A REMARK ON ORBITAL FREE ENTROPY

YOSHIMICHI UEDA

Abstract. A lower estimate of the orbital free entropy χ_{orb} under unitary conjugation is proved, and it together with Voiculescu's observation shows that the conjectural exact formula relating χ_{orb} to the free entropy χ breaks in general in contrast to the case when given random multi-variables are all hyperfinite.

1. Introduction

Voiculescu's theory of free entropy (see [10]) has two alternative approaches; the microstate free entropy χ and the microstate-free free entropy χ^*, both of which are believed to define the *same* free entropy (at least under the R^ω-embeddability assumption). Similarly to the microstate free entropy χ, the orbital free entropy $\chi_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n)$ of given random self-adjoint multi-variables $\mathbf{X}_i$ was also constructed based on an appropriate notion of microstates (called 'orbital microstates', i.e., the 'unitary orbit part' of the usual matricial microstates appearing in the definition of χ) in [3],[5]. The free entropy should be understood, in some senses, as the 'size' of a given set of non-commutative random variables, while $\chi_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n)$ *precisely* measures how far the positional relation among the $W^*(\mathbf{X}_i)$ are from the freely independent positional relation in the ambient tracial W^*-probability space. In fact, we have known that $\chi_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n)$ is non-positive and equals zero if and only if the $W^*(\mathbf{X}_i)$ are freely independent (modulo the R^ω-embeddability assumption). This fact and the other general properties of χ_{orb} suggest that the minus orbital free entropy $-\chi_{\mathrm{orb}}$ is a microstate variant of Voiculescu's free mutual information i^* whose definition is indeed 'microstate-free'. Hence it is natural to expect that those two quantities have the same properties.

In [9] Voiculescu (implicitly) proved that

$$i^*(v_1A_1v_1^*;\dots;v_nA_nv_n^*:B) \leq -(\Sigma(v_1)+\cdots+\Sigma(v_n))$$

holds for unital $*$-subalgebras $A_1,\dots,A_n,B$ of a tracial W^*-probability space and a freely independent family of unitaries $v_1,\dots,v_n$ in the same W^*-probability space such that the family is freely independent of $A_1\vee\cdots\vee A_n\vee B$. Here, we set $\Sigma(v_i) := \int_{\mathbb{T}}\int_{\mathbb{T}}\log|\zeta_1-\zeta_2|\,\mu_{v_i}(d\zeta_1)\mu_{v_i}(d\zeta_2)$ with the spectral distribution measure μ_{v_i} of v_i with respect to τ. In fact, this inequality immediately follows from [9, Proposition 9.4] (see Proposition 10.11 in the same paper). Its natural 'orbital counterpart' should be

$$\chi_{\mathrm{orb}}(v_1\mathbf{X}_1v_1^*,\dots,v_n\mathbf{X}_nv_n^*) \geq \Sigma(v_1)+\cdots+\Sigma(v_n)$$

with regarding $A_i = W^*(\mathbf{X}_i)$ $(1\leq i\leq n)$ and $B=\mathbb{C}$. We will prove a slightly improved inequality (Theorem 3.1). The inequality is nothing but a further evidence about the unification conjecture between i^* and χ_{orb}. However, more importantly, the inequality together with Voiculescu's discussion [9, §§14.1] answers, in the negative, the question on the expected relation

Date: Feb. 20th, 2017.
2010 *Mathematics Subject Classification.* 46L54, 52C17, 28A78, 94A17.
Key words and phrases. Free independence; Free entropy; Free mutual information; Orbital free entropy.
Supported by Grant-in-Aid for Challenging Exploratory Research 16K13762.

between χ_{orb} and χ. Namely, the main formula in [3] (see (7) below), which we call the exact formula relating χ_{orb} to χ, does not hold without any additional assumptions.

In the final part of this short note we also give an observation about the question of whether or not there is a variant of χ_{orb} satisfying both the 'W^*-invariance' for each given random self-adjoint multi-variable and the exact formula relating χ_{orb} to χ in general. Here it is fair to mention two other attempts due to Biane–Dabrowski [1] and Dabrowski [2], but this question is not yet resolved at the moment of this writing.

2. Preliminaries

Throughout this note, $(\mathcal{M},\tau)$ denotes a tracial W^*-probability space, that is, $\mathcal{M}$ is a finite von Neumann algebra and τ a faithful normal tracial state on $\mathcal{M}$. We denote the $N\times N$ self-adjoint matrices by $M_N(\mathbb{C})^{\mathrm{sa}}$ and the Haar probability measure on the $N\times N$ unitary group $\mathrm{U}(N)$ by $\gamma_{\mathrm{U}(N)}$.

2.1. **Orbital free entropy.** ([3],[5].) Let $\mathbf{X}_i=(X_{i1},\dots,X_{ir(i)})$, $1\le i\le n$, be arbitrary random self-adjoint multi-variables in $(\mathcal{M},\tau)$. We recall an expression of $\chi_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n)$ that we will use in this note. Let $R>0$ be given possibly with $R=\infty$, and $m\in\mathbb{N}$ and $\delta>0$ be arbitrarily given. For given multi-matrices $\mathbf{A}_i=(A_{ij})_{j=1}^{r(i)}\in(M_N(\mathbb{C})^{\mathrm{sa}})^{r(i)}$, $1\le i\le n$, the set of orbital microstates $\Gamma_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n:(\mathbf{A}_i)_{i=1}^n\,;N,m,\delta)$ is defined to be all $(U_i)_{i=1}^n\in\mathrm{U}(N)^n$ such that

