

ON PEAK PHENOMENA FOR NON-COMMUTATIVE H^∞

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Dedicated to Professor Fumio Hiai on the occasion of his 60th birthday

ABSTRACT. A non-commutative extension of Amar and Lederer's peak set result is given. As its simple applications it is shown that any non-commutative H^∞ -algebra $H^\infty(M, \tau)$ has unique predual, and moreover some restriction in some of the results of Blecher and Labuschagne are removed, making them hold in full generality.

1. INTRODUCTION

Let $H^\infty(\mathbb{D})$ be the Banach algebra of all bounded analytic functions on the unit disk \mathbb{D} equipped with the supremum norm $\|\cdot\|_\infty$. It is known (but non-trivial) that $H^\infty(\mathbb{D})$ can be regarded as a closed subalgebra of $L^\infty(\mathbb{T})$ by $f(e^{\sqrt{-1}\theta}) := \lim_{r \nearrow 1} f(re^{\sqrt{-1}\theta})$ a.e. θ . Then, $L^\infty(\mathbb{T})$ is isometrically isomorphic to $C(X)$ with a certain compact Hausdorff space X via the Gel'fand representation $f \mapsto \hat{f}$, and the linear functional $f \in H^\infty(\mathbb{D}) \mapsto \frac{1}{2\pi} \int_0^{2\pi} f(e^{\sqrt{-1}\theta}) d\theta$ is known to admit a unique representing measure m on X so that $\frac{1}{2\pi} \int_0^{2\pi} f(e^{\sqrt{-1}\theta}) d\theta = \int_X \hat{f}(x) m(dx)$ holds. In this setup, Amar and Lederer [3] proved that any closed subset $F \subset X$ with $m(F) = 0$ admits $f \in H^\infty(\mathbb{D})$ with $\|f\|_\infty \leq 1$ such that $P := \{x \in X : \hat{f}(x) = 1\} = \{x \in X : |\hat{f}(x)| = 1\}$ contains F and $m(P) = 0$ still holds. This is a key in any existing proof of the uniqueness of predual of $H^\infty(\mathbb{D})$. The reader can find some information on Amar and Lederer's result in [20, §6] and also see [6].

The main purpose of these notes is to provide an analogous fact of the above-mentioned result of Amar and Lederer for non-commutative H^∞ -algebras introduced by Arveson [5] in the 60's under the name of finite maximal subdiagonal algebras. Here a non-commutative H^∞ -algebra means a σ -weakly closed (possibly non-self-adjoint) unital subalgebra A of a finite von Neumann algebra M with a faithful normal tracial state τ satisfying the following conditions:

- the unique τ -preserving (i.e., $\tau \circ E = \tau$) conditional expectation $E : M \rightarrow D := A \cap A^*$ is multiplicative on A ;
- the σ -weak closure of $A + A^*$ is exactly M ,

where $A^* := \{a^* \in M : a \in A\}$. (Remark here that an important work due to Exel [11] plays an important rôle behind this simple definition.) In what follows we write $A = H^\infty(M, \tau)$ and call D the diagonal subalgebra. Recently, in their series of papers Blecher and Labuschagne established many fundamental properties of these non-commutative H^∞ -algebras, analogous to classical theories modeled after $H^\infty(\mathbb{D})$, all of which are nicely summarized in [8]. The reader can also find a nice exposition (especially, on the non-commutative Hilbert transform in the framework of $H^\infty(M, \tau)$) in Pisier and Xu's survey on non-commutative L^p -spaces [23, §8].

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More precisely, what we want to prove here is that for any non-zero singular $\varphi \in M^*$ in the sense of Takesaki [29] one can find a “peak” projection p for A in the sense of Hay [16] such that p dominates the (right) support projection of φ but is smaller than the central support projection $z_s \in M^{**}$ of the singular part $M^* \ominus M_*$. This is not exactly same as Amar and Lederer’s result, but is enough for usual applications (even in classical theory for $H^\infty(\mathbb{D})$). Indeed, we will demonstrate it by proving that any non-commutative H^∞ -algebra $A = H^\infty(M, \tau)$ has the unique predual M_*/A_\perp with $A_\perp := \{\psi \in M_* : \psi|_A = 0\}$. Proving it is our initial motivation; in fact, it can be regarded as an affirmative answer to the following natural (at least for us) question: Is the relative topology on A induced from $\sigma(M, M_*)$, which is most important, an intrinsic one of A ? Also, our unique predual result may provide a new perspective in the direction of establishing the uniqueness of preduals by Grothendieck [15] for L^∞ -spaces, by Dixmier [10] and Sakai [25] for von Neumann algebras or W^* -algebras, and then by Ando [4] and also a little bit later but independent work due to Wojtaszczyk [32] for $H^\infty(\mathbb{D})$. In particular, our result can be regarded as a simultaneous generalization of those classical results. Moreover, our result is an affirmative answer to a question posed by Godefroy stated in [8], and more importantly it covers any existing generalization like [9],[14] of the above-mentioned work for $H^\infty(\mathbb{D})$ as a particular case. A natural “Lebesgue decomposition” or “normal/singular decomposition” for the dual of $H^\infty(M, \tau)$ is also given. The decomposition was first given by our ex-student Shintaro Sewatari in his master thesis [27] as a simple application of the non-commutative F. and M. Riesz theorem recently established by Blecher and Labuschagne [7] so that the finite dimensionality assumption for the diagonal subalgebra D was necessary there. Here it is established in full generality based on our Amar–Lederer type result instead of the non-commutative F. and M. Riesz theorem. After the completion of the presented work, the author found the paper [22] of H. Pfizner, where it is shown that any separable L -embedded Banach space X becomes the unique predual of its dual X^* . This means that establishing the Lebesgue decomposition is enough to show the uniqueness of predual for any non-commutative H^∞ -algebra $A = H^\infty(M, \tau)$ with M_* separable.

