

FREE TRANSPORTATION COST INEQUALITIES FOR NON-COMMUTATIVE MULTI-VARIABLES

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ABSTRACT. We prove the free analogue of the transportation cost inequality for tracial distributions of non-commutative self-adjoint (also unitary) multi-variables based on random matrix approximation procedure.

INTRODUCTION

The transportation cost inequality (TCI) gives an upper bound for the quadratic Wasserstein distance by the square roof of the relative entropy. For probability measures on a Polish space \mathcal{X} , the relative entropy is $S(\mu, \nu) = \int_{\mathcal{X}} \log \frac{d\mu}{d\nu} d\mu$ if $\mu \ll \nu$ (otherwise, $S(\mu, \nu) = +\infty$) while the quadratic Wasserstein distance is defined as

$$W_2(\mu, \nu) := \inf_{\pi} \sqrt{\iint_{\mathcal{X} \times \mathcal{X}} d(x, y)^2 d\pi(x, y)}, \quad (0.1)$$

where $d(x, y)$ is the metric on \mathcal{X} and π runs over the probability measures on $\mathcal{X} \times \mathcal{X}$ with marginals μ and ν . In 1996, M. Talagrand [20] obtained the celebrated TCI $W_2(\mu, \nu) \leq \sqrt{2S(\mu, \nu)}$ for probability measures on \mathbb{R}^n , where ν is the standard Gaussian measure on \mathbb{R}^n . Since then, the TCI has been received a lot of attention. It was shown by F. Otto and C. Villani [18] that, in the Riemannian manifold setting, the TCI follows from the logarithmic Sobolev inequality (LSI) of D. Bakry and M. Emery [1]. The LSI gives a lower bound for the relative Fisher information by the relative entropy, which has played important roles in several contexts. Recent developments in both LSI and TCI are found in [16, 21] for example.

On the other hand, Ph. Biane and D. Voiculescu [4] proved the free analogue of Talagrand's TCI for compactly supported measures on \mathbb{R} , where the relative entropy is replaced by its free analogue and the Gaussian measure by the semicircular one. In [11, 12] we developed the random matrix approximation method to obtain a slight generalization of Biane and Voiculescu's free TCI as well as its counterpart on the circle \mathbb{T} . The free analogues of the LSI's on \mathbb{R} and on \mathbb{T} were also obtained in [3] and [11, 13] by the same method.

Recently, M. Ledoux [17] used a similar random matrix technique to prove the free analogue of the Brunn-Minkowski inequality for measures on \mathbb{R} , from which (together with the Hamilton-Jacobi approach) he gave short proofs of the free TCI and LSI for measures on \mathbb{R} . Furthermore, his approach was shown in [10] to be still applicable for getting the free TCI in [12] for measures on \mathbb{T} .

The free TCI's and LSI's so far are restricted to measures on \mathbb{R} or \mathbb{T} and are not truly non-commutative. However, Voiculescu's free entropy for multi-variables was well developed in [23] (see also [25] and [9, Chap. 6]) and the Wasserstein distance was also introduced in [4]

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for multi-variables in the C^* -algebra setting; so we must be in a good position to extend the free TCI to non-commutative multi-variables. This is what we are going to do here. In fact, we will show the truly non-commutative free TCI when the “reference distribution” is chosen to be that associated with freely independent (self-adjoint or unitary) random variables. It of course includes the above-mentioned free analogue of Talagrand’s TCI. However, the present work is still in a very beginning in this direction of the subject matter. For example, it is interesting to seek for a non-commutative generalization of the above-mentioned Ledoux’s approach, which probably brings a new insight into free probability theory.

In this paper, after preliminaries on the Wasserstein distance in §1 following [4], we obtain in §2 the free TCI for non-commutative tracial distributions of self-adjoint multi-variables with respect to a certain free product distribution (see Theorem 2.2). In §3 we present a sharper TCI (see Theorem 3.1) by replacing the free entropy with another free entropy-like quantity (introduced in [7] from the viewpoint of statistical mechanics) but tracial distributions are rather restricted. Furthermore, the counterparts of these free TCI’s in the unitary setting are sketched in §4 without much details for proofs.

1. PRELIMINARIES

1.1. Notations. When \mathcal{A} is a unital C^* -algebra, \mathcal{A}^{sa} stands for the set of self-adjoint elements of \mathcal{A} , and we denote by $S(\mathcal{A})$ the state space of \mathcal{A} and by $TS(\mathcal{A})$ the tracial state space of \mathcal{A} , i.e., the set of all $\varphi \in S(\mathcal{A})$ such that $\varphi(ab) = \varphi(ba)$, $a, b \in \mathcal{A}$. The universal free product C^* -algebra of two copies of \mathcal{A} is denoted by $\mathcal{A} \star \mathcal{A}$, and σ_1 and σ_2 stand for the canonical embedding maps of \mathcal{A} into the left and right copies of \mathcal{A} in $\mathcal{A} \star \mathcal{A}$, respectively. Moreover, the universal free product C^* -algebra of n copies of \mathcal{A} is simply written as $\mathcal{A}^{\star n}$. A pair (\mathcal{A}, τ) with $\tau \in TS(\mathcal{A})$ is called a tracial C^* -probability space, and when \mathcal{A} is a von Neumann algebra and τ is a faithful normal tracial state it is called a tracial W^* -probability space.

The usual non-normalized trace on the $N \times N$ complex matrix algebra $M_N(\mathbb{C})$ is denoted by Tr_N , and $\|A\|_{HS}$ is the Hilbert-Schmidt norm of $A \in M_N(\mathbb{C})$, i.e., $\|A\|_{HS} := \sqrt{\text{Tr}_N(A^*A)}$ while $\|A\|$ is the operator norm of A . We denote by M_N^{sa} the set of all self-adjoint $A \in M_N(\mathbb{C})$ and by Λ_N the Lebesgue measure on M_N^{sa} with the obvious Euclidean structure. As usual, $U(N)$ and $SU(N)$ are the unitary and special unitary groups of order N . We denote by γ_N^U and γ_N^{SU} the Haar probability measures on $U(N)$ and $SU(N)$, respectively. We also denote by $\mathcal{P}(\mathcal{X})$ the set of all Borel probability measures on a Polish space \mathcal{X} .

1.2. Non-commutative distributions. Slightly unlike the usual, we will employ the scheme in [7] to deal with “non-commutative distributions.” Let us fix $n \in \mathbb{N}$ and $R > 0$. An underlying C^* -algebra we adopt is the universal free product C^* -algebra $\mathcal{A}_R^{(n)} := C([-R, R])^{\star n}$ with norm $\|\cdot\|_R$ and a canonical set of self-adjoint generators $X_i(t) = t$ in the i th copy of $C([-R, R])$, $1 \leq i \leq n$. Each $\varphi \in S(\mathcal{A}_R^{(n)})$ provides a *distribution* or *law* of X_1, \dots, X_n whose (non-commutative) *moments* are given by $\varphi(X_{i_1} \cdots X_{i_m})$ ’s. Any distribution in the C^* -algebra setting can be indeed realized in this way. More precisely, if a_1, \dots, a_n are self-adjoint variables in a C^* -probability space (\mathcal{A}, φ) with operator norm $\|a_i\| \leq R$, then one has a (unique) $*$ -homomorphism Ψ from $\mathcal{A}_R^{(n)}$ into \mathcal{A} sending each X_i to a_i so that the distribution of X_1, \dots, X_n under $\varphi \circ \Psi \in S(\mathcal{A}_R^{(n)})$ coincides with that of a_1, \dots, a_n under φ . Our main objects in the paper are the Wasserstein distance and the free entropy, which have been well developed only in terms of tracial states. Thus, in what follows, we will restrict our consideration only to tracial distributions, i.e., elements in $TS(\mathcal{A}_R^{(n)})$.

The (microstates) *free entropy* χ introduced by Voiculescu [23] is defined in our context for every $\tau \in TS(\mathcal{A}_R^{(n)})$ as follows: Let π_τ be the GNS representation of $\mathcal{A}_R^{(n)}$ associated with τ ; then we have the tracial W^* -probability space $(\pi_\tau(\mathcal{A}_R^{(n)})'', \tilde{\tau})$ with the normal extension $\tilde{\tau}$ of τ together with the self-adjoint variables $\pi_\tau(X_1), \dots, \pi_\tau(X_n)$. Then, the free entropy of τ at our disposal is

$$\chi(\tau) := \chi(\pi_\tau(X_1), \dots, \pi_\tau(X_n)) = \chi_R(\pi_\tau(X_1), \dots, \pi_\tau(X_n))$$

(see [23] and also [9, 6.3.6] for the latter equality). By definition the free entropy $\chi(\tau)$ is determined only by the moments of τ in (X_1, \dots, X_n) independently of a particular choice of $R > 0$. (This is also the case for the Wasserstein distance as will be seen in §§1.3.)

