Combinatorial properties of cluster algebras associated with marked surfaces

(点付き曲面に付随する団代数の組合せ的性質)

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Preface

Cluster algebras, introduced by Fomin and Zelevinsky in 2002 [FZ02], are commutative algebras with generators called *cluster variables*. The certain tuples of cluster variables are called *clusters*. Their original motivation was to study total positivity of semisimple Lie groups and canonical bases of quantum groups. In recent years, it has interacted with various subjects in mathematics, for example, representation theory of quivers, Poisson geometry, integrable systems, and so on.

By Laurent phenomenon, any cluster variable in a cluster algebra is expressed by a Laurent polynomial of the initial cluster variables $(x_1, \dots, x_n)$ and coefficients $(x_{n+1}, \dots, x_m)$

$$x = \frac{f(x_1, \dots, x_m)}{x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}}$$

where $f(x_1, \dots, x_m) \in \mathbb{Z}[x_1, \dots, x_m]$ and $d_i \in \mathbb{Z}_{\geq 0}$ [FZ02, FZ07]. An explicit formula for the Laurent polynomials of cluster variables is called a *cluster expansion formula*. In [LS13], Lee and Schiffler gave a cluster expansion formula for the cluster algebras of rank 2 in terms of Dyck paths, and as an application, they proved in [LS15] the positivity conjecture of Fomin and Zelevinsky [FZ02]. Another well-known cluster expansion formula is Caldero-Chapoton formula given by categorification [CC, CK, Pa, Pl11a]. It was also applied to solve several conjectures of Fomin and Zelevinsky (see e.g. [CKLP]). Thus cluster expansion formulas are basic and important ingredients to study cluster algebras. In this thesis, we give a cluster expansion formula for cluster algebras given from *marked surfaces* that were developed in [FoG06, FoG09, FST, FoT, GSV].

For a marked surface (S, M) and the associated cluster algebra $\mathcal{A}(S, M)$, its cluster complex is identified with a connected component of the tagged arc complex of (S, M) [FST]. In this way, cluster variables correspond to tagged arcs, and clusters correspond to tagged triangulations. Many properties of $\mathcal{A}(S, M)$ can be shown by using this correspondence (see e.g. [FeST, FST, FoT, Lab09, M]). For a tagged triangulation T of (S, M) and the corresponding seed Σ_T in $\mathcal{A}(S, M)$, we denote by $\mathcal{A}(T)$ the cluster algebra $\mathcal{A}(S, M)$ with respect to initial seed Σ_T. For a tagged arc δ of (S, M), we denote by x_δ the corresponding cluster variable in $\mathcal{A}(T)$. Musiker, Schiffler and Williams [MSW11] gave a cluster expansion formula in terms of perfect matchings of snake graphs. Using it, they proved the positivity conjecture [MSW11] and constructed two bases [MSW13] for these cluster algebras. In this thesis, we introduce the notion of *maximal independent sets of angles* and give a cluster expansion formula in terms of them. Our (cluster expansion) formula simplifies one of [MSW11] as we will discuss later (see Subsection 7.1). Moreover, it has some applications.

As an application, it generalizes the cluster expansion formula in type A given by Carroll and Price [CaP], which is in terms of perfect matchings of a bipartite graph. For any tagged arc δ, we construct a bipartite graph B_δ. It is easy to show that there is a natural bijection between maximal independent sets of angles and perfect matchings of B_δ. Therefore, our formula provides the cluster expansion formula in terms of perfect matchings of B_δ for any $\mathcal{A}(T)$.

As another application, we study numerical invariants of cluster variables, called *f-vectors* and *g-vectors*. From our formula, it follows that for a tagged arc δ:

- the f-vector of x_δ coincides with the tuple of intersection numbers of δ and tagged arcs of T;
- the g-vector of x_δ coincides with the tuple of shear coordinates of δ with respect to T.

By giving some properties of intersection numbers and shear coordinates, we also obtain similar arguments for f-vectors and g-vectors, respectively.

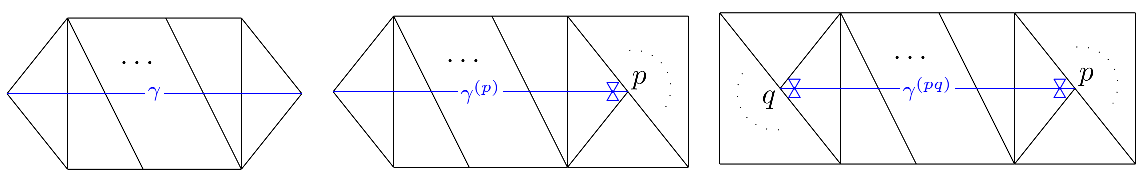
Now we describe the organization of this thesis and more details of our results briefly. This thesis consists of five parts. In Part 1, we prepare the basic notions we need throughout this thesis. Parts 2, 3, 4 and 5 are based on the papers [Y19a], [Y19c], [GY] and [Y19b], respectively.

In Parts 2 and 3, we give a cluster expansion formula in terms of maximal independent sets of angles for $\mathcal{A}(T)$. In Part 2, we assume that (S, M) is an n-gon and $\mathcal{A}(T)$ has coefficients arising from boundary segments of (S, M). This cluster algebra $\mathcal{A}(T)$ coincides with the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}(2, n+3)$. In Part 3, we assume that (S, M) is an arbitrary marked surface and $\mathcal{A}(T)$ has principal coefficients, which were introduced in [FZ07]. In the preface, we state the result of Part 3 for an ideal triangulation T without self-folded triangles and $\mathcal{A}(T)$ with coefficient-free. If (S, M) is a polygon, then it coincides with the result of Part 2 for $\mathcal{A}(T)$ with coefficient-free. Moreover, we also assume that δ is a tagged arc of (S, M) such that the underlying arc γ of δ is not in T, where *tagged arcs* of (S, M) are certain curves in S whose endpoints are in M and each end is tagged in one of two ways, *plain* or *notched* (see Section 2). In this thesis, we represent tagged arcs as follows:

We call a tagged arc δ

- a *plain arc* if its both ends are tagged plain;
- a 1-*notched arc* if an end of δ is tagged plain and the other end is tagged notched;
- a 2-*notched arc* if its both ends are tagged notched.

Let p and q be the endpoints of γ and $\gamma^{(p)}$ be the 1-notched arc obtained from γ by tagging its end p notched. Similarly, we define the 2-notched arc $\gamma^{(pq)}$ with both ends tagged notched. In particular, $\delta = \gamma$, $\gamma^{(p)}$ or $\gamma^{(pq)}$. Our cluster expansion formula for x_γ (resp., $x_{\gamma^{(p)}}$, $x_{\gamma^{(pq)}}$) comes down to type A (resp., D, $\widetilde{D}$) corresponding to polygons with no punctures (resp., one puncture, two punctures). We construct a triangulated polygon T_δ associated with δ as follows (see Subsection 7.1 for details). First, γ crosses triangles of T and we obtain the triangulation T_γ of an polygon by gluing their copies in the order. Similarly, we construct $T_{\gamma^{(p)}}$ (resp., $T_{\gamma^{(pq)}}$) by adjoining to T_γ copies of all triangles incident to p (resp., p and q). See Figure 1.



T_γ (type A) $T_{\gamma^{(p)}}$ (type D) $T_{\gamma^{(pq)}}$ (type $\widetilde{D}$)

FIGURE 1. Triangulated polygon T_δ for each tagged arc δ

We consider the graph whose vertices are all angles of T_δ incident to at least one arc of T_δ, and whose edges are given by cliques containing all angles incident to a given vertex of T_δ and cliques containing all angles incident to a given triangle of T_δ. We call a maximal independent set of the graph a *maximal independent set of angles in* T_δ. We denote by $\mathbb{A}(T_\delta)$ the set of maximal independent sets of angles in T_δ. For an arc or boundary segment τ of T_δ, we denote $x_\tau = x_{\tau'}$ if τ corresponds to a non-boundary segment τ' of T and we denote $x_\tau = 1$ otherwise. Then, for an angle a of T_δ, $x_a := x_\tau$, where τ is the side opposite to a in the triangle containing a. The result Theorem 7.5 gives a cluster expansion formula for cluster algebras with principal coefficients defined from arbitrary triangulated surfaces. Here, we specialize it to the case satisfying above assumptions.

Theorem 0.1 (Theorem 7.3, Theorem 7.5)**.** *On the above assumptions, we have*

$$x_\delta = \frac{1}{\prod_{\tau \in T_\delta} x_\tau} \sum_{A \in \mathbb{A}(T_\delta)} \prod_{a \in A} x_a.$$

Moreover, we give bijections between several different combinatorial objects containing maximal independent sets of angles and perfect matchings of the bipartite graph B_δ associated with δ (Theorem 7.4). This correspondence gives a generalization of the cluster expansion formula in [CaP] (see (7.2)).

In the rest of the preface, we state the results obtained from our formula and their applications for f-vectors and g-vectors.

First, we consider f-vectors in $\mathcal{A}(T)$. Using our formula, we show that for a tagged arc δ, the f-vector of x_δ is equal to a vector, called its *intersection vector*, whose entries are intersection numbers of δ and tagged arcs of T (Theorem 7.8). In Part 4, we study intersection vectors of tagged arcs and apply them to f-vectors and F*-matrices* via Theorem 7.8, where the F-matrix of a cluster $\mathbf{x}$ is a matrix whose columns are f-vectors of cluster variables in $\mathbf{x}$ [FuG]. Our main result shows that a tagged triangulation T' of (S, M) is uniquely determined by the intersection vectors of tagged arcs of T' (Theorem 13.1). In particular, it gives the following result.

Theorem 0.2 (Corollary 16.3)**.** *For a tagged triangulation T of (S, M), clusters in $\mathcal{A}(T)$ are uniquely determined by their F-matrices.*

Moreover, we also characterize a tagged triangulation T such that non-initial cluster variables in $\mathcal{A}(T)$ are uniquely determined by their f-vectors.

Proposition 0.3 (Proposition 16.5)**.** *Let T be a tagged triangulation of (S, M). Then non-initial cluster variables in $\mathcal{A}(T)$ are uniquely determined by their f-vectors if and only if either of the following conditions holds:*

- *(S, M) is a closed surface with exactly one puncture;*
- *there are no tagged arcs δ and ϵ of T connecting two (possibly same) common punctures such that their underlying arcs are different.*

Next, we consider g-vectors. For a cluster algebra $\mathcal{A}$ with rank n, g-vectors are integer vectors in $\mathbb{Z}^n$. For a cluster $\mathbf{x}$ of $\mathcal{A}$, its *g-vector cone* is a cone in $\mathbb{R}^n$ spanned by g-vectors of cluster variables in $\mathbf{x}$. If the number of clusters of $\mathcal{A}$ is finite, the union of g-vector cones associated with all clusters of $\mathcal{A}$ is equal to $\mathbb{R}^n$. In general, it is not true. In Part 5, we study g-vector cones associated with clusters of $\mathcal{A}(T)$, where T is a tagged triangulation of (S, M) with n tagged arcs. We determine the closure $\overline{g\mathrm{Cone(T)}}$ of the union of g-vector cones associated with all clusters of $\mathcal{A}(T)$.

Theorem 0.4 (Theorem 20.2)**.** *If (S, M) is a closed surface with exactly one puncture, then we have*

$$\overline{g\mathrm{Cone(T)}} = \Big\{(a_i)_{1 \le i \le n} \in \mathbb{R}^n \mid \sum_{1 \le i \le n} a_i \ge 0\Big\}.$$

Otherwise, we have

$$\overline{g\mathrm{Cone(T)}} = \mathbb{R}^n\,.$$

Our main ingredients are laminations on (S, M), their shear coordinates and their asymptotic behavior under Dehn twists (see Section 21 for details). More precisely, for a tagged arc δ of (S, N), we have its shear coordinate $b_T(\delta)$ with respect to T. It is known that $b_T(\delta)$ coincides with the g-vector of x_δ, which was proved in [R14b] (see also [FeT]) by using the tropical duality given in [NZ]. Our formula can directly show it although we don't give the proof in this thesis.

Finally, we apply Theorem 0.4 to representation theory. Let $\mathcal{C}$ be a Hom-finite Krull-Schmidt 2-Calabi-Yau triangulated category, for example, the cluster category associated with an acyclic quiver [BMRRT]. Then $\mathcal{C}$ has cluster tilting objects and there are mutations between them. In general, we don't know how many connected components are there in the exchange graph of cluster tilting objects in $\mathcal{C}$. We give an answer of this problem for triangulated categories associated with marked surfaces. For a tagged triangulation T of (S, M), we can construct a 2-Calabi-Yau triangulated category $\mathcal{C}(T)$ with cluster tilting objects, which was constructed in [A]. Then it is also possible to define g-vectors of indecomposable rigid objects and g-vector cones of basic cluster tilting objects in $\mathcal{C}(T)$ (see Subsection 23.1). Note that these g-vector cones and their faces form a fan [DIJ]. Such a fan plays an important role in the study of scattering diagrams and their wall-chamber structures (see e.g. [B, BST, GHKK, GS, KS, Y18]). By the correspondence between g-vectors in

$\mathcal{A}(T)$ and $\mathcal{C}(T)$ (Theorem 23.3, Corollary 23.4), Theorem 0.4 means that g-vector cones are dense in the scattering diagram. It gives the following application.