Our Amar–Lederer type result also enables us to remove the finite dimensionality assumption for the diagonal subalgebra D from the results in [7] numbered 3.5, 4.1, 4.2 and 4.3 there, including the non-commutative Gleason–Whitney theorem. Moreover, it gives a nice variant of Blecher and Labuschagne’s non-commutative F. and M. Riesz theorem. Thus, it unexpectedly brings the current theory of non-commutative H^p -spaces due to Blecher and Labuschagne (see [8]), which was already somewhat complete and satisfying, to an even more perfect and satisfactory form, though the presented work was initially aimed to prove the unique predual result for $H^\infty(M, \tau)$ as mentioned above.

In closing, we should note that a bit different syntax has been (and will be) used for dual spaces. For a Banach space X we denote by X^* and X_* its dual and predual instead of the usual X^* and X_* , while X^* stands for the set of adjoints of elements in X when X is a subset of a C^* -algebra.

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2. AMAR–LEDERER TYPE RESULT FOR $H^\infty(M, \tau)$

Let $A = H^\infty(M, \tau)$ be a non-commutative H^∞ -algebra with a finite von Neumann algebra M and a faithful normal tracial state τ on M .

Theorem 2.1. *For any non-zero singular $\varphi \in M^*$ there is a contraction $a \in A$ and a projection $p \in M^{**}$ such that*

- (2.1.1) a^n converges to p in the w^* -topology $\sigma(M^{**}, M^*)$ as $n \rightarrow \infty$;
- (2.1.2) $\langle |\varphi|, p \rangle = |\varphi|(1)$;
- (2.1.3) $\langle \psi, p \rangle = 0$ for all $\psi \in M_*$ (regarded as a subspace of M^*), or equivalently a^n converges to 0 in $\sigma(M, M_*)$ as $n \rightarrow \infty$. This, in particular, shows that $p \leq z_s$.

Here, $\langle \cdot, \cdot \rangle : M^* \times M^{**} \rightarrow \mathbf{C}$ is the dual pairing and $|\varphi|$ denotes the absolute value of φ with the polar decomposition $\varphi = v \cdot |\varphi|$ due to Sakai [26] and Tomita [31], when regarding φ as an element in the predual of the enveloping von Neumann algebra M^{**} by $(M^{**})_* = M^*$.

Proof. Note that $|\varphi|$ is still singular. In fact, $|\varphi| = v^* \cdot \varphi \in v^* z_s M^* \subset z_s M^*$ since z_s is a central projection. Here z_s stands for the central support projection of $M^* \ominus M_*$ as in §1. The orthogonal families of non-zero projections in $\text{Ker}|\varphi|$ clearly form an inductive set by inclusion, and then Zorn's lemma ensures the existence of a maximal family $\{q_k\}$, which is at most countable since M is σ -finite. Let $q_0 := \sum_k q_k$ in M . If $q_0 \neq 1$, then Takesaki's criterion [30] shows the existence of a non-zero projection $r \in \text{Ker}|\varphi|$ with $r \leq 1 - q_0$, a contradiction to the maximality. Thus, $q_0 = 1$. Moreover, if $\{q_k\}$ is a finite set, then $|\varphi|(1) = \sum_k |\varphi|(q_k) = 0$, a contradiction. Therefore, $\{q_k\}$ is a countably infinite family with $\sum_k q_k = 1$ in M . Letting $p_n := 1 - \sum_{k \leq n} q_k$ we have $p_n \rightarrow 0$ σ -weakly as $n \rightarrow \infty$ but $|\varphi|(p_n) = |\varphi|(1)$ for all n . Set $p_0 := \bigwedge_n p_n$ in M^{**} . Then, $\langle |\varphi|, p_0 \rangle = \lim_n \langle |\varphi|, p_n \rangle = \lim_n |\varphi|(p_n) = |\varphi|(1) \neq 0$, and in particular, $p_0 \neq 0$.

