On Asymptotics of Nonlocal p-Laplacian Operators and Related Eigenvalues

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Summary

We present here a brief overview of the contents of this thesis.

The main topic of this thesis is Nonlocal p-Laplacian equations. The thesis is divided into two parts:

(I) Asymptotic Behaviour of Fractional p-Laplacian Functionals under Dirichlet conditions;

(II) On Relative-Nonlocal p-Rayleigh Quotients.

Nonlocal p-Laplacian equations arise naturally in the study of stochastic process with jumps, and more precisely of Lévy process. This type of process are particular interest in Finance and Physics etc. Moreover nonlocal p-Laplacian equations appear naturally in other contexts such as Geometry, Fluid Mechanics, and Image Processing. The classical nonlocal p-Laplacian operator is defined as

$$(-\Delta_p)^s u(x) := 2 \lim_{\epsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy, \ \ p \in (1,+\infty), \ \ s \in (0,1), \tag{1}$$

and the variational forms usually defined as

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy, \ \ p \in (1,+\infty), \ \ s \in (0,1). \tag{2}$$

In the preliminary part, we mainly introduce some basic tools and spaces to be used later, including Γ-convergence and *relative-nonlocal* Sobolev space $\widetilde{W}^{s,p}_{0,tR}(\Omega)$ ($t > 1$ and R is the diameter of Ω).

In chapter I of this thesis, we mainly use Γ-convergence to analysis the behaviour of fractional p-Laplacian functionals with non-homogeneous Dirichlet boundary as $p \to +\infty$, and the asymptotic behaviour triggered by varying s with homogeneous Dirichlet boundary.

On the asymptotic behaviour as $p \to +\infty$, there have been many researches in this direction. According to the results of Chambolle, Lindgren and Monneau (see [29]), there exists one optimal Hölder extension of the fractional ∞-Lpalacian equations

$$\begin{cases} \sup_{y \in \overline{\Omega}, y \neq x} \frac{u(y)-u(x)}{|y-x|^\alpha} + \inf_{y \in \overline{\Omega}, y \neq x} \frac{u(y)-u(x)}{|y-x|^\alpha} = 0 & for \ x \in \Omega, \\ u = g & on \ \partial\Omega. \end{cases} \tag{3}$$

And the unique minimizer to the functional (2) restricted on Ω with non-homogeneous Dirichlet boundary conditions converges to the solution of ∞-Laplacian equations in viscosity sense. In our result, by Γ-convergence we establish the convergence of the minimizers of (2) restricted on Ω with non-homogeneous Dirichlet boundary conditions to one minimizer of the functional $\|\frac{u(x)-u(y)}{|x-y|^\alpha}\|_{L^\infty(\Omega\times\Omega)}$ under the same Dirichlet boundary condition.

Also in the De Giorgi-sense, we investigate the asymptotic behaviour of the minimizers of the operator (2) with varying s in homogeneous Dirichlet boundary condition. We will see that it is totally different for the behaviour from below compared with the case from above. And as a byproduct, we also establish an equivalent form of the sobolev space $W_0^{s,p}(\Omega)$ for arbitrary open bounded set Ω.

In chapter II of this dissertation, we investigate the asymptotic behaviour of nonlocal p-Rayleigh quotient as s varies, which is defined by

$$\inf_{u\in X\setminus\{0\}} \frac{\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy}{\int_\Omega |u|^p dx}.$$

We will see that the behaviour from below and above exhibit a different phenomenon. And we also studied the convergence of corresponding nonlocal eigenfunctions. In order to investigate the asymptotic behaviour of the nonlocal p-Rayleigh quotients, we work in the relative-nonlocal space $\widetilde{W}_{0,tR}^{s,p}(\Omega)$, equivalent to the usual $\widetilde{W_0^{s,p}}(\Omega)$. In this part, we also establish some equivalent forms of the nonlocal Sobolev space $\widetilde{W}_{0,tR}^{s,p}(\Omega)$. This result can be merged with the similar results obtained in chapter I in De Giorgi-sense, which are essentially the same. This part of work is mainly inspired by the former work by P. Lindqvist, M. Degiovanni and M. Marzocchi, etc., who investigated the asymptotic behaviour of p-Rayleigh quotients with varying p.

As a partial result, we also proved that the operator $(-\Delta_p)^s$ is a homeomorphism of $\widetilde{W}_{0,tR}^{s,p}(\Omega)$ and its dual space $\widetilde{W}_{tR}^{-s,q}(\Omega)$ with $\frac{1}{p}+\frac{1}{q}=1$. This part lies in Appendix A1.

This thesis is divide into two chapters. Each chapter corresponds to a preprint paper, as follows.
Chapter I:

- Raphael Feng LI, Asymptotics of Dirichlet Problems to Fractional p-Laplacian Functionals: Approach in De Giorgi Sense[1].

Chapter II:

- Raphael Feng LI, On Relative-Nonlocal p-Rayleigh Quotients[2].

[1] [50]
[2] [51]

Preliminary

In this chapter, we mainly introduce some useful tools and workspace we utilize later: The Γ-convergence and *relative-nonlocal* Sobolev spaces.

0.1 A Glimpse of Γ-convergence

The notion of Γ-convergence has become, over the more than forty years after its introduction by Ennio De Giorgi, the commonly-recognized notion of convergence for variational problems. It has been used to such as, homogenization theory, phase transitions, singular perturbations, boundary value problems in wildly perturbed domains, approximation of variational problems, and non-smooth analysis. Among variational convergence, De Giorgi's Γ-convergence plays a central role for its compactness. Especially all other variational convergence can be easily expressed in the language of Γ-convergence.

The Γ-convergence is defined as:

Definition 0.1.1 (Γ-convergence)**.** *Let X be a metric space. A sequence $\{E_n\}$ of functionals $E_n : X \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is said to $\Gamma(X)$-convergence to $E_\infty : X \to \bar{\mathbb{R}}$, and we write $\Gamma(X)\text{-}\lim\limits_{n\to+\infty} E_n = E_\infty$, if the following hold:*
(i) for every $u \in X$ and $\{u_n\} \subset X$ such that $u_n \to u$ in X, we have

$$E_\infty(u) \leq \liminf_{n\to+\infty} E_n(u_n);$$

(ii) for every $u \in X$ there exists a sequence $\{u_n\} \subset X$(called a recovery sequence) such that $u_n \to u$ in X and

$$E_\infty(u) \geq \limsup_{n\to+\infty} E_n(u_n).$$

For further information, one can refer to [30][23][63]. Here we list some basic properties of Γ-convergence.

Proposition 0.1.1 ([63] Proposition 13.2)**.**

$$F_\infty = \Gamma - \lim_{h\to\infty} F_h$$

is lower semi-continuous.

Definition 0.1.2. *For every function $F : X \to \overline{R}$, the lower semi-continuous envelop (or relaxed function) $sc^- F$ of F is defined for every $x \in X$ by*

$$(sc^- F)(x) = \sup_{G \in \mathcal{G}(F)} G(x),$$

where $\mathcal{G}(F)$ is the set of all lower semi-continuous functions G on X such that $G(y) \leq F(y)$ for every $y \in X$.

Proposition 0.1.2 ([30] Proposition 5.4)**.** *If (F_h) is an increasing sequence, then*

$$\Gamma - \lim_{h\to\infty} F_h = \lim_{h\to\infty} sc^- F_h = \sup_{h \in \mathbb{N}} sc^- F_h.$$

Remark 0.1.1. *If (F_h) is an increasing sequence of lower semi-continuous functions which converges pointwise to a function F, then F is lower semi-continuous and (F_h) Γ-converges to F by Proposition 0.1.2. This property does not hold if the functions F_h are not lower semi-continuous.*

We give a counterexample as follows:

Example 0.1.1. *In $X = \mathbb{R}$ define for $k \in \mathbb{N}$,*

$$F_k[x] := -\delta_{-1/k}(x) + \delta_{1/k}(x) = \begin{cases} \pm 1 & if\ x = \pm 1/k, \\ 0 & otherwise, \end{cases}$$

and

$$F_\infty[x] := -\delta_0(x) = \begin{cases} -1 & if\ x = 0, \\ 0 & otherwise. \end{cases}$$

We can see that in this example $\Gamma\text{-}\lim_k F_k = F_\infty$, but $\lim_k F_k = 0$ in pointwise.

0.2 *Relative-Nonlocal* Sobolev spaces

In order to neatly present the subject, we first need some definitions.

Let $0 < s < 1$ and $1 < p < +\infty$. For every $\Omega \subset \mathbb{R}^N$ open and bounded set, the natural homogeneous setting for equations involving the operator $(-\Delta_p)^s$ restricted on Ω is the space $W_0^{s,p}(\Omega)$, defined as the completion of $C_0^\infty(\Omega)$ with respect to the standard Gagliardo semi-norm

$$[u]_{W^{s,p}(\Omega)} := \left(\int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{1/p}. \tag{4}$$

For the basic properties of Gagliardo semi-norm we refer the reader to [35].

Let us recall that the usual admissible nonlocal space for operator $(-\Delta_p)^s$, the Sobolev space $\widetilde{\mathbf{W}}_{\mathbf{0}}^{\mathbf{s,p}}(\mathbf{\Omega})$, defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\widetilde{W}_0^{s,p}(\Omega)} := \left\{ \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dxdy \right\}^{\frac{1}{p}}, \ \forall\ u \in C_0^\infty(\Omega).$$

In fact this space is equivalently defined by taking the completion of $C_0^\infty(\Omega)$ with respect to the full norm

$$\left(\int_\Omega |u|^p\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{\frac{1}{p}},$$

see Remark 2.5 in [17]. And if the boundary $\partial\Omega$ regular enough, such as Lipschitz, the space $\widetilde{W}_0^{s,p}(\Omega)$ is in coincidence with $W_0^{s,p}(\Omega)$, i.e., Ω can be extensible. For more information on this topic, one can refer to [72].

Let $t>1$, for our requirement, we define a new semi-norm $W_{tR}^{s,p}(\Omega)$ by

$$[u]_{W_{tR}^{s,p}(\Omega)} := \left\{\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\right\}^{\frac{1}{p}} \tag{5}$$

for any measurable function u in $L^p(\Omega)$, in which,

$$R = diam(\Omega) := \sup\{|x-y| : \ \forall\ x,y\in\Omega\}, \tag{6}$$

and $B_{tR}(\Omega)$ is define as the N-dimensional ball with diameter tR located at the same center as the smallest ball containing Ω.

Now we consider the **relative-nonlocal** Sobolev space $\mathbf{\widetilde{W}_{0,tR}^{s,p}(\Omega)}$ defined as the completion of $C_0^\infty(\Omega)$ with respect to the semi-norm (5)

$$\|u\|_{\widetilde{W}_{0,tR}^{s,p}(\Omega)} := \left\{\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\right\}^{\frac{1}{p}},\ \forall\ u\in C_0^\infty(\Omega). \tag{7}$$

This is a reflexive Banach space for $1<p<+\infty$.

As in $\widetilde{W}_0^{s,p}(\Omega)$, the semi-norm $[u]_{\widetilde{W}_{0,tR}^{s,p}(\Omega)}$ can be equivalently defined by taking the full norm

$$\left(\int_\Omega |u|^p\right)^{\frac{1}{p}} + \left(\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{\frac{1}{p}},$$

for any admissible function u (see Proposition 0.2.1).

We point out that the Sobolev space $\widetilde{W}_{0,tR}^{s,p}(\Omega)$ is equivalently defined as the one $\widetilde{W}_0^{s,p}(\Omega)$. Indeed, obviously $\widetilde{W}_0^{s,p}(\Omega)$ is contained in $\widetilde{W}_{0,tR}^{s,p}(\Omega)$, and for the reverse inclusion relationship we refer the reader to Theorem 0.2.2. As in the space $\widetilde{W}_0^{s,p}(\Omega)$, if the boundary $\partial\Omega$ regular enough, such as Lipschitz, the space $\widetilde{W}_{0,tR}^{s,p}(\Omega)$ is in coincidence with $W_0^{s,p}(\Omega)$, i.e., Ω can be extensible under the norm $W_{tR}^{s,p}(\Omega)$. In fact the equivalence is also a direct result by the sufficient and necessary condition for extensible domain (see [72]). Since the ball $B_{tR}(\Omega)$ clearly fits for the condition in [72], then we can directly conclude that $\|u\|_{\widetilde{W}_0^{s,p}(\Omega)} \le C\|u\|_{\widetilde{W}_{0,tR}^{s,p}(\Omega)}$.

Remark 0.2.1. *As denoted in [17][20], If $sp\neq 1$ and $\partial\Omega$ is smooth enough, $\widetilde{W}_0^{s,p}(\Omega)$ coincides with the usual on $W_0^{s,p}(\Omega)$, which is defined as the completion of $C_0^\infty(\Omega)$ with* *respect to the norm*

$$\left(\int_{\Omega}|u|^{p}\right)^{\frac{1}{p}}+\left(\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p}}{|x-y|^{N+sp}}dxdy\right)^{\frac{1}{p}}.$$

See for example Proposition B.1 in [17]. Without the smoothness assumption, obviously $\widetilde{W}_0^{s,p}(\Omega)\subset W_0^{s,p}(\Omega)$.

If in the borderline case $sp=1$, *one has the strict inclusion* $\widetilde{W}_0^{s,p}(\Omega)\subsetneq W_0^{s,p}(\Omega)$. *For the details we refer the reader to Remark 2.1 of [20] and the references therein.*

Owing to the equivalence suggested in Theorem 0.2.2, the statement above is also available for our space $\widetilde{W}_{0,tR}^{s,p}(\Omega)$.

Remark 0.2.2. *As usual setting on the nonlocal problems, the Sobolev space* $\widetilde{W}_0^{s,p}(\Omega)$ *is an admissible space (see [13][17][37][20][22][32] etc.). However, in this work we try to investigate the fractional problem on a general bounded open set in* $\mathbb{R}^N$. *As we will see, if we were to work on the space* $W_0^{s,p}(\Omega)$, *we would get seldom information on the boundary data, which is definitely important to us. Since we have no any regularity assumption on the boundary* $\partial\Omega$, *we could not even get any useful compactness results or even Poincaré-type inequalities. So the Sobolev space* $W_0^{s,p}(\Omega)$ *is too large to us. And it is proved that the Euler-Lagrange equations of the fractional Laplacian is ill-posedness in space* $W^{s,p}(\Omega)$ *with the Dirichlet boundary condition* $g|_{\partial\Omega}$ *(see [40][70]). On the other hand, if we utilize the usual space* $\widetilde{W}_0^{s,p}(\Omega)$, *we can get enough information on the boundary data; but due to our special problem setting, it seems difficult for us to do any precise comparison on the eigenvalues for varying* s. *This means the points too far away from the boundary become a burden to us. So we define a relative-nonlocal Sobolev space* $\widetilde{W}_{0,tR}^{s,p}(\Omega)$.

For an improvement preparation, we also define the space $\widetilde{\mathbf{W}}_{\mathbf{tR}}^{\mathbf{s,p}}(\mathbf{\Omega})$ by

$$\widetilde{W}_{tR}^{s,p}(\Omega):=\left\{u\in W^{s,p}(\Omega):[u]_{W_{tR}^{s,p}(\Omega)}<+\infty,\ \text{and}\ u=0\ \text{on}\ B_{tR}(\Omega)\setminus\Omega\right\},\tag{8}$$

which is a completion of $C^{\infty}(\Omega)$ under the norm $W_{tR}^{s,p}(\Omega)$. Obviously the space $\widetilde{W}_{tR}^{s,p}(\Omega)$ is also a reflexive Banach space.

We want mention here that in fact one can choose any $t>1$ in the multiplication pair tR of the ball diameter based on the definition.

(**Watch out!**) tR here means the diameter of the ball, not as in the usual ball as the radius. Anyway, this is only for the special case in the definition of *relative-nolocal* spaces. And we will still use $B_r(x)$ to denote a ball with radius r centered at point x for the common ball. We also want to emphasis that t in above definitions is independent of Ω.

Throughout this thesis, we use $\mathcal{L}^N(U)$ to denote the N-dimensional Lebesgue measure of the set U in $\mathbb{R}^N$. We prove the following Poincaré-type inequality. For the case on space $\widetilde{W}_0^{s,p}(\Omega)$ one can see Lemma 2.4 in [17].

Proposition 0.2.1. *Let $1 < p < +\infty$ and $0 < s < 1$, $\Omega \subset \mathbb{R}^N$ be an open bounded set. There holds*

$$\|u\|^p_{L^p(\Omega)} \leq \mathcal{I}_{N,s,p}(\Omega)[u]_{W^{s,p}_{tR}(\Omega)}, \ for\ every\ u \in C_0^\infty(\Omega), \tag{9}$$

where the geometric quantity $\mathcal{I}_{N,s,p}(\Omega)$ is defined by

$$\mathcal{I}_{N,s,p}(\Omega) = \min\left\{\frac{diam(\Omega \cup B)^{N+sp}}{\mathcal{L}^N(B)} : B \subset B_{tR}(\Omega) \setminus \Omega \ is\ a\ ball\right\}.$$

Proof. Suppose any $u \in C_0^\infty(\Omega)$ and $B_r \subset B_{4R}(\Omega) \setminus \Omega$, i.e. a ball of radius r contained in the relative complement of Ω in $B_{4R}(\Omega)$. Let $x \in \Omega$ and $y \in B_r$ we then have

$$|u(x)|^p = \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}|x-y|^{N+sp},$$

from which we can infer

$$\mathcal{L}^N(B_r)|u(x)|^p \leq \sup_{x\in\Omega, y\in B_r} |x-y|^{N+sp} \int_{B_r} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dy.$$

We perform an integration on Ω with respect to x to obtain

$$\int_\Omega |u|^p dx \leq \frac{diam(\Omega \cup B_r)^{N+sp}}{\mathcal{L}^N(B_r)} \int_{B_r}\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy,$$

which concludes the result. □

Let us recall some imbedding properties for fractional Sobolev spaces.

Proposition 0.2.2 ([32], Proposition 2.1)**.** *Let $p \in (1, +\infty)$ and $0 < s \leq s' < 1$. Let Ω be an open set in $\mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ be a measurable function. Then*

$$\|u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{s',p}(\Omega)}$$

for some suitable positive constant $C = C(N, s, p) \geq 1$. In particular

$$W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega).$$

Remark 0.2.3. *We want to mention that in the Proposition above we did not assume any regular property on the boundary data $\partial\Omega$. Anyway, in the case $W^{1,p}(\Omega) \subseteq W^{s,p}(\Omega)$, the boundary $\partial\Omega$ should satisfy some Lipschitz continuity; otherwise, there exists a counterexample for the failure of the imbedding, i.e. there is the function $u \in W^{1,p}(\Omega)$ but $u \notin W^{s,p}(\Omega)$ (see [32] Example 9.1).*

Theorem 0.2.1. *Let $1 < p < +\infty$ and $s \in (0,1)$, let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Let $\{u_n\}_{n\in\mathbb{N}} \subset \widetilde{W}^{s,p}_{0,tR}(\Omega)$ be a bounded sequence, i.e.*

$$\sup_{n\in\mathbb{N}} \|u_n\|_{\widetilde{W}^{s,p}_{0,tR}(\Omega)} < +\infty. \tag{10}$$

Then there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ converging in $L^p(\Omega)$ to a function u, and $u \in \widetilde{W}^{s,p}_{0,tR}(\Omega)$.

Proof. We use the strategy in the proof of Theorem 2.7 in [17]. For completeness we give the detail below.

We first observe that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^p(\Omega)$, thanks to (10) and the Poincaré inequality (9). Then we can extend the function u_n to $B_{tR}(\Omega)$ by zero. Then in order to use the classical Riesz-Fréchet-Kolmogorov compactness theorem we need to check that

$$\lim_{|h|\to 0}\left(\sup_{n\in\mathbb{N}}\int_{B_{tR}(\Omega)} |u_n(x+h)-u_n(x)|^p dx\right) = 0.$$

By Lemma 0.2.1 and (10) we have

$$\begin{aligned}&\textstyle\int_{B_{tR}(\Omega)} |u_n(x+h)-u_n(x)|^p dx = |h|^{sp}\int_{B_{tR}(\Omega)} \frac{|u_n(x+h)-u_n(x)|^p}{|h|^{sp}} dx\\ &\le C|h|^{sp}[u]^p_{W^{s,p}_{tR}(\Omega)} \le C'|h|^{sp},\end{aligned}$$

for every $|h| < 1$. This establishes the uniform continuity desired, and we get the convergence of $\{u_{n_k}\}$ to u in $L^p(\Omega)$. As the space $\widetilde{W}^{s,p}_{0,tR}(\Omega)$ is a reflexive Banach space, so we can use the compactness to get the conclusion. □

Here we also give the imbedding in the case $N > sp$. The proof is essentially the same as Proposition 2.9 in [17]. The only difference lies in that we are working on the space $\widetilde{W}^{s,p}_{0,tR}(\Omega)$. Of course, this also works for the space $\widetilde{W}^{s,p}_{tR}(\Omega)$.

Proposition 0.2.3. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Let $s \in (0,1)$ and $p \in (1,+\infty)$ such that $N < sp$. Then for every $u \in \widetilde{W}^{s,p}_{0,tR}(\Omega)$ there holds $u \in C^{0,\gamma}(\overline{\Omega})$ with $\gamma = s - N/p$. Moreover, we have*

$$|u(x)-u(y)| \le (\beta_{N,s,p}\|u\|_{\widetilde{W}^{s,p}_{0,tR}(\Omega)})|x-y|^\gamma,\ \forall\ x,y \in B_{tR}(\Omega),$$

and

$$\|u(x)\|_{L^\infty(\Omega)} \le (\beta_{N,s,p}\|u\|_{\widetilde{W}^{s,p}_{0,tR}(\Omega)})R^\gamma,\ \forall\ x \in \Omega,$$

in which R is the diameter of Ω, defined in (6).