Corollary 0.5 (Corollary 20.4)**.** *Let T be a tagged triangulation of (S, M). If (S, M) is a closed surface with exactly one puncture, the exchange graph of cluster tilting objects in $\mathcal{C}(T)$ has precisely two connected components. Otherwise, it is connected.*

Corollary 20.4 was given by Qiu and Zhou [QZ] for marked surfaces with non-empty boundary. But our proof is entirely different from theirs since a key point of their proof is to show that for a certain triangulation of (S, M), the associated Jacobian algebra is skewed-gentle.

Part 1. Preliminary

1. Cluster algebras with coefficients from ice quivers

We begin with recalling the definition of cluster algebras with coefficients associated with ice quivers [K10]. We denote by $[1,n]$ the interval $\{1,2,\dots,n\}$. For positive integers $n \le m$, an *ice quiver of type* (n,m) is a quiver Q with vertices $Q_0=[1,m]$ such that there are no arrows between vertices in $[n+1,m]$. The elements of $[n+1,m]$ are called *frozen vertices*. We also denote by Q_1 the set of arrows in Q.

To define cluster algebras with coefficients from ice quivers, we need to prepare some notations. Let $\mathcal{F} := \mathbb{Q}(t_1,\dots,t_m)$ be the field of rational functions in m variables over $\mathbb{Q}$.

Definition 1.1. [FZ07, Definition 2.3, Definition 2.4, (2.15)] (1) A *seed* is a pair $(\mathbf{x},Q)$ consisting of the following data:
(i) $\mathbf{x}=(x_1,\dots,x_m)$ is a free generating set of $\mathcal{F}$ over $\mathbb{Q}$.
(ii) Q is an ice quiver of type (n,m) without loops and 2-cycles.
Then we refer to $\mathbf{x}$ as the *cluster*, to each x_i as a *cluster variable* for $i\in[1,n]$ and a *coefficient* for $i\in[n+1,m]$.
(2) For a seed $(\mathbf{x},Q)$, the *mutation* $\mu_k(\mathbf{x},Q)=(\mathbf{x}',Q')$ in direction k $(1\le k\le n)$ is defined as follows.
(i) $\mathbf{x}'=(x'_1,\dots,x'_m)$ is defined by

$$x_kx'_k=\prod_{(j\to k)\in Q_1}x_j+\prod_{(j\leftarrow k)\in Q_1}x_j \quad \text{and} \quad x'_i=x_i \quad \text{if} \quad i\neq k.$$

(ii) Q' is the ice quiver obtained from Q by the following steps:
(1) For any path $i\to k\to j$, add an arrow $i\to j$.
(2) Reverse all arrows incident to k.
(3) Remove a maximal set of disjoint 2-cycles.
(4) Remove all arrow connecting two frozen vertices.

We remark that μ_k is an involution, that is, we have $\mu_k\mu_k(\mathbf{x},Q)=(\mathbf{x},Q)$. Moreover, it is elementary that $\mu_k(\mathbf{x},Q)$ is also a seed. Now we define cluster algebras with coefficients. For an ice quiver Q, we fix a seed $(\mathbf{x}=(x_1,\dots,x_m),Q)$ which we call an *initial seed*. We also call each x_i an *initial cluster variable*.

Definition 1.2. [FZ07, Definition 2.11] The *cluster algebra* $\mathcal{A}(Q)=\mathcal{A}(\mathbf{x},Q)$ *with coefficients* for the initial seed $(\mathbf{x},Q)$ is a $\mathbb{Z}$-subalgebra of $\mathcal{F}$ generated by the cluster variables and the coefficients obtained by all sequences of mutations from $(\mathbf{x},Q)$.

One of the remarkable properties of cluster algebras is the *Laurent phenomenon*.

Theorem 1.3. [FZ02, Theorem 3.1][FZ03, Proposition 11.2] *Every element of the cluster algebra $\mathcal{A}(Q)$ is a Laurent polynomial over $\mathbb{Z}[x_{n+1},\dots,x_m]$ in the initial cluster variables, that is, $\mathcal{A}(Q)$ is contained in $\mathbb{Z}[x_1^{\pm1},\dots,x_n^{\pm1},x_{n+1},\dots,x_m]$.*

We also recall cluster algebras with principal coefficients [FZ07, Definition 3.1]. Let Q be an ice quiver of type (n,n), that is a quiver simply, without loops and 2-cycles whose vertices are $Q_0=\{1,\dots,n\}$. The *framed quiver* associated with Q is the quiver $\hat{Q}$ obtained from Q by adding vertices $\{1',\dots,n'\}$ and arrows $\{i\to i' \mid 1\le i\le n\}$. Then $\mathcal{A}(\hat{Q})$ is called the *cluster algebra with principal coefficients* for the initial seed $((x_1,\dots,x_n,y_1=x_{n+1},\dots,y_n=x_{2n}),\hat{Q})$.

2. Ideal and tagged triangulations

We start with recalling the notions of [FST]. Let S be a connected compact oriented Riemann surface with (possibly empty) boundary ∂S and M a non-empty finite set of marked points on S

with at least one marked point on each boundary component. We call the pair (S, M) a *marked surface*. Any marked point in the interior of S is called a *puncture*. For technical reasons, we assume that (S, M) is neither a monogon with at most one puncture, a digon without punctures, a triangle without punctures, nor a sphere with at most three punctures.

An *arc* γ of (S, M) is a curve in S with endpoints in M, considered up to isotopy, such that the following conditions are satisfied:

- γ does not intersect itself except at its endpoints;
- γ is disjoint from M and ∂S except at its endpoints;
- γ does not cut out an unpunctured monogon or an unpunctured digon.

An arc with two identical endpoints is called a *loop*. A curve homotopic to a boundary component between two marked points is called a *boundary segment*.

Two arcs are called *compatible* if they don't intersect in the interior of S. When we consider intersections of curves γ and δ, we assume that γ and δ intersect transversally in a minimum number of points. An *ideal triangulation* is a maximal collection of distinct pairwise compatible arcs. A triangle with only two distinct sides is called *self-folded* (see Figure 2). For an ideal triangulation

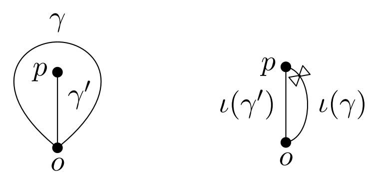


FIGURE 2. A self-folded triangle and the corresponding tagged arcs

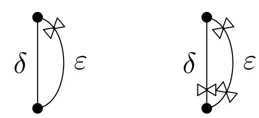


FIGURE 3. Pairs of conjugate arcs (δ, ε)

T, a *flip* at an arc $\gamma \in T$ replaces γ with another arc $\gamma' \notin T$ such that $(T \setminus \{\gamma\}) \cup \{\gamma'\}$ is an ideal triangulation. Notice that an arc inside a self-folded triangle can not be flipped. To make flip always possible, the notion of tagged arcs was introduced in [FST].

A *tagged arc* δ of (S, M) is an arc whose each end is tagged in one of two ways, *plain* or *notched*, such that the following conditions are satisfied:

- δ does not cut out a monogon with exactly one puncture;
- If an endpoint of δ lie on ∂S, then it is tagged plain;
- If δ is a loop, then the both ends are tagged in the same way.

For an arc γ of (S, M), we define a tagged arc $\iota(\gamma)$ as follows:

- If γ does not cut out a monogon with exactly one puncture, then $\iota(\gamma)$ is the tagged arc obtained from γ by tagging both ends plain;
- If γ is a loop at $o \in M$ cutting out a monogon with exactly one puncture p, then there is a unique arc γ' that connects o and p and does not intersect γ. And then $\iota(\gamma)$ is the tagged arc obtained by tagging γ' plain at o and notched at p (see Figure 2).

A *pair of conjugate arcs* is, for a self-folded triangle $\{\gamma, \gamma'\}$, $(\iota(\gamma), \iota(\gamma'))$ or a pair obtained from $(\iota(\gamma), \iota(\gamma'))$ by simultaneous changing tags at each endpoint (see Figure 3).

For a tagged arc δ, we denote by δ° the arc obtained from δ by forgetting its tags. Two tagged arcs δ and ϵ are called *compatible* if the following conditions are satisfied:

- The arcs δ° and ϵ° are compatible;
- If $\delta^\circ = \epsilon^\circ$, then at least one end of ϵ is tagged in the same way as the corresponding end of δ;
- If $\delta^\circ \neq \epsilon^\circ$ and they have a common endpoint o, then the ends of δ and ϵ at o are tagged in the same way.

A *partial tagged triangulation* is a collection of distinct pairwise compatible tagged arcs. If a partial tagged triangulation is maximal, then it is called a *tagged triangulation*. The number of tagged arcs in a tagged triangulation of (S, M) is constant [FST, Theorem 7.9]. We denote by $\mathbb{T}$ the set of tagged triangulations of (S, M). We can define *flips* of tagged triangulations in the same way as ones of ideal triangulations. In particular, any tagged arc can be flipped.

Theorem 2.1. [FST, Theorem 7.9, Proposition 7.10] *If (S, M) is a closed surface with exactly one puncture, the exchange graph of $\mathbb{T}$ has exactly two isomorphic components: one in which all ends of tagged arcs are plain and one in which they are notched. Otherwise, it is connected, that is, any two tagged triangulations of (S, M) are connected by a sequence of flips.*

3. Cluster algebras from triangulated surfaces

Let T be a tagged triangulation of (S, M). Fomin, Shapiro and Thurston [FST] constructed a quiver Q_T without loops and 2-cycles as follows: Any tagged triangulation is obtained by gluing together a number of puzzle pieces in Table 1 and by simultaneous changing all tags at some punctures (see [FST, Remark 4.2] for details). The vertices of Q_T are arcs of T and its arrows are obtained as in Table 1 for puzzle pieces of T, where we remove arrows incident to ∂S. Thus

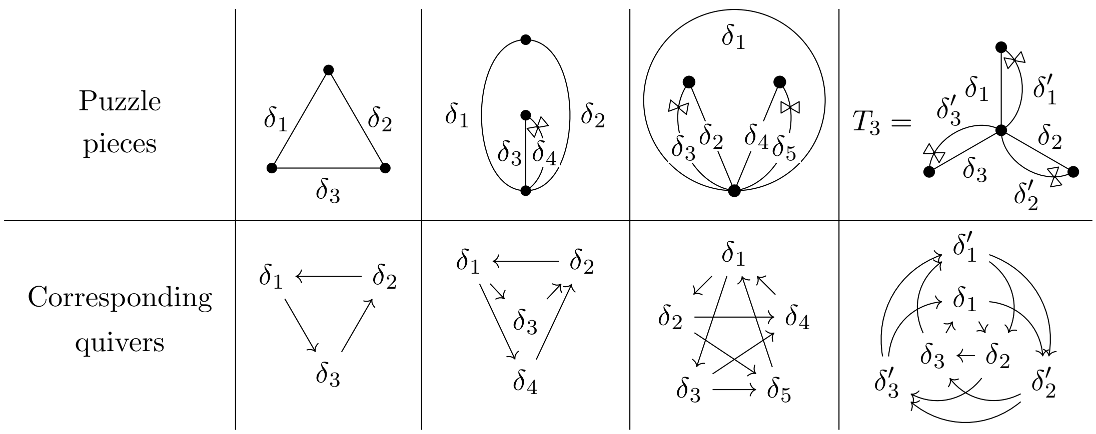


Table 1. Puzzle pieces and the corresponding quivers

we have the cluster algebra $\mathcal{A}(T) := \mathcal{A}(\hat{Q_T})$ with principal coefficients associated with T. We say that a puzzle piece in the first (resp., second, third) diagram from the left on Figure 1 is a *triangle piece* (resp., a 1-*puncture piece*, a 2-*puncture piece*). This cluster algebra $\mathcal{A}(T)$ has the following properties.

