RIGIDITY OF FREE PRODUCT VON NEUMANN ALGEBRAS

CYRIL HOUDAYER AND YOSHIMICHI UEDA

Abstract. Let I be any nonempty set and $(M_i, \varphi_i)_{i \in I}$ any family of nonamenable factors, endowed with arbitrary faithful normal states, that belong to a large class $\mathcal{C}_{\text{anti-free}}$ of (possibly type III) von Neumann algebras including all nonprime factors, all nonfull factors and all factors possessing Cartan subalgebras. For the free product $(M, \varphi) = \ast_{i \in I}(M_i, \varphi_i)$, we show that the free product von Neumann algebra M retains the cardinality $|I|$ and each nonamenable factor M_i up to stably inner conjugacy, after permutation of the indices. Our main theorem unifies all previous Kurosh-type rigidity results for free product type II_1 factors and is new for free product type III factors. It moreover provides new rigidity phenomena for type III factors.

1. Introduction and statement of the Main Theorem

In his seminal article [Oz04], Ozawa obtained the first Kurosh-type rigidity results for free product type II_1 factors. Among other things, he showed that whenever $m \geq 1$ and $M_1, \dots, M_m$ are weakly exact nonamenable *nonprime* type II_1 factors, the tracial free product von Neumann algebra $M_1 \ast \cdots \ast M_m$ retains the integer m and each factor M_i up to inner conjugacy, after permutation of the indices. Ozawa's approach to Kurosh-type rigidity for II_1 factors was based on a combination of his C*-algebraic techniques [Oz03] and of Popa's intertwining techniques [Po01, Po03] (see also [OP03]). Shortly after, using Popa's deformation/rigidity theory, Ioana–Peterson–Popa obtained in [IPP05] Kurosh-type rigidity results for tracial free products of *weakly rigid* type II_1 factors, that is, II_1 factors possessing regular diffuse von Neumann subalgebras with relative property (T) in the sense of [Po01]. These Kurosh-type rigidity results for II_1 factors were then unified and further generalized by Peterson in [Pe06], using his L^2-rigidity techniques, to cover tracial free products of nonamenable L^2-*rigid* type II_1 factors. In [As09], Asher extended Ozawa's original result [Oz04] to free products of weakly exact nonamenable nonprime type II_1 factors with respect to nontracial states.

Regarding the structure of free product von Neumann algebras, the questions of factoriality, type classification and fullness for arbitrary free product von Neumann algebras were recently completely solved by Ueda in [Ue10]. For any free product $(M, \varphi) = (M_1, \varphi_1) \ast (M_2, \varphi_2)$ with $\dim_{\mathbf{C}} M_i \geq 2$ and $(\dim_{\mathbf{C}} M_1, \dim_{\mathbf{C}} M_2) \neq (2, 2)$, the free product von Neumann algebra M splits as a direct sum $M = M_c \oplus M_d$ where M_c is a full factor of type II_1 or of type III_λ (with $0 < \lambda \leq 1$) and $M_d = 0$ or M_d is a multimatrix algebra. Moreover, Chifan–Houdayer showed in [CH08] (see also [Ue10]) that M_c is always a *prime* factor (see Peterson [Pe06] for the previous work in the tracial case) and Boutonnet–Houdayer–Raum showed in [BHR12] that M_c has no Cartan subalgebra (see Ioana [Io12] for the the previous work in the tracial case). Very recently, in our joint work [HU15], we completely settled the questions of maximal amenability and maximal property Gamma of the inclusion $M_1 \subset M$ in arbitrary free product von Neumann algebras. In view of these recent structural results obtained in full generality, it

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is thus natural to seek for Kurosh-type rigidity results for *arbitrary* free product von Neumann algebras.

In this paper, we unify and generalize all the previous Kurosh-type rigidity results to *arbitrary* free products $(M, \varphi) = *_{i \in I}(M_i, \varphi_i)$ over *arbitrary* index sets I, where all M_i are nonamenable factors that belong to a large class of (possibly type III) factors that we call *anti-freely decomposable*. In order to state our Main Theorem, we will use the following terminology.

Definition. We will say that a nonamenable factor M with separable predual is *anti-freely decomposable* if at least one of the following conditions holds.

(i) M is *not prime*, that is, $M = M_1 \mathbin{\overline{\otimes}} M_2$ where M_1 and M_2 are diffuse factors (e.g. M is McDuff).
(ii) M has *property Gamma*, that is, the central sequence algebra $M' \cap M^\omega$ is diffuse for some nonprincipal ultrafilter $\omega \in \beta(\mathbf{N})\setminus\mathbf{N}$ (e.g. M is of type III_0; see [Co74, Proposition 3.9]).
(iii) M possesses an amenable finite von Neumann subalgebra A with expectation such that $A' \cap M = \mathcal{Z}(A)$ and $\mathcal{N}_M(A)'' = M$ (e.g. M possesses a Cartan subalgebra).
(iv) M is a II_1 factor that possesses a regular diffuse von Neumann subalgebra with relative property (T) in the sense of [Po01, Definition 4.2.1] (e.g. M is a II_1 factor with property (T) [CJ85]).

We will denote by $\mathcal{C}_{\text{anti-free}}$ the class of nonamenable factors with separable predual that are anti-freely decomposable in the sense of the above definition.

Recall that the *Kurosh isomorphism theorem* for discrete groups (see e.g. [CM82, pp 105]) says that any discrete group can uniquely (up to permutation of components) be decomposed into a free product of *freely indecomposable* subgroups. It is not clear at all how to capture *freely indecomposable von Neumann algebras* practically. However, all the known *general* structural results on free product von Neumann algebras suggest that $\mathcal{C}_{\text{anti-free}}$ is indeed a natural large class of freely indecomposable factors. Hence the Main Theorem of this paper stated below is indeed a von Neumann algebra counterpart of the Kurosh isomorphism theorem and unifies all the previous counterparts.

Main Theorem. *Let I and J be any nonempty sets and $(M_i)_{i \in I}$ and $(N_j)_{j \in J}$ any families of nonamenable factors in the class $\mathcal{C}_{\text{anti-free}}$. For each $i \in I$ and each $j \in J$, choose any faithful normal states $\varphi_i \in (M_i)_*$ and $\psi_j \in (N_j)_*$. Denote by $(M, \varphi) = *_{i \in I}(M_i, \varphi_i)$ and $(N, \psi) = *_{j \in J}(N_j, \psi_j)$ the corresponding free products.*

(1) *Assume that M and N are isomorphic. Then $|I| = |J|$ and there exists a bijection $\alpha : I \to J$ such that M_i and $N_{\alpha(i)}$ are stably isomorphic for all $i \in I$.*
(2) *Assume that M and N are isomorphic and identify $M = N$. Assume moreover that M_i is a type* III *factor for all $i \in I$. Then there exists a unique bijection $\alpha : I \to J$ such that M_i and $N_{\alpha(i)}$ are inner conjugate for all $i \in I$.*

Our Main Theorem is new for free products of type III factors. In that case (see item (2)), our statement is as sharp as all previous Kurosh-type rigidity results for free products of type II_1 factors. We point out that for *tracial* free products, our Main Theorem is still new in cases (i), (ii) and (iii) when the index set I is *infinite* (compare with [Oz04, Pe06, Io12]).

We now briefly explain the strategy of the proof of the Main Theorem. We refer to Sections 4 and 5 for further details. As we will see, the proof builds upon the tools and techniques we developed in our previous work [HU15] on the *asymptotic structure* of free product von Neumann algebras. Using the very recent generalization of Popa's intertwining techniques in [HI15, §4], it suffices, modulo some technical things, to prove the existence of a bijection

$\alpha : I \to J$ such that $M_i \preceq_M N_{\alpha(i)}$ and $N_{\alpha(i)} \preceq_M M_i$ for all $i \in I$. To simplify the discussion, fix $i \in I$. We need to show that there exists $j \in J$ such that $M_i \preceq_M N_j$.

Firstly, assume that M_i is in case (i) or (ii). Exploiting the anti-free decomposability property of M_i (in case (i)) and a new characterization of property Gamma for arbitrary von Neumann algebras (in case (ii)) (see Theorem 3.1) together with various technical results from our previous work [HU15], it suffices to prove that for a well-chosen diffuse abelian subalgebra $A \subset M_i$ with expectation whose relative commutant $A' \cap M_i$ is nonamenable, there exists $j \in J$ such that $A \preceq_M N_j$. This is achieved in Theorem 4.4 by using a combination of Popa's spectral gap argument [Po06] together with Connes–Takesaki's structure theory for type III von Neumann algebras [Co72, Ta03] and Houdayer–Isono's recent intertwining theorem [HI15]. Secondly, assume that M_i is in case (iii). Then it suffices again to prove that there exists $j \in J$ such that $A \preceq_M N_j$. The proof is slightly more involved (see Theorem 4.6) and relies on Vaes's recent dichotomy result for normalizers inside tracial amalgamated free product von Neumann algebras [Va13] (improving Ioana's previous result [Io12] and involving Popa–Vaes's striking dichotomy result [PV11]) instead of Popa's spectral gap argument [Po06] (see Appendix A). Thirdly, assume that M_i is in case (iv). Then it suffices to prove that there exists $j \in J$ such that $A \preceq_M N_j$ where $A \subset M_i$ is a diffuse regular subalgebra with relative property (T). This is achieved in Theorem 4.8 by reconstructing [IPP05, Theorem 4.3] in the semifinite setting.

In Section 6, we prove further new results regarding the structure of free product von Neumann algebras. In particular, we obtain a complete characterization of *solidity* [Oz03] for free products with respect to arbitrary faithful normal states and over arbitrary index sets.

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Contents

2. Preliminaries

For any von Neumann algebra M, we will denote by $\mathcal{Z}(M)$ the centre of M, by $z_M(e)$ the central support of a projection $e \in M$, by $\mathcal{U}(M)$ the group of unitaries in M, by $\mathrm{Ball}(M)$ the unit ball of M with respect to the uniform norm $\|\cdot\|_\infty$ and by $(M, \mathrm{L}^2(M), J^M, \mathfrak{P}^M)$ the standard form of M. We will say that an inclusion of von Neumann algebras $P \subset 1_P M 1_P$ is *with expectation* if there exists a faithful normal conditional expectation $\mathrm{E}_P : 1_P M 1_P \to P$. We will say that a σ-finite von Neumann algebra M is *tracial* if it is endowed with a faithful normal tracial state τ.

Background on σ-finite von Neumann algebras. Let M be any σ-finite von Neumann algebra with unique predual M_* and $\varphi \in M_*$ any faithful state. We will write $\|x\|_\varphi = \varphi(x^*x)^{1/2}$ for every $x \in M$. Recall that on $\mathrm{Ball}(M)$, the topology given by $\|\cdot\|_\varphi$ coincides with the σ-strong topology. Denote by $\xi_\varphi \in \mathfrak{P}^M$ the unique representing vector of φ. The mapping $M \to \mathrm{L}^2(M) : x \mapsto x\xi_\varphi$ defines an embedding with dense image such that $\|x\|_\varphi = \|x\xi_\varphi\|_{\mathrm{L}^2(M)}$ for all $x \in M$.

We denote by σ^φ the modular automorphism group of the state φ. The *centralizer* M^φ of the state φ is by definition the fixed point algebra of (M, σ^φ). The *continuous core* of M with respect to φ, denoted by $\mathrm{c}_\varphi(M)$, is the crossed product von Neumann algebra $M \rtimes_{\sigma^\varphi} \mathbf{R}$. The natural inclusion $\pi_\varphi : M \to \mathrm{c}_\varphi(M)$ and the unitary representation $\lambda_\varphi : \mathbf{R} \to \mathrm{c}_\varphi(M)$ satisfy the *covariance* relation

$$\lambda_\varphi(t)\pi_\varphi(x)\lambda_\varphi(t)^* = \pi_\varphi(\sigma_t^\varphi(x)) \quad \text{for all } x \in M \text{ and all } t \in \mathbf{R}.$$

Put $\mathrm{L}_\varphi(\mathbf{R}) = \lambda_\varphi(\mathbf{R})''$. There is a unique faithful normal conditional expectation $\mathrm{E}_{\mathrm{L}_\varphi(\mathbf{R})} : \mathrm{c}_\varphi(M) \to \mathrm{L}_\varphi(\mathbf{R})$ satisfying $\mathrm{E}_{\mathrm{L}_\varphi(\mathbf{R})}(\pi_\varphi(x)\lambda_\varphi(t)) = \varphi(x)\lambda_\varphi(t)$ for all $x \in M$ and all $t \in \mathbf{R}$. The faithful normal semifinite weight defined by $f \mapsto \int_{\mathbf{R}} \exp(-s)f(s)\,\mathrm{d}s$ on $\mathrm{L}^\infty(\mathbf{R})$ gives rise to a faithful normal semifinite weight Tr_φ on $\mathrm{L}_\varphi(\mathbf{R})$ *via* the Fourier transform. The formula $\mathrm{Tr}_\varphi = \mathrm{Tr}_\varphi \circ \mathrm{E}_{\mathrm{L}_\varphi(\mathbf{R})}$ extends it to a faithful normal semifinite trace on $\mathrm{c}_\varphi(M)$.

Because of Connes's Radon–Nikodym cocycle theorem [Co72, Théorème 1.2.1] (see also [Ta03, Theorem VIII.3.3]), the semifinite von Neumann algebra $\mathrm{c}_\varphi(M)$ together with its trace Tr_φ does not depend on the choice of φ in the following precise sense. If $\psi \in M_*$ is another faithful state, there is a canonical surjective $*$-isomorphism $\Pi_{\varphi,\psi} : \mathrm{c}_\psi(M) \to \mathrm{c}_\varphi(M)$ such that $\Pi_{\varphi,\psi} \circ \pi_\psi = \pi_\varphi$ and $\mathrm{Tr}_\varphi \circ \Pi_{\varphi,\psi} = \mathrm{Tr}_\psi$. Note however that $\Pi_{\varphi,\psi}$ does not map the subalgebra $\mathrm{L}_\psi(\mathbf{R}) \subset \mathrm{c}_\psi(M)$ onto the subalgebra $\mathrm{L}_\varphi(\mathbf{R}) \subset \mathrm{c}_\varphi(M)$ (and hence we use the symbol $\mathrm{L}_\varphi(\mathbf{R})$ instead of the usual $\mathrm{L}(\mathbf{R})$).

We start with a rather technical lemma.

Lemma 2.1. *Let M be any σ-finite von Neumann algebra endowed with any faithful state $\varphi \in M_*$. Then for any projection $p \in M$, there exists a projection $q \in M^\varphi$ such that $p \sim q$ in M.*

Proof. Replacing M with $Mz_M(p)$ with the central support $z_M(p)$, we may and will assume that $z_M(p) = 1$. By [KR97, Proposition 6.3.7], one can decompose $p = p_1 + p_2$ along $M = M_1 \oplus M_2$ so that p_1 is finite and p_2 is properly infinite. Since M is σ-finite, p_2 is equivalent to 1_{M_2}, which clearly belongs to M^φ. Hence we may and will assume that p is finite with $z_M(p) = 1$ and hence M is semifinite. Write $\varphi = \mathrm{Tr}(h\,\cdot\,)$ for some nonsingular positive selfadjoint operator h affiliated with M and take a MASA $A \subset M$ that contains $\{h^{\mathrm{i}t} \,|\, t \in \mathbf{R}\}''$. We have $A \subset M^\varphi$. Since A is a MASA with expectation, A is generated by finite projections in M (see e.g. [To71, Proposition 4.4] but this case can be proved without such a general assertion). We will prove that p is equivalent in M to a projection in A. Thanks to [Ka82, Corollaries 3.8, 3.13], we may and will assume, by decomposing M into the components of type I_n, II_1 and II_∞, that M is of type II_∞. Here is a claim.

Claim. For any nonzero finite projection $e \in M$ and any nonzero projection $f \in A$ such that $z_M(e)z_M(f) \neq 0$, there exist nonzero projections $e' \in eMe$ and $f' \in Af$ such that e' is equivalent to f' in M.

Proof of the Claim. As we observed before, there is an increasing sequence of projections $r_n \in A$ that are finite in M and such that $r_n \to 1$ σ-strongly. By assumption, there exists $x \in M$ such that $exf \neq 0$. Then there exists n_0 so that $exfr_{n_0} \neq 0$. Taking the polar decomposition of the element $exfr_{n_0}$, we can find a nonzero subprojection e' of e such that e' is equivalent in M to a subprojection s of fr_{n_0}. Observe that s may not be in A. Hence we have to work further.

Consider the MASA Afr_{n_0} in the type II_1 von Neumann subalgebra $fr_{n_0}Mfr_{n_0}$. By [Ka82, Proposition 3.13], we can find a projection $f' \in Afr_{n_0} \subset A$ that is equivalent to s in M. □

By Zorn's lemma, let $((p_i, q_i))_{i\in I}$ be a maximal family of pairs of projections such that $(p_i)_{i\in I}$ and $(q_i)_{i\in I}$ are families of pairwise orthogonal projections, all p_i are subprojections of p, all q_i are in A and $p_i \sim q_i$ for every $i \in I$. Suppose that $e := p - \sum_{i\in I} p_i \neq 0$ and put $f := 1 - \sum_{i\in I} q_i$. Observe that the central support of f must be equal to 1 since $\sum_{i\in I} q_i \sim \sum_{i\in I} p_i \le p$ is finite and M is of type II_∞ and hence properly infinite. Therefore, by the above claim, there exist nonzero projections p_0, q_0 such that $p_0 \le e$, $q_0 \le f$, $q_0 \in A$ and $p_0 \sim q_0$, a contradiction to the maximality of the family $((p_i, q_i))_{i\in I}$. Consequently, $p = \sum_{i\in I} p_i \sim \sum_{i\in I} q_i \in A$. Hence we are done. □

The following simple application of the previous lemma will turn out to be useful for Popa's intertwining techniques in the type III setting.

Proposition 2.2. *Let $A \subset M$ be any unital inclusion of σ-finite von Neumann algebras with expectation and $p \in A' \cap M$ any nonzero projection. Then $Ap \subset pMp$ is also with expectation.*

Proof. By assumption we may choose a faithful state $\psi \in M_*$ such that A is globally invariant under the modular automorphism group σ^ψ and, in particular, so is $A' \cap M$. Put $\varphi := \psi|_{A'\cap M}$ and observe that $(A' \cap M)^\varphi \subset M^\psi$. Applying Lemma 2.1 to $p \in A' \cap M$ with φ we obtain a partial isometry $v \in A' \cap M$ such that $vv^* = p$ and $v^*v \in (A' \cap M)^\varphi \subset M^\psi$, the latter of which shows that $Av^*v \subset v^*vMv^*v$ is with expectation. Since $v \in A' \cap M$, the inclusions $Av^*v \subset v^*vMv^*v$ and $Ap \subset pMp$ are conjugate to each other via $\mathrm{Ad}(v)$ and hence $Ap \subset pMp$ is with expectation. □

Recall that for any inclusion of von Neumann algebras $A \subset M$, the *group of normalizing unitaries* is defined by

$$\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}.$$

The von Neumann algebra $\mathcal{N}_M(A)''$ is called the *normalizer of A inside M*. The next result is a variation on [Po03, Lemma 3.5] and will be used in the proof of Theorem 4.6.

Proposition 2.3. *Let M be any σ-finite von Neumann algebra and $A \subset M$ any von Neumann subalgebra. Assume moreover that $A' \cap M = \mathcal{Z}(A)$. Then for any nonzero projection $p \in \mathcal{Z}(A)$, we have*

$$\mathcal{N}_{pMp}(Ap)'' = p\big(\mathcal{N}_M(A)''\big)p.$$

Proof. For any $u \in \mathcal{N}_{pMp}(Ap)$, we have $v := u + (1-p) \in \mathcal{N}_M(A)$ and $pvp = u$. Thus, $\mathcal{N}_{pMp}(Ap) \subset p\mathcal{N}_M(A)p$ and hence the inclusion $(\subset)$ holds without taking double commutant. Therefore, it suffices to prove the reverse inclusion relation.

Write $N := \mathcal{N}_M(A)''$ for simplicity. Let $u \in \mathcal{N}_M(A)$ be an arbitrary element. Set $v := pup$. Since $\mathrm{Ad}(u)|_A$ gives a unital $*$-automorphism of A, we have $u\mathcal{Z}(A)u^* = \mathcal{Z}(A)$ so that $v^*v = u^*pu\,p$ and $vv^* = upu^*\,p$ are projections in $\mathcal{Z}(A)$. In particular, v is a partial isometry. Moreover, it is plain to see that for each $a \in A$, we have

$$vav^* = (uau^*)(upu^*)p \in Avv^* \quad \text{and} \quad v^*av = (u^*au)(u^*pu)p \in Av^*v.$$

Hence, $vAv^* = Avv^*$ and $v^*Av = Av^*v$. Observe that $\mathcal{Z}(N) \subset A' \cap M = \mathcal{Z}(A)$.

Claim. There exists a partial isometry $w \in N$ such that (i) $w^*w, ww^* \in \mathcal{Z}(A)p$, (ii) $wAw^* = Aww^*$, $w^*Aw = Aw^*w$, (iii) $wv^*v = v = vv^*w$, and moreover that (iv) with letting $z := z_N(w^*w) = z_N(ww^*) \in \mathcal{Z}(A)p$ (see the notation at the beginning of this section), there exist orthogonal projections $z_1, z_2, z_3 \in \mathcal{Z}(N)$ with $z_1 + z_2 + z_3 = z$ so that

- $w^*wz_1 = z_1$ but $ww^*z_1 \lneqq z_1$,

- $w^*wz_2 \lneqq z_2$ but $ww^*z_2 = z_2$ and
- $w^*wz_3 = z_3 = ww^*z_3$.

Proof of the Claim. To this end, choose a maximal family of partial isometries $w_i \in M$ such that $(w_i^*w_i)_i$ and $(w_iw_i^*)_i$ are families of pairwise orthogonal projections, $w_iAw_i^* = Aw_iw_i^*$, $w_i^*Aw_i = Aw_i^*w_i$, $w_i^*w_i \le p - v^*v$ and $w_iw_i^* \le p - vv^*$. Then $w := v + \sum_i w_i$ clearly enjoys (i)–(iii).