Proof. Let $\forall\ x_0 \in B_{tR}(\Omega)$, and $\delta > 0$ such that $B_\delta(x_0) \subset B_{tR}(\Omega)$. Then we estimate

$$\int_{B_\delta(x_0)} |u(x)-\overline{u}_{x_0,\delta}|^p dx \le \frac{1}{\mathcal{L}^N(B_\delta(x_0))}\int_{B_\delta(x_0)\times B_\delta(x_0)} |u(x)-u(y)|^p dxdy,$$

where $\overline{u}_{x_0,\delta}$ denotes the average of u on $B_\delta(x_0)$. Observing that $|x-y|\le 2\delta$ for every $x,y\in B_\delta(x_0)$ and using $\mathcal{L}^N(B_\delta(x_0))=\omega_N\delta^N$, we have

$$\int_{B_\delta(x_0)}|u(x)-\overline{u}_{x_0,\delta}|^p dx\le C\delta^{sp}[u]^p_{\widetilde{W}^{s,p}_{tR}(\Omega)},$$

namely

$$\mathcal{L}^N(B_\delta(x_0))^{-\frac{sp}{N}}\int_{B_\delta(x_0)}|u(x)-\overline{u}_{x_0,\delta}|^p dx\le C[u]^p_{\widetilde{W}^{s,p}_{tR}(\Omega)},$$

which implies that u belongs to the Campanato space (see Theorem 2.9, [39]), which is isomorphic to $C^{0,\gamma}$ with $\gamma=s-p/N$. For the last statement, just moving variable y out of Ω, then we conclude the desired result. □

Remark 0.2.4. *In the statement of Theorem 2.9 in [39], there is the assumption "without external cusps" on $\partial\Omega$, however, it is automatically satisfied in our setting, since we are working in the ball $B_{tR}(\Omega)$, not Ω itself.*

Now we give a simple proof of the imbedding of $\widetilde{W}^{s,p}_{0,tR}(\Omega)$ into $\widetilde{W}^{s,p}_0(\Omega)$.

Theorem 0.2.2 (Equivalence Theorem)**.** *Let $0<s<1$, $1<p<+\infty$ and $\Omega\subset\mathbb{R}^N$ be a bounded open set. Then there exists a constant $C=C(N,s,p,\Omega)$ such that*

$$[u]_{\widetilde{W}^{s,p}_0(\Omega)}\le C[u]_{\widetilde{W}^{s,p}_{0,4R}(\Omega)},\ \ for\ \forall\ u\in C^\infty_0(\Omega).$$

Proof. Since

$$[u]^p_{\widetilde{W}^{s,p}_0(\Omega)}=\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy,$$

we separate it into two parts as

$$V=\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy,$$

and

$$W=2\int_{(\mathbb{R}^N\setminus B_{4R}(\Omega))}\int_\Omega\frac{|u(x)|^p}{|x-y|^{N+sp}}dxdy,$$

during which, for the definition of R and $B_{4R}(\Omega)$ one can refer to (7).

Obviously V part is just the definition of $[u]_{\widetilde{W}^{s,p}_{0,4R}(\Omega)}$, then we also perform a separation on $[u]_{\widetilde{W}^{s,p}_{0,4R}(\Omega)}$, that is,

$$[u]_{\widetilde{W}^{s,p}_{0,4R}(\Omega)}=X+Y,$$

in which,

$$X=\int_{B_{\frac{3}{2}R}(\Omega)\times B_{\frac{3}{2}R}(\Omega)}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy,$$

and

$$Y = 2\int_{(B_{4R}(\Omega)\backslash B_{\frac{3}{2}R}(\Omega))}\int_{\Omega}\frac{|u(x)|^p}{|x-y|^{N+sp}}dxdy.$$

Then we mainly compare W and Y. So for W we have

$$\begin{aligned}\int_{(\mathbb{R}^N\backslash B_{4R}(\Omega))}\int_{\Omega}\tfrac{|u(x)|^p}{|x-y|^{N+sp}}dxdy &\leq N\omega^N\int_{\Omega}|u|^p dx\int_{\frac{R}{2}}^{+\infty}\tfrac{r^{N-1}}{r^{N+sp}}dr\\ &= \tfrac{N\omega^N}{sp}(\tfrac{2}{R})^{sp}\int_{\Omega}|u|^p dx,\end{aligned}$$

and for Y

$$\begin{aligned}\int_{(B_{4R}(\Omega)\backslash B_{\frac{3}{2}R}(\Omega))}\int_{\Omega}\tfrac{|u(x)|^p}{|x-y|^{N+sp}}dxdy &\geq N\omega^N\int_{\Omega}|u|^p dx\int_{\frac{5R}{4}}^{\frac{3R}{2}}\tfrac{r^{N-1}}{r^{N+sp}}dr\\ &= \tfrac{N\omega^N}{sp}\tfrac{(\frac{4}{5})^{sp}-(\frac{2}{3})^{sp}}{R^{sp}}\int_{\Omega}|u|^p dx\\ &\geq C(s,p)\tfrac{N\omega^N}{sp}(\tfrac{2}{R})^{sp}\int_{\Omega}|u|^p dx.\end{aligned}$$

Then we have $W \leq C(s,p)Y$, so we have established that $[u]_{\widetilde{W}_0^{s,p}(\Omega)} \leq C[u]_{\widetilde{W}_{0,2R}^{s,p}(\Omega)}$. $\square$

We recall the following lemma established in [17], which is also available here in our setting due to the equivalence between $\widetilde{W}_0^{s,p}(\Omega)$ and $\widetilde{W}_{0,4R}^{s,p}(\Omega)$.

Lemma 0.2.1. *Let $1 \leq p < +\infty$ and $0 < s < 1$, let $\Omega \subset \mathbb{R}^N$ be an open bounded set. For every $u \in W_0^{s,p}(\mathbb{R}^N)$ there holds*

$$\sup_{|h|>0}\int_{\mathbb{R}^N}\frac{|u(x+h)-u(x)|^p}{|h|^{sp}}dx \leq C[u]^p_{W^{s,p}(\mathbb{R}^N)},$$

for a constant $C = C(N,p,s) > 0$.

Chapter 1

Asymptotic Behaviour of Dirichlet Problems

In this chapter, we will use Γ-convergence to analyze the fractional p-Laplacian operator $(-\Delta_p)^s$. First we give a review of the research on p-Laplacian equations.

For the p-Laplacian equations, G. Aronsson initiated the research of the equations

$$\Delta_\infty = \Sigma_{i,j=1,\ldots,N} u_{x_i x_j} u_{x_i} u_{x_j} = 0 \text{ } on \text{ } \Omega. \tag{1.1}$$

It is shown that if u_p minimizes the integrals $\int_\Omega |\nabla u|^p$, then $u_p \to u$ as u solves the equation (1.1). So what happens if we replace the space $W^{1,p}(\Omega)$ by $W^{s,p}(\Omega)$ with $s \in (0,1)$? And what if s is varying?

In section 1.1 of this chapter, we study the limit of minimizers of the fractional $W^{\alpha,p}$-norms as $p \to +\infty$ in De Giorgi sense. In particular, we analyzed the Γ-convergence of non-homogeneous Dirichlet boundary problem for fractional p-Laplacian in this approximation process, and proved that as $p \to +\infty$ the minimizers of fractional p-Laplacian with Dirichlet boundary Γ-converges to a minimizer of fractional ∞-Laplacian under the same Dirichlet boundary condition.

In section 1.2, we first investigate the asymptotic behaviour of non-homogeneous fractional p-functionals with homogeneous Dirichlet boundary condition when $k \to s$ from above; then we study the approximation process of a free fractional p-functional as $k \to s$ from below, during which we will find some special phenomenon different from the case from above. Both of the way to dispose these two asymptotic directions are in the De Giorgi sense.

1.1 Asymptotics as p goes to infinity

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. It's well-known that the minimizers u_p of the integrals

$$\int_\Omega |\nabla u|^p,$$

under suitable conditions, as p→ $+\infty$, approximate to the minimizer u of the equation

$$\Delta_\infty u = \sum_{i,j=1,2,\dots,N} u_{ij}u_iu_j = 0 \ on\ \Omega,$$

with $u_i = \frac{\partial u}{\partial x_i}$ and $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, which is usually referred to as ∞-Laplacian equation, introduced by Aronsson in the fundamental work [5][6] as the Euler-Lagrange equations associated to the functional

$$\|\nabla u\|_{L^\infty(\Omega)}.$$

Here, the weak solution u to the ∞-Laplacian equation is understood in the viscosity sense. One can refer to, for instance, [3][4][5][6][7][12][1][15] [16] for the limitation discussion as $p \to +\infty$. Moreover, u is known as a local minimizer up to a Lipschitz extension, for which, one can refer to [1]. One can notice that the approximation process above is pointwise, and in [19][18] one can find another approximation approach based on Γ-convergence, which is also our concentration here.

In this section, we are concerned with the fractional case.

We study the Dirichlet problem and the minimizers of the functional

$$\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}}dxdy, \tag{1.2}$$

for $p\alpha > N$ (N is the dimension of $\mathbb{R}^N$) with $\alpha \in (0,1)$, and Ω being a bounded domain in $\mathbb{R}^N$. For the fractional Sobolev semi-norm $W^{s,p}(\Omega)$ ($s \in (0,1)$) defined as

$$[u]^p_{W^{s,p}(\Omega)} := \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy,$$

in the limit case as $p \to +\infty$, the fractional functional approximates to, formally,

$$\|\frac{|u(x)-u(y)|}{|x-y|^s}\|_{L^\infty(\Omega\times\Omega)}. \tag{1.3}$$

In general, the Euler-Lagrange equations of the fractional functional (1.2) is

$$\mathcal{L}^\alpha_p u(x) := \int_\Omega |\frac{u(x)-u(y)}{|x-y|^\alpha}|^{p-1}\frac{sgn(u(x)-u(y))}{|x-y|^\alpha}dy = 0 \ in\ \Omega. \tag{1.4}$$

In viscosity sense, as $p \to +\infty$, the equation (1.4) should converge to the Hölder ∞-Laplacian equation (refer to [29]), defined as

$$L^\alpha u = 0 \ in\ \Omega, \tag{1.5}$$

with the definition of operator L^α

$$L^\alpha u(x) := \sup_{y\in\overline{\Omega}, y\neq x} \frac{u(y)-u(x)}{|y-x|^\alpha} + \inf_{y\in\overline{\Omega}, y\neq x} \frac{u(y)-u(x)}{|y-x|^\alpha} \ for\ x \in \Omega. \tag{1.6}$$

For the research on the Dirichlet problem of Euler-Lagrange equations of functional (1.2)

$$\begin{cases} \mathcal{L}_p^\alpha u(x) = f(x) & in\ \Omega, \\ u = g & on\ \partial\Omega. \end{cases} \tag{1.7}$$

one can refer to [29][56][52]. We also want to mention that the boundary condition can be changed to the fully nonlocal case, that is $u = g$ *on* $\mathbb{R}^N \setminus \Omega$, and then we would work on the space $\widetilde{W}_0^{s,p}(\Omega)$ defined as the complete closure of $C_0^\infty(\Omega)$ under the norm $W^{s,p}(\mathbb{R}^N)$. For the research in this direction, one can refer to [37][32][59], and a final generalization comment in [29].

For the Dirichlet problem of Hölder ∞-Laplacian equations, we denote

$$\begin{cases} L^\alpha u = f\ in\ \Omega, \\ \ u = g\ on\ \partial\Omega. \end{cases} \tag{1.8}$$

In [29] one can see that under suitable conditions when $p \to +\infty$ is large enough, the weak solutions of Dirichlet problem of (2.3) converge to the weak solutions of the equations (1.8) in the viscosity sense. For the readers' convenience, we list the results below without proof.

Theorem 1.1.1 ([29] Theorem 1.1, limit equation as $p \to +\infty$)**.** *Let $\alpha \in (0, 1]$ and if $\alpha = 1$ assume $N \geq 2$. Consider a bounded Lipschitz domain Ω in $\mathbb{R}^N$, and boundary data $g \in C^{0,\alpha}(\partial\Omega)$. For any $p > 2N/\alpha$, there exists a unique minimizer u_p of (1.2) satisfying $u = g$ on $\partial\Omega$. Moreover, as $p \to +\infty$, we have $u_p \to u_\infty$ uniformly in $\bar{\Omega}$ and $u_\infty \in C^{0,\alpha}(\bar{\Omega})$ is a viscosity solution of (1.5).*

One can see that under suitable conditions the minimizers u exhibit α-Hölder continuity up to the boundary. So it is safe to assume that the boundary value $g|_{\partial\Omega}$ is α-Hölder continuous when p is large enough.

And the first half of this chapter is to investigate the convergence of fractional functional (1.2) to the infinity functional (1.3) when $p \to +\infty$ in De Giorgi sense. Then based on this, we also investigate the compatibility of non-homogeneous Dirichlet problems during the process $p \to +\infty$ of the functional (1.2).

We want seize the chance to mention the following implicit representation of viscosity solution to (1.8) when $f = 0$. We just give the statement of the theorem, and for the proof details one can refer to [29].

Theorem 1.1.2 ([29] Theorem 1.5, existence for general f, partial uniqueness)**.** *Let $\alpha \in (0, 1]$, Ω be a bounded open domain, $g \in C(\partial\Omega)$ and $f \in C(\Omega) \cap L^\infty(\Omega)$.*

- *(Existence) Then there exists a viscosity solution $u \in C(\bar{\Omega})$ of (1.8).*
- *(Partial uniqueness) Assume $f = 0$. Then the viscosity solution $u \in C(\bar{\Omega})$ of (1.8) is unique and is defined implicitly by the following:*

$$u(x) = \begin{cases} g(x) & if\ x \in \partial\Omega, \\ a\ with\ \ell_x(a) = 0 & if\ x \in \Omega, \end{cases} \tag{1.9}$$

where

$$\ell_x(a) = \sup_{y \in \partial\Omega} \frac{g(y) - a}{|y - x|^\alpha} + \inf_{y \in \partial\Omega} \frac{g(y) - a}{|y - x|^\alpha}.$$

1.1.1 Main Results

In order to neatly present the subject, we first need some definitions. The natural setting for variational functional of the operator $\mathcal{L}_p^\alpha$ in the domain $\Omega \subseteq \mathbb{R}^N$ is the space $W_0^{s,p}(\Omega)$ with $s = \alpha - N/p$, $\alpha \in (0,1)$ and $p\alpha > N$, defined as the completion of $C_0^\infty(\Omega)$ with respect to the standard Gagliardo semi-norm

$$[u]_{W^{s,p}(\Omega)} := \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy\right)^{\frac{1}{p}};$$

if $p = \infty$, the semi-norms $W^{s,\infty}(\overline{\Omega})$ and $W^{s,\infty}(\Omega)$ are respectively defined by

$$[u]_{W^{s,\infty}(\overline{\Omega})} := \sup_{x\neq y, x,y\in\overline{\Omega}} \left|\frac{u(x)-u(y)}{|x-y|^s}\right|,$$

and

$$[u]_{W^{s,\infty}(\Omega)} := \sup_{x\neq y, x,y\in\Omega} \left|\frac{u(x)-u(y)}{|x-y|^s}\right|.$$

In all that follows, for $\alpha \in (0,1)$ and $q\alpha > N$, we define $E_{\alpha,p} : L^q(\Omega) \to [0,\infty]$ by

$$E_{\alpha,p}(u) = \begin{cases} \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} dxdy\right)^{\frac{1}{p}} & if\ u \in W^{s,p}(\Omega)\ (s = \alpha - N/q), \\ \infty & otherwise. \end{cases}$$

Define $\overline{E}_{\alpha,\infty} : L^q(\Omega) \to [0,\infty]$ by

$$\overline{E}_{\alpha,\infty}(u) = \begin{cases} \sup\limits_{x\neq y, x,y\in\overline{\Omega}} \left|\frac{u(x)-u(y)}{|x-y|^\alpha}\right| & if\ u \in W^{s,\infty}(\overline{\Omega})\ (s = \alpha), \\ \infty & otherwise; \end{cases}$$

and $E_{\alpha,\infty} : L^q(\Omega) \to [0,\infty]$ by

$$E_{\alpha,\infty}(u) = \begin{cases} \sup\limits_{x\neq y, x,y\in\Omega} \left|\frac{u(x)-u(y)}{|x-y|^\alpha}\right| & if\ u \in W^{s,\infty}(\Omega)\ (s = \alpha), \\ \infty & otherwise. \end{cases}$$

The first result concerns the $\Gamma(L^q(\Omega))$-convergence of the functional

$$E_{\alpha,q}$$

to the α-infinity functional

$$E_{\alpha,\infty} \quad and \quad \overline{E}_{\alpha,\infty}$$

respectively, as $u_p \to u$ in $L^q(\Omega)$ strongly for different suitable $q > \frac{N}{\alpha}$.

Theorem 1.1.3 (Asymptotic behaviour of $p \to +\infty$). *Let $\alpha \in (0,1)$ and Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. We consider $\{p\}_p$ as a strictly increasing sequence going to $+\infty$. Then we have*

(i) $\Gamma(L^q(\Omega))\text{-}\lim\limits_{p\to+\infty} E_{\alpha,p} = E_{\alpha,\infty}$ *with some* $q > \frac{N}{\alpha}$*;*

(ii) $\Gamma(L^q(\Omega))\text{-}\lim\limits_{p\to+\infty} E_{\alpha,p} = \overline{E}_{\alpha,\infty}$ *with some* $q > \frac{2N}{\alpha}$*.*

The proof of this theorem follows from Proposition 1.1.1 and Proposition 1.1.2 below.

Remark 1.1.1. *The reason why we utilize here Ω being a domain, not more general as an open bounded set in $\mathbb{R}^N$, is that we would use the compact imbedding theorem for fractional sobolev space $W^{s,p}(\Omega)$, which to our best knowledge, is valid only for domain (see [29][32].*

Remark 1.1.2. *For recent application of Γ-convergence in other situations of the fractional case, one can refer to [61][2][22]. For a general introduction of Γ-convergence, one can refer to [30][23].*

Remark 1.1.3. [1] *One can also find a similar result in [21], which established a approximation to Hölder infinity Laplacian equation by Γ-convergence by Orlicz fractional Laplacians (see Theorem 5.2 therein).*

We may apply the Γ-limit of "free" energy results above to minimum of the form

$$m_\epsilon = \inf\left\{\int_\Omega f_\epsilon(x,u,D^s u)dx - \int_\Omega \langle g,u\rangle dx : u = \varphi \text{ on } \partial\Omega\right\}, \tag{1.10}$$

during which Ω stands for a bounded (smooth enough) domain of $\mathbb{R}^N$ and $s \in (0,1)$, and D^s denotes a fractional differential operator.

Applications of Γ-convergence to PDEs can be generally related to the behavior of the Euler-Lagrange equations. Notice that the possibility of defining a Γ-limit related to these problems will not be linked to the properties (or even the existence) of the solutions of the related Euler-Lagrange equations ([23]). So the existence and uniqueness of the minimizer of the limitation energy does not imply corresponding uniqueness of the solutions to the limitation Euler-Lagrange equations (Theorem 1.1.2). In fact, by [40][70], one can see that the equations

$$\begin{cases} (-\Delta)^s u = 0, & \text{in } \Omega,\\ u = g, & \text{on } \partial\Omega,\end{cases} \tag{1.11}$$

is ill-posedness, but of course we can find a minimizer of its variational form

$$\min_{u|_{\partial\Omega}=g} \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy.$$

In particular in this chapter, we can only state that the minimizer sequence would convergence to a minimizer of the limitation functional, but there are also many other extensions and characterizations of the minimizer (see, e.g., [8][28][29]).

[1] This notification is attributed to Professor Terasawa.

We can see that in the functional (1.10) there exist two other terms: the force term g and the boundary φ. Anyway even if we have established the Γ-convergence for the functional $E_{\alpha,p}(u)$, we can only get immediately the same convergence result for the minimizers of such functionals in the same space, but not for the minimum problems with non-homogeneous Dirichlet boundary conditions. So we have to verify the compatibility of the condition $u=\varphi$ on $\partial\Omega$, which is our next main result in this section.

For the preparation of the investigation of the compatibility of the Dirichlet boundary conditions, we give some definitions first. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, $0<\alpha<1$ and $p>\frac{2N}{\alpha}$. Now with $\varphi \in C^{0,\alpha}(\Omega)$ we define some admissible function sets

$$X^{\varphi}_{\alpha,p}(\Omega) := \{u : \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} dxdy\right)^{\frac{1}{p}} < +\infty,\ u=\varphi \ on\ \partial\Omega\},$$

and

$$X^{\varphi}_{\alpha,\infty}(\overline{\Omega}) := \{u : \sup_{x,y\in\overline{\Omega}, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} < +\infty,\ u=\varphi \ on\ \partial\Omega\}.$$

The energy integrals are defined as follows:

$$E^{\varphi}_{\alpha,p}(u) = \begin{cases} \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} dxdy\right)^{\frac{1}{p}}, & if\ u\in X^{\varphi}_{\alpha,p}(\Omega), \\ \infty & otherwise; \end{cases} \tag{1.12}$$

and

$$\overline{E}^{\varphi}_{\alpha,\infty}(u) = \begin{cases} \sup_{x,y\in\overline{\Omega}, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}}, & if\ u\in X^{\varphi}_{\alpha,\infty}(\overline{\Omega}), \\ \infty & otherwise. \end{cases} \tag{1.13}$$

Since when p is large enough, we have $W^{s,p}(\Omega)$ imbedded in $C^{0,s-\frac{N}{p}}(\overline{\Omega})$ compactly, so functions in $W^{s,p}(\Omega)$ become continuous automatically up to the boundary. Then on the existence and uniqueness of minimizers for functionals $E^{\varphi}_{\alpha,p}(u_p)$ $(p>\frac{2N}{\alpha})$, one can refer to ([29], Lemma 6.3). For the completeness, we state the lemma here without proof.