Theorem 3.1. [FST, Theorem 7.11][FoT, Theorem 6.1] *Let T be a tagged triangulation of (S, M).*

(1) *If (S, M) is not a closed surface with exactly one puncture, the tagged arcs δ of (S, M) correspond bijectively with the cluster variables x_δ in $\mathcal{A}(T)$. This induces that the tagged triangulations T' of (S, M) correspond bijectively with the clusters $\mathbf{x}_{T'}$ in $\mathcal{A}(T)$. Moreover, the tagged triangulation obtained from T' by flipping at $\delta \in T'$ corresponds the cluster obtained from $\mathbf{x}_{T'}$ by mutating at x_δ.*

(2) *If (S, M) is a closed surface with exactly one puncture, the plain arcs δ of (S, M) correspond bijectively with the cluster variables x_δ in $\mathcal{A}(T)$. This induces that the tagged triangulations T' which consist of plain arcs δ of (S, M) correspond bijectively with the clusters $\mathbf{x}_{T'}$ in $\mathcal{A}(T)$. Moreover, the tagged triangulation obtained from T' by flipping at $\delta \in T'$ corresponds the cluster obtained from $\mathbf{x}_{T'}$ by mutating at x_δ.*

For a tagged arc t and a puncture p of (S, M), we denote by $t^{(p)}$ the tagged arc obtained from t by changing tags at p, where $t^{(p)} = t$ if p is not an endpoint of t.

Proposition 3.2. [MSW11, Proposition 3.15] *Let T be a tagged triangulation of (S, M) consisting of tagged arcs $t_1, \dots, t_N$. We denote by $T^{(p)}$ the tagged triangulation consisting of $t_1^{(p)}, \dots, t_N^{(p)}$. Let $\Sigma_T = (\mathbf{x}, Q_T)$ and $\Sigma_{T^{(p)}} = (\mathbf{x}^{(p)}, Q_{T^{(p)}})$ be the corresponding initial seeds of $\mathcal{A}(T)$ and $\mathcal{A}(T^{(p)})$, respectively. Then for a tagged arc δ, we have*

$$[x_{\delta^{(p)}}]^{\mathcal{A}(T^{(p)})}_{\Sigma_{T^{(p)}}} = [x_\delta]^{\mathcal{A}(T)}_{\Sigma_T}|_{x \leftarrow x^{(p)}, y \leftarrow y^{(p)}},$$

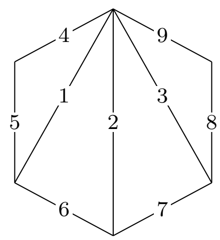
where $[x_\delta]^{\mathcal{A}(T)}_{\Sigma_T}$ is the Laurent expansion of x_δ with respect to Σ_T in $\mathcal{A}(T)$.

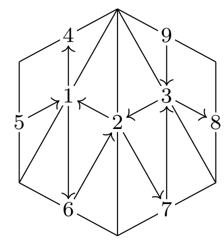
Part 2. Cluster expansion formulas in type A

4. Introduction

This part is based on the paper [Y19a]. We study cluster algebras of type A_n with $n+3$ coefficients. More precisely, it is the cluster algebra structure of the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}(2, n+3)$ introduced in [FZ03].

For a triangulated $(n+3)$-gon T with n diagonals, we define a new quiver $\overline{Q}_T$ from Q_T by adding a vertex for each boundary segment of T and arrows between original vertices and new vertices following the above rule. We give an example of this construction:

T :


$\overline{Q}_T$:


Recall that any quiver of type A_n is isomorphic to Q_T for some triangulated $(n+3)$-gon T [CCS]. Let T be a triangulated $(n+3)$-gon such that $Q := Q_T$ is an acyclic quiver with underlying graph

$$1 \text{ --- } 2 \text{ --- } 3 \text{ --- } \cdots \text{ --- } n.$$

We regard $\overline{Q} := \overline{Q}_T$ as an ice quiver of type $(n, 2n+3)$ with frozen vertices $[n+1, 2n+3]$ and denote by $\mathcal{A}(\overline{Q})$ the corresponding cluster algebra (see Definition 1.2). It is known that non-initial cluster variables of $\mathcal{A}(\overline{Q})$ are indexed by pairs (i, j) of integers such that $1 \le i \le j \le n$ [FZ03, ST]. More precisely, (i, j) is associated with a cluster variable

$$\frac{f^{[i,j]}}{x_i x_{i+1} \cdots x_j},$$

where $f^{[i,j]} \in \mathbb{Z}[x_1, \ldots, x_{2n+3}]$ is not divisible by $x_1, \ldots, x_n$.

Our main result is the cluster expansion formula for $f^{[i,j]}$ using the following notion.

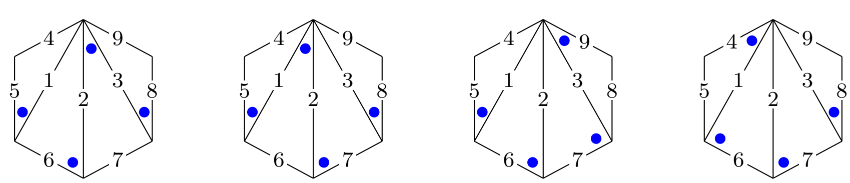
We consider the graph whose vertices are all angles of T incident to at least one arc, and whose edges are given by cliques containing all angles incident to a given vertex of T and cliques containing all angles incident to a given triangle of T. We call a maximal independent set of the graph a *maximal independent set of angles in* T. Note that a maximal independent set of angles in T was called a perfect matching of angles in [Y19a], which is a selection of marked angles such that:

(1) each vertex v incident to at least a diagonal is matched to one marked angle incident to v,

(2) each triangle of T has exactly one marked angle.

We denote by $\mathbb{A}(T)$ the set of maximal independent sets of angles in T.

Clearly, any maximal independent set of angles $A \in \mathbb{A}(T)$ satisfies $|A| = n+1$. For example, we provide the complete list of $\mathbb{A}(T)$ for the above case T:

(4.1)


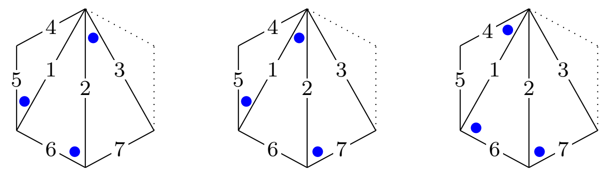
We denote by $T^{[i,j]}$ the triangulated subpolygon of T such that the set of diagonals of $T^{[i,j]}$ is $[i, j]$. We give our main result using maximal independent sets of angles in $T^{[i,j]}$.

Theorem 4.1. *For $1 \leq i \leq j \leq n$, we have*

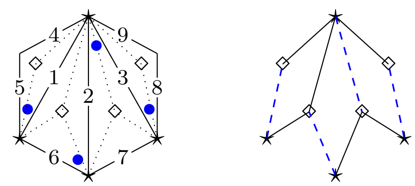
$$f^{[i,j]} = \sum_{A \in \mathbb{A}(T^{[i,j]})} \prod_{a \in A} x_a,$$

where x_a is the initial cluster variable corresponding to the opposite side of the angle a.

In the above example (4.1), we obtain $f^{[1,3]} = x_1x_4x_7x_9 + x_3x_4x_6x_9 + x_1x_2x_4x_8 + x_2x_3x_5x_9$ by Theorem 4.1. Also, we obtain $f^{[1,2]} = x_1x_4x_7 + x_3x_4x_6 + x_2x_3x_5$ since the complete list of $\mathbb{A}(T^{[1,2]})$ is the following:



Remark 4.2. There is a natural bijection between maximal independent sets of angles in a triangulated polygon and perfect matchings of the corresponding bipartite graph introduced by Carroll and Price [CaP] as follows:



Our formula in Theorem 4.1 can be regarded as an analog of the computation of a cluster variable in terms of perfect matching enumeration in a weighted bipartite graph (see [CeP, Pr]).

Next, we give another cluster expansion formula in type A_n in terms of quivers. We introduce a new notion of *discrete subsets* D of an ice quiver, where 'discrete' means that two arrows in D can not be a connected by a path.

Definition 4.3. Let $R = (R_0, R_1, s, t)$ be an ice quiver, and R' the full subquiver of R consisting of all non-frozen vertices. A subset D of R_1 is called *discrete* if for any $\alpha, \beta \in D$ there is no path in R' from $t(\alpha)$ to $s(\beta)$. We denote by $\mathbb{D}(R)$ the set of all maximal discrete subsets of R.

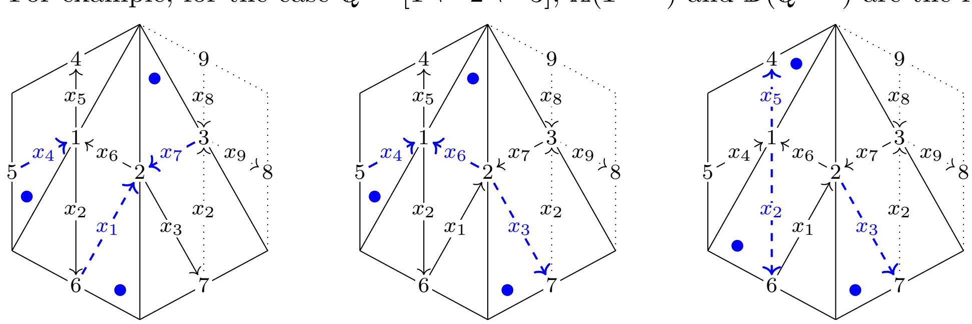
We denote by $\overline{Q}^{[i,j]}$ the ice quiver obtained from $T^{[i,j]}$.

Corollary 4.4. *For $1 \leq i \leq j \leq n$, we have*

$$f^{[i,j]} = \sum_{D \in \mathbb{D}(\overline{Q}^{[i,j]})} \prod_{\alpha \in D} x_\alpha,$$

where x_α is the initial cluster variable corresponding to the third side of the triangle in T with sides $s(\alpha)$ and $t(\alpha)$.

We prove Corollary 4.4 by giving a natural bijection between $\mathbb{A}(T^{[i,j]})$ and $\mathbb{D}(\overline{Q}^{[i,j]})$ (see Proposition 6.6). For example, for the case $Q = [1 \leftarrow 2 \leftarrow 3]$, $\mathbb{A}(T^{[1,2]})$ and $\mathbb{D}(\overline{Q}^{[1,2]})$ are the following:



Thus we have $f^{[1,2]} = x_1x_4x_7 + x_3x_4x_6 + x_2x_3x_5$.

5. The cluster expansion formula of Musiker and Schiffler

Musiker and Schiffler [MS] gave a cluster expansion formula using a *snake graph* and its *perfect matchings*. Let T be a triangulated $(n+3)$-gon such that Q_T is acyclic. We recall the construction of Musiker and Schiffler. For more details, see [MS]. The construction can be followed step by step in Example 5.3. As Q_T is acyclic, at most two sides of each triangle of T are diagonals. The snake graph $\overline{G} := \overline{G}_T$ is obtained from T by *unfolding* along the third side of each of triangles of T, two sides of which are diagonals (see Figure 4). We label all edges of $\overline{G}$ by the corresponding arcs or boundary segments of T.

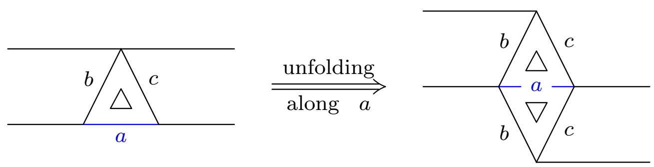


FIGURE 4. Unfolding $\triangle$, where a is boundary segment, while b and c are not

Let $\boxbslash_k$ be the square of $\overline{G}$ with diagonal k, and $\overline{G}^{[i,j]}$ the minimal subgraph of $\overline{G}$ containing $\boxbslash_i$, $\boxbslash_{i+1}$, ..., $\boxbslash_j$ for $1 \leq i \leq j \leq n$. Let $G := G_T$ be the graph obtained from $\overline{G}$ by removing the diagonal in each $\boxbslash_k$. Similarly, let $\square_k$ and $G^{[i,j]}$ be the graphs obtained from $\boxbslash_k$ and $\overline{G}^{[i,j]}$ in the same way as G.

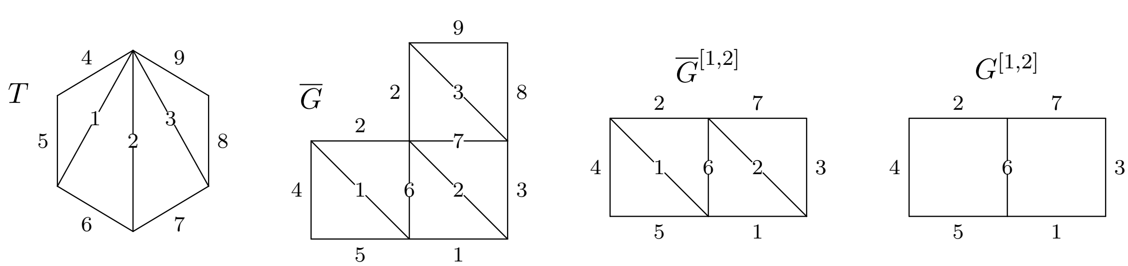
Definition 5.1. A *perfect matching* in $G^{[i,j]}$ is a subset P of the edges in $G^{[i,j]}$ such that each vertex is contained in exactly one edge on P. We denote by $\mathbb{P}(G^{[i,j]})$ the set of all perfect matchings in $G^{[i,j]}$.