Here, let us introduce a certain class of non-commutative distributions coming from so-called matrix integrals, which will play an important role in the paper. For each $N \in \mathbb{N}$ and $A_1, \dots, A_n \in M_N^{sa}$ with $\|A_i\| \leq R$ we have the “non-commutative functional calculus”

$$h \in \mathcal{A}_R^{(n)} \mapsto h(A_1, \dots, A_n) \in M_N(\mathbb{C})$$

that is the canonical $*$ -homomorphism from $\mathcal{A}_R^{(n)}$ into $M_N(\mathbb{C})$ sending each X_i to A_i . Let r_R be the retraction of \mathbb{R} onto $[-R, R]$, i.e.,

$$r_R(t) := \begin{cases} -R & \text{if } t < -R, \\ t & \text{if } -R \leq t \leq R, \\ R & \text{if } t > R. \end{cases}$$

The next lemma is quite easy to show from the obvious inequality $|r_R(\alpha) - r_R(\beta)| \leq |\alpha - \beta|$ for $\alpha, \beta \in \mathbb{R}$; so we omit the proof.

Lemma 1.1. *We have $\|r_R(A) - r_R(B)\|_{HS} \leq \|A - B\|_{HS}$ for every $A, B \in M_N^{sa}$.*

Hence, a usual approximation argument shows that the function $(A_1, \dots, A_n) \mapsto h(r_R(A_1), \dots, r_R(A_n))$ is continuous on $(M_N^{sa})^n \cong \mathbb{R}^{N^2n}$ with respect to the Euclidean structure for each fixed $h \in \mathcal{A}_R^{(n)}$. Thus, each probability measure $\lambda \in \mathcal{P}((M_N^{sa})^n)$ gives rise to the tracial distribution $\hat{\lambda}_R \in TS(\mathcal{A}_R^{(n)})$ defined by

$$\hat{\lambda}_R(h) := \int_{(M_N^{sa})^n} \frac{1}{N} \text{Tr}_N(h(r_R(A_1), \dots, r_R(A_n))) d\lambda(A_1, \dots, A_n), \quad h \in \mathcal{A}_R^{(n)}.$$

We call this $\hat{\lambda}_R$ the *random matrix distribution* associated with λ . When the measure λ is supported in $(M_{N,R}^{sa})^n$ where $M_{N,R}^{sa} := \{A \in M_N^{sa} : \|A\| \leq R\}$, the retraction r_R is of course not needed in the above definition so that $\hat{\lambda}_R$ is simply defined by integrating over $(M_{N,R}^{sa})^n$.

1.3. Wasserstein distance. This part is from [4] with slight modifications. Let (a_1, \dots, a_n) and (b_1, \dots, b_n) be n -tuples of non-commutative variables in tracial C^* -probability spaces (\mathcal{A}_1, τ_1) and (\mathcal{A}_2, τ_2) , respectively. Here, it may be emphasized that a_i 's as well as b_i 's are not necessarily self-adjoint (even not normal). We write $(a_1, \dots, a_n) \sim (b_1, \dots, b_n)$ if the $*$ -distributions (or $*$ -moments) of (a_1, \dots, a_n) and of (b_1, \dots, b_n) are same, i.e.,

$$\tau_1(a_{i_1}^{\varepsilon_1} \cdots a_{i_m}^{\varepsilon_m}) = \tau_2(b_{i_1}^{\varepsilon_1} \cdots b_{i_m}^{\varepsilon_m})$$

for all $m \in \mathbb{N}$, $i_1, \dots, i_m \in \{1, \dots, n\}$ and $\varepsilon_1, \dots, \varepsilon_m \in \{1, *\}$. For $1 \leq p < \infty$, the p -Wasserstein distance introduced in [4] is defined by

$$W_p((a_1, \dots, a_n), (b_1, \dots, b_n)) := \inf \left\{ \left(\sum_{i=1}^n \tau(|a'_i - b'_i|^p) \right)^{1/p} \right\}, \quad (1.1)$$

where infimum is taken over all $2n$ -tuples $(a'_1, \dots, a'_n, b'_1, \dots, b'_n)$ in some tracial C^* -probability space (\mathcal{A}, τ) such that $(a'_1, \dots, a'_n) \sim (a_1, \dots, a_n)$ and $(b'_1, \dots, b'_n) \sim (b_1, \dots, b_n)$. The definition itself says that the quantity (1.1) depends only on the $*$ -moments of (a_1, \dots, a_n) and (b_1, \dots, b_n) .

Another definition of W_p was also introduced in [4], which is a bit more tractable than the above. Let \mathcal{A} be a unital C^* -algebra with a specified n -tuple (a_1, \dots, a_n) of generators. For a given pair $\tau_1, \tau_2 \in TS(\mathcal{A})$ we define the set of (non-commutative tracial) *joining states* between τ_1 and τ_2 by

$$TS(\mathcal{A} \star \mathcal{A} | \tau_1, \tau_2) := \{\tau \in TS(\mathcal{A} \star \mathcal{A}) : \tau \circ \sigma_1 = \tau_1, \tau \circ \sigma_2 = \tau_2\}.$$

For $1 \leq p < \infty$, the p -Wasserstein distance between τ_1 and τ_2 is defined by

$$W_p(\tau_1, \tau_2) := \inf \left\{ \left(\sum_{i=1}^n \tau(|\sigma_1(a_i) - \sigma_2(a_i)|^p) \right)^{1/p} : \tau \in TS(\mathcal{A} \star \mathcal{A} | \tau_1, \tau_2) \right\}. \quad (1.2)$$

As remarked in [4, §§1.2], the two definitions (1.1) and (1.2) give the same quantity in the following way.

Proposition 1.2. *For every $\tau_1, \tau_2 \in TS(\mathcal{A})$, let (a'_1, \dots, a'_n) and (a''_1, \dots, a''_n) be (a_1, \dots, a_n) in (\mathcal{A}, τ_1) and in (\mathcal{A}, τ_2) , respectively. Then we have*

$$W_p(\tau_1, \tau_2) = W_p((a'_1, \dots, a'_n), (a''_1, \dots, a''_n)).$$

The proof is easily done by manipulating appropriate GNS representations so that we leave it to the reader. An important consequence of the proposition is that $W_p(\tau_1, \tau_2)$ in (1.2) is independent of a particular choice of \mathcal{A} with a specified n -tuple (a_1, \dots, a_n) ; namely, it is determined only by the $*$ -moments of τ_1 and τ_2 in (a_1, \dots, a_n) .

Basic properties of W_p are in order.

- 1° $W_p(\tau_1, \tau_2)$ is a metric on $TS(\mathcal{A})$ (see [4, Theorem 1.3]).
- 2° $W_p(\tau_1, \tau_2)$ is jointly lower semi-continuous in $(\tau_1, \tau_2) \in TS(\mathcal{A}) \times TS(\mathcal{A})$ in weak*-topology (see [4, Proposition 1.4]).
- 3° $W_p(\tau_1, \tau_2)^p$ is jointly convex in $(\tau_1, \tau_2) \in TS(\mathcal{A}) \times TS(\mathcal{A})$. (This is easy to prove though not included in [4].)
- 4° If a_1, \dots, a_n are self-adjoint (or more generally normal) and mutually commuting, then $W_p(\tau_1, \tau_2)$ coincides with the usual p -Wasserstein distance $W_p(\mu_1, \mu_2)$ (see (0.1)), where $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}^n)$ (or $\mathcal{P}(\mathbb{C}^n)$) are the spectral distribution measures of the n -tuple (a_1, \dots, a_n) constructed via the GNS representations associated with τ_1, τ_2 , respectively (see [4, Theorem 1.5]).

We will treat the (quadratic) 2-Wasserstein distance $W_2(\tau_1, \tau_2)$ for tracial distributions of self-adjoint random variables in §2, §3 and for those of unitary random variables in §4. In the self-adjoint case, we will take the universal $\mathcal{A}_R^{(n)}$ with the specified self-adjoint generators X_1, \dots, X_n . This is indeed universal in the sense that when a_1, \dots, a_n are self-adjoint in any \mathcal{A} , any tracial distribution of (a_1, \dots, a_n) can be realized via some $\tau \in TS(\mathcal{A}_R^{(n)})$ as long as $R \geq \|a_i\|$, $1 \leq i \leq n$ (see §§1.2).