Lemma 1.1.1 ([29] Lemma 6.3, existence and uniqueness of minimizer)**.** *let $\alpha\in(0,1]$ and assume that Ω is a bounded Lipschitz domain. Consider $\varphi\in C^{0,\alpha}(\partial\Omega)$ and define the set*

$$X_{\varphi}(\Omega) := \{u\in C(\overline{\Omega}),\ u=\varphi \ on\ \partial\Omega\}.$$

Define the minimization problem

$$I = \inf_{u\in X_{\varphi}(\Omega)} E_p(u),$$

where

$$E_p(u) = \int_{\Omega\times\Omega} |\frac{u(x)-u(y)}{|x-y|^{\alpha}}|^p dxdy.$$

Then for any $p > \frac{2N}{\alpha}$, problem I has a unique minimizer u_p. Moreover, for any function $\phi \in C_c^\infty(\Omega)$, we have

$$\int_{\Omega\times\Omega} |\tfrac{u_p(x)-u_p(y)}{|x-y|^\alpha}|^{p-1} \left\{ \tfrac{sgn(u_p(y)-u_p(x))}{|y-x|^\alpha} \right\} (\phi(y)-\phi(x))dxdy = 0.$$

Now we give another main result in this section:

Theorem 1.1.4 (Compatibility of Dirichlet boundary)**.** *Let $\alpha \in (0,1)$ and $q > \frac{2N}{\alpha}$. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and $\varphi \in C^{0,\alpha}(\partial\Omega)$. Then we have*

(i) $\Gamma(L^q(\Omega))\text{-}\lim_{p\to+\infty} E^\varphi_{\alpha,p} = \overline{E}^\varphi_{\alpha,\infty}$;

(ii)[2] Let $\{u_p\}_p$ is the minimizer sequence of the functional sequence $\{E^\varphi_{\alpha,p}\}_p$. If $u_p \to u$ in $L^q(\Omega)$ strongly and

$$\Gamma(L^q(\Omega)) - \lim_{p\to+\infty} E^\varphi_{\alpha,p} = \overline{E}^\varphi_{\alpha,\infty},$$

then u is a minimizer of $\overline{E}^\varphi_{\alpha,\infty}$ in $X^\varphi_{\alpha,\infty}(\overline{\Omega})$.

1.1.2 Proof of Theorem 1.1.3

Proposition 1.1.1 ($\Gamma - \limsup$ inequality)**.** *Let $\alpha \in (0,1)$ and $q > \frac{N}{\alpha}$, and let $u \in L^q(\Omega)$. Let $\{p\}$ be a sequence of strictly increasing positive numbers going to $+\infty$. Then there exists a sequence $\{u_p\}_p$ converging to u in $L^q(\Omega)$ such that*

$$\limsup_{p\to+\infty} E_{\alpha,p}(u_p) \le E_{\alpha,\infty}(u) \le \overline{E}_{\alpha,\infty}(u).$$

Proof. If $\overline{E}_{\alpha,\infty}(u) = +\infty$, the inequality is satisfied automatically, so there is noting to prove. Thus let us take $\overline{E}_{\alpha,\infty}(u) < +\infty$.

Now we will find a "recovery sequence" to verify the condition (ii) of the Γ-convergence equality. Let us consider the sequence $\{u_p\}_p \subset L^q(\Omega)$, where $u_p := u$ for all $p \ge 1$. Then we have

$$\begin{aligned} &\limsup_{p\to+\infty} \left(\int_{\Omega\times\Omega} \tfrac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} dxdy\right)^{\frac{1}{p}} \\ &\le \lim_{p\to+\infty} \left(\int_{\Omega\times\Omega} \Big(\sup_{x\neq y, x,y\in\Omega} \tfrac{|u(x)-u(y)|}{|x-y|^\alpha}\Big)^p dxdy \right)^{\frac{1}{p}}, \end{aligned}$$

and then

$$\begin{aligned} \limsup_{p\to+\infty} \left(\int_{\Omega\times\Omega} \tfrac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} dxdy\right)^{\frac{1}{p}} &\le \sup_{x\neq y, x,y\in\Omega} \tfrac{|u(x)-u(y)|}{|x-y|^\alpha} = E_{\alpha,\infty}(u) \\ &\le \sup_{x\neq y, x,y\in\overline{\Omega}} |\tfrac{u(x)-u(y)}{|x-y|^\alpha}| = \overline{E}_{\alpha,\infty}(u), \end{aligned}$$

which concludes the desired result. ∎

[2] Many thanks to Professor Terasawa for pointing out the already existing reference on this result. See [45] Corollary 6.1.1, which is more general.

Now we attempt to verify the condition (i) in the Definition (0.1.1).

In this thesis, we use $\mathcal{L}^N(U)$ to denote the N-dimensional Lebesgue measure of the measurable set $U \subset \mathbb{R}^N$.

Proposition 1.1.2 ($\Gamma - \liminf$ inequality)**.** *Let $\alpha \in (0,1)$, $q > \frac{N}{\alpha}$, $u \in L^q(\Omega)$, and let $\{p\}$ be a sequence of strictly increasing positive numbers going to $+\infty$. Consider any sequence $\{u_p\}_p \subset L^q(\Omega)$ converging to u in $L^q(\Omega)$, then we have*

$$E_{\alpha,\infty}(u) \leq \liminf_{p\to+\infty} E_{\alpha,p}(u_p).$$

And if $q > \frac{2N}{\alpha}$, we have

$$\overline{E}_{\alpha,\infty}(u) \leq \liminf_{p\to+\infty} E_{\alpha,p}(u_p).$$

Proof. **Step 1.** If $\liminf\limits_{p\to+\infty} E_{\alpha,p}(u_p) = \infty$, the inequality is satisfied automatically, so there is nothing to prove.

Then let us suppose that $\liminf\limits_{p\to+\infty} E_{\alpha,p}(u_p) < \infty$. Then we can infer that there exists $L > 0$ such that $\liminf\limits_{p\to+\infty} E_{\alpha,p}(u_p)$ is uniformly bounded, i.e.

$$\liminf_{p\to+\infty} E_{\alpha,p}(u_p) \leq L,$$

and based on a subsequence $\{u_{p_n}\}$ of $\{u_p\}$ we have

$$\lim_{n\to+\infty} E_{\alpha,p_n}(u_{p_n}) = \liminf_{p\to+\infty} E_{\alpha,p}(u_p) \leq L,$$

For convenience we still denote the sequence $\{p_n\}$ by $\{p\}$.

Since $p \to +\infty$, then for p large enough there holds $p > q$. Then by Hölder inequality, we have that

$$\begin{aligned}\left(\int_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^q}{|x-y|^{\alpha q}} dxdy\right)^{1/q} &\leq C(\Omega,N)\left(\int_{\Omega\times\Omega} \frac{|u_p(x)-u_p(y)|^p}{|x-y|^{\alpha p}} dxdy\right)^{1/p}\\ &\leq C(\Omega,N)L\end{aligned}$$

As $u_p \to u$ strongly in $L^q(\Omega)$ for every p, in view of Poincaré-Wirtinger inequality, we have that u_p is uniformly bounded in $W^{\alpha-\frac{N}{q},q}(\Omega)$. Then we can extract a subsequence u_p (not relabelled) such that

$$u_p \rightharpoonup u$$

weakly in $W^{\alpha-\frac{N}{q},q}(\Omega)$. Then by the lower semi-continuity of $W^{\alpha-\frac{N}{q},q}(\Omega)$ and Hölder inequality we have

$$\begin{aligned}\int_{\Omega\times\Omega} \left|\frac{u(x)-u(y)}{|x-y|^{\alpha}}\right|^q dxdy &\leq \lim_{n\to+\infty} \int_{\Omega\times\Omega} \left|\frac{u_p(x)-u_p(y)}{|x-y|^{\alpha}}\right|^q dxdy\\ &\leq \lim_{n\to+\infty} \mathcal{L}^N(\Omega)^2 \left(\int_{\Omega\times\Omega} \left|\frac{u_p(x)-u_p(y)}{|x-y|^{\alpha}}\right|^p dxdy\right)^{\frac{q}{p}}.\end{aligned}$$

This means

$$\left(\int_{\Omega\times\Omega}|\frac{u(x)-u(y)}{|x-y|^{\alpha}}|^{q}dxdy\right)^{\frac{1}{q}} \leq \mathcal{L}^N(\Omega)^{2/q}\lim_{n\to+\infty}\left(\int_{\Omega\times\Omega}|\frac{u(x)-u(y)}{|x-y|^{\alpha}}|^{p}dxdy\right)^{\frac{1}{p}}.$$

Therefore letting $q\to+\infty$ we obtain

$$E_{\alpha,\infty}(u)\leq\lim_{p\to+\infty}E_{\alpha,p}(u_p),$$

which implies for the original sequence $\{p\}$

$$E_{\alpha,\infty}(u)\leq\liminf_{p\to+\infty}E_{\alpha,p}(u_p).$$

Then if $q>2N/\alpha$, then by Sobolev compact imbedding theorem, we have

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})}\leq C\|u\|_{W^{\alpha-\frac{N}{q},q}(\Omega)},$$

where $\gamma=\alpha-\frac{2N}{q}$, and

$$C^{0,\gamma}(\overline{\Omega}):=\{f\in C(\overline{\Omega}),\ \|f\|_{L^\infty(\overline{\Omega})}+\sup_{x\neq y,x,y\in\overline{\Omega}}\frac{|f(x)-f(y)|}{|x-y|^{\gamma}}<+\infty\}.$$

So u is continuous up to the boundary, then for every boundary point x_0 on $\partial\Omega$, we can find a sequence $\{x_m\}\subset\Omega$ ($m\in\mathbb{N}$) such that $\lim_{m\to+\infty}|x_m-x_0|=0$ and $|u(x_m)-u(x_0)|<\epsilon$ for $\forall\ \epsilon>0$ when m is large enough. Then for any $y\in\Omega$

$$\begin{aligned}\frac{|u(x_0)-u(y)|}{|x_0-y|^{\alpha}} &= \frac{|u(x_0)-u(x_m)+u(x_m)-u(y)|}{|x_0-y|^{\alpha}}\\ &\leq \frac{|u(x_0)-u(x_m)|}{|x_0-y|^{\alpha}}+\frac{|u(x_m)-u(y)|}{|x_0-y|^{\alpha}}\\ &\leq \epsilon+\frac{|u(x_m)-u(y)|}{|x_0-y|^{\alpha}}\\ &\leq \sup_{x\neq y,x,y\in\Omega}|\frac{u(x)-u(y)}{|x-y|^{\alpha}}|+\epsilon.\end{aligned}$$

Since ϵ is arbitrary, we conclude that

$$\sup_{x\neq y,x,y\in\overline{\Omega}}|\frac{u(x)-u(y)}{|x-y|^{\alpha}}|\leq\sup_{x\neq y,x,y\in\Omega}|\frac{u(x)-u(y)}{|x-y|^{\alpha}}|,$$

and obviously

$$\sup_{x\neq y,x,y\in\Omega}|\frac{u(x)-u(y)}{|x-y|^{\alpha}}|\leq\sup_{x\neq y,x,y\in\overline{\Omega}}|\frac{u(x)-u(y)}{|x-y|^{\alpha}}|,$$

which concludes the desired result

$$\overline{E}_{\alpha,\infty}(u)\leq\liminf_{p\to+\infty}E_{\alpha,p}(u_p).$$

□

1.1.3 Proof of Theorem 1.1.4

In this section, based on the Γ-convergence results established in Theorem 1.1.3, we verify the compatibility of non-homogeneous of Dirichlet condition for the fractional Laplacian functional (1.2) as $p \to +\infty$.

Proof. **Step 1.** Now we firstly verify the lim inf inequality in the definition of Γ-convergence for

$$\Gamma(L^q(\Omega)) - \lim_{p\to+\infty} E^{\varphi}_{\alpha,p} = E^{\varphi}_{\alpha,\infty}.$$

So let $u_p \to u$ in $L^q(\Omega)$. Then if

$$\liminf_{p\to+\infty} E^{\varphi}_{\alpha,p}(u_p) = +\infty,$$

there is nothing to prove. So we may directly assume that for a sequence $\{u_p\} \subset L^q(\Omega)$ such that

$$\liminf_{p\to+\infty} E^{\varphi}_{\alpha,p}(u_p) < +\infty.$$

Then we can extract a subsequence (not relabelled) $\{u_p\}$ and there exists some $L > 0$ such that

$$\lim_{p\to+\infty} E^{\varphi}_{\alpha,p}(u_p) < L,$$

which implies that sequence $\{u_p\}$ is uniformly bounded in $W^{\alpha-\frac{N}{q},q}(\Omega)$ by Sobolev imbedding theorem.

Then as in the step 1 of proof to Proposition 1.1.2, since $u_p \to u$ in $L^q(\Omega)$ strongly, by Poincaré-Wirtinger inequality, we have $u_p \rightharpoonup u$ weakly in $W^{\alpha-\frac{N}{q},q}(\Omega)$. Since for any $p \in (q, +\infty)$, $u_p \in X^{\varphi}_{\alpha,p}(\Omega)$ and by Sobolev imbedding theorem $X^{\varphi}_{\alpha,p}(\Omega) \hookrightarrow X^{\varphi}_{\alpha,q}$, we infer that $\{u_p\} \subset X^{\varphi}_{\alpha,q}$ for $p > q$. Since $X^{\varphi}_{\alpha,q}$ is closed in $W^{\alpha-\frac{N}{q},q}(\Omega)$, then by the reflexivity of $W^{\alpha-\frac{N}{q},q}(\Omega)$, we get that $u \in X^{\varphi}_{\alpha,q}(\Omega)$. Then following the same process as step 2 in the proof of Proposition 1.1.2, we have that for any sequence $u_p \to u$ in $L^q(\Omega)$

$$\overline{E}^{\varphi}_{\alpha,\infty}(u) \leq \liminf_{p\to+\infty} E^{\varphi}_{\alpha,p}(u_p).$$

The only difference from Proposition 1.1.2 is that the function space $X^{\varphi}_{\alpha,q}(\Omega)$: since $q > \frac{2N}{\alpha}$, by Sobolev imbedding theorem $X^{\varphi}_{\alpha,q}(\Omega) \hookrightarrow C(\overline{\Omega})$, so we can directly get the estimates for $\overline{E}^{\varphi}_{\alpha,\infty}(u)$.

Step 2. Now we are in the position to verify the recovery sequence condition in the Γ-convergence definition for

$$\Gamma(L^q(\Omega)) - \lim_{p\to+\infty} E^{\varphi}_{\alpha,p} = \overline{E}^{\varphi}_{\alpha,\infty}.$$

that is, to find a sequence $\{u_p\} \subset X^{\varphi}_{\alpha,p}(\Omega)$ such that for any $u \in L^q(\Omega)$

$$\overline{E}^{\varphi}_{\alpha,\infty}(u) \geq \limsup_{p\to+\infty} E^{\varphi}_{\alpha,p}(u_p), \tag{1.14}$$

and

$$u_p \to u \ in \ L^q(\Omega).$$

In fact, as in Proposition 1.1.1, we can directly let $u_p := u$ in $X^{\varphi}_{\alpha,p}(\Omega)$. Indeed, as $u \in X^{\varphi}_{\alpha,\infty}(\overline{\Omega})$, then $u \in X_{\varphi}(\Omega)$ (defined in Lemma 1.1.1), and by Sobolev imbedding theorem we infer that $E_{\alpha,p}(u)$ is bounded. Because $u \in X^{\varphi}_{\alpha,\infty}(\overline{\Omega})$, which is up to the boundary, and when $p > \frac{2N}{\alpha}$, $u \in C^{0,\alpha-\frac{2N}{p}}(\overline{\Omega})$, we can infer that $u \in X^{\varphi}_{\alpha,p}(\Omega)$. Then

$$\begin{aligned}\left\{\limsup_{p\to+\infty} \int_{\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} dxdy\right\}^{1/p} &\leq \limsup_{p\to+\infty} (\mathcal{L}^N(\Omega))^{\frac{2}{p}} \left\{[u]^p_{W^{\alpha,\infty}(\overline{\Omega})}\right\}^{1/p} \\ &= [u]_{W^{\alpha,\infty}(\overline{\Omega})},\end{aligned}$$

which concludes the results together with step 1.

Step 3. Next we prove (ii) of Theorem 1.1.4. We claim that $u \in X^{\varphi}_{\alpha,\infty}(\overline{\Omega})$ is a minimizer of functional $\overline{E}^{\varphi}_{\alpha,\infty}$ given any $v \in X^{\varphi}_{\alpha,\infty}(\overline{\Omega})$.

Supposing that the sequence $\{E^{\varphi}_{\alpha,p}\}_p$ $\Gamma(L^q(\Omega))$-converges to $\overline{E}^{\varphi}_{\alpha,\infty}$ at v, then by the definition of Γ-convergence, there exists a sequence $\{\omega_p\}_p$ such that

$$\omega_p \to v \ in \ L^q(\Omega), \ as \ p \to +\infty,$$

and

$$\limsup_{p\to+\infty} E^{\varphi}_{\alpha,p}(\omega_p) \leq E^{\varphi}_{\alpha,\infty}(v). \tag{1.15}$$

Since by assumption the sequence $\{u_p\}_p$ are the minimizers of $E^{\varphi}_{\alpha,p}$ in $X_{\varphi}(\Omega)$ for corresponding p, we infer that

$$E^{\varphi}_{\alpha,p}(u_p) \leq E^{\varphi}_{\alpha,p}(\omega_p).$$

Thus we have

$$\liminf_{p\to+\infty} E^{\varphi}_{\alpha,p}(u_p) \leq \limsup_{p\to+\infty} E^{\varphi}_{\alpha,p}(u_p) \leq \limsup_{p\to+\infty} \overline{E}^{\varphi}_{\alpha,\infty}(\omega_p). \tag{1.16}$$

So combining (1.15) and (1.16) yields that

$$\overline{E}^{\varphi}_{\alpha,\infty}(u) \leq \overline{E}^{\varphi}_{\alpha,\infty}(v),$$

which concludes the proof. □

1.2 Asymptotic Behaviour on Varying s

In this section, by Γ-convergence we investigate the behaviour of the operator $(-\Delta_p)^s$ under homogeneous Dirichlet boundary condition as s varies. For the case $k \to s$ from above, we assume that $0 < s < k < 1$, $p \in (1, +\infty)$ and Ω being an open bounded set in $\mathbb{R}^N$

without regularity assumption on $\partial\Omega$. In order to investigate the asymptotics smoothly, we utilize the *relative-nonlocal* Sobolev space $\widetilde{W}^{s,p}_{0,tR}(\Omega)$ introduced in preliminary part, in which, $t>1$ and R is the diameter of Ω. And $B_{tR}(\Omega)$ is define as the N-dimensional ball with diameter tR located at the same center as the smallest ball containing Ω. Then in some admissible space Y, we investigate the asymptotic behaviours of the functionals

$$\min_{u\in Y}\left(\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\frac{|u(x)-u(y)|^p}{|x-y|^{N+kp}}dxdy+\int_\Omega fudx\right),$$

when k decreases to some $s\in(0,1)$.

Then under the case $k\to s$ from below, we assume that $0<k<s<1$, $p\in(1,+\infty)$ and Ω being an open bounded set in $\mathbb{R}^N$, without further regularity assumption on $\partial\Omega$. Then inspired by [34] (see also [51]), we construct a space $W^{s^-,p}_0(\Omega)$, to study the free functional

$$\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{N+kp}}dxdy$$

when k increases to some $s\in(0,1)$. We will see that we can not get a ideal result as in the case approximating from above. And as a byproduct, we give a description of the equivalence between the spaces $W^{s^-,p}_0(\Omega)$ and $W^{s,p}_0(\Omega)$ in De Giorgi sense. For more information on this topic, one can see [34][51].

Let $0<s<1$, $p\in(1,+\infty)$, and $\Omega\subset\mathbb{R}^N$ be an open bounded set. We consider the nonlocal nonlinear operator $(-\Delta_p)^s u$ interpreted as

$$(-\Delta_p)^s u(x):=2\lim_{\epsilon\searrow0}\int_{\mathbb{R}^N\setminus B_\epsilon(x)}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}dy,\ x\in\mathbb{R}^N.$$

For more information on this operator we refer the reader to [2][17][37][56][22][32][59].

Firstly, we give a glimpse of the operator $(-\Delta_p)^s$ acting on the space $W^{s,p}_0(\Omega)$, which is defined as the closure of $C_0^\infty(\Omega)$ under the semi-norm

$$[u]_{W^{s,p}(\Omega)}=\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy,$$

which is in fact also a norm in $W^{s,p}_0(\Omega)$. If $p=2$, the operator becomes liner case $(-\Delta)^s$ and the corresponding equations is fractional Laplacian, denoted as

$$\begin{cases}(-\Delta)^s u=f, & in\ \Omega,\\ u=g, & on\ \partial\Omega.\end{cases}\tag{1.17}$$

However, by [40][70], even for the homogeneous case $f=0$, the equation (1.17) is not well-posedness compared with the well-posedness non-homogeneous Dirichlet boundary condition given by

$$\begin{cases}(-\Delta)^s u=0, & in\ \Omega,\\ u=g, & on\ \mathbb{R}^N\setminus\Omega.\end{cases}\tag{1.18}$$

In other words, the ill-posedness of (1.17) and the well-posedness of (1.18) show that an $(-\Delta)^s$ function in a domain Ω cannot be determined only by its value on the boundary $\partial\Omega$, but depends on its value on the whole area $\mathbb{R}^N \setminus \Omega$. For more details in this direction, one can see such as [14][10][64][65][66][67] etc. and references therein.

Then based on the information above, we utilize the admissible space for the operator $(-\Delta_p)^s$, the nonlocal Sobolev space $\widetilde{W}_0^{s,p}(\Omega)$ defined as the closure of $C_0^\infty(\Omega)$ under the semi-norm

$$[u]_{W^{s,p}(\mathbb{R}^N)} = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy,$$

which is in fact also a norm in $\widetilde{W}_0^{s,p}(\Omega)$. As it is declared in the preliminary if the boundary $\partial\Omega$ regular enough, such as Lipschitz, the space $\widetilde{W}_0^{s,p}(\Omega)$ is in coincidence with $W_0^{s,p}(\Omega)$, i.e., Ω can be extensible. And in this section, we do not assume any regularity on $\partial\Omega$.

