = (D, \theta) * L(\mathbb{F}_\infty) \cong L(\mathbb{F}_\infty),$$

by the results of Ken Dykema [5].

Now, $M = M_1 *_D \big(W^*(L_k + L_k^*, k > 0)\big)$. Since M_1 is a factor and L_k are ω-creation operators over M, where $\omega_{ij} = \delta_{ij} C_i p_i \theta\big((v^*)^i \, dv^i\big)$, we get by [19], Proposition 5.4 that $M \cong M_1 * L(\mathbb{F}_t)$, where $t = \sum_i \theta\big(v^i (v^*)^i\big)^2$. Hence $M \cong L(\mathbb{F}_\infty)$. $\square$

3. Quantum SU_q actions

The main goal of this section is to analyze inclusions of type III factors coming from a minimal free product type action of $\mathrm{SU}_q(2)$ on a von Neumann algebra.

Let (A, δ) be the Hopf-von Neumann algebra of Woronowicz's quantum group $\mathrm{SU}_q(2)$ [29], [28], and denote by h the canonical Haar state on A. Let V be the multiplicative unitary, i.e., the unitary on $L^2(A) \otimes L^2(A)$ defined by

$$V\big(\Lambda_h(a) \otimes \zeta\big) = \delta(a)\big(\Lambda_h(1) \otimes \zeta\big),$$

and let W be the fundamental unitary, i.e., the unitary on $L^2(A) \otimes L^2(A)$ defined by

$$W\big(\eta \otimes \Lambda_h(a)\big) = \delta(a)\big(\eta \otimes \Lambda_h(1)\big),$$

where $\Lambda_h \colon A \to L^2(A)$ is the canonical injection associated with the Haar state h. The dual Hopf-von Neumann algebra $\hat{A}$ is the von Neumann algebra acting on $L^2(A)$ generated by elements of the form:

$$(\mathrm{Id} \otimes \omega)(V), \quad \omega \in B\big(L^2(A)\big)_*.$$

Let m be an integer $\geqq 2$ or $+\infty$ and set

$$M = \big(L(\mathbb{F}_m), \tau\big) * (A, h)$$

and let $\Gamma \colon M \to M \otimes A$ be the free product of the trivial action of $\mathrm{SU}_q(2)$ on $L(\mathbb{F}_m)$, and the left regular representation action δ of $\mathrm{SU}_q(2)$ on A [24], §3.

Recall that the crossed-product $M \rtimes_\Gamma \mathrm{SU}_q(2)$ is the von Neumann subalgebra of $M \otimes B\big(L^2(A)\big)$ generated by $\Gamma(M)$ and $\mathbb{C}1 \otimes \hat{A}$.

For each irreducible finite-dimensional unitary representation $\pi \colon V_\pi \to V_\pi \otimes A$ of $\mathrm{SU}_q(2)$, consider the Wassermann-type inclusion [26] of von Neumann algebras

$$\mathcal{N} = \mathbb{C}1 \otimes M^\Gamma \subset \big(B(V_\pi) \otimes M\big)^{\mathrm{Ad}\,\pi \otimes \Gamma} = \mathcal{M}.$$

It turns out that the negative (normalized) q-trace $\tau_q^{(\pi,-)}$ gives rise to a finite-index conditional expectation $\mathscr{E} \colon \mathcal{M} \to \mathcal{N}$ (see [24], eq. (3.6)). We collect below some facts about this inclusion, most of which are from [24].

Theorem 3.1. *Let π be an irreducible representation of* $\mathrm{SU}_q(2)$ *and let $\mathcal{N} \subset \mathcal{M}$ be as above, taken with the conditional expectation $\mathscr{E}$. Then*:

(a) *$\mathcal{N}$ and $\mathcal{M}$ are type III_{q^2} factors and $\mathcal{N} \cong \mathcal{M}$.*

(b) *The inclusion $\mathcal{N} \subset \mathcal{M}$ is irreducible.*

(c) *The index of $\mathcal{E}$ is the square of the q-dimension of π, $(\dim_q \pi)^2$.*

(d) *Let ϕ denote the restriction of the free product state $\tau * h$ to $\mathcal{N} = M^\Gamma \subset M$. Then the centralizers $\hat{\mathcal{N}} = \mathcal{N}^\phi$ and $\hat{\mathcal{M}} = \mathcal{M}^{\phi\circ\mathcal{E}}$ are both factors of type II_1.*

(e) *The inclusion $\mathcal{N} \subset \mathcal{M}$ is "essentially type II". More precisely, the restriction of $\mathcal{E}$ to the inclusion $N \subset M$ is trace-preserving, and*

$$\hat{\mathcal{N}} = \mathcal{N}^\phi \subset \mathcal{M}^{\phi\circ\mathcal{E}} = \hat{\mathcal{M}}$$

is an inclusion of type II_1 factors with the same index and the same system of higher relative commutants as $\mathcal{N} \subset \mathcal{M}$. In particular, $\hat{\mathcal{N}} \subset \hat{\mathcal{M}}$ is irreducible.