In this subsection, we provide an inequality between the free and usual 2-Wasserstein distances for random matrix distributions introduced in §§1.2, which will be one of the keys in our later discussions. The inequality corresponds to that in [12, Lemmas 2.6 and 2.8]; however the argument here is simpler than there because we do not (indeed cannot) treat the “eigenvalue distributions.”

Lemma 1.3. *For every pair $\lambda_1, \lambda_2 \in \mathcal{P}((M_N^{sa})^n)$ and every $R > 0$, let $\widehat{\lambda}_{1,R}, \widehat{\lambda}_{2,R} \in TS(\mathcal{A}_R^{(n)})$ be the corresponding random matrix distributions. Then we have*

$$W_2(\widehat{\lambda}_{1,R}, \widehat{\lambda}_{2,R}) \leq \frac{1}{\sqrt{N}} W_2(\lambda_1, \lambda_2),$$

where $W_2(\lambda_1, \lambda_2)$ is the usual 2-Wasserstein distance between λ_1, λ_2 defined by

$$\inf_{\pi} \sqrt{\iint_{(M_N^{sa})^n \times (M_N^{sa})^n} \sum_{i=1}^n \|A_i - B_i\|_{HS}^2 d\pi}$$

over the joining measures π on $(M_N^{sa})^n \times (M_N^{sa})^n$ of λ_1, λ_2 , i.e., measures whose marginals are λ_1, λ_2 (see also (0.1)).

Proof. For each n -tuple $\mathbf{A} = (A_1, \dots, A_n) \in (M_N^{sa})^n$ one has the $*$ -homomorphism

$$\Psi_R^{\mathbf{A}} : h \in \mathcal{A}_R^{(n)} \mapsto h(r_R(A_1), \dots, r_R(A_n)) \in M_N(\mathbf{C})$$

sending X_i to $r_R(A_i)$ (see §§1.2), and moreover for each $\mathbf{A}, \mathbf{B} \in (M_N^{sa})^n$ there is a unique $*$ -homomorphism

$$\Psi_R^{\mathbf{A}, \mathbf{B}} := \psi_R^{\mathbf{A}} \star \psi_R^{\mathbf{B}} : \mathcal{A}_R^{(n)} \star \mathcal{A}_R^{(n)} \rightarrow M_N(\mathbf{C})$$

determined by

$$\Psi_R^{\mathbf{A}, \mathbf{B}} \circ \sigma_1 = \Psi_R^{\mathbf{A}}, \quad \Psi_R^{\mathbf{A}, \mathbf{B}} \circ \sigma_2 = \Psi_R^{\mathbf{B}}.$$

As in §§1.2, the function $(\mathbf{A}, \mathbf{B}) \mapsto \Psi_R^{\mathbf{A}, \mathbf{B}}(h)$ is continuous with respect to the Hilbert-Schmidt norms for each fixed $h \in \mathcal{A}_R^{(n)} \star \mathcal{A}_R^{(n)}$; hence every joining measure π of λ_1, λ_2 gives rise to the tracial distribution $\widehat{\pi}_R \in TS(\mathcal{A}_R^{(n)} \star \mathcal{A}_R^{(n)})$ defined by

$$\widehat{\pi}_R(h) := \iint_{(M_N^{sa})^n \times (M_N^{sa})^n} \frac{1}{N} \text{Tr}_N(\Psi_R^{\mathbf{A}, \mathbf{B}}(h)) d\pi(\mathbf{A}, \mathbf{B}), \quad h \in \mathcal{A}_R^{(n)} \star \mathcal{A}_R^{(n)},$$

which satisfies

$$\widehat{\pi}_R \circ \sigma_1 = \widehat{\lambda}_{1,R}, \quad \widehat{\pi}_R \circ \sigma_2 = \widehat{\lambda}_{2,R}.$$

Therefore, we have $\widehat{\pi}_R \in TS(\mathcal{A}_R^{(n)} \star \mathcal{A}_R^{(n)} | \widehat{\lambda}_{1,R}, \widehat{\lambda}_{2,R})$ so that

$$\begin{aligned} W_2(\widehat{\lambda}_{1,R}, \widehat{\lambda}_{2,R})^2 &\leq \widehat{\pi}_R \left(\sum_{i=1}^n (\sigma_1(X_i) - \sigma_2(X_i))^2 \right) \\ &= \iint \sum_{i=1}^n \frac{1}{N} \text{Tr}_N(\Psi_R^{\mathbf{A}, \mathbf{B}}((\sigma_1(X_i) - \sigma_2(X_i))^2)) d\pi(\mathbf{A}, \mathbf{B}) \\ &= \iint \sum_{i=1}^n \frac{1}{N} \text{Tr}_N((r_R(A_i) - r_R(B_i))^2) d\pi(\mathbf{A}, \mathbf{B}) \\ &= \iint \sum_{i=1}^n \frac{1}{N} \|r_R(A_i) - r_R(B_i)\|_{HS}^2 d\pi(\mathbf{A}, \mathbf{B}) \\ &\leq \frac{1}{N} \iint \sum_{i=1}^n \|A_i - B_i\|_{HS}^2 d\pi(\mathbf{A}, \mathbf{B}), \end{aligned}$$

where the latter inequality is due to Lemma 1.1. Hence, the desired inequality follows by taking the infimum of the last integral over the joining measures π of λ_1, λ_2 . \square

Finally, we remark that the 2-Wasserstein distance is sometimes defined with the cost function of the form $\frac{1}{2} \times (\text{distance})^2$. In fact, in [10, 11, 12] we adopted the definition with a $\frac{1}{2}$ -multiple constant so that the bounds of TCI's there and in the present paper are 2 times different.

2. FREE TCI FOR χ

We will obtain the free TCI for non-commutative tracial distributions with respect to the distribution of freely independent random variables, including a natural free analogue of celebrated Talagrand's TCI [20] with respect to the standard Gaussian measure on \mathbb{R}^n .

Let $\mathbf{Q} = (Q_1, \dots, Q_n)$ be an n -tuple of real-valued continuous functions on \mathbb{R} with

$$\lim_{|x| \rightarrow \infty} \exp(-\varepsilon Q_i(x)) = 0 \quad \text{for every } \varepsilon > 0. \quad (2.1)$$

Then, for each Q_i we define the $N \times N$ self-adjoint random matrix $\lambda_N(Q_i) \in \mathcal{P}(M_N^{sa})$ by

$$d\lambda_N(Q_i)(A) := \frac{1}{Z_N(Q_i)} \exp(-N \text{Tr}_N(Q_i(A))) d\Lambda_N(A)$$

with a normalization constant $Z_N(Q_i)$, whose mean eigenvalue distribution on \mathbb{R} is denoted by $\widehat{\lambda}_N(Q_i)$. With $Q := Q_i$, a fundamental result in the theory of weighted potentials (see [19, I.1.3]) tells us that the functional

$$-\Sigma(\mu) + \mu(Q) := - \iint_{\mathbb{R}^2} \log|x-y| d\mu(x) d\mu(y) + \int_{\mathbb{R}} Q(x) d\mu(x), \quad \mu \in \mathcal{P}(\mathbb{R}),$$

has a unique minimizer μ_Q which is compactly supported and called the *equilibrium measure* associated with Q . For example, when $Q(x) = x^2/2$, μ_Q is the $(0, 1)$ -semicircular distribution $d\gamma_{0,2}(x) := \frac{1}{2\pi} \sqrt{4-x^2} dx$ supported on $[-2, 2]$. Furthermore, the large deviation principle for self-adjoint random matrices (see [2], [9, 5.4.3]) shows that $\widehat{\lambda}_N(Q)$ weakly converges to the equilibrium measure μ_Q . Let $R_0 > 0$ be the smallest such that all μ_{Q_i} 's are supported in $[-R_0, R_0]$; for example, $R_0 = 2$ when $Q_i(x) = x^2/2$ for all i . We then notice that

$$\lambda_N(Q_i)(M_{N,R_0}^{sa}) = \widehat{\lambda}_N(Q_i)([-R_0, R_0]) \longrightarrow \mu_{Q_i}([-R_0, R_0]) = 1 \quad (2.2)$$

as $N \rightarrow \infty$ for every $1 \leq i \leq n$. Let us consider the product measure

$$\lambda_N(\mathbf{Q}) := \bigotimes_{i=1}^n \lambda_N(Q_i) \in \mathcal{P}((M_N^{sa})^n),$$

that is,

$$d\lambda_N(\mathbf{Q})(A_1, \dots, A_n) = \frac{1}{Z_N(\mathbf{Q})} \exp\left(-N \sum_{i=1}^n \text{Tr}_N(Q_i(A_i))\right) d\Lambda_N^{\otimes n}(A_1, \dots, A_n)$$

with $Z_N(\mathbf{Q}) := \prod_{i=1}^n Z_N(Q_i)$, and $\widehat{\lambda}_{N,R}(\mathbf{Q}) \in TS(\mathcal{A}_R^{(n)})$ denotes the random matrix distribution associated with $\lambda_N(\mathbf{Q})$ (see §§1.2). Furthermore, when $R \geq R_0$, define the tracial distribution $\tau_{\mathbf{Q}} \in TS(\mathcal{A}_R^{(n)})$ to be the free product state $\star_{i=1}^n \mu_{Q_i}$ on $\mathcal{A}_R^{(n)} = C([-R, R])^{\star n}$, where each μ_{Q_i} is meant a state on $C([-R, R])$ defined by integration. (Note that the moments of $\tau_{\mathbf{Q}}$ is independent of a choice of $R \geq R_0$.)