However, if we work in the space $\widetilde{W}_0^{s,p}(\Omega)$, we would find that it seems a little difficult to get uniform comparison estimations for a pair of s and s', not to mention a sequence of s_j.

Then for our special problem setting here, we utilize a *relative-nonlocal* Sobolev space denoted as $\widetilde{W}_{0,tR}^{s,p}(\Omega)$, in which, t is large than 1, and R denotes the diameter of Ω, defined by

$$R := \sup_{x,y\in\Omega} \{|x-y| : \forall\ x, y \in \Omega\}.$$

This is a reflexive Banach space for $1 < p < +\infty$. For more details of this space, we refer the reader to the preliminary part.

1.2.1 Γ-convergence as $s_j \to s$ from Above

In this subsection, we use Γ-convergence to investigate the asymptotic behaviour of the following equations with varying s,

$$\begin{cases} (-\Delta_p)^s u = f, & in\ \Omega, \\ u = 0, & on\ B_{tR}(\Omega) \setminus \Omega, \end{cases} \tag{1.19}$$

in the weak sense as

$$\begin{cases} u \in \widetilde{W}_{0,tR}^{s,p}(\Omega), \\ \int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \frac{|u(x)-u(y)|^{p-s}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} dxdy \\ = \int_\Omega fvdx,\ for\ every\ v \in \widetilde{W}_{0,tR}^{s,p}(\Omega), \end{cases}$$

for which, the variational form is

$$\min_{u\in \widetilde{W}_{0,tR}^{s,p}(\Omega)} \left(\frac{1}{p} \int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy + \int_\Omega fudx \right).$$

For the existence and uniqueness of solutions to this equation, one can refer to [37][56][32]. In fact, it is a very standard approach based on the direct method and strict convexity of the semi-norm $W_{tR}^{s,p}(\Omega)$.

For every $0 < s < 1$ and $p \in (1, +\infty)$, let us define the functional $F_s(u)$ as

$$F_s(u) = \frac{1}{p} \int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy + \int_\Omega f u dx.$$

Let $\forall\ 0 < \epsilon \ll 1$, $0 < s < 1$ and $p \in (1, +\infty)$, and let $(\widetilde{W}^{s+\epsilon,p}_{0,tR}(\Omega))^*$ denote the usual dual space of the space $\widetilde{W}^{s+\epsilon,p}_{0,tR}(\Omega)$. Since $\widetilde{W}^{s+\epsilon,p}_{0,tR}(\Omega) \hookrightarrow \widetilde{W}^{s,p}_{0,tR}(\Omega)$, we have

$$(\widetilde{W}^{s,p}_{0,tR}(\Omega))^* \hookrightarrow (\widetilde{W}^{s+\epsilon,p}_{0,tR}(\Omega))^*.$$

A sequence $\{F_k\}_k$ is said to be $equi-coercive$ if there exist a compact set $K \subset X$ such that

$$\inf_X F_k = \inf_K F_k$$

for each $k \in \mathbb{N}$ (see [30][23]).

In the following theorem, we give the Γ-convergence on functionals $F_s(u)$.

Theorem 1.2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $0 < s < 1$, $p \in (1, +\infty)$, and for $f \in (\widetilde{W}^{s,p}_{0,tR}(\Omega))^*$. If $\{s_j\}_j \subset (0,1)$ be non-increasing sequence converging to s, then the sequence $\{F_{s_j}\}_j$ defined on $L^p(\Omega)$ is equi-coercive in $L^p(\Omega)$, and $F_{s_j}(u)$ Γ-converges to $F_s(u)$ in $L^p(\Omega)$ at every $u \in L^p(\Omega)$ which satisfies*

$$\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+(s+\epsilon)p}} dxdy < +\infty.$$

Proof. We observe that it is obviously that the infimum of each F_{s_j} is attained in $\widetilde{W}^{s_j,p}_{0,tR}(\Omega)$. It is well known that the Fréchet derivative of F_{s_j} (i.e. Euler-Lagrange forms) is the functional on $\widetilde{W}^{s_j,p}_{0,tR}(\Omega)$ given by

$$v \to \int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+s_jp}} dxdy + \langle f, v\rangle,$$

in which $\langle f, v\rangle$ denotes the usual dual product. The unique minimizer of F_{s_j} in $\widetilde{W}^{s_j,p}_{0,tR}(\Omega)$ is just the solution u_{s_j} to (2.3) (see e.g. [37][56][59]). Due to the Rellich-Kondrachov compact embedding theorems, the closure $\overline{K}$ in $L^p(\Omega)$ of the set $K := \{u_{s_j}, j \in \mathbb{N}\}$ is compact. Again from the discussion above we infer that

$$\inf_{L^p(\Omega)} F_{s_j} = F(u_{s_j}) = \inf_{\overline{K}} F_{s_j}$$

for each $j \in \mathbb{N}$. Then the sequence $\{F_{s_j}\}_j$ is equi-coercive. For more information on other equivalent conditions one can see Chapter 2 and 7 in [30].

Now we consider a sequence $\{w_{s_j}\}_j$ in $L^p(\Omega)$ that converges to w in $L^p(\Omega)$. If

$$\liminf_{j\to+\infty} \left(\frac{1}{p} \int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \frac{|w_{s_j}(x)-w_{s_j}(y)|^p}{|x-y|^{N+s_jp}} dxdy + \langle f, w_{s_j}\rangle \right) < +\infty,$$

one can extract a subsequence (not relabelled) $\{w_{s_j}\}_j$ for which

$$\begin{aligned}&\lim_{j\to+\infty}\tfrac{1}{p}\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\tfrac{|w_{s_j}(x)-w_{s_j}(y)|^p}{|x-y|^{N+s_jp}}dxdy+\langle f,w_{s_j}\rangle\\&=\liminf_{j\to+\infty}\left(\tfrac{1}{p}\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\tfrac{|w_{s_j}(x)-w_{s_j}(y)|^p}{|x-y|^{N+s_jp}}dxdy+\langle f,w_{s_j}\rangle\right)\\&=L<+\infty.\end{aligned}$$

Since $s\le s_j$ and $f\in(\widetilde{W}^{s,p}_{0,tR}(\Omega))^*$, one can easily have that

$$\frac{1}{p}\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\frac{|w_{s_j}(x)-w_{s_j}(y)|^p}{|x-y|^{N+s_jp}}dxdy\le C\left(1+\|f\|_{(\widetilde{W}^{s_j,p}_{0,tR}(\Omega))^*}\|w_{s_j}\|_{\widetilde{W}^{s_j,p}_{0,tR}(\Omega)}\right)$$

for some positive constant C and each $j\in\mathbb{N}$. Then by Young's inequality we have that the sequence $\{w_{s_j}\}$ is uniformly bounded in $\widetilde{W}^{s,p}_{0,tR}(\Omega)$ by Sobolev-type embedding (see [35][51]). Then thanks to the reflexivity of the space $\widetilde{W}^{s,p}_{0,tR}(\Omega)$ together with again the Sobolev-type embedding we have that $w\in\widetilde{W}^{s,p}_{0,tR}(\Omega)$. Then without loss of generality one can consider the weak convergence in $\widetilde{W}^{s,p}_{0,tR}(\Omega)$

$$w_{s_j}\rightharpoonup w.$$

For simplicity, we denote the diameter of Ω by R. Then by the weak lower semi-continuity one has

$$\begin{aligned}&\tfrac{1}{p}\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\tfrac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}dxdy\\&\le\liminf_{j\to+\infty}\tfrac{1}{p}\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\tfrac{|w_{s_j}(x)-w_{s_j}(y)|^p}{|x-y|^{N+sp}}dxdy\\&\le\liminf_{j\to+\infty}\tfrac{1}{p}tR^{(s_j-s)p}\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\tfrac{|w_{s_j}(x)-w_{s_j}(y)|^p}{|x-y|^{N+s_jp}}dxdy\\&=\liminf_{j\to+\infty}(\tfrac{1}{p}tR^{(s_j-s)p}\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\tfrac{|w_{s_j}(x)-w_{s_j}(y)|^p}{|x-y|^{N+s_jp}}dxdy\\&+\langle f,w_{s_j}\rangle-\langle f,w_{s_j}\rangle)\\&=L-\langle f,w\rangle.\end{aligned}$$

In fact, if we check the process above carefully, let $F_s(w)=+\infty$, then

$$\liminf_{j\to+\infty}\frac{1}{p}\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\frac{|w_{s_j}(x)-w_{s_j}(y)|^p}{|x-y|^{N+s_jp}}dxdy+\langle f,w_{s_j}\rangle=+\infty;$$

if it is this case, then obviously

$$F_s(w)=+\infty=\liminf_{j\to+\infty}\frac{1}{p}\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\frac{|w_{s_j}(x)-w_{s_j}(y)|^p}{|x-y|^{N+s_jp}}dxdy+\langle f,w_{s_j}\rangle.$$

So it follows from above arguments that if $w_{s_j}\to w$ in $L^p(\Omega)$, then we have

$$F_{s_j}(w_{s_j})\to F_s(w)\ in\ [0,+\infty].$$

We can complete the proof of the Γ-convergence by observing that for each $u \in L^p(\Omega)$,

$$\begin{aligned}&\lim_{j\to+\infty} F_{s_j}(u) = \lim_{j\to+\infty} \tfrac{1}{p}\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \tfrac{|u(x)-u(y)|^p}{|x-y|^{N+s_jp}}dxdy + \langle f, u\rangle \\ &= \tfrac{1}{p}\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \tfrac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy + \langle f, u\rangle = F_s(u).\end{aligned}$$

□

1.2.2 Γ-convergence as $s_j \to s$ from Below

In this subsection, we just give the Γ-convergence result of some free functionals to express some special characters of the asymptotic behaviours from below. We can see there is something different from the case converging from above. And in order to make the difference clear, we do not use the *relative-nonlocal* setting. We just work on the usual Sobolev space $W^{s,p}(\Omega)$, and this does not change the intrinsic quality, since we do not use any compact imbedding properties, which needs the extension assumption of $\partial\Omega$ (see [72]).

Now we should make some modifications on the space we work on. Let $1 < p < +\infty$, $0 < s < 1$ and $\Omega \subset \mathbb{R}^N$ be an open bounded set. We set

$$W_0^{s^-,p}(\Omega) = W^{s,p}(\Omega) \cap \left(\bigcap_{0<k<s} W_0^{k,p}(\Omega)\right) = \bigcap_{0<k<s}\left(W^{s,p}(\Omega)\cap W_0^{k,p}(\Omega)\right),$$

where $W_0^{s,p}(\Omega)$ is the complete closure of $C_0^\infty(\Omega)$ under the semi-norm $W^{s,p}(\Omega)$ defined by

$$W^{s,p}(\Omega) := \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy.$$

We can clearly see that $W_0^{s^-,p}(\Omega)$ is a closed vector space of $W^{s,p}(\Omega)$ satisfying

$$W_0^{s,p}(\Omega) \subset W_0^{s^-,p}(\Omega).$$

But when $sp > N$, $W_0^{s^-,p}(\Omega) \neq W_0^{s,p}(\Omega)$, since the boundary $\partial\Omega$ is not regular enough (see Theorem 8.2 in [35]).

We define two functionals $\mathcal{E}^s$ and $\underline{\mathcal{E}}^s$, mapping $L^p(\Omega)$ to $[0, +\infty]$ as

$$\mathcal{E}^s = \begin{cases} \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy, & if u \in W_0^{s,p}(\Omega), \\ +\infty & otherwise. \end{cases}$$
$$\underline{\mathcal{E}}^s = \begin{cases} \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy, & if u \in W_0^{s^-,p}(\Omega), \\ +\infty & otherwise. \end{cases}$$

For preparation we need the following definition.

Definition 1.2.1. *For every function $F : X \to \overline{\mathbb{R}}$ the lower semi-continuous envelop (or relaxed function) $sc^- F$ of F is defined for every $x \in X$ by*

$$(sc^- F)(x) = \sup_{G \in \mathcal{G}(F)} G(x),$$

where $\mathcal{G}(F)$ is the set of all lower semi-continuous functions G on X such that $G(y) \le F(y)$ for every $y \in X$.

We can see that in fact $sc^- F$ is the greatest lower semi-continuous function majorized by F. For more information on the relax function and the relations with Γ-convergence function one can see Chapter $3-5$ in [30].

Now we introduce the following proposition.

Proposition 1.2.1 ([30] Proposition 5.4)**.** *If (F_h) is an increasing sequence, then*

$$\Gamma - \lim_{h \to +\infty} F_h = \lim_{h \to +\infty} sc^- F_h = \sup_{h \in \mathbb{N}} sc^- F_h.$$

Theorem 1.2.2. *For every sequence $\{s_j\}_j \subset (0,1)$ strictly increasing to $s \in (0,1)$, $1 < p < +\infty$, let Ω be an open bounded set in $\mathbb{R}^N$, then it holds*

$$\Gamma - \lim_{j \to +\infty} \underline{\mathcal{E}}^{s_j} = \Gamma - \lim_{j \to +\infty} \mathcal{E}^{s_j} = \underline{\mathcal{E}}^{s}.$$

Proof. Let R denote the diameter of Ω. Define F^s and F^{s_j} as mapping $L^p(\Omega)$ to $[0, +\infty]$ by

$$F^s(u) = R^{sp} \mathcal{E}^s(u), \ \underline{F}^s(u) = R^{sp} \underline{\mathcal{E}}^s(u).$$

Then clearly F^s and $\underline{F}^s$ are lower semi-continuous, and the sequences $\{F^{s_j}\}$ and $\{\underline{F}^{s_j}\}$ are both increasing and pointwise convergent to $\underline{F}^s$. Indeed, for $0 < k \le s < 1$, this is just a simple calculation as

$$\begin{aligned} \int_{\Omega \times \Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+kp}} dxdy &\le \int_{\Omega \times \Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp+(k-s)p}} dxdy \\ &\le R^{(s-k)p} \int_{\Omega \times \Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy. \end{aligned}$$

Then thanks to Proposition 1.2.1, we infer that

$$\Gamma - \lim_{j \to +\infty} \underline{F}^{s_j} = \Gamma - \lim_{j \to +\infty} F^{s_j} = \underline{F}^s,$$

and then the assertion easily follows. $\square$

From the Theorem above, we can see that the result is not as smooth as the case in Theorem 1.2.1 to get the accumulation function belong to the ideal space $W_0^{s,p}(\Omega)$, but a wider space $W_0^{s^-,p}(\Omega)$. And as a byproduct we immediately establish the following results.

Corollary 1.2.1. *For every $s \in (0,1)$, $1 < p < +\infty$, let Ω be an open bounded set in $\mathbb{R}^N$, then the following conditions are equivalent:*

(i) For every sequence $\{s_j\}_j \subset (0,1)$ strictly increasing to $s \in (0,1)$, it holds

$$\Gamma - \lim_{j \to +\infty} \mathcal{E}^{s_j} = \mathcal{E}^s;$$

(ii) $W_0^{s^-,p}(\Omega) = W_0^{s,p}(\Omega)$.

Remark 1.2.1. *We want to mention that we can also establish similar result like Corollary 1.2.1 in our relative-nonlocal setting. For other equivalent forms one can refer to [51] or the results in chapter 2 (see Theorem 2.6.2), in which, we have also established some other equivalent forms of the space $\widetilde{W}_{0,tR}^{s^-,p}(\Omega)$ (see chapter 2) in the relative-nonlocal setting under no regularity assumptions on $\partial\Omega$.*

Chapter 2

On Nonlocal p-Rayleigh Quotients

2.1 Background and Motivation

In the non-fractional case, let Ω be a connected and bounded open set of $\mathbb{R}^N$, and let $1 < p < \infty$. The study of the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u(x) = \lambda |u|^{p-2} u & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{2.1}$$

namely,

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx = \lambda \int_\Omega |u|^{p-2} uv dx, \quad \forall v \in W_0^{1,p}(\Omega), \end{cases}$$

has been the object of many researchers for a long time motivated by Lindqvist in the fundamental paper [53]. In [54], it has been proved that the first eigenvalue is simple and is the unique eigenvalue which admits a unique positive (or negative) eigenfunction on the domain.

There have been ample research results on the asymptotic behaviour of the p-Laplacian equations on varying p. For the research on this topic we refer the reader to [53][54][55][27] [60][33][34], during which the main topic is on the p-Rayleigh quotient

$$\lambda_p^1 = \inf_{u \neq 0} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx}. \tag{2.2}$$

Here we denote by λ_p^1 the first eigenvalue and by u_p the associated positive eigenfunction such that

$$\int_\Omega u_p^p dx = 1.$$

As in [55], about the behavior of λ_p^1 and u_p with respect to p one has from the right in full generality

$$\lim_{q \to p^+} \lambda_q^1 = \lambda_p^1,$$
$$\lim_{q \to p^+} \int_\Omega |\nabla u_q - \nabla u_p|^p dx = 0,$$

while the corresponding assertions from the left

$$\lim_{q\to p^-} \lambda_q^1 = \lambda_p^1,$$
$$\lim_{q\to p^-} \int_\Omega |\nabla u_q - \nabla u_p|^q dx = 0,$$

are true under some further assumption about $\partial\Omega$. Also in [55] a counterexample shows that otherwise in general they are false. Without any regularity assumption on $\partial\Omega$, in [55] it is proved that if

$$\lim_{q\to p^-} \int_\Omega |\nabla u_q - \nabla u_p|^q dx = 0,$$

then

$$\lim_{q\to p^-} \lambda_q^1 = \lambda_p^1.$$

While in [34] (Theorem 3.2) the authors proposed an auxiliary method which allows to describe the behavior of λ_q^1 and u_q as $q \to p^-$. In the same paper several equivalent characterizations of the fact that

$$\lim_{q\to p^-} \lambda_q^1 = \lambda_p^1$$

was provided in Theorem 4.1, and in Corollary 4.4 the authors proved that if

$$\lim_{q\to p^-} \lambda_q^1 = \lambda_p^1,$$

then

$$\lim_{q\to p^-} \int_\Omega |\nabla u_q - \nabla u_p|^q dx = 0,$$

without any assumption on $\partial\Omega$, which closed the open problem proposed in [55].

What interests us is the nonlocal setting, let $0 < s < 1$ and $p > 1$, let Ω be an open bounded (may be not connected) set in $\mathbb{R}^N$. We define

$$\begin{cases} (-\Delta_p)^s u(x) = \lambda |u|^{p-2}u & \textit{in } \Omega, \\ u = 0 & \textit{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.3}$$

where

$$(-\Delta_p)^s u(x) := 2 \lim_{\delta\to 0^+} \int_{\{y\in\mathbb{R}^N : |y-x|\geq\delta\}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy,$$

is the *fractional p-Laplacian*. Here the solutions of (2.3) are always understood in the weak sense (see (2.5)).

For the motivation leading to the research of such equations, we refer the readers to the contribution of Caffarelli in [25]. Since then, many efforts have been devoted to the study of this operator, among which we mention eigenvalue problems [37][44][56][20][22], and the regularity theory [43][46][47][32]. For a existence proof via Morse theory one can refer to [42].

The operator $(-\Delta_p)^s$ aries as the first variation of the fractional Dirichlet intergral

$$\Phi_{s,p}(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy, \tag{2.4}$$

and therefore the counterpart of the p-Laplacian operator defined in (2.1). Up to a homogeneity it is not difficult to see that (s,p)-eigenvalues correspond to the critical points of the functional (2.4) restricted to the manifold $\mathcal{S}_{s,p}(\Omega)$ containing the functional with unitary L^p-norm.

Let us briefly recall that the first eigenvalue $\lambda^1_{s,p}(\Omega)$ has a variational characterization in the nonlocal space $\widetilde{W}^{s,p}_0(\Omega)$ defined as a completion of $C^\infty_0(\Omega)$ under the norm $W^{s,p}(\mathbb{R}^N)$, as it corresponds to the minimum of $\Phi_{s,p}$ on $\mathcal{S}_p(\Omega)$, defined as

$$\lambda^1_{s,p}(\Omega) := \inf_{u\in\widetilde{W}^{s,p}_0(\Omega)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy}{\int_\Omega |u|^p dx},$$

which is the well-known *nonlocal p-Rayleigh quotients*, or equivalently

$$\lambda^1_{s,p}(\Omega) := \inf_{u\in\mathcal{S}_{s,p}(\Omega)} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy.$$

For a detailed investigation on the first eigenvalue and second eigenvalue for the operator $(-\Delta_p)^s$ one can refer to [37][44][17][20][22] etc.

2.2 Nonlocal p-Eigenvalues

What we dispose here is asymptotic behaviour of the nonlocal eigenvalues, with respect to the regular exponent s, in the *relative-nonlocal* Sobolev space $\widetilde{W}^{s,p}_{0,tR}(\Omega)$ (see (7)). For $u\in\widetilde{W}^{s,p}_{0,tR}(\Omega)$, the first variation of the functional (2.4) is expressed in the following weak sense,

$$\begin{aligned}\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} &\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}dy \\ &= \lambda\int_\Omega |u|^{p-2}uvdy \ in\ \Omega,\end{aligned} \tag{2.5}$$

for every $v\in\widetilde{W}^{s,p}_{0,tR}(\Omega)$.