The formula of Musiker and Schiffler is the following cluster expansion formula obtained by the perfect matchings in $G^{[i,j]}$.

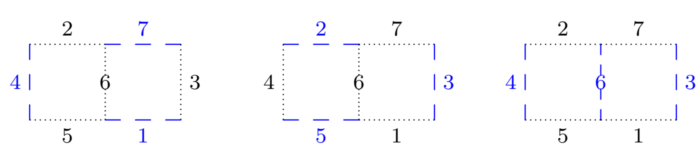
Theorem 5.2. [MS] *For $1 \leq i \leq j \leq n$, we have*

$$f^{[i,j]} = \sum_{P \in \mathbb{P}(G^{[i,j]})} \prod_{e \in P} x_e.$$

Example 5.3. For the quiver $Q = [1 \leftarrow 2 \leftarrow 3]$, we have



Then $G^{[1,2]}$ has the following three perfect matchings:

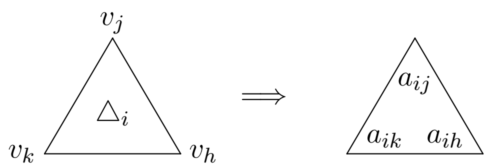


Thus we have $f^{[1,2]} = x_1x_4x_7 + x_2x_3x_5 + x_3x_4x_6$.

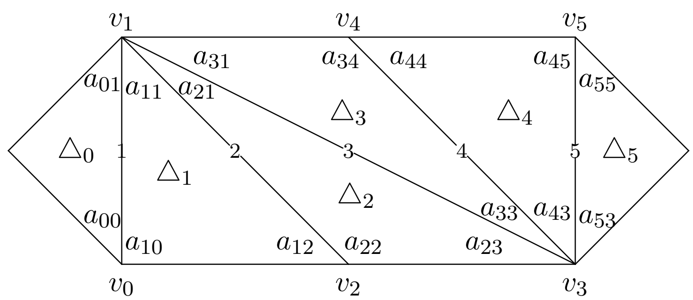
6. Proofs of Theorem 4.1 and Corollary 4.4

We will show Theorem 4.1 and Corollary 4.4 only for $[i,j] = [1,n]$ since $f^{[i,j]}$ in $\mathcal{A}(\overline{Q})$ coincides with $f^{[i,j]}$ in $\mathcal{A}(\overline{Q}^{[i,j]})$ by Theorem 5.2. Let T be a triangulated $(n+3)$-gon such that $Q := Q_T$ is an acyclic quiver, that is of type A_n.

6.1. **Proof of Theorem 4.1.** We prepare some notations to consider maximal independent sets of angles in the triangulated $(n+3)$-gon T.
(a) As Q is of type A_n, it is possible, in a unique way, to label the triangles appearing in T as $\triangle_0, \triangle_1, \ldots, \triangle_n$, such that two sides of $\triangle_i$ (resp., one side of $\triangle_0$, $\triangle_n$) are the diagonals i and $i+1$ (resp., the diagonal 1, the diagonal n) for $i \in [1, n-1]$.
(b) Label $n+1$ vertices of T by $v_0, \ldots, v_n$ such that v_0 is incident to the diagonal 1 and not incident to the diagonal 2, and v_k is incident to the diagonal k for $k \in [1,n]$.
(c) Label by a_{ij} the angle of $\triangle_i$ at the vertex v_j. Let $A(T)$ be the set of all angles in T that are labeled in this way.



For example, the labellings of T for the case $Q = [1 \leftarrow 2 \leftarrow 3 \rightarrow 4 \rightarrow 5]$ are the following:

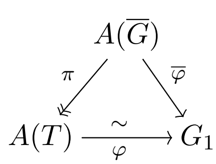


Thus $A(T) = \{a_{00}, a_{01}, a_{10}, a_{11}, a_{12}, a_{20}, a_{22}, a_{23}, a_{30}, a_{33}, a_{34}, a_{43}, a_{44}, a_{45}, a_{53}, a_{55}\}$.
Under the above parametrization of the angles in T, the following is clear.

Lemma 6.1. *A subset A of $A(T)$ is a maximal independent set of angles if and only if it satisfies the following conditions.*
(A1) For any $i \in [0,n]$, there is a unique $j \in [0,n]$ such that $a_{ij} \in A$,
(A2) For any $j \in [0,n]$, there is a unique $i \in [0,n]$ such that $a_{ij} \in A$.

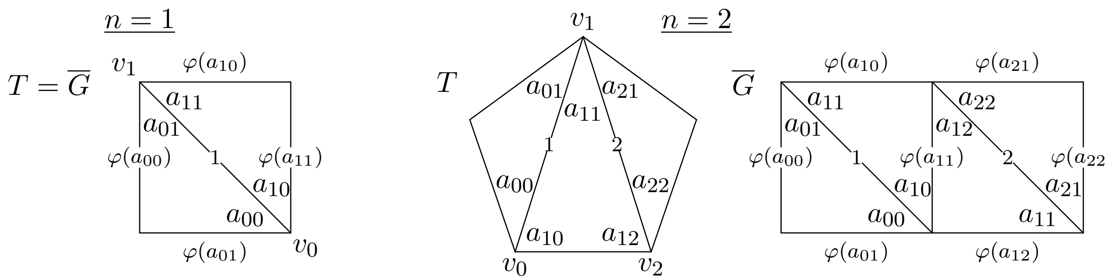
We denote by G_1 the set of edges of G. Let $A(\overline{G})$ be the set of angles between a diagonal and a side of a $\boxslash_k$ in $\overline{G}$, and $\overline{\varphi} : A(\overline{G}) \to G_1$ the map sending $a \in A(\overline{G})$ to the side that is opposite to a. Clearly, $\overline{\varphi} : A(\overline{G}) \to G_1$ is surjective. By definition of the unfolding process (see Section 5), there is a canonical surjection $\pi : A(\overline{G}) \to A(T)$ compatible with the construction of $\overline{G}$.

Lemma 6.2. *There exists a bijection $\varphi : A(T) \to G_1$ making the following diagram commutative:*

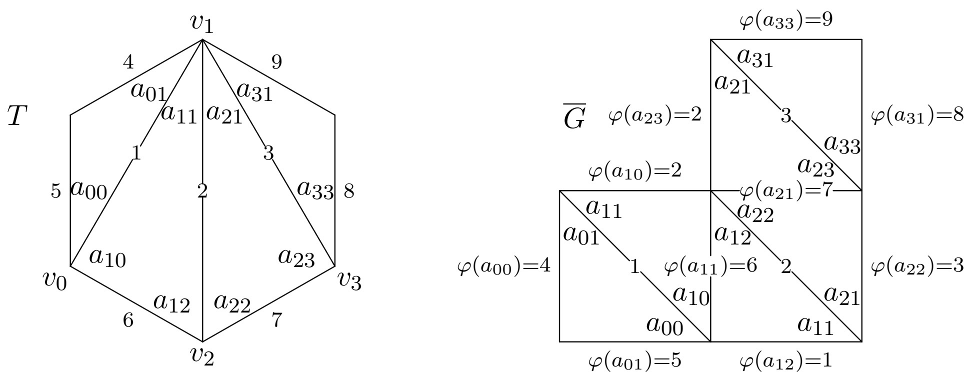


Before proving Lemma 6.2, we give simple examples.

Example 6.3. ($n = 1$) For the case $Q = [1]$, we have a natural identification of $\overline{G}$ with T.
($n = 2$) For the case $Q = [1 \leftarrow 2]$, the following diagonals illustrate the bijection $\varphi : A(T) \to G_1$:



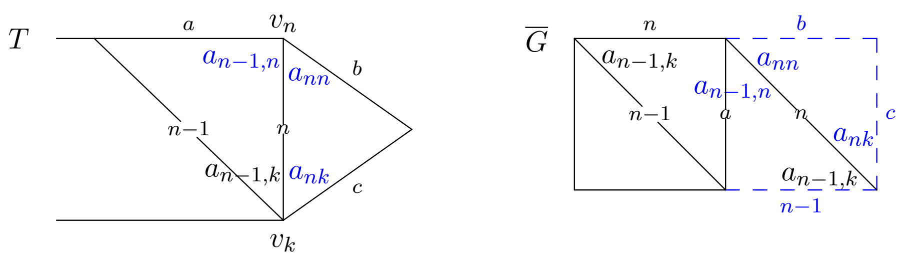
($n = 3$) For the case $Q = [1 \leftarrow 2 \leftarrow 3]$, the following diagonals illustrate the bijection $\varphi : A(T) \to G_1$:



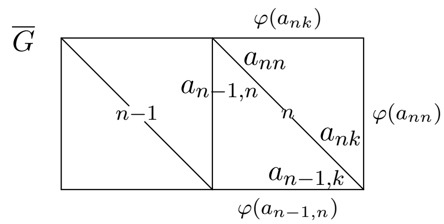
Proof of Lemma 6.2. We construct φ by induction on n. For $n = 1$, we have a natural identification of $\overline{G}$ with T which gives the desired map. Assume that we constructed a bijection $\varphi : A(T^{[1,n-1]}) \to G_1^{[1,n-1]}$ making the following diagram commutative:

$$\begin{array}{ccc} & A(\overline{G}^{[1,n-1]}) & \\ {\scriptstyle \pi}\swarrow & & \searrow{\scriptstyle \overline{\varphi}} \\ A(T^{[1,n-1]}) & \xrightarrow[\varphi]{\sim} & G_1^{[1,n-1]} \end{array}$$

Then $A(T)$ has 3 additional angles labeled $a_{n-1,n}$, a_{nk} and a_{nn}, and G_1 has 3 additional edges labeled $n-1$, b and c, as shown below in blue:



We extend the map $\varphi : A(T^{[1,n-1]}) \to G_1^{[1,n-1]}$ to $\varphi : A(T) \to G_1$ by assigning $\varphi(a_{n-1,n})$, $\varphi(a_{nk})$ and $\varphi(a_{nn})$ to the edges of G_1 shown in the following diagram:



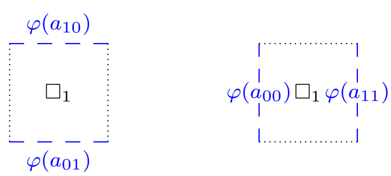
$\square$

The following is a key proposition to show Theorem 4.1.

Proposition 6.4. *The map φ of Lemma 6.2 induces a bijection $\varphi : \mathbb{A}(T) \to \mathbb{P}(G)$.*

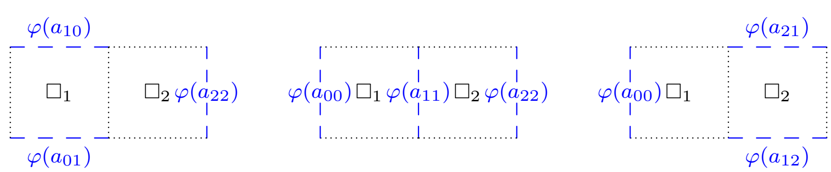
To prove Proposition 6.4, we consider the set of all perfect matchings of G.

(1) If $n = 1$, the following is a complete list of $\mathbb{P}(G)$:



Then the map $\varphi : A(T) \to G_1$ of Lemma 6.2 induces a bijection $\varphi : \mathbb{A}(T) \to \mathbb{P}(G)$ by Lemma 6.1 (see Example 6.3).

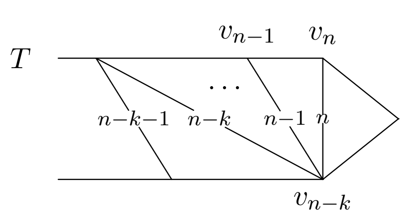
(2) If $n = 2$, the following is a complete list of $\mathbb{P}(G)$:

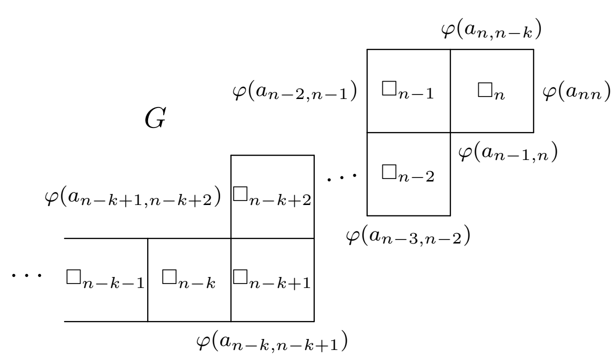


Then the map $\varphi : A(T) \to G_1$ of Lemma 6.2 induces a bijection $\varphi : \mathbb{A}(T) \to \mathbb{P}(G)$ by Lemma 6.1 (see Example 6.3).