Choose a maximal orthogonal family of projections $z_{3i} \in \mathcal{Z}(N)z$ such that $w^*wz_{3i} = z_{3i} = ww^*z_{3i}$. Set $z_3 := \sum_i z_{3i}$. Choose a maximal orthogonal family of projections $z_{2j} \in \mathcal{Z}(N)(z - z_3)$ such that $ww^*z_{2j} = z_{2j}$. Set $z_2 := \sum_j z_{2j}$. By construction, we have $w^*wz_3 = z_3 = ww^*z_3$ and $w^*wz_2 \lneqq z_2 = ww^*z_2$. Set $z_1 := z - z_2 - z_3$. Assume that $z_1 \neq 0$; otherwise we are already done. By the maximality of the families $(z_{2j})_j$ and $(z_{3i})_i$, observe that no nonzero projection $z' \in \mathcal{Z}(N)z_1$ enjoys $ww^*z' = z'$. This means that the central support of $z_1 - ww^*z_1$ in N is equal to z_1. Suppose that $w^*wz_1 \lneqq z_1$. Then $(z_1 - ww^*z_1)N(z_1 - w^*wz_1) \neq \{0\}$ must hold. Hence there exists $x \in \mathcal{N}_M(A)$ such that $(z_1 - ww^*z_1)x(z_1 - w^*wz_1) \neq 0$. Observe that $z_1 \in \mathcal{Z}(N) \subset A' \cap M = \mathcal{Z}(A)$. Thus the first part (dealing with the v) shows that $w_0 := (z_1 - ww^*z_1)x(z_1 - w^*wz_1) \in N$ is a new nonzero partial isometry such that $w_0^*w_0 \in \mathcal{Z}(A)(z_1 - w^*wz_1)$, $w_0w_0^* \in \mathcal{Z}(A)(z_1 - ww^*z_1)$, $w_0Aw_0^* = Aw_0w_0^*$ and $w_0^*Aw_0 = Aw_0^*w_0$, a contradiction due to the maximality of the family $(w_i)_i$. Hence $w^*wz_1 = z_1$ (and $ww^*z_1 \lneqq z_1$). Thus we have proved the claim. □

Write $w_k := wz_k$, $k = 1, 2, 3$. Observe that $\mathcal{Z}(N) \subset A' \cap M = \mathcal{Z}(A)$ and hence each w_k, in place of w, satisfies (i)–(ii) in the above claim. We will first deal with w_1 when it is nonzero. Set $e_1 := z_1 - w_1w_1^* \neq 0$ and $e_i := w_1^{i-1}e_1w_1^{*i-1}$, $i = 2, 3, \dots$. Observe that all the projections e_n are in $\mathcal{Z}(A)z_1$, since $\mathrm{Ad}(w_1)|_{Az_1}$ defines a unital $*$-isomorphism between Az_1 and $Aw_1w_1^*$ with $w_1w_1^* \in \mathcal{Z}(A)$. We claim that the projections e_n are pairwise orthogonal. Indeed, if $i \lneqq j$, we have $0 \le e_ie_j = w_1^{i-1}e_1w_1^{*i-1}w_1^{j-1}e_1w_1^{*j-1} = w_1^{i-1}(e_1w_1^{j-i}e_1w_1^{*j-i})w_1^{*i-1} \le w_1^{i-1}((z_1 - w_1w_1^*)(w_1w_1^*))w_1^{*i-1} = 0$ so that $e_ie_j = 0$. We also claim that $w_1fw_1^* = f$ with $f := z_1 - \sum_{n\ge1} e_n$. Indeed, $w_1fw_1^* = w_1w_1^* - \sum_{n\ge2} e_n = z_1 - (z_1 - w_1w_1^*) - \sum_{n\ge2} e_n = z_1 - \sum_{n\ge1} e_n = f$. Put $w_1(n) := w_1(\sum_{i=1}^{n-1} e_i) + w_1^{*n-1}e_n + \sum_{i\ge n+1} e_i + w_1f + (p - z_1)$. Clearly, all the elements $w_1(n)$ are in $\mathcal{N}_{pMp}(Ap)$ and $w_1(n)z_1 = z_1w_1(n)$ converges to w_1 as $n \to \infty$ and hence $w_1 \in \mathcal{N}_{pMp}(Ap)''$. Similarly, we can prove that $w_2^* \in \mathcal{N}_{pMp}(Ap)''$, implying that $w_2 \in \mathcal{N}_{pMp}(Ap)''$. Finally, it is trivial that $w_3 + (p - z_3) \in \mathcal{N}_{pMp}(Ap)$, implying that $w_3 \in \mathcal{N}_{pMp}(Ap)''$. Consequently, we have $v = vv^*w = vv^*(w_1 + w_2 + w_3) \in \mathcal{N}_{pMp}(Ap)''$. Hence we are done. □

We point out that we do not need to assume the inclusion $A \subset M$ to be with expectation in Proposition 2.3.

Popa's intertwining techniques. To fix notation, let M be any σ-finite von Neumann algebra, 1_A and 1_B any nonzero projections in M, $A \subset 1_AM1_A$ and $B \subset 1_BM1_B$ any von Neumann subalgebras. Popa introduced his powerful *intertwining-by-bimodules techniques* in [Po01] in the case when M is finite and more generally in [Po03] in the case when M is endowed with an almost periodic faithful normal state φ for which $1_A \in M^\varphi$, $A \subset 1_AM^\varphi1_A$ and $1_B \in M^\varphi$, $B \subset 1_BM^\varphi1_B$. It was showed in [HV12, Ue12] that Popa's intertwining techniques extend to the case when B is finite and with expectation in 1_BM1_B and $A \subset 1_AM1_A$ is any von Neumann subalgebra.

In this paper, we will need the following generalization of [Po01, Theorem A.1] in the case when $A \subset 1_AM1_A$ is any finite von Neumann subalgebra with expectation and $B \subset 1_BM1_B$ is any von Neumann subalgebra with expectation.

Theorem 2.4 ([HI15, Theorem 4.3]). *Let M be any σ-finite von Neumann algebra, 1_A and 1_B any nonzero projections in M, $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ any von Neumann subalgebras with faithful normal conditional expectations $\mathrm{E}_A : 1_A M 1_A \to A$ and $\mathrm{E}_B : 1_B M 1_B \to B$ respectively. Assume moreover that A is a finite von Neumann algebra.*

Then the following conditions are equivalent:

(1) *There exist projections $e \in A$ and $f \in B$, a nonzero partial isometry $v \in eMf$ and a unital normal $*$-homomorphism $\theta : eAe \to fBf$ such that the inclusion $\theta(eAe) \subset fBf$ is with expectation and $av = v\theta(a)$ for all $a \in eAe$.*
(2) *There exist $n \geq 1$, a projection $q \in \mathbf{M}_n(B)$, a nonzero partial isometry $v \in \mathbf{M}_{1,n}(1_A M)q$ and a unital normal $*$-homomorphism $\pi \colon A \to q\mathbf{M}_n(B)q$ such that the inclusion $\pi(A) \subset q\mathbf{M}_n(B)q$ is with expectation and $av = v\pi(a)$ for all $a \in A$.*
(3) *There exists no net $(w_i)_{i\in I}$ of unitaries in $\mathcal{U}(A)$ such that $\lim_i \mathrm{E}_B(b^* w_i a) = 0$ σ-strongly for all $a, b \in 1_A M 1_B$.*

If one of the above conditions is satisfied, we will say that A embeds with expectation into B inside M *and write $A \preceq_M B$.*

Moreover, [HI15, Theorem 4.3] asserts that when $B \subset 1_B M 1_B$ is a *semifinite* von Neumann subalgebra endowed with any fixed faithful normal semifinite trace Tr, then $A \preceq_M B$ if and only if there exist a projection $e \in A$, a Tr-finite projection $f \in B$, a nonzero partial isometry $v \in eMf$ and a unital normal $*$-homomorphism $\theta : eAe \to fBf$ such that $av = v\theta(a)$ for all $a \in eAe$. Hence, in that case, the notation $A \preceq_M B$ is consistent with [Ue12, Proposition 3.1]. In particular, the projection $q \in \mathbf{M}_n(B)$ in Theorem 2.4 (2) is chosen to be finite under the trace $\mathrm{Tr} \otimes \mathrm{tr}_n$, when B is semifinite with any fixed faithful normal semifinite trace Tr. We refer to [HI15, Section 4] for further details.

Remark 2.5. Keep the notation of Theorem 2.4.

(1) Proposition 2.2 gives the following useful additional facts to Theorem 2.4: The inclusions $eAe\, vv^* \subset vv^* M vv^*$ and $\theta(eAe)\, v^*v \subset v^*v M v^*v$ in (2) are also with expectation. Likewise, the inclusions $Aww^* \subset ww^* M ww^*$ and $\pi(A)w^*w \subset w^*w\mathbf{M}_n(M)w^*w$ in (3) are also with expectation.
(2) Assume that there exist $k \geq 1$ and a nonzero partial isometry $u \in \mathbf{M}_{1,k}(M)$ such that $uu^* \in A' \cap 1_A M 1_A$ and $u^*Au \preceq_{\mathbf{M}_k(M)} \mathbf{M}_k(B)$. Then $A \preceq_M B$ holds. Indeed, there exist $n \geq 1$, a projection $q \in \mathbf{M}_n(\mathbf{M}_k(M))$, a nonzero partial isometry $w \in \mathbf{M}_{1,n}(u^*u\mathbf{M}_k(M))q$ and a unital normal $*$-homomorphism $\pi : u^*Au \to q\mathbf{M}_n(\mathbf{M}_k(B))q$ such that the unital inclusion $\pi(u^*Au) \subset q\mathbf{M}_n(\mathbf{M}_k(B))q$ is with expectation and $yw = w\pi(y)$ for all $y \in u^*Au$. Define the unital normal $*$-homomorphism $\iota : A \to u^*Au : a \mapsto u^*au$. Then a simple computation shows that $a\, uw = uw\, (\pi \circ \iota)(a)$ for all $a \in A$, where $uw \in \mathbf{M}_{1,nk}(1_A M)q$ and $uw \neq 0$, $\pi \circ \iota : A \to q\mathbf{M}_{nk}(B)q$ is a unital normal $*$-homomorphism and the unital inclusion $(\pi \circ \iota)(A) \subset q\mathbf{M}_{nk}(B)q$ is with expectation. Therefore, we obtain $A \preceq_M B$.

We are also going to use the following useful technical lemma. This is a generalization of [Va07, Remark 3.8].

Lemma 2.6. *Keep the notation of Theorem 2.4. Let $B \subset P \subset 1_P M 1_P$ be any intermediate von Neumann subalgebra with expectation. Assume that $A \preceq_M P$ and $A \npreceq_M B$.*

Then there exist $k \geq 1$, a projection $q \in \mathbf{M}_k(P)$, a nonzero partial isometry $w \in \mathbf{M}_{1,k}(1_A M)q$ and a unital normal $*$-homomorphism $\pi : A \to q\mathbf{M}_k(P)q$ such that the unital inclusion $\pi(A) \subset q\mathbf{M}_k(P)q$ is with expectation, $\pi(A) \npreceq_{\mathbf{M}_k(P)} \mathbf{M}_k(B)$ and $aw = w\pi(a)$ for all $a \in A$.

Proof. Since $A \preceq_M P$, there exist $k \geq 1$, a projection $q \in \mathbf{M}_k(P)$, a nonzero partial isometry $w \in \mathbf{M}_{1,k}(1_A M)q$ and a unital normal $\ast$-homomorphism $\pi : A \to q\mathbf{M}_k(P)q$ such that the unital inclusion $\pi(A) \subset q\mathbf{M}_k(P)q$ is with expectation and $aw = w\pi(a)$ for all $a \in A$. We have $w^*w \in \pi(A)' \cap q\mathbf{M}_k(M)q$. Following [Va07, Remark 3.8], denote by q_0 the support projection (belonging to $q\mathbf{M}_k(P)q$) of the element $\mathrm{E}_{q\mathbf{M}_k(P)q}(w^*w)$ and observe that $q_0 \in \pi(A)' \cap q\mathbf{M}_k(P)q$. Observe that $\mathrm{E}_{q\mathbf{M}_k(P)q}((q-q_0)w^*w(q-q_0)) = 0$, and hence $w(q-q_0) = 0$, that is, $w = wq_0$. Thanks to Proposition 2.2, replacing q and π with q_0 and $\pi(\cdot)q_0$, respectively, we may assume without loss of generality that q is equal to the support projection of the element $\mathrm{E}_{q\mathbf{M}_k(P)q}(w^*w)$.

We claim that we have $\pi(A) \npreceq_{\mathbf{M}_k(P)} \mathbf{M}_k(B)$. Indeed, otherwise there exist $n \geq 1$, a projection $r \in \mathbf{M}_n(\mathbf{M}_k(B))$, a nonzero partial isometry $u \in \mathbf{M}_{1,n}(q\mathbf{M}_k(P))r$ and a unital normal $\ast$-homomorphism $\theta : \pi(A) \to r\mathbf{M}_n(\mathbf{M}_k(B))r$ such that the unital inclusion $(\theta \circ \pi)(A) \subset r\mathbf{M}_n(\mathbf{M}_k(B))r$ is with expectation and $bu = u\theta(b)$ for all $b \in \pi(A)$. We moreover have $a\,wu = wu\,(\theta\circ\pi)(a)$ for all $a \in A$. Observe that $wu \neq 0$. Indeed, otherwise we have $wu = 0$ and hence

$$\mathrm{E}_{\mathbf{M}_k(P)}(w^*w)u = \mathrm{E}_{\mathbf{M}_n(\mathbf{M}_k(P))}(w^*w\,u) = 0.$$

Since q is equal to the support projection of the element $\mathrm{E}_{q\mathbf{M}_k(P)q}(w^*w)$ and since $u \in \mathbf{M}_{1,n}(q\mathbf{M}_k(P))r$, this implies that $qu = 0$ and hence $u = 0$, a contradiction. Therefore, we have $wu \neq 0$ and hence $A \preceq_M B$, a contradiction. Consequently, we obtain $\pi(A) \npreceq_{\mathbf{M}_k(P)} \mathbf{M}_k(B)$. ☐

We point out that when $P \subset 1_P M 1_P$ is a *semifinite* von Neumann subalgebra endowed with a faithful normal semifinite trace Tr, we may choose the nonzero projection $q \in \mathbf{M}_k(P)$ appearing in Lemma 2.6 to be of finite trace with respect to the faithful normal trace $\mathrm{Tr} \otimes \mathrm{tr}_k$.

Amalgamated free product von Neumann algebras. Let I be any nonempty set and $(B \subset M_i)_{i\in I}$ any family of inclusions of σ-finite von Neumann algebras with faithful normal conditional expectations $\mathrm{E}_i : M_i \to B$. The *amalgamated free product* $(M, \mathrm{E}) = \ast_{B,i\in I}(M_i, \mathrm{E}_i)$ is the unique pair of von Neumann algebra M generated by $(M_i)_{i\in I}$ and faithful normal conditional expectation $\mathrm{E} : M \to B$ such that $(M_i)_{i\in I}$ is *freely independent* with respect to E:

$$\mathrm{E}(x_1 \cdots x_n) = 0 \;\text{ whenever }\; x_j \in M_{i_j}^\circ,\; i_1, \dots, i_n \in I \;\text{ and }\; i_1 \neq \cdots \neq i_n.$$

Here and in what follows, we denote by $M_i^\circ := \ker(\mathrm{E}_i)$. We call the resulting M the *amalgamated free product von Neumann algebra* of $(M_i, E_i)_{i\in I}$ over B. We refer to the product $x_1 \cdots x_n$ where $x_j \in M_{i_j}^\circ$, $i_1, \dots, i_n \in I$ and $i_1 \neq \cdots \neq i_n$ as a *reduced word* in $M_{i_1}^\circ \cdots M_{i_n}^\circ$ of *length* $n \geq 1$. The linear span of B and of all the reduced words in $M_{i_1}^\circ \cdots M_{i_n}^\circ$ where $n \geq 1$, $i_1, \dots, i_n \in I$ and $i_1 \neq \cdots \neq i_n$ forms a unital σ-strongly dense $\ast$-subalgebra of M.

When $B = \mathbf{C}1$, $\mathrm{E}_i = \varphi_i(\cdot)1$ for all $i \in I$ and $\mathrm{E} = \varphi(\cdot)1$, we will simply write $(M, \varphi) = \ast_{i\in I}(M_i, \varphi_i)$ and call the resulting M the *free product von Neumann algebra* of $(M_i, \varphi_i)_{i\in I}$.

When B is a semifinite von Neumann algebra with faithful normal semifinite trace Tr and the weight $\mathrm{Tr} \circ \mathrm{E}_i$ is tracial on M_i for every $i \in I$, the weight $\mathrm{Tr} \circ \mathrm{E}$ is tracial on M (see [Po90, Proposition 3.1] for the finite case and [Ue98a, Theorem 2.6] for the general case). In particular, M is a semifinite von Neumann algebra. In that case, we will refer to $(M, \mathrm{E}) = \ast_{B,i\in I}(M_i, \mathrm{E}_i)$ as a *semifinite* amalgamated free product.

Let $\varphi \in B_*$ be any faithful state. Then for all $t \in \mathbf{R}$, we have $\sigma_t^{\varphi\circ\mathrm{E}} = \ast_{i\in I}\sigma_t^{\varphi\circ\mathrm{E}_i}$ (see [Ue98a, Theorem 2.6]). By [Ta03, Theorem IX.4.2], for every $i \in I$, there exists a unique $\varphi \circ \mathrm{E}$-preserving conditional expectation $\mathrm{E}_{M_i} : M \to M_i$. Moreover, we have $\mathrm{E}_{M_i}(x_1 \cdots x_n) = 0$ for all the reduced words $x_1 \cdots x_n$ that contain at least one letter from M_j° for some $j \in I \setminus \{i\}$ (see e.g. [Ue10, Lemma 2.1]). We will denote by $M \ominus M_i := \ker(\mathrm{E}_{M_i})$. For more on (amalgamated) free product von Neumann algebras, we refer the reader to [BHR12, Po90, Ue98a, Ue10, Ue12, Vo85, VDN92].

The next lemma is a variant of [HU15, Lemma 2.6].

Lemma 2.7. *For each $i \in \{1,2\}$, let $B \subset M_i$ be any inclusion of σ-finite von Neumann algebras with faithful normal conditional expectation $\mathrm{E}_i : M_i \to B$. Denote by $(M, \mathrm{E}) = (M_1, \mathrm{E}_1) *_B (M_2, \mathrm{E}_2)$ the corresponding amalgamated free product.*

Let $\psi \in M_*$ be any faithful state such that $\psi = \psi \circ \mathrm{E}_{M_1}$. Let $(u_j)_{j \in J}$ be any net in $\mathrm{Ball}((M_1)^\psi)$ such that $\lim_j \mathrm{E}_1(b^* u_j a) = 0$ σ-strongly for all $a, b \in M_1$. Then for all $x, y \in M$, we have $\lim_j \mathrm{E}_{M_2}(y^* u_j x) = 0$ σ-strongly.

Proof. We first prove the σ-strong convergence when $x, y \in M_1 \cup M_1 M_2^\circ \cdots M_2^\circ M_1$ are *words* of the form $x = ax'c$ or $x = a$ and $y = by'd$ or $y = b$ with $a, b, c, d \in M_1$ and $x', y' \in M_2^\circ \cdots M_2^\circ$. By free independence, for all $j \in J$, we have

$$\mathrm{E}_{M_2}(y^* u_j x) = \begin{cases} \mathrm{E}_{M_2}(d^* y'^* \, \mathrm{E}_1(b^* u_j a) \, x'c) & (x = ax'c, y = by'd), \\ \mathrm{E}_{M_2}(d^* y'^*) \mathrm{E}_1(b^* u_j a) & (x = a, y = by'd), \\ \mathrm{E}_1(b^* u_j a) \mathrm{E}_{M_2}(x'c) & (x = ax'c, y = b), \\ \mathrm{E}_1(b^* u_j a) & (x = a, y = b). \end{cases}$$

Since $\lim_j \mathrm{E}_1(b^* u_j a) = 0$ σ-strongly, we have $\lim_j \mathrm{E}_{M_2}(y^* u_j x) = 0$ σ-strongly.

We combine now the same pattern of approximation as in the proof of [HU15, Lemma 2.6] with a trick using standard forms as in the proof of [HU15, Theorem 3.1]. Namely, we will work with the standard form $(M, \mathrm{L}^2(M), J^M, \mathfrak{P}^M)$ and denote by e_{M_2} the Jones projection determined by E_{M_2}. Choose a faithful state $\varphi \in M_*$ with $\varphi = \varphi \circ \mathrm{E}$. Denote by $\xi_\varphi, \xi_\psi \in \mathfrak{P}^M$ the unique representing vectors of φ, ψ respectively. Observe that $\varphi = \varphi \circ \mathrm{E}_{M_2}$ and hence $e_{M_2} x \xi_\varphi = \mathrm{E}_{M_2}(x) \xi_\varphi$ holds for every $x \in M$ (though we do *not* have $e_{M_2} x \xi_\psi = \mathrm{E}_{M_2}(x) \xi_\psi$). The rest of the proof is divided into three steps.

(First step) We first prove that $\lim_j \|e_{M_2} y^* u_j \xi\|_{\mathrm{L}^2(M)} = 0$ for any $\xi \in \mathrm{L}^2(M)$ and any *word* $y \in M_1 \cup M_1 M_2^\circ \cdots M_2^\circ M_1$. Indeed, we may choose a sequence $(x_k)_k$, where each x_k is a finite linear combination of *words* in $M_1 \cup M_1 M_2^\circ \cdots M_2^\circ M_1$, and such that $\|\xi - x_k \xi_\varphi\|_{\mathrm{L}^2(M)} \to 0$ as $k \to \infty$, since those linear combinations of words form a σ-strongly dense $*$-subalgebra of M. Then, for all $j \in J$ and $k \in \mathbf{N}$, we have

$$\begin{aligned} \|e_{M_2} y^* u_j \xi\|_{\mathrm{L}^2(M)} &\leq \|e_{M_2} y^* u_j x_k \xi_\varphi\|_{\mathrm{L}^2(M)} + \|e_{M_2} y^* u_j (\xi - x_k \xi_\varphi)\|_{\mathrm{L}^2(M)} \\ &\leq \|\mathrm{E}_{M_2}(y^* u_j x_k) \xi_\varphi\|_{\mathrm{L}^2(M)} + \|y\|_\infty \|\xi - x_k \xi_\varphi\|_{\mathrm{L}^2(M)}. \end{aligned}$$

The first part of the proof implies that $\limsup_j \|e_{M_2} y^* u_j \xi\|_{\mathrm{L}^2(M)} \leq \|y\|_\infty \|\xi - x_k \xi_\varphi\|_{\mathrm{L}^2(M)}$ for all $k \in \mathbf{N}$ and hence $\lim_j \|e_{M_2} y^* u_j \xi\|_{\mathrm{L}^2(M)} = 0$.