Let us introduce the admissible space $\mathcal{S}_{s,p}(\Omega)$ for eigenvalues as

$$\mathcal{S}_{s,p}(\Omega) = \{u\in\widetilde{W}^{s,p}_{0,tR}(\Omega) : \|u\|_{L^p(\Omega)} = 1\},$$

and we also define the *m-th* (*variational*) *eigenvalues* of (2.3)

$$\lambda^m_{s,p} := \inf_{K\in\mathcal{W}^{s,p}_m(\Omega)} \max_{u\in K} [u]^p_{W^{s,p}_{tR}(\Omega)}, \tag{2.6}$$

in which we define for $0<s<1$

$$\mathcal{W}^{s,p}_m(\Omega) = \{K\subset\mathcal{S}_{s,p}(\Omega) : K\ symmetric\ and\ compact,\ i(K)\geq m\}, \tag{2.7}$$

where $i(K) \geq 1$ is an integer and defined whenever K is nonempty, compact and symmetric subset of a topological vector space such that $0 \notin K$. Well-known examples are $Krasnosel'ski\breve{\imath}\ genus$ (see [48][62][9][71][69]). Following the setting in [22] we recall that for every nonempty and symmetric subset $A \subset X$ of a Banach space, its $Krasnosel'ski\breve{\imath}\ genus$ index defined by

$$i(A) = \inf\{k \in \mathbb{N} : \exists\ a\ continuous\ odd\ map\ f : A \to \mathbb{S}^{k-1}\},$$

with the convention that $i(A) = \infty$, if no such an integer k exists.

For $m = 1$ the definition coincides with

$$\lambda^1_{s,p}(\Omega) = \min_{u \in \mathcal{S}_{s,p}(\Omega)} [u]^p_{W^{s,p}_{tR}(\Omega)},\ global\ minimum,$$

and for completeness we also mention that for $m = 2$ it coincides with

$$\lambda^2_{s,p}(\Omega) = \inf_{\gamma \in \Sigma(u_1, -u_1)} \max_{u \in \gamma([0,1])} [u]^p_{W^{s,p}_{tR}(\Omega)},\ mountain\ pass\ lemma,$$

where u_1 is a minimizer associated with $\lambda^1_{s,p}(\Omega)$ and $\Sigma(u_1, -u_1)$ is the set of continuous paths on $\mathcal{S}_{s,p}(\Omega)$ connecting u_1 and $-u_1$ (see [26], Corollary 3.2 for the local case, and [20], Theorem 5.3 for the nonlocal one).

Remark 2.2.1. *The asymptotic problem of the variational eigenvalues λ^m_p with respect to p of (2.1) has been first studied by Lindqvist [55] and Huang [41] in the case of first and second eigenvalue respectively. In the more general setting the problems are tackled in [27][60][58]. In [33] the case of presence of weights and unbounded sets has been considered under the Γ-convergence approach. In particular, we want to mention the result in [22], which analyzed the limit behavior as $s \to 1$ using Γ-convergence.*

Throughout this chapter, we use

$$(s,p) - eigenvalues$$

to denote the fractional p-eigenvalues, and

$$(s,p) - eigenfunctions$$

to denote the corresponding fractional p-eigenfunctions.

Here we would like to recall a basic but interesting properties of the fractional first eigenvalue, although not used in our thesis, the following Faber-Krahn inequality.

Theorem 2.2.1 (Faber-Krahn inequality, [17] Theorem 3.5)**.** *Let $1 < p < \infty$ and $0 < s < 1$. For every $\Omega \subset \mathbb{R}^N$ open and bounded, we have*

$$(\mathcal{L}^N(\Omega))^{\frac{sp}{N}} \lambda^1_{s,p}(\Omega) \geq (\mathcal{L}^N(B))^{\frac{sp}{N}} \lambda^1_{s,p}(B), \tag{2.8}$$

where B is any N-dimensional ball. Moreover, if equality holds in (2.8) then Ω is a ball. In other words, balls uniquely minimizes the first eigenvalue $\lambda^1_{s,p}$ among sets with given N-dimensional Lebesgue measure.

We recall the existing global boundedness and continuity of the first (s,p)-eigenfunctions. In [20], the authors give a global L^∞ bound for the solutions u to the nonlocal p-Laplacian equations in the sense that

$$\begin{cases} (-\Delta_p)^s u(x) = F(x) \ in \ \Omega, \\ \qquad\qquad u = 0 \ in \ \mathbb{R}^N \setminus \Omega. \end{cases}$$

And the solution is in the space $\widetilde{W}_0^{s,p}(\Omega)$.

The boundedness result is as follows, here Ω is an open bounded set in $\mathbb{R}^N$.

Theorem 2.2.2 ([20] Theorem 3.1, Global L^∞ bound)**.** *Let $1 < p < \infty$ and $0 < s < 1$ be such that $sp < N$. If $F \in L^q(\Omega)$ for $q > N/(sp)$, then $u \in L^\infty(\Omega)$. Moreover, we have the scaling invariant estimate*

$$\|u\|_{L^\infty(\Omega)} \leq (C\chi^{\frac{1}{\chi-1}})^{\frac{\chi}{\chi-1}}(T_{s,p}|\Omega|^{\frac{sp}{N}-\frac{1}{q}}\|F\|_{L^q(\Omega)})^{\frac{1}{p-1}},$$

where $C = C(p) > 0$, $\chi = \frac{p^*}{pq'}$, and $T_{s,p}$ is the sharp Sobolev constant defined by

$$T_{s,p} := \sup_{v \in W_0^{s,p}(\mathbb{R}^N)} \left\{ \left(\int_{\mathbb{R}^N} |v|^{p^*} dx \right)^{\frac{p}{p^*}} : \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+sp}} dxdy = 1 \right\} < +\infty.$$

For the case $sp > N$ we can directly use the Sobolev type imbedding $\widetilde{W}_0^{s,p}(\Omega) \hookrightarrow L^\infty \cap C^{0,s-N/p}(\Omega)$ (refer to [17], Proposition 2.9). And if $sp = N$, where $F \in L^q(\Omega)$ for $q > 1$, it is exactly the same process as in $sp < N$. In fact the same proof process is also available for the eigenfunctions of the operator $(-\Delta_p)^s$. It is obvious for the case $sp > N$, and for the case $sp \leq N$, we have $u \in L^\infty(\Omega)$. By interpolation one can get the estimates for (s,p)-eigenfunctions when $sp < N$

$$\|u\|_{L^\infty(\Omega)} \leq [\widetilde{C}_{N,s,p}\lambda^1_{s,p}]^{\frac{N}{sp}}\|u\|_{L^1(\Omega)},$$

where

$$\widetilde{C}_{N,s,p} = T_{s,p}(\frac{p^*}{p})^{\frac{N-sp}{s}\frac{p-1}{p}}.$$

One can also refer to Remark 3.2 in [20].

Theorem 2.2.3 ([20] Corollary 3.14, Continuity of Eigenfunctions)**.** *Let $1 < p < \infty$ and $0 < s < 1$. Every (s,p)-eigenfunction of the open bounded set $\Omega \subset \mathbb{R}^N$ is continuous.*

Theorem 2.2.4 ([37] Theorem 4.2, Proportionality of Eigenfunctions)**.** *Let $s \in (0,1)$ and $p > 1$. Then all the positive eigenfunctions corresponding to $\lambda^1_{s,p}$ are proportional.*

Remark 2.2.2. *There are some differences between the proportionality of first eigenfunctions to operators p-Laplacian and nonlocal p-Laplacian, i.e. for the no sign-changing and proportional properties, there is no need to let Ω be connected in the nonlocal setting. For the details one can see e.g. [53][17][37][20][34].*

Throughout our thesis, the problem settings are on the space $\widetilde{W}^{s,p}_{0,tR}(\Omega)$ without any regular assumption on $\partial\Omega$. Anyway we want to point out that in the proof of three properties of eigenfunctions above (Theorem 2.2.2, 2.2.3 and 2.2.4), no regularity assumptions were exerted on the boundary data $\partial\Omega$. If we check the proof of three theorems mentioned just now carefully (see the details in [20][37]), it's convenient for us applying for the proof process directly without any essential modification (but some minor adjustment on the constant only depending on N, s and p) to get the same estimate results, thanks to the equivalence between $\widetilde{W}^{s,p}_{0}(\Omega)$ and $\widetilde{W}^{s,p}_{0,tR}(\Omega)$.

2.3 Main Results

In this chapter we mainly analysis the asymptotic behaviour of $\lambda^1_{s,p}$ with varying s. We firstly state that in the nonlocal case, we only assume that $\Omega \subset \mathbb{R}^N$ is an open bounded set, no any connection or regularity assumption *a priori*.

In order to investigate the comparison of different $\lambda^1_{s,p}$, we will work in the *relative-nonlocal* Sobolev space $\widetilde{W}^{s,p}_{0,tR}(\Omega)$ $(t > 1)$ introduced in subsection 0.2; then in the this subsection, we review the definitions and some basic properties of the first (s,p)-eigenvalues and corresponding eigenfunctions. Then in our relative-nonlocal settings we also call the first (s,p)-eigenvalue as *relative-nonlocal p-Rayleigh quotient*.

In the following sections, we give the asymptotic behaviours in the process $k \to s$, and the behaviour is quite different from the left-hand side and from the right-hand side. In section 2.4, we prove a general result (see Theorem 2.4.1 below) as

$$\lim_{k\to s^-} \lambda^1_{k,p} \le \lambda^1_{s,p} = \lim_{k\to s^+} \lambda^1_{k,p},$$

and the corresponding eigenfunctions' convergence behaviour

$$\lim_{k\to s^+} [u_k - u_s]_{W^{s,p}_{4R}(\Omega)} = 0.$$

Different from the behaviour of $k \to s^+$, we give in section 2.5 that if the following convergence holds true

$$\lim_{k\to s^-} [u_k - u_s]_{W^{s,p}_{4R}(\Omega)} = 0,$$

then the convergence

$$\lim_{k\to s^-} \lambda^1_{k,p} = \lambda^1_{s,p}$$

holds true for every open bounded set. We also show that even $\lim\limits_{k\to s^-} \lambda^1_{k,p} = \lambda^1_{s,p}$, there only holds

$$\lim_{k_j\to s^-} [u_{k_j} - u]_{W^{k_j,p}_{4R}(\Omega)} = 0,$$

for $u \in \widetilde{W}^{s,p}_{4R}(\Omega)$ but may not being in $\widetilde{W}^{s,p}_{0,4R}(\Omega)$. While if $u \in \widetilde{W}^{s,p}_{0,4R}(\Omega)$, then $u = u_s$. In any case we have

$$\lambda^1_{s,p} = \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy$$

with $\|u\|_{L^p(\Omega)} = 1$.

Since we cannot exclude the probability that functions in $\widetilde{W}^{k,p}_{0,tR}(\Omega)$ may not belong to the spaces $\widetilde{W}^{s,p}_{0,tR}(\Omega)$ for any $0 < k < s < 1$. So inspired by the approach in [34], in section 2.6 we introduce a larger relative-nonlocal space $\widetilde{W}^{s^-,p}_{0,tR}(\Omega)$ for the special left-hand side convergence of $k \to s^-$. If we use $\underline{\lambda}^1_{s,p}$ and $\underline{u}_s$ to denote the first (s,p)-eigenvalue and first (s,p)-eigenfunction respectively in $\widetilde{W}^{s^-,p}_{0,tR}(\Omega)$, we show that

$$\lim_{k\to s^-} \underline{\lambda}^1_{k,p} = \lim_{k\to s^-} \lambda^1_{k,p} = \underline{\lambda}^1_{s,p},$$

and

$$\lim_{k\to s^-} [\underline{u}_k - \underline{u}_s]_{W^{k,p}_{4R}(\Omega)} = \lim_{k\to s^-} [u_k - \underline{u}_s]_{W^{k,p}_{4R}(\Omega)} = 0.$$

Utilizing the strategy as in [34], we also give some equivalent characterizations of $\underline{\lambda}^1_{s,p} = \lambda^1_{s,p}$ in the last section.

In this chapter we work on the set Ω without extension property, which is the main source of the singularities happening. In fact, for the case $s = 1$, if there is some extension assumption on $\partial\Omega$, then $\lim\limits_{k\to s^-} (1-k)\lambda^1_{k,p} = \lambda^1_{s,p}$. One can see a case for $s \to 1$ in [22] by Brasco, Parini and Squassina. But we still have no idea whether this would happen in the fractional case. To the best of our knowledge, it seems hopeless due to the nonlocal nature.

2.4 General Approximation Behavior

Although we can define the *relative-nonlocal* Sobolev space $\widetilde{W}^{s,p}_{0,tR}(\Omega)$ for any $t > 1$, in our problem setting here, we directly make $\mathbf{t = 4}$ for some convenience in the latter calculation, of course there are infinitely many other choices.

As we have mentioned before, we do not assume any regularity on Ω. Most results in this section are derived in an elementary way, only by functional analysis in Sobolev spaces but no deep properties of eigenfunctions. The fractional first eigenvalue is simple (see [37][20][22]), and associated eigenfunction u_p is unique both up to a multiplication of some constant and choice of sign. We normalize the first (s,p)-eigenfunctions by $\|u_p\|_{L^p(\Omega)} = 1$ so that

$$\lambda^1_{s,p} = \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|u_p(x) - u_p(y)|^p}{|x-y|^{N+sp}} dxdy.$$

By simple calculations we observe that $\lambda^1_{s,p}(\Omega)$ enjoys a scaling law

$$\lambda^1_{s,p}(t\Omega) = t^{-sp}\lambda^1_{s,p}(\Omega),\ t > 0.$$

Note that if $\Omega_1 \subset \Omega_2$, we have $\lambda^1_{s,p}(\Omega_1) \geq \lambda^1_{s,p}(\Omega_2)$. This is a direct conclusion from the nonlocal p-Rayleigh quotient.

Lemma 2.4.1. *For any open bounded set $\Omega \subset \mathbb{R}^N$ and $0 < s \leq k < 1$, we have*

$$(\frac{5R}{2})^{sp}\lambda^1_{s,p} \leq (\frac{5R}{2})^{kp}\lambda^1_{k,p},$$

where R denotes the diameter of Ω, defined in (6).

Proof. Let u be in the admissible space $\mathcal{S}_{k,p}(\Omega)$, then by Hölder inequality and

$$\begin{aligned} \lambda^1_{s,p} &\leq \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \\ &= \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+kp+(s-k)p}} dxdy \\ &\leq (\tfrac{5R}{2})^{(k-s)p} \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+kp}} dxdy. \end{aligned}$$

As $u \in \mathcal{S}_{k,p}(\Omega)$, we have from the inequality above

$$\begin{aligned} \lambda^1_{s,p} &\leq (\tfrac{5R}{2})^{(k-s)p} \inf_{u\in\mathcal{S}_{k,p}(\Omega)} \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+kp}} dxdy \\ &= (\tfrac{5R}{2})^{(k-s)p}\lambda^1_{k,p}. \end{aligned}$$

Then we have $(\frac{5R}{2})^{sp}\lambda^1_{s,p} \leq (\frac{5R}{2})^{kp}\lambda^1_{k,p}$. □

Theorem 2.4.1.

$$\lim_{k\to s^-} \lambda^1_{k,p} \leq \lambda^1_{s,p} = \lim_{k\to s^+} \lambda^1_{k,p}.$$

Proof. According to Lemma 2.4.1, by the monotony of $(\frac{5R}{2})^{sp}\lambda^1_{s,p}$ and the continuity of $(\frac{5R}{2})^{sp}$ on s, letting $k \to s^+$ and $k \to s^-$ respectively, we have

$$\lim_{k\to s^-} \lambda^1_{k,p} \leq \lambda^1_{s,p} \leq \lim_{k\to s^+} \lambda^1_{k,p}.$$

For the other direction of the equality, by letting $\{k_i\}_i$ be a sequence decreasing to s as $i \to +\infty$, we notice the fact that $C_0^\infty(\Omega)$ is dense in both $\widetilde{W}^{k_i,p}_{0,4R}(\Omega)$ and $\widetilde{W}^{s,p}_{0,4R}(\Omega)$ $(\widetilde{W}^{k_i,p}_{0,4R}(\Omega) \hookrightarrow \widetilde{W}^{s,p}_{0,4R}(\Omega))$. We have then for any $\phi \in C_0^\infty(\Omega)$ with unitary $L^p(\Omega)$-norm such that

$$\lambda^1_{k_i,p} \leq \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+k_ip}} dxdy;$$

then letting $i \to \infty$ we have

$$\lim_{i\to\infty} \lambda^1_{k_i,p} \leq \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+sp}} dxdy.$$

Taking the infimum over all admissible function $\phi \in \mathcal{S}_{s,p}(\Omega)$ we find that

$$\lim_{i\to\infty} \lambda^1_{k_i,p} \leq \lambda^1_{s,p},$$

which concludes $\lim_{i\to\infty} \lambda^1_{k_i,p} = \lambda^1_{s,p}$. □

Remark 2.4.1. *Anyway we can not exclude the possibility that there exist some functions* $u \in \widetilde{W}^{k,p}_{0,4R}(\Omega)$ *but not in* $\widetilde{W}^{s,p}_{0,4R}(\Omega)$ *for* $\forall\ k < s$*; even the limitation function of the eigenfunctions sequence* $\{u_k\}$ *would belong to* $\widetilde{W}^{s,p}_{4R}(\Omega)$ *and* $\widetilde{W}^{s-\varepsilon,p}_{0,4R}(\Omega)$ $(\varepsilon > 0)$ *as proved in Lemma 2.5.1.*

Theorem 2.4.2. *The strong convergence of the eigenfunctions* u_k *to* u_s

$$\lim_{k\to s^+} [u_k - u_s]_{W^{s,p}_{4R}(\Omega)} = 0 \tag{2.9}$$

is valid for any bounded open set Ω.

Proof. **Step 1**. Up to a normalization $\|u_k\|_{L^p(\Omega)} = 1$, we have for $s \le k$ that

$$\begin{aligned}
&\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \tfrac{|u_k(x)-u_k(y)|^p}{|x-y|^{N+sp}} dxdy \\
&\le \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \tfrac{|u_k(x)-u_k(y)|^p}{|x-y|^{N+kp+(s-k)p}} dxdy \\
&\le (\tfrac{5R}{2})^{(k-s)p} \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \tfrac{|u_k(x)-u_k(y)|^p}{|x-y|^{N+kp}} dxdy \\
&= (\tfrac{5R}{2})^{(k-s)p} \lambda^1_{k,p},
\end{aligned} \tag{2.10}$$

implying the uniform boundedness of $[u_k]_{\widetilde{W}^{s,p}_{4R}(\Omega)}$. So as $k \to s^+$ we can extract a subsequence $\{u_{k_i}\}$ converging weakly in the space $\widetilde{W}^{s,p}_{4R}(\Omega)$ to a function u in space $\widetilde{W}^{s,p}_{4R}(\Omega)$. The limit function u is in $\widetilde{W}^{s,p}_{0,4R}(\Omega)$ as every u_{k_i} is in $\widetilde{W}^{s,p}_{0,4R}(\Omega)$. Then by Poincaré inequality (see (9)) $\|u_{k_i} - u\|_{L^p(\Omega)} \to 0$, so we have normalization $\|u\|_{L^p(\Omega)} = 1$. This implies $u \in \mathcal{S}_{s,p}(\Omega)$.

Now let identify $u = u_s$. By the weak lower semi-continuity we have

$$\begin{aligned}
&\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \tfrac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \\
&\le \liminf_{i\to\infty} \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \tfrac{|u_{k_i}(x)-u_{k_i}(y)|^p}{|x-y|^{N+sp}} dxdy \\
&= \liminf_{i\to\infty} \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \tfrac{|u_{k_i}(x)-u_{k_i}(y)|^p}{|x-y|^{N+k_ip+(s-k_i)p}} dxdy \\
&\le \liminf_{i\to\infty} \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \tfrac{|u_{k_i}(x)-u_{k_i}(y)|^p}{|x-y|^{N+k_ip+(s-k_i)p}} dxdy \\
&\le \liminf_{i\to\infty} (\tfrac{5R}{2})^{(k_i-s)p} \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \tfrac{|u_{k_i}(x)-u_{k_i}(y)|^p}{|x-y|^{N+k_ip}} dxdy
\end{aligned}$$

and up to a normalization we have

$$\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \le \liminf_{i\to\infty} \lambda^1_{k_i,p} = \lambda^1_{s,p},$$

in which the last equality is by Theorem 2.4.1. As u is an admissible function in the p-Rayleigh quotient for $\lambda^1_{s,p}$, by the uniqueness of the first eigenfunction we have that $u = u_s$.

Step 2. Now let us concern on the strong convergence (2.9). For the case $p \ge 2$, as

$$(u_k - u_s)(x) - (u_k - u_s)(y) = u_k(x) - u_k(y) - (u_s(x) - u_s(y))$$

we introduce the Clarkson's inequality obtaining

$$\begin{aligned} &|\tfrac{(u_k-u_s)(x)-(u_k-u_s)(y)}{2}|^p + |\tfrac{u_k(x)-u_k(y)+u_s(x)-u_s(y)}{2}|^p \\ &= |\tfrac{u_k(x)-u_k(y)-(u_s(x)-u_s(y))}{2}|^p + |\tfrac{u_k(x)-u_k(y)+u_s(x)-u_s(y)}{2}|^p \\ &\leq \tfrac{1}{2}|u_k(x)-u_k(y)|^p + \tfrac{1}{2}|u_s(x)-u_s(y)|^p, \end{aligned} \tag{2.11}$$

since u_k and u_s are in the admissible space for first eigenvalue $\lambda^1_{s,p}$, then we obtain

$$\lambda^1_{s,p} \leq \frac{\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|\frac{(u_k+u_s)(x)}{2}-\frac{(u_k+u_s)(y)}{2}|^p}{|x-y|^{N+sp}}dxdy}{\int_\Omega |\frac{u_k+u_s}{2}|^p dx},$$

and by (2.10) we have

$$\limsup_{k\to s+} \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|u_k(x)-u_k(y)|^p}{|x-y|^{N+sp}}dxdy \leq \lambda^1_{s,p}.$$

Then after divided by $|x-y|^{N+sp}\int_\Omega |\frac{u_k+u_s}{2}|^p dx$ and performing the double integral on $B_{4R}(\Omega)\times B_{4R}(\Omega)$ on (2.11) we have

$$\lim_{k\to s^+} \frac{[u_k-u_s]^p_{W^{s,p}_{4R}(\Omega)}}{2^p} + \lambda^1_{s,p} \leq \frac{\lambda^1_{s,p}}{2} + \frac{\lambda^1_{s,p}}{2},$$

by recalling that

$$\lim_{k\to s^+} \int_\Omega |\frac{u_k+u_s}{2}|^p dx = \int_\Omega |u_s|^p dx = 1.$$

Then we conclude the desired result for $p \geq 2$.