Choosing a subsequence if necessary, we may and do assume $\tau(p_n) \leq n^{-6}$. Then we can define an element $g := \sum_{n=1}^\infty n p_n \in L^2(M, \tau)$, the non-commutative L^2 -space associated with (M, τ) , since $\sum_{n=1}^\infty \|n p_n\|_{2, \tau} \leq \sum_{n=1}^\infty n^{-2} < +\infty$. By the non-commutative Riesz theorem [24, Theorem 1] and [18, Theorem 5.4] there is an element $\tilde{g} = \tilde{g}^* \in L^2(M, \tau)$ such that $f := g + \sqrt{-1}\tilde{g}$ falls in the closure $[A]_{2, \tau}$ of A in $L^2(M, \tau)$ via the canonical embedding $M \hookrightarrow L^2(M, \tau)$. We can regard $g, \tilde{g}, f \in L^2(M, \tau)$ as unbounded operators, affiliated with M , on the Hilbert space $\mathcal{H} := L^2(M, \tau)$ with a common core \mathcal{D} . Let $\xi \in \mathcal{D}$ be chosen arbitrary. Since $g \geq 0$ and $\tilde{g} = \tilde{g}^*$, one has $\|(1+f)\xi\|_{2, \tau} \|\xi\|_{2, \tau} \geq |((1+f)\xi|\xi)_\tau| = |(\xi|\xi)_\tau + (g\xi|\xi)_\tau + \sqrt{-1}(\tilde{g}\xi|\xi)_\tau| \geq \|\xi\|_{2, \tau}^2$ and similarly $\|(1+f)^*\xi\|_{2, \tau} \|\xi\|_{2, \tau} \geq \|\xi\|_{2, \tau}^2$, and hence $(1+f)^{-1} \in M$ exists and $\|(1+f)^{-1}\|_\infty \leq 1$. Also, similarly one has $\|(1+f)\xi\|_{2, \tau} \|\xi\|_{2, \tau} \geq |((1+f)\xi|f\xi)_\tau| = |(\xi|f\xi)_\tau + (f\xi|f\xi)_\tau| = |(\xi|g\xi)_\tau - \sqrt{-1}(\xi|\tilde{g}\xi)_\tau + (f\xi|f\xi)_\tau| \geq \|f\xi\|_{2, \tau}^2$ so that $\|f\xi\|_{2, \tau} \leq \|(1+f)\xi\|_{2, \tau}$ holds. Therefore, $\|f(1+f)^{-1}\zeta\|_{2, \tau} \leq \|\zeta\|_{2, \tau}$ for all $\zeta \in \mathcal{H}$, and thus $b := f(1+f)^{-1} \in M$ is a contraction.

We will then prove that b actually falls in A . First, recall the following standard but non-trivial fact: any bounded element in the closure $[A]_{p, \tau}$ of A in $L^p(M, \tau)$, the non-commutative L^p -space, falls in A . In fact, let $x \in [A]_{p, \tau}$ be a bounded element, i.e., $x \in M$, and then there is a sequence $\{a_n\}$ in A with $\|a_n - x\|_{p, \tau} \rightarrow 0$ as $n \rightarrow \infty$. For each $y \in A$ with $E(y) = 0$ one has $\|a_n y - xy\|_{p, \tau} \rightarrow 0$ as $n \rightarrow \infty$ so that $\tau(xy) = \lim_n \tau(a_n y) = 0$ implying $x \in A$, where we use $A = \{x \in M : \tau(xy) = 0 \text{ for all } y \in A \text{ with } E(y) = 0\}$ due to Arveson [5]. (It seems that this fact is used but not mentioned explicitly in the final step of the proof of [24, Lemma 2] that we need here). Letting $g_N := \sum_{n=1}^N n p_n \in M$ with its conjugate \tilde{g}_N we have $f_N = g_N + \sqrt{-1}\tilde{g}_N \rightarrow f$ in $L^2(M, \tau)$ as $N \rightarrow \infty$ thanks to the non-commutative Riesz theorem [24, Theorem 1] and [18, Theorem 5.4]. As before, for each N one has $(1+f_N)^{-1} \in M$ and $\|(1+f_N)^{-1}\|_\infty \leq 1$, and moreover the discussion in [24, Lemma 2] shows that $(1+f_N)^{-1}$ indeed falls in A . Since $(1+f)^{-1} \in M$ and $\|(1+f)^{-1}\|_\infty \leq 1$ as shown before, we have, for each $\xi \in M \subset L^2(M, \tau)$ (a right-bounded vector in $L^2(M, \tau)$), $\|((1+f_N)^{-1} - (1+f)^{-1})\xi\|_{2, \tau} = \|(1+f_N)^{-1}(f - f_N)(1+f)^{-1}\xi\|_{2, \tau} \leq \|\xi\|_\infty \|f - f_N\|_{2, \tau} \rightarrow 0$ as $N \rightarrow \infty$ so that $(1+f)^{-1} =$

$\lim_N(1 + f_N)^{-1} \in A$ in strong operator topology, implying $b = f(1 + f)^{-1} \in M \cap [A]_{2,\tau} = A$ as claimed above.