(Second step) We next prove that $\lim_j \|e_{M_2} y^* u_j x \xi_\psi\|_{\mathrm{L}^2(M)} = 0$ for any *analytic element* $x \in M$ with respect to the modular automorphism group σ^ψ and any element $y \in M$. Indeed, we may choose a sequence $(y_k)_k$, where each y_k is a finite linear combination of *words* in $M_1 \cup M_1 M_2^\circ \cdots M_2^\circ M_1$, and such that $\lim_{k \to \infty} \|y^* \xi_\psi - y_k^* \xi_\psi\|_{\mathrm{L}^2(M)} = 0$. Then, for all $j \in J$ and $k \in \mathbf{N}$, we have

$$\begin{aligned} \|e_{M_2} y^* u_j x \xi_\psi\|_{\mathrm{L}^2(M)} &\leq \|e_{M_2} y_k^* u_j x \xi_\psi\|_{\mathrm{L}^2(M)} + \|e_{M_2} (y^* - y_k^*) u_j x \xi_\psi\|_{\mathrm{L}^2(M)} \\ &\leq \|e_{M_2} y_k^* u_j x \xi_\psi\|_{\mathrm{L}^2(M)} + \|(y^* - y_k^*) u_j x \xi_\psi\|_{\mathrm{L}^2(M)} \\ &= \|e_{M_2} y_k^* u_j x \xi_\psi\|_{\mathrm{L}^2(M)} + \|J^M \sigma_{\mathrm{i}/2}^\psi(x)^* u_j^* J^M (y^* \xi_\psi - y_k^* \xi_\psi)\|_{\mathrm{L}^2(M)} \\ &\leq \|e_{M_2} y_k^* u_j x \xi_\psi\|_{\mathrm{L}^2(M)} + \|\sigma_{\mathrm{i}/2}^\psi(x)\|_\infty \|y^* \xi_\psi - y_k^* \xi_\psi\|_{\mathrm{L}^2(M)}, \end{aligned}$$

since $u_j \in (M_1)^\psi$. The first step implies that $\limsup_j \|e_{M_2} y^* u_j x \xi_\psi\|_{\mathrm{L}^2(M)} \leq \|\sigma^\psi_{\mathrm{i}/2}(x)\|_\infty \|y^*\xi_\psi - y_k^*\xi_\psi\|_{\mathrm{L}^2(M)}$ for all $k \in \mathbf{N}$ and hence $\lim_j \|e_{M_2} y^* u_j x \xi_\psi\|_{\mathrm{L}^2(M)} = 0$.

(Final step) We finally prove that $\lim_j \|\mathrm{E}_{M_2}(y^* u_j x)\xi_\varphi\|_{\mathrm{L}^2(M)} = 0$ for any elements $x, y \in M$. Indeed, we may choose a sequence $(x_k)_k$ in M of analytic elements with respect to the modular automorphism group σ^ψ such that $\lim_{k\to\infty} \|x\xi_\varphi - x_k\xi_\psi\|_{\mathrm{L}^2(M)} = 0$. Then, for all $j \in J$ and $k \in \mathbf{N}$, we have

$$\begin{aligned}\|\mathrm{E}_{M_2}(y^* u_j x)\xi_\varphi\|_{\mathrm{L}^2(M)} &= \|e_{M_2} y^* u_j x \xi_\varphi\|_{\mathrm{L}^2(M)} \\ &\leq \|e_{M_2} y^* u_j x_k \xi_\psi\|_{\mathrm{L}^2(M)} + \|e_{M_2} y^* u_j (x\xi_\varphi - x_k\xi_\psi)\|_{\mathrm{L}^2(M)} \\ &\leq \|e_{M_2} y^* u_j x_k \xi_\psi\|_{\mathrm{L}^2(M)} + \|y\|_\infty \|x\xi_\varphi - x_k\xi_\psi\|_{\mathrm{L}^2(M)}.\end{aligned}$$

The second step implies that $\limsup_j \|\mathrm{E}_{M_2}(y^* u_j x)\xi_\varphi\|_{\mathrm{L}^2(M)} \leq \|y\|_\infty \|x\xi_\varphi - x_k\xi_\psi\|_{\mathrm{L}^2(M)}$ for all $k \in \mathbf{N}$ and hence $\lim_j \|\mathrm{E}_{M_2}(y^* u_j x)\xi_\varphi\|_{\mathrm{L}^2(M)} = 0$. Hence we are done. □

The next lemma will be used in the proof of the Main Theorem. This can be regarded as a variant of [Po83, Corollary 4.3], [Ge95, Lemma 5.1] (in the tracial case), [Ue98b, Proposition 6] (in the non-tracial case) and also part of [IPP05, Theorem 1.1] (in the tracial amalgamated free product case).

Lemma 2.8. *For each $i \in \{1, 2\}$, let (M_i, φ_i) be any σ-finite von Neumann algebra endowed with any faithful normal state. Denote by $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ the corresponding free product.*

Let $1_Q \in M$ be any nonzero projection and $Q \subset 1_Q M_1 1_Q$ be any diffuse von Neumann subalgebra with expectation. Let $n \geq 1$. If a partial isometry $v \in \mathbf{M}_{1,n}(M)$ with $vv^* \in Q$ or $vv^* \in Q' \cap 1_Q M 1_Q$ satisfies that $v^* Q v \subset \mathbf{M}_n(M_2)$, then $1_Q\, v = 0$. In particular, when $vv^* \in Q$, we have $v = 0$.

Proof. When $vv^* \in Q$, replacing Q with vv^*Qvv^* we may and will assume that $vv^* = 1_Q$. Hence, since $vv^* = 1_Q \in Q$ or $vv^* \in Q' \cap 1_Q M 1_Q$, we may think of the map $Q \to \mathbf{M}_n(M_2) : x \mapsto v^* x v$ as a normal (non-unital) $*$-homomorphism.

Since $Q \subset 1_Q M_1 1_Q$ is with expectation, we may choose a faithful state $\psi \in M_*$ such that $\psi = \psi \circ \mathrm{E}_{M_1}$, $1_Q \in (M_1)^\psi$, $Q \subset 1_Q M 1_Q$ is globally invariant under the modular automorphism group σ^{ψ_Q} and $Q^{\psi_Q} \subset 1_Q (M_1)^\psi 1_Q$ is diffuse where $\psi_Q := \frac{\psi(1_Q \cdot 1_Q)}{\psi(1_Q)}$. See e.g. the proof of [HU15, Lemma 2.1].

Write $v = [v_1 \cdots v_n] \in \mathbf{M}_{1,n}(M)$ and denote by tr_n the canonical normalized trace on $\mathbf{M}_n(\mathbf{C})$. Since Q^{ψ_Q} is diffuse, we can choose a sequence of unitaries $(u_k)_k$ in $\mathcal{U}(Q^{\psi_Q})$ with $\lim_{k\to\infty} u_k = 0$ σ-weakly. By Lemma 2.7, we have

$$\lim_{k\to\infty} \|\mathrm{E}_{\mathbf{M}_n(M_2)}(v^* u_k v)\|^2_{\varphi\otimes\mathrm{tr}_n} = \lim_{k\to\infty} \sum_{i,j=1}^{n} \|\mathrm{E}_{M_2}(v_i^* u_k v_j)\|^2_\varphi = 0.$$

Since $v^* u_k v \in \mathcal{U}(v^* Q v) \subset \mathbf{M}_n(M_2)$, we have

$$\|v^* 1_Q v\|_{\varphi\otimes\mathrm{tr}_n} = \|v^* u_k v\, v^* 1_Q v\|_{\varphi\otimes\mathrm{tr}_n} = \|v^* u_k v\|_{\varphi\otimes\mathrm{tr}_n} = \|\mathrm{E}_{\mathbf{M}_n(M_2)}(v^* u_k v)\|_{\varphi\otimes\mathrm{tr}_n} \to 0$$

as $k \to \infty$, implying that $1_Q\, v = 0$. □

We point out that the above way of proof is applicable even to amalgamated free products over non-trivial subalgebras under suitable assumptions. Similarly, the same can be said about [HU15, Proposition 2.7].

Ultraproduct von Neumann algebras. Let M be any σ-finite von Neumann algebra and $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ any nonprincipal ultrafilter. Define

$$\mathcal{I}_\omega(M) = \{(x_n)_n \in \ell^\infty(\mathbf{N}, M) : x_n \to 0 \ \ast\text{-strongly as } n \to \omega\}$$
$$\mathcal{M}^\omega(M) = \{(x_n)_n \in \ell^\infty(\mathbf{N}, M) : (x_n)_n \, \mathcal{I}_\omega(M) \subset \mathcal{I}_\omega(M) \text{ and } \mathcal{I}_\omega(M)\,(x_n)_n \subset \mathcal{I}_\omega(M)\}.$$

The *multiplier algebra* $\mathcal{M}^\omega(M)$ is a C*-algebra and $\mathcal{I}_\omega(M) \subset \mathcal{M}^\omega(M)$ is a norm closed two-sided ideal. Following [Oc85, §5.1], we define the *ultraproduct von Neumann algebra* M^ω by $M^\omega := \mathcal{M}^\omega(M)/\mathcal{I}_\omega(M)$, which is indeed known to be a von Neumann algebra. We denote the image of $(x_n)_n \in \mathcal{M}^\omega(M)$ by $(x_n)^\omega \in M^\omega$.

For every $x \in M$, the constant sequence $(x)_n$ lies in the multiplier algebra $\mathcal{M}^\omega(M)$. We will then identify M with $(M + \mathcal{I}_\omega(M))/\mathcal{I}_\omega(M)$ and regard $M \subset M^\omega$ as a von Neumann subalgebra. The map $\mathrm{E}_\omega : M^\omega \to M : (x_n)^\omega \mapsto \sigma\text{-weak} \lim_{n\to\omega} x_n$ is a faithful normal conditional expectation. For every faithful state $\varphi \in M_\ast$, the formula $\varphi^\omega := \varphi \circ \mathrm{E}_\omega$ defines a faithful normal state on M^ω. Observe that $\varphi^\omega((x_n)^\omega) = \lim_{n\to\omega} \varphi(x_n)$ for all $(x_n)^\omega \in M^\omega$.

Following [Co74, §2], we define

$$\mathcal{M}_\omega(M) := \left\{(x_n)_n \in \ell^\infty(\mathbf{N}, M) : \lim_{n\to\omega} \|x_n\varphi - \varphi x_n\| = 0, \forall \varphi \in M_\ast\right\}.$$

We have $\mathcal{I}_\omega(M) \subset \mathcal{M}_\omega(M) \subset \mathcal{M}^\omega(M)$. The *asymptotic centralizer* M_ω is defined by $M_\omega := \mathcal{M}_\omega(M)/\mathcal{I}_\omega(M)$. We have $M_\omega \subset M^\omega$. Moreover, by [Co74, Proposition 2.8] (see also [AH12, Proposition 4.35]), we have $M_\omega = M' \cap (M^\omega)^{\varphi^\omega}$ for every faithful state $\varphi \in M_\ast$.

Let $Q \subset M$ be any von Neumann subalgebra with faithful normal conditional expectation $\mathrm{E}_Q : M \to Q$. Choose a faithful state $\varphi \in M_\ast$ in such a way that $\varphi = \varphi \circ \mathrm{E}_Q$. We have $\ell^\infty(\mathbf{N}, Q) \subset \ell^\infty(\mathbf{N}, M)$, $\mathcal{I}_\omega(Q) \subset \mathcal{I}_\omega(M)$ and $\mathcal{M}^\omega(Q) \subset \mathcal{M}^\omega(M)$. We will then identify $Q^\omega = \mathcal{M}^\omega(Q)/\mathcal{I}_\omega(Q)$ with $(\mathcal{M}^\omega(Q) + \mathcal{I}_\omega(M))/\mathcal{I}_\omega(M)$ and be able to regard $Q^\omega \subset M^\omega$ as a von Neumann subalgebra. Observe that the norm $\|\cdot\|_{(\varphi|_Q)^\omega}$ on Q^ω is the restriction of the norm $\|\cdot\|_{\varphi^\omega}$ to Q^ω. Observe moreover that $(\mathrm{E}_Q(x_n))_n \in \mathcal{I}_\omega(Q)$ for all $(x_n)_n \in \mathcal{I}_\omega(M)$ and $(\mathrm{E}_Q(x_n))_n \in \mathcal{M}^\omega(Q)$ for all $(x_n)_n \in \mathcal{M}^\omega(M)$. Therefore, the mapping $\mathrm{E}_{Q^\omega} : M^\omega \to Q^\omega : (x_n)^\omega \mapsto (\mathrm{E}_Q(x_n))^\omega$ is a well-defined conditional expectation satisfying $\varphi^\omega \circ \mathrm{E}_{Q^\omega} = \varphi^\omega$. Hence, $\mathrm{E}_{Q^\omega} : M^\omega \to Q^\omega$ is a faithful normal conditional expectation. For more on ultraproduct von Neumann algebras, we refer the reader to [AH12, Oc85].

We give a useful result showing how Popa's intertwining techniques behave with respect to taking ultraproduct von Neumann algebras.

Proposition 2.9. *Let M be any σ-finite von Neumann algebra, 1_A and 1_B any nonzero projections in M, $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ any von Neumann subalgebras with faithful normal conditional expectations $\mathrm{E}_A : 1_A M 1_A \to A$ and $\mathrm{E}_B : 1_B M 1_B \to B$ respectively. Assume moreover that A is a finite von Neumann algebra.*

Let $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ be any nonprincipal ultrafilter. Define $A^\omega \subset (1_A M 1_A)^\omega = 1_A M^\omega 1_A$ and $B^\omega \subset (1_B M 1_B)^\omega = 1_B M^\omega 1_B$. If $A^\omega \preceq_{M^\omega} B^\omega$, then $A \preceq_M B$.

Proof. The proof uses an idea of [Io12, Lemma 9.5]. Choose a faithful state $\varphi \in M_\ast$ in such a way that $1_B \in M^\varphi$ and $\varphi_B \circ \mathrm{E}_B = \varphi_B$ with $\varphi_B := \frac{\varphi(1_B \cdot 1_B)}{\varphi(1_B)}$. Assume that $A^\omega \preceq_{M^\omega} B^\omega$. By Theorem 2.4, there exist $\delta > 0$ and a finite subset $\mathcal{F} \subset 1_A M^\omega 1_B$ such that

$$\sum_{a,b\in\mathcal{F}} \|\mathrm{E}_{B^\omega}(b^\ast u a)\|_{\varphi^\omega}^2 > \delta, \forall u \in \mathcal{U}(A^\omega). \tag{2.1}$$

For each $a \in \mathcal{F}$, write $a = (a_n)^\omega$ with a fixed sequence $(a_n)_n \in 1_A \mathcal{M}^\omega(M) 1_B$.

We next claim that there exists $n \in \mathbf{N}$ such that

$$\sum_{a,b\in\mathcal{F}} \|\mathrm{E}_{B^\omega}(b_n^\ast u a_n)\|_{\varphi^\omega}^2 \geq \delta, \forall u \in \mathcal{U}(A^\omega). \tag{2.2}$$

Assume by contradiction that this is not the case. Then for every $n \in \mathbf{N}$, there exists $u_n \in \mathcal{U}(A^\omega)$ such that

$$\sum_{a,b \in \mathcal{F}} \|\mathrm{E}_{B^\omega}(b_n^* u_n a_n)\|_{\varphi^\omega}^2 < \delta.$$

Since A is a finite von Neumann algebra, we may write $u_n = (u_m^{(n)})^\omega$ with a sequence $(u_m^{(n)})_m \in \ell^\infty(\mathbf{N}, A)$ such that $u_m^n \in \mathcal{U}(A)$ for all $m \in \mathbf{N}$. Then we have

$$\lim_{m \to \omega} \sum_{a,b \in \mathcal{F}} \|\mathrm{E}_B(b_n^* u_m^{(n)} a_n)\|_\varphi^2 < \delta$$

for all $n \in \mathbf{N}$. Thus, we may choose $m_n \in \mathbf{N}$ large enough so that $v_n := u_{m_n}^{(n)} \in \mathcal{U}(A)$ satisfies

$$\sum_{a,b \in \mathcal{F}} \|\mathrm{E}_B(b_n^* v_n a_n)\|_\varphi^2 < \delta.$$

Since A is finite, we may define $v := (v_n)^\omega \in \mathcal{U}(A^\omega)$ and we obtain

$$\sum_{a,b \in \mathcal{F}} \|\mathrm{E}_{B^\omega}(b^* v a)\|_{\varphi^\omega}^2 = \lim_{n \to \omega} \sum_{a,b \in \mathcal{F}} \|\mathrm{E}_B(b_n^* v_n a_n)\|_\varphi^2 \leq \delta. \tag{2.3}$$

Equations (2.1) and (2.3) give a contradiction. This shows that Equation (2.2) holds. Therefore, up to replacing the finite subset $\mathcal{F} \subset 1_A M^\omega 1_B$ with $\{a_n : a \in \mathcal{F}\} \subset 1_A M 1_B$, we may assume that $\mathcal{F} \subset 1_A M 1_B$ in Equation (2.1). In particular, we obtain

$$\sum_{a,b \in \mathcal{F}} \|\mathrm{E}_B(b^* u a)\|_\varphi^2 \geq \delta, \forall u \in \mathcal{U}(A).$$

This finally implies that $A \preceq_M B$. □

3. A characterization of von Neumann algebras with property Gamma

In this section, we generalize Popa's characterization of property Gamma for *tracial* von Neumann algebras (see [Oz03, Proposition 7] with $\mathcal{N}_0 = \mathcal{M}$) to *arbitrary* von Neumann algebras. This generalization is an unpublished result due to Houdayer–Raum, which they obtained through their recent work [HR14].

Theorem 3.1. *Let M be any diffuse von Neumann algebra with separable predual and $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ any nonprincipal ultrafilter. The following conditions are equivalent.*

- (i) *The central sequence algebra $M' \cap M^\omega$ is diffuse.*
- (ii) *The asymptotic centralizer M_ω is diffuse.*
- (iii) *There exists a faithful state $\psi \in M_*$ such that $M' \cap (M^\psi)^\omega$ is diffuse.*
- (iv) *There exists a decreasing sequence $(A_n)_n$ of diffuse abelian von Neumann subalgebras of M with expectation such that $M = \bigvee_{n \in \mathbf{N}}((A_n)' \cap M)$.*

Proof. Let $z_k \in \mathcal{Z}(M)$ be a sequence of central projections such that $\sum_k z_k = 1$, Mz_0 has a diffuse center and Mz_k is a diffuse factor for all $k \geq 1$. The equivalences (i) ⇔ (ii) ⇔ (iii) ⇔ (iv) are all obvious for Mz_0, since all conditions actually hold true. Indeed, in order to obtain (iv), observe that it suffices to take $A_n = \mathcal{Z}(Mz_0)$ for every $n \in \mathbf{N}$. It remains to prove the equivalences for each Mz_k with $k \geq 1$. Therefore, in order to prove the result and without loss of generality, we may assume that M is a diffuse factor.

(i) ⇒ (ii) (c.f. [HR14, Corollary 2.6].) Fix a faithful state $\varphi \in M_*$. By [Co74, Proposition 2.8] (see also [AH12, Proposition 4.35]), we have $M_\omega = M' \cap (M^\omega)^{\varphi^\omega}$. Then M_ω is diffuse by [HR14, Theorem 2.3] (see also [Co74, Corollary 3.8]).

(ii) ⇒ (iii) Fix a faithful state $\varphi \in M_*$. Since $M' \cap (M^\omega)^{\varphi^\omega} = M_\omega$ is diffuse, we may choose a projection $e \in M_\omega$ such that $\varphi^\omega(e) = 2^{-1}$. Since M is diffuse, we may write $e = (e_n)^\omega$

with a sequence of projections $(e_n)_n \in \mathcal{M}^\omega(M)$ such that $\varphi(e_n) = 2^{-1}$ for all $n \in \mathbf{N}$ (see [HR14, Proposition 2.2]). Observe that σ-weak $\lim_{n\to\omega} e_n = 2^{-1}1_M$, since M is a factor. Fix a countable $\|\cdot\|_\varphi$-dense subset $Y = \{y_n : n \in \mathbf{N}\} \subset M$.

Since $e \in M_\omega = M' \cap (M^\omega)^{\varphi^\omega}$, there exists $n \in \mathbf{N}$ large enough so that the projection $p_0 := e_n \in M$ satisfies $\varphi(p_0) = 2^{-1}$, $\|y_0p_0 - p_0y_0\|_\varphi \leq 2^{-1}$ and $\|\varphi p_0 - p_0\varphi\| \leq 2^{-1}$. Next, $ep_0 \in (M' \cap M^\omega)p_0 \subset (p_0Mp_0)^\omega$ is a projection satisfying $\varphi^\omega(ep_0) = \lim_{n\to\omega} \varphi(e_np_0) = 2^{-2}$ because of σ-weak $\lim_{n\to\omega} e_n = 2^{-1}1_M$. Since p_0Mp_0 is diffuse, we may write $ep_0 = (r_n)^\omega$ with a sequence of projections $(r_n)_n \in \mathcal{M}^\omega(p_0Mp_0)$ such that $\varphi(r_n) = 2^{-2}$ for all $n \in \mathbf{N}$. Likewise, we may write $ep_0^\perp = (s_n)^\omega$ for a sequence of projections $(s_n)_n \in \mathcal{M}^\omega(p_0^\perp Mp_0^\perp)$ such that $\varphi(s_n) = 2^{-2}$ for all $n \in \mathbf{N}$. Observe that $e = ep_0 + ep_0^\perp = (r_n + s_n)^\omega \in M' \cap (M^\omega)^{\varphi^\omega}$. Then there exists $n \in \mathbf{N}$ large enough so that $p_1 := r_n + s_n$ satisfies $\varphi(p_1) = 2^{-1}$, $p_0p_1 = p_1p_0$, $\varphi(p_0p_1) = 2^{-2}$, $\|y_jp_1 - p_1y_j\| \leq 2^{-2}$ for all $0 \leq j \leq 1$ and $\|\varphi p_1 - p_1\varphi\| \leq 2^{-2}$.

Repeating the above procedure, we construct by induction a sequence of projections $(p_n)_n$ in M satisfying the following properties:

(P1) $\varphi(p_n) = 2^{-1}$ for all $n \in \mathbf{N}$.
(P2) $p_jp_n = p_np_j$ for all $j, n \in \mathbf{N}$.
(P3) $\varphi(p_{i_1} \cdots p_{i_r}) = 2^{-r}$ for all $r \geq 1$ and all r-tuples $(i_1, \ldots, i_r)$ of pairwise distinct integers.
(P4) $\|y_jp_n - p_ny_j\|_\varphi \leq 2^{-(n+1)}$ for all $0 \leq j \leq n$.
(P5) $\|\varphi p_n - p_n\varphi\| \leq 2^{-(n+1)}$ for all $n \in \mathbf{N}$.

It follows that $(p_n)_n \in \mathcal{M}^\omega(M)$ and $p := (p_n)^\omega \in M' \cap (M^\omega)^{\varphi^\omega}$ satisfies $\varphi^\omega(p) = 2^{-1}$.