(f) *The algebras $\mathcal{N} = M^\Gamma$ and $M \rtimes_\Gamma \mathrm{SU}_q(2)$ are isomorphic.*

Proof. The fact that $\mathcal{N}$ is of type III_{q^2} was proved in [24]; this fact also follows from the explicit description of $\mathcal{N}$ given in Appendix I.

For the convenience of the reader we give an expanded proof of the isomorphism $\mathcal{N} \cong \mathcal{M}$ (see [24], Remark 10). Let n be the dimension of V_π as a vector space. Choose a standard orthonormal system of vectors ξ_i, $1 \leqq i \leqq n$ for the space V_π so that the (co)-representation π is given by

$$\pi(\xi_j) = \sum_{i=1}^n \xi_i \otimes u_{ij} \in V_\pi \otimes A,$$

for some $u_{ij} \in A$ satisfying $\sum_k u_{ki}^* u_{kj} = \delta_{ij}$.

Since $\mathcal{N} = M^\Gamma \subset M$ is of type III, we can find isometries $S_i \in M^\Gamma$, $i = 1, \dots, n$ so that $\sum S_i S_i^* = 1$ and $S_i^* S_j = \delta_{ij}$. Set

$$v_j = \sum_{i=1}^n S_i \cdot u_{ij} \in M^\Gamma \cdot A \subset M, \quad H = \mathrm{span}_{\mathbb{C}}\{v_j \colon 1 \leqq j \leqq n\}.$$

Then $v_i^* v_j = \delta_{ij}$ and $\sum v_i v_i^* = 1$. Moreover, since $S_i \in M^\Gamma$ are fixed by Γ, we get (since Γ is an action and $\Gamma|_A = \delta$)

$$\begin{aligned}
\Gamma(v_j) &= \sum_i (S_j \otimes 1) \cdot \Gamma(u_{ij}) \\
&= \sum_i (S_j \otimes 1) \cdot \delta(u_{ij}) \\
&= \sum_{i,k} (S_j \otimes 1) \cdot (u_{ki} \otimes u_{ij}) \\
&= \sum_i V_i \otimes u_{ij}.
\end{aligned}$$

It follows that the isomorphism $H \ni v_i \mapsto \xi_i \in V_\pi$ is equivariant with respect to the restriction of Γ to H and the (co)representation π on V. It follows that the isomorphism

$$\Phi\colon B(V_\pi) \otimes M \to M, \quad \Phi\big((q_{ij})_{ij}\big) = \sum_{ij} v_i q_{ij} v_j^*$$

is equivariant with respect to the action $\mathrm{Ad}\,\pi \otimes \Gamma$ on $B(V_\pi) \otimes M$ and the action Γ on M. Thus Φ restricts to an isomorphism of $\mathcal{N} = M^\Gamma$ and $\mathcal{M} = \big(B(V_\pi) \otimes M\big)^{(\mathrm{Ad}\,\pi \otimes \Gamma)}$. Thus (a) holds.

Statements (b)–(e) were proved in [24] (Theorem 8 and the discussion on pp. 43–49).

For part (f), it is known that

$$M \rtimes_\Gamma \mathrm{SU}_q(2) = \big(M \otimes B\big(L^2(A)\big)\big)^{\tilde{\Gamma}},$$

where $\tilde{\Gamma} := \mathrm{Ad}\big(1 \otimes \Sigma(W^*)\big) \circ (\mathrm{Id} \otimes \Sigma) \circ (\Gamma \otimes \mathrm{Id})$, and where Σ is the flip map. (See [3], Remark 20 for a simple proof, which also works in the W^*-algebraic setting without any change. For the reader's convenience, we should mention that the definition of "$\lambda(\omega)$" in that paper involves a typographical error.) Since (M, Γ) contains (A, δ), we see that

$$\big(M \otimes B\big(L^2(A)\big)\big)^{\tilde{\Gamma}} \cong M^\Gamma \otimes B\big(L^2(A)\big)$$

(see [23], Lemma 4.2) so that the crossed-product $M \rtimes_\Gamma \mathrm{SU}_q(2)$ and the fixed-point algebra M^Γ are isomorphic to each other, both being of type III. □