$$\left|\mathrm{tr}_N(h((U_i\mathbf{A}_iU_i^*)_{i=1}^n))-\tau(h((\mathbf{X}_i)_{i=1}^n))\right|<\delta$$

holds whenever h is a $*$-monomial in $(r(1)+\cdots+r(n))$ indeterminates of degree not greater than m. Similarly, $\Gamma_R(\mathbf{X}_i;N,m,\delta)$ denotes the set of all $\mathbf{A}\in((M_N(\mathbb{C})^{\mathrm{sa}})_R)^{r(i)}$ such that

$$\left|\mathrm{tr}_N(h(\mathbf{A}))-\tau(h(\mathbf{X}_i))\right|<\delta$$

holds whenever h is a $*$-monomial in $r(i)$ indeterminates of degree not greater than m. It is rather trivial that if some $\mathbf{A}_i$ sits in $((M_N(\mathbb{C})^{\mathrm{sa}})_R)^{r(i)}\setminus\Gamma_R(\mathbf{X}_i\,;N,m,\delta)$, then $\Gamma_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n:(\mathbf{A}_i)_{i=1}^n\,;N,m,\delta)$ must be the empty set. Hence we define

$$\begin{aligned}&\bar\chi_{\mathrm{orb},R}(\mathbf{X}_1,\dots,\mathbf{X}_n\,;N,m,\delta)\\&:=\sup_{\mathbf{A}_i\in(M_N(\mathbb{C})^{\mathrm{sa}})_R^{r(i)}}\log\left(\gamma_{\mathrm{U}(N)}^{\otimes n}\left(\Gamma_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n:(\mathbf{A}_i)_{i=1}^n\,;N,m,\delta)\right)\right)\qquad(1)\\&=\sup_{\mathbf{A}_i\in\Gamma_R(\mathbf{X}_i\,;N,m,\delta)}\log\left(\gamma_{\mathrm{U}(N)}^{\otimes n}\left(\Gamma_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n:(\mathbf{A}_i)_{i=1}^n\,;N,m,\delta)\right)\right)\end{aligned}$$

(defined to be $-\infty$ if some $\Gamma_R(\mathbf{X}_i\,;N,m,\delta)=\emptyset$), and we define

$$\chi_{\mathrm{orb},R}(\mathbf{X}_1,\dots,\mathbf{X}_n):=\lim_{\substack{m\to\infty\\ \delta\searrow0}}\varlimsup_{N\to\infty}\frac{1}{N^2}\bar\chi_{\mathrm{orb},R}(\mathbf{X}_1,\dots,\mathbf{X}_n\,;N,m,\delta).\qquad(2)$$

It is known, see [5, Corollary 2.7], that $\chi_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n):=\sup_{R>0}\chi_{\mathrm{orb},R}(\mathbf{X}_1,\dots,\mathbf{X}_n)=\chi_{\mathrm{orb},R}(\mathbf{X}_1,\dots,\mathbf{X}_n)$ holds whenever $R\ge\max\{\|X_{ij}\|_\infty\,|\,1\le i\le n,1\le j\le r(i)\}$.

Let $\mathbf{v}=(v_1,\dots,v_s)$ be an s-tuple of unitaries in $(\mathcal{M},\tau)$. For given multi-matrices $\mathbf{A}_i=(A_{ij})_{j=1}^{r(i)}\in(M_N(\mathbb{C})^{\mathrm{sa}})^{r(i)}$, $1\le i\le n$, $\Gamma_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n:(\mathbf{A}_i)_{i=1}^n:\mathbf{v}\,;N,m,\delta)$ in presence of $\mathbf{v}$ is defined to be all $(U_i)_{i=1}^n\in\mathrm{U}(N)^n$ such that there exists $\mathbf{V}=(V_1,\dots,V_s)\in U(N)^s$ so that

$$\left|\mathrm{tr}_N(h((U_i\mathbf{A}_iU_i^*)_{i=1}^n,\mathbf{V}))-\tau(h((\mathbf{X}_i)_{i=1}^n,\mathbf{v}))\right|<\delta$$

holds whenever h is a $*$-monomial in $(r(1)+\cdots+r(n)+s)$ indeterminates of degree not greater than m. Then $\chi_{\mathrm{orb},R}(\mathbf{X}_1,\dots,\mathbf{X}_n:\mathbf{v})$ can be obtained in the same way as above with $\Gamma_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n:(\mathbf{A}_i)_{i=1}^n:\mathbf{v}\,;N,m,\delta)$ in place of $\Gamma_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n:(\mathbf{A}_i)_{i=1}^n\,;N,m,\delta)$.

Remark that $\chi_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n:\mathbf{v}) := \sup_{R>0}\chi_{\mathrm{orb},R}(\mathbf{X}_1,\dots,\mathbf{X}_n:\mathbf{v}) = \chi_{\mathrm{orb},R}(\mathbf{X}_1,\dots,\mathbf{X}_n:\mathbf{v})$ also holds if $R \geq \max\{\|X_{ij}\|_\infty \,|\, 1\leq i\leq n, 1\leq j\leq r(i)\}$. Moreover, $\chi_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n:\mathbf{v}) \leq \chi_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n)$ trivially holds.