We begin by restating the so-called asymptotic freeness due to Voiculescu [22] in our situation.

Lemma 2.1. *Whenever $R \geq R_0$ we have*

$$\lim_{N \rightarrow \infty} \widehat{\lambda}_{N,R}(\mathbf{Q}) = \tau_{\mathbf{Q}} \quad \text{weakly}^*.$$

Proof. Fix an arbitrary $R \geq R_0$. By (2.2) we get

$$\lambda_N(\mathbf{Q})((M_{N,R}^{sa})^n) = \prod_{i=1}^n \widehat{\lambda}_N(Q_i)([-R, R]) \longrightarrow 1 \quad \text{as } N \rightarrow \infty.$$

For any non-commutative polynomial p in X_1, \dots, X_n ($\in \mathcal{A}_R^{(n)}$) we have

$$\begin{aligned} \widehat{\lambda}_{N,R}(\mathbf{Q})(p) &= \int \frac{1}{N} \text{Tr}_N(p(r_R(A_1), \dots, r_R(A_n))) d\lambda_N(\mathbf{Q})(A_1, \dots, A_n) \\ &= \int_{(M_{N,R}^{sa})^n} \frac{1}{N} \text{Tr}_N(p(A_1, \dots, A_n)) d\lambda_N(\mathbf{Q})(A_1, \dots, A_n) \\ &\quad + \int_{(M_N^{sa})^n \setminus (M_{N,R}^{sa})^n} \frac{1}{N} \text{Tr}_N(p(r_R(A_1), \dots, r_R(A_n))) d\lambda_N(\mathbf{Q})(A_1, \dots, A_n). \end{aligned}$$

Since

$$\begin{aligned} &\left| \int_{(M_N^{sa})^n \setminus (M_{N,R}^{sa})^n} \frac{1}{N} \text{Tr}_N(p(r_R(A_1), \dots, r_R(A_n))) d\lambda_N(\mathbf{Q})(A_1, \dots, A_n) \right| \\ &\leq \|p\|_R (1 - \lambda_N(\mathbf{Q})((M_{N,R}^{sa})^n)) \longrightarrow 0 \quad \text{as } N \rightarrow \infty, \end{aligned}$$

the desired assertion follows from the naturally expected fact that

$$\begin{aligned} \tau_{\mathbf{Q}}(p) &= \lim_{N \rightarrow \infty} \int_{(M_{N,R}^{sa})^n} \frac{1}{N} \text{Tr}_N(p(A_1, \dots, A_n)) d\lambda_{N,R}(\mathbf{Q})(A_1, \dots, A_n) \\ &= \lim_{N \rightarrow \infty} \int_{(M_{N,R}^{sa})^n} \frac{1}{N} \text{Tr}_N(p(A_1, \dots, A_n)) d\lambda_N(\mathbf{Q})(A_1, \dots, A_n), \end{aligned}$$

where

$$\begin{aligned} \lambda_{N,R}(\mathbf{Q}) &:= \frac{1}{\lambda_N(\mathbf{Q})((M_{N,R}^{sa})^n)} \lambda_N(\mathbf{Q}) \Big|_{(M_{N,R}^{sa})^n} = \bigotimes_{i=1}^n \lambda_{N,R}(Q_i), \\ \lambda_{N,R}(Q_i) &:= \frac{1}{\lambda_N(Q_i)(M_{N,R}^{sa})} \lambda_N(Q_i) \Big|_{M_{N,R}^{sa}}. \end{aligned}$$

Indeed, this is a simple consequence of an asymptotic freeness result in [9, 4.3.5], slightly generalizing Voiculescu's original in [22] to the setup in almost sure sense as well as to general unitarily invariant self-adjoint random matrices. Also, one should note that $\lambda_{N,R}(Q_i)$ still weakly converges to μ_{Q_i} thanks to (2.2). \square

We are now in a position to state the main result of this section.

Theorem 2.2. *Assume that there exists a constant $\rho > 0$ such that all $Q_i(x) - \frac{\rho}{2}x^2$, $1 \leq i \leq n$, are convex on \mathbf{R} (so that the condition (2.1) automatically holds). Assume $R \geq R_0$ with R_0 given above. Then we have*

$$W_2(\tau, \tau_{\mathbf{Q}}) \leq \sqrt{\frac{2}{\rho} \left(-\chi(\tau) + \tau \left(\sum_{i=1}^n Q_i(X_i) \right) + B_{\mathbf{Q}} \right)}$$

for every $\tau \in TS(\mathcal{A}_R^{(n)})$, where

$$B_{\mathbf{Q}} := \lim_{N \rightarrow \infty} \left(\frac{1}{N^2} \sum_{i=1}^n \log Z_N(Q_i) + \frac{n}{2} \log N \right). \quad (2.3)$$

Since the equilibrium measure with respect to $Q(x) = x^2/2$ is the $(0, 1)$ -semicircular distribution $\gamma_{0,2}$ and

$$\lim_{N \rightarrow \infty} \left(\frac{1}{N^2} \log Z_N(Q) + \frac{1}{2} \log N \right) = \frac{1}{2} \log 2\pi$$

(see e.g. [9, 4.4.6 and pp. 185–186]), the above theorem includes the free analogue of Talagrand's TCI as follows.

Corollary 2.3. *If $R \geq 2$ and $\gamma_{0,2}^{\star n} \in TS(\mathcal{A}_R^{(n)})$ is the (non-commutative) distribution of a standard semicircular system, then*

$$W_2\left(\tau, \gamma_{0,2}^{\star n}\right) \leq \sqrt{2 \left(-\chi(\tau) + \tau \left(\frac{1}{2} \sum_{i=1}^n X_i^2 \right) + \frac{n}{2} \log 2\pi \right)}$$

for every $\tau \in TS(\mathcal{A}_R^{(n)})$.

To prove Theorem 2.2, we need the following:

Lemma 2.4. *Assume the same assumption for Q_i 's with a constant $\rho > 0$. Then, for every $N \in \mathbb{N}$ and every $\lambda \in \mathcal{P}((M_N^{sa})^n)$, we have*

$$W_2(\lambda, \lambda_N(\mathbf{Q})) \leq \sqrt{\frac{2}{\rho N} S(\lambda, \lambda_N(\mathbf{Q}))},$$

where $S(\lambda, \lambda_N(\mathbf{Q}))$ is the relative entropy of λ with respect to $\lambda_N(\mathbf{Q})$.