As before we have $\|(1 + f)\xi\|_{2,\tau}\|\xi\|_{2,\tau} \geq |((1 + f)\xi|\xi)_\tau| \geq (g\xi|\xi)_\tau \geq n(p_n\xi|\xi)_\tau = n\|p_n\xi\|_{2,\tau}^2$ for each $\xi \in \mathcal{D}$. Here the inequality $(g\eta|\eta)_\tau \geq n(p_n\eta|\eta)_\tau$ for η in the domain of g is used. (This can be easily checked when η is in $M \subset L^2(M, \tau)$, and $M \subset L^2(M, \tau)$ is known to form a core of g thanks to a classical result, see, e.g. [28, Theorem 9.8]). Thus, letting $\xi := (1 + f)^{-1}\zeta$ for each $\zeta \in \mathcal{H}$ we get $\|p_n(1 + f)^{-1}\zeta\|_{2,\tau}^2 \leq n^{-1}\|\zeta\|_{2,\tau}\|(1 + f)^{-1}\zeta\|_{2,\tau} \leq n^{-1}\|\zeta\|_{2,\tau}^2$ so that $\|p_n - p_nb\|_\infty = \|p_n(1 + f)^{-1}\|_\infty \leq n^{-1/2}$. In the universal representation $M \curvearrowright \mathcal{H}_u$ we have $\|(p_0 - p_0b)\zeta\|_{\mathcal{H}_u} \leq \|p_0\zeta - p_n\zeta\|_{\mathcal{H}_u} + \|p_n - p_nb\|_\infty\|\zeta\|_{\mathcal{H}_u} + \|p_n(b\zeta) - p_0(b\zeta)\|_{\mathcal{H}_u} \leq \|p_0\zeta - p_n\zeta\|_{\mathcal{H}_u} + n^{-1/2}\|\zeta\|_{\mathcal{H}_u} + \|p_n(b\zeta) - p_0(b\zeta)\|_{\mathcal{H}_u} \rightarrow 0$ as $n \rightarrow \infty$ for each $\zeta \in \mathcal{H}_u$ since $p_0 = \bigwedge_n p_n$ in $M^{**} = M''$ on \mathcal{H}_u . Since b is a contraction, we get $p_0 = p_0b = bp_0 = p_0bp_0$. Then, by [16, Lemma 3.7] the new contraction $a := (1 + b)/2$ satisfies that a^n converges to a certain projection $p \in M^{**}$ in $\sigma(M^{**}, M^*)$ as $n \rightarrow \infty$, and $p_0 \leq p$ so that $\langle |\varphi|, p \rangle = |\varphi|(1)$. If a vector $\xi \in \mathcal{H}$ satisfies $\|a\xi\|_{2,\tau} = \|\xi\|_{2,\tau}$, then $2\|\xi\|_{2,\tau} = \|\xi + b\xi\|_{2,\tau} \leq \|\xi\|_{2,\tau} + \|b\xi\|_{2,\tau} \leq 2\|\xi\|_{2,\tau}$, which implies $\|b\xi\|_{2,\tau} = \|\xi\|_{2,\tau}$ and $\|\xi + b\xi\|_{2,\tau} = \|\xi\|_{2,\tau} + \|b\xi\|_{2,\tau}$. Then, it is plain to see that these two norm conditions imply $b\xi = \xi$. However, $(1 + f)^{-1}\xi = (1 - b)\xi = 0$ so that $\xi = 0$. Therefore, there is no reducing subspace of a in \mathcal{H} , on which a acts as a unitary. Hence the so-called Foguel decomposition ([12]) shows that $a^n \rightarrow 0$ σ -weakly as $n \rightarrow \infty$. In particular, $\langle \psi, p \rangle = \lim_n \langle \psi, a^n \rangle = \lim_n \psi(a^n) = 0$ for all $\psi \in M_\star$. \square

Choose $\varphi \in M^*$, and decompose it into the normal and singular parts $\varphi = \varphi_n + \varphi_s$ with $\varphi_n := (1 - z_s) \cdot \varphi \in M_\star$ and $\varphi_s := z_s \cdot \varphi \in M^* \ominus M_\star$. Assume that $\varphi_s \neq 0$, and let $p \in M^{**}$ be a projection for φ_s as in Theorem 2.1. By (2.1.2) and the polar decomposition $\varphi_s = v \cdot |\varphi_s|$ we have $|\langle \varphi_s, (1 - p)x \rangle| = |\langle v \cdot |\varphi_s|, (1 - p)x \rangle| \leq \langle |\varphi_s|, 1 - p \rangle^{1/2} \langle |\varphi_s|, v^*x^*xv \rangle^{1/2} = 0$ for every $x \in M^{**}$ so that $\varphi_s \cdot (1 - p) = 0$, i.e., $\varphi_s = \varphi_s \cdot p$. Moreover, by (2.1.3) a similar estimate shows $\varphi_n \cdot p = 0$. Hence, we get $\varphi_s = \varphi \cdot p$. Therefore we have the following corollary:

Corollary 2.2. *If $\varphi \in M^*$ has the non-zero singular part $\varphi_s \in M^* \ominus M_\star$, then there is a contraction $a \in A$ and a projection $p \in M^{**}$ such that $a^n \rightarrow p$ in $\sigma(M^{**}, M^*)$, $a^n \rightarrow 0$ in $\sigma(M, M_\star)$ as $n \rightarrow \infty$ and $\varphi_s = \varphi \cdot p$.*

We next examine the contraction a and the projection p in Theorem 2.1 and/or Corollary 2.2. By [16, Lemma 3.6], a peaks at p and moreover $(a^*a)^n \searrow p$ in $\sigma(M^{**}, M^*)$ as $n \rightarrow \infty$ so that p is a closed projection in the sense of Akemann [1],[2]. For any positive $\psi \in M^*$ one has $\sum_{n=2}^N |\psi((a^*a)^n - (a^*a)^{n-1})| = -\sum_{n=2}^N \psi((a^*a)^n - (a^*a)^{n-1}) = \psi(a^*a) - \psi((a^*a)^N) \rightarrow \langle \psi, a^*a - p \rangle$ as $N \rightarrow \infty$, from which one easily observes that the sequence $\{(a^*a)^n\}$ is weakly unconditionally convergent, see, e.g. [13, Définition 1]. This fact is necessary in the course of proving that M_\star/A_\perp is the unique predual of A .

3. FIRST APPLICATIONS: PREDUAL OF $H^\infty(M, \tau)$

We first establish the following theorem:

Theorem 3.1. *M_\star/A_\perp is the unique predual of $A = H^\infty(M, \tau)$.*

A Banach space E is said to have a unique predual when the following property holds: If the duals F^* and G^* of two other Banach spaces F and G are isometrically isomorphic to E , then $F = G$ must hold in the dual E^* via the canonical embeddings. Our discussion will be done in the line presented in [14, IV] so that what we will actually prove is that M_\star/A_\perp has property (X) in the sense of Godefroy and Talagrand and the desired assertion immediately follows from their result, see [13, Définition 3, Théorème 5].

Proof. Choose $\varphi \in A^*$, and then one can extend it to $\tilde{\varphi} \in M^*$ by the Hahn–Banach extension theorem. Decompose $\tilde{\varphi}$ into the normal and singular parts $\tilde{\varphi} = \tilde{\varphi}_n + \tilde{\varphi}_s$. It suffices to show the following: If $\lim_n \varphi(x_n) = 0$ for any weakly unconditionally convergent sequence $\{x_n\}$ in A with $x_n \rightarrow 0$ in $\sigma(A, M_\star/A_\perp)$ or the relative topology from $\sigma(M, M_\star)$ as $n \rightarrow \infty$, then $\tilde{\varphi}_s|_A = 0$, that is, $\varphi = \tilde{\varphi}_n|_A$ must hold. We may assume $\tilde{\varphi}_s \neq 0$. By Corollary 2.2 together with the discussion just below it, we can find two sequences $\{a_n\}$ and $\{b_n\}$ and a projection $p \in M^{\star\star}$ such that (i) the a_n 's are in A ; (ii) the b_n 's are in M and $\{b_n\}$ is weakly (in $\sigma(M, M^{\star\star})$) unconditionally convergent; (iii) both a_n and b_n converge to p in $\sigma(M^{\star\star}, M^{\star\star})$ but to 0 in $\sigma(M, M_\star)$; (iv) $\tilde{\varphi}_s = \tilde{\varphi} \cdot p$. Then, as same as in [14, Théorème 33] (by using a trick in [17, the proof of Proposition 1.c.3 in p.32]) we may and do assume that $\{a_n\}$ is also weakly unconditionally convergent. Let $x \in A$ be chosen arbitrary, and then $\{a_n x\}$ clearly becomes weakly unconditionally convergent. Moreover, it trivially holds that $a_n x \rightarrow 0$ in $\sigma(M, M_\star)$ as $n \rightarrow \infty$. Therefore, we have $\tilde{\varphi}_s(x) = \langle \tilde{\varphi}, px \rangle = \lim_n \langle \tilde{\varphi}, a_n x \rangle = \lim_n \varphi(a_n x) = 0$ by the assumption here. \square

As is well-known the predual M_\star of a von Neumann algebra M can be naturally embedded to the dual M^* as the range of an L -projection, see [29]. Hence it is natural to ask whether the predual M_\star/A_\perp of $A = H^\infty(M, \tau)$ can be also embedded to the dual A^* as the range of an L -projection. This is indeed true in general. Here we will explain it as an application of our Amar–Lederer type result.