2.2. **Microstate free entropy for unitaries.** (See [4, §6.5].) Let $\mathbf{v}=(v_1,\dots,v_n)$ be an n-tuple of unitaries in $\mathcal{M}$. We recall the microstate free entropy $\chi_u(\mathbf{v})$. Let $m\in\mathbb{N}$ and $\delta>0$ be arbitrarily given. For every $N\in\mathbb{N}$ we define $\Gamma_u(\mathbf{v};N,m,\delta)$ to be the set of all $\mathbf{V}=(V_1,\dots,V_n)\in\mathrm{U}(N)^n$ such that $\big|\mathrm{tr}_N(h(\mathbf{V}))-\tau(h(\mathbf{v}))\big|<\delta$ holds whenever h is a $*$-monomial in n indeterminates of degree not greater than m. Then

$$\chi_u(\mathbf{v}) := \lim_{\substack{m\to\infty\\ \delta\searrow 0}} \varlimsup_{N\to\infty}\frac{1}{N^2}\gamma_{\mathrm{U}(N)}^{\otimes n}\big(\Gamma_u(\mathbf{v};N,m,\delta)\big). \tag{3}$$

Note that $\chi_u(\mathbf{v})=\sum_{i=1}^n\chi_u(v_i)$ holds when $v_1,\dots,v_n$ are freely independent and that $\chi_u(\mathbf{v})=0$ if $\mathbf{v}$ is a freely independent family of Haar unitaries. Moreover, when $n=1$, $\chi_u(v_1)=\Sigma(v_1)$ holds.

2.3. **Voiculescu's measure concentration result.** ([8]) Let $(\mathfrak{A},\phi)$ be a non-commutative probability space, and $(\Omega_i)_{i\in I}$ be a family of subsets of $\mathfrak{A}$. Denote by $(\mathfrak{A}^{\star I},\phi^{\star I})$ the reduced free product of copies of $(\mathfrak{A},\phi)$ indexed by I, and by λ_i the canonical map of $\mathfrak{A}$ onto the i-th copy of $\mathfrak{A}$ in $\mathfrak{A}^{\star I}$. For each $\varepsilon>0$ and $m\in\mathbb{N}$ we say that $(\Omega_i)_{i\in I}$ are (m,ε)-free (in $(\mathfrak{A},\phi)$) if

$$\big|\phi(a_1\cdots a_k)-\phi^{\star I}(\lambda_{i_1}(a_1)\cdots\lambda_{i_k}(a_k))\big|<\varepsilon$$

for all $a_j\in\Omega_{i_j}$, $i_j\in I$ with $1\leq j\leq k$ and $1\leq k\leq m$.

Lemma 2.1. (Voiculescu [8, Corollary 2.13]) *Let $R>0$, $\varepsilon>0$, $\theta>0$ and $m\in\mathbb{N}$ be given. Then there exists $N_0\in\mathbb{N}$ such that*

$$\begin{aligned}\gamma_{\mathrm{U}(N)}^{\otimes p}\big(\{(U_1,\dots,U_p)\in\mathrm{U}(N)^p : \{T_1^{(0)},\dots,T_{q_0}^{(0)}\},\{U_1T_1^{(1)}U_1^*,\dots,U_1T_{q_1}^{(1)}U_1^*\},\\ \dots,\{U_pT_1^{(p)}U_p^*,\dots,U_pT_{q_p}^{(p)}U_p^*\}\ \textit{are } (m,\varepsilon)\textit{-free}\}\big)>1-\theta\end{aligned}$$

whenever $N\geq N_0$ and $T_j^{(i)}\in M_N(\mathbb{C})$ with $\|T_j^{(i)}\|_\infty\leq R$, $1\leq p\leq m$, $1\leq q_i\leq m$, $1\leq i\leq p$, $1\leq j\leq q_i$. $0\leq i\leq p$.

3. Lower estimate of χ_{orb} under unitary conjugation

This section is devoted to proving the following:

Theorem 3.1. *Let $\mathbf{X}_1,\dots,\mathbf{X}_{n+1}$ be random self-adjoint multi-variables in $(\mathcal{M},\tau)$ and $\mathbf{v}=(v_1,\dots,v_n)$ be an n-tuple of unitaries in $\mathcal{M}$. Assume that $\mathbf{X}:=\mathbf{X}_1\sqcup\cdots\sqcup\mathbf{X}_{n+1}$ has f.d.a. in the sense of Voiculescu* [8, Definition 3.1] *(or equivalently, $W^*(\mathbf{X})$ is R^ω-embeddable) and that $\mathbf{X}$ and $\mathbf{v}$ are freely independent. Then*

$$\begin{aligned}\chi_{\mathrm{orb}}(v_1\mathbf{X}_1v_1^*,\dots,v_n\mathbf{X}_nv_n^*,\mathbf{X}_{n+1}) &\geq \chi_{\mathrm{orb}}(v_1\mathbf{X}_1v_1^*,\dots,v_n\mathbf{X}_nv_n^*,\mathbf{X}_{n+1}:\mathbf{v})\\ &\geq \chi_{\mathrm{orb}}(v_1\mathbf{X}v_1^*,\dots,v_n\mathbf{X}v_n^*,\mathbf{X}:\mathbf{v})\\ &\geq \chi_u(\mathbf{v}).\end{aligned}\tag{4}$$

Proof. The first inequality in (4) is trivial, and the second follows from (the conditional variant of) [5, Theorem 2.6(6)]. Hence it suffices only to prove the third inequality in (4). We may and do also assume that $\chi_u(\mathbf{v})>-\infty$; otherwise the desired inequality trivially holds.