We would like to point out that the n-th step $\mathcal{M}_n$ in the basic construction associated to $\mathcal{N} \subset \mathcal{M}$ is rather easy to describe:

$$\mathcal{M}_n \cong \big(B(V_{\rho_n}) \otimes M\big)^{\mathrm{Ad}\,\rho_n \otimes \Gamma}$$

where

$$\rho_{2s+1} = \hat{\pi} \otimes \rho_{2s},$$
$$\rho_{2s} = \pi \otimes \rho_{2s-1},$$
$$\rho_0 = \pi,$$

and $\hat{\pi}$ is the canonical dual of π. The conditional expectation $\mathcal{E}_n\colon \mathcal{M}_n \to \mathcal{M}_{n-1}$ is given by

$$\mathcal{E}_{2s+1} = \tau_q^{(\hat{\pi}, -)} \otimes \mathrm{Id} \otimes \cdots,$$
$$\mathcal{E}_{2s} = \tau_q^{(\pi, -)} \otimes \mathrm{Id} \otimes \cdots.$$

This allows one to compute the λ-lattice of higher relative commutants of $\mathcal{N} \subset \mathcal{M}$ (and thus of $\hat{\mathcal{N}} \subset \hat{\mathcal{M}}$) in terms of the representation theory of $\mathrm{SU}_q(2)$. In particular, one can show that if π is taken to be the fundamental representation, then the principal graph of $\hat{\mathcal{N}} \subset \hat{\mathcal{M}}$ is A_∞, and the index is $(q + q^{-1})^2$.

For $0 < \lambda < 1$, let T_λ denote the (unique) free Araki-Woods factor of type III_λ (see [18] and Section 2).

Proposition 3.2. *The factor* $\big((L(\mathbb{F}_m), \tau) * (A, h)\big) \rtimes_\Gamma \mathrm{SU}_q(2)$ *is isomorphic to the free Araki-Woods factor* T_{q^2}. *Hence* $\mathcal{M} \cong \mathcal{N} \cong T_{q^2}$.

Proof. It is known that the fundamental unitary W lies in $A \otimes \hat{A}'$ (acting on $L^2(A) \otimes L^2(A)$), and hence the mapping $\Phi\colon x \in B\big(L^2(A)\big) \mapsto W(1 \otimes x)W^*$ gives rise to an isomorphism from $B\big(L^2(A)\big)$ onto $A \rtimes_\delta \mathrm{SU}_q(2) = A \rtimes_\delta \hat{A}$ (see e.g. [1]).

Moreover, the crossed product structure and the Haar state h on A give rise to a normal faithful conditional expectation

$$E\colon B\big(L^2(A)\big) = A \rtimes_\delta \hat{A} \to \hat{A}$$

given by

$$E(x) = (h \otimes \mathrm{Id})\big(W(1 \otimes x)W^*\big), \quad x \in B\big(L^2(A)\big).$$

We claim that

$$M \rtimes_\Gamma \mathrm{SU}_q(2) \cong \big(L(\mathbb{F}_m) \otimes \hat{A}, \tau \otimes \mathrm{id}\big) *_{\hat{A}} \big(B\big(L^2(A)\big), E\big).$$

This isomorphism is obtained by extending the map

$$\Psi := \Phi^{-1}\colon A \rtimes_\delta \mathrm{SU}_q(2) \to B\big(L^2(A)\big)$$

to

$$\big[\big(L(\mathbb{F}_m), \tau\big) * (A, h)\big] \rtimes_\Gamma \mathrm{SU}_q(2) \supset A \rtimes_\delta \mathrm{SU}_q(2)$$

by mapping elements in $L(\mathbb{F}_m)$ to the canonical copy of $L(\mathbb{F}_m)$ inside

$$\big(L(\mathbb{F}_m) \otimes \hat{A}, \tau \otimes \mathrm{id}\big) *_{\hat{A}} \big(B\big(L^2(A)\big), E\big).$$

Since $\Phi(\hat{a}) = 1 \otimes \hat{a}$, $\hat{a} \in \hat{A}$ and $(h \otimes \mathrm{Id}) \circ \Phi = E$, $\Psi = \Phi^{-1}$ extends to the desired isomorphism (a freeness condition needs to be checked, see [24], Proposition 1 and [25]).

(If instead of a quantum group action, we would have an action of an ordinary group G, the isomorphism above would be the content of the general identity

$$(R, \phi) * (S, \psi) \rtimes_{\mathrm{id} * \alpha} G = \big(R \otimes L(G), \phi \otimes \mathrm{id}\big) *_{L(G)} (S \rtimes_\alpha G, E_G),$$

where E_G is the canonical $L(G)$-valued conditional expectation on $S \rtimes_\alpha G$ associated to the state $\psi\colon S \to \mathbb{C}$.)