In the case $1 < p < 2$ one also have the Clarkson's inequality

$$\begin{aligned} &\left\{|\tfrac{(u_k-u_s)(x)-(u_k-u_s)(y)}{2}|^p\right\}^{\frac{1}{p-1}} + \left\{|\tfrac{u_k(x)-u_k(y)+u_s(x)-u_s(y)}{2}|^p\right\}^{\frac{1}{p-1}} \\ &= \left\{|\tfrac{u_k(x)-u_k(y)-(u_s(x)-u_s(y))}{2}|^p\right\}^{\frac{1}{p-1}} + \left\{|\tfrac{u_k(x)-u_k(y)+u_s(x)-u_s(y)}{2}|^p\right\}^{\frac{1}{p-1}} \\ &\leq \left\{\tfrac{1}{2}|u_k(x)-u_k(y)|^p + \tfrac{1}{2}|u_s(x)-u_s(y)|^p\right\}^{\frac{1}{p-1}}, \end{aligned}$$

then performing the same process as in the case $p \geq 2$ we get the desired result (2.9). □

2.5 Behaviour from Below

There are some essential differences between the approximations from above and from below. When k approaches s from below, it is almost impossible for us to get a uniform bound for the functions sequence in the norm $\widetilde{W}^{s,p}_{0,4R}(\Omega)$. So we cannot get a strong convergence result of the approximating sequence.

Theorem 2.5.1. *The convergence*

$$\lim_{k\to s^-} \lambda^1_{k,p} = \lambda^1_{s,p}$$

holds true for any bounded open set if the following convergence holds true

$$\lim_{k\to s^-} [u_k - u_s]_{W^{s,p}_{4R}(\Omega)} = 0. \tag{2.12}$$

Proof. Supposing that (2.12) holds true and by Proposition 0.2.2, for any $\varepsilon > 0$ satisfying $s-\varepsilon \le k$ we have that

$$[u_k - u_s]_{W^{s-\varepsilon,p}_{4R}(\Omega)} \le C[u_k - u_s]_{W^{k,p}_{4R}(\Omega)}.$$

And we also have

$$\begin{aligned}&\textstyle\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|u_k(x)-u_k(y)|^p}{|x-y|^{N+(s-\varepsilon)p}}dxdy\\ &= \textstyle\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|u_k(x)-u_k(y)|^p}{|x-y|^{N+kp-kp+(s-\varepsilon)p}}dxdy\\ &\le (\frac{5R}{2})^{k-s+\varepsilon} \lim_{k\to s^-} \textstyle\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|u_k(x)-u_k(y)|^p}{|x-y|^{N+kp}}dxdy.\end{aligned}$$

Since

$$[u_s]_{W^{s-\varepsilon,p}_{4R}(\Omega)} \le [u_s - u_k]_{W^{s-\varepsilon,p}_{4R}(\Omega)} + [u_k]_{W^{s-\varepsilon,p}_{4R}(\Omega)},$$

then letting k approximating s^- and $\varepsilon \to 0$ we have that

$$[u_s]_{W^{s,p}_{4R}(\Omega)} \le \lim_{k\to s^-} [u_k]_{W^{k,p}_{4R}(\Omega)}.$$

Then up to a normalization we have $\lambda^1_{s,p} \le \lim\limits_{k\to s^-} \lambda^1_{k,p}$. Thanks to Corollary 2.4.1 this concludes that $\lambda^1_{s,p} = \lim\limits_{k\to s^-} \lambda^1_{k,p}$. □

In the next lemma, we give the behavior of u_k and u_s when $\lambda^1_{k,p} \to \lambda^1_{s,p}$. As it is shown, the limiting function of the eigenfunctions may not be the "corresponding" eigenfunction, only if some further assumption is satisfied.

Lemma 2.5.1. *Suppose that $\lim\limits_{k\to s^-} \lambda^1_{k,p} = \lambda^1_{s,p}$. Then up to a subsequence $\{k_j\}$ in the process of k tending to s from below, we have that there exists some function $u \in \widetilde{W}^{s,p}_{4R}(\Omega)$ such that the following formula holds true:*

$$\lim_{k_j\to s^-} [u_{k_j} - u]_{W^{k_j,p}_{4R}(\Omega)} = 0.$$

If $u \in \widetilde{W}^{s,p}_{0,4R}(\Omega)$, then $u = u_s$. In any case

$$\lambda^1_{s,p} = \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy$$

with $\|u\|_{L^p(\Omega)} = 1$.

Proof. From the assumption we know $[u_k]_{W^{k,p}_{4R}(\Omega)}$ is uniformly bounded, so by the same process as in Theorem 2.5.1 we have $a\ fortiori$ the uniform bound for $[u_k]_{W^{s-\varepsilon,p}_{4R}(\Omega)}$ for any $\varepsilon > 0$. Then we can find a limitation function $u \in \widetilde{W}^{s-\varepsilon,p}_{0,4R}(\Omega)$ by Theorem 0.2.1, and up to a subsequence of k (denoted by k_j) such that

(i) $[u_{k_j} - u_s]_{W^{s-\varepsilon,p}_{4R}(\Omega)} \to 0$ weakly as $j \to \infty$;

(ii) $\|u_{k_j} - u\|_{L^p(\Omega)} \to 0$ strongly (by Poincaré inequality (9)),

where in (ii) we have the normalization of $\|u\|_{L^p(\Omega)} = 1$ and

$$\lim_{j\to\infty} \|\frac{u + u_{k_j}}{2}\|_{L^p(\Omega)} = 1. \tag{2.13}$$

In particular we have

$$\begin{aligned}
[u]^p_{W^{s-\varepsilon,p}_{4R}(\Omega)} &\leq \liminf_{j\to\infty}[u_{k_j}]^p_{W^{s-\varepsilon,p}_{4R}(\Omega)} \\
&\leq (\tfrac{5R}{2})^{k_j-s+\varepsilon} \liminf_{j\to\infty}[u_{k_j}]^p_{W^{k_j,p}_{4R}(\Omega)} \\
&= (\tfrac{5R}{2})^{k_j-s+\varepsilon}\lambda^1_{s,p}.
\end{aligned}$$

Thus letting $\varepsilon \to 0$ and $j \to \infty$ we have $u \in \widetilde{W}^{s,p}_{4R}(\Omega)$ and $[u]^p_{W^{s,p}_{4R}(\Omega)} \leq \lambda^1_{s,p}$.

Again, as $k_j < s$, we infer that

$$\begin{aligned}
&\textstyle\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+k_jp}}dxdy \leq \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp+(k_j-s)p}}dxdy \\
&\leq (\tfrac{5R}{2})^{(s-k_j)p}\textstyle\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy = (\tfrac{5R}{2})^{(s-k_j)p}[u]^p_{W^{s,p}_{4R}(\Omega)},
\end{aligned}$$

which implies that $\lim\limits_{k_j\to s^-} \lambda^1_{k_j,p} \leq [u]^p_{W^{s,p}_{4R}(\Omega)}$ as $j \to +\infty$ together with the fact that $\|u\|_{L^p(\Omega)} = 1$. Since $\lim\limits_{k\to s^-} \lambda^1_{k,p} = \lambda^1_{s,p}$ and $[u]^p_{W^{s,p}_{4R}(\Omega)} \leq \lambda^1_{s,p}$, thus we have $\lambda^1_{s,p} = [u]^p_{W^{s,p}_{4R}(\Omega)}$. In fact, if we apply for the assumption $u \in \widetilde{W}^{s,p}_{0,4R}(\Omega)$, then by the uniqueness of eigenfunction we have that $u = u_s$.

Now we start to verify the convergence of eigenfunctions $\{u_{k_j}\}$ to u. In fact we just need to reproduce the same process as in the proof of Theorem 2.4.2 together with the help of Clarkson's inequality for both the case $p \geq 2$

$$\begin{aligned}
&|\tfrac{(u_k-u)(x)-(u_k-u)(y)}{2}|^p + |\tfrac{u_k(x)-u_k(y)+u(x)-u(y)}{2}|^p \\
&\leq \tfrac{1}{2}|u_k(x) - u_k(y)|^p + \tfrac{1}{2}|u_s(x) - u_s(y)|^p,
\end{aligned}$$

and the case $1 \leq p \leq 2$

$$\begin{aligned}
&\left\{|\tfrac{(u_k-u)(x)-(u_k-u)(y)}{2}|^p\right\}^{\frac{1}{p-1}} + \left\{|\tfrac{u_k(x)-u_k(y)+u(x)-u(y)}{2}|^p\right\}^{\frac{1}{p-1}} \\
&\leq \left\{\tfrac{1}{2}|u_k(x) - u_k(y)|^p + \tfrac{1}{2}|u(x) - u(y)|^p\right\}^{\frac{1}{p-1}},
\end{aligned}$$

and by recalling the normalization (2.13). Then we conclude that

$$\lim_{k_j\to s^-}[u_{k_j} - u]_{W^{k_j,p}_{4R}(\Omega)} = 0.$$

□

Remark 2.5.1. *If working on the open bounded set Ω without extension property, during the establishment of condition (ii) in the proof above, we can not use the Rellich-Kondrachov-type compactness theorem and a uniform Poincaré-type inequality for the functions in $W^{k,p}(\Omega)$, even for the space $W_0^{k,p}(\Omega)$, since we know very few information about the boundary data. And there is no corresponding compact imbedding results existing, except that $\partial\Omega$ satisfies some extension property (see [72]) and Ω being a domain. That is also the reason why we define the relative-nonlocal space $\widetilde{W}^{s,p}_{0,4R}(\Omega)$.*

2.6 Behaviour from Below in a Larger space

Inspired by [34], this section is mainly concerned with an improvement argument to the asymptotic behaviours triggered by the convergence of the first (s,p)-eigenvalues as $k \to s^-$.

2.6.1 Definitions and Basic Properties

As we have noticed, in the case $k \to s^-$ there are no corresponding ideal results as in the case $k \to s^+$, because we can not exclude the blow-up probability of a function transforming from the space $\widetilde{W}^{k,p}_{0,4R}(\Omega)$ to a more regular space $\widetilde{W}^{s,p}_{0,4R}(\Omega)$ $(k < s)$. We try to construct a larger admissible space to investigate the asymptotic behaviour when $k \to s^-$.

Let Ω denote a bounded open subset in $\mathbb{R}^N$, $0 < s < 1$ and $1 < p < +\infty$. No assumption will be imposed on *a priori* on the regularity of $\partial\Omega$. We set

$$\widetilde{W}^{s^-,p}_{0,4R}(\Omega) := \widetilde{W}^{s,p}_{4R}(\Omega) \cap \left(\bigcap_{0<k<s} \widetilde{W}^{k,p}_{0,4R}(\Omega) \right) = \bigcap_{0<k<s} \left(\widetilde{W}^{s,p}_{4R}(\Omega) \cap \widetilde{W}^{k,p}_{0,4R}(\Omega) \right)$$

Proposition 2.6.1. *We have the following facts for the space $\widetilde{W}^{s^-,p}_{0,4R}(\Omega)$*

(i) $\widetilde{W}^{s^-,p}_{0,4R}(\Omega)$ is a closed vector subspace of $\widetilde{W}^{s,p}_{4R}(\Omega)$ satisfying

$$\widetilde{W}^{s,p}_{0,4R}(\Omega) \subseteq \widetilde{W}^{s^-,p}_{0,4R}(\Omega);$$

(ii) if $sp < N$, we have $\widetilde{W}^{s^-,p}_{0,4R}(\Omega) \subseteq L^{p^*}(\Omega)$ and

$$\inf \left\{ \frac{[u]^p_{W^{s,p}_{4R}(\Omega)}}{\left(\int_\Omega |u|^{p^*} dx\right)^{p/p^*}} : u \in \widetilde{W}^{s^-,p}_{0,4R}(\Omega) \setminus 0 \right\}$$

$$= \inf \left\{ \frac{[u]^p_{W^{s,p}_{4R}(\Omega)}}{\left(\int_\Omega |u|^{p^*} dx\right)^{p/p^*}} : u \in \widetilde{W}^{s,p}_{0,4R}(\Omega) \setminus 0 \right\},$$

where $p^* = \frac{Np}{N-sp}$;

(iii) if $sp > N$, we have $\widetilde{W}^{s^-,p}_{0,4R}(\Omega) = \widetilde{W}^{s,p}_{0,4R}(\Omega)$.

Proof. The following proof essentially follows the methods in [34] except some adjustment to the fractional case with varying s.

It is obvious that $\widetilde{W}^{s,p}_{4R}(\Omega) \cap \widetilde{W}^{k,p}_{0,4R}(\Omega)$ is a closed vector subspace of $\widetilde{W}^{s,p}_{4R}(\Omega)$ containing $\widetilde{W}^{s,p}_{0,4R}(\Omega)$, so we establish (i).

If $sp < N$, let U be a bounded open subset of $\mathbb{R}^N$ with $\overline{\Omega} \subseteq U$, and we can suppose U to be a domain with extension property. Let $u \in \widetilde{W}^{k,p}_{0,4R}(\Omega)$. As u obtains value 0 on $B_{4R}(\Omega) \setminus \Omega$, so is the same on $B_{4R}(\Omega) \setminus U$. Then we have $u \in \widetilde{W}^{s,p}_{0,4R}(U)$, particularly we have $\widetilde{W}^{s,p}_{0,4R}(U) \subset L^{p^*}(B_{4R}(\Omega))$ (see [32], Theorem 6.5). Since

$$\frac{[u]^p_{W^{s,p}_{4R}(U)}}{\left(\int_U |u|^{p^*} dx\right)^{p/p^*}} = \frac{[u]^p_{W^{s,p}_{4R}(\Omega)}}{\left(\int_\Omega |u|^{p^*} dx\right)^{p/p^*}},$$

and the value of

$$\inf \left\{ \frac{[u]^p_{W^{s,p}_{4R}(\Omega)}}{\left(\int_\Omega |u|^{p^*} dx\right)^{p/p^*}} : u \in \widetilde{W}^{s^-,p}_{0,4R}(\Omega) \setminus 0 \right\}$$

is independent of Ω and U (indeed it is the best Sobolev imbedding constant, see [37] and Remark 3.4 in [17]), so we conclude (ii).

If $sp > N$ and $u \in \widetilde{W}^{s^-,p}_{0,4R}(\Omega)$, we can always find ϵ small enough such that $(s-\epsilon)p > N$, then by Proposition 0.2.3 we have $u \in C(\overline{\Omega}) \cap \widetilde{W}^{s,p}_{4R}(\Omega)$, which implies that $u = 0$ on $\partial\Omega$. So we have $u \in \widetilde{W}^{s,p}_{0,4R}(\Omega)$ (see Theorem 9.17 in [24], the regularity of $\partial\Omega$ not used in the proof $(i) \Rightarrow (ii)$, and this also works for the fractional case). $\square$

Now we define

$$\underline{\lambda}^1_{s,p} = \inf \left\{ [u]^p_{\widetilde{W}^{s,p}_{4R}(\Omega)} : \ u \in \widetilde{W}^{s^-,p}_{0,4R}(\Omega) \setminus \{0\}, \ and \ \|u\|_{L^p(\Omega)} = 1 \right\},$$

where the semi-norm is defined by (8) and (5). We define the admissible spaces for first (s,p)-eigenfunction of $\underline{\lambda}^1_{s,p}$, denoted as $\underline{\mathcal{S}}_{s,p}(\Omega)$, and

$$\underline{\mathcal{S}}_{s,p}(\Omega) := \left\{ u : u \in \widetilde{W}^{s^-,p}_{0,4R}(\Omega), \ \|u\|_{L^p(\Omega)} = 1 \right\}.$$

As an eigenvalue of nonlocal p-Laplacian equations, $\underline{\lambda}^1_{s,p}$ is understood in the following weak sense

$$\begin{cases} u \in \widetilde{W}^{s^-,p}_{0,tR}(\Omega), \\ \int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \frac{|u(x)-u(y)|^{p-s}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} dxdy \\ = \underline{\lambda}^1_{s,p} \int_\Omega |u|^{p-2} uv dx, \ for \ every \ v \in \widetilde{W}^{s^-,p}_{0,tR}(\Omega). \end{cases}$$

We can see that $\underline{\lambda}^1_{s,p}$ is well-defined thanks to Theorem 0.2.1 and Proposition 2.6.1. Although the proof of Theorem 0.2.1 therein is on the space $\widetilde{W}^{s,p}_{0,4R}(\Omega)$, it works also for the space $\widetilde{W}^{s,p}_{4R}(\Omega)$ only by replacing the approximation function space $C_0^\infty(\Omega)$ with $C^\infty(\Omega)$, together with the fact that $\widetilde{W}^{s^-,p}_{0,4R}(\Omega)$ is a closed subspace. Obviously we have

$$0 < \underline{\lambda}^1_{s,p} \leq \lambda^1_{s,p}.$$

Now we list some basic properties of the corresponding first (s,p)-eigenfunction, denoted by $\underline{u}_s$,

- there exists exactly only one strictly positive (or strictly negative) (even Ω disconnected) $\underline{u}_s \in \widetilde{W}^{s^-,p}_{0,4R}(\Omega)$ such that

$$\int_\Omega |\underline{u}_s|^p dx = 1, \ [\underline{u}_s]_{W^{s,p}_{4R}(\Omega)} = \underline{\lambda}^1_{s,p};$$

- $\underline{u}_s \in L^\infty(\Omega) \cap C(\Omega)$;

- the positive (or negative) eigenfunctions of $\underline{\lambda}^1_{s,p}$ are proportional.

We emphasis that if we check the proofs of the same properties of $\lambda^1_{s,p}$ and u_s, we would find that we can also use them directly to the proofs of $\underline{\lambda}^1_{s,p}$ and $\underline{u}_s$. Since we are working in the nonlocal spaces, and we can always get the tools, such as Poincaré-type inequality and Rellich-type compactness, which are necessary.

Proof. In fact, the proof of the properties is standard base on the Proposition 2.6.1. The existence of $\underline{u}_s$ is a consequence of Theorem 0.2.1, and the uniqueness basically follows from the strict convexity of the norm $W^{s,p}_{4R}(\Omega)$ (see e.g. [37] Theorem 4.1). And the boundedness and continuity of the first (s,p)-eigenfunction follows from Theorem 2.2.2 and Theorem 2.2.3. For the details one can refer to such as [37][42][56][17][20][22] etc. And for the proportionality of all the positive (or negative) eigenfunctions to $\underline{\lambda}^1_{s,p}$ one can refer to Theorem 2.2.4 in section 2.2 and corresponding references therein. □

2.6.2 Asymptotic Behaviour from Below

Theorem 2.6.1. *Let $0 < k < s < 1$ and $1 < p < +\infty$, let Ω be an open bounded set in $\mathbb{R}^N$. We have*

$$\lim_{k\to s^-} \underline{\lambda}^1_{k,p} = \lim_{k\to s^-} \lambda^1_{k,p} = \underline{\lambda}^1_{s,p},$$

and

$$\lim_{k\to s^-} [\underline{u}_k - \underline{u}_s]_{W^{k,p}_{4R}(\Omega)} = \lim_{k\to s^-} [u_k - \underline{u}_s]_{W^{k,p}_{4R}(\Omega)} = 0.$$

Proof. We utilize the strategy as in the proof of Theorem 3.2 in [34].

Now we start to prove the convergence of the eigenvalues $\underline{\lambda}^1_{k,p}$ as $k \to s$ in step 1 and step 2.

Step 1. Suppose any $u \in \widetilde{W}^{s^-,p}_{0,4R}(\Omega)$ with $\|u\|_{L^p(\Omega)} = 1$, we have that

$$\begin{aligned}
&\lambda^1_{k,p} \le \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \tfrac{|u(x)-u(y)|^p}{|x-y|^{N+kp}} dxdy \\
&= \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \tfrac{|u(x)-u(y)|^p}{|x-y|^{N+sp-sp+kp}} dxdy \\
&\le (\tfrac{5R}{2})^{(s-k)p} \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \tfrac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy,
\end{aligned}$$

then by the arbitrariness of k as $k \to s^-$, we infer that

$$\lim_{k\to s^-} \lambda^1_{k,p} \le \underline{\lambda}^1_{s,p}.$$

Step 2. Since we already know that $\underline{\lambda}^1_{s,p} \le \lambda^1_{s,p}$ for $\forall\ 0 < s < 1$, we only need to verify that $\underline{\lambda}^1_{s,p} \le \lim\limits_{k\to s^-} \underline{\lambda}^1_{k,p}$.

Let $\{k\} \subset (0, s)$ be a strictly increasing sequence to s, and let $v_k \in \widetilde{W}^{k^-,p}_{0,4R}(\Omega)$ with

$$v_k > 0,\ \|v_k\|_{L^p} = 1,\ [v_k]^p_{W^{k,p}_{4R}(\Omega)} = \underline{\lambda}^1_{k,p},$$

Of course we can make $v_k < 0$, the rest are the same.