Write $\mathbf{X}=(X_1,\dots,X_r)$ for simplicity. Set $R:=\max\{\|X_j\|_\infty \,|\, 1\leq j\leq r\}$, and let $m\in\mathbb{N}$ and $\delta>0$ be arbitrarily given. We can choose $\delta'>0$ in such a way that $\delta'\leq\delta$ and that, for every $N \in \mathbb{N}$, if $\mathbf{A} \in \Gamma_R(\mathbf{X}; N, m, \delta')$ and $\mathbf{V} = (V_1, \dots, V_n) \in \Gamma_u(\mathbf{v}; N, 2m, \delta')$ are $(3m, \delta')$-free, then

$$\left|\mathrm{tr}_N(h(V_1\mathbf{A}V_1 \sqcup \cdots \sqcup V_n\mathbf{A}V_n^* \sqcup \mathbf{A} \sqcup \mathbf{V})) - \tau(h(v_1\mathbf{X}v_1^* \sqcup \cdots \sqcup v_n\mathbf{X}v_n^* \sqcup \mathbf{v}))\right| < \delta$$

whenever h is a $*$-monomial of $(n+1)r + n$ indeterminates of degree not greater than m. For such a $\delta' > 0$ the assumptions here ensure that there exists $N_0 \in \mathbb{N}$ so that $\Gamma_R(\mathbf{X}\,;\, N, m, \delta') \neq \emptyset$ and the probability measure

$$\nu_N := \frac{1}{\gamma_{\mathrm{U}(N)}^{\otimes n}(\Gamma_u(\mathbf{v}; N, 2m, \delta'))} \gamma_{\mathrm{U}(N)}^{\otimes n} \restriction_{\Gamma_u(\mathbf{v};N,2m,\delta')}$$

is well-defined whenever $N \geq N_0$. Let $\Xi(N) \in \Gamma_R(\mathbf{X}; N, m, \delta')$ be arbitrarily chosen for each $N \geq N_0$. Note that $\Xi(N)$ also falls in $\Gamma_R(\mathbf{X}; N, m, \delta) = \Gamma(v_i\mathbf{X}v_i^*; N, m, \delta)$ since $\delta' < \delta$. Then we define

$$\Theta(N, 3m, \delta') := \{(V_1, \dots, V_n, U) \in \mathrm{U}(N)^{n+1} \,|\, \{V_1, \dots, V_n\} \text{ and } U\Xi(N)U^* \text{ are } (3m, \delta')\text{-free}\}.$$

By what we have remarked at the beginning of this paragraph, we see that

$$\begin{aligned} &(V_1, \dots, V_n, U) \in \Theta(N, 3m, \delta') \cap (\Gamma_u(\mathbf{v}; N, 3m, \delta') \times \mathrm{U}(N)) \\ &\Longrightarrow (V_1U, \dots, V_nU, U) \in \Gamma_{\mathrm{orb}}(v_1\mathbf{X}v_1^*, \dots, v_n\mathbf{X}v_n^*, \mathbf{X} : (\Xi(N), \dots, \Xi(N)) : \mathbf{v}\,;\, N, m, \delta). \end{aligned} \tag{5}$$

By Lemma 2.1 there exists $N_1 \geq N_0$ so that

$$\gamma_{\mathrm{U}(N)}\big(\{U \in \mathrm{U}(N) \,|\, (V_1, \dots, V_n, U) \in \Theta(N, 3m, \delta')\}\big) > \frac{1}{2}$$

for every $N \geq N_1$ and every $(V_1, \dots, V_n) \in \mathrm{U}(N)^n$. Consequently, we have

$$(\nu_N \otimes \gamma_{\mathrm{U}(N)})\big(\Theta(N, 3m, \delta')\big) > \frac{1}{2}$$

whenever $N \geq N_1$. Therefore, for every $N \geq N_1$ we have

$$\begin{aligned} &\frac{1}{2}\gamma_{\mathrm{U}(N)}^{\otimes n}\big(\Gamma_u(\mathbf{v}; N, 2m, \delta')\big) \\ &< \gamma_{\mathrm{U}(N)}^{\otimes n}\big(\Gamma_u(\mathbf{v}; N, 2m, \delta')\big) \times (\nu_N \otimes \gamma_{\mathrm{U}(N)})\big(\Theta(N, 3m, \delta')\big) \\ &= \gamma_{\mathrm{U}(N)}^{\otimes(n+1)}\big(\Theta(N, 3m, \delta') \cap (\Gamma_u(\mathbf{v}; N, 3m, \delta') \times \mathrm{U}(N))\big) \\ &\leq \gamma_{\mathrm{U}(N)}^{\otimes(n+1)}\big(\{(V_1, \dots, V_n, U) \in \mathrm{U}(N)^{n+1} \,| \\ &\qquad (V_1U, \dots, V_nU, U) \in \Gamma_{\mathrm{orb}}(v_1\mathbf{X}v_1^*, \dots, v_n\mathbf{X}v_n^*, \mathbf{X} : (\Xi(N), \dots, \Xi(N)) : \mathbf{v}\,;\, N, m, \delta)\}\big) \\ &= \int_{\mathrm{U}(N)} \gamma_{\mathrm{U}(N)}(dU)\, \gamma_{\mathrm{U}(N)}^{\otimes n}\big(\{(V_1, \dots, V_n) \in \mathrm{U}(N)^n \,| \\ &\qquad (V_1U, \dots, V_nU, U) \in \Gamma_{\mathrm{orb}}(v_1\mathbf{X}v_1^*, \dots, v_n\mathbf{X}v_n^*, \mathbf{X} : (\Xi(N), \dots, \Xi(N)) : \mathbf{v}\,;\, N, m, \delta)\}\big) \\ &= \int_{\mathrm{U}(N)} \gamma_{\mathrm{U}(N)}(dU)\, \gamma_{\mathrm{U}(N)}^{\otimes n}\big(\{(U_1, \dots, U_n) \in \mathrm{U}(N)^n \,| \\ &\qquad (U_1, \dots, U_n, U) \in \Gamma_{\mathrm{orb}}(v_1\mathbf{X}v_1^*, \dots, v_n\mathbf{X}v_n^*, \mathbf{X} : (\Xi(N), \dots, \Xi(N)) : \mathbf{v}\,;\, N, m, \delta)\}\big) \\ &= \gamma_{\mathrm{U}(N)}^{\otimes(n+1)}\big(\Gamma_{\mathrm{orb}}(v_1\mathbf{X}v_1^*, \dots, v_n\mathbf{X}v_n^*, \mathbf{X} : (\Xi(N), \dots, \Xi(N)) : \mathbf{v}\,;\, N, m, \delta)\big), \end{aligned} \tag{6}$$