Applying Theorem 2.2, and noting that $M \rtimes_\Gamma \mathrm{SU}_q(2)$ is of type III_{q^2} (Theorem 3.1 (f) and (a)), we find that

$$\mathscr{M} \cong \mathscr{N} \cong \left(L(\mathbb{F}_m) \otimes \hat{A}, \tau \otimes \mathrm{id}\right) *_{\hat{A}} \left(B(H), E\right)$$

is isomorphic to the unique type III_{q^2} free Araki-Woods factor T_{q^2}. □

Remark 3.3. Since the free Araki-Woods factor T_{q^2} is prime (i.e., cannot be written as a tensor product of two diffuse von Neumann algebras, see [21]), it follows that the inclusion $\mathscr{N} \subset \mathscr{M}$ cannot be decomposed as a tensor product of a type II inclusion with a fixed type III factor, although the λ-lattices of the type III and type II inclusions coincide. (This remark can be also obtained from the results of Ge [7].)

Lemma 3.4. *With the assumptions of Theorem* 3.1 (d), *$\hat{\mathscr{M}}$ and $\hat{\mathscr{N}}$ are both isomorphic to $L(\mathbb{F}_\infty)$.*

Proof. Since $\hat{N} = \mathscr{N}^{\phi}$ is a factor (Theorem 3.1 (d)) and $\mathscr{N} = M^{\Gamma}$ is isomorphic to T_{q^2}, $\hat{\mathscr{N}} \cong L(\mathbb{F}_\infty)$ (see Theorem 2.3). Similarly, since $\mathscr{M} \cong T_{q^2}$ and $\hat{\mathscr{M}} = \mathscr{M}^{\phi \circ \mathscr{E}}$ is a factor, $\hat{\mathscr{M}} \cong L(\mathbb{F}_\infty)$. □

For a general finite dimensional representation π the factor map structure associated with the type III inclusion $\mathscr{M} \supseteq \mathscr{N}$ (see [9]) can be described as follows. Decompose

$$\pi = n_1 \cdot \pi_{\ell_1} \oplus \cdots \oplus n_k \cdot \pi_{\ell_k}$$

into multiples of irreducible spin ℓ representations π_ℓ (see [10]) with $\ell_i \neq \ell_j$ $(i \neq j)$.

Set $X := \{1, 2, \ldots, k\} \times [0, -\log q^2)$ and $F_t(j, s) := (j, s + t)$, and let $(X_{\mathscr{M}}, F_t^{\mathscr{M}})$ and $(X_{\mathscr{N}}, F_t^{\mathscr{N}})$ be the flows of weights associated with $\mathscr{M}$ and $\mathscr{N}$, respectively, which are both identified with $([0, -\log q^2),$ translation by $t)$. Let us define two factor maps

$$\pi^{\mathscr{M}} \colon X \to X_{\mathscr{M}} = [0, -\log q^2), \quad \pi^{\mathscr{N}} \colon X \to X_{\mathscr{N}} = [0, -\log q^2)$$

by

$$\pi^{\mathscr{M}}(j, s) = s - \ell_j \log q^2 \mod -\log q^2, \quad \pi^{\mathscr{N}}(j, s) = s.$$

These $\pi^{\mathscr{M}}$, $\pi^{\mathscr{N}}$ describe the factor map structure. As a consequence, we get:

Remark 3.5. The inclusion $\mathscr{M} \supseteq \mathscr{N}$ is essentially type II if and only if all the set

$$\{\ell \colon \pi_\ell \text{ occurs in the decomposition of } \pi\}$$

either consists entirely of half-integers or consists entirely of integers. Under this assumption, all statements in Theorem 3.1 still hold, with the exception of statement (b) (the resulting inclusion is of course no longer irreducible).

This means that our method of constructing irreducible type II_1 subfactors can be applied to the special standard λ-lattices arising from representations of the quantum group $\mathrm{SU}_q(2)$ (see [2]).

We are now ready to prove our main result:

Theorem 3.6. *For every $\lambda \in \mathscr{I}$, there exists an irreducible subfactor N of $M = L(\mathbb{F}_\infty)$ of index λ, so that $N \cong L(\mathbb{F}_\infty)$.*

Proof. The case $\lambda \in \mathscr{I} \cap [0, 4]$ was obtained by Radulescu in [16].