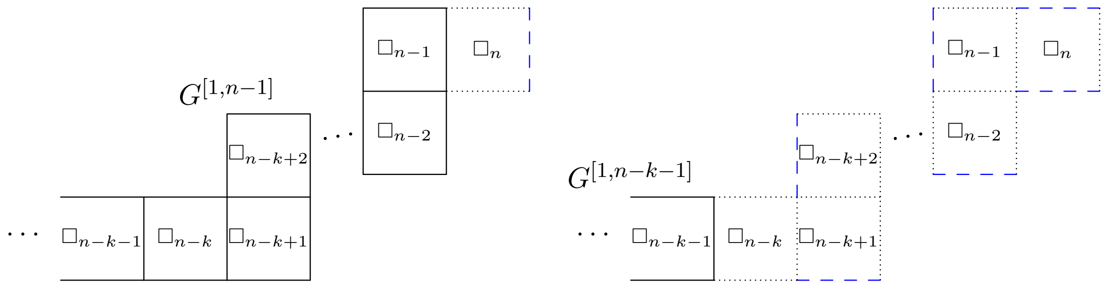
(3) If $n \geq 3$, there are the following cases. A triple $(\square_{i-1}, \square_i, \square_{i+1})$ is called *straight* if its squares lie in one column or one row, and *zigzag* if not. Triangulations of $(n+3)$-gon are divided into the following $n-1$ types.

• Type k $(1 \leq k \leq n-1)$: If v_{n-k} is incident to the diagonal n in T, $(\square_{n-k-1}, \square_{n-k}, \square_{n-k+1})$ is straight and $(\square_{n-i-1}, \square_{n-i}, \square_{n-i+1})$ is zigzag for any $i \in [1, k-1]$.

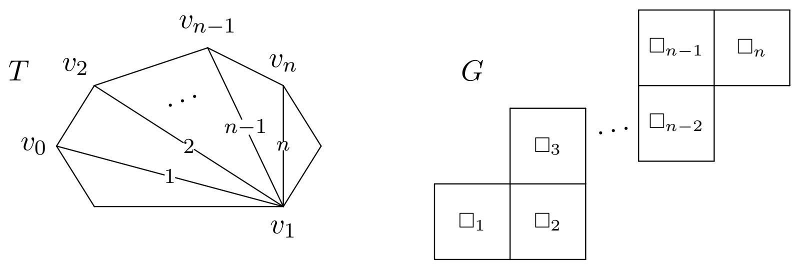




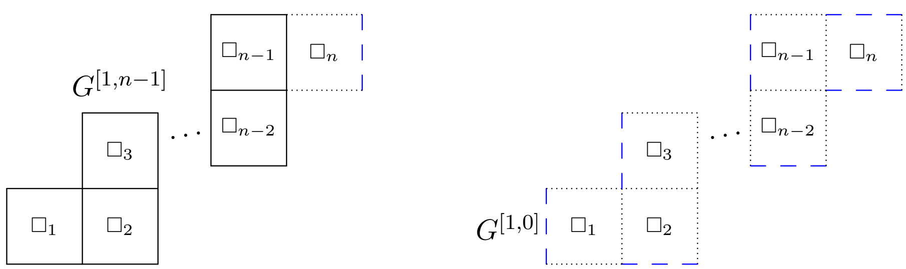
Then there is a natural bijection $\mathbb{P}\big(G^{[1,n-1]}\big) \sqcup \mathbb{P}\big(G^{[1,n-k-1]}\big) \to \mathbb{P}(G)$ explained in the following diagram:



Note that, in type $n-1$, $(\square_{i-1}, \square_i, \square_{i+1})$ is zigzag for any $i \in [2, n-1]$.

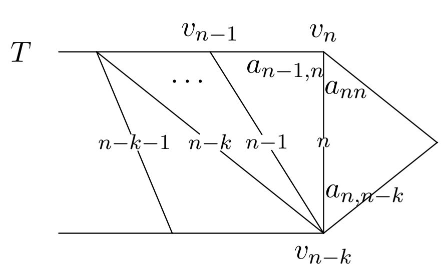


In this case, we define $G^{[1,0]}$ to be the graph consisting of the edge of $\square_1$ not incident to $\square_2$. Then there is also a natural bijection $\mathbb{P}\big(G^{[1,n-1]}\big) \sqcup \mathbb{P}\big(G^{[1,0]}\big) \to \mathbb{P}(G)$ explained in the following diagram:

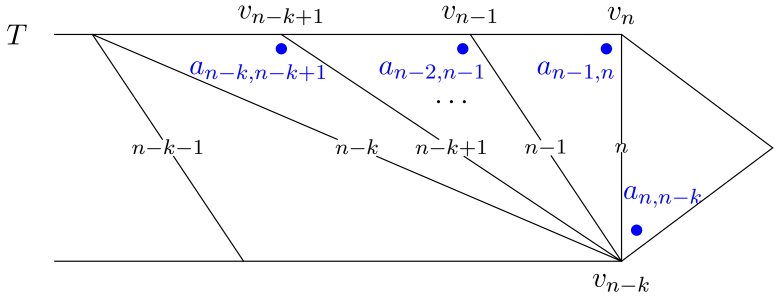


Proof of Proposition 6.4. We prove it by induction on n. For $n = 1, 2$, the assertion follows from the above observations (1) and (2). Assume $n \geq 3$. Then $\mathbb{A}(T)$ is written as a disjoint union $\mathbb{A}(T) = \mathbb{A}'(T) \sqcup \mathbb{A}''(T)$, where $\mathbb{A}'(T)$ consists of all $A \in \mathbb{A}(T)$ containing a_{nn}. Then the natural inclusion $A(T^{[1,n-1]}) \to A(T)$ induces a bijection $\mathbb{A}(T^{[1,n-1]}) \to \mathbb{A}'(T)$ given by $A \mapsto A \sqcup \{a_{nn}\}$. We consider $n-1$ types in the above observation (3).

- Type k $(1 \leq k \leq n-2)$: In this case, any $A \in \mathbb{A}''(T)$ contains $a_{n,n-k}$ and $a_{n-1,n}$ by Lemma 6.1.



Thus the natural inclusion $A(T^{[1,n-k-1]}) \to A(T)$ induces a bijection $\mathbb{A}(T^{[1,n-k-1]}) \to \mathbb{A}''(T)$ given by $A \mapsto A \sqcup \{a_{n,n-k}, a_{n-1,n}, a_{n-2,n-1}, \ldots, a_{n-k,n-k+1}\}$.

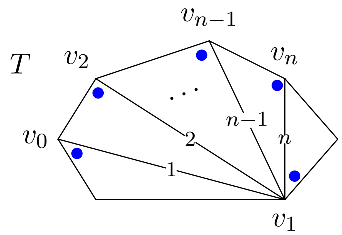


By induction on n, φ induces a bijection

$$\mathbb{A}(T) \simeq \mathbb{A}(T^{[1,n-1]}) \sqcup \mathbb{A}(T^{[1,n-k-1]}) \simeq \mathbb{P}(G^{[1,n-1]}) \sqcup \mathbb{P}(G^{[1,n-k-1]}) \simeq \mathbb{P}(G).$$

By construction, it is easy to check that it is again compatible with φ.

• Type $n-1$: In this case, any element of $\mathbb{A}''(T)$ contains a_{n1} and $a_{n-1,n}$, thus $\mathbb{A}''(T) = \{A_0 := \{a_{00}, a_{n1}, a_{12}, \ldots, a_{n-1,n}\}\}$ by Lemma 6.1.



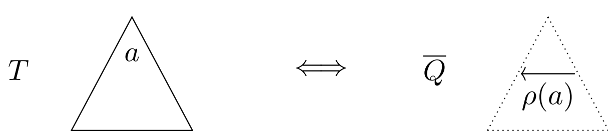
Then φ induces a bijection

$$\mathbb{A}(T) \simeq \mathbb{A}(T^{[1,n-1]}) \sqcup \{A_0\} \simeq \mathbb{P}(G^{[1,n-1]}) \sqcup \mathbb{P}\big(G^{[1,0]}\big) \simeq \mathbb{P}(G).$$

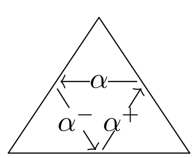
By construction, it is easy to check that it is again compatible with φ. □

Proof of Theorem 4.1. The assertion follows immediately from Theorem 5.2 and Proposition 6.4. □

6.2. **Proof of Corollary 4.4.** We have a natural bijection $\rho : A(T) \to \overline{Q}_1$ given by the following picture:

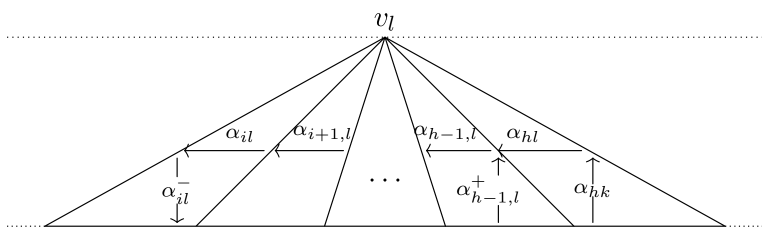


We denote $\rho(a_{ij})$ by α_{ij}. Moreover, we denote other arrows in this triangle by the following, if exists:



Lemma 6.5. *For any $A \in \mathbb{A}(T)$, we have $\rho(A) \in \mathbb{D}(\overline{Q})$.*

Proof. Firstly, we show that $\rho(A)$ is a discrete subset of $\overline{Q}$. Assume that $a_{ij} \neq a_{hk}$ are two elements of A such that there is a path $\alpha_{il}\alpha_{i+1,l} \cdots \alpha_{h-1,l}\alpha_{hl}$ in Q from $t(\alpha_{hk})$ to $s(\alpha_{ij})$. Since $a_{ij}, a_{hk} \in A$, $i \neq h$ and $j \neq k$ hold by Lemma 6.1. It follows from $j \neq k$ that at least one of α_{ij} and α_{hk} does not belong to Q. Without loss of generality, we assume that $s(\alpha_{hk})$ is a boundary segment of T. Then α_{ij} is either α_{il} or α_{il}^-.



Since α_{hl}^- does not belong to $\rho(A)$ and A is a maximal independent set of angles, $\alpha_{h-1,l}^+$ belongs to $\rho(A)$. Repeating the same argument, α_{sl}^+ belongs to $\rho(A)$ for any $s \in [i, h]$. This is a contradiction since α_{ij} and α_{il}^+ belong to $\rho(A)$.

To prove that $\rho(A)$ is maximal discrete, take $a_{ij} \in A(T) \setminus A$. By (A1), there exists $a_{ik} \in A$ with $k \neq j$. In this case, there exists a path in Q of length 0 or 1 either from $t(\alpha_{ij})$ to $s(\alpha_{ik})$, or from $t(\alpha_{ik})$ to $s(\alpha_{ij})$. In both cases, $\rho(A) \sqcup \{\alpha_{ij}\}$ is not discrete. Thus the assertion follows. □

Proposition 6.6. *The bijection* $\rho : A(T) \to \overline{Q}_1$ *induces a bijection* $\rho : \mathbb{A}(T) \to \mathbb{D}(\overline{Q})$.

Proof. By Lemma 6.5, we only have to show that for any $D \in \mathbb{D}(\overline{Q})$, we have $\rho^{-1}(D) \in \mathbb{A}(T)$.

Firstly, we assume that $\rho^{-1}(D)$ does not satisfy (A1). Then there exists i such that one of the following conditions holds.

(1) There are two elements $\alpha_{ij} \neq \alpha_{ik}$ in D.

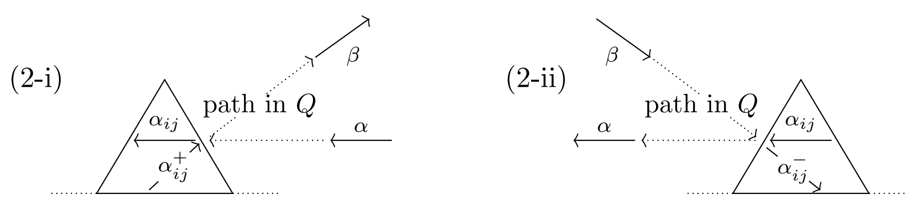
(2) There are no j such that $\alpha_{ij} \in D$.

In the case (1), there exists a path in Q of length 0 or 1 either from $t(\alpha_{ij})$ to $s(\alpha_{ik})$, or from $t(\alpha_{ik})$ to $s(\alpha_{ij})$. Thus D is not discrete.

In the case (2), we take an arrow α_{ij} such that $t(\alpha_{ij}^-) = s(\alpha_{ij}^+)$ is a boundary segment of T. Since D is maximal discrete and $\alpha_{ij} \notin D$, one of the following conditions hold.