For each pair $0 \leq m \leq n$, put $\varphi_{m,n} := \sum_{j_m,\ldots,j_n \in \{1,\perp\}} p_m^{j_m} \cdots p_n^{j_n}\, \varphi\, p_m^{j_m} \cdots p_n^{j_n} \in M_*$ and observe that φ_{mn} is a faithful normal state. For any pair $0 \leq m \leq n$, using the triangle inequality with (P2) and (P5), we have

$$\|\varphi_{m,n} - \varphi_{m,n+1}\| \leq \left\| \varphi - \sum_{j_{n+1} \in \{1,\perp\}} p_{n+1}^{j_{n+1}}\, \varphi\, p_{n+1}^{j_{n+1}} \right\| \leq 2\, \|\varphi p_{n+1} - p_{n+1}\varphi\| \leq 2^{-(n+1)}.$$

This implies that for each $m \in \mathbf{N}$, the sequence $(\varphi_{m,n})_n$ is Cauchy and hence convergent in M_*. Put $\Phi_m = \lim_{n\to\infty} \varphi_{m,n} \in M_*$ and observe that Φ_m is a normal state. We moreover have

$$\begin{aligned}\|\varphi - \Phi_m\| &\leq \|\varphi - \varphi_{m,n}\| + \|\varphi_{m,n} - \Phi_m\| \\ &\leq 2\,\|\varphi p_m - p_m\varphi\| + \sum_{n \geq m} \|\varphi_{m,n} - \varphi_{m,n+1}\| \\ &\leq 2^{-m} + \sum_{n\geq m} 2^{-(n+1)} = 2^{-(m-1)}.\end{aligned}$$

This implies that $\lim_{m\to\infty} \Phi_m = \varphi$. Observe that $\Phi_m\, p_n = p_n\, \Phi_m$ for all $0 \leq m \leq n$.

We next claim that Φ_m is a *faithful* normal state for all $m \in \mathbf{N}$. Indeed, fix $m \in \mathbf{N}$ and let $x \in M^+$ such that $\Phi_m(x) = 0$. We prove by induction over $n \geq m$ that $\Phi_n(x) = 0$. By assumption, we have $\Phi_m(x) = 0$. Assume that $\Phi_n(x) = 0$ for some $n \geq m$. Observe that $0 = \Phi_n(x) = \Phi_{n+1}(p_nxp_n + p_n^\perp xp_n^\perp)$. Denote by $q \in M$ the support of the normal state Φ_{n+1}. We have $qp_nxp_nq = 0 = qp_n^\perp xp_n^\perp q$. This implies that $x^{1/2}p_nq = 0 = x^{1/2}p_n^\perp q$ and hence $x^{1/2}q = 0$, that is, $qxq = 0$. Thus, $\Phi_{n+1}(x) = 0$. Therefore, we have $\Phi_n(x) = 0$ for all $n \geq m$ and hence $\varphi(x) = \lim_{n\to\infty} \Phi_n(x) = 0$. Since φ is faithful, we obtain $x = 0$. This shows that $\Phi_m \in M_*$ is faithful for every $m \in \mathbf{N}$.

Letting $\psi := \Phi_0$, we have $p_n \in M^\psi$ for all $n \in \mathbf{N}$ and hence $p = (p_n)^\omega \in M' \cap (M^\psi)^\omega$. Since $\varphi^\omega(p) = 2^{-1}$, we have $p \neq 0, 1$. This implies that $M' \cap (M^\psi)^\omega$ is diffuse. Indeed, proceeding as in the proof of [Co74, Corollary 3.8], let $f \in M' \cap (M^\psi)^\omega$ be any projection such that $\psi^\omega(f) = \lambda$ with $\lambda \neq 0, 1$. Write $f = (f_n)^\omega$ where $f_n \in M^\psi$ is a projection for every $n \in \mathbf{N}$. Observe that since M is a factor, we have σ-weak $\lim_{n\to\omega} f_n = \lambda 1_M$. We can construct by induction an increasing sequence of integers $k_n \in \mathbf{N}$ satisfying the following properties:

(P1) $|\psi(f_n f_{k_n}) - \lambda\psi(f_n)| \leq (n+1)^{-1}$ for all $n \in \mathbf{N}$.
(P2) $\|f_n f_{k_n} - f_{k_n} f_n\|_\psi \leq (n+1)^{-1}$ for all $n \in \mathbf{N}$.
(P3) $\|y_j f_{k_n} - f_{k_n} y_j\|_\psi \leq (n+1)^{-1}$ for all $0 \leq j \leq n$.

It follows that $r := (f_n f_{k_n})^\omega \in M' \cap (M^\psi)^\omega$ is a projection satisfying $r \leq f$ and $\psi(r) = \lambda^2$. This shows that $f \in M' \cap (M^\psi)^\omega$ is not a minimal projection and hence $M' \cap (M^\psi)^\omega$ is diffuse.

(iii) $\Rightarrow$ (iv) The proof of this implication is entirely analogous to the one of [Oz03, Proposition 7] with $\mathcal{N}_0 = \mathcal{M}$ but we give the details for the sake of completeness. Fix a countable $\|\cdot\|_\psi$-dense subset $Y = \{y_n : n \in \mathbf{N}\} \subset M$. Since $M' \cap (M^\psi)^\omega$ is diffuse (note that M^ψ is also diffuse), the proof of (ii) $\Rightarrow$ (iii) shows that we can construct by induction a sequence of projections $p_n \in M^\psi$ satisfying the following properties:

(P1) $\psi(p_n) = 2^{-1}$ for all $n \in \mathbf{N}$.
(P2) $p_j p_n = p_n p_j$ for all $j, n \in \mathbf{N}$.
(P3) $\psi(p_{i_1} \cdots p_{i_r}) = 2^{-r}$ for all $r \geq 1$ and all r-tuples $(i_1, \dots, i_r)$ of pairwise distinct integers.
(P4) $\|y_j p_n - p_n y_j\|_\psi \leq 2^{-(n+1)}$ for all $0 \leq j \leq n$.

For each $k \in \mathbf{N}$, define $D_k := \mathbf{C}p_k \oplus \mathbf{C}p_k^\perp$. For each pair $0 \leq m \leq n$, define $A_{m,n} := \bigvee_{m \leq k \leq n} D_k$ and $A_m := \bigvee_{m \leq k} D_k = \bigvee_{m \leq n} A_n$. Observe that $A_{m,n}, A_m \subset M^\psi$ for all $0 \leq m \leq n$. We also have that $(A_m)_m$ is a decreasing sequence of diffuse abelian von Neumann subalgebras of M^ψ by (P2) and (P3). Fix $j \in \mathbf{N}$ and let $n \geq m \geq j$. Whenever $C \subset M$ is a von Neumann subalgebra globally invariant under the modular automorphism group σ^ψ, denote by $\mathrm{E}^\psi_C : M \to C$ the unique ψ-preserving conditional expectation. We have $A_{m,n+1} = A_{m,n} \vee D_{n+1} \subset M^\psi$, $\mathrm{E}^\psi_{(A_{m,n+1})' \cap M} = \mathrm{E}^\psi_{(A_{m,n})' \cap M} \circ \mathrm{E}^\psi_{(D_{n+1})' \cap M}$ (see e.g. [Po83, Lemma 1.2.2]) and

$$\begin{aligned}\left\|\mathrm{E}^\psi_{(A_{m,n+1})'\cap M}(y_j) - \mathrm{E}^\psi_{(A_{m,n})'\cap M}(y_j)\right\|_\psi &= \left\|\mathrm{E}^\psi_{(A_{m,n})'\cap M}\left(\mathrm{E}^\psi_{(D_{n+1})'\cap M}(y_j) - y_j\right)\right\|_\psi \\ &\leq \left\|\mathrm{E}^\psi_{(D_{n+1})'\cap M}(y_j) - y_j\right\|_\psi \\ &\leq 2\,\|p_{n+1} y_j - y_j p_{n+1}\|_\psi \leq 2^{-(n+1)}.\end{aligned}$$

By [Po81, Lemma 1.2 1°], we have $\|y_j - \mathrm{E}^\psi_{(A_m)'\cap M}(y_j)\|_\psi = \lim_{n\to\infty} \|y_j - \mathrm{E}^\psi_{(A_{m,n})'\cap M}(y_j)\|_\psi$ and hence

$$\begin{aligned}\left\|y_j - \mathrm{E}^\psi_{(A_m)'\cap M}(y_j)\right\|_\psi &= \lim_n \left\|y_j - \mathrm{E}^\psi_{(A_{m,n})'\cap M}(y_j)\right\|_\psi \\ &\leq \left\|y_j - \mathrm{E}^\psi_{(A_{m,n})'\cap M}(y_j)\right\|_\psi + \sum_{n\geq m}\left\|\mathrm{E}^\psi_{(A_{m,n+1})'\cap M}(y_j) - \mathrm{E}^\psi_{(A_{m,n})'\cap M}(y_j)\right\|_\psi \\ &\leq 2\|p_m y_j - y_j p_m\|_\psi + \sum_{n\geq m}\left\|\mathrm{E}^\psi_{(A_{m,n+1})'\cap M}(y_j) - \mathrm{E}^\psi_{(A_{m,n})'\cap M}(y_j)\right\|_\psi \\ &\leq 2^{-m} + 2^{-m} = 2^{-(m-1)}.\end{aligned}$$

It follows that $\lim_m \|y_j - \mathrm{E}^\psi_{(A_m)'\cap M}(y_j)\|_\psi = 0$ for all $j \in \mathbf{N}$. Since $Y \subset M$ is $\|\cdot\|_\psi$-dense, this implies that $\lim_m \|y - \mathrm{E}^\psi_{(A_m)'\cap M}(y)\|_\psi = 0$ for all $y \in M$ and hence $M = \bigvee_{n\in\mathbf{N}}((A_n)' \cap M)$.

(iv) $\Rightarrow$ (i) For every $n \in \mathbf{N}$, choose a projection $p_n \in A_n \subset A_0$ such that $\varphi(p_n) = 2^{-1}$. Then $p := (p_n)^\omega \in M' \cap M^\omega$ and $\varphi^\omega(p) = 2^{-1}$. Therefore, $M' \cap M^\omega \neq \mathbf{C}1$ and hence $M' \cap M^\omega$ is diffuse since M is a factor (see e.g. [HR14, Corollary 2.6], or the final part of the proof of (ii) $\Rightarrow$ (iii)). $\square$

4. Structure of AFP von Neumann algebras over arbitrary index sets

In this section, we prove key results regarding the position of finite von Neumann subalgebras with expectation and with either nonamenable relative commutant (see Theorem 4.4) or nonamenable normalizer (see Theorem 4.6) inside arbitrary free product von Neumann algebras over arbitrary index sets.

Semifinite AFP von Neumann algebras over arbitrary index sets. We will be using the following notation throughout this section.

Notation 4.1. Let I be any nonempty set and $(\mathcal{B} \subset \mathcal{M}_i)_{i \in I}$ any family of inclusions of semifinite σ-finite von Neumann algebras with faithful normal conditional expectations $\mathrm{E}_i : \mathcal{M}_i \to \mathcal{B}$, where $\mathcal{B}$ has a faithful normal semifinite trace Tr such that $\mathrm{Tr} \circ \mathrm{E}_i$ is tracial on $\mathcal{M}_i$ for every $i \in I$. Assume moreover that $\mathcal{B}$ is amenable. Denote by $(\mathcal{M}, \mathrm{E}) = *_{\mathcal{B}, i \in I}(\mathcal{M}_i, \mathrm{E}_i)$ the corresponding semifinite amalgamated free product. For every nonempty subset $\mathcal{G} \subset I$, put $(\mathcal{M}_{\mathcal{G}}, \mathrm{E}_{\mathcal{G}}) = *_{\mathcal{B}, i \in \mathcal{G}}(\mathcal{M}_i, \mathrm{E}_i)$. By convention, put $\mathcal{M}_{\emptyset} := \mathcal{B}$. In this context, any trace means (an amplification $\mathrm{Tr}_n := \mathrm{Tr} \otimes \mathrm{tr}_n$ of) the trace $\mathrm{Tr} := \mathrm{Tr} \circ \mathrm{E}$ or $\mathrm{Tr} \circ \mathrm{E}_{\mathcal{G}}$.

The next proposition will be used to reduce the problem of locating subalgebras inside arbitrary semifinite amalgamated free product von Neumann algebras over *arbitrary* index sets to *finite* index sets.

Proposition 4.2. *Keep Notation 4.1. Let $p \in \mathcal{M}$ be any nonzero finite trace projection and $\mathcal{Q} \subset p\mathcal{M}p$ any von Neumann subalgebra. Assume that for any nonzero projection $z \in \mathcal{Q}' \cap p\mathcal{M}p$ and any nonempty finite subset $\mathcal{F} \subset I$, we have $\mathcal{Q}z \preceq_{\mathcal{M}} \mathcal{M}_{\mathcal{F}^c}$. Then $\mathcal{Q}$ is amenable.*

Proof. The proof uses an idea due to Ioana. By contradiction, assume that $\mathcal{Q}$ is not amenable. Up to cutting down by a nonzero central projection in $\mathcal{Z}(\mathcal{Q})$ if necessary, we may assume without loss of generality that $\mathcal{Q}$ has no amenable direct summand and that for any nonzero projection $z \in \mathcal{Q}' \cap p\mathcal{M}p$ and any nonempty finite subset $\mathcal{F} \subset I$, we have $\mathcal{Q}z \preceq_{\mathcal{M}} \mathcal{M}_{\mathcal{F}^c}$. By assumption and using [HI15, Lemma 4.11], for every nonempty finite subset $\mathcal{F} \subset I$, there exist $n_{\mathcal{F}} \geq 1$, a finite trace projection $q_{\mathcal{F}} \in \mathbf{M}_{n_{\mathcal{F}}}(\mathcal{M}_{\mathcal{F}^c})$, a nonzero partial isometry $w_{\mathcal{F}} \in \mathbf{M}_{1,n_{\mathcal{F}}}(p\mathcal{M})q_{\mathcal{F}}$ and a unital normal $*$-homomorphism $\pi_{\mathcal{F}} : \mathcal{Q} \to q_{\mathcal{F}}\mathbf{M}_{n_{\mathcal{F}}}(\mathcal{M}_{\mathcal{F}^c})q_{\mathcal{F}}$ such that $aw_{\mathcal{F}} = w_{\mathcal{F}}\pi_{\mathcal{F}}(a)$ for all $a \in \mathcal{Q}$ and $\lim_{\mathcal{F}} w_{\mathcal{F}}w_{\mathcal{F}}^* = p = 1_{\mathcal{Q}}$. Observe that $w_{\mathcal{F}}w_{\mathcal{F}}^* \in \mathcal{Q}' \cap p\mathcal{M}p$ and $w_{\mathcal{F}}^*w_{\mathcal{F}} \in \pi_{\mathcal{F}}(\mathcal{Q})' \cap q_{\mathcal{F}}\mathbf{M}_{n_{\mathcal{F}}}(\mathcal{M})q_{\mathcal{F}}$. Since $\pi_{\mathcal{F}}(\mathcal{Q})$ has no amenable direct summand and $\mathcal{B}$ is amenable, we have $\pi_{\mathcal{F}}(\mathcal{Q}) \npreceq_{\mathbf{M}_{n_{\mathcal{F}}}(\mathcal{M})} \mathbf{M}_{n_{\mathcal{F}}}(\mathcal{B})$ and hence [BHR12, Theorem 2.5] shows $w_{\mathcal{F}}^*w_{\mathcal{F}} \in q_{\mathcal{F}}\mathbf{M}_{n_{\mathcal{F}}}(\mathcal{M}_{\mathcal{F}^c})q_{\mathcal{F}}$ for all $\mathcal{F}$. Thus, we may assume that $q_{\mathcal{F}} = w_{\mathcal{F}}^*w_{\mathcal{F}} \in \mathbf{M}_{n_{\mathcal{F}}}(\mathcal{M}_{\mathcal{F}^c})$ for all $\mathcal{F}$. It follows that $w_{\mathcal{F}}^*\mathcal{Q}w_{\mathcal{F}} \subset q_{\mathcal{F}}\mathbf{M}_{n_{\mathcal{F}}}(\mathcal{M}_{\mathcal{F}^c})q_{\mathcal{F}}$ for all $\mathcal{F}$.

Put $\widetilde{\mathcal{M}} := \mathcal{M} *_{\mathcal{B}} \mathcal{M}$, where we regard the left-hand copy of $\mathcal{M}$ as the original $\mathcal{M}$, and denote by $\Theta \in \mathrm{Aut}(\widetilde{\mathcal{M}})$ the free flip (trace preserving) automorphism. Likewise, for every $\mathcal{F}$, put $\widetilde{\mathcal{M}}_{\mathcal{F}} := \mathcal{M}_{\mathcal{F}} *_{\mathcal{B}} \mathcal{M}_{\mathcal{F}}$ and denote by $\Theta_{\mathcal{F}} \in \mathrm{Aut}(\widetilde{\mathcal{M}}_{\mathcal{F}})$ the free flip (trace preserving) automorphism. Regard $\Theta_{\mathcal{F}} \in \mathrm{Aut}(\widetilde{\mathcal{M}})$ by letting $\Theta_{\mathcal{F}}|_{\widetilde{\mathcal{M}}_{\mathcal{F}^c}} = \mathrm{id}_{\widetilde{\mathcal{M}}_{\mathcal{F}^c}}$ where $\widetilde{\mathcal{M}}_{\mathcal{F}^c} := \mathcal{M}_{\mathcal{F}^c} *_{\mathcal{B}} \mathcal{M}_{\mathcal{F}^c}$. We have $\lim_{\mathcal{F}} \Theta_{\mathcal{F}} = \Theta$ in $\mathrm{Aut}(\widetilde{\mathcal{M}})$. Observe that since $w_{\mathcal{F}}^*\mathcal{Q}w_{\mathcal{F}} \subset q_{\mathcal{F}}\mathbf{M}_{n_{\mathcal{F}}}(\mathcal{M}_{\mathcal{F}^c})q_{\mathcal{F}}$, we have $(\mathrm{id}_{n_{\mathcal{F}}} \otimes \Theta_{\mathcal{F}})(w_{\mathcal{F}}^*aw_{\mathcal{F}}) = w_{\mathcal{F}}^*aw_{\mathcal{F}}$ for all $a \in \mathcal{Q}$. Letting $\xi_{\mathcal{F}} := (\mathrm{id}_{n_{\mathcal{F}}} \otimes \Theta_{\mathcal{F}})(w_{\mathcal{F}})w_{\mathcal{F}}^*$, for all $a \in \mathcal{Q}$, we have

$$\begin{aligned}\Theta_{\mathcal{F}}(a)\,\xi_{\mathcal{F}} &= \Theta_{\mathcal{F}}(a)\,(\mathrm{id}_{n_{\mathcal{F}}}\otimes\Theta_{\mathcal{F}})(w_{\mathcal{F}})w_{\mathcal{F}}{}^* \\ &= (\mathrm{id}_{n_{\mathcal{F}}}\otimes\Theta_{\mathcal{F}})(aw_{\mathcal{F}})w_{\mathcal{F}}{}^* \\ &= (\mathrm{id}_{n_{\mathcal{F}}}\otimes\Theta_{\mathcal{F}})(w_{\mathcal{F}}\,w_{\mathcal{F}}{}^*aw_{\mathcal{F}})w_{\mathcal{F}}{}^* \\ &= (\mathrm{id}_{n_{\mathcal{F}}}\otimes\Theta_{\mathcal{F}})(w_{\mathcal{F}})\,(\mathrm{id}_{n_{\mathcal{F}}}\otimes\Theta_{\mathcal{F}})(w_{\mathcal{F}}{}^*aw_{\mathcal{F}})\,w_{\mathcal{F}}{}^* \\ &= (\mathrm{id}_{n_{\mathcal{F}}}\otimes\Theta_{\mathcal{F}})(w_{\mathcal{F}})\,w_{\mathcal{F}}{}^*aw_{\mathcal{F}}\,w_{\mathcal{F}}{}^* \\ &= (\mathrm{id}_{n_{\mathcal{F}}}\otimes\Theta_{\mathcal{F}})(w_{\mathcal{F}})w_{\mathcal{F}}{}^*\,a \\ &= \xi_{\mathcal{F}}\,a.\end{aligned}$$

Endow $\mathcal{H} := \mathrm{L}^2(\widetilde{\mathcal{M}})$ with the $\mathcal{M}$-$\mathcal{M}$-bimodule structure given by $x\cdot\eta\cdot y := \Theta(x)\,\eta\,y$ for all $x, y \in \mathcal{M}$ and all $\eta \in \mathrm{L}^2(\widetilde{\mathcal{M}})$. By construction and using [Ue98a, Section 2], there exists a $\mathcal{B}$-$\mathcal{B}$-bimodule $\mathcal{L}$ such that we have the following isomorphism

$$\mathcal{H} \cong \mathrm{L}^2(\mathcal{M}) \otimes_{\mathcal{B}} \mathcal{L} \otimes_{\mathcal{B}} \mathrm{L}^2(\mathcal{M})$$

as $\mathcal{M}$-$\mathcal{M}$-bimodules. (Indeed, for any amalgamated free product $(M, E) = (M_1, E_1) *_B (M_2, E_2)$ we have $\mathrm{L}^2(M) \cong \mathrm{L}^2(M_2) \otimes_B \mathcal{K} \otimes_B \mathrm{L}^2(M_1)$ as M_2-M_1-bimodule with $\mathcal{K} := \mathrm{L}^2(B) \oplus (\mathrm{L}^2(M_1^\circ) \otimes_B \mathrm{L}^2(M_2^\circ)) \oplus \cdots \oplus (\mathrm{L}^2(M_1^\circ) \otimes_B \cdots \otimes_B \mathrm{L}^2(M_2^\circ)) \oplus \cdots$.) Since $\mathcal{B}$ is amenable, [AD93, Lemma 1.7] shows that the $\mathcal{M}$-$\mathcal{M}$-bimodule $\mathcal{H}$ is weakly contained in the coarse $\mathcal{M}$-$\mathcal{M}$-bimodule $\mathrm{L}^2(\mathcal{M}) \otimes \mathrm{L}^2(\mathcal{M})$. This implies (see the proof of [CH08, Proposition 3.1]) that the $p\mathcal{M}p$-$p\mathcal{M}p$-bimodule $p\cdot\mathcal{H}\cdot p$ is weakly contained in the coarse $p\mathcal{M}p$-$p\mathcal{M}p$-bimodule $\mathrm{L}^2(p\mathcal{M}p) \otimes \mathrm{L}^2(p\mathcal{M}p)$.