Denote by A_n^\star the set of all $\varphi \in A^*$ that can be extended to $\tilde{\varphi} \in M_\star$, and also by A_s^\star the set of all $\psi \in A^*$ that can be extended to $\tilde{\psi} \in M^* \ominus M_\star$. This definition agrees with [4, p.35]. For any $\varphi \in A^*$, by the Hahn–Banach extension theorem one can extend it to $\tilde{\varphi} \in M^*$. Then, decompose $\tilde{\varphi}$ into the normal and singular parts $\tilde{\varphi} = \tilde{\varphi}_n + \tilde{\varphi}_s$. We set $\varphi_n := \tilde{\varphi}_n|_A \in A_n^\star$ and $\varphi_s := \tilde{\varphi}_s|_A \in A_s^\star$. Then we call $\varphi = \varphi_n + \varphi_s$ an “ $(M \supset A)$ -Lebesgue decomposition” of φ . On first glance, it is likely that this decomposition depends on the particular choice of the extension $\tilde{\varphi}$. However, we have:

Proposition 3.2. *The following hold true:*

$$(3.4.1) \quad A_n^\star \cap A_s^\star = \{0\}.$$

(3.4.2) *The notion of $(M \supset A)$ -Lebesgue decomposition $\varphi = \varphi_n + \varphi_s$ of $\varphi \in A^*$ is well-defined, that is, φ_n and φ_s are uniquely determined by φ . Moreover, $\|\varphi\| = \|\varphi_n\| + \|\varphi_s\|$ holds.*

Proof. (3.4.1) On contrary, suppose that there is a non-zero $\varphi \in A_n^\star \cap A_s^\star$, and then one can choose $\tilde{\varphi}_n \in M_\star$ and $\tilde{\varphi}_s \in M^* \ominus M_\star$ in such a way that $\varphi = \tilde{\varphi}_n|_A = \tilde{\varphi}_s|_A$. Since $\varphi \neq 0$ implies $\tilde{\varphi}_s \neq 0$, one can find, by Corollary 2.2, a contraction $a \in A$ and a projection $p \in M^{\star\star}$ so that $a^n \rightarrow p$ in $\sigma(M^{\star\star}, M^{\star\star})$, $a^n \rightarrow 0$ in $\sigma(M, M_\star)$ as $n \rightarrow \infty$ and $\tilde{\varphi}_s = \tilde{\varphi}_s \cdot p$. Let $x \in A$ be arbitrary, and $a^n x \rightarrow 0$ in $\sigma(M, M_\star)$ clearly holds. Then one has $\varphi(x) = \tilde{\varphi}_s(x) = \langle \tilde{\varphi}_s, px \rangle = \lim_n \langle \tilde{\varphi}_s, a^n x \rangle = \lim_n \varphi(a^n x) = \lim_n \tilde{\varphi}_n(a^n x) = 0$, a contradiction.

(3.4.2) Assume that we have two $(M \supset A)$ -Lebesgue decompositions $\varphi = \varphi_{n1} + \varphi_{s1} = \varphi_{n2} + \varphi_{s2}$. Then $\varphi_{n1} - \varphi_{n2} = \varphi_{s2} - \varphi_{s1} \in A_n^\star \cap A_s^\star = \{0\}$ by (3.4.1) so that $\varphi_{n1} = \varphi_{n2}$ and $\varphi_{s1} = \varphi_{s2}$. Hence the $(M \supset A)$ -Lebesgue decomposition is well-defined. Let $\tilde{\varphi} \in M^*$ be the Hahn–Banach extension of φ , i.e., $\|\tilde{\varphi}\| = \|\varphi\|$. By definition we have $\varphi_n = \tilde{\varphi}_n|_A$ and $\varphi_s = \tilde{\varphi}_s|_A$. Then one has $\|\varphi\| = \|\tilde{\varphi}\| = \|\tilde{\varphi}_n\| + \|\tilde{\varphi}_s\| \geq \|\varphi_n\| + \|\varphi_s\| \geq \|\varphi_n + \varphi_s\| = \|\varphi\|$ so that the desired norm equation follows. \square

Corollary 3.3. *The predual M_\star/A_\perp of $A = H^\infty(M, \tau)$ is the range of an L -projection from A^* . Hence M_\star/A_\perp has Pełczyński's property (V^*) , and, in particular, is sequentially weakly complete.*

Proof. The first part is immediate from the above proposition since $A_n^* = M_\star/A_\perp$ trivially holds. The latter half is due to Pfitzner’s theorem [21] and an observation of Pełczyński [19, Proposition 6]. \square

It seems a natural question to find an “intrinsic characterization” of singularity for elements in A^* like Takesaki’s criterion [30]. It seems that there is no such result even in the classical theory.