where the fourth line is obtained by (5) and the sixth due to the right-invariance of the Haar probability measure $\gamma_{\mathrm{U}(N)}$. Hence

$$\chi_u(\mathbf{v}) \leq \varlimsup_{N\to\infty} \frac{1}{N^2} \log\Big(\frac{1}{2}\gamma_{\mathrm{U}(N)}^{\otimes n}\big(\Gamma_u(\mathbf{v}; N, 2m, \delta')\big)\Big)$$

$$\leq \varlimsup_{N\to\infty} \frac{1}{N^2} \log \gamma_{\mathrm{U}(N)}^{\otimes(n+1)}\big(\Gamma_{\mathrm{orb}}(v_1\mathbf{X}v_1^*, \dots, v_n\mathbf{X}v_n^*, \mathbf{X} : (\Xi(N), \dots, \Xi(N)) : \mathbf{v}\,; N, m, \delta)\big)$$

$$\leq \varlimsup_{N\to\infty} \frac{1}{N^2} \bar{\chi}_{\mathrm{orb},R}(v_1\mathbf{X}v_1^*, \dots, v_n\mathbf{X}v_n^*, \mathbf{X} : \mathbf{v}; N, m, \delta),$$

implying the desired inequality since m, δ are arbitrary. □

Remark 3.2. Inequality (4) is not optimal as follows. Assume that $(\mathcal{M}, \tau) = (L(\mathbb{F}_r), \tau_{\mathbb{F}_r}) \star (L(\mathbb{Z}_m), \tau_{\mathbb{Z}_m})$ and that $\mathbf{X}$ is the canonical free semicircular generators of $L(\mathbb{F}_r)$ and v is a canonical generator of $L(\mathbb{Z}_m)$. Since $\tau(v) = 0$, one easily confirms that $v\mathbf{X}v^*$ and $\mathbf{X}$ are freely independent so that $\chi_{\mathrm{orb}}(v\mathbf{X}v^*, \mathbf{X}) = 0$. On the other hand, we know that $\chi_u(v) = -\infty$, since the spectral measure of v has an atom.

Remark 3.3. The proof of Theorem 3.1 (actually, the idea of obtaining the second equality in (6)) gives an alternative representation of $\bar{\chi}_{\mathrm{orb},R}(\mathbf{X}_1, \dots, \mathbf{X}_n\,; N, m, \delta)$:

$$\begin{aligned}
&\bar{\chi}_{\mathrm{orb},R}(\mathbf{X}_1, \dots, \mathbf{X}_n\,; N, m, \delta) \\
&\quad= \sup_{\mathbf{A}_i \in (M_N(\mathbb{C})^{\mathrm{sa}})_R^{r(i)}} \log\Big(\gamma_{\mathrm{U}(N)}^{\otimes n-1}\big(\big\{(U_i)_{i=1}^{n-1} \in \mathrm{U}(N)^{n-1} \mid \\
&\qquad\qquad (U_1, \dots, U_{n-1}, I_N) \in \Gamma_{\mathrm{orb}}(\mathbf{X}_1, \dots, \mathbf{X}_n : (\mathbf{A}_i)_{i=1}^n\,; N, m, \delta)\big\}\big)\Big) \\
&\quad= \sup_{\mathbf{A}_i \in \Gamma_R(\mathbf{X}_i\,; N, m, \delta)} \log\Big(\gamma_{\mathrm{U}(N)}^{\otimes n-1}\big(\big\{(U_i)_{i=1}^{n-1} \in \mathrm{U}(N)^{n-1} \mid \\
&\qquad\qquad (U_1, \dots, U_{n-1}, I_N) \in \Gamma_{\mathrm{orb}}(\mathbf{X}_1, \dots, \mathbf{X}_n : (\mathbf{A}_i)_{i=1}^n\,; N, m, \delta)\big\}\big)\Big),
\end{aligned}$$

when $n \geq 2$. This corresponds to [9, Remarks 10.2(c)].

4. Discussions

4.1. **Negative observation.** In [3, Theorem 2.6] the following formula was shown when all $\mathbf{X}_i$ are singletons:

$$\chi(\mathbf{X}_1 \sqcup \cdots \sqcup \mathbf{X}_n) = \chi_{\mathrm{orb}}(\mathbf{X}_1, \dots, \mathbf{X}_n) + \sum_{i=1}^n \chi(\mathbf{X}_i). \tag{7}$$