Regard $\xi_{\mathcal{F}} \in \mathcal{H}$ and put $\eta_{\mathcal{F}} := p\cdot\xi_{\mathcal{F}}\cdot p \in p\cdot\mathcal{H}\cdot p$. First, we have

$$\begin{aligned}\|\eta_{\mathcal{F}} - \xi_{\mathcal{F}}\|_2 &= \|\Theta(p)\,\xi_{\mathcal{F}} - \xi_{\mathcal{F}}\|_2 && (\text{since } \eta_{\mathcal{F}} = \Theta(p)\,\xi_{\mathcal{F}}) \\ &\leq \|(\Theta(p) - \Theta_{\mathcal{F}}(p))\,\xi_{\mathcal{F}}\|_2 && (\text{since } \Theta_{\mathcal{F}}(p)\,\xi_{\mathcal{F}} = \xi_{\mathcal{F}}\,p = \xi_{\mathcal{F}}) \\ &\leq \|\Theta(p) - \Theta_{\mathcal{F}}(p)\|_2\,\|\xi_{\mathcal{F}}\|_\infty \\ &\leq \|\Theta(p) - \Theta_{\mathcal{F}}(p)\|_2 \to 0 \quad \text{as} \quad \mathcal{F}\to\infty.\end{aligned}$$

Then we have

$$\begin{aligned}\|\xi_{\mathcal{F}}\|_2^2 &= \mathrm{Tr}(w_{\mathcal{F}}\,(\mathrm{id}_{n_{\mathcal{F}}}\otimes\Theta_{\mathcal{F}})(w_{\mathcal{F}}{}^*w_{\mathcal{F}})\,w_{\mathcal{F}}{}^*) \\ &= \mathrm{Tr}_{n_{\mathcal{F}}}((\mathrm{id}_{n_{\mathcal{F}}}\otimes\Theta_{\mathcal{F}})(w_{\mathcal{F}}{}^*w_{\mathcal{F}})\,w_{\mathcal{F}}{}^*w_{\mathcal{F}}) \\ &= \mathrm{Tr}_{n_{\mathcal{F}}}(w_{\mathcal{F}}{}^*w_{\mathcal{F}}) \qquad (\text{since } w_{\mathcal{F}}{}^*w_{\mathcal{F}} \in \mathbf{M}_{n_{\mathcal{F}}}(\widetilde{\mathcal{M}}_{\mathcal{F}^c})) \\ &= \mathrm{Tr}(w_{\mathcal{F}}w_{\mathcal{F}}{}^*) \to \mathrm{Tr}(p) \quad \text{as} \quad \mathcal{F}\to\infty.\end{aligned}$$

Since $\lim_{\mathcal{F}}\|\eta_{\mathcal{F}} - \xi_{\mathcal{F}}\|_2 = 0$, this implies that $\lim_{\mathcal{F}}\|\eta_{\mathcal{F}}\|_2 = \|p\|_2$. For all $x \in p\mathcal{M}p$ and all $\mathcal{F}$, we have

$$\|x\cdot\eta_{\mathcal{F}}\|_2 = \|\Theta(x)\,\eta_{\mathcal{F}}\|_2 \leq \|\Theta(x)\|_2\,\|\eta_{\mathcal{F}}\|_\infty \leq \|x\|_2.$$

For every $a \in \mathcal{Q}$, we have

$$\begin{aligned}\|a\cdot\xi_{\mathcal{F}} - \xi_{\mathcal{F}}\cdot a\|_2 &= \|\Theta(a)\,\xi_{\mathcal{F}} - \xi_{\mathcal{F}}\,a\|_2 \\ &\leq \|(\Theta(a) - \Theta_{\mathcal{F}}(a))\,\xi_{\mathcal{F}}\|_2 \qquad (\text{since } \Theta_{\mathcal{F}}(a)\,\xi_{\mathcal{F}} = \xi_{\mathcal{F}}\,a) \\ &\leq \|\Theta(a) - \Theta_{\mathcal{F}}(a)\|_2\,\|\xi_{\mathcal{F}}\|_\infty \\ &\leq \|\Theta(a) - \Theta_{\mathcal{F}}(a)\|_2,\end{aligned}$$

and hence $\lim_{\mathcal{F}}\|a\cdot\xi_{\mathcal{F}} - \xi_{\mathcal{F}}\cdot a\|_2 = 0$. Since $\lim_{\mathcal{F}}\|\eta_{\mathcal{F}} - \xi_{\mathcal{F}}\|_2 = 0$, this implies that $\lim_{\mathcal{F}}\|a\cdot\eta_{\mathcal{F}} - \eta_{\mathcal{F}}\cdot a\|_2 = 0$ for all $a \in \mathcal{Q}$. By Connes's characterization of amenability [Co75] applied to the finite von Neumann algebra $\mathcal{Q}$ and the net $(\eta_{\mathcal{F}})_{\mathcal{F}}$ in $p\cdot\mathcal{H}\cdot p$ (see also [Io12, Lemma 2.3]), it follows that $\mathcal{Q}$ has an amenable direct summand, a contradiction. □

Relative commutants inside AFP von Neumann algebras. We begin by studying relative commutants inside *semifinite* amalgamated free product von Neumann algebras.

Theorem 4.3. *Keep Notation 4.1. Let $p \in \mathcal{M}$ be any nonzero finite trace projection and $\mathcal{Q} \subset p\mathcal{M}p$ any von Neumann subalgebra with no amenable direct summand and such that $\mathcal{Q}' \cap p\mathcal{M}p \npreceq_{\mathcal{M}} \mathcal{B}$. Then there exists $i \in I$ such that $\mathcal{Q} \preceq_{\mathcal{M}} \mathcal{M}_i$.*

Proof. Since $\mathcal{Q}' \cap p\mathcal{M}p \npreceq_{\mathcal{M}} \mathcal{B}$, we have $(\mathcal{Q}' \cap p\mathcal{M}p)^\omega \npreceq_{\mathcal{M}^\omega} \mathcal{B}^\omega$ by Proposition 2.9. Since $(\mathcal{Q}' \cap p\mathcal{M}p)^\omega \subset \mathcal{Q}' \cap (p\mathcal{M}p)^\omega$, we also have $\mathcal{Q}' \cap (p\mathcal{M}p)^\omega \npreceq_{\mathcal{M}^\omega} \mathcal{B}^\omega$. For each nonempty finite subset $\mathcal{F} \subset I$, regard $\mathcal{M} = \mathcal{M}_{\mathcal{F}} *_{\mathcal{B}} \mathcal{M}_{\mathcal{F}^c}$. By [HU15, Corollary 4.2], for every nonzero projection $z \in \mathcal{Q}' \cap p\mathcal{M}p$, we have $\mathcal{Q}z \preceq_{\mathcal{M}} \mathcal{M}_{\mathcal{F}}$ or $\mathcal{Q}z \preceq_{\mathcal{M}} \mathcal{M}_{\mathcal{F}^c}$. Since $\mathcal{Q}$ has no amenable direct summand, there exists a nonzero projection $z \in \mathcal{Q}' \cap p\mathcal{M}p$ and a nonempty finite subset $\mathcal{F}$ such that $\mathcal{Q}z \preceq_{\mathcal{M}} \mathcal{M}_{\mathcal{F}}$ by Proposition 4.2. Therefore, we have $\mathcal{Q} \preceq_{\mathcal{M}} \mathcal{M}_{\mathcal{F}}$. Put $\mathcal{P} := \mathcal{Q} \vee (\mathcal{Q}' \cap p\mathcal{M}p)$. Since $\mathcal{Q} \npreceq_{\mathcal{M}} \mathcal{B}$, we also have that $\mathcal{P} \preceq_{\mathcal{M}} \mathcal{M}_{\mathcal{F}}$ by [BHR12, Proposition 2.7].

Then there exist $n \geq 1$, a finite trace projection $q \in \mathbf{M}_n(\mathcal{M}_{\mathcal{F}})$, a nonzero partial isometry $w \in \mathbf{M}_{1,n}(p\mathcal{M})q$ and a unital normal $*$-homomorphism $\pi : \mathcal{P} \to q\mathbf{M}_n(\mathcal{M}_{\mathcal{F}})q$ such that $aw = w\pi(a)$ for all $a \in \mathcal{P}$. Observe that $ww^* \in \mathcal{P}' \cap p\mathcal{M}p = \mathcal{Z}(\mathcal{P})$ and $w^*w \in \pi(\mathcal{P})' \cap q\mathbf{M}_n(\mathcal{M})q$. Since $\mathcal{P}$ has no amenable direct summand, we have $\pi(\mathcal{P}) \npreceq_{\mathbf{M}_n(\mathcal{M}_{\mathcal{F}})} \mathbf{M}_n(\mathcal{B})$. This implies that $w^*w \in q\mathbf{M}_n(\mathcal{M}_{\mathcal{F}})q$ by [BHR12, Theorem 2.5] and hence we may assume that $q = w^*w$. We obtain $w^*\mathcal{P}w \subset q\mathbf{M}_n(\mathcal{M}_{\mathcal{F}})q$. Observe that $w^*\mathcal{Q}w$ and $w^*(\mathcal{Q}' \cap p\mathcal{M}p)w$ are commuting unital subalgebras of $w^*\mathcal{P}w$ such that $w^*\mathcal{Q}w$ has no amenable direct summand and $w^*(\mathcal{Q}' \cap p\mathcal{M}p)w \npreceq_{\mathbf{M}_n(\mathcal{M}_{\mathcal{F}})} \mathbf{M}_n(\mathcal{B})$ by Remark 2.5 (2) (recall that $\mathcal{Q}' \cap p\mathcal{M}p \npreceq_{\mathcal{M}} \mathcal{B}$). Observe that for each $i \in \mathcal{F}$, $w^*\mathcal{Q}w \preceq_{\mathbf{M}_n(\mathcal{M}_{\mathcal{F}})} \mathbf{M}_n(\mathcal{M}_i)$ leads to $\mathcal{Q} \preceq_{\mathcal{M}} \mathcal{M}_i$ by Remark 2.5 (2). Therefore, we have showed that in order to prove Theorem 4.3, we may assume that the index set I is finite.

When the index set I is finite, a straightforward induction procedure over $k := |I|$ using a combination of the above reasoning with [HU15, Corollary 4.2] proves the result. Indeed, assume that the result is true for any set I such that $|I| = k$ with $k \geq 1$. Next, let I be any set such that $|I| = k + 1$. Simply denote $I = \{1, \dots, k+1\}$. Regard $\mathcal{M} = \mathcal{M}_{\mathcal{F}} *_{\mathcal{B}} \mathcal{M}_{k+1}$ where $\mathcal{F} = \{1, \dots, k\}$. The same reasoning as in the first paragraph above shows that $\mathcal{Q} \preceq_{\mathcal{M}} \mathcal{M}_{\mathcal{F}}$ or $\mathcal{Q} \preceq_{\mathcal{M}} \mathcal{M}_{k+1}$. If $\mathcal{Q} \preceq_{\mathcal{M}} \mathcal{M}_{k+1}$, we are done. If $\mathcal{Q} \preceq_{\mathcal{M}} \mathcal{M}_{\mathcal{F}}$, the same reasoning as in the second paragraph above shows that with letting $\mathcal{P} := \mathcal{Q} \vee (Q' \cap p\mathcal{M}p)$ there exists $n \geq 1$, a finite trace projection $q \in \mathbf{M}_n(\mathcal{M}_{\mathcal{F}})$, a nonzero partial isometry $w \in \mathbf{M}_{1,n}(p\mathcal{M})q$ and a unital normal $*$-homomorphism $\pi : \mathcal{P} \to q\mathbf{M}_n(\mathcal{M}_{\mathcal{F}})q$ such that $aw = w\pi(a)$ for all $a \in \mathcal{P}$. We may moreover assume that $w^*w = q$. Then $w^*\mathcal{Q}w$ and $w^*(\mathcal{Q}' \cap p\mathcal{M}p)w$ are commuting unital subalgebras of $w^*\mathcal{P}w$ such that $w^*\mathcal{Q}w$ has no amenable direct summand and $w^*(\mathcal{Q}' \cap p\mathcal{M}p)w \npreceq_{\mathbf{M}_n(\mathcal{M}_{\mathcal{F}})} \mathbf{M}_n(\mathcal{B})$. Using the induction hypothesis, there exists $i \in \mathcal{F} = \{1, \dots, k\}$ such that $w^*\mathcal{Q}w \preceq_{\mathbf{M}_n(\mathcal{M}_{\mathcal{F}})} \mathbf{M}_n(\mathcal{M}_i)$. Then $\mathcal{Q} \preceq_{\mathcal{M}} \mathcal{M}_i$ holds by Remark 2.5 (2). This finishes the proof of the induction procedure and completes the proof of Theorem 4.3. $\square$

We now prove a general result locating finite subalgebras with expectation and with nonamenable relative commutant inside *arbitrary* amalgamated free product von Neumann algebras. This result will be used in the proof of the Main Theorem (Cases (i) and (ii)).

Theorem 4.4. *Let I be any nonempty set and $(B \subset M_i)_{i \in I}$ any family of inclusions of σ-finite von Neumann algebras with faithful normal conditional expectations $\mathrm{E}_i : M_i \to B$. Assume moreover that B is amenable. Denote by $(M, \mathrm{E}) = *_{B, i \in I}(M_i, \mathrm{E}_i)$ the corresponding amalgamated free product.*

Let $1_A \in M$ be any nonzero projection and $A \subset 1_A M 1_A$ any finite von Neumann subalgebra with expectation. Then at least one of the following conditions holds true:

- *There exists $i \in I$ such that $A \preceq_M M_i$.*
- *The von Neumann subalgebra $A' \cap 1_A M 1_A$ is amenable.*

Proof. Put $\widetilde{A} = A \oplus \mathbf{C}(1_M - 1_A)$ and denote by $\mathrm{E}_{\widetilde{A}} : M \to \widetilde{A}$ a faithful normal conditional expectation. Choose a faithful trace $\tau_{\widetilde{A}} \in \widetilde{A}_*$ and put $\psi = \tau_{\widetilde{A}} \circ \mathrm{E}_{\widetilde{A}}$. Observe that $1_A \in M^\psi$ and the von Neumann subalgebras A and $A' \cap 1_A M 1_A$ are globally invariant under the modular automorphism group σ^{ψ_A} of $\psi_A := \frac{\psi(1_A \cdot 1_A)}{\psi(1_A)}$.

Assume that $A' \cap 1_A M 1_A$ is not amenable. Observe that if $A \preceq_M B$, we are done. Hence we may further assume that $A \npreceq_M B$. Choose a nonzero central projection $z \in \mathcal{Z}(A' \cap 1_A M 1_A)$ such that $(A' \cap 1_A M 1_A)z$ has no amenable direct summand. Observe that $z \in M^\psi$ and the von Neumann subalgebras Az and $(A' \cap 1_A M 1_A)z$ are globally invariant under the modular automorphism group σ^{ψ_z} of $\psi_z := \frac{\psi(z \cdot z)}{\psi(z)}$. Then $\mathrm{c}_{\psi_z}((A' \cap 1_A M 1_A)z)$ has no amenable direct summand by [BHR12, Proposition 2.8].

Fix a faithful state $\varphi \in B_*$ and put $\mathcal{B} := \mathrm{c}_\varphi(B)$, $\mathcal{M} := \mathrm{c}_{\varphi\circ\mathrm{E}}(M)$ and $\mathcal{M}_i := \mathrm{c}_{\varphi\circ\mathrm{E}_i}(M_i)$ for every $i \in I$. Let $q \in \mathrm{L}_\psi(\mathbf{R})$ be any nonzero finite trace projection and put $p := \Pi_{\varphi,\psi}(q)$ and $\mathcal{Q} := \Pi_{\varphi,\psi}(q\mathrm{c}_{\psi_z}((A' \cap 1_A M 1_A)z)q)$. Then $\mathcal{Q} \subset p\mathcal{M}p$ has no amenable direct summand. Since $A \npreceq_M B$, we have $Az \npreceq_M B$ by [HI15, Remark 4.2 (2)]. By [HU15, Lemma 2.4] we obtain $\Pi_{\varphi,\psi}(\pi_\psi(Az)q) \npreceq_{\mathcal{M}} \mathcal{B}$. Since $\Pi_{\varphi,\psi}(\pi_\psi(Az)q) \subset \mathcal{Q}' \cap p\mathcal{M}p$, we conclude that $\mathcal{Q}' \cap p\mathcal{M}p \npreceq_{\mathcal{M}} \mathcal{B}$.

By Theorem 4.3, there exists $i \in I$ such that $\mathcal{Q} \preceq_{\mathcal{M}} \mathcal{M}_i$. Since $\mathcal{Q} \npreceq_{\mathcal{M}} \mathcal{B}$ (recall that $c_{\psi_z}((A' \cap 1_A M 1_A)z)$ has no amenable direct summand), we also have $\mathcal{Q}' \cap p\mathcal{M}p \preceq_{\mathcal{M}} \mathcal{M}_i$ by [BHR12, Proposition 2.7] and hence $\Pi_{\varphi,\psi}(\pi_\psi(Az)q) \preceq_{\mathcal{M}} \mathcal{M}_i$. By [HU15, Lemma 2.4], this implies that $Az \preceq_M M_i$ and hence $A \preceq_M M_i$ by [HI15, Remark 4.2 (2)]. □

Normalizers inside AFP von Neumann algebras. We begin by studying normalizers inside *semifinite* amalgamated free product von Neumann algebras. For a technical reason, we only deal with amalgamated free products of type II_∞ factors. This result will be sufficient for our purposes.

Theorem 4.5. *Keep Notation 4.1. Assume moreover that $\mathcal{B}$ is a diffuse subalgebra and $\mathcal{M}_\mathcal{G}$ is a type II_∞ factor for every nonempty subset $\mathcal{G} \subset I$. Let $p \in \mathcal{M}$ be any nonzero finite trace projection and $\mathcal{A} \subset p\mathcal{M}p$ any amenable von Neumann subalgebra such that $\mathcal{A} \npreceq_{\mathcal{M}} \mathcal{B}$. Put $\mathcal{Q} := \mathcal{N}_{p\mathcal{M}p}(\mathcal{A})''$ and assume that $\mathcal{Q}$ has no amenable direct summand. Then there exists $i \in I$ such that $\mathcal{Q} \preceq_{\mathcal{M}} \mathcal{M}_i$.*

Proof. For each nonempty finite subset $\mathcal{F} \subset I$, regard $\mathcal{M} = \mathcal{M}_\mathcal{F} *_\mathcal{B} \mathcal{M}_{\mathcal{F}^c}$. Let $z \in \mathcal{Q}' \cap p\mathcal{M}p$ be any nonzero projection. Since $\mathcal{M}$ is a type II_∞ factor and since $\mathcal{B} \subset \mathcal{M}$ is a diffuse subalgebra with trace preserving conditional expectation, there exists $u \in \mathcal{U}(\mathcal{M})$ such that $uzu^* \in \mathcal{B}$. Since the unital inclusion $u\mathcal{A}zu^* \subset u\mathcal{Q}zu^*$ is regular and since $u\mathcal{A}zu^* \npreceq_{\mathcal{M}} \mathcal{B}$ by assumption (and Remark 2.5 (2)), we have $u\mathcal{Q}zu^* \preceq_{\mathcal{M}} \mathcal{M}_\mathcal{F}$ or $u\mathcal{Q}zu^* \preceq_{\mathcal{M}} \mathcal{M}_{\mathcal{F}^c}$ by Theorem A.4 (together with the comment following it) and [BHR12, Proposition 2.7]. Accordingly, we have $\mathcal{Q}z \preceq_{\mathcal{M}} \mathcal{M}_\mathcal{F}$ or $\mathcal{Q}z \preceq_{\mathcal{M}} \mathcal{M}_{\mathcal{F}^c}$ (by Remark 2.5 (2)). Since $\mathcal{Q}$ has no amenable direct summand, Proposition 4.2 ensures that there exist a nonzero projection $z \in \mathcal{Q}' \cap p\mathcal{M}p$ and a nonempty finite subset $\mathcal{F}$ such that $\mathcal{Q}z \preceq_{\mathcal{M}} \mathcal{M}_\mathcal{F}$. Therefore, we have that $\mathcal{Q} \preceq_{\mathcal{M}} \mathcal{M}_\mathcal{F}$.

Then there exist $n \geq 1$, a finite trace projection $q \in \mathbf{M}_n(\mathcal{M}_\mathcal{F})$, a nonzero partial isometry $w \in \mathbf{M}_{1,n}(p\mathcal{M})q$ and a unital normal $*$-homomorphism $\pi : \mathcal{Q} \to q\mathbf{M}_n(\mathcal{M}_\mathcal{F})q$ such that $aw = w\pi(a)$ for all $a \in \mathcal{Q}$. Observe that $ww^* \in \mathcal{Q}' \cap p\mathcal{M}p$ and $w^*w \in \pi(\mathcal{Q})' \cap q\mathbf{M}_n(\mathcal{M})q$. Since $\mathcal{Q}$ has no amenable direct summand, we have $\pi(\mathcal{Q}) \npreceq_{\mathbf{M}_n(\mathcal{M}_\mathcal{F})} \mathbf{M}_n(\mathcal{B})$. Then [BHR12, Theorem 2.5] implies that $w^*w \in q\mathbf{M}_n(\mathcal{M}_\mathcal{F})q$ and hence we may assume that $q = w^*w$. We obtain $w^*\mathcal{Q}w \subset q\mathbf{M}_n(\mathcal{M}_\mathcal{F})q$.

Since $ww^* \in \mathcal{Q}' \cap p\mathcal{M}p$, it follows that the unital inclusion $w^*\mathcal{A}w \subset w^*\mathcal{Q}w$ is regular, $w^*\mathcal{Q}w$ has no amenable direct summand, and $w^*\mathcal{A}w \npreceq_{\mathbf{M}_n(\mathcal{M}_\mathcal{F})} \mathbf{M}_n(\mathcal{B})$ (by Remark 2.5 (2) and $\mathcal{A} \npreceq_{\mathcal{M}} \mathcal{B}$). By Remark 2.5 (2), $w^*\mathcal{Q}w \preceq_{\mathbf{M}_n(\mathcal{M}_\mathcal{F})} \mathbf{M}_n(\mathcal{M}_i)$ implies that $\mathcal{Q} \preceq_{\mathcal{M}} \mathcal{M}_i$ for every $i \in \mathcal{F}$.

Therefore, since $\mathbf{M}_n(\mathcal{M}_\mathcal{G})$ is a type II_∞ factor for every nonempty subset $\mathcal{G} \subset \mathcal{F}$ and since $\mathbf{M}_n(\mathcal{B})$ is diffuse, we have showed that in order to prove Theorem 4.3, we may assume that the index set I is finite.

When the index set I is finite, a straightforward induction procedure over $k := |I|$ using a combination of the above reasoning with the assumptions that $\mathcal{M}_\mathcal{G}$ is a type II_∞ factor for every nonempty subset $\mathcal{G} \subset I$ and $\mathcal{B}$ is diffuse and with Theorem A.4 proves the result (see the last paragraph of the proof of Theorem 4.3). ☐

We now prove a result locating certain finite subalgebras with expectation and with nonamenable normalizer inside *arbitrary* free products of σ-finite factors. This result will be used in the proof of the Main Theorem (Case (iii)).