4. SECOND APPLICATIONS: NONCOMMUTATIVE FUNCTION ALGEBRA THEORY

In this section we will explain how our Amar–Lederer type result nicely complements the non-commutative function algebra theory due to Blecher and Labuschagne [7]. The key is the following variant of Blecher and Labuschagne’s F. and M. Riesz theorem, which was given implicitly in the previous version of these notes. The current, explicit formulation was suggested by the referee.

Theorem 4.1. *Any non-commutative H^∞ -algebra $A = H^\infty(M, \tau)$ satisfies the following property: Whenever $\varphi \in M^*$ annihilates A , the normal and singular parts φ_n and φ_s annihilate A separately.*

Proof. Although the proof is essentially same as that of Proposition 3.2, we do give it for the sake of completeness. Let us choose $\varphi \in M^*$ in such a way that $\varphi|_A = 0$, and decompose it into the normal and singular parts $\varphi = \varphi_n + \varphi_s$. On contrary, we assume that $\varphi_n|_A \neq 0$ or $\varphi_s|_A \neq 0$. If there existed $x \in A$ with $\varphi_n(x) \neq 0$, then it would follow that $\varphi_s(x) = -\varphi_n(x) \neq 0$. Thus we may assume that $\varphi_s|_A \neq 0$. Then, by Corollary 2.2 one can find a contraction $a \in A$ and a projection $p \in M^{**}$ so that $a^n \rightarrow p$ in $\sigma(M^{**}, M^*)$, $a^n \rightarrow 0$ in $\sigma(M, M_\star)$ as $n \rightarrow \infty$ and $\varphi_s = \varphi_s \cdot p$. For any $x \in A$, the $a^n x$ ’s still fall in A but $\varphi(a^n x) \rightarrow \langle \varphi, px \rangle = \varphi_s(x)$, and consequently $\varphi_s(x) = 0$, a contradiction. \square

Blecher and Labuschagne’s F. and M. Riesz theorem [7, §3], which is originated in classical theory, asserts a quite similar property, that is, whenever $\varphi \in M^*$ annihilates $A_0 := \{a \in A : E(a) = 0\}$ the normal and singular parts φ_n and φ_s annihilate A_0 and A , respectively, and moreover its necessary and sufficient requirement is that D is finite dimensional. Note that this F. and M. Riesz property is apparently stronger than the consequence of Theorem 4.1 here, and it should be remarked that the proofs of Corollary 3.5, Theorem 4.1, Theorem 4.2 and Corollary 4.3 in [7] need only the consequence of the above theorem but do not use Blecher and Labuschagne’s F. and M. Riesz theorem itself. Thus, they all hold true without any assumption. Consequently, we get the next theorem.

Theorem 4.2. *Any non-commutative H^∞ -algebra $A = H^\infty(M, \tau)$ enjoys the following:*

- (4.2.1) *If $\varphi \in M^*$ annihilates $A + A^*$, then the φ must be singular. (cf. [7, Corollary 3.5].)*
- (4.2.2) *Every Hahn–Banach extension to M of any normal (i.e., continuous in the relative topology induced from $\sigma(M, M_\star)$) functional on A must fall in M_\star . (cf. the second part of [7, Theorem 4.1].)*
- (4.2.3) *Any $\varphi \in M_\star$ is the unique Hahn–Banach extension of its restriction to $A + A^*$. In particular, $\|\varphi\| = \|\varphi|_{A+A^*}\|$ for any $\varphi \in M_\star$. (cf. [7, Theorem 4.2].)*
- (4.2.4) *Any element in M can be σ -weakly approximated by a norm-bounded net consisting of elements in $A + A^*$. (cf. [7, Corollary 4.3].)*

The above (4.2.2), called the non-commutative Gleason–Whitney theorem, might sound a contradiction to what Pełczyński pointed out in [20, Proposition 6.3], a comment to Amar and Lederer’s result. However, this is not the case since the (4.2.2) mentions only Hahn–Banach extensions.

Remarks 4.3. Following the referee's suggestion let us call a subalgebra A of a finite von Neumann algebra M with a faithful normal tracial state an F. and M. Riesz algebra if it satisfies the consequence of Theorem 4.1 of these notes. (Clearly, $H^\infty(M, \tau)$ with D finite dimensional has been an example of F. and M. Riesz algebra since the appearance of [7].) Then, any F. and M. Riesz algebra has (GW1) of [7], and furthermore does (GW) of [7] if $A + A^*$ is σ -weak dense in M . Also, Corollary 3.5, Theorem 4.1, Theorem 4.2 and Corollary 4.3 in [7] hold true if $A + A^*$ is σ -weak dense in M too. The proofs in [7] still work without any change. The referee communicated to us that he or she had noticed in 2007 these observations together with the fact that any F. and M. Riesz algebra with separable predual has unique predual.