Obviously there holds that

$$\sup_{k<s} [v_k]^p_{W^{k,p}_{4R}(\Omega)} < +\infty. \tag{2.14}$$

Let $0 < t < s$. Then up to a subsequence $\{v_k\}$ (not relabelled) and thanks to Theorem 0.2.1, for $t < k$ we get some $u \in \widetilde{W}^{t,p}_{0,4R}(\Omega)$ such that $v_k \rightharpoonup u$ weakly in $\widetilde{W}^{t,p}_{0,4R}(\Omega)$ and $v_k \to u$ strongly in $L^p(\Omega)$. Let $k \to s$, then we have the sequence $\{v_k\}$ is bounded in $\widetilde{W}^{t,p}_{0,4R}(\Omega)$ for any $t \in (0, s)$, so it holds that

$$u \in \bigcap_{0<t<s} \widetilde{W}^{t,p}_{0,4R}(\Omega),$$

and

$$u > 0\ a.e.\ in\ \Omega,\ \|u\|_{L^p(\Omega)} = 1.$$

Moreover, for every $t < s$, there holds by the lower semi-continuity

$$\begin{aligned}
[u]^p_{W^{t,p}_{4R}(\Omega)} &\le \liminf_{k\to s} [v_k]^p_{W^{t,p}_{4R}(\Omega)} \le \liminf_{k\to s} (\tfrac{5R}{2})^{(k-t)p} [v_k]^p_{W^{k,p}_{4R}(\Omega)} \\
&= \lim_{k\to s} (\tfrac{5R}{2})^{(k-t)p} \underline{\lambda}^1_{k,p} = (\tfrac{5R}{2})^{(s-t)p} \lim_{k\to s} \underline{\lambda}^1_{k,p}.
\end{aligned}$$

Then by the arbitrariness of t and (2.14), we infer that $u \in W^{s,p}_{4R}(\Omega)$, hence

$$u \in W^{s^-,p}_{0,4R}(\Omega),$$

and the fact

$$\underline{\lambda}^1_{s,p} \le [u]^p_{W^{s,p}_{4R}(\Omega)} \le \lim_{k\to s} \underline{\lambda}^1_{k,p}.$$

Then together with step 1 and the fact that $\underline{\lambda}^1_{k,p} \leq \lambda^1_{k,p}$, it follows that

$$\lim_{k\to s} \underline{\lambda}^1_{k,p} = \lim_{k\to s} \lambda^1_{k,p} = \underline{\lambda}^1_{s,p}.$$

Step 3. Now we start to prove the convergence of the eigenfunctions in the semi-norm $W^{k,p}_{4R}(\Omega)$ $(k < s)$.

By the uniqueness of first (s,p)-eigenfunctions (up to the normalization and choice of the sign), we infer from step 2 that $v_k = \underline{u}_k$ and $u = \underline{u}_s$, and

$$\lim_{k\to s^-} \underline{u}_k = \underline{u}_s \ weakly\ in\ \widetilde{W}^{t,p}_{0,4R}(\Omega),\ \ for\ \forall\ t < s.$$

Since they keep the normalization by

$$\lim_{k\to s^-} \|\frac{\underline{u}_k + \underline{u}_s}{2}\|_{L^p(\Omega)} = 1,$$

then

$$\liminf_{k\to s^-} [\frac{\underline{u}_k + \underline{u}_s}{2}]^p_{W^{k,p}_{4R}(\Omega)} \geq \underline{\lambda}^1_{s,p}.$$

Then again applying for the classical Clarkson's inequalities and the same process as in Theorem 2.4.2, we obtain for $2 < p < +\infty$

$$[\frac{\underline{u}_k + \underline{u}_s}{2}]^p_{W^{k,p}_{4R}(\Omega)} + [\frac{\underline{u}_k - \underline{u}_s}{2}]^p_{W^{k,p}_{4R}(\Omega)} \leq \frac{1}{2}[\underline{u}_k]^p_{W^{k,p}_{4R}(\Omega)} + \frac{1}{2}[\underline{u}_s]^p_{W^{k,p}_{4R}(\Omega)},$$

and for $1 < p \leq 2$

$$[\frac{\underline{u}_k + \underline{u}_s}{2}]^{\frac{p}{p-1}}_{W^{k,p}_{4R}(\Omega)} + [\frac{\underline{u}_k - \underline{u}_s}{2}]^{\frac{p}{p-1}}_{W^{k,p}_{4R}(\Omega)} \leq \frac{1}{2}[\underline{u}_k]^{\frac{p}{p-1}}_{W^{k,p}_{4R}(\Omega)} + \frac{1}{2}[\underline{u}_s]^{\frac{p}{p-1}}_{W^{k,p}_{4R}(\Omega)},$$

then together with the fact established in step 2, we conclude that

$$\lim_{k\to s^-} [\underline{u}_k - \underline{u}_s]_{W^{k,p}_{4R}(\Omega)} = 0.$$

Similarly starting from the fact $\lim_{k\to s^-} \lambda^1_{k,p} = \underline{\lambda}^1_{s,p}$, it also holds

$$\lim_{k\to s^-} [u_k - \underline{u}_s]_{W^{k,p}_{4R}(\Omega)} = 0.$$

□

2.6.3 A Glance at Dual space

For $s \in (0,1)$, $p \in (1,+\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$, following the symbol setting in [17] we denote the dual space of $\widetilde{W}^{s,p}_{0,tR}(\Omega)$ as $\widetilde{W}^{-s,q}_{tR}(\Omega)$ defined by

$$\widetilde{W}^{-s,q}_{tR}(\Omega) := \{F : \widetilde{W}^{s,p}_{0,tR}(\Omega) \to \mathbb{R},\ \ F\ linear\ and\ continuous\}.$$

We define the space $L^q(\Omega\times\Omega)$ for $1\le q<+\infty$ by

$$L^q(\Omega\times\Omega):=\{u(x,y):\left\{\int_{\Omega\times\Omega}|u(x,y)|^q dxdy\right\}^{\frac{1}{q}}<+\infty\}. \tag{2.15}$$

Following the definition in [17], we defines the linear and continuous operator

$$R_{s,p}:\widetilde{W}^{s,p}_{0,tR}(\Omega)\to L^p(B_{tR}(\Omega)\times B_{tR}(\Omega))$$

by

$$R_{s,p}(u)(x,y)=\frac{u(x)-u(y)}{|x-y|^{N/p+s}},\ for\ every\ u\in\widetilde{W}^{s,p}_{0,tR}(\Omega).$$

Lemma 2.6.1. *The operator $R^*_{s,p}:L^q(B_{tR}(\Omega)\times B_{tR}(\Omega))\to\widetilde{W}^{-s,q}_{tR}(\Omega)$ defined by*

$$\langle R^*_{s,p}(\phi),u\rangle:=\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\phi(x,y)\frac{u(x)-u(y)}{|x-y|^{N/p+s}}dxdy,\ for\ every\ u\in\widetilde{W}^{s,p}_{0,tR}(\Omega),$$

is linear and continuous. Moreover, $R^*_{s,p}$ is the adjoint of $R_{s,p}$.

Proof. For the proof of this lemma, one can refer to *Lemma* 8.1 in [17]. There is no essential difference. □

Remark 2.6.1. *The operator $R^*_{s,p}$ has to be thought of as a sort of nonlocal divergence. Observe that by performing a* **discrete integration by parts**, *$R^*_{s,p}$ can be formally written as*

$$R^*_{s,p}(\phi)(x)=\int_{B_{tR}(\Omega)}\frac{\phi(x,y)-\phi(y,x)}{|x-y|^{N/p+s}}dy,\ x\in B_{tR}(\Omega),$$

so that

$$\langle R^*_{s,p}(\phi),u\rangle=\int_\Omega\left(\int_{B_{tR}(\Omega)}\frac{\phi(x,y)-\phi(y,x)}{|x-y|^{N/p+s}}dy\right)u(x)dx,\ \ u\in\widetilde{W}^{s,p}_{0,tR}(\Omega).$$

Indeed, by using this formula

$$\begin{aligned}&\textstyle\int_{B_{tR}(\Omega)}u(x)R^*_{s,p}(\phi)(x)dx\\&=\textstyle\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}u(x)\frac{\phi(x,y)}{|x-y|^{N/p+s}}dydx-\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}u(x)\frac{\phi(y,x)}{|x-y|^{N/p+s}}dydx,\end{aligned}$$

and exchanging the role of x and y in the second integral in the down line, we obtain that this is formally equivalent to the formula in Lemma 2.6.1.

Lemma 2.6.2. *For every $f\in\widetilde{W}^{-s,q}_{tR}(\Omega)$, one has*

$$\begin{aligned}&\|f\|_{\widetilde{W}^{-s,q}_{tR}(\Omega)}\\&=\min_{\phi\in L^q(B_{tR}(\Omega)\times B_{tR}(\Omega))}\left\{\|\phi\|_{L^q(B_{tR}(\Omega)\times B_{tR}(\Omega))}:R^*_{s,p}(\phi)=f\ in\ B_{tR}(\Omega)\right\}.\end{aligned}$$

Proof. For the details to get this one can refer to *Proposition* 8.3 and *Corollary* 8.4 in [17], and there is no essential difference from here. □

Remark 2.6.2. *By Lemma 2.6.2, we know that for every $f \in \widetilde{W}_{tR}^{-s,q}(\Omega)$, we have one representation function $\phi \in L^q(B_{tR}(\Omega) \times B_{tR}(\Omega))$, s.t. $R_{s,p}^*(\phi) = f$. Obviously, the definition and Lemma 2.6.1 and 2.6.2 here also work for the space $\widetilde{W}_{tR}^{s,p}(\Omega)$, and of course the space $\widetilde{W}_{0,tR}^{s^-,p}(\Omega)$.*

In fact, we have established a homeomorphism between the space $\widetilde{W}_{0,tR}^{s,p}(\Omega)$ and its dual space $\widetilde{W}_{tR}^{-s,q}(\Omega)$ by the mapping $(-\Delta_p)^s$, which will be used later. For detailed information, one can see section A1.

2.6.4 Some Equivalent Characterizations

Under no assumptions on $\partial\Omega$, we give some equivalent characterizations for the space $\widetilde{W}_{0,4R}^{s^-,p}(\Omega)$, aiming to characterize the behaviour of

$$\lim_{k\to s^-} \lambda_{k,p}^1 = \lambda_{s,p}^1.$$

Here are the paralleling results with the similar strategy as in the section 4 of [34], which describes the behaviour of p-Rayleigh quotients with varying p of the equations (2.1).

Theorem 2.6.2. *Let $1 < p < +\infty$ and $0 < s < 1$, let Ω be an open bounded set in $\mathbb{R}^N$, the following facts are equivalent:*
(a) $\lim\limits_{k\to s^-} \lambda_{k,p}^1 = \lambda_{s,p}^1$;
(b) $\widetilde{W}_{0,4R}^{s^-,p}(\Omega) = \widetilde{W}_{0,4R}^{s,p}(\Omega)$;
(c) $\underline{\lambda}_{s,p}^1 = \lambda_{s,p}^1$;
(d) $\underline{u}_s = u_s$;
(e) $\underline{u}_s \in \widetilde{W}_{0,4R}^{s,p}(\Omega)$;
(f) the solution $u \in \widetilde{W}_{0,4R}^{s^-,p}(\Omega)$ of the equation

$$\begin{aligned}\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \tfrac{|u(x)-u(y)|^{p-2}}{|x-y|^{N+sp}}&(u(x)-u(y))(v(x)-v(y))dxdy\\ &= \int_\Omega v dx,\ \forall\ v \in \widetilde{W}_{0,4R}^{s^-,p}(\Omega),\end{aligned}$$

given by Lemma 2.6.3 belongs to $\widetilde{W}_{0,4R}^{s,p}(\Omega)$.

Here we want to mention another result in chapter 1 (see Corollary 1.2.1), where we utilize Γ-convergence to give an equivalent form of the space $W_0^{s^-,p}(\Omega)$ with no regularity assumption on $\partial\Omega$. Of course, it also does work in the *relative-nonlocal* settings here.

Before we prove the Theorem 2.6.2, we need to establish the following comparison lemma:

Lemma 2.6.3 (Comparison Lemma)**.** *Let $1 < p < +\infty$ and $s \in (0,1)$, let q satisfy $\frac{1}{p}+\frac{1}{q}=1$, then the following facts hold:*
(i) for every $f \in L^q(\Omega)$ and every $F(x,y) \in L^q(B_{4R}(\Omega)\times B_{4R}(\Omega))$, there exists one and only one solution $w \in \widetilde{W}^{s^-,p}_{0,4R}(\Omega)$ such that for every $v \in \widetilde{W}^{s^-,p}_{0,4R}(\Omega)$

$$\begin{aligned}&\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \tfrac{|w(x)-w(y)|^{p-2}}{|x-y|^{N+sp}}(w(x)-w(y))(v(x)-v(y))dxdy\\ &=\int_\Omega fvdx + \int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} F(x,y)\tfrac{v(x)-v(y)}{|x-y|^{N/p+s}}dxdy,\end{aligned}$$

and the map

$$\begin{array}{rcl} L^q(\Omega)\times L^q(B_{4R}(\Omega)\times B_{4R}(\Omega)) & \to & W^{s,p}_{4R}(\Omega)\\ (f,F) & \mapsto & w\end{array}$$

is continuous;
(ii) if $F_1, F_2 \in L^q(\Omega)$ with $F_1 \le F_2$ a.e. in Ω and $w_1, w_2 \in \widetilde{W}^{s^-,p}_{0,4R}(\Omega)$ are the solutions of

$$\begin{aligned}&\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \tfrac{|w_t(x)-w_t(y)|^{p-2}}{|x-y|^{N+sp}}(w_t(x)-w_t(y))(v(x)-v(y))dxdy\\ &\qquad\qquad\qquad\qquad = \int_\Omega F_t v dx,\ \forall\ v \in \widetilde{W}^{s^-,p}_{0,4R}(\Omega),\end{aligned}$$

then it holds $w_1 \le w_2$ a.e. in Ω.

Remark 2.6.3. *In fact, we can establish stronger results than (i) in Lemma 2.6.3. We establish that the operator $(-\Delta_p)^s$ is a homeomorphism of $\widetilde{W}^{s,p}_{0,tR}(\Omega)$ onto its dual space $\widetilde{W}^{-s,q}_{tR}(\Omega)$. For details, we refer the reader to section A1. Also, we can see that by the same process as in section A1, the operator $(-\Delta_p)^s$ also is a homeomorphism of $\widetilde{W}^{s,p}_{tR}(\Omega)$ onto the corresponding dual space $(\widetilde{W}^{s,p}_{tR}(\Omega))^*$.*

Proof. The only difference between this lemma and the Theorem 7.1 in [34] is the nonlocal setting here.

We can see that (i) is a direct result of Proposition 2.6.1. Indeed just by Hölder inequality and Young's inequality respectively to conclude a coercive result, this gets the existence of the solution; then by a strictly convexity property of the semi-norm $W^{s,p}_{4R}(\Omega)$ the uniqueness is determined.

Now we attempt to prove (ii). Since $(w_1-w_2)^+ \in \widetilde{W}^{s^-,p}_{0,4R}(\Omega)$, we have

$$\begin{aligned}&\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \tfrac{|w_1(x)-w_1(y)|^{p-2}}{|x-y|^{N+sp}}(w_1(x)-w_1(y))\\ &\times((w_1-w_2)^+(x)-(w_1-w_2)^+(y))dxdy = \int_\Omega F_1(x)(w_1-w_2)^+(x)dx,\\ &\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \tfrac{|w_2(x)-w_2(y)|^{p-2}}{|x-y|^{N+sp}}(w_2(x)-w_2(y))\\ &\times((w_1-w_2)^+(x)-(w_1-w_2)^+(y))dxdy = \int_\Omega F_2(x)(w_1-w_2)^+(x)dx,\end{aligned}$$

hence,

$$\begin{aligned}&0 \le \int_{\{w_1>w_2\}\times\{w_1>w_2\}}\\ &\Big(\tfrac{|w_1(x)-w_1(y)|^{p-2}}{|x-y|^{N+sp}}(w_1(x)-w_1(y)) - \tfrac{|w_2(x)-w_2(y)|^{p-2}}{|x-y|^{N+sp}}(w_2(x)-w_2(y))\Big)\\ &\times((w_1-w_2)^+(x)-(w_1-w_2)^+(y))dxdy\\ &=\int_\Omega (F_1-F_2)(x)(w_1-w_2)^+(x)dx \le 0,\end{aligned}$$

it follows that $w_1 \le w_2$ a.e. in Ω. $\square$

Now we are prepared to prove the Theorem 2.6.2.

Proof of Theorem 2.6.2. Obviously $(a) \Leftrightarrow (c)$.

Now we consider the assertions from (b) to (f). Clearly we have $(b) \Rightarrow (c)$.

If $\underline{\lambda}^1_{s,p} = \lambda^1_{s,p}$, we infer that $u_s \in \widetilde{W}^{s,p}_{0,4R}(\Omega) \subset \widetilde{W}^{s^-,p}_{0,4R}(\Omega)$ satisfies

$$u_s > 0 \; a.e. \; in \; \Omega, \; \int_\Omega {u_s}^p dx = 1, \; and \; [u_s]^p_{W^{s,p}_{4R}(\Omega)} = \underline{\lambda}^1_{s,p}.$$

By the uniqueness of corresponding eigenfunction of $\underline{\lambda}^1_{s,p}$, we have that $u_s = \underline{u}_s$, namely $(c) \Rightarrow (d)$.

Of course, $(d) \Rightarrow (e)$.

If $\underline{u}_s \in \widetilde{W}^{s,p}_{0,4R}(\Omega)$, let

$$f_k = \min\{\underline{\lambda}^1_{s,p}(k\underline{u}_s)^{p-1}, 1\}$$

and let w_k be the solution of

$$\begin{aligned}\textstyle\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|w_k(x)-w_k(y)|^{p-2}}{|x-y|^{N+sp}}&(w_k(x)-w_k(y))(v(x)-v(y))dxdy \\ &= \textstyle\int_\Omega f_k v dx, \; \forall \; v \in \widetilde{W}^{s^-,p}_{0,4R}(\Omega)\end{aligned}$$

by Lemma 2.6.3. Since $0 \le w_k \le \underline{\lambda}^1_{s,p}(k\underline{u}_s)^{p-1}$ a.e. in Ω, we have $w_k \le k\underline{u}_s$ a.e. in Ω according to (ii) of Lemma 2.6.3. Because $w_k \in W^{s,p}_{4R}(\Omega)$ and $k\underline{u}_s \in \widetilde{W}^{s,p}_{0,4R}(\Omega)$, we infer that $w_k \in \widetilde{W}^{s,p}_{0,4R}(\Omega)$.

Then letting $k \to +\infty$, we have $\{f_k\}$ converge to 1 in $L^p(\Omega)$. Hence from (i) of Lemma 2.6.3 we infer that

$$\lim_{k\to+\infty} [w_k - u]^p_{W^{s,p}_{4R}(\Omega)} = 0,$$

whence $u \in \widetilde{W}^{s,p}_{0,4R}(\Omega)$. Then $(e) \Rightarrow (f)$.

Now let us suppose that (f) holds and u is the solution in (f). If $F \in L^\infty(\Omega)$ and $w \in \widetilde{W}^{s^-,p}_{0,4R}(\Omega)$ is the solution of

$$\begin{aligned}\textstyle\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|w(x)-w(y)|^{p-2}}{|x-y|^{N+sp}}&(w(x)-w(y))(v(x)-v(y))dxdy \\ &= \textstyle\int_\Omega F v dx, \; \forall \; v \in \widetilde{W}^{s^-,p}_{0,4R}(\Omega),\end{aligned}$$

we have that $-M^{p-1} \le F \le M^{p-1}$ for some $M > 0$, whence $-Mu \le w \le Mu$ a.e. in Ω. It follows $w \in \widetilde{W}^{s,p}_{0,4R}(\Omega)$.

Now suppose that $w \in \widetilde{W}^{s^-,p}_{0,4R}(\Omega)$. Thanks to Theorem A1.3, we have a unique $F \in \widetilde{W}^{-s^-,q}_{4R}(\Omega)$ such that

$$\begin{aligned}\textstyle\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \frac{|w(x)-w(y)|^{p-2}}{|x-y|^{N+sp}}&(w(x)-w(y))(v(x)-v(y))dxdy \\ &= \textstyle\int_\Omega F v dx, \; \forall \; v \in \widetilde{W}^{s,p}_{4R}(\Omega).\end{aligned}$$

Then due to Lemma 2.6.2 and Lemma 2.6.1, we know there exists one representation function $\phi(x,y) \in L^q(B_{4R}(\Omega) \times B_{4R}(\Omega))$ such that

$$\begin{aligned}&\langle \phi, R_{s,p}(v)\rangle_{(L^q(B_{4R}(\Omega)\times B_{4R}(\Omega)),L^p(B_{4R}(\Omega)\times B_{4R}(\Omega)))} = \langle F, v\rangle_{(\widetilde{W}_{4R}^{-s^-,q}(\Omega),\widetilde{W}_{0,4R}^{s^-,p}(\Omega))}\\ &= \langle R_{s,p}^*(\phi), v\rangle_{(\widetilde{W}_{4R}^{-s^-,q}(\Omega),\widetilde{W}_{0,4R}^{s^-,p}(\Omega))} := \int_\Omega R_{s,p}^*(\phi)(x)v(x)dx.\end{aligned}$$

Then by the density of $C_c^\infty(B_{4R}(\Omega) \times B_{4R}(\Omega))$ in $L^q(B_{4R}(\Omega) \times B_{4R}(\Omega))$ (see (2.15)), let $\{f_k\} \subset C_c^\infty(B_{4R}(\Omega) \times B_{4R}(\Omega))$ be the sequence converging to ϕ in $L^q(B_{4R}(\Omega) \times B_{4R}(\Omega))$. So for every $v \in \widetilde{W}_{0,4R}^{s^-,p}(\Omega)$ we have

$$\begin{aligned}0 &= \lim_{k\to+\infty} \langle \phi - f_k, R_{s,p}(v)\rangle_{(L^q(B_{4R}(\Omega)\times B_{4R}(\Omega)),L^p(B_{4R}(\Omega)\times B_{4R}(\Omega)))}\\ &= \int_\Omega R_{s,p}^*(\phi - f_k)(x)v(x)dx.\end{aligned}$$

Since $f_k \in C_c^\infty(B_{4R}(\Omega) \times B_{4R}(\Omega))$, we have

$$L^\infty(B_{4R}(\Omega)) \ni R_{s,p}^*(f_k)(x) = \int_{B_{tR}(\Omega)} \frac{f_k(x,y) - f_k(y,x)}{|x-y|^{N/p+s}} dy.$$

Then there exists unique $w_k \in \widetilde{W}_{0,4R}^{s,p}(\Omega)$ such that

$$\begin{aligned}&\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \tfrac{|w_k(x)-w_k(y)|^{p-2}}{|x-y|^{N+sp}}(w_k(x) - w_k(y))(v(x) - v(y))dxdy\\ &\qquad = \int_\Omega R_{s,p}^*(f_k)vdx, \ \forall\ v \in \widetilde{W}_{0,4R}^{s^-,p}(\Omega).\end{aligned}$$

Since

$$\begin{aligned}&\int_{B_{4R}(\Omega)\times B_{4R}(\Omega)} \tfrac{|w(x)-w(y)|^{p-2}}{|x-y|^{N+sp}}(w(x) - w(y))(v(x) - v(y))dxdy\\ &\qquad = \int_\Omega Fvdx = \int_\Omega R_{s,p}^*(\phi)vdx, \ \forall\ v \in \widetilde{W}_{0,4R}^{s^-,p}(\Omega),\end{aligned}$$

it follows from (i) of Lemma 2.6.3

$$\lim_{k\to+\infty} [w_k - w]_{W_{4R}^{s,p}(\Omega)}^p = 0,$$

whence $w \in \widetilde{W}_{0,4R}^{s,p}(\Omega)$. Therefore $(f) \Rightarrow (b)$. $\square$

Appendix

A1 Homeomorphism

By adapting the settings in section 2.6.3, here we mimic the strategy in [38][57] to establish the homeomorphism of the operator $(-\Delta_p)^s$ from the space $\widetilde{W}^{s,p}_{0,tR}(\Omega)$ to its dual space $\widetilde{W}^{-s,q}_{tR}(\Omega)$.