Let $\lambda > 4$, and choose q so that $\lambda = (q + q^{-1})^2$. Let π be the fundamental representation of $\mathrm{SU}_q(2)$. By Theorem 3.1 (e) and Lemma 3.4, for each $0 < q < 1$, we obtain an irreducible inclusion of type II_1 factors

$$L(\mathbb{F}_\infty) \cong \hat{\mathscr{N}} = \mathscr{N}^{\phi} \subset \mathscr{M}^{\phi\circ\mathscr{E}} = \hat{\mathscr{M}} \cong L(\mathbb{F}_\infty)$$

of index $(q + q^{-1})^2$. □

Remark 3.7. In the same way, we see that any irreducible representation π of $\mathrm{SU}_q(2)$ gives rise to an irreducible subfactor $\hat{\mathscr{N}} \subset \hat{\mathscr{M}}$, $\hat{\mathscr{N}} \cong \hat{\mathscr{M}} \cong L(\mathbb{F}_\infty)$, of index $(\dim_q \pi)^2$.

Appendix I

We give here a concrete system of generators for the von Neumann algebra M^Γ appearing in the proof of the main theorem. This gives also a concrete example illustrating the proof of Theorem 2.1 works.

Let H be a Hilbert space, given as a direct sum

$$H = \bigoplus_k H_k \otimes K_k,$$

with H_k and K_k finite-dimensional or infinite-dimensional. Let $\hat{A}$ be the subalgebra of $B(H)$ given by

$$\bigoplus_k B(H_k) \otimes \mathrm{Id}_{K_k}.$$

For each k, let θ_k be a normal faithful state on $B(K_k)$ given by

$$\theta_k(T) = \mathrm{Tr}(D_k T),$$

where $D_k \in B(K_k)$ is a fixed positive matrix. Then

$$E = \bigoplus_k \mathrm{id} \otimes \theta_k(\cdot) 1_{B(K_k)}$$

is a normal faithful conditional expectation from $B(H)$ onto $\hat{A}$.

Let $d_k = \dim K_k$. Choose a basis for K_k, so that D_k is diagonal, with eigenvalues $\lambda_1(k), \ldots, \lambda_{d(k)}(k)$. Write $e_{ij}(k)$, $1 \leqq i, j \leqq d(k)$ for the matrix units generating $B(K_k)$, associated to this basis.

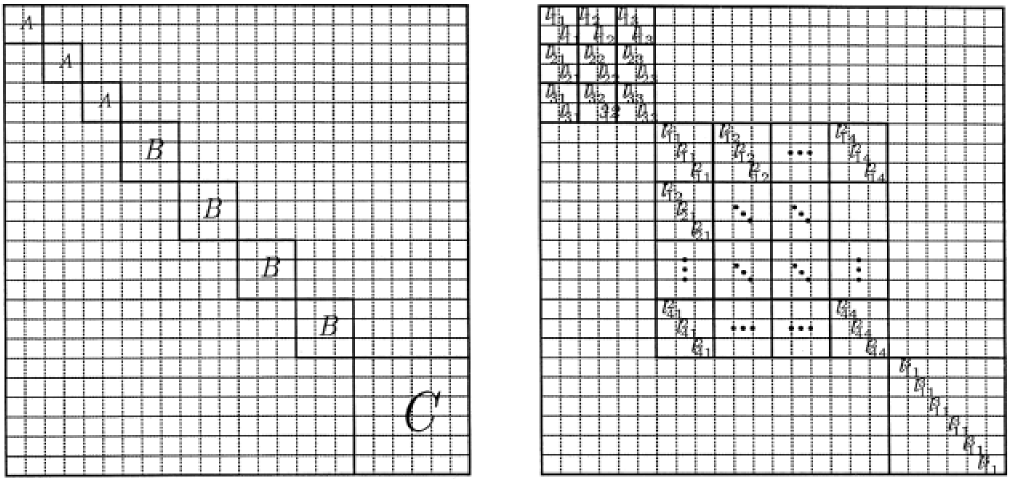

Figure 3.1. $\hat{A} \subset B(H)$ and L. Here the dotted squares are matrix units of $B(H)$; large solid-line squares are matrix units in $B(K_j)$; $A \in B(H_1)$, $B \in B(H_2)$, $C \in B(H_3)$, and $d(1) = 3$, $d(2) = 4$ and $d(3) = 1$.

Let $l_{ij}^k(p)$, $p, k = 1, 2, \ldots$, $1 \leqq i, j \leqq d(k)$ be isometries satisfying

$$l_{ij}^k(p) \cdot l_{i'j'}^{k'}(p') = \delta_{pp'}\delta_{ii'}\delta_{jj'}\delta_{kk'}1,$$

and generating the Toeplitz extension $\mathscr{T}$ of the Cuntz algebra. Let ψ be the canonical vacuum state on $\mathscr{T}$ (see [25]). Consider the tensor product conditional expectation $\psi \otimes E$ on $P = B\big(L^2(\mathscr{T}, \psi)\big) \otimes B(H)$.