Proof. Since all $Q_i(x) - \frac{\rho}{2}x^2$ are convex on \mathbf{R} , so is

$$(A_1, \dots, A_n) \in (M_N^{sa})^n \mapsto N \sum_{i=1}^n \text{Tr}_N \left(Q_i(A_i) - \frac{\rho}{2} A_i^2 \right).$$

(This is the reason why the multiple constant $1/N$ appears, see [12, p. 212].) Hence, the TCI for measures on Euclidean spaces (see [16, Theorem 6.5]) slightly generalizing Talagrand's original implies the desired inequality with regarding $(M_N^{sa})^n$ as $\mathbb{R}^{N^2 n}$. \square

Proof of Theorem 2.2. First, note that the existence of the limit in (2.3) is in [9, 5.4.3]. When $\chi(\tau) = -\infty$ nothing has to be done so that let us assume $\chi(\tau) > -\infty$. Recall that

$$\begin{aligned} \chi(\tau) &= \chi_R(\pi_\tau(X_1), \dots, \pi_\tau(X_n)) \\ &= \lim_{\substack{m \rightarrow \infty \\ \varepsilon \searrow 0}} \limsup_{N \rightarrow \infty} \left(\frac{1}{N^2} \log \Lambda_N^{\otimes n}(\Gamma_R(\pi_\tau(X_1), \dots, \pi_\tau(X_n); N, m, \varepsilon)) + \frac{n}{2} \log N \right), \end{aligned}$$

where $\Gamma_R(\pi_\tau(X_1), \dots, \pi_\tau(X_n); N, m, \varepsilon)$ is the set of all n -tuples $(A_1, \dots, A_n) \in (M_{N,R}^{sa})^n$ such that

$$\left| \frac{1}{N} \text{Tr}_N(A_{i_1} \cdots A_{i_r}) - \tau(X_{i_1} \cdots X_{i_r}) \right| = \left| \frac{1}{N} \text{Tr}_N(A_{i_1} \cdots A_{i_r}) - \tilde{\tau}(\pi_\tau(X_{i_1}) \cdots \pi_\tau(X_{i_r})) \right| < \varepsilon$$

for all possible i_1, \dots, i_r with $1 \leq r \leq m$. A suitable subsequence $N(1) < N(2) < \dots$ can be chosen in such a way that letting

$$\Gamma_R(\tau; k) := \Gamma_R(\pi_\tau(X_1), \dots, \pi_\tau(X_n); N(k), k, 1/k)$$

we get

$$\chi(\tau) = \lim_{k \rightarrow \infty} \left(\frac{1}{N(k)^2} \log \Lambda_{N(k)}^{\otimes n}(\Gamma_R(\tau; k)) + \frac{n}{2} \log N(k) \right). \quad (2.4)$$

Looking at this, we introduce the random matrix distribution $\widehat{\lambda}_{N(k),R} \in TS(\mathcal{A}_R^{(n)})$ associated with the probability measure

$$\lambda_{N(k)} := \frac{1}{\Lambda_{N(k)}^{\otimes n}(\Gamma_R(\tau; k))} \Lambda_{N(k)}^{\otimes n} |_{\Gamma_R(\tau; k)} \in \mathcal{P}((M_{N(k),R}^{sa})^n).$$

Let h be an arbitrary monomial $X_{i_1} \cdots X_{i_r} \in \mathcal{A}_R^{(n)}$. As long as $r \leq k$, we get

$$\begin{aligned} & \left| \frac{1}{N(k)} \text{Tr}_{N(k)}(h(A_1, \dots, A_n)) - \tau(h) \right| \\ &= \left| \frac{1}{N(k)} \text{Tr}_{N(k)}(A_{i_1} \cdots A_{i_r}) - \tau(X_{i_1} \cdots X_{i_r}) \right| < \frac{1}{k} \end{aligned}$$

for all $(A_1, \dots, A_n) \in \Gamma_R(\tau; k)$, and hence

$$\begin{aligned} & \left| \widehat{\lambda}_{N(k),R}(h) - \tau(h) \right| \\ & \leq \int_{\Gamma_R(\tau; k)} \left| \frac{1}{N(k)} \text{Tr}_{N(k)}(h(A_1, \dots, A_n)) - \tau(h) \right| d\lambda_k(A_1, \dots, A_n) < \frac{1}{k}. \end{aligned}$$

This shows that $\widehat{\lambda}_{N(k),R}(h) \rightarrow \tau(h)$ as $k \rightarrow \infty$ for all monomials h so that we get

$$\lim_{k \rightarrow \infty} \widehat{\lambda}_{N(k),R} = \tau \quad \text{weakly}^*. \quad (2.5)$$

By Lemmas 1.3 and 2.4 we have

$$\begin{aligned} & W_2(\widehat{\lambda}_{N(k),R}, \widehat{\lambda}_{N(k),R}(\mathbf{Q}))^2 \\ & \leq \frac{1}{N(k)} W_2(\lambda_{N(k)}, \lambda_{N(k)}(\mathbf{Q}))^2 \\ & \leq \frac{2}{\rho N(k)^2} S(\lambda_{N(k)}, \lambda_{N(k)}(\mathbf{Q})) \\ & = \frac{2}{\rho N(k)^2} \int_{(M_{N(k),R}^{sa})^n} \log \frac{d\lambda_{N(k)}}{d\lambda_{N(k)}(\mathbf{Q})}(A_1, \dots, A_n) d\lambda_{N(k)}(A_1, \dots, A_n) \\ & = \frac{2}{\rho N(k)^2} \int_{(M_{N(k),R}^{sa})^n} \left(-\log \Lambda_{N(k)}^{\otimes n}(\Gamma_R(\tau; k)) \right. \\ & \quad \left. + N(k) \sum_{i=1}^n \text{Tr}_{N(k)}(Q_i(A_i)) + \sum_{i=1}^n \log Z_{N(k)}(Q_i) \right) d\lambda_{N(k)}(A_1, \dots, A_n) \\ & = \frac{2}{\rho} \left\{ -\left(\frac{1}{N(k)^2} \log \Lambda_{N(k)}^{\otimes n}(\Gamma_R(\tau; k)) + \frac{n}{2} \log N(k) \right) \right. \\ & \quad \left. + \widehat{\lambda}_{N(k),R} \left(\sum_{i=1}^n Q_i(A_i) \right) + \left(\frac{1}{N(k)^2} \sum_{i=1}^n \log Z_{N(k)}(Q_i) + \frac{n}{2} \log N(k) \right) \right\}. \end{aligned}$$

The above last formula converges to

$$\frac{2}{\rho} \left(-\chi(\tau) + \tau \left(\sum_{i=1}^n Q_i(A_i) \right) + B_{\mathbf{Q}} \right)$$

as $k \rightarrow \infty$ thanks to (2.4) and (2.5). On the other hand, by the joint lower semi-continuity of W_2 (see 2° in §§1.3), Lemma 2.1 and (2.5), we have

$$W_2(\tau, \tau_{\mathbf{Q}}) \leq \liminf_{k \rightarrow \infty} W_2(\widehat{\lambda}_{N(k), R}, \widehat{\lambda}_{N(k), R}(\mathbf{Q})),$$

completing the proof. \square

3. FREE TCI FOR η

We first recall the free pressure π_R and the free entropy-like quantity η_R (the Legendre transform of π_R) introduced in [7]. For $R > 0$ fixed let $(\mathcal{A}_R^{(n)})^{sa}$ and $(M_{N,R}^{sa})^n$ be as in §1. For each $h \in (\mathcal{A}_R^{(n)})^{sa}$ the *free pressure* $\pi_R(h)$ of h is defined by

$$\pi_R(h) := \limsup_{N \rightarrow \infty} \left(\frac{1}{N^2} P_{N,R}(h) + \frac{n}{2} \log N \right),$$

where the (microstates) pressure function $P_{N,R}(h)$ is given as

$$P_{N,R}(h) := \log \int_{(M_{N,R}^{sa})^n} \exp(-N \text{Tr}_N(h(A_1, \dots, A_n))) d\Lambda_N^{\otimes n}(A_1, \dots, A_n).$$

Note that π_R is a convex function on $(\mathcal{A}_R^{(n)})^{sa}$ such that $|\pi_R(h_1) - \pi_R(h_2)| \leq \|h_1 - h_2\|_R$ for all $h_1, h_2 \in (\mathcal{A}_R^{(n)})^{sa}$. For $\tau \in TS(\mathcal{A}_R^{(n)})$ the quantity $\eta_R(\tau)$ is defined by

$$\eta_R(\tau) := \inf \left\{ \tau(h) + \pi_R(h) : h \in (\mathcal{A}_R^{(n)})^{sa} \right\}.$$

We then have

$$\pi_R(h) = \max \left\{ -\tau(h) + \eta_R(\tau) : \tau \in TS(\mathcal{A}_R^{(n)}) \right\}$$

so that π_R on $(\mathcal{A}_R^{(n)})^{sa}$ and η_R on $TS(\mathcal{A}_R^{(n)})$ are the Legendre transforms of each other with respect to the Banach space duality between $(\mathcal{A}_R^{(n)})^{sa}$ and $(\mathcal{A}_R^{(n)})^{*,sa}$ ($\supset TS(\mathcal{A}_R^{(n)})$). We say that $\tau \in TS(\mathcal{A}_R^{(n)})$ is an *equilibrium tracial state* associated with $h \in (\mathcal{A}_R^{(n)})^{sa}$ if the equality

$$\pi_R(h) = -\tau(h) + \eta_R(\tau) \tag{3.1}$$

holds. This equality is a kind of variational principle.