Theorem 4.6. *Let I be any nonempty set and $(M_i, \varphi_i)_{i\in I}$ any family of σ-finite factors endowed with any faithful normal states. Denote by $(M, \varphi) = *_{i\in I}(M_i, \varphi_i)$ the corresponding free product.*

Let $1_A \in M$ be any nonzero projection and $A \subset 1_A M 1_A$ be any amenable finite von Neumann subalgebra with expectation such that $A' \cap 1_A M 1_A = \mathcal{Z}(A)$. Then at least one of the following conditions holds true:

- *There exists $i \in I$ such that $A \preceq_M M_i$.*
- *The von Neumann subalgebra $\mathcal{N}_{1_A M 1_A}(A)''$ is amenable.*

Proof. Denote by R_∞ the unique type III_1 AFD factor. Put $\widetilde{B} = \mathbf{C}1_M \mathbin{\overline{\otimes}} R_\infty$, $\widetilde{M} = M \mathbin{\overline{\otimes}} R_\infty$ and $\widetilde{\mathrm{E}} = \varphi \otimes \mathrm{id}_{R_\infty}$, $\widetilde{M_i} = M_i \mathbin{\overline{\otimes}} R_\infty$ and $\widetilde{\mathrm{E}}_i = \varphi_i \otimes \mathrm{id}_{R_\infty}$ for every $i \in I$. We may and will naturally regard the pair $(\widetilde{M}, \widetilde{\mathrm{E}})$ as

$$(\widetilde{M}, \widetilde{\mathrm{E}}) = *_{\widetilde{B}, i\in I}(\widetilde{M}_i, \widetilde{\mathrm{E}}_i).$$

Observe that $\widetilde{M_i} = M_i \mathbin{\overline{\otimes}} R_\infty$ is a type III_1 factor for every $i \in I$. (This is well-known without explicit reference and can be confirmed by computing the (smooth) flow of weights; see [CT76, Corollary 6.8].) For every nonempty subset $\mathcal{G} \subset I$, $M_\mathcal{G}$ is a factor by [Ue10, Theorem 4.1], and hence $\widetilde{M}_\mathcal{G} = M_\mathcal{G} \mathbin{\overline{\otimes}} R_\infty$ is a type III_1 factor by the same reasoning as above.

Fix an irreducible type II_1 subfactor $R \subset R_\infty$ with expectation (whose existence is explained in e.g. [Ha85, Example 1.6]). Put $\widetilde{A} = (A \oplus \mathbf{C}(1_M - 1_A)) \mathbin{\overline{\otimes}} R$ and denote by $\mathrm{E}_{\widetilde{A}} : \widetilde{M} \to \widetilde{A}$ a faithful normal conditional expectation. Choose a faithful trace $\tau_{\widetilde{A}} \in \widetilde{A}_*$ and put $\psi = \tau_{\widetilde{A}} \circ \mathrm{E}_{\widetilde{A}}$. We will simply denote $D := (1_A \otimes 1_R)\widetilde{A}(1_A \otimes 1_R)$ and $1_D := 1_A \otimes 1_R$. Observe that $D' \cap 1_D \widetilde{M} 1_D = \mathcal{Z}(D)$, the unital inclusion $(\mathcal{N}_{1_A M 1_A}(A)'' \oplus \mathbf{C}(1_M - 1_A)) \mathbin{\overline{\otimes}} \mathbf{C}1_R \subset \widetilde{M}$ is with expectation and also so is

$$\mathcal{N}_{1_A M 1_A}(A)'' \mathbin{\overline{\otimes}} \mathbf{C}1_R = 1_D((\mathcal{N}_{1_A M 1_A}(A)'' \oplus \mathbf{C}(1_M - 1_A)) \mathbin{\overline{\otimes}} \mathbf{C}1_R)1_D \subset \mathcal{N}_{1_D \widetilde{M} 1_D}(D)''.$$

Moreover, we have $1_D \in \widetilde{M}^\psi$ and the von Neumann subalgebras D and $\mathcal{N}_{1_D \widetilde{M} 1_D}(D)''$ are globally invariant under the modular automorphism group σ^{ψ_D} of $\psi_D := \frac{\psi(1_D \cdot 1_D)}{\psi(1_D)}$.

Observe that we have

$$\mathrm{c}_{\psi_D}(\mathcal{N}_{1_D \widetilde{M} 1_D}(D)'') \subset \mathcal{N}_{\mathrm{c}_{\psi_D}(1_D \widetilde{M} 1_D)}(\mathrm{c}_{\psi_D}(D))''.$$

Indeed, let $u \in \mathcal{N}_{1_D \widetilde{M} 1_D}(D)$ and $t \in \mathbf{R}$. For every $a \in D$, we have

$$u\sigma_t^{\psi_D}(u^*)\, a = u\, \sigma_t^{\psi_D}(u^* a u)\, \sigma_t^{\psi_D}(u)^* = u\, u^* a u\, \sigma_t^{\psi_D}(u)^* = a\, u \sigma_t^{\psi_D}(u)^*.$$

This shows that $u\sigma_t^{\psi_D}(u)^* \in D' \cap 1_D \widetilde{M} 1_D = \mathcal{Z}(D)$ and hence we have $\pi_{\psi_D}(u)\lambda_{\psi_D}(t)\pi_{\psi_D}(u)^* \in \pi_{\psi_D}(\mathcal{Z}(D))\lambda_{\psi_D}(t)$. Therefore we obtain that $\mathrm{c}_{\psi_D}(\mathcal{N}_{1_D \widetilde{M} 1_D}(D)'') \subset \mathcal{N}_{\mathrm{c}_{\psi_D}(1_D \widetilde{M} 1_D)}(\mathrm{c}_{\psi_D}(D))''$ and this inclusion is with trace preserving conditional expectation.

Assume that $\mathcal{N}_{1_A M 1_A}(A)''$ is not amenable. Observe that if $A \preceq_M B$, we are done. Hence we may further assume that $A \not\preceq_M B$. Then $\mathcal{N}_{1_D \widetilde{M} 1_D}(D)''$ is not amenable (since it contains $\mathcal{N}_{1_A M 1_A}(A)'' \mathbin{\overline{\otimes}} \mathbf{C}1_R$ with expectation) and $D \not\preceq_{\widetilde{M}} \widetilde{B}$ by [HI15, Lemma 4.6]. Choose a nonzero projection $z \in \mathcal{Z}(\mathcal{N}_{1_D \widetilde{M} 1_D}(D)'')$ such that $\mathcal{N}_{1_D \widetilde{M} 1_D}(D)'' z$ has no amenable direct summand. Observe that $z \in \widetilde{M}^\psi$, $Dz \subset z\widetilde{M}^\psi z$ and $\mathcal{N}_{z\widetilde{M}z}(Dz)'' = \mathcal{N}_{1_D \widetilde{M} 1_D}(D)'' z$ is globally invariant under the modular automorphism group σ^{ψ_z} of $\psi_z = \frac{\psi(z \cdot z)}{\psi(z)}$. Then $\mathrm{c}_{\psi_z}(\mathcal{N}_{z\widetilde{M}z}(Dz)'')$ has no amenable direct summand by [BHR12, Proposition 2.8].

Fix a faithful state $\chi \in (R_\infty)_*$ and put $\mathcal{B} := \mathrm{c}_\chi(\widetilde{B})$, $\mathcal{M} := \mathrm{c}_{\varphi \otimes \chi}(\widetilde{M})$ and $\mathcal{M}_i := \mathrm{c}_{\varphi_i \otimes \chi}(\widetilde{M_i})$ for every $i \in I$. Observe that $\mathcal{B} \subset \mathcal{M}$ is a diffuse subalgebra with trace preserving conditional expectation and $\mathcal{M}_\mathcal{G}$ is a type II_∞ factor for every nonempty subset $\mathcal{G} \subset I$. Let $q \in \mathrm{L}_\psi(\mathbf{R})$ be any nonzero finite trace projection. Put $p := \Pi_{\varphi\otimes\chi,\psi}(q) \in \mathcal{M}$, $\mathcal{A} := \Pi_{\varphi\otimes\chi,\psi}(\mathrm{c}_{\psi_z}(Dz)q)$ and $\mathcal{Q} := \Pi_{\varphi\otimes\chi,\psi}(q\mathrm{c}_{\psi_z}(\mathcal{N}_{z\widetilde{M}z}(Dz)'')q)$. Since $Dz \subset z\widetilde{M}^\psi z$ and $\mathrm{L}(\mathbf{R})$ is a MASA in $\mathbf{B}(\mathrm{L}^2(\mathbf{R}))$, we have $\mathrm{c}_{\psi_z}(Dz)' \cap \mathrm{c}_{\psi_z}(z\widetilde{M}z) = \mathcal{Z}(\mathrm{c}_{\psi_z}(Dz))$ and hence

$$q\big(\mathcal{N}_{\mathrm{c}_{\psi_z}(z\widetilde{M}z)}(\mathrm{c}_{\psi_z}(Dz))''\big)q = \mathcal{N}_{q\mathrm{c}_{\psi_z}(z\widetilde{M}z)q}(\mathrm{c}_{\psi_z}(Dz)q)''$$

by Proposition 2.3. Then we have $\mathcal{Q} = \mathcal{N}_{p\mathcal{M}p}(\mathcal{A})''$, $\mathcal{Q}$ has no amenable direct summand, and $\mathcal{A} \not\preceq_\mathcal{M} \mathcal{B}$ by [HU15, Lemma 2.4] since $Dz \not\preceq_{\widetilde{M}} \widetilde{B}$. By Theorem 4.5, there exists $i \in I$ such that $\mathcal{A} \preceq_\mathcal{M} \mathcal{M}_i$. Then [HU15, Lemma 2.4] shows that $Dz \preceq_{\widetilde{M}} \widetilde{M}_i$ and hence $D \preceq_{\widetilde{M}} \widetilde{M}_i$. Finally, [HI15, Lemma 4.6] guarantees $A \preceq_M M_i$. □

We point out that when dealing with *tracial* free product von Neumann algebras, Theorem 4.6 holds true for any family $(M_i, \tau_i)_{i\in I}$ of tracial von Neumann algebras and any amenable von Neumann subalgebra $A \subset 1_A M 1_A$.

Relative property (T) subalgebras inside AFP von Neumann algebras. Recall from [Po01, Definition 4.2.1] that an inclusion of tracial von Neumann algebras $A \subset (N, \tau)$ is said to have *relative property (T)* if for every net $(\Phi_i : N \to N)_{i\in I}$ of subtracial subunital completely positive maps such that $\lim_i \|\Phi_i(x) - x\|_2 = 0$ for all $x \in N$, we have

$$\lim_i \sup_{y \in \mathrm{Ball}(A)} \|\Phi_i(y) - y\|_2 = 0.$$

We begin by locating relative property (T) subalgebras inside *semifinite* amalgamated free product von Neumann algebras. This is a semifinite analogue of [IPP05, Theorem 4.3].

Theorem 4.7. *Keep Notation 4.1. Let $p \in \mathcal{M}$ be any nonzero finite trace projection and $\mathcal{A} \subset p\mathcal{M}p$ any von Neumann subalgebra with relative property (T). Then there exists $i \in I$ such that $\mathcal{A} \preceq_\mathcal{M} \mathcal{M}_i$.*

Proof. For each nonempty finite subset $\mathcal{F} \subset I$, regard $\mathcal{M} = \mathcal{M}_\mathcal{F} *_\mathcal{B} \mathcal{M}_{\mathcal{F}^c}$ and denote by $\mathrm{E}_{\mathcal{M}_\mathcal{F}} : \mathcal{M} \to \mathcal{M}_\mathcal{F}$ the unique trace preserving conditional expectation. Define the net $\Phi_\mathcal{F} : p\mathcal{M}p \to p\mathcal{M}p$ of subtracial subunital unital completely positive maps by $\Phi_\mathcal{F}(x) = p\mathrm{E}_{\mathcal{M}_\mathcal{F}}(x)p$ for all $x \in p\mathcal{M}p$. Observe that $\lim_\mathcal{F} \|\Phi_\mathcal{F}(x) - x\|_2 = 0$ for all $x \in p\mathcal{M}p$.

By relative property (T) of the inclusion $\mathcal{A} \subset p\mathcal{M}p$, there exists a nonempty finite subset $\mathcal{F} \subset I$ such that

$$\|\mathrm{E}_{\mathcal{M}_\mathcal{F}}(u)\|_2 \geq \|p\mathrm{E}_{\mathcal{M}_\mathcal{F}}(u)p\|_2 = \|\Phi_\mathcal{F}(u)\|_2 \geq \frac{1}{2}\|p\|_2, \forall u \in \mathcal{U}(\mathcal{A}).$$

If $\mathcal{A} \not\preceq_\mathcal{M} \mathcal{M}_\mathcal{F}$, then by Theorem 2.4, there exists a net $(u_j)_{j\in J}$ in $\mathcal{U}(\mathcal{A})$ with $\lim_j \|\mathrm{E}_{\mathcal{M}_\mathcal{F}}(u_j)\|_2 = 0$; contradicting the above inequality. Hence we obtain $\mathcal{A} \preceq_\mathcal{M} \mathcal{M}_\mathcal{F}$.

If $\mathcal{A} \preceq_\mathcal{M} \mathcal{B}$, then $\mathcal{A} \preceq_\mathcal{M} \mathcal{M}_i$ for any $i \in I$ and we are done. If $\mathcal{A} \not\preceq_\mathcal{M} \mathcal{B}$, then Theorem 2.4 and Lemma 2.6 altogether enable us to choose $n \geq 1$, a finite trace projection $q \in \mathbf{M}_n(\mathcal{M}_\mathcal{F})$, a nonzero partial isometry $w \in \mathbf{M}_{1,n}(p\mathcal{M})q$ and a unital normal $*$-homomorphism $\pi : \mathcal{A} \to q\mathbf{M}_n(\mathcal{M}_\mathcal{F})q$ such that $\pi(\mathcal{A}) \not\preceq_{\mathbf{M}_n(\mathcal{M}_\mathcal{F})} \mathbf{M}_n(\mathcal{B})$ and $aw = w\pi(a)$ for all $a \in \mathcal{A}$. By [BHR12, Theorem 2.5], we have $\pi(\mathcal{A})' \cap q\mathbf{M}_n(\mathcal{M})q = \pi(\mathcal{A})' \cap q\mathbf{M}_n(\mathcal{M}_\mathcal{F})q$. Then we have $w^*w \in \pi(\mathcal{A})' \cap q\mathbf{M}_n(\mathcal{M}_\mathcal{F})q$ and so we may assume that $q = w^*w$ and $w^*\mathcal{A}w = \pi(\mathcal{A}) \subset q\mathbf{M}_n(\mathcal{M}_\mathcal{F})q$.

By relative property (T) of the inclusion $\mathcal{A} \subset p\mathcal{M}p$ and [Po01, Proposition 4.7], the unital inclusion $w^*\mathcal{A}w \subset q\mathbf{M}_n(\mathcal{M})q$ has relative property (T). Consider

$$\mathbf{M}_n(\mathcal{M}) = \left(*_{\mathbf{M}_n(\mathcal{B}), i \in \mathcal{F}} \mathbf{M}_n(\mathcal{M}_i)\right) *_{\mathbf{M}_n(\mathcal{B})} \mathbf{M}_n(\mathcal{M}_{\mathcal{F}^c}).$$

Since $\mathcal{A} \not\preceq_\mathcal{M} \mathcal{B}$, we have $w^*\mathcal{A}w \not\preceq_{\mathbf{M}_n(\mathcal{M})} \mathbf{M}_n(\mathcal{B})$ by Remark 2.5 (2). Since moreover $w^*\mathcal{A}w \subset q\mathbf{M}_n(\mathcal{M}_\mathcal{F})q$, we have $w^*\mathcal{A}w \not\preceq_{\mathbf{M}_n(\mathcal{M})} \mathbf{M}_n(\mathcal{M}_{\mathcal{F}^c})$ by Theorem 2.4 with the help of Lemma 2.7. Since the unital inclusion $w^*\mathcal{A}w \subset q\mathbf{M}_n(\mathcal{M})q$ moreover has relative property (T), [BHR12, Theorem 3.3] (whose proof works well for semifinite amalgamated free products of finitely many algebras) shows that there exists $i \in \mathcal{F}$ such that $w^*\mathcal{A}w \preceq_{\mathbf{M}_n(\mathcal{M})} \mathbf{M}_n(\mathcal{M}_i)$. By Remark 2.5 (2), this implies that $\mathcal{A} \preceq_\mathcal{M} \mathcal{M}_i$. ◻

We point out that we do not need to assume $\mathcal{B}$ to be amenable in Theorem 4.7. We finally deduce the following result that will be used in the proof of the Main Theorem (Case (iv)).

Theorem 4.8. *Let I be any nonempty set and $(B \subset M_i)_{i \in I}$ any family of inclusions of σ-finite von Neumann algebras with faithful normal conditional expectations $\mathrm{E}_i : M_i \to B$. Denote by $(M, \mathrm{E}) = *_{B, i \in I}(M_i, \mathrm{E}_i)$ the corresponding amalgamated free product.*

Let $1_Q \in M$ be any nonzero projection and $Q \subset 1_Q M 1_Q$ any finite von Neumann subalgebra with expectation that possesses a von Neumann subalgebra $A \subset Q$ with relative property (T). Then there exists $i \in I$ such that $A \preceq_M M_i$.

Proof. Put $\widetilde{Q} = Q \oplus \mathbf{C}(1_M - 1_Q)$ and let $\mathrm{E}_{\widetilde{Q}} : M \to \widetilde{Q}$ be a faithful normal conditional expectation. Choose a faithful trace $\tau_{\widetilde{Q}} \in (\widetilde{Q})_*$ and put $\psi = \tau_{\widetilde{Q}} \circ \mathrm{E}_{\widetilde{Q}}$. Observe that $1_Q \in M^\psi$ and $Q \subset 1_Q M^\psi 1_Q$.

Fix a faithful state $\varphi \in B_*$ and put $\mathcal{B} := \mathrm{c}_\varphi(B)$, $\mathcal{M} := \mathrm{c}_{\varphi \circ \mathrm{E}}(M)$ and $\mathcal{M}_i := \mathrm{c}_{\varphi \circ \mathrm{E}_i}(M_i)$ for every $i \in I$. Fix a nonzero finite trace projection $q \in \mathrm{L}_\psi(\mathbf{R})$ and put $p := \Pi_{\varphi,\psi}(q) \in \mathcal{M}$ and $\mathcal{A} := \Pi_{\varphi,\psi}(\pi_\psi(A)q) \subset p\mathcal{M}p$. Since the inclusion $A \subset Q$ has relative property (T) and since

$$(\Pi_{\varphi,\psi}(\pi_\psi(A)q) \subset \Pi_{\varphi,\psi}(\pi_\psi(Q)q)) \cong (A \subset Q)$$

and

$$\mathcal{A} = \Pi_{\varphi,\psi}(\pi_\psi(A)q) \subset \Pi_{\varphi,\psi}(\pi_\psi(Q)q) \subset p\mathcal{M}p,$$

the inclusion $\mathcal{A} \subset p\mathcal{M}p$ also has relative property (T). By Theorem 4.7, there exists $i \in I$ such that $\mathcal{A} \preceq_\mathcal{M} \mathcal{M}_i$, and hence $A \preceq_M M_i$ by [HU15, Lemma 2.4]. ◻

5. Proof of the Main Theorem

Assume that M and N are isomorphic and identify $M = N$. Note however that we cannot identify φ with ψ.

Fix an arbitrary $i \in I$. We first prove the following intermediate assertion:

($\diamondsuit$) There exist $j = \alpha(i) \in J$, $n_j \geq 1$ and a nonzero partial isometry $v_j \in \mathbf{M}_{1,n_j}(M)$ such that $v_j^* v_j \in \mathbf{M}_{n_j}(N_j)$, $v_j v_j^* \in M_i$ and $v_j^* M_i v_j \subset v_j^* v_j \mathbf{M}_{n_j}(N_j) v_j^* v_j$. Observe that the unital inclusion $v_j v_j^* M_i v_j v_j^* \subset v_j v_j^* M v_j v_j^*$ is with expectation and hence so is the unital inclusion $v_j^* M_i v_j \subset v_j^* v_j \mathbf{M}_{n_j}(M) v_j^* v_j$. Therefore the unital inclusion $v_j^* M_i v_j \subset v_j^* v_j \mathbf{M}_{n_j}(N_j) v_j^* v_j$ is with expectation. When N_j is semifinite, we will be able to choose the partial isometry $v_j \in \mathbf{M}_{1,n_j}(M)$ in such a way that $v_j^* v_j \in \mathbf{M}_{n_j}(N_j)$ has finite trace. This is because we are going to use Theorem 2.4 and show, as a crucial step, that $A \preceq_M N_j$ for a well-chosen *finite* von Neumann subalgebra $A \subset M_i$ with expectation.

We will treat cases (i), (ii), (iii), (iv) separately as follows.

Case (i). Assume that M_i is not prime, and hence we may write $M_i = P_1 \mathbin{\overline{\otimes}} P_2$ with diffuse factors P_1 and P_2. We may assume without loss of generality that P_2 is not amenable. Choose a faithful state $\chi_1 \in (P_1)_*$ such that $(P_1)^{\chi_1}$ is diffuse (see [HS90, Theorem 11.1]) and a faithful state $\chi_2 \in (P_2)_*$, and put $\chi = \chi_1 \otimes \chi_2$. Observe that there exists a diffuse abelian von Neumann subalgebra $A_1 \subset (P_1)^{\chi_1}$, and put $A = A_1 \otimes \mathbf{C}1$. Since P_2 is not amenable and since the unital inclusion $\mathbf{C}1 \otimes P_2 \subset A' \cap M$ is with expectation (observe that $\mathbf{C}1 \otimes P_2 \subset M$ is with expectation), $A' \cap M$ is not amenable either. By Theorem 4.4, there exists $j = \alpha(i) \in J$ such that $A \preceq_M N_j$.

There exist $n_j \geq 1$, a projection $q_j \in \mathbf{M}_{n_j}(N_j)$, a nonzero partial isometry $v_j \in \mathbf{M}_{1,n_j}(M)$ and a unital normal $*$-homomorphism $\pi : A \to q_j \mathbf{M}_{n_j}(N_j) q_j$ such that the unital inclusion $\pi(A) \subset q_j \mathbf{M}_{n_j}(N_j) q_j$ is with expectation and $a v_j = v_j \pi(a)$ for all $a \in A$. By Proposition 2.2 (see Remark 2.5 (1)), the unital inclusions $A v_j v_j^* \subset v_j v_j^* M v_j v_j^*$ and $v_j^* A v_j = \pi(A) v_j^* v_j \subset v_j^* v_j \mathbf{M}_{n_j}(M) v_j^* v_j$ are with expectation. By [HU15, Proposition 2.7 (2)], we have $v_j v_j^* \in A' \cap M = A' \cap M_i$, $v_j^* v_j \in \mathbf{M}_{n_j}(N_j)$ and $v_j^*(A' \cap M_i) v_j \subset v_j^* v_j \mathbf{M}_{n_j}(N_j) v_j^* v_j$.