Let

$$L_p = \sum_k \sum_{i,j=1}^{d(k)} \sqrt{\lambda_i(k)}\, l_{ij}^k(p) \otimes \big(1_{B(H_k)} \otimes e_{ij}(k)\big) \in P$$

(see Figure 3.1). Then for all $T \in B(H)$, we get

$$\begin{aligned} L_p^* T L_q &= \delta_{pq}\left(\sum_k \sum_{i,j=1}^{d(k)} \sqrt{\lambda_i(k)}\, l_{ij}^k(p) \otimes \big(1_{B(H_k)} \otimes e_{ij}(k)\big)\right)^* T \\ &\quad \cdot \sum_{k'} \sum_{i',j'=1}^{d(k')} \sqrt{\lambda_{i'}(k')}\, l_{i'j'}^{k'}(p) \otimes \big(1_{B(H_{k'})} \otimes e_{i'j'}(k)\big) \\ &= \delta_{pq} \sum_k \sum_{i,j}^{d(k)} \lambda_i(k) \big(1_{B(H_k)} \otimes e_{ji}(k)\big) T \big(1_{B(H_k)} \otimes e_{ij}(k)\big) \\ &= \delta_{pq} E(T). \end{aligned}$$

Hence by [19], Theorem 2.3, $\{L_p : p = 1, 2, \ldots\}$ are $*$-free with respect to $\psi \otimes E : P \to B(H)$ from $B(H)$ with amalgamation over $\hat{A}$. Moreover, since $L_p^* a L_q = \delta_{pq} a$ for all $a \in \hat{A}$, we get that $W^*(\hat{A}, L_p + L_p^* : p \geqq 1) \cong W^*(L_p + L_p^* : p \geqq 1) \otimes \hat{A} \cong L(\mathbb{F}_\infty) \otimes \hat{A}$ (cf. [20], Examples 3.2 and 3.3(b)). We conclude that

$$M = W^*\big(B(H), L_p + L_p^* : p = 1, 2, \ldots\big) \cong \big(L(\mathbb{F}_\infty) \otimes \hat{A}, \tau \otimes \mathrm{id}\big) *_{\hat{A}} \big(B(H), E\big).$$

Now, fix a minimal projection $p \in B(H_1) \otimes e_{11}(1)$. Then the compression pMp is generated as a von Neumann algebra by the entries of $\{L_p + L_p^*\colon p \geqq 1\}$ viewed as matrices. These entries have the form $\sqrt{\lambda_i(k)}\, l_{ij}^k(p) + \sqrt{\lambda_j(k)}\big(l_{ji}^k(p)\big)^*$, $k = 1, 2, \ldots,\ 1 \leqq i \leqq j \leqq d(k)$, $p \geqq 1$. The algebra generated by such elements is isomorphic to a free Araki-Woods factor (see [18], Section 5); the classifying *Sd* invariant for this type of factors is the multiplicative subgroup of $\mathbb{R}$ generated by the ratios $\lambda_i(k)/\lambda_j(k)$, $k = 1, 2, \ldots,\ 1 \leqq i, j \leqq d(k)$.

We now turn to the particular case of $\hat{A}$ arising from a quantum group action, where one can describe the inclusion $B\big(L^2(A)\big) \supseteqq \hat{A}$ and the conditional expectation E explicitly. We omit the details of the computation, but give only the consequence for the reader's convenience.

Let $w_{ij}^{(\ell)}$, $i, j \in I_\ell = \{-\ell, \ell + 1, \ldots, \ell\}$, be the matrix elements of the spin ℓ $(\in (1/2)\mathbb{N}_0)$ representation as given in [10]. The Peter-Weyl type theorem says that the set

$$\left\{ \left(\frac{1 - q^{2(2\ell+1)}}{(1 - q^2) q^{2(\ell+i)}} \right)^{1/2} \Lambda_h(w_{ij}^\ell) \right\}_{i, j \in I_\ell,\, \ell \in (1/2)\mathbb{N}_0}$$

forms a complete orthonormal basis of $L^2(A)$. Let us denote the matrix units with respect to the basis by $e_{(i_1, j_1, \ell_1)(i_2, j_2, \ell_2)}$'s. The set of elements of the form:

$$\hat{E}_{i_1 i_2}^{(\ell)} = \sum_{j \in I_\ell} e_{(i_1, j, \ell)(i_2, j, \ell)}$$

forms a complete system of matrix units of $\hat{A}$, and the conditional expectation E is computed as follows:

$$E(e_{(i_1, j_1, \ell_1)(i_2, j_2, \ell_2)}) = \delta_{(i_1, \ell_1)(i_2, \ell_2)} \frac{1}{1 + q^2 + \cdots + (q^2)^{2\ell_1}} q^{2(\ell_1 - i_1)} \hat{E}_{j_1 j_2}^{(\ell_1)}.$$