In this section we will prove the next TCI for non-commutative tracial distributions with η_R in place of χ . Since $\chi_R(\tau) \leq \eta_R(\tau)$ [7, Theorem 4.5], this TCI is sharper than that given in Theorem 2.2 though τ becomes restrictive here. But it is worth noting (see [6], also [14, V.1.1]) that the set of $\tau \in TS(\mathcal{A}_R^{(n)})$ satisfying the assumption in the theorem is norm-dense in $\{\tau \in TS(\mathcal{A}_R^{(n)}) : \eta_R(\tau) > -\infty\}$.

Theorem 3.1. *Let $\mathbf{Q} = (Q_1, \dots, Q_n)$ be an n -tuple of real-valued continuous functions on \mathbb{R} , and assume that there exists a constant $\rho > 0$ such that all $Q_i(x) - \frac{\rho}{2}x^2$, $1 \leq i \leq n$, are convex on \mathbb{R} . Assume $R \geq R_0$ with R_0 given in §2. If $\tau \in TS(\mathcal{A}_R^{(n)})$ is an equilibrium tracial state associated with some $h \in (\mathcal{A}_R^{(n)})^{sa}$, then*

$$W_2(\tau, \tau_{\mathbf{Q}}) \leq \sqrt{\frac{2}{\rho} \left(-\eta_R(\tau) + \tau \left(\sum_{i=1}^n Q_i(X_i) \right) + B_{\mathbf{Q}} \right)}, \tag{3.2}$$

where $B_{\mathbf{Q}}$ is the constant in (2.3).

The essence of the proof is same as that of Theorem 2.2 based on the random matrix approximation procedure. For each $n \in \mathbb{N}$ and $h \in (\mathcal{A}_R^{(n)})^{sa}$ define $\lambda_{N,R}(h) \in \mathcal{P}((M_{N,R}^{sa})^n)$ by

$$\frac{d\lambda_{N,R}(h)}{d\Lambda_N^{\otimes n}}(A_1, \dots, A_n) = \frac{1}{Z_{N,R}(h)} \exp(-N \text{Tr}_N(h(A_1, \dots, a_n))) \chi_{(M_{N,R}^{sa})^n}(A_1, \dots, A_n)$$

with the normalization constant $Z_{N,R}(h) := \exp(P_{N,R}(h))$. This $\lambda_{N,R}(h)$ is a unique probability measure on $(M_{N,R}^{sa})^n$ satisfying the (microstates) Gibbs variational principle

$$P_{N,R}(h) = -N^2 \widehat{\lambda}_{N,R}(h)(h) + S(\lambda_{N,R}(h)) \quad (3.3)$$

with the Boltzmann-Gibbs entropy

$$S(\lambda_{N,R}(h)) := - \int_{(M_{N,R}^{sa})^n} \frac{d\lambda_{N,R}(h)}{d\Lambda_N^{\otimes n}} \log \frac{d\lambda_{N,R}(h)}{d\Lambda_N^{\otimes n}} d\Lambda_N^{\otimes n}.$$

Proof of Theorem 3.1. We may prove that

$$\frac{1}{2} W_2(\tau_0, \tau_{\mathbf{Q}})^2 \leq \frac{1}{\rho} \left(-\pi_R(h_0) + \tau_0 \left(\sum_{i=1}^n Q_i(X_i) - h_0 \right) + B_{\mathbf{Q}} \right) \quad (3.4)$$

when $\tau_0 \in TS(\mathcal{A}_R^{(n)})$ and $h_0 \in (\mathcal{A}_R^{(n)})^{sa}$ satisfy the variational equality (3.1). Let us first assume that τ_0 is a unique equilibrium tracial state associated with h_0 (equivalently, π_R is differentiable at h_0). Choose a subsequence $N(1) < N(2) < \dots$ such that

$$\pi_R(h_0) = \lim_{k \rightarrow \infty} \left(\frac{1}{N(k)^2} P_{N(k),R}(h_0) + \frac{n}{2} \log N(k) \right) \quad (3.5)$$

and $\widehat{\lambda}_{N(k),R}(h_0)$ weakly* converges to some $\tau_1 \in TS(\mathcal{A}_R^{(n)})$. For every $h \in (\mathcal{A}_R^{(n)})^{sa}$ we get

$$\begin{aligned} & \widehat{\lambda}_{N(k),R}(h_0)(h) + \frac{1}{N(k)^2} P_{N(k),R}(h) \\ & \geq \frac{1}{N(k)^2} S(\lambda_{N(k),R}(h_0)) = \widehat{\lambda}_{N(k),R}(h_0)(h_0) + \frac{1}{N(k)^2} P_{N(k),R}(h_0) \end{aligned}$$

thanks to (3.3). From this and (3.5) as well as the weak* convergence of $\widehat{\lambda}_{N(k),R}(h_0)$ it is easy to see that

$$\tau_1(h) + \pi_R(h) \geq \tau_1(h_0) + \pi_R(h_0)$$

so that τ_1 is an equilibrium tracial state associated with h_0 . Therefore,

$$\widehat{\lambda}_{N(k),R}(h_0) \longrightarrow \tau_0 \text{ weakly* as } k \rightarrow \infty. \quad (3.6)$$

For every $N \in \mathbb{N}$ let $\lambda_N(\mathbf{Q})$, $Z_N(\mathbf{Q})$ and $\widehat{\lambda}_{N,R}(\mathbf{Q})$ be defined as in §2. By Lemmas 1.3 and 2.4, as in the proof of Theorem 2.2 we have

$$\begin{aligned} & \frac{1}{2}W_2(\widehat{\lambda}_{N,R}(h_0), \widehat{\lambda}_{N,R}(\mathbf{Q})) \\ & \leq \frac{1}{\rho N^2} \int_{(M_{N,R}^{sa})^n} \log \frac{d\lambda_{N,R}(h_0)}{d\lambda_N(\mathbf{Q})} d\lambda_{N,R}(h_0) \\ & = \frac{1}{\rho} \left(-\frac{1}{N^2} S(\lambda_{N,R}(h_0)) + \widehat{\lambda}_{N,R}(h_0) \left(\sum_{i=1}^n Q_i(X_i) \right) + \frac{1}{N^2} \log Z_N(\mathbf{Q}) \right) \\ & = \frac{1}{\rho} \left(-\frac{1}{N^2} P_{N,R}(h_0) + \widehat{\lambda}_{N,R}(h_0) \left(\sum_{i=1}^n Q_i(X_i) - h_0 \right) + \frac{1}{N^2} \log Z_N(\mathbf{Q}) \right) \end{aligned}$$

thanks to (3.1). Now, restrict the above estimates to the subsequence $N(1) < N(2) < \dots$ and apply (3.5) as well as (2.3). By 2° in §§1.3 together with Lemma 2.1 and (3.6), we then obtain

$$\begin{aligned} \frac{1}{2}W_2(\tau_0, \tau_{\mathbf{Q}})^2 & \leq \liminf_{k \rightarrow \infty} \frac{1}{2}W_2(\widehat{\lambda}_{N(k),R}(h_0), \widehat{\lambda}_{N(k),R}(\mathbf{Q}))^2 \\ & \leq -\pi_R(h_0) + \tau_0 \left(\sum_{i=1}^n Q_i(X_i) - h_0 \right) + B_{\mathbf{Q}}. \end{aligned}$$

Next, assume that τ_0 is a not necessarily unique equilibrium tracial state associated with h_0 . According to [15] (also [5, 6.2.43]), τ_0 belongs to the weakly* closed convex hull of the set \mathcal{T}_0 of $\tau \in TS(\mathcal{A}_R^{(n)})$ for which there exist $h_k \in (\mathcal{A}_R^{(n)})^{sa}$ and $\tau_k \in TS(\mathcal{A}_R^{(n)})$ such that τ_k is a unique equilibrium tracial state associated with h_k for each $k \in \mathbb{N}$, $\|h_k - h_0\|_R \rightarrow 0$ and $\tau_k \rightarrow \tau_0$ weakly*. To show (3.4) for τ_0 and h_0 , it suffices thanks to 2° and 3° in §§1.3 to prove it for every $\tau \in \mathcal{T}_0$ and h_0 . Let h_k and τ_k be as in the description of the set \mathcal{T}_0 . Then, the above-proven case implies that

$$\frac{1}{2}W_2(\tau_k, \tau_{\mathbf{Q}})^2 \leq -\pi_R(h_k) + \tau_k \left(\sum_{i=1}^n Q_i(X_i) - h_k \right) + B_{\mathbf{Q}}$$

for all $k \in \mathbb{N}$. Hence (3.4) for τ and h_0 is obtained by letting $k \rightarrow \infty$ in view of 2° in §§1.3 and the norm-continuity of π_R ; thus the proof is completed. \square