Observe that $A' \cap M_i = ((A_1)' \cap P_1) \mathbin{\overline{\otimes}} P_2$. By the same reasoning as in the proof of [HI15, Lemma 4.13] and by Lemma 2.1, there exist nonzero projections $p_1 \in (A_1)' \cap (P_1)^{\chi_1}$ and $p_2 \in (P_2)^{\chi_2}$ such that $p_1 p_2 \precsim v_j v_j^*$ in $A' \cap M_i$. Let $u \in A' \cap M_i$ be a partial isometry such that $uu^* = p_1 p_2$ and $u^* u \leq v_j v_j^*$. We have $a\, u v_j = u v_j\, \pi(a)$ for all $a \in A$ and $(u v_j)(u v_j)^* = u v_j v_j^* u^* = p_1 p_2$ and $(u v_j)^*(u v_j) = v_j^* u^* u v_j \in \mathbf{M}_{n_j}(N_j)$. So, up to replacing v_j with $u v_j$, we may assume that $v_j v_j^* = p_1 p_2$.

Observe that the unital inclusion $p_2 P_2 p_2 p_1 \subset p_1 p_2 M p_1 p_2$ is with expectation and so is the unital inclusion $v_j^* P_2 p_1 v_j \subset v_j^* v_j \mathbf{M}_{n_j}(N_j) v_j^* v_j$ (recall that $v_j^*(A' \cap M_i) v_j \subset v_j^* v_j \mathbf{M}_{n_j}(N_j) v_j^* v_j$). Then [HU15, Proposition 2.7 (1)] shows that

$$\begin{aligned} v_j^* P_1 p_2 v_j &= (v_j^* P_2 p_1 v_j)' \cap (v_j^* M v_j) \\ &= (v_j^* P_2 p_1 v_j)' \cap v_j^* v_j \mathbf{M}_{n_j}(M) v_j^* v_j \\ &= (v_j^* P_2 p_1 v_j)' \cap v_j^* v_j \mathbf{M}_{n_j}(N_j) v_j^* v_j \\ &\subset v_j^* v_j \mathbf{M}_{n_j}(N_j) v_j^* v_j. \end{aligned}$$

Since $v_j^* M_i v_j = v_j^* P_1 p_2 v_j \vee v_j^* P_2 p_1 v_j$, we obtain $v_j^* M_i v_j \subset v_j^* v_j \mathbf{M}_{n_j}(N_j) v_j^* v_j$.

Case (ii). Assume that M_i has property Gamma, that is, the central sequence algebra $(M_i)' \cap (M_i)^\omega$ is diffuse. By Theorem 3.1, there exists a decreasing sequence $(A_n)_n$ of diffuse abelian subalgebras of M_i with expectation such that $M_i = \bigvee_n ((A_n)' \cap M_i)$. Since M_i is not amenable, there exists $n \in \mathbf{N}$ such that $(A_n)' \cap M_i$ is not amenable. Observe that $(A_n)' \cap M_i = (A_n)' \cap M$ by [HU15, Proposition 2.7 (1)]. By Theorem 4.4, there exists $j = \alpha(i) \in J$ such that $A_n \preceq_M N_j$.

There exist $n_j \geq 1$, a projection $q_j \in \mathbf{M}_{n_j}(N_j)$, a nonzero partial isometry $v_j \in \mathbf{M}_{1,n_j}(M)$ and a unital normal $*$-homomorphism $\pi : A_n \to q_j \mathbf{M}_{n_j}(N_j) q_j$ such that the unital inclusion $\pi(A_n) \subset q_j \mathbf{M}_{n_j}(N_j) q_j$ is with expectation and $a v_j = v_j \pi(a)$ for all $a \in A_n$. By [HU15, Proposition 2.7 (2)], we have $v_j v_j^* \in (A_n)' \cap M = (A_n)' \cap M_i$, $v_j^* v_j \in \mathbf{M}_{n_j}(N_j)$ and $v_j^*((A_n)' \cap M_i) v_j \subset v_j^* v_j \mathbf{M}_{n_j}(N_j) v_j^* v_j$. Observe that $\pi(A_k) \subset \pi(A_n)$ is with expectation for every $k \geq n$ (since A_n is abelian). Hence, the inclusion $\pi(A_k) \subset q_j \mathbf{M}_{n_j}(N_j) q_j$ is with expectation for every $k \geq n$. As in the second paragraph in case (i) we observe that $v_j^*((A_k)' \cap M_i) v_j \subset v_j^* v_j \mathbf{M}_{n_j}(N_j) v_j^* v_j$ for every $k \geq n$. Since $M_i = \bigvee_{n \in \mathbf{N}} ((A_n)' \cap M_i)$, we finally obtain $v_j^* M_i v_j \subset v_j^* v_j \mathbf{M}_{n_j}(N_j) v_j^* v_j$.

Case (iii). Assume that M_i possesses an amenable finite von Neumann subalgebra A with expectation such that $A' \cap M_i = \mathcal{Z}(A)$ and $\mathcal{N}_{M_i}(A)'' = M_i$. Since M_i is not a type I factor, it is easy to see that A is necessarily diffuse and hence [HU15, Proposition 2.7] shows that $A' \cap M =$

$A' \cap M_i = \mathcal{Z}(A)$ and $\mathcal{N}_M(A)'' = \mathcal{N}_{M_i}(A)'' = M_i$. By Theorem 4.6, there exists $j = \alpha(i) \in J$ such that $A \preceq_M N_j$. Namely, there exist $n_j \geq 1$, a projection $q_j \in \mathbf{M}_{n_j}(N_j)$, a nonzero partial isometry $v_j \in \mathbf{M}_{1,n_j}(M)$ and a unital normal $*$-homomorphism $\pi : A \to q_j\mathbf{M}_{n_j}(N_j)q_j$ such that the unital inclusion $\pi(A) \subset q_j\mathbf{M}_{n_j}(N_j)q_j$ is with expectation and $av_j = v_j\pi(a)$ for all $a \in A$. By [HU15, Proposition 2.7], we have $v_jv_j^* \in A' \cap M = A' \cap M_i$ and $v_j^*v_j \in \mathbf{M}_{n_j}(N_j)$, and hence $v_j^*M_iv_j \subset v_j^*v_j\mathbf{M}_{n_j}(N_j)v_j^*v_j$.

Case (iv). Assume that M_i is a II_1 factor that possesses a regular diffuse von Neumann subalgebra $A \subset M_i$ with relative property (T). By Theorem 4.8, there exists $j = \alpha(i) \in J$ such that $A \preceq_M N_j$. In the exactly same way as in the proof of case (iii), we conclude that there exist $n_j \geq 1$ and a nonzero partial isometry $v_j \in \mathbf{M}_{1,n_j}(M)$ such that $v_j^*v_j \in \mathbf{M}_{n_j}(N_j)$, $v_jv_j^* \in M_i$ and $v_j^*M_iv_j \subset v_j^*v_j\mathbf{M}_{n_j}(N_j)v_j^*v_j$.

We have completed the proof of the desired intermediate assertion ($\diamondsuit$).

By symmetry, for any given $j \in J$, there exist $i = \beta(j) \in I$, $m_i \geq 1$ and a nonzero partial isometry $w_i \in \mathbf{M}_{1,m_i}(M)$ such that $w_i^*w_i \in \mathbf{M}_{m_i}(M_i)$, $w_iw_i^* \in N_j$ and $w_i^*N_jw_i \subset w_i^*w_i\mathbf{M}_{m_i}(M_i)w_i^*w_i$. Moreover, the unital inclusion $w_i^*N_jw_i \subset w_i^*w_i\mathbf{M}_{m_i}(M_i)w_i^*w_i$ is with expectation.

For every $i \in I$, put $w_i^{(n_{\alpha(i)})} := w_i \otimes 1_{n_{\alpha(i)}} \in \mathbf{M}_{1,m_i}(M) \otimes \mathbf{M}_{n_{\alpha(i)}}(\mathbf{C}) = \mathbf{M}_{n_{\alpha(i)},n_{\alpha(i)}m_i}(M)$. Observe that $w_i^{(n_{\alpha(i)})}\big(w_i^{(n_{\alpha(i)})}\big)^* = w_iw_i^* \otimes 1_{n_{\alpha(i)}} \in \mathbf{M}_{n_{\alpha(i)}}(N_{\alpha(i)})$, $\big(w_i^{(n_{\alpha(i)})}\big)^*w_i^{(n_{\alpha(i)})} = w_i^*w_i \otimes 1_{n_{\alpha(i)}} \in \mathbf{M}_{n_{\alpha(i)}m_i}(M_{\beta(\alpha(i))})$ and

$$\begin{aligned}\big(w_i^{(n_{\alpha(i)})}\big)^*\mathbf{M}_{n_{\alpha(i)}}(N_{\alpha(i)})w_i^{(n_{\alpha(i)})} &= w_i^*N_{\alpha(i)}w_i \otimes \mathbf{M}_{n_{\alpha(i)}}(\mathbf{C}) \\ &\subset w_i^*w_i\mathbf{M}_{m_i}(M_{\beta(\alpha(i))})w_i^*w_i \otimes \mathbf{M}_{n_{\alpha(i)}}(\mathbf{C}) \\ &= \big(w_i^{(n_{\alpha(i)})}\big)^*w_i^{(n_{\alpha(i)})}\mathbf{M}_{n_{\alpha(i)}m_i}(M_{\beta(\alpha(i))})\big(w_i^{(n_{\alpha(i)})}\big)^*w_i^{(n_{\alpha(i)})}.\end{aligned}$$

Since the inclusion $w_i^*N_{\alpha(i)}w_i \subset w_i^*w_i\mathbf{M}_{m_i}(M_{\beta(\alpha(i))})w_i^*w_i$ is with expectation, so is the above inclusion.

Since M_i and $N_{\alpha(i)}$ are diffuse factors and since the projection $(v_{\alpha(i)})^*v_{\alpha(i)} \in \mathbf{M}_{n_{\alpha(i)}}(N_{\alpha(i)})$ has finite trace if $N_{\alpha(i)}$ is semifinite as claimed in the first paragraph of the proof, up to shrinking $v_{\alpha(i)}(v_{\alpha(i)})^* \in M_i$ if necessary, we may further choose the partial isometry $v_{\alpha(i)} \in \mathbf{M}_{1,n_{\alpha(i)}}(M)$ so that $(v_{\alpha(i)})^*v_{\alpha(i)} \precsim w_i^{(n_{\alpha(i)})}\big(w_i^{(n_{\alpha(i)})}\big)^*$ in $\mathbf{M}_{n_{\alpha(i)}}(N_{\alpha(i)})$. Since $N_{\alpha(i)}$ is a factor, we can find a nonzero partial isometry $u \in \mathbf{M}_{n_{\alpha(i)}}(N_{\alpha(i)})$ such that $uu^* = (v_{\alpha(i)})^*v_{\alpha(i)}$ and $u^*u \leq w_i^{(n_{\alpha(i)})}\big(w_i^{(n_{\alpha(i)})}\big)^*$. Then $v := v_{\alpha(i)}uw_i^{n_{\alpha(i)}}$ is a nonzero partial isometry in $\mathbf{M}_{1,n_{\alpha(i)}m_i}(M)$ such that

$$\begin{aligned}vv^* &= v_{\alpha(i)}uw_i^{(n_{\alpha(i)})}\big(w_i^{(n_{\alpha(i)})}\big)^*u^*(v_{\alpha(i)})^* = v_{\alpha(i)}uu^*(v_{\alpha(i)})^* = v_{\alpha(i)}(v_{\alpha(i)})^* \in M_i \\ v^*v &= \big(w_i^{(n_{\alpha(i)})}\big)^*u^*(v_{\alpha(i)})^*v_{\alpha(i)}uw_i^{(n_{\alpha(i)})} = \big(w_i^{(n_{\alpha(i)})}\big)^*u^*uw_i^{(n_{\alpha(i)})} \in \mathbf{M}_{n_{\alpha(i)}m_i}(M_{\beta(\alpha(i))})\end{aligned}$$

and

$$\begin{aligned}v^*M_iv &= \big(w_i^{(n_{\alpha(i)})}\big)^*u^*(v_{\alpha(i)})^*\, M_i\, v_{\alpha(i)}u\, w_i^{(n_{\alpha(i)})} \\ &\subset \big(w_i^{(n_{\alpha(i)})}\big)^*\, u^*\, \mathbf{M}_{n_{\alpha(i)}}(N_{\alpha(i)})\, u\, w_i^{(n_{\alpha(i)})} \\ &\subset v^*v\mathbf{M}_{n_{\alpha(i)}m_i}(M_{\beta(\alpha(i))})v^*v.\end{aligned} \tag{5.1}$$

Note that the inclusions in (5.1) are with expectation.

By Lemma 2.8, we have $\beta(\alpha(i)) = i$ for every $i \in I$. Since the inclusions in (5.1) are with expectation and since $vv^* \in M_i$ and $v^*v \in \mathbf{M}_{n_{\alpha(i)}m_i}(M_i)$, we necessarily have $v \in \mathbf{M}_{1,n_{\alpha(i)}m_i}(M_i)$ by [HU15, Proposition 2.7 (1)]. Therefore, (5.1) must be equality with $\beta(\alpha(i)) = i$. This implies that $v_{\alpha(i)}u = v\big(w_i^{(n_{\alpha(i)})}\big)^* \in \mathbf{M}_{1,n_{\alpha(i)}}(M)$ with $(v_{\alpha(i)}u)(v_{\alpha(i)}u)^* = v\big(w_i^{(n_{\alpha(i)})}\big)^* w_i^{(n_{\alpha(i)})} v^* \in M_i$, $(v_{\alpha(i)}u)^*(v_{\alpha(i)}u) = u^*u \in \mathbf{M}_{n_{\alpha(i)}}(N_{\alpha(i)})$ and $u^*(v_{\alpha(i)})^* M_i v_{\alpha(i)} u = u^*u\mathbf{M}_{n_{\alpha(i)}}(N_{\alpha(i)})u^*u$. By symmetry, we have $\alpha(\beta(j)) = j$ for every $j \in J$. This shows that $\alpha : I \to J$ is indeed a bijection and M_i and $N_{\alpha(i)}$ are stably isomorphic to each other for every $i \in I$. Hence we have proved item (1) of the Main Theorem.

Assume moreover that M_i is a type III factor for every $i \in I$. This forces N_j to be a type III factor for every $j \in J$. Therefore, up to conjugating by partial isometries in M_i and $N_{\alpha(i)}$, we may assume that $n_{\alpha(i)} = 1$ and that there exists a unitary $u_i \in \mathcal{U}(M)$ such that $u_i M_i u_i^* = N_{\alpha(i)}$ for every $i \in I$. The uniqueness of the bijection $\alpha : I \to J$ as in item (2) of the Main Theorem is guaranteed by Lemma 2.8. Therefore, we have completed the proof of the Main Theorem.

6. Further results

Following [Oz03, VV05], we say that a σ-finite diffuse von Neumann algebra M is *solid* if for any diffuse von Neumann subalgebra $A \subset M$ with expectation, the relative commutant $A' \cap M$ is amenable. More generally, we will say that a σ-finite (not necessarily diffuse) von Neumann algebra M is solid if either M is atomic or if its nonzero diffuse direct summand is solid. Recall that whenever M is a diffuse solid von Neumann algebra, $p\mathbf{M}_n(M)p$ is also solid for all $n \geq 1$ and all nonzero projections $p \in \mathbf{M}_n(M)$ (see e.g. [HR14, Proposition 3.2] for a similar statement and its proof). The class of solid von Neumann algebras includes bi-exact group von Neumann algebras [BO08, Oz03], free quantum group von Neumann algebras [VV05] and free Araki-Woods factors [Ho07].

Part of the technology provided for proving the Main Theorem also enables us to prove the following characterization of solidity for free products with respect to arbitrary faithful normal states and over arbitrary index sets. It moreover generalizes the main result of [GJ07].

Theorem 6.1. *Let I be any nonempty set and $(M_i, \varphi_i)_{i\in I}$ be any family of von Neumann algebras endowed with any faithful normal states. Then, for the corresponding free product $(M, \varphi) = *_{i\in I}(M_i, \varphi_i)$, the free product von Neumann algebra M is solid if and only if so are all M_i.*

Proof. (The only if part) Assume that some M_i is not solid. By definition, there exist a nonzero projection $z \in \mathcal{Z}(M_i)$ and a diffuse von Neumann subalgebra $P \subset M_i z$ with expectation such that the relative commutant $P' \cap M_i z$ is nonamenable. Since the unital inclusion $P' \cap M_i z \subset zM_i z$ is with expectation so is the unital inclusion $P' \cap M_i z \subset zMz$. This implies that the unital inclusion $P' \cap M_i z \subset P' \cap zMz$ is with expectation and hence $P' \cap zMz$ is nonamenable. Therefore, zMz is not solid and neither is M.

(The if part) Assume that all M_i are solid. Suppose on the contrary that M is not solid. Then there exist a diffuse von Neumann subalgebra $Q \subset 1_Q M 1_Q$ with expectation such that the relative commutant $Q' \cap 1_Q M 1_Q$ is nonamenable. As in the proof of [HU15, Lemma 2.1], choose a faithful state $\psi \in M_*$ such that $1_Q \in M^\psi$, Q is globally invariant under the modular automorphism group σ^{ψ_Q} where $\psi_Q = \frac{\psi(1_Q \cdot 1_Q)}{\psi(1_Q)}$ and $A := Q^{\psi_Q}$ is diffuse. Since $Q' \cap 1_Q M 1_Q \subset A' \cap 1_Q M 1_Q$ with $1_Q = 1_A$ is with expectation, $A' \cap 1_A M 1_A$ is also nonamenable. Up to cutting down by a suitable nonzero central projection $z \in \mathcal{Z}(A' \cap 1_A M 1_A)$, for which $(A' \cap 1_A M 1_A)z$ has no amenable direct summand and up to replacing A with Az (note that $z \in M^\psi$ and $Az \subset zMz$ is with expectation), we may further assume without loss of generality that the relative commutant $A' \cap 1_A M 1_A$ has no amenable direct summand. By Theorem 4.4, there exists $i \in I$ such that $A \preceq_M M_i$.

Then there exist $n \geq 1$, a projection $q \in \mathbf{M}_n(M_i)$, a nonzero partial isometry $w \in \mathbf{M}_{1,n}(1_A M)q$ and a unital normal $*$-homomorphism $\pi : A \to q\mathbf{M}_n(M_i)q$ such that the unital inclusion

$\pi(A) \subset q\mathbf{M}_n(M_i)q$ is with expectation and $aw = w\pi(a)$ for all $a \in A$. By Remark 2.5 (1) both of the inclusions $Aww^* \subset ww^*Mww^*$ and $\pi(A)w^*w \subset w^*w\mathbf{M}_n(M)w^*w$ are with expectation. Proceeding as in the proof of the Main Theorem (case (i)), we have $w^*w \in q\mathbf{M}_n(M_i)q$ and w^*Aw and $w^*(A' \cap 1_A M 1_A)w$ are commuting subalgebras of $w^*w\mathbf{M}_n(M_i)w^*w$ with expectation. Since w^*Aw is diffuse and $w^*(A' \cap 1_A M 1_A)w$ is not amenable, $w^*w\mathbf{M}_n(M_i)w^*w$ is not solid. This however contradicts the fact that M_i is solid. □

The first part of the above proof actually shows that any von Neumann subalgebra of a solid von Neumann algebra with expectation must be solid.

Remark 6.2. Recall that a tracial von Neumann algebra M is *strongly solid* if for any amenable diffuse von Neumann subalgebra $A \subset M$, the normalizer $\mathcal{N}_M(A)''$ is amenable. Using a combination of the proofs of Theorem 6.1 and [Io12, Theorem 1.8] with Theorem 4.6 (for tracial von Neumann algebras; see the remark after its proof) in place of Theorem 4.4, we can also show that a given tracial free product von Neumann algebra over an *arbitrary* index set is strongly solid if and only if so are all the component algebras.

We point out that we can then obtain examples of strongly solid II_1 factors that do not have the weak* completely bounded approximation property (CBAP). Indeed, for every $n \geq 1$, take a lattice $\Gamma_n < \mathrm{Sp}(n, 1)$ and denote by $(M, \tau) = *_{n \in \mathbf{N} \setminus \{0\}}(\mathrm{L}(\Gamma_n), \tau_{\Gamma_n})$ the canonical tracial free product II_1 factor. By [CS11, Theorem B] and the above fact, M is a strongly solid II_1 factor. Moreover, it follows from [CH88] that M does not have the weak* CBAP.

Remark 6.3. Any diffuse solid von Neumann algebra M with property Gamma (and with separable predual) is necessarily amenable. Indeed, by Theorem 3.1, there exists a decreasing sequence $(A_n)_n$ of diffuse abelian von Neumann subalgebras of M with expectation such that $M = \bigvee_{n \in \mathbf{N}}((A_n)' \cap M)$. By solidity, $(A_n)' \cap M$ is amenable for every $n \in \mathbf{N}$ and hence M is amenable.

Appendix A. Normalizers inside semifinite AFP von Neumann algebras

Ozawa–Popa's relative amenability in the semifinite setting. Let (M, Tr) be any semifinite σ-finite von Neumann algebra endowed with a faithful normal semifinite trace and $B \subset M$ any von Neumann subalgebra with trace preserving conditional expectation $\mathrm{E}_B : M \to B$. Denote by $\langle M, B \rangle$ the basic extension associated with E_B and by e_B the canonical Jones projection. Then there exists a faithful normal semifinite operator-valued weight, called the *dual* operator-valued weight, $\widehat{\mathrm{E}}_B : \langle M, B \rangle_+ \to \widehat{M}_+$ satisfying $\widehat{\mathrm{E}}_B(e_B) = 1$ (see e.g. [ILP96, §2.1]). Moreover, the linear span of Me_BM forms a σ-strongly dense $*$-subalgebra of $\langle M, B \rangle$ and $\sigma_t^{\mathrm{Tr} \circ \widehat{\mathrm{E}}_B}(e_B) = e_B$ for all $t \in \mathbf{R}$. Thus, $\mathrm{Tr}_{\langle M, B \rangle} := \mathrm{Tr} \circ \widehat{\mathrm{E}}_B$ becomes a faithful normal semifinite trace on $\langle M, B \rangle$.