(2-i) There exists $\alpha \in D$ such that there is a path in Q from $t(\alpha)$ to $s(\alpha_{ij})$.

(2-ii) There exists $\alpha \in D$ such that there is a path in Q from $t(\alpha_{ij})$ to $s(\alpha)$.



We consider the case (2-i). In this case, there exists the arrow α_{ij}^+. Since D is maximal discrete and $\alpha_{ij}^+ \notin D$, there exists $\beta \in D$ such that there is a path in Q from $t(\alpha)$ to $s(\beta)$ factoring through $t(\alpha_{ij}^+) = s(\alpha_{ij})$. Thus there is a path in Q from $t(\alpha)$ to $s(\beta)$, a contradiction. In the case (2-ii), we have a contradiction by the same argument. Consequently, $\rho^{-1}(D)$ satisfies (A1).

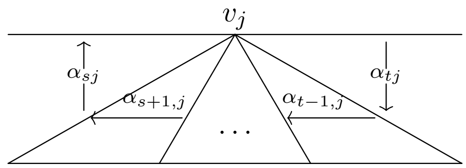
Secondly, we assume that $\rho^{-1}(D)$ does not satisfy (A2). Then there exists j such that one of the following conditions holds.

(1) There are two elements $\alpha_{ij} \neq \alpha_{hj}$ in D.

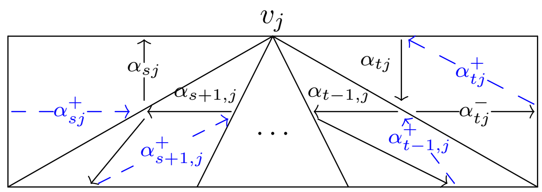
(2) There are no i such that $\alpha_{ij} \in D$.

In the case (1), there is a path in Q either from $t(\alpha_{ij})$ to $s(\alpha_{hj})$, or from $t(\alpha_{hj})$ to $s(\alpha_{ij})$. Thus $\{\alpha_{ij}, \alpha_{hj}\}$ is not discrete.

In the case (2), all angles at the vertex v_j are labeled as $a_{sj}, a_{s+1,j}, \dots, a_{t-1,j}, a_{tj}$.

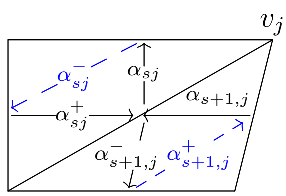


Since $\rho^{-1}(D)$ satisfies (A1), for any $i \in [s,t]$, precisely one of α_{ij}^- and α_{ij}^+ belongs to D. Assume $\alpha_{ij}^+ \in D$ for some $i \in [s,t-1]$. Then $\alpha_{i+1,j}^-$ does not belong to D since $t(\alpha_{ij}^+) = s(\alpha_{i+1,j}^-)$. Thus $\alpha_{i+1,j}^+$ exists and belongs to D. Repeating the same argument, α_{hj}^+ exists and belongs to D for any $h \in [i,t]$. This is a contradiction since the arrow $\alpha_{tj}^- : t(\alpha_{t-1,j}^+) \to s(\alpha_{tj}^+)$ belongs to Q.



Assume $\alpha_{ij}^- \in D$ for some $i \in [s+1,t]$. Then we have a contradiction by the same argument.

It remains to consider the case $s+1 = t$, and α_{sj}^- and $\alpha_{s+1,j}^+$ belong to D. In this case, there is a path $\alpha_{s+1,j}^- \alpha_{sj}^+ : t(\alpha_{sj}^-) \to s(\alpha_{s+1,j}^+)$ in Q, a contradiction.



Consequently, $\rho^{-1}(D)$ satisfies (A2). □

Proof of Corollary 4.4. The assertion follows from Theorem 4.1 and Proposition 6.6. □

Part 3. Combinatorial cluster expansion formulas from triangulated surfaces

7. Introduction

This part is based on the paper [Y19c]. We extend cluster expansion formulas in the previous part for cluster algebras defined from arbitrary triangulated surfaces. In particular, we assume that cluster algebras have principal coefficients. In the case of triangulated polygons, Carroll and Price [CaP] gave a cluster expansion formula in terms of perfect matchings of bipartite graphs (see also [CeP]). Using it, Propp [Pr] studied the Conway and Coxeter frieze patterns and Markov numbers. In general case, Musiker, Schiffler and Williams gave a cluster expansion formula in terms of perfect matchings of snake graphs. Using it, they proved the positivity conjecture [MSW11] and constructed two bases [MSW13] for these cluster algebras. The first aim of this part is to give a cluster expansion formula for these cluster algebras in terms of maximal independent sets of angles (Theorem 7.5). This simplifies their formula as we will discuss later. The second aim of this part is to give bijections between several different combinatorial objects containing perfect matchings of snake graphs (Theorem 7.4). This correspondence gives a generalization of the cluster expansion formula in [CaP].

This part is organized as follows. In the rest of this section, we give our results and some examples. For simplicity, we first specialize Theorem 7.5 to the coefficient-free case, that is $y_i = 1$ for all i (Theorem 7.3). Using Theorem 7.5, we also study *f-vectors* of cluster variables. In Section 8, we recall the cluster expansion formula of Musiker-Schiffler-Williams. We prove Theorem 7.5 and a part of Theorem 7.4 simultaneously in Section 9. We prove our results for the corresponding bipartite graphs in Section 10 and study minimal cuts of the corresponding quivers with potential in Sections 11. Finally, some elements in $\mathcal{A}(T)$ correspond to *essential loops* in T (see Section 12 for details). In the case of a marked surface without punctures, it is known that these elements and cluster variables form a base of $\mathcal{A}(T)$ (see Theorem 12.3). We give the formula for these elements in terms of good maximal independent sets of angles in Theorem 12.5.

7.1. Our results in the coefficient-free case.

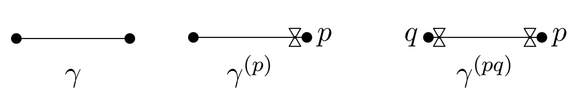
Let (S,M) be a marked surface and T a tagged triangulation of (S,M). Let $\mathcal{A}(T)$ be the cluster algebra with principal coefficients defined from T (see Section 3). For simplicity, in this part, we assume that if (S,M) is a closed surface with exactly one puncture, all tagged arcs are plain arcs. For a tagged arc δ, we denote by x_δ the corresponding cluster variable in $\mathcal{A}(T)$. We index the tagged arcs of T by $[1,N]$. In particular, x_i (resp., y_i) is the corresponding initial cluster variable (resp., coefficient) in $\mathcal{A}(T)$ for $i \in [1,N]$.

We obtain a tagged triangulation $\hat{T}$ from T by simultaneous changing all tags at some punctures, in such a way that there is an ideal triangulation T^0 satisfying $\hat{T} = \iota(T^0)$ (see [MSW11, Remark 3.11]). In view of Proposition 3.2, it is enough to consider a tagged triangulation T satisfying $T = \hat{T}$. In particular, in the rest of this part, we assume the following assumption.

Assumption 7.1. *The initial tagged triangulation T consists of plain arcs and 1-notched arcs, with at most one 1-notched arc incident to each puncture.*

In this case, for each 1-notched arc of T, the corresponding plain arc is also in T.

For a tagged arc δ of (S,M), we denote by $\overline{\delta}$ the plain arc corresponding to δ. Now, we only consider the case of $\gamma := \overline{\delta} \notin T$. Let p and q be the endpoints of γ. Let $\gamma^{(p)}$ be the 1-notched arc obtained from γ by tagging its end p notched. Similarly, we define the 2-notched arc $\gamma^{(pq)}$ with both ends tagged notched:



In particular, $\delta = \gamma$, $\gamma^{(p)}$ or $\gamma^{(pq)}$. To give cluster expansion formulas, we can make the following assumption by Proposition 3.2.

Assumption 7.2. *If $\delta = \gamma^{(p)}$ (resp., $\delta = \gamma^{(pq)}$), there is no 1-notched arc incident to p (resp., p or q) in T.*

Our cluster expansion formula for x_γ (resp., $x_{\gamma^{(p)}}$, $x_{\gamma^{(pq)}}$) comes down to type A (resp., D, $\widetilde{D}$) corresponding to polygons with no punctures (resp., one puncture, two punctures). We construct a triangulated polygon T_δ associated with δ as follows.

Let $\tau_1, \dots, \tau_n$ be the arcs of T^0 crossing γ in order of occurrence along γ (we can have $\tau_i = \tau_j$ even if $i \neq j$). Hence γ crosses $n+1$ triangles $\triangle_0, \dots, \triangle_n$, in this order. Suppose first that none of these triangles is self-folded. Then for $i \in [0, n]$, let $\triangle_{\gamma,i}$ be a copy of the oriented triangle $\triangle_i$, hence $\triangle_{\gamma,i}$ contains the sides τ_i and τ_{i+1} (τ_1 only if $i = 0$, and τ_n only if $i = n$). Then T_γ is the triangulation of an $(n+3)$-gon obtained by gluing these triangles along the edges τ_i. Similarly, we construct $T_{\gamma^{(p)}}$ (resp., $T_{\gamma^{(pq)}}$) by adjoining to T_γ copies of all triangles incident to p (resp., p and q) if none of them is self-folded. See Figure 5. If γ crosses self-folded triangles or there are self-folded triangles incident to p or q, we adapt the construction using the local transformations of Figure 6. Note that, by Assumption 7.2, it is not necessary to consider the case, where the end of δ in the middle of Figure 6 is tagged notched.

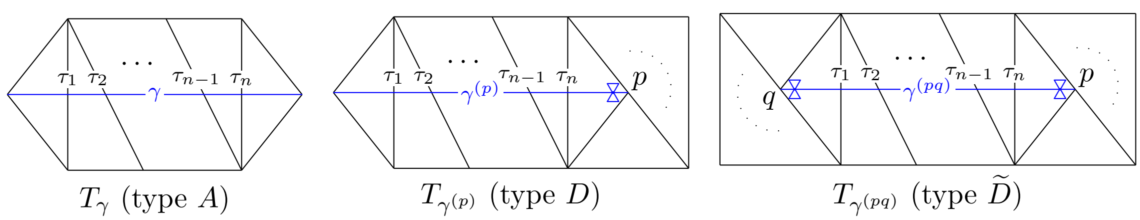


FIGURE 5. Triangulated polygon T_δ for each tagged arc δ

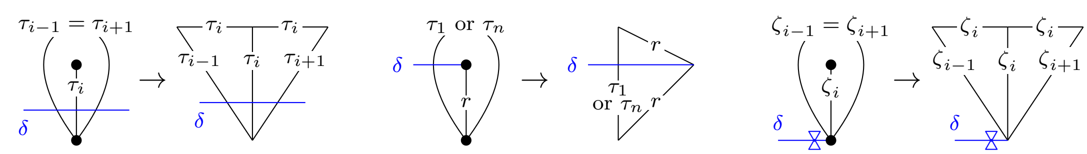


FIGURE 6. Replacing self-folded triangles

We call interior arcs of each polygon T_δ *diagonals* and non-interior arcs of T_δ *boundary segments*. We consider the graph whose vertices are all angles of T_δ incident to at least one diagonal, and whose edges are given by cliques containing all angles incident to a given vertex of T_δ and cliques containing all angles incident to a given triangle of T_δ. We call a maximal independent set of the graph a *maximal independent set of angles in* T_δ. We denote by $\mathbb{A}(T_\delta)$ the set of maximal independent sets of angles in T_δ. It is easy to see that $\mathbb{A}(T_\delta) \neq \emptyset$ (e.g. see Figure 7).

For a diagonal or boundary segment τ of T_δ, we denote $x_\tau = x_{\tau'}$ if τ corresponds to a non-boundary segment τ' of T and we denote $x_\tau = 1$ otherwise. Then, for an angle a of T_δ, $x_a := x_\tau$, where τ is the side opposite to a in the triangle containing a. Using Assumption 7.1, we define a ring homomorphism

$$\Phi : \mathbb{Z}[x_1^{\pm 1}, \dots, x_N^{\pm 1}] \to \mathbb{Z}[x_1^{\pm 1}, \dots, x_N^{\pm 1}] \tag{7.1}$$

by

$$\Phi(x_j) := \begin{cases} x_j x_k & \text{if } j \text{ is a 1-notched arc, where } k \text{ is the plain arc of } T \text{ corresponding to } j, \\ x_j & \text{otherwise,} \end{cases}$$

for any $j \in [1, N]$. Our main result Theorem 7.5 gives a cluster expansion formula for cluster algebras with principal coefficients defined from triangulated surfaces. In this subsection, we specialize it to the coefficient-free case.