Corollary 3.2. *Let $\mathbf{Q} = (Q_1, \dots, Q_n)$ be an n -tuple of real-valued continuous functions on \mathbb{R} with the same assumption as in Theorem 3.1 for some $\rho > 0$, and let $R > 0$ be as in Theorem 3.1. Then $\tau_{\mathbf{Q}}$ is a unique equilibrium tracial state associated with $\sum_{i=1}^n Q_i(X_i) \in (\mathcal{A}_R^{(n)})^{sa}$.*

Proof. If $\tau_0 \in TS(\mathcal{A}_R^{(n)})$ is an equilibrium tracial state associated with $\sum_{i=1}^n Q_i(X_i)$, then the right-hand side of (3.2) (or (3.4)) is zero so that $\tau_0 = \tau_{\mathbf{Q}}$. \square

In particular, let $Q_i(x) = x^2/2$, $1 \leq i \leq n$, and $R \geq 2$. If $\tau \in TS(\mathcal{A}_R^{(n)})$ is an equilibrium tracial state associated with some $h \in (\mathcal{A}_R^{(n)})^{sa}$, then

$$W_2\left(\tau, \gamma_{0,2}^{\star n}\right) \leq \sqrt{2 \left(-\eta_R(\tau) + \tau \left(\frac{1}{2} \sum_{i=1}^n X_i^2 \right) + \frac{n}{2} \log 2\pi \right)},$$

where $\gamma_{0,2}^{\star n}$ denotes the distribution of a standard semicircular system. Hence, $\gamma_{0,2}^{\star n}$ is a unique equilibrium tracial state associated with $\frac{1}{2} \sum_{i=1}^n X_i^2$. This also says that η_R admits a unique

maximizer $\gamma_{0,2}^{\star n}$ when restricted on $\{\tau \in TS(\mathcal{A}_R^{(n)}) : \tau(\sum_{i=1}^n X_i^2) \leq n\}$, which is a refinement of the same result for χ in [24].

The question whether the TCI (3.2) holds or not for any $\tau \in TS(\mathcal{A}_R^{(n)})$ without the equilibrium assumption is still left open (and seems very important to obtain an in-depth understanding of η_R).

4. THE UNITARY CASE

4.1. TCI for χ_u . For each $n \in \mathbb{N}$, the universal free product C^* -algebra $C(\mathbb{T})^{\star n}$, where \mathbb{T} is the unit circle, is nothing but the universal group C^* -algebra $C^*(\mathbb{F}_n)$ of the free group \mathbb{F}_n of n generators. Let g_1, \dots, g_n denote the canonical n unitary generators of $C^*(\mathbb{F}_n)$. For each $\tau \in TS(C^*(\mathbb{F}_n))$ take the W^* -probability space $(\pi_\tau(C^*(\mathbb{F}_n))'', \tilde{\tau})$ via the GNS representation π_τ and define the *free entropy* (unitary version) $\chi_u(\tau)$ by

$$\chi_u(\tau) := \chi_u(\pi_\tau(g_1), \dots, \pi_\tau(g_n))$$

(see [9, §6.5] for the precise definition of the microstates free entropy for n -tuples of unitaries).

On the other hand, the p -Wasserstein distance $W_p(\tau_1, \tau_2)$ between $\tau_1, \tau_2 \in TS(C^*(\mathbb{F}_n))$ is defined by (1.2) with (g_1, \dots, g_n) in place of (a_1, \dots, a_n) . (Note that a_1, \dots, a_n were not necessarily self-adjoint in §§1.3.)

For each real-valued continuous function Q on \mathbb{T} , the functional

$$-\Sigma(\mu) + \mu(Q) := - \iint_{\mathbb{T}^2} \log |\zeta - \eta| d\mu(\zeta) d\mu(\eta) + \int_{\mathbb{T}} Q(\zeta) d\mu(\zeta), \quad \mu \in \mathcal{P}(\mathbb{T}),$$

has a unique minimizer μ_Q called the *equilibrium measure* associated with Q (see [19]). When $\mathbf{Q} = (Q_1, \dots, Q_n)$ is an n -tuple of real-valued continuous functions on \mathbb{T} , we define $\tau_{\mathbf{Q}} \in TS(C^*(\mathbb{F}_n))$ as the free product of μ_{Q_i} 's, i.e., $\tau_{\mathbf{Q}} := \star_{i=1}^n \mu_{Q_i}$ on $C^*(\mathbb{F}_n) = C(\mathbb{T})^{\star n}$.

The next theorem is the counterpart of Theorem 2.2 in the unitary setting.

Theorem 4.1. *Assume that there exists a constant $\rho > -\frac{1}{2}$ such that all $Q_i(e^{\sqrt{-1}t}) - \frac{\rho}{2}t^2$, $1 \leq i \leq n$, are convex on \mathbb{R} . Then we have*

$$W_2(\tau, \tau_{\mathbf{Q}}) \leq \sqrt{\frac{4}{1+2\rho} \left(-\chi_u(\tau) + \tau \left(\sum_{i=1}^n Q_i(g_i) \right) + B_{\mathbf{Q}} \right)}$$

for every $\tau \in TS(C^*(\mathbb{F}_n))$, where

$$B_{\mathbf{Q}} := \chi_u(\tau_{\mathbf{Q}}) - \tau_{\mathbf{Q}} \left(\sum_{i=1}^n Q_i(g_i) \right)$$

(See also 3° below for the constant $B_{\mathbf{Q}}$). Furthermore, $\tau_{\mathbf{Q}}$ is a unique minimizer of $-\chi_u(\tau) + \tau(\sum_{i=1}^n Q_i(g_i))$ for $\tau \in TS(C^*(\mathbb{F}_n))$.

In the special case where Q_i 's are all zero and so $\rho = 0$, the above inequality becomes

$$W_2(\tau, \gamma_0^{\star n}) \leq 2\sqrt{-\chi_u(\tau)}, \quad \tau \in TS(C^*(\mathbb{F}_n)),$$

where the free product state $\gamma_0^{\star n}$ is the distribution of a standard Haar unitary system of n variables.

A key idea in proving the theorem is to apply the classical TCI in the Riemannian setting in a certain random matrix approximation. Here, by a geometric reason on Ricci curvature tensors, random matrices at our disposal are special unitary ones instead of unitary. Some important facts needed in the proof are in order.

1° **The SU-microstates free entropy.** Even when $U(N)$ is replaced by $SU(N)$ in the definition of $\chi_u(u_1, \dots, u_n)$ [9, §6.5], the microstates free entropy introduced is the same. To prove this, define $\xi : \mathbb{T}^N \rightarrow \{(\zeta_1, \dots, \zeta_N) \in \mathbb{T}^N : \zeta_1 \cdots \zeta_N = 1\}$ by

$$\xi(\zeta_1, \dots, \zeta_N) := (\zeta_1(\zeta_1 \cdots \zeta_N)^{-1/N}, \dots, \zeta_N(\zeta_1 \cdots \zeta_N)^{-1/N}),$$

where $\zeta^{1/N}$ for $\zeta \in \mathbb{T}$ is the principal N th root and $\zeta^{-1/N} := (\zeta^{1/N})^{-1}$, and define $\Xi : U(N) \rightarrow SU(N)$ by $\Xi(U) := V \text{diag} \xi(\zeta_1, \dots, \zeta_N) V^*$ under a diagonalization $U = V \text{diag}(\zeta_1, \dots, \zeta_N) V^*$ with $V \in U(N)$ and $(\zeta_1, \dots, \zeta_N) \in \mathbb{T}^N$. Then, Ξ is a well-defined Borel measurable map and we have $\gamma_N^U \circ \Xi^{-1} = \gamma_N^{SU}$ (see §§1.1 for notations). Now, the above-mentioned fact can be directly shown by using the forms of γ_N^U and γ_N^{SU} under diagonalizations (see e.g. [12, §§1.5]).