Definition A1. *Let X be a Banach space. An operator $T : X \to X^*$ is said to be of* **type M** *if for any weakly-convergent sequence $x_n \rightharpoonup x$ such that $T(x_n) \rightharpoonup f$ and*

$$\limsup \langle x_n, T(x_n)\rangle \leq \langle x, f\rangle, \tag{16}$$

one has $T(x) = f$. T is said to be **hemi-continuous** *if for any fixed $x, y \in X$, the real-valued function*

$$s \mapsto \langle y, T(x + sy)\rangle$$

is continuous.

Theorem A1.1 ([68], Chapter 2, Lemma 2.1)**.** *Let X be a reflexive Banach space and $T : X \to X^*$ be a hemi-continuous and monotone operator. Then T is of type M.*

Proof. For any fixed $y \in X$, (x_n), x and f as in Definition A1, the assumed monotonicity of T yields

$$0 \leq \langle x_n - y, T(x_n) - T(y)\rangle$$

for all n; hence, from (16), we have

$$\langle x - y, T(y)\rangle \leq \langle x - y, f\rangle.$$

In particular, for any $z \in X$ and $n \in \mathbb{N}$,

$$\langle z, T(x - (\frac{z}{n}))\rangle \leq \langle z, f\rangle,$$

which, in conjunction with hemi-continuity, immediately yields

$$\langle z, T(x)\rangle \leq \langle z, f\rangle$$

for all $z \in X$. This implies $T(x) = f$, as claimed. $\square$

Theorem A1.2 ([68], Chapter 2, Theorem 2.1)**.** *Let X be a separable and reflexive Banach space, and let $T : X \to X^*$ be of type M and bounded. If for some $f \in X^*$ there exists $\epsilon > 0$ for which $\langle x, T(x)\rangle > \langle x, f\rangle$ for every $x \in X$ with $\|x\|_X > \epsilon$, then f belongs to the range of T.*

Lemma A1. *For $x, y \in \mathbb{R}^N$ and a constant p, we have*

$$\begin{aligned}\frac{1}{2}[(|x|^{p-2} - |y|^{p-2})(|x|^2 - |y|^2) + (|x|^{p-2} + |y|^{p-2})|x-y|^2]\\ = (|x|^{p-2}x - |y|^{p-2}y) \cdot (x-y).\end{aligned}$$

Proof. It is by a straight calculation by writing

$$|x|^2 - |y|^2 = (x+y) \cdot (x-y)$$

and

$$|x-y|^2 = (x-y) \cdot (x-y)$$

on the left-hand side of the equality. □

Let $u, v \in \widetilde{W}^{s,p}_{0,tR}(\Omega)$, then we define the inner product $\langle u, (-\Delta_p)^s v\rangle$ by

$$\langle u, (-\Delta_p)^s v\rangle := \int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(u(x)-u(y))}{|x-y|^{N+sp}} dxdy,$$

which is well-defined by Hölder inequality.

Lemma A2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $t > 1$, $0 < s < 1$, $p \in (1, +\infty)$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then the operator*

$$(-\Delta_p)^s : \widetilde{W}^{s,p}_{0,tR} \to \widetilde{W}^{-s,q}_{tR}(\Omega)$$

is bounded, hemi-continuous and monotone. Also, $(-\Delta_p)^s$ is of type M.

Proof. Let $S \subset \widetilde{W}^{s,p}_{0,tR}(\Omega)$ be bounded, namely $\sup\{\|u\|_{\widetilde{W}^{s,p}_{tR}(\Omega)}, u \in S\} \le C$. For $u \in S$ and w in the unit ball of $\widetilde{W}^{s,p}_{0,tR}(\Omega)$, we have

$$\langle w, (-\Delta_p)^s u\rangle = \int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+sp}} dxdy.$$

Then via Hölder inequality it is clear that

$$\sup\{\|(-\Delta_p)^s u\|_{\widetilde{W}^{-s,q}_{tR}(\Omega)}, u \in S\} \le C,$$

which shows that $(-\Delta_p)^s$ is bounded.

For the proof of the hemi-continuity, let $t \in \mathbb{R}$ fixed. For $1 < p \leq 2$,

$$|u+tv|^{p-1} \leq |u|^{p-1} + |t|^{p-1}|v|^{p-1}, \tag{17}$$

while for $p > 2$,

$$|u+tv|^{p-1} \leq 2^{p-2}(|u|^{p-1} + |t|^{p-1}|v|^{p-1}). \tag{18}$$

At the same time, it follows from the definition that

$$\begin{aligned}&\langle v, (-\Delta_p)^s(u+tv)\rangle \\ &= \int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \frac{|(u+tv)(x)-(u+tv)(y)|^{p-2}((u+tv)(x)-(u+tv)(y))(v(x)-v(y))}{|x-y|^{N+sp}}dxdy.\end{aligned} \tag{19}$$

In view of

$$(u+tv)(x) - (u+tv)(y) = u(x) - u(y) + t(v(x) - v(y)),$$

together with (17) and (18), the integrand in (19) is bounded by

$$\begin{aligned}&|(u+tv)(x) - (u+tv)(y)|^{p-2}((u+tv)(x) - (u+tv)(y))(v(x) - v(y)) \\ &\leq \max\{1, 2^{p-2}\}(|u(x) - u(y)|^{p-1}|v(x) - v(y)| + |t|^{p-1}|v(x) - v(y)|^p),\end{aligned}$$

which is integrability by Hölder inequality. Then by Lebesgue Dominated Convergence Theorem we obtain the hemi-continuity of operator $(-\Delta_p)^s$.

The proof of monotonicity need the help of Lemma A1. In fact, for $p \geq 2$ and $\xi, \eta \in \mathbb{R}^N$,

$$|\xi - \eta|^p = |\xi - \eta|^{p-2}(\xi - \eta)^2 \leq 2^{p-3}|\xi - \eta|^2(|\xi|^{p-2} + |\eta|^{p-2}),$$

combined with the identity in Lemma A1, yields the estimate

$$|\xi - \eta|^p \leq 2^{p-2}(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta). \tag{20}$$

On the other hand, for $1 < p \leq 2$ ($\xi \neq 0, \eta \neq 0$), we utilize the following inequality from [49]

$$(p-1)|\xi - \eta|^p \leq [(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta)](|\xi|^p + |\eta|^p)^{\frac{2-p}{p}}. \tag{21}$$

Then by the definition of operator $(-\Delta_p)^s$ and letting u and v fixed in $\widetilde{W}^{s,p}_{0,tR}(\Omega)$, we have

$$\begin{aligned}&\langle u - v, (-\Delta_p)^s(u) - (-\Delta_p)^s(v)\rangle \\ &= \int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}(|u(x) - u(y)|^{p-2}(u(x) - u(y)) - |v(x) - v(y)|^{p-2}(v(x) - v(y))) \\ &\times ((u-v)(x) - (u-v)(y)) \frac{dxdy}{|x-y|^{N+sp}} \\ &= \int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}(|u(x) - u(y)|^{p-2}(u(x) - u(y)) - |v(x) - v(y)|^{p-2}(v(x) - v(y))) \\ &\times ((u(x) - u(y)) - (v(x) - v(y))) \frac{dxdy}{|x-y|^{N+sp}}.\end{aligned} \tag{22}$$

Then we denote $u(x) - u(y)$ as W, and $v(x) - v(y)$ as V. The integrand in (22) becomes

$$(|W|^{p-2}W - |V|^{p-2}V)(W - V),$$

Which, due to (20) and (21), leads to the monotonicity of $(-\Delta_p)^s$.

Since the relative-nonlocal space $\widetilde{W}^{s,p}_{0,tR}(\Omega)$ is reflexive and separable, thanks to Theorem A1.1 we obtain that $(-\Delta_p)^s$ is of type M, which concludes the desired result. $\square$

Now we establish our main result on the homeomorphism of operator $(-\Delta_p)^s$.

Theorem A1.3. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Let $0 < s < 1$ and $p, q \in (1, +\infty)$ such that $1/p + 1/q = 1$. Then the operator $(-\Delta_p)^s$ is a homeomorphism of $\widetilde{W}^{s,p}_{0,tR}(\Omega)$ onto its dual $\widetilde{W}^{-s,q}_{tR}(\Omega)$.*

Proof. We have already proved the continuity of operator $(-\Delta_p)^s$ in Lemma A2, then we need to prove respectively the surjectivity, injectivity and the continuity of the operator $(-\Delta_q)^{-s}$, which is the reverse operator of $(-\Delta_p)^s$.

Step 1. Firstly, we prove the surjectivity of $(-\Delta_p)^s$. Fix $f \in \widetilde{W}^{-s,q}_{tR}(\Omega)$. For $u \in \widetilde{W}^{s,p}_{0,tR}(\Omega)$ with

$$[u]_{W^{s,p}_{tR}(\Omega)} > \max\{1, \|f\|^{\frac{1}{p-1}}_{\widetilde{W}^{-s,q}_{tR}(\Omega)}\};$$

thus for such u, we have

$$\begin{aligned}
&\langle u, (-\Delta_p)^s u\rangle = \int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \tfrac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \\
&= [u]^p_{W^{s,p}_{tR}(\Omega)} = [u]^{p-1}_{W^{s,p}_{tR}(\Omega)}[u]_{W^{s,p}_{tR}(\Omega)} \\
&> \|f\|_{\widetilde{W}^{-s,q}_{tR}(\Omega)}[u]_{W^{s,p}_{tR}(\Omega)},
\end{aligned}$$

from which, together with Theorem A1.2, we can infer that f is in the range of $(-\Delta_p)^s$, namely, $(-\Delta_p)^s$ is surjective.

Step 2. Now we are prepared to prove the injectivity of $(-\Delta_p)^s$.

Now we consider $u, v \in \widetilde{W}^{s,p}_{0,tR}(\Omega)$ such that $(-\Delta_p)^s(u) = (-\Delta_p)^s(v)$. Then we estimate the semi-norm $W^{s,p}_{tR}(\Omega)$ of $u - v$ in space $\widetilde{W}^{s,p}_{0,tR}(\Omega)$. If $1 < p < 2$, we utilize the inequality (21) established in Lemma A2, then by denoting $I := u(x) - u(y)$ and $J := v(x) - v(y)$ we have the following process:

$$\begin{aligned}
&[u-v]^p_{W^{s,p}_{tR}(\Omega)} \\
&= \int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \tfrac{|(u-v)(x)-(u-v)(y)|^p}{|x-y|^{N+sp}} dxdy \\
&= \int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \tfrac{|(u(x)-u(y))-(v(x)-v(y))|^p}{|x-y|^{N+sp}} dxdy \\
&\leq \tfrac{1}{p-1}\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)} \tfrac{(|I|^{p-2}I-|J|^{p-2}J)(I-J)}{|x-y|^{N+sp}} (|I|^p + |J|^p)^{\frac{2-p}{p}} dxdy,
\end{aligned}$$

during which, we used the inequality (21); since $1 < p < 2$, we set $\frac{2p-2}{p} + \frac{2-p}{p} = 1$ as a conjugate pair, then via the Hölder inequality we proceed the inequality process above as

$$\begin{aligned}
&[u-v]^p_{W^{s,p}_{tR}(\Omega)}\\
&\le \tfrac{1}{p-1}\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\tfrac{\{(|I|^{p-2}I-|J|^{p-2}J)(I-J)\}^{\frac{2p-2}{p}}}{|x-y|^{N+sp}}\\
&\times\{(|I|^{p-2}I-|J|^{p-2}J)(I-J)\}^{\frac{2-p}{p}}(|I|^p+|J|^p)^{\frac{2-p}{p}}dxdy\\
&\le \tfrac{1}{p-1}\left(\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\tfrac{(|I|^{p-2}I-|J|^{p-2}J)(I-J)}{|x-y|^{N+sp}}dxdy\right)^{\frac{p}{2p-2}}\\
&\times\left(\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\tfrac{(|I|^{p-2}I-|J|^{p-2}J)(I-J)(|I|^p+|J|^p)}{|x-y|^{N+sp}}dxdy\right)^{\frac{p}{2-p}}\\
&=\tfrac{1}{p-1}\langle u-v,(-\Delta_p)^s(u)-(-\Delta_p)^s(v)\rangle^{\frac{p}{2p-2}}\\
&\times\left(\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\tfrac{(|I|^{p-2}I-|J|^{p-2}J)(I-J)(|I|^p+|J|^p)}{|x-y|^{N+sp}}dxdy\right)^{\frac{p}{2-p}},
\end{aligned}$$

in which, the last integrand can be controlled by

$$\frac{(|I|^p+|J|^p+|I|^{p-1}|J|+|J|^{p-1}|I|)(|I|^p+|J|^p)}{|x-y|^{N+sp}}:=C(u,v).$$

Since $(-\Delta_p)^s(u)=(-\Delta_p)^s(v)$, we have from above process that $[u-v]_{W^{s,p}_{tR}(\Omega)}=0$, then by Poincaré-type inequality, we have $\|u-v\|_{L^p(\Omega)}=0$.

For the case $p\ge 2$, we just utilize (20) directly getting the injectivity of operator $(-\Delta_p)^s$.

Step 3. Now we only need to verify the continuity of reverse operator $(-\Delta_q)^{-s}$. For simplicity, we denote $(-\Delta_q)^{-s}$ by T. Let $T(v_n)\to T(u)$ for $\{v_n\}_n\subset \widetilde{W}^{s,p}_{0,tR}(\Omega)$. We claim that the sequence $\{v_n\}_n$ is bounded.

Indeed, if the sequence $\{v_n\}_n$ is unbounded, one could extract a subsequence $\{u_n\}_n$ with $\|u_n\|_{L^p(\Omega)}>n$. Then set $w_n=\frac{u_n}{\|u_n\|_{L^p(\Omega)}}$ and notice that for arbitrary $\phi\in\widetilde{W}^{s,p}_{0,tR}(\Omega)$ with $[\phi]_{\widetilde{W}^{s,p}_{0,tR}(\Omega)}\le 1$, the equality

$$\begin{aligned}
&|\langle\phi,T(w_n)\rangle|\\
&=\tfrac{1}{[u_n]^{p-1}_{\widetilde{W}^{s,p}_{0,tR}(\Omega)}}\left|\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\tfrac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}dxdy\right|\\
&\le\tfrac{1}{[u_n]^{p-1}_{\widetilde{W}^{s,p}_{0,tR}(\Omega)}}\|T(u_n)\|_{\widetilde{W}^{-s,q}_{tR}(\Omega)}.
\end{aligned}$$

So by let $n\to+\infty$, since $T(u_n)\to T(u)$ and $[u_n]_{\widetilde{W}^{s,p}_{0,tR}(\Omega)}\ge\|u_n\|_{L^p(\Omega)}>n$ by Poincaré-type inequality (see (9)), we infer that

$$\|T(w_n)\|_{\widetilde{W}^{-s,q}_{tR}(\Omega)}\to 0 \tag{23}$$

as $n\to+\infty$.

On the other hand, by the definition of w_n, we directly infer that

$$\begin{aligned}
&\|T(w_n)\|_{\widetilde{W}^{-s,q}_{tR}(\Omega)}\ge\langle w_n,T(w_n)\rangle=\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\tfrac{|w_n(x)-w_n(y)|^p}{|x-y|^{N+sp}}dxdy\\
&=\tfrac{1}{[u_n]^p_{\widetilde{W}^{s,p}_{0,tR}(\Omega)}}\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\tfrac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}dxdy=1,
\end{aligned}$$

which contradicts (23). Then we get that $\{v_n\}_n$ is bounded in $\widetilde{W}^{s,p}_{0,tR}(\Omega)$.

Now we proceed as in step 2 by letting $1<p<2$ and $p\geq 2$ respectively. For the case $p\geq 2$, we directly use (20) to get that

$$[v_n-u]^p_{W^{s,p}_{tR}(\Omega)} \leq s^{p-2}\langle v_n-u, T(v_n)-T(u)\rangle \leq [v_n-u]^p_{W^{s,p}_{tR}(\Omega)}\|T(v_n)-T(u)\|^p_{\widetilde{W}^{-s,q}_{tR}(\Omega)},$$

which implies that $\|v_n-u\|_{L^p(\Omega)}\to 0$ by Poincaré-type inequality as $n\to+\infty$.

On the other hand, if $1<p<2$, we need a small modification of the inequality (21), i.e., for arbitrary $\xi,\eta\in\mathbb{R}^N$ and $\forall\ \epsilon>0$

$$\begin{array}{l}
(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)\cdot(\xi-\eta)=(\xi-\eta)\cdot\int_0^1\frac{d}{dt}\left(|\eta+t(\xi-\eta)|^{p-2}(\eta+t(\xi-\eta))\right)dt\\
=|\xi-\eta|^2\int_0^1|\eta+t(\xi-\eta)|^{p-2}dt\\
+(p-2)\int_0^1|\eta+t(\xi-\eta)|^{p-4}\left((\eta+t(\xi-\eta))\cdot(\xi-\eta)\right)^2dt\\
\geq(p-1)|\xi-\eta|^2\int_0^1|\eta+t(\xi-\eta)|^{p-2}dt\\
\geq(p-1)|\xi-\eta|^2(\epsilon+|\xi|+|\eta|)^{p-2},
\end{array}$$

namely,

$$(p-1)|\xi-\eta|^2(\epsilon+|\xi|+|\eta|)^{p-2}\leq(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)\cdot(\xi-\eta). \tag{24}$$

Then by denoting $I:=v_n(x)-v_n(y)$ and $J:=u(x)-u(y)$, we can write $[v_n-u]^p_{\widetilde{W}^{s,p}_{0,tR}(\Omega)}$ as

$$\begin{array}{l}
\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\frac{|I-J|^p}{(1+|I|+|J|)^{p(2-p)/2}}(\epsilon+|I|+|J|)^{p(2-p)/2}\frac{dxdy}{|x-y|^{N+sp}}\\
\leq\left(\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}(\epsilon+|I|+|J|)^p\frac{dxdy}{|x-y|^{N+sp}}\right)^{1-p/2}\\
\times\left(\int_{B_{tR}(\Omega)\times B_{tR}(\Omega)}\frac{|I-J|^2}{(\epsilon+|I|+|J|)^{2-p}}\frac{dxdy}{|x-y|^{N+sp}}\right)^{p/2}\\
=: X_n+Y_n.
\end{array}$$

In the first term X_n, since $\epsilon>0$ is arbitrary, we set $\epsilon=|x-y|^{N/p+s}$ in X_n, then due to the boundedness of v_n and u in $\widetilde{W}^{s,p}_{0,tR}(\Omega)$, we have that X_n is bounded.

For the term Y_n, again by inequality (24) we have

$$\begin{array}{l}
Y_n\leq\frac{1}{p-1}|\langle v_n-u,T(v_n)-T(u)\rangle|\\
\leq\frac{1}{p-1}\|T(v_n)-T(u)\|_{\widetilde{W}^{-s,q}_{tR}(\Omega)}\|v_n-u\|_{\widetilde{W}^{s,p}_{0,tR}(\Omega)}\\
\leq\frac{1}{p-1}\|T(v_n)-T(u)\|_{\widetilde{W}^{-s,q}_{tR}(\Omega)}\left(\sup_n\|v_n\|_{\widetilde{W}^{s,p}_{0,tR}(\Omega)}+\|u\|_{\widetilde{W}^{s,p}_{0,tR}(\Omega)}\right),
\end{array}$$

which implies that $Y_n\to 0$ as $n\to+\infty$, thanks to the fact that v_n and u is bounded in $\widetilde{W}^{s,p}_{0,tR}(\Omega)$, and the assumption $T(v_n)\to T(u)$ in $\widetilde{W}^{-s,q}_{tR}(\Omega)$.

By all above, we infer that $\|v_n-u\|_{\widetilde{W}^{s,p}_{0,tR}(\Omega)}\to 0$ as $n\to+\infty$. Thus

$$(-\Delta_q)^{-s}:\widetilde{W}^{-s,q}_{tR}(\Omega)\to\widetilde{W}^{s,p}_{0,tR}(\Omega)$$

is continuous.

Then we conclude the results that $(-\Delta_p)^s$ is a homeomorphism of $\widetilde{W}^{s,p}_{0,tR}(\Omega)$ onto $\widetilde{W}^{-s,q}_{tR}(\Omega)$. $\square$

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