Theorem A.1. ([OP07, Theorem 2.1]) *Let $p \in M$ be any nonzero projection with $\mathrm{Tr}(p) < +\infty$ and $A \subset pMp$ any von Neumann subalgebra. Write $\tau := \frac{1}{\mathrm{Tr}(p)}\mathrm{Tr}|_{pMp}$. The following conditions are equivalent:*

1. *There exists an A-central state φ on $p\langle M, B \rangle p$ such that $\varphi|_{pMp} = \tau$.*
2. *There exists an A-central state φ on $p\langle M, B \rangle p$ such that $\varphi|_{pMp}$ is normal and such that $\varphi|_{\mathcal{Z}(A' \cap pMp)}$ is faithful.*
3. *There exists a conditional expectation $\Phi : p\langle M, B \rangle p \to A$ such that $\Phi|_{pMp}$ gives the unique τ-preserving conditional expectation from pMp onto A.*
4. *There exists a net $(\xi_i)_{i \in I}$ of vectors in $\mathrm{L}^2(\langle M, B \rangle, \mathrm{Tr}_{\langle M, B \rangle})$ such that*
   - *$p\xi_i p = \xi_i$ for all $i \in I$,*
   - *$\lim_i \langle x\xi_i, \xi_i \rangle_{\mathrm{Tr}_{\langle M, B \rangle}} = \tau(x)$ for all $x \in pMp$ and*
   - *$\lim_i \|a\xi_i - \xi_i a\|_{2, \mathrm{Tr}_{\langle M, B \rangle}} = 0$ for all $a \in A$.*

We will say that A is amenable relative to B inside M *if one of the above equivalent conditions holds.*

Proof. Observe that we have a natural identification of $\mathrm{L}^2(p\langle M,B\rangle p, \mathrm{Tr}_{\langle M,B\rangle}|_{p\langle M,B\rangle p})$ with $p \cdot \mathrm{L}^2(\langle M,B\rangle, \mathrm{Tr}_{\langle M,B\rangle}) \cdot p$ as pMp-pMp-bimodules. Then the proof of [OP07, Theorem 2.1] applies *mutatis mutandis.* □

Lemma A.2. ([OP07, Corollary 2.3] and [Io12, Lemma 2.3]) *Let* $p \in M$ *be any nonzero projection with* $\mathrm{Tr}(p) < +\infty$ *and* $A \subset pMp$ *any von Neumann subalgebra. Let* $\mathcal{L}$ *be any* B-M-*bimodule. Assume that there exists a net* $(\xi_i)_{i\in I}$ *of vectors in* $p\mathcal{H}$ *with* $\mathcal{H} := \mathrm{L}^2(M, \mathrm{Tr}) \otimes_B \mathcal{L}$ *such that the following conditions hold:*

- $\limsup_i \|x\xi_i\|_{\mathcal{H}} \le \|x\|_{2,\tau}$ *for all* $x \in pMp$,
- $\limsup_i \|\xi_i\|_{\mathcal{H}} > 0$ *and*
- $\lim_i \|a\xi_i - \xi_i a\|_{\mathcal{H}} = 0$ *for all* $a \in A$.

Then there exists a nonzero projection $z \in \mathcal{Z}(A' \cap pMp)$ *such that* Az *is amenable relative to* B *inside* M.

Proof. Observe that $\langle M,B\rangle = (J^M B J^M)' \cap \mathbf{B}(\mathrm{L}^2(M))$ also acts naturally on $\mathcal{H}$ in this semifinite setting, where J^M is the modular conjugation on the standard form $\mathrm{L}^2(M)$. Then the proof of [Io12, Lemma 2.3] applies *mutatis mutandis* to obtain item (2) in Theorem A.1. □

Vaes's dichotomy result in the semifinite setting. Let $(M, \mathrm{E}) = (M_1, \mathrm{E}_1) *_B (M_2, \mathrm{E}_2)$ be any *semifinite* amalgamated free product von Neumann algebra endowed with a faithful normal semifinite trace Tr such that $\mathrm{Tr} \circ \mathrm{E} = \mathrm{Tr}$. Let $q \in B$ be any nonzero projection such that $\mathrm{Tr}(q) < +\infty$. Up to replacing Tr with $\frac{1}{\mathrm{Tr}(q)}\mathrm{Tr}$ if necessary, we may and will assume that $\mathrm{Tr}(q) = 1$.

Denote by $\mathbf{F}_2 = \langle \gamma_1, \gamma_2\rangle$ the free group on two generators and put

$$(\widetilde{M}, \widetilde{\mathrm{E}}) = (M, \mathrm{E}) *_B (B \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{F}_2), \mathrm{id} \otimes \tau_{\mathbf{F}_2}),$$
$$(\widetilde{M}_i, \widetilde{\mathrm{E}}_i) = (M_i, \mathrm{E}_i) *_B (B \mathbin{\overline{\otimes}} \mathrm{L}(\langle\gamma_i\rangle), \mathrm{id} \otimes \tau_{\langle\gamma_i\rangle}), \quad i \in \{1,2\}.$$

Denote by $\mathbf{F}_2 \to \mathrm{L}(\mathbf{F}_2) : \gamma \mapsto \lambda_\gamma$ the canonical unitary representation and regard $\mathrm{L}(\mathbf{F}_2) \cong \mathbf{C}1_B \otimes \mathrm{L}(\mathbf{F}_2) \subset \widetilde{M}$. Then we can naturally identify $(\widetilde{M}, \widetilde{\mathrm{E}}) = (\widetilde{M}_1, \widetilde{\mathrm{E}}_1) *_B (\widetilde{M}_2, \widetilde{\mathrm{E}}_2)$. Following [IPP05, §2], we can construct a 1-parameter unitary group u_i^t in $\mathrm{L}(\langle\gamma_i\rangle) \subset \widetilde{M}_i \subset \widetilde{M}$ such that $u_i^1 = \lambda_{\gamma_i}$ and $\tau_{\langle\gamma_i\rangle}(u_i^t) = \frac{\sin(\pi t)}{\pi t}$ for all $t \in \mathbf{R}$.

Fix an arbitrary faithful state $\chi \in B_*$. Then $\sigma_t^{\chi\circ\widetilde{\mathrm{E}}_i} = \sigma_t^{\chi\circ\mathrm{E}_i} * (\sigma_t^\chi \otimes \mathrm{id})$ (see [Ue98a, Theorem 2.6]) and hence u_i^t lies in the centralizer of $\chi \circ \widetilde{\mathrm{E}}_i$ for all $t \in \mathbf{R}$. Therefore, we have $\chi \circ \widetilde{\mathrm{E}}_i = (\chi \circ \widetilde{\mathrm{E}}_i) \circ \mathrm{Ad}(u_i^t)$, implying that $\widetilde{\mathrm{E}}_i = \widetilde{\mathrm{E}}_i \circ \mathrm{Ad}(u_i^t)$ for all $t \in \mathbf{R}$. Consequently, $\theta_t := \mathrm{Ad}(u_1^t) * \mathrm{Ad}(u_2^t) \in \mathrm{Aut}(\widetilde{M})$ is well-defined for all $t \in \mathbf{R}$. A similar consideration shows that $\sigma_t^{\mathrm{Tr}_B\circ\widetilde{\mathrm{E}}} = \sigma_t^{\mathrm{Tr}} * (\sigma_t^{\mathrm{Tr}_B} \otimes \mathrm{id}) = \mathrm{id}$ with $\mathrm{Tr}_B := \mathrm{Tr}|_B$ so that $\widetilde{\mathrm{Tr}} := \mathrm{Tr}_B \circ \widetilde{\mathrm{E}}$ gives a faithful normal semifinite trace on $\widetilde{M}$ extending Tr naturally. The triple $(M \subset \widetilde{M}, \widetilde{\mathrm{Tr}}, \theta_t)$ is the semifinite analogue of Popa's malleable deformation for *tracial* amalgamated free product von Neumann algebras as defined in [IPP05, §2]. The basic inequalities such as [Va13, Eq.(3.1),(3.2)] hold true as they are (see e.g. [BHR12, §3.1] with the necessary refinement along [Va13, §3.1]). Observe that $\theta_t(q) = q$ for every $t \in \mathbf{R}$ so that $\theta_t(p) \le q$ for every projection $p \in qMq$ and every $t \in \mathbf{R}$.

Recall that the key observation of [Io12] is that the von Neumann algebra $N := \bigvee_{\gamma\in\mathbf{F}_2} \lambda_\gamma M \lambda_\gamma^*$ is identified with the amalgamated free product of infinitely many copies of (M, E) over $\mathbf{F}_2$ as index set and that $\widetilde{M}$ admits the crossed product decomposition $\widetilde{M} = N \rtimes_{\mathrm{Ad}(\lambda)} \mathbf{F}_2$ whose canonical conditional expectation is denoted by $\mathrm{E}_N : \widetilde{M} \to N$. Moreover, $q\widetilde{M}q = qNq \rtimes_{\mathrm{Ad}(\lambda)} \mathbf{F}_2$ holds naturally and the canonical conditional expectation $\mathrm{E}_{qNq} : q\widetilde{M}q \to qNq$ coincides with the restriction of $\mathrm{E}_N : \widetilde{M} \to N$ to qNq since $q \in B \subset N \subset \widetilde{M}$ and thus $[\lambda_\gamma, q] = 0$ for all $\gamma \in \mathbf{F}_2$.

Theorem A.3 ([Va13, Theorem 3.2]). *Let $p \in qMq$ be any nonzero projection and $A \subset pMp$ any von Neumann subalgebra. Assume that for all $t \in (0,1)$, $\theta_t(A)$ is amenable relative to qNq inside $q\widetilde{M}q$. Then at least one of the following conditions holds:*

- *Either $A \preceq_M M_1$ or $A \preceq_M M_2$ holds.*
- *A is amenable relative to B inside M.*

Proof. The proof is identical to the one of [Va13, Theorem 3.2] with only minor modifications. This is why we will only sketch it. The most essential part of Vaes's proof is done at the Hilbert space level and hence it suffices to explain how to provide the right framework to modify the proof accordingly.

The functional $\tau := \widetilde{\mathrm{Tr}}|_{q\widetilde{M}q}$ defines a faithful normal tracial state on $q\widetilde{M}q = qNq \rtimes_{\mathrm{Ad}(\lambda)} \mathbf{F}_2$, since $\mathrm{Tr}(q) = 1$. Denote by $\langle q\widetilde{M}q, qNq\rangle$ the basic extension of $q\widetilde{M}q$ by $\mathrm{E}_{qNq} : q\widetilde{M}q \to qNq$ with Jones projection e_{qNq}. To simplify the notation, we will simply write $\mathrm{Tr} := \tau \circ \widehat{\mathrm{E}}_{qNq}$ where

$$\widehat{\mathrm{E}}_{qNq} : \langle q\widetilde{M}q, qNq\rangle_+ \to \widehat{q\widetilde{M}q}_+$$

is the dual faithful normal semifinite operator-valued weight satisfying $\widehat{\mathrm{E}}_{qNq}(e_{qNq}) = 1_{qNq} = q$.

Let I be the set of all the quadruplets $i = (X, Y, \delta, t)$ with finite subsets $X \subset q\widetilde{M}q$ and $Y \subset \mathcal{U}(A)$, $0 < \delta < 1$ and $0 < t < 1$. The set I becomes a directed set with the order relation $(X, Y, \delta, t) \leq (X', Y', \delta', t')$ defined by $X \subset X'$, $Y \subset Y'$, $\delta \geq \delta'$ and $t \geq t'$. Since $\theta_t(A)$ is amenable relative to qNq inside $q\widetilde{M}q$, for each $i = (X, Y, \delta, t) \in I$, [OP07, Theorem 2.1 and the remark following it] enables us to find a vector $\xi_i \in \mathrm{L}^2(\langle q\widetilde{M}q, qNq\rangle)$ in such a way that $\|\xi_i\|_{2,\mathrm{Tr}} \leq 1$,

$$\begin{aligned} |\langle x\xi_i, \xi_i\rangle_{\mathrm{Tr}} - \tau(x)| &\leq \delta \quad \text{for all } x \in X \cup \{(\theta_t(y) - y)^*(\theta_t(y) - y) \mid y \in Y\}, \\ \|\theta_t(y)\xi_i - \xi_i\theta_t(y)\|_{2,\mathrm{Tr}} &\leq \delta \quad \text{for all } y \in Y. \end{aligned}$$

Observe that $\lim_i \langle x\xi_i, \xi_i\rangle_{\mathrm{Tr}} = \tau(x)$ for all $x \in q\widetilde{M}q$ and $\lim_i \|y\xi_i - \xi_i y\|_{2,\mathrm{Tr}} = 0$ for all $y \in A$.

Denote by $\mathcal{K} \subset \mathrm{L}^2(\langle q\widetilde{M}q, qNq\rangle)$ the closed linear subspace generated by $\{x\lambda_\gamma e_{qNq}\lambda_\gamma^* \mid x \in qMq, \gamma \in \mathbf{F}_2\}$ and by $e : \mathrm{L}^2(\langle q\widetilde{M}q, qNq\rangle) \to \mathcal{K}$ the orthogonal projection. Note that $e \in (qMq)' \cap \mathbf{B}(\mathrm{L}^2(q\widetilde{M}q))$. Thus, the net $\xi_i' := p(1-e)\xi_i$ satisfies $\limsup_i \|x\xi_i'\|_{2,\mathrm{Tr}} \leq \|x\|_{2,\tau}$ for all $x \in pMp$ and $\lim_i \|a\xi_i' - \xi_i' a\|_{2,\mathrm{Tr}} = 0$ for all $a \in A$.

Suppose that $A \not\preceq_M M_1$ and $A \not\preceq_M M_2$. What we have to show is that A is amenable relative to B inside M. By contradiction and proceeding as in the first paragraph of the proof of [Va13, Theorem 3.2], we may and do assume that *no corner of A is amenable relative to B inside M*, that is, Az is not amenable relative to B inside M for any nonzero projection $z \in \mathcal{Z}(A' \cap pMp)$.

Observe that $\langle q\widetilde{M}q, qNq\rangle = q\langle \widetilde{M}, N\rangle q$ with $e_{qNq} = qe_N$ $(= e_N q)$, where $\langle \widetilde{M}, N\rangle$ is the basic extension of $\widetilde{M}$ by the canonical trace preserving conditional expectation $\mathrm{E}_N : \widetilde{M} \to N$ and also that the traces Tr on $\langle q\widetilde{M}q, qNq\rangle$ and $\widetilde{\mathrm{Tr}} \circ \widehat{\mathrm{E}}_N$ on $\langle \widetilde{M}, N\rangle$ with the dual operator-valued weight $\widehat{\mathrm{E}}_N$ agree since $\widehat{\mathrm{E}}_N(qe_N) = q\widehat{\mathrm{E}}_N(e_N) = q$. (It is then natural to denote the latter trace by the same symbol Tr.) Thus, $\mathrm{L}^2(\langle q\widetilde{M}q, qNq\rangle)$ can be identified with $q \cdot \mathrm{L}^2(\langle \widetilde{M}, N\rangle) \cdot q$. If we identify M with the γth free product component $\lambda_\gamma M \lambda_\gamma^*$, then we have the decomposition $\mathrm{L}^2(N) = \mathrm{L}^2(M) \oplus (\mathrm{L}^2(M) \otimes_B \mathcal{X} \otimes_B \mathrm{L}^2(M))$ as M-M-bimodules for some B-B-bimodule $\mathcal{X}$ (see [Ue98a,

§2]). Then we see, in the same way as in the proof of [Io12, Lemma 4.2], that $\mathrm{L}^2(\langle q\widetilde{M}q, qNq\rangle)\ominus\mathcal{K}$ is identified, as qMq-qMq-bimodule, with $q\cdot(\mathrm{L}^2(M)\otimes_B\mathcal{L})\cdot q\subset \mathrm{L}^2(M)\otimes_B\mathcal{L}$ for some B-M-bimodule $\mathcal{L}$. Thus, Lemma A.2 implies that $\lim_i\|\xi_i'\|_{2,\mathrm{Tr}}=0$, namely $\lim_i\|p\xi_i-ep\xi_i\|_{2,\mathrm{Tr}}=0$.

As in the proof of [Va13, Theorem 3.4], we can construct an isometry $U:\mathrm{L}^2(qMq)\otimes\ell^2(\mathbf{F}_2)\to\mathrm{L}^2(\langle q\widetilde{M}q,qNq\rangle)$ in such a way that $UU^*=e$ and that $U((x\otimes 1)\eta(y\otimes 1))=x(U\eta)y$ for all $x,y\in qMq$ and all $\eta\in\mathrm{L}^2(qMq)\otimes\ell^2(\mathbf{F}_2)$. Put $\zeta_i:=U^*p\xi_i\in p\mathrm{L}^2(qMq)\otimes\ell^2(\mathbf{F}_2)$ for every $i\in I$. Since $\mathrm{L}^2(qMq)\otimes\ell^2(\mathbf{F}_2)\subset\mathrm{L}^2(q\widetilde{M}q)\otimes\ell^2(\mathbf{F}_2)=(q\otimes1)\cdot(\mathrm{L}^2(\widetilde{M})\otimes\ell^2(\mathbf{F}_2))\cdot(q\otimes1)\subset\mathrm{L}^2(\widetilde{M})\otimes\ell^2(\mathbf{F}_2)$, we can follow line by line the rest of the proof of [Va13, Theorem 3.4, pp 704–709] inside $\mathrm{L}^2(\widetilde{M})\otimes\ell^2(\mathbf{F}_2)$ with the following remarks:

1°. $\mathrm{L}^2(\widetilde{M})=\mathrm{L}^2(M)\oplus(\mathrm{L}^2(M)\otimes_B\mathcal{Y}\otimes_B\mathrm{L}^2(M))$ as M-M-bimodules for some B-B-bimodule $\mathcal{Y}$ and hence $\mathrm{L}^2(\widetilde{M})\ominus\mathrm{L}^2(M)=\mathrm{L}^2(M)\otimes_B\mathcal{L}'$ with some B-M-bimodule $\mathcal{L}'$.
2°. A key formula [Va13, Lemma 3.2] (essentially due to Ioana) holds even in the semifinite setting (whose proof goes along that of [BHR12, Lemma 3.5]).
3°. The semifinite counterpart of [Va13, Theorem 3.1] (that is essentially due to Ioana–Peterson–Popa [IPP05]) was already provided by Boutonnet–Houdayer–Raum [BHR12, Theorem 3.3] and we need to use it in place of [Va13, Theorem 3.1].

Following line by line the proof of [Va13, Theorem 3.4, pp 704–709], we can then reach a contradiction. Giving the full details is just a task of understanding Vaes's argument modulo the above three remarks. $\square$

Let $p\in qMq$ be any nonzero projection and $A\subset pMp$ any von Neumann subalgebra. Assume that A is amenable relative to M_i inside M for some $i\in\{1,2\}$. Then, by checking Theorem A.1 (4) and regarding $\mathrm{L}^2(M)$ as a subspace of $\mathrm{L}^2(\widetilde{M})$ naturally, we see that A is amenable relative to M_i inside $\widetilde{M}$. For every $t\in(0,1)$, $\theta_t(A)$ is amenable relative to $\theta_t(M_i)=u_i^tM_iu_i^{t*}$ inside $\widetilde{M}$. The Jones projection $e_{\theta_t(M_i)}$ coincides with $u_i^te_{M_i}u_i^{t*}$ so that $\langle\widetilde{M},\theta_t(M_i)\rangle=\langle\widetilde{M},M_i\rangle$ and hence $\theta_t(A)$ is amenable relative to M_i and also to N since $M_i\subset N$. Since $\langle q\widetilde{M}q,qNq\rangle=q\langle\widetilde{M},N\rangle q$ (see the proof of Theorem A.3), $\theta_t(A)$ is amenable relative to qNq inside $q\widetilde{M}q$ thanks to Theorem A.1 (1). Applying Popa–Vaes's dichotomy result [PV11, Theorem 1.6 and Remark 6.3] to $\theta_t(A)\subset q\widetilde{M}q=qNq\rtimes_{\mathrm{Ad}(\lambda)}\mathbf{F}_2$, we have that at least one of the following conditions holds: $\theta_t(A)\preceq_{q\widetilde{M}q}qNq$ or $\theta_t(\mathcal{N}_{pMp}(A)'')$ $(\subset\mathcal{N}_{\theta_t(p)\widetilde{M}\theta_t(p)}(\theta_t(A))'')$ is amenable relative to qNq inside $q\widetilde{M}q$. Since this is true for every $t\in(0,1)$, at least one of the following conditions holds:

(A) $\theta_t(A)\preceq_{q\widetilde{M}q}qNq$, and hence $\theta_t(A)\preceq_{\widetilde{M}}N$ for some $t\in(0,1)$ or
(B) $\theta_t(\mathcal{N}_{pMp}(A)'')$ is amenable relative to qNq inside $q\widetilde{M}q$ for every $t\in(0,1)$.

In case (A), we use [BHR12, Theorem 3.4] (whose proof actually works even when the projection p there lies in $\mathrm{Proj}_{\mathrm{f}}(\mathcal{M})$ rather than $\mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$) and the consequence is that $A\preceq_M B$ or $\mathcal{N}_{pMp}(A)''\preceq_M M_i$ for some $i\in\{1,2\}$. In case (B), Theorem A.3 implies that $\mathcal{N}_{pMp}(A)''\preceq_M M_i$ for some $i\in\{1,2\}$ or $\mathcal{N}_{pMp}(A)''$ is amenable relative to B inside M. Consequently, we obtain the following result.

Theorem A.4. *Let $p\in qMq$ be any nonzero projection and $A\subset pMp$ any von Neumann subalgebra. Assume that A is amenable relative to one of the M_i inside M. Then at least one of the following holds:*

- *$A\preceq_M B$.*
- *Either $\mathcal{N}_{pMp}(A)''\preceq_M M_1$ or $\mathcal{N}_{pMp}(A)''\preceq_M M_2$ holds.*
- *$\mathcal{N}_{pMp}(A)''$ is amenable relative to B inside M.*

Suppose that A is amenable. Then A is amenable relative to any von Neumann subalgebra with expectation inside M. Hence the above dichotomy holds. Suppose moreover that B is also amenable but $\mathcal{N}_{pMp}(A)''$ is *not*. Then, it is impossible that $\mathcal{N}_{pMp}(A)''$ is amenable relative to B inside M. In fact, there exists a (non-normal) conditional expectation from $\mathbf{B}(p\mathrm{L}^2(M))$ onto $p\langle M, B\rangle p$ since B is amenable and thus $\mathcal{N}_{pMp}(A)''$ must be amenable, a contradiction. Therefore, the dichotomy becomes that one of $A \preceq_M B$, $\mathcal{N}_{pMp}(A)'' \preceq_M M_1$ and $\mathcal{N}_{pMp}(A)'' \preceq_M M_2$ holds true.

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Laboratoire de Mathématiques d'Orsay, Université Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay, France

E-mail address: cyril.houdayer@math.u-psud.fr

Graduate School of Mathematics, Kyushu University, Fukuoka, 819-0395, Japan

E-mail address: ueda@math.kyushu-u.ac.jp