In other words, in the notation of the first part of the appendix, we have

$$\dim H_\ell = \dim K_\ell = d(\ell) = 2\ell + 1,$$

and the eigenvalues of θ_ℓ are given by

$$\lambda_i(\ell) = \frac{1}{1 + q^2 + \cdots + (q^2)^{2\ell}} q^{2(\ell - i)}.$$

Since the *Sd* invariant of the associated type III factor is the multiplicative subgroup of positive reals generated by the ratios $\lambda_i(\ell)/\lambda_j(\ell)$, we see that in this case $Sd = q^{2\mathbb{Z}}$, i.e., the factor is of type III_{q^2}.

Appendix II

For the reader's convenience, we give here a direct argument showing that the conditional expectation $E_{B(H)}\colon \Phi\big(B(H),\eta\big) \to B(H)$ in Theorem 2.2 is faithful. We use the notations given in the beginning of §2. For more on this, interested readers are referred to [20].

Lemma (cf. [19], Lemma 4.2). *Suppose that* $\eta_{ij} = \delta_{ij}E$, *where* $E\colon M \to B$ *is a faithful normal conditional expectation. Fix a faithful normal state* φ *on* B. *Then the state* $\theta := \varphi \circ E \circ E_M$ *on* $\Phi(M,\eta)$ *satisfies*

$$\theta(f_0 X_{i_1} f_1 \cdots X_{i_{n-1}} f_{n-1} X_{i_n} f_n) = \theta\big(\sigma_i^{\varphi\circ E}(f_n) f_0 X_{i_1} f_1 \cdots X_{i_{n-1}} f_{n-1} X_{i_n}\big)$$
$$= \theta\big(X_{i_n} \sigma_i^{\varphi\circ E}(f_n) f_0 X_{i_1} f_1 \cdots X_{i_{n-1}} f_{n-1}\big)$$

for all analytic elements $f_0, f_1, \ldots, f_n \in M$ *with respect to the modular action* $\sigma^{\varphi\circ E}$.

Proof. The proof is a straightforward modification of [19], Lemma 4.2. Note that $\varphi \circ E\big(xE(y)\big) = \varphi \circ E\big(E(x)y\big)$ $(x, y \in M)$, so we use the following identities (instead of the trace property of $\varphi \circ E$ used in Lemma 4.2):

(1) $\sigma_z^{\varphi\circ E} \circ E(x) = E \circ \sigma_z^{\varphi\circ E}(x)$ for analytic $x \in M$ and $z \in \mathbb{C}$.

(2) For analytic $x, y \in M$

$$\theta(xy) = \varphi \circ E(xy) = \varphi \circ E\big(\sigma_i^{\varphi\circ E}(y)x\big) = \theta\big(\sigma_i^{\varphi\circ E}(y)x\big). \quad \square$$

Since $\sigma_t^{\varphi\circ E} \circ E = E \circ \sigma_t^{\varphi\circ E}$ and $\varphi \circ E \circ \sigma_t^{\varphi\circ E} = \varphi \circ E$ $(t \in \mathbb{R})$, there exists a 1-parameter group of automorphisms σ_t $(t \in \mathbb{R})$ of $\Phi(M,\eta)$, for which

$$\sigma_t|_M = \sigma_t^{\varphi\circ E}, \quad \sigma_t(X_i) = X_i$$

(note that $\theta \circ \sigma_t = \theta$). Hence the above lemma implies the following:

Proposition. *With the assuputions and notations in the Lemma, the action* σ_t $(t \in \mathbb{R})$ *satisfies the modular condition for* θ, *i.e.*, $\sigma_t = \sigma_t^\theta$, *and hence* θ *is faithful. In particular, the canonical conditional expectation* $E_M\colon \Phi(M,\eta) \to M$ *is faithful.*

References

[1] *S. Baaj* and *G. Skandalis*, Unitaires multiplicatifs et dualité pour les produits croisés de C*-algèbres, Ann. Sci. Éc. Norm. Sup. (4) **26** (1993), no. 4, 425–488.

[2] *T. Banica*, Representations of compact quantum groups and subfactors, J. reine angew. Math. **509** (1999), 167–198.

[3] *F. Boca*, Ergodic actions of compact matrix pseudogroups on C*-algebras, Recent advances in operator algebras (Orléans, 1992), Astérisque **232** (1995), 93–109.

[4] *A. Connes*, Une classification des facteurs de type III, Ann. Sci. Éc. Norm. Sup. **6** (1973), 133–252.