Theorem 7.3. *Let δ be a tagged arc of (S, M).*
(1) If $\overline{\delta} \notin T$, we have

$$x_\delta = \Phi\left(\frac{1}{\operatorname{cross}(T,\delta)} \sum_{A \in \mathbb{A}(T_\delta)} x(A) \right), \quad \textit{where} \quad \operatorname{cross}(T,\delta) := \prod_{\tau \in T_\delta} x_\tau \quad \textit{and} \quad x(A) := \prod_{a \in A} x_a.$$

(2) Suppose that $\overline{\delta} \in T$ and $\delta \notin T$. Let p and q be the endpoints of $\overline{\delta}$. If p (resp., q) is a puncture, we denote by ℓ_p (resp., ℓ_q) the loop with endpoint q (resp., p) cutting out a monogon containing only p (resp., q). We can define triangulated polygons T_{ℓ_p} and T_{ℓ_q} in the same way as for plain arcs. Then, for $s = p$ or q, we have

$$x_\delta = \begin{cases} \frac{x_{\ell_s}}{x_{\overline{\delta}}} & \textit{if } \delta = \overline{\delta}^{(s)} \\ \frac{x_{\ell_p} x_{\ell_q} + 1}{x_{\overline{\delta}}} & \textit{if } \delta = \overline{\delta}^{(pq)} \end{cases}, \quad \textit{where} \quad x_{\ell_s} = \Phi\left(\frac{1}{\operatorname{cross}(T,\ell_s)} \sum_{A \in \mathbb{A}(T_{\ell_s})} x(A) \right).$$

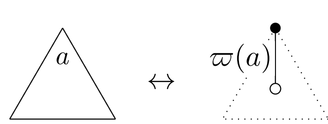
There are two key steps to prove Theorem 7.5.

The first step is the cluster expansion formula given by Musiker-Schiffler-Williams [MSW11]. A *perfect matching* in a graph G is a set P of edges of G such that each vertex of G is contained in exactly one edge in P. One can construct a snake graph G_δ associated with T_δ. Musiker-Schiffler-Williams gave a cluster expansion formula in terms of perfect matchings of G_δ (see Section 8). Note that perfect matchings of $G_{\gamma^{(p)}}$ and $G_{\gamma^{(pq)}}$ are different from general perfect matchings of graphs, that are also called *symmetric perfect matchings* and *compatible perfect matchings*, respectively (see Definitions 8.6 and 8.9).

The second step is Theorem 7.4 below. It gives bijections between several different combinatorial objects, that we introduce now. The bipartite graph B_δ associated with T_δ is defined as follows: The set of black vertices consists of vertices incident to at least one diagonal of T_δ and the set of white vertices consists of triangles of T_δ. Edges are drawn between the white vertex corresponding to a triangle ABC and the three black vertices corresponding to A, B and C if they exist. On the other hand, we associate to δ a quiver with potential $(\overline{Q}_\delta, \overline{W}_\delta)$ in Subsection 11.1, and we define *minimal cuts* of $(\overline{Q}_\delta, \overline{W}_\delta)$ in Definition 11.4.

Theorem 7.4. *There are bijections between the following objects:*
(1) Maximal independent sets of angles in T_δ, *(2) Perfect matchings of G_δ,*
(3) Perfect matchings of B_δ, *(4) Minimal cuts of $(\overline{Q}_\delta, \overline{W}_\delta)$,*
for any tagged arc δ of (S, M) such that $\overline{\delta} \notin T$.

By Theorem 7.4, we also obtain cluster expansion formulas in terms of perfect matchings of bipartite graphs and minimal cuts of quivers with potential. More precisely, the bijection between (1) and (3) in Theorem 7.4 is induced by a natural bijection ϖ between the set of angles incident to at least one diagonal of T_δ and the set of edges of B_δ given by the following picture:



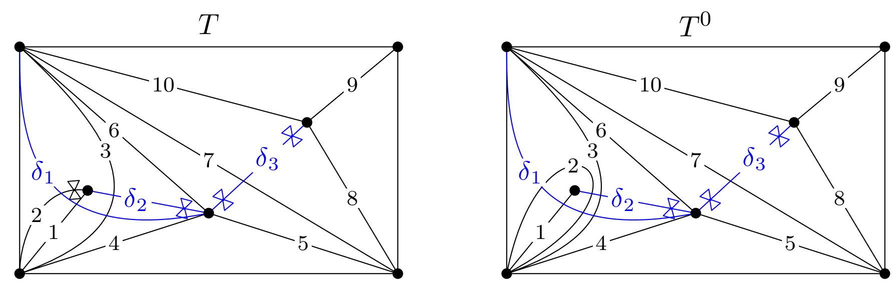
For a side e of B_δ, we denote $x_e = x_{\varpi^{-1}(e)}$. For a tagged arc δ of (S,M) with $\overline{\delta} \notin T$, we have

$$x_\delta = \Phi\left(\frac{1}{\mathrm{cross}(T,\delta)} \sum_E x(E)\right), \quad \text{where} \quad x(E) := \prod_{e \in A} x_e,$$

and E runs over all perfect matchings of B_δ. Similarly, we obtain a cluster expansion formula in terms of minimal cuts of quivers with potential (see Corollary 11.5).

Our main result Theorem 7.5 is obtained from the bijection between (1) and (2) in Theorem 7.4 and the cluster expansion formula of Musiker-Schiffler-Williams by showing that the bijection preserves the corresponding initial cluster variables. Notice that the construction of T_δ is simpler than the one of G_δ. Moreover, the definition of a maximal independent set of angles is more uniform than the definition of a perfect matching of G_δ, where three cases need to be distinguished depending of the tags attached to δ. Therefore, our new formula simplifies the formula of Musiker-Schiffler-Williams.

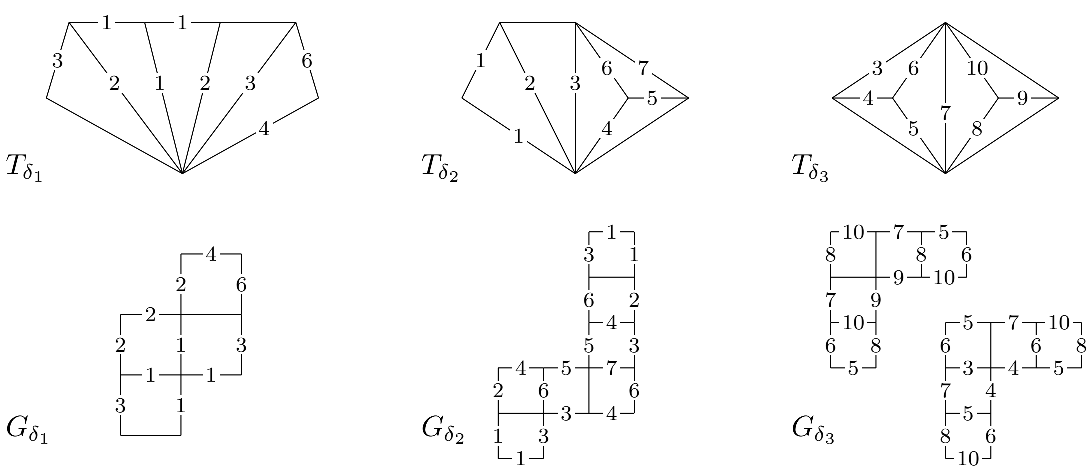
7.2. **Example in the coefficient-free case.** Let (S,M) be a square with three punctures. We consider the following tagged triangulation T and the corresponding ideal triangulation T^0:

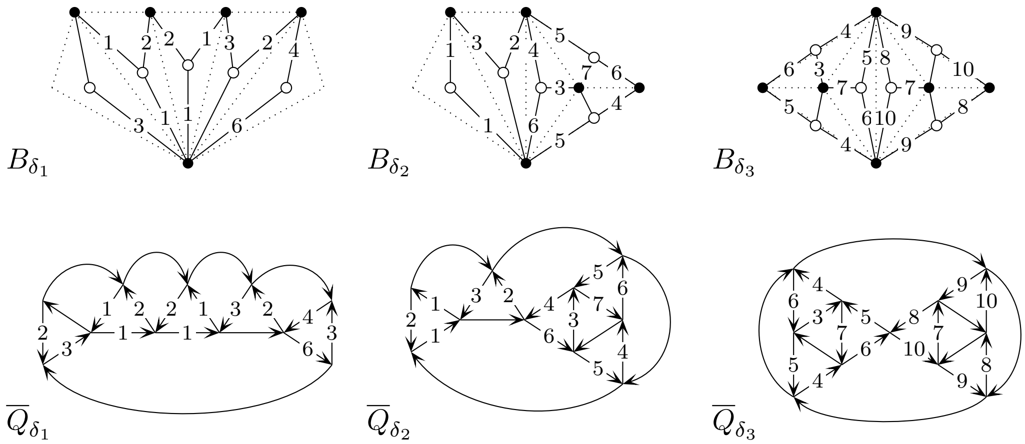


The cluster algebra $\mathcal{A}(T)$ has initial cluster variables $x_1, \ldots, x_{10}$. The ring homomorphism $\Phi : \mathbb{Z}[x_1^{\pm1}, \ldots, x_{10}^{\pm1}] \to \mathbb{Z}[x_1^{\pm1}, \ldots, x_{10}^{\pm1}]$ is given by

$$\Phi(x_i) = \begin{cases} x_1 x_2 & \text{if } i = 2, \\ x_i & \text{otherwise.} \end{cases}$$

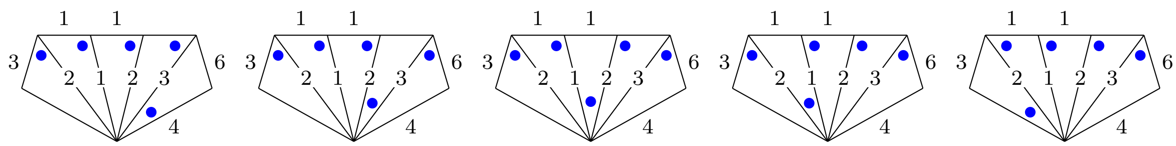
The combinatorial data corresponding to the above three tagged arcs δ_1, δ_2 and δ_3 are given as follows:





We use Theorem 7.3 to obtain the cluster expansions of x_{δ_1}, x_{δ_2} and x_{δ_3} with respect to the initial cluster variables $x_1, \ldots, x_{10}$ in $\mathcal{A}(T)$.

(1) δ_1: There are five maximal independent sets of angles in T_{δ_1}, corresponding to five monomials as follows:

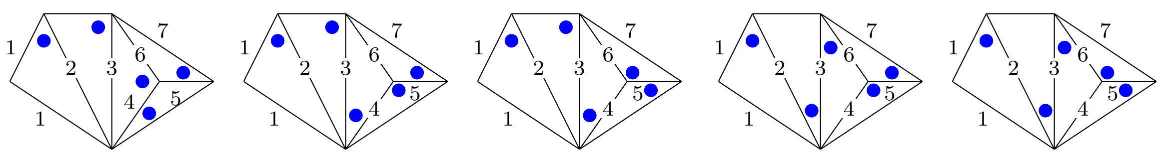


$x_1x_2^2x_6$ $\quad$ $x_1x_2x_4$ $\quad$ $x_1x_2x_3x_4$ $\quad$ $x_1x_2x_3x_4$ $\quad$ $x_1x_2x_3^2x_4$

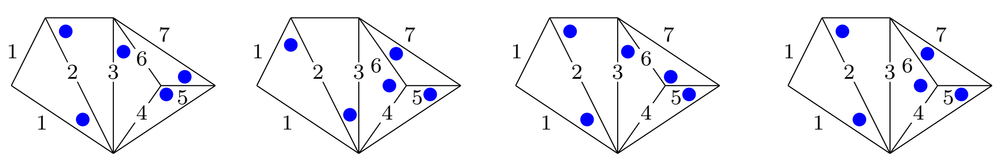
Since $\operatorname{cross}(T, \delta_1) = x_1x_2^2x_3$, the corresponding cluster variable is

$$x_{\delta_1} = \Phi\left(\frac{1}{x_2x_3}(x_2x_6 + x_4 + x_3x_4 + x_3x_4 + x_3^2x_4)\right) = \frac{1}{x_1x_2x_3}(x_1x_2x_6 + x_4 + 2x_3x_4 + x_3^2x_4).$$

(2) δ_2: There are nine maximal independent sets of angles in T_{δ_2}, corresponding to nine monomials as follows:



$x_1x_2x_3x_5x_6$ $\quad$ $x_1x_2x_6^2y_4$ $\quad$ $x_1x_2x_4x_6x_7y_4y_5$ $\quad$ $x_1x_4x_6y_3y_4$ $\quad$ $x_1x_4^2x_7y_3y_4y_5$