2° **A key inequality.** For each $\lambda \in \mathcal{P}(SU(N)^n)$ define the distribution $\widehat{\lambda} \in TS(C^*(\mathbb{F}_n))$ by

$$\widehat{\lambda}(h) := \int_{SU(N)^n} \frac{1}{N} \text{Tr}_N(h(U_1, \dots, U_n)) d\lambda(U_1, \dots, U_n), \quad h \in C^*(\mathbb{F}_n),$$

where $h \in C^*(\mathbb{F}_n) \mapsto h(U_1, \dots, U_n) \in M_N(\mathbb{C})$ is the $*$ -homomorphism (“non-commutative functional calculus”) sending each g_i to U_i for each $(U_1, \dots, U_n) \in SU(N)$. For every $\lambda_1, \lambda_2 \in \mathcal{P}(SU(N)^n)$ we have

$$W_2(\widehat{\lambda}_1, \widehat{\lambda}_2) \leq \frac{1}{\sqrt{N}} W_{2, HS}(\lambda_1, \lambda_2) \leq \frac{1}{\sqrt{N}} W_{2, \text{geod}}(\lambda_1, \lambda_2),$$

where $W_{2, HS}(\lambda_1, \lambda_2)$ is the Wasserstein distance with respect to the distance on $SU(N)^n$ induced by the Hilbert-Schmidt norm, while $W_{2, \text{geod}}(\lambda_1, \lambda_2)$ with respect to the geodesic distance. The proof of the first inequality is similar to that of Lemma 1.3 while the second is obvious.

3° **Asymptotic freeness for SU-random matrices.** Let $\mathbf{Q} = (Q_1, \dots, Q_n)$ be real-valued continuous functions on \mathbb{T} . For each $n \in \mathbb{N}$ define $\lambda_N(\mathbf{Q}) \in \mathcal{P}(SU(N)^n)$ by the product measure $\lambda_N(\mathbf{Q}) := \bigotimes_{i=1}^n \lambda_N(Q_i)$ of

$$d\lambda_N(Q_i)(U) := \frac{1}{Z_N(Q_i)} \exp(-N \text{Tr}_N(Q_i(U))) d\gamma_N^{SU}(U)$$

with a normalization constant $Z_N(Q_i)$. The asymptotic freeness for unitary random matrices due to [22] remains valid for special unitary random matrices. In fact, a stronger result on the almost sure asymptotic freeness for independent special unitary random matrices can be shown by modifying the proof in [9, 4.3.5]. Also, as a consequence of the large deviation theorem [13, Theorem 2.1], it follows that the mean eigenvalue distribution of $\lambda_N(Q_i)$ converges to μ_{Q_i} for $1 \leq i \leq n$. We thus see that

$$\widehat{\lambda}_N(\mathbf{Q}) \longrightarrow \tau_{\mathbf{Q}} = \star_{i=1}^n \mu_{Q_i} \text{ weakly}^*.$$

Moreover, since

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log Z_N(Q_i) = \Sigma(\mu_{Q_i}) - \mu_{Q_i}(Q_i)$$

(see [13, Theorem 2.1]), we notice that the constant $B_{\mathbf{Q}}$ in Theorem 4.1 can be expressed as

$$B_{\mathbf{Q}} = \sum_{i=1}^n (\Sigma(\mu_{Q_i}) - \mu_{Q_i}(Q_i)) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{i=1}^n \log Z_N(Q_i).$$

4° **TCI on $SU(N)^n$.** Let $\mathbf{Q} = (Q_1, \dots, Q_n)$ be as in Theorem 4.1 and assume further that all Q_i 's are C^2 -functions. Then, thanks to [12, Lemmas 1.2 and 1.3], the function

$$\Psi_N(U_1, \dots, U_n) := N \text{Tr}_N \left(\sum_{i=1}^n Q_i(U_i) \right)$$

is a C^2 -function on $SU(N)^n$ with the Hessian $\text{Hess}(\Psi_N) \geq N\rho I_{(N^2-1)n}$. Also, note that the Ricci curvature tensor of $SU(N)^n$ is $\text{Ric}(SU(N)^n) = \frac{N}{2} I_{(N^2-1)n}$. Hence, by the TCI in the Riemannian manifold setting due to [18] combined with [1], we obtain

$$W_{2, \text{geod}}(\lambda, \lambda_N(\mathbf{Q})) \leq \sqrt{\frac{4}{N(1+2\rho)} S(\lambda, \lambda_N(\mathbf{Q}))}$$

for every $\lambda \in \mathcal{P}(SU(N)^n)$.

Now, the proof of Theorem 4.1 based on the above facts 1°–4° is analogous to that of Theorem 2.2; so the details are left to the reader. But, it is worthwhile to note one more point. As in the proof of [12, Theorem 2.7], the regularization technique by the use of Poisson integrals enables us to assume that all Q_i 's are smooth functions on \mathbb{T} so that one can go through with 4°.

4.2. **TCI for η_u .** For each $h \in C^*(\mathbb{F}_n)^{sa}$ we introduce the *free pressure* (unitary version) $\pi_u(h)$ by

$$\begin{aligned} \pi_u(h) &:= \limsup_{N \rightarrow \infty} \left(\frac{1}{N^2} \log \int_{U(N)^n} \exp(-N \text{Tr}_N(h(U_1, \dots, U_n))) d(\gamma_N^U)^{\otimes n}(U_1, \dots, U_n) \right) \\ &= \limsup_{N \rightarrow \infty} \left(\frac{1}{N^2} \log \int_{SU(N)^n} \exp(-N \text{Tr}_N(h(U_1, \dots, U_n))) d(\gamma_N^{SU})^{\otimes n}(U_1, \dots, U_n) \right). \end{aligned}$$

The equality of the two lim sup's can be shown from the fact stated in the above 1°. As in the self-adjoint setting [7, Proposition 2.3], π_u is convex on $C^*(\mathbb{F}_n)^{sa}$ and $|\pi_u(h_1) - \pi_u(h_2)| \leq \|h_1 - h_2\|$ for all $h_1, h_2 \in C^*(\mathbb{F}_n)^{sa}$. It is seen as in [7, Theorem 3.4] (or (3.1)) that π_u is the converse Legendre transform of η_u as

$$\pi_u(h) = \max\{-\tau(h) + \eta_u(\tau) : \tau \in TS(C^*(\mathbb{F}_n))\}, \quad h \in C^*(\mathbb{F}_n)^{sa},$$

and we say that τ is an *equilibrium tracial state* associated with h if $\pi_u(h) = -\tau(h) + \eta_u(h)$ holds.

In particular when $N = 1$ and $\mu \in \mathcal{P}(\mathbb{T})$ ($= TS(C(\mathbb{T}))$) we have $\eta_u(\mu) = \chi_u(\mu)$ ($= \Sigma(\mu)$) (see [8, §6]). The proof of the next theorem is similar to that of [7, Theorem 4.5].

Theorem 4.2. *We have $\chi_u(\tau) \leq \eta_u(\tau)$ for every $\tau \in TS(C^*(\mathbb{F}_n))$. Moreover, if τ is a free product tracial state (i.e., g_1, \dots, g_n are $*$ -free with respect to τ), then $\chi_u(\tau) = \eta_u(\tau)$.*

Furthermore, by considering the minimal C^* -tensor product $C^*(\mathbb{F}_n) \otimes_{\min} C^*(\mathbb{F}_n)$ as in [7, §6], the definition of η_u can be modified so that the modified $\tilde{\eta}_u(\tau)$ is equal to $\chi_u(\tau)$ for all $\tau \in TS(C^*(\mathbb{F}_n))$. This result is of considerable importance but it is not directly related to the free TCI in Theorem 4.3; so we omit the details.

Finally, we state the counterpart of Theorem 3.1 in the unitary setting; the TCI is sharper than that in Theorem 4.1 though τ is rather restricted. The structure of the proof is quite parallel with that of Theorem 3.1 and the details are again left to the reader.

Theorem 4.3. *Let $\mathbf{Q} = (Q_1, \dots, Q_n)$ be real-valued continuous functions on \mathbb{T} satisfying the same assumption as in Theorem 4.1 with a constant $\rho > -\frac{1}{2}$. If $\tau \in TS(C^*(\mathbb{F}_n))$ is an equilibrium tracial state associated with some $h \in C^*(\mathbb{F}_n)^{sa}$, then*

$$W_2(\tau, \tau_{\mathbf{Q}}) \leq \sqrt{\frac{4}{1+2\rho} \left(-\eta_u(\tau) + \tau \left(\sum_{i=1}^n Q_i(g_i) \right) + B_{\mathbf{Q}} \right)},$$

where $B_{\mathbf{Q}}$ is the same constant as in Theorem 4.1. Furthermore, $\tau_{\mathbf{Q}}$ is a unique equilibrium tracial state associated with $\sum_{i=1}^n Q_i(g_i) \in C^*(\mathbb{F}_n)^{sa}$.

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