[5] *K. Dykema*, Interpolated free group factors, Pacific J. Math. **163** (1994), 123–135.

[6] *K. Dykema*, Free products of finite dimensional and other von Neumann algebras with respect to non-tracial states, Free Probability (D.-V. Voiculescu, ed.), Fields Inst. Commun. **12**, American Mathematical Society (1997), 41–88.

[7] *L. Ge*, Applications of free entropy to finite von Neumann algebras. II, Ann. Math. (2) **147** (1998), 1, 143–157.
[8] *V. F. R. Jones*, Index for subfactors, Invent. Math. **72** (1983), 1–25.
[9] *H. Kosaki*, Index theory for type III factors, Mappings of operator algebras (Philadelphia, PA, 1988), Birkhäuser Boston, Boston, MA (1990), 227–231.
[10] *T. Masuda*, *K. Mimachi*, *Y. Nakagami*, *M. Noumi*, and *K. Ueno*, Representations of the quantum group $SU_q(2)$ and the little q-Jacobi polynomials, J. Funct. Anal. **99** (1991), no. 2, 357–386.
[11] *M. Pimsner*, A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$, Free Probability (D.-V. Voiculescu, ed.), Fields Inst. Commun. **12**, American Mathematical Society (1997), 189–212.
[12] *S. Popa*, Markov traces on universal Jones algebras and subfactors of finite index, Invent. Math. **111** (1993), 375–405.
[13] *S. Popa*, An axiomatization of the lattice of higher relative commutants of a subfactor, Invent. Math. **120** (1995), no. 3, 427–445.
[14] *S. Popa*, Universal constructions of subfactors, preprint 1999.
[15] *F. Rădulescu*, A one parameter group of automorphisms of $L(\mathbb{F}_\infty) \otimes B(\mathscr{H})$ scaling the trace, C.R. Acad. Sci. Paris **314** (1992), no. 1, 1027–1032.
[16] *F. Rădulescu*, Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group, of noninteger index, Invent. math. **115** (1994), 347–389.
[17] *F. Rădulescu*, A type III_λ factor with core isomorphic to the von Neumann algebra of a free group tensor $B(H)$, Recent advances in operator algebras (Orléans, 1992), Astérisque **232** (1995), 203–209.
[18] *D. Shlyakhtenko*, Free quasi-free states, Pacific J. Math. **177** (1997), 329–368.
[19] *D. Shlyakhtenko*, Some applications of freeness with amalgamation, J. reine angew. Math. **500** (1998), 191–212.
[20] *D. Shlyakhtenko*, A-valued semicircular systems, J. Funct. Anal. **166** (1999), 1–47.
[21] *D. Shlyakhtenko*, Prime type III factors, PNAS **97** (2000), 12439–12441.
[22] *R. Speicher*, Combinatorial theory of the free product with amalgamation and operator-valued free probability theory, Mem. Amer. Math. Soc. **132** (1998), x+88.
[23] *Y. Ueda*, A minimal action of the compact quantum group $SU_q(n)$ on a full factor, J. Math. Soc. Japan **51** (1999), no. 2, 449–461.
[24] *Y. Ueda*, On the fixed-point algebra under a minimal free product type action of the quantum group $SU_q(2)$, IMRN (2000), no. 1, 35–56.
[25] *D.-V. Voiculescu*, *K. Dykema*, and *A. Nica*, Free random variables, CRM monogr. ser. **1**, American Mathematical Society, 1992.
[26] *A. Wassermann*, Coactions and Yang-Baxter equations for ergodic actions and subfactors, Operator algebras and applications, Vol. 2, Cambridge Univ. Press, Cambridge (1988), 203–236.
[27] *H. Wenzl*, C^* tensor categories from quantum groups, J. Amer. Math. Soc. **11** (1998), no. 2, 261–282.
[28] *S. L. Woronowicz*, Compact matrix pseudogroups, Comm. Math. Phys. **111** (1987), no. 4, 613–665.
[29] *S. L. Woronowicz*, Twisted SU(2) group. An example of a noncommutative differential calculus, Publ. Res. Inst. Math. Sci. **23** (1987), no. 1, 117–181.
[30] *F. Xu*, Standard λ-lattices from quantum groups, Invent. Math. **134** (1998), no. 3, 455–487.

Department of Mathematics, UCLA, Los Angeles, CA 90095, USA
e-mail: shlyakht@math.ucla.edu

Graduate School of Mathematics, Kyushu University,
Fukuoka, 810-8560 Japan
e-mail: ueda@math.kyushu-u.ac.jp

Eingegangen 31. Oktober 2000, in revidierter Fassung 28. Mai 2001