Note that the same formula trivially holds true (as $-\infty = -\infty$) even when one replaces each singleton $\mathbf{X}_i$ with a hyperfinite non-singleton $\mathbf{X}_i$, that is, $W^*(\mathbf{X}_i)$ is hyperfinite and $\mathbf{X}_i$ consists of at least two elements. Beyond the hyperfiniteness situation, inequality ($\leq$) in (7) still holds (see [5, Proposition 2.8]), but equality unfortunately does not in general as follows. The following argument is attributed to Voiculescu [9, §§14.1]. Let $\mathbf{X} = (X_1, X_2)$ be a semicircular system in $\mathcal{M}$ and $v \in \mathcal{M}$ be a unitary such that $\tau(v) \neq 0$, $\chi_u(v) > -\infty$, and that $\mathbf{X}$ and v are $*$-freely independent. Set $Y_i := vX_iv^*$, $i = 1, 2$, and $\mathbf{Y} := (Y_1, Y_2)$. By [9, Proposition 2.5] $W^*(X_1, X_2, Y_1, Y_2) = W^*(X_1, X_2, Y_1) = W^*(X_1, X_2, v)$, and hence by [7, Proposition 3.8]

$$\chi(X_1, X_2, Y_1, Y_2) = \chi(X_1, X_2, Y_1, I) \leq \chi(X_1, X_2, Y_1) + \chi(I) = -\infty,$$

where I denotes the unit of $\mathcal{M}$. On the other hand, by Theorem 3.1 $\chi_{\mathrm{orb}}(\mathbf{X}, \mathbf{Y}) \geq \chi_u(v) > -\infty$, implying that

$$\chi(\mathbf{X} \sqcup \mathbf{Y}) = -\infty < \chi_{\mathrm{orb}}(\mathbf{X}, \mathbf{Y}) + \chi(\mathbf{X}) + \chi(\mathbf{Y}).$$

In particular, the quantity "C^ω" (or probably "C" too) in [5, Remark 2.9] does not coincide with χ_{orb} in general. An interesting question is whether or not $\chi(\mathbf{X}_1 \sqcup \cdots \sqcup \mathbf{X}_n) > -\infty$ is enough to make the exact formula relating χ_{orb} to χ hold. Note that $\chi_{\mathrm{orb}} = \widetilde{\chi}_{\mathrm{orb}}$ (Biane–Dabrowski's variant [1]) holds under the assumption. Moreover, the orbital free entropy dimension $\delta_{0,\mathrm{orb}}(\mathbf{X}, \mathbf{Y})$

must be zero in this case thanks to [5, Proposition 4.3(5)], since $\chi_{\mathrm{orb}}(\mathbf{X},\mathbf{Y}) > -\infty$. Also $\delta_0(\mathbf{X}) = \delta_0(\mathbf{Y}) = 2$ is trivial. Note that $\chi_u(v) > -\infty$ forces that the probability distribution of v has no atom. Thus, it is likely (if one believes that δ_0 gives a W^*-invariant) that

$$\delta_0(\mathbf{X}\sqcup\mathbf{Y}) \overset{?}{=} 3 < 4 = \delta_{0,\mathrm{orb}}(\mathbf{X},\mathbf{Y}) + \delta_0(\mathbf{X}) + \delta_0(\mathbf{Y})$$

is expected. This means that if $\delta_0(\mathbf{X}_1 \sqcup\cdots\sqcup \mathbf{X}_n) = \delta_{0,\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n) + \sum_{i=1}^n \delta_0(\mathbf{X}_i)$ held in general, then the W^*-invariance problem of δ_0 would be resolved negatively. Hence it seems still interesting only to ask whether $\delta_0(\mathbf{X}\sqcup\mathbf{Y}) \lneqq 4$ or not.

4.2. **Other possible variants of χ_{orb}.** The above discussion tells us that if a variant of χ_{orb} satisfies Theorem 3.1, then the variant does not satisfy the exact formula relating χ_{orb} to χ in general. Following our previous work [3] with Hiai and Miyamoto one may consider the following variant of χ_{orb}: For each $1\le i\le n$, we select an (operator norm-)bounded sequence $\{\Xi_i(N)\}_{N\in\mathbb{N}}$ with $\Xi_i(N) \in (M_N(\mathbb{C})^{\mathrm{sa}})^{r(i)}$ such that the joint distribution of $\Xi_i(N)$ under tr_N converges to that of $\mathbf{X}_i$ under τ as $N\to\infty$. Then we replace $\bar\chi_{\mathrm{orb},R}(\mathbf{X}_1,\dots,\mathbf{X}_n\,;N,m,\delta)$ in the definition of χ_{orb} with

$$\begin{aligned}&\chi_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n : (\Xi_i(N))_{i=1}^n\,; N,m,\delta)\\ &\quad := \log\Big(\gamma_{\mathrm{U}(N)}^{\otimes n}\big(\Gamma_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n : (\Xi_i(N))_{i=1}^n\,; N,m,\delta)\big)\Big),\end{aligned}$$

and define

$$\begin{aligned}&\chi_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n : (\Xi_i(N))_{i=1}^n)\\ &\quad := \lim_{\substack{m\to\infty\\ \delta\searrow 0}} \varlimsup_{N\to\infty} \frac{1}{N^2}\chi_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n : (\Xi_i(N))_{i=1}^n\,; N,m,\delta).\end{aligned}$$

The conditional variant $\chi_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n : (\Xi_i(N))_{i=1}^n : \mathbf{v})$ is defined exactly in the same fashion as $\chi_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n : \mathbf{v})$. Then we may consider their supremum all over the possible choices of $(\Xi_i(N))_{i=1}^n$ under some suitable constraint as a variant of χ_{orb}.