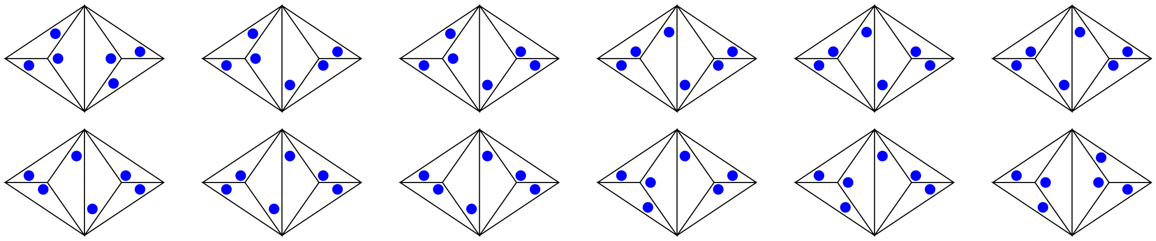


$x_1x_3x_4x_6y_2y_3y_4$ $\quad$ $x_1x_3x_4x_5y_3y_4y_5y_6$ $\quad$ $x_1x_3x_4^2x_7y_2y_3y_4y_5$ $\quad$ $x_1x_3^2x_4x_5y_2y_3y_4y_5y_6$

Since $\operatorname{cross}(T, \delta_2) = x_2x_3x_4x_5x_6$, the corresponding cluster variable is

$$\begin{aligned} x_{\delta_2} &= \Phi\left(\frac{1}{x_2x_3x_4x_5x_6}\begin{pmatrix} x_1x_2x_3x_5x_6 + x_1x_2x_6^2 + x_1x_2x_4x_6x_7 + x_1x_4x_6 + x_1x_4^2x_7 \\ +x_1x_3x_4x_6 + x_1x_3x_4x_5 + x_1x_3x_4^2x_7 + x_1x_3^2x_4x_5 \end{pmatrix}\right) \\ &= \frac{1}{x_2x_3x_4x_5x_6}\begin{pmatrix} x_1x_2x_3x_5x_6 + x_1x_2x_6^2 + x_1x_2x_4x_6x_7 + x_4x_6 + x_4^2x_7 \\ +x_3x_4x_6 + x_3x_4x_5 + x_3x_4^2x_7 + x_3^2x_4x_5 \end{pmatrix}. \end{aligned}$$

(3) δ_3: There are 12 maximal independent sets of angles in T_{δ_3} as follows:



and 6 others obtained by rotation of angle π from the bottom row. Then the corresponding cluster variable is

$$x_{\delta_3} = \frac{1}{x_4x_5x_6x_7x_8x_9x_{10}} \begin{pmatrix} x_4x_5x_7^2x_9x_{10} + x_4x_5x_7x_{10}^2 + x_3x_5^2x_7x_9x_{10} \\ +x_4x_5x_7x_8x_{10} + x_5x_6x_9x_{10} + x_3x_5^2x_{10}^2 \\ +x_3x_5^2x_8x_{10} + x_5x_6x_{10}^2 + x_3x_5x_6x_8x_{10} \\ +x_5x_6x_8x_{10} + x_3x_5x_6x_8^2 + x_6^2x_8x_{10} \\ +x_6^2x_8^2 + x_4x_6x_7x_8x_{10} + x_3x_5x_6x_7x_8x_9 \\ +x_4x_6x_7x_8^2 + x_6^2x_7x_8x_9 + x_4x_6x_7^2x_8x_9 \end{pmatrix},$$

which is not affected by Φ since x_2 don't appear.

For the case (2), we illustrate Theorem 7.4 in Examples 8.8, 10.2 and 11.6.

7.3. **Our results in the principal coefficients case.** We keep the notations of Subsection 7.1. Let $\zeta_1, \ldots, \zeta_m$ (resp., $\xi_1, \ldots, \xi_\ell$) be the diagonals of T_δ incident to p (resp., q) winding counterclockwisely around p (resp., q) such that τ_n, ζ_1, and ζ_m (resp., τ_1, ξ_1, and ξ_ℓ) are contained in the same triangle (see Figure 7). We define an element $A_-(T_\delta) \in \mathbb{A}(T_\delta)$, which we call the *minimal matching* of T_δ, satisfying the following *min-condition*: For each boundary vertex v of T_δ that is incident to at least one diagonal of T_δ, the angle $a \in A_-(T_\delta)$ at v comes first in the counterclockwise order around v. Clearly, the minimal matching is uniquely determined (see Figure 7).

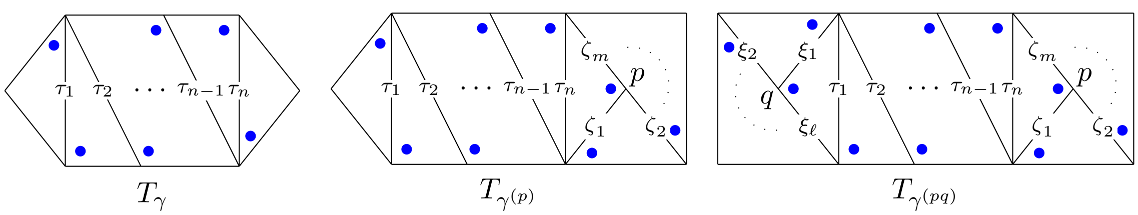


FIGURE 7. Minimal matchings

We expand the ring homomorphism (7.1) into

$$\Phi : \mathbb{Z}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}, y_1^{\pm 1}, \ldots, y_N^{\pm 1}] \to \mathbb{Z}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}, y_1^{\pm 1}, \ldots, y_N^{\pm 1}]$$

by

$$\Phi(y_j) := \begin{cases} \frac{y_j}{y_k} & \text{if } j \text{ is plain and corresponds to the 1-notched arc } k \text{ of } T, \\ y_j & \text{otherwise,} \end{cases}$$

for any $j \in [1, N]$. For two sets A and B, we denote by $A \triangle B$ the symmetric difference $(A \cup B) \setminus (A \cap B)$. An *exterior angle* of T_δ is an angle between a boundary segment and a diagonal of T_δ. Let $A \in \mathbb{A}(T_\delta)$. We denote by $Y'(A)$ the set of diagonals of T_δ that are sides of at least one exterior angle in $A_-(T_\delta) \triangle A$. We define the set

$$Y(A) := \begin{cases} Y'(A) \sqcup \{\tau_1\} & \text{if } \delta = \gamma^{(pq)},\ n = 1, \text{ and } A \text{ contains at least one of the four angles} \\ & \text{between } \zeta_m \text{ or } \xi_\ell \text{ and } \tau_1 \text{ or a boundary segment of } T_{\gamma^{(pq)}}, \\ Y'(A) & \text{otherwise.} \end{cases}$$

We are ready to state the main theorem of this part.

Theorem 7.5. *Let δ be a tagged arc of (S,M).*

(1) If $\overline{\delta}\notin T$, we have

$$x_\delta=\Phi\left(\frac{1}{\operatorname{cross}(T,\delta)}\sum_{A\in\mathbb{A}(T_\delta)}x(A)y(A)\right),\quad \textit{where}\ \ y(A):=\prod_{\tau\in Y(A)}y_\tau.$$

(2) Suppose that $\overline{\delta}\in T$ and $\delta\notin T$. Let r and s be the endpoints of $\overline{\delta}$. Then, for $s=p$ or q, we have

$$x_\delta=\begin{cases}\frac{x_{\ell_s}}{x_{\overline{\delta}}} & \textit{if } \delta=\overline{\delta}^{(s)},\\ \frac{x_{\ell_p}x_{\ell_q}y_{\overline{\delta}}+(1-\prod_{\tau\in T}y_\tau^{e_p(\tau)})(1-\prod_{\tau\in T}y_\tau^{e_q(\tau)})}{x_{\overline{\delta}}} & \textit{if } \delta=\overline{\delta}^{(pq)},\end{cases}$$

where $e_s(\tau)$ is the number of ends of τ incident to s, and

$$x_{\ell_s}=\Phi\left(\frac{1}{\operatorname{cross}(T,\ell_s)}\sum_{A\in\mathbb{A}(T_{\ell_s})}x(A)y(A)\right).$$

Since Theorem 7.5(2) follows from [FoT, Lemma 8.2, Theorem 8.6] and [MSW11, Proposition 4.21], we only prove Theorem 7.5(1) in Section 9.

In the rest of this section, we consider the bipartite graph B_δ. We define the *minimal matching* of B_δ by $E_-(B_\delta):=\varpi^{-1}(A_-(T_\delta))\in\mathbb{P}(B_\delta)$, where $\mathbb{P}(B_\delta)$ the set of perfect matchings of B_δ. For a diagonal τ of T_δ, there are exactly two triangles $\triangle$, $\triangle'$ of T_δ with edge τ. We label by τ the square of B_δ whose vertices are two white vertices corresponding to $\triangle$, $\triangle'$ and two black vertices corresponding to endpoints of τ.

Proposition 7.6. *For $E\in\mathbb{P}(B_\delta)$, the set $E_-(B_\delta)\triangle E$ consists of all boundary edges of some (possibly empty or disconnected) subgraph B_E of B_δ that is a union of squares.*

We denote by $I(E)$ the set of the squares of B_δ contained in B_E.

Proposition 7.7. *For $E\in\mathbb{P}(B_\delta)$, $I(E)=Y(\varpi^{-1}(E))$ holds.*

By Theorem 7.5 and Proposition 7.7, for a tagged arc δ of (S,M) such that $\overline{\delta}\notin T$, we have

$$x_\delta=\Phi\left(\frac{1}{\operatorname{cross}(T,\delta)}\sum_{E\in\mathbb{P}(B_\delta)}x(E)y(E)\right),\quad\text{where}\ \ y(E):=\prod_{i\in I(E)}y_i. \tag{7.2}$$

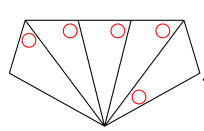
This formula is a generalization of the cluster expansion formula in type A given by Carroll and Price [CaP] (see also [CeP, Pr]). We prove Propositions 7.6 and 7.7 in Section 10.

7.4. **Example in the principal coefficients case.** We consider the square (S,M) with three punctures and the tagged triangulation T of (S,M) given in Subsection 7.2. The cluster algebra $\mathcal{A}(T)$ has initial cluster variables $x_1,\dots,x_{10}$ and initial principal coefficients $y_1,\dots,y_{10}$. The ring homomorphism $\Phi:\mathbb{Z}[x_1^{\pm1},\dots,x_{10}^{\pm1},y_1^{\pm1},\dots,y_{10}^{\pm1}]\to\mathbb{Z}[x_1^{\pm1},\dots,x_{10}^{\pm1},y_1^{\pm1},\dots,y_{10}^{\pm1}]$ is given by

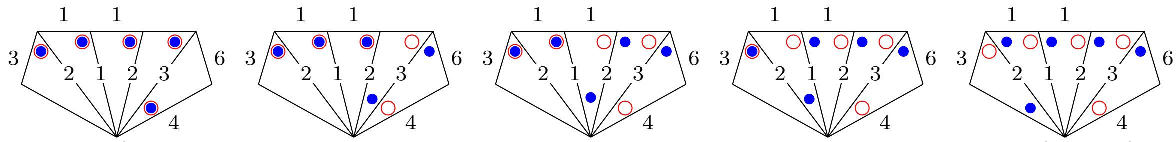
$$\Phi(x_i)=\begin{cases}x_1x_2 & \text{if } i=2,\\ x_i & \text{otherwise,}\end{cases}\qquad \Phi(y_i)=\begin{cases}\frac{y_1}{y_2} & \text{if } i=1,\\ y_i & \text{otherwise.}\end{cases}$$

We use Theorem 7.5 to obtain the cluster expansions of δ_1, δ_2 and δ_3 given in Subsection 7.2 with respect to the initial cluster variables $x_1,\dots,x_{10}$ and coefficients $y_1,\dots,y_{10}$ in $\mathcal{A}(T)$.

(1) δ_1: The minimal matching is

$A_-(T_{\delta_1})=$ .

Then there are five maximal independent sets of angles in T_{δ_1}, corresponding to five monomials as follows:

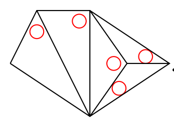


$x_1x_2^2x_6$ $\quad$ $x_1x_2x_4y_3$ $\quad$ $x_1x_2x_3x_4y_2y_3$ $\quad$ $x_1x_2x_3x_4y_1y_2y_3$ $\quad$ $x_1x_2x_3^2x_4y_1y_2^2y_3$

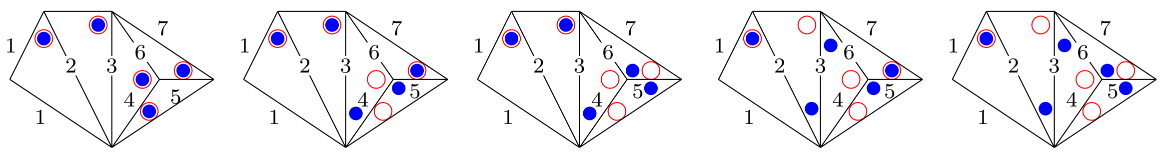
Since $\