REFERENCES

- [1] C.A. Akemann, The general Stone-Weierstrass problem, *J. Funct. Anal.*, **4** (1969), 277–294.
- [2] C.A. Akemann, Left ideal structure of C^* -algebras, *J. Funct. Anal.*, **6** (1970), 305–317.
- [3] E. Amar and A. Lederer, Points exposés de la boule unité de $H_\infty(D)$, *C. R. Acad. Sci. Paris, Série A*, **272** (1971), 1449–1452.
- [4] T. Ando, On the predual of H^∞ , *Comment. Math.* Special issue **1** (1978), 33–40.
- [5] W. B. Arveson, Analyticity in operator algebras, *Amer. J. Math.*, **89** (1967), 578–642.
- [6] K. Barbey, Ein Satz über abstrakte analytische Funktionen, *Arch. Math. (Basel)* **26** (1975), 521–527.
- [7] D.P. Blecher and L.E. Labuschagne, Noncommutative function theory and unique extensions, *Studia Math.*, **178** (2007), 177–195.
- [8] D.P. Blecher and L.E. Labuschagne, von Neumann algebraic H^p theory in *Function spaces*, 89–114, *Contemp. Math.*, **435**, Amer. Math. Soc., Providence, RI, 2007.
- [9] J. Chaumat, Unicité du prédual, *C. R. Acad. Sc. Paris*, **288** (1979), 411–414.
- [10] J. Dixmier, Formes linéaires sur un anneau d'opérateurs, *Bull. Soc. Math. France*, **81** (1953), 9–39.
- [11] R. Exel, Maximal subdiagonal algebras, *Amer. J. Math.*, **110** (1988), 775–782.
- [12] S.R. Foguel, Power of contraction in Hilbert space, *Pacific J. Math.*, **13** (1963), 551–562.
- [13] G. Godefroy and M. Talagrand, Classes d'espaces de Banach à prédual unique, *C. R. Acad. Sci. Paris Sér. I Math.*, **292** (1981), 323–325.
- [14] G. Godefroy, Sous-espaces bien disposés de L^1 -applications, *Trans. Amer. Math. Soc.*, **286** (1984), 227–249.
- [15] A. Grothendieck, Une caractérisation vectorielle-mérique des espaces L^1 , *Canad. J. Math.* **7** (1955), 552–561.
- [16] D. M. Hay, Closed projections and peak interpolation for operator algebras, *Integr. equ. oper. theory*, **57** (2007), 491–512.
- [17] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. II. Function spaces*, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, **97**, Springer-Verlag, Berlin-New York, 1979. x+243 pp.
- [18] M. Marsalli and G. West, Noncommutative H^p spaces, *J. Operator Theory*, **40** (1998), 339–355.
- [19] A. Pelczyński, Banach spaces on which every unconditionally converging operator is weakly compact, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, **10** (1962), 641–648
- [20] A. Pelczyński, *Banach spaces of analytic functions and absolutely summing operators*, CBMS Regional Conference Series in Mathematics, No. **30**, Amer. Math. Soc., Providence, R.I., 1977.
- [21] H. Pfizner, L -summands in their biduals have Pelczyński's property (V^*), *Studia Math.*, **104** (1993), 91–98.
- [22] H. Pfizner, Separable L -embedded Banach spaces are preduals, *Bull. London Math. Soc.*, **39** (2007), 1039–1044.
- [23] G. Pisier and Q. Xu, Non-commutative L^p -spaces, in Vol.2. of *Handbook on Banach spaces*, Ed. W.B. Johnson and J. Lindenstrauss, North-Holland, Amsterdam, 2003.
- [24] N. Randrianantoanina, Hilbert transform associated with finite maximal subdiagonal algebras, *J. Austral. Math. Soc. Ser., A*, **65** (1998), no. 3, 388–404.
- [25] S. Sakai, A characterization of W^* -algebras, *Pacific J. Math.* **6** (1956), 763–773.
- [26] S. Sakai, On linear functionals of W^* -algebras, *Proc. Japan Acad.*, **34** (1958), 571–574.
- [27] S. Sewatari, Master thesis, Kyushu University, Jan. 2007.
- [28] Ş. Strătilă and L. Zsidó, *Lectures on von Neumann algebras*, Abacus Press, Tunbridge Wells, 1979.
- [29] M. Takesaki, On the conjugate space of operator algebras, *Tohoku Math. Jour.*, **10** (1958), 194–203.
- [30] M. Takesaki, On the singularity of a positive linear functional on operator algebra, *Proc. Japan Acad.*, **30** (1959), 365–366.
- [31] M. Tomita, Spectral theory of operator algebras, I, *Math. J. Okayama Univ.*, **9** (1959), 63–98.

- [32] P. Wojtaszczyk, On projections in spaces of bounded analytic functions with applications, *Studia Math.*, **65** (1979), 147–173.

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