Even if the constraint of selecting sequences of multi-matrices is chosen to be the way of approximating to the freely independent copies of given random self-adjoint multi-variables, then the resulting variant of χ_{orb} still satisfies Theorem 3.1, and in turn does not satisfy the exact formula relating χ_{orb} to χ in general. More precisely we can prove the following:

Proposition 4.1. *Let $\mathbf{X} = (X_j)_{j=1}^r$ be a random self-adjoint multi-variables in $(\mathcal{M},\tau)$ and $\mathbf{v} = (v_1,\dots,v_n)$ be an n-tuple of unitaries in $\mathcal{M}$. Assume that $\mathbf{X}$ has f.d.a. (see Theorem 3.1) and that $\mathbf{X}$ and $\mathbf{v}$ are freely independent. Then there exists a bounded sequence $\{(\Xi_i(N))_{i=1}^{n+1}\}_{N\in\mathbb{N}}$ with $\Xi_i(N)\in(M_N(\mathbb{C})^{\mathrm{sa}})^r$ such that the joint distribution of $\Xi_1(N)\sqcup\cdots\sqcup\Xi_{n+1}(N)$ under tr_N converges to the freely independent $n+1$ copies $\mathbf{X}_1^f\sqcup\cdots\sqcup\mathbf{X}_{n+1}^f$ of $\mathbf{X}$ (n.b., the joint distribution of $\mathbf{X}$ is identical to that of every $v_i\mathbf{X}v_i^*$) under τ as $N\to\infty$, and moreover that*

$$\begin{aligned}&\chi_{\mathrm{orb}}(v_1\mathbf{X}v_1^*,\dots,v_n\mathbf{X}v_n^*,\mathbf{X} : (\Xi_i(N))_{i=1}^{n+1})\\ &\quad \ge \chi_{\mathrm{orb}}(v_1\mathbf{X}v_1^*,\dots,v_n\mathbf{X}v_n^*,\mathbf{X} : (\Xi_i(N))_{i=1}^{n+1}) : \mathbf{v}) \ge \chi_u(\mathbf{v}).\end{aligned}$$

Proof. Let $R>0$ be sufficiently large. Since $\mathbf{X}$ has f.d.a., Lemma 2.1 shows that for each $m\in\mathbb{N}$ and $\delta>0$ one has $\{((U_i)_{i=1}^{n+1},\mathbf{A}) \in \mathrm{U}(N)^{n+1}\times((M_N(\mathbb{C})^{\mathrm{sa}})_R)^r \mid (U_i\mathbf{A}U_i^*)_{i=1}^{n+1} \in \Gamma_R(\mathbf{X}_1^f\sqcup\cdots\sqcup\mathbf{X}_{n+1}^f\,; N,m,\delta)\} \neq \emptyset$ for all sufficiently large $N\in\mathbb{N}$. By using this fact, it is easy to choose a bounded sequence $\Xi(N)\in((M_N(\mathbb{C})^{\mathrm{sa}})_R)^r$ and a sequence $(W_i(N))_{i=1}^{n+1}\in\mathrm{U}(N)^{n+1}$ in such a way that both the joint distributions of $\Xi(N)$ and of $W_1(N)\Xi(N)W_1(N)^*\sqcup\cdots\sqcup W_{n+1}(N)\Xi(N)W_{n+1}(N)^*$ under tr_N converge to those of $\mathbf{X}$ and of $\mathbf{X}_1^f\sqcup\cdots\sqcup\mathbf{X}_{n+1}^f$, respectively, under τ as $N\to\infty$. Set $\Xi_i(N) := W_i(N)\Xi(N)W_i(N)^*$, $1\le i\le n+1$, and we will prove that $(\Xi_i(N))_{i=1}^{n+1}$ is a desired sequence.

For given $m \in \mathbb{N}$ and $\delta > 0$, we choose $0 < \delta' < \delta$ as in the proof of Theorem 3.1. Let ν_N and $\Theta(N, 3m, \delta')$ be also chosen exactly in the same way as in the proof of Theorem 3.1. We can choose $N_0 \in \mathbb{N}$ in such a way that $\Xi(N) \in \Gamma_R(\mathbf{X}\,;\, N, m, \delta')$ and ν_N is well-defined as long as $N \geq N_0$. By the same reasoning as in the proof of Theorem 3.1 we have

$$\begin{aligned}
&(V_1, \dots, V_n, U) \in \Theta(N, 3m, \delta') \cap (\Gamma_u(\mathbf{v}; N, 3m, \delta') \times \mathrm{U}(N)) \\
&\Longrightarrow (V_1 U, \dots, V_n U, U) \in \Gamma_{\mathrm{orb}}(v_1 \mathbf{X} v_1^*, \dots, v_n \mathbf{X} v_n^*, \mathbf{X} : (\Xi(N), \dots, \Xi(N)) : \mathbf{v}\,;\, N, m, \delta) \\
&\Longleftrightarrow (V_1 U W_1(N)^*, \dots, V_n U W_n(N)^*, U W_{n+1}(N)^*) \\
&\qquad\qquad\qquad \in \Gamma_{\mathrm{orb}}(v_1 \mathbf{X} v_1^*, \dots, v_n \mathbf{X} v_n^*, \mathbf{X} : (\Xi_i(N))_{i=1}^{n+1} : \mathbf{v}\,;\, N, m, \delta).
\end{aligned} \tag{8}$$

As in the proof of Theorem 3.1 again, Lemma 2.1 shows that there exists $N_1 \geq N_0$ so that

$$(\nu_N \otimes \gamma_{\mathrm{U}(N)})\big(\Theta(N, 3m, \delta')\big) > \frac{1}{2}$$

whenever $N \geq N_1$. Therefore, for every $N \geq N_1$ we have

$$\begin{aligned}
&\frac{1}{2}\gamma_{\mathrm{U}(N)}^{\otimes n}\big(\Gamma_u(\mathbf{v}; N, 2m, \delta')\big) \\
&< \gamma_{\mathrm{U}(N)}^{\otimes n}\big(\Gamma_u(\mathbf{v}; N, 2m, \delta')\big) \times (\nu_N \otimes \gamma_{\mathrm{U}(N)})\big(\Theta(N, 3m, \delta')\big) \\
&= \gamma_{\mathrm{U}(N)}^{\otimes (n+1)}\big(\Theta(N, 3m, \delta') \cap (\Gamma_u(\mathbf{v}; N, 3m, \delta') \times \mathrm{U}(N))\big) \\
&\leq \gamma_{\mathrm{U}(N)}^{\otimes (n+1)}\big(\{(V_1, \dots, V_n, U) \in \mathrm{U}(N)^{n+1} \mid \\
&\qquad\qquad (V_1 U W_1(N)^*, \dots, V_n U W_n(N)^*, U W_{n+1}(N)^*) \\
&\qquad\qquad\qquad \in \Gamma_{\mathrm{orb}}(v_1 \mathbf{X} v_1^*, \dots, v_n \mathbf{X} v_n^*, \mathbf{X} : (\Xi_i(N))_{i=1}^{n+1} : \mathbf{v}\,;\, N, m, \delta)\}\big) \\
&= \int_{\mathrm{U}(N)} \gamma_{\mathrm{U}(N)}(dU)\, \gamma_{\mathrm{U}(N)}^{\otimes n}\big(\{(V_1, \dots, V_n) \in \mathrm{U}(N)^n \mid \\
&\qquad (V_1 U W_1(N)^*, \dots, V_n U W_n(N)^*, U W_{n+1}(N)^*) \\
&\qquad\qquad\qquad \in \Gamma_{\mathrm{orb}}(v_1 \mathbf{X} v_1^*, \dots, v_n \mathbf{X} v_n^*, \mathbf{X} : (\Xi_i(N))_{i=1}^{n+1} : \mathbf{v}\,;\, N, m, \delta)\}\big) \\
&= \int_{\mathrm{U}(N)} \gamma_{\mathrm{U}(N)}(dU)\, \gamma_{\mathrm{U}(N)}^{\otimes n}\big(\{(V_1, \dots, V_n) \in \mathrm{U}(N)^n \mid \\
&\qquad (V_1 U W_{n+1}(N) W_1(N)^*, \dots, V_n U W_{n+1}(N) W_n(N)^*, U) \\
&\qquad\qquad\qquad \in \Gamma_{\mathrm{orb}}(v_1 \mathbf{X} v_1^*, \dots, v_n \mathbf{X} v_n^*, \mathbf{X} : (\Xi_i(N))_{i=1}^{n+1} : \mathbf{v}\,;\, N, m, \delta)\}\big) \\
&= \int_{\mathrm{U}(N)} \gamma_{\mathrm{U}(N)}(dU)\, \gamma_{\mathrm{U}(N)}^{\otimes n}\big(\{(U_1, \dots, U_n) \in \mathrm{U}(N)^n \mid \\
&\quad (U_1, \dots, U_n, U) \in \Gamma_{\mathrm{orb}}(v_1 \mathbf{X} v_1^*, \dots, v_n \mathbf{X} v_n^*, \mathbf{X} : (\Xi_i(N))_{i=1}^{n+1} : \mathbf{v}\,;\, N, m, \delta)\}\big) \\
&= \gamma_{\mathrm{U}(N)}^{\otimes (n+1)}\big(\Gamma_{\mathrm{orb}}(v_1 \mathbf{X} v_1^*, \dots, v_n \mathbf{X} v_n^*, \mathbf{X} : (\Xi_i(N))_{i=1}^{n+1} : \mathbf{v}\,;\, N, m, \delta)\big),
\end{aligned}$$

where the fourth line is obtained by (8) and both the sixth and the seventh due to the right-invariance of the Haar probability measure $\gamma_{\mathrm{U}(N)}$. Hence the desired inequality follows as in the proof of Theorem 3.1. □

In view of our work [6] and Voiculescu's liberation theory [9], a candidate constraint of selecting sequences of multi-matrices may be the way of approximating to $\mathbf{X}_1 \sqcup \cdots \sqcup \mathbf{X}_n$ globally, though it probably does not satisfy the exact formula relating χ_{orb} to χ in general.

Acknowledgment

The author would like to thank the referee for his or her careful reading and pointing out several typos.

References

[1] Ph. Biane and Y. Dabrowski, Concavification of free entropy. *Adv. Math.*, **234** (2013), 667–696.
[2] Y. Dabrowski, A Laplace principle for hermitian Brownian motion and free entropy. arXiv:1604.06420.
[3] F. Hiai, T. Miyamoto and Y. Ueda, Orbital approach to microstate free entropy. *Internat. J. Math.*, **20** (2009), 227–273.
[4] F. Hiai and D. Petz, *The Semicircle Law, Free Random Variables and Entropy.* Mathematical Surveys and Monographs, Vol. 77, Amer. Math. Soc., Providence, 2000.
[5] Y. Ueda, Orbital free entropy, revisited. *Indiana Univ. Math. J.*, **63** (2014), 551–577.
[6] Y. Ueda, Matrix liberation process I: Large deviation upper bound and almost sure convergence. arXiv:1610.04101.
[7] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory II. *Invent. Math.*, **118** (1994), 411–440.
[8] D. Voiculescu, A strengthened asymptotic freeness result for random matrices with applications to free entropy. *Int. Math. Res. Not.*, **1998** (1998), 41–63.
[9] D. Voiculescu, The analogue of entropy and of Fisher's information measure in free probability theory VI: Liberation and mutual free information. *Adv. Math.*, **146** (1999), 101–166.
[10] D. Voiculescu, Free entropy. *Bull. London Math. Soc.*, **34** (2002), 257–278.

Graduate School of Mathematics, Kyushu University, Fukuoka, 810-8560, Japan
E-mail address: ueda@math.kyushu-u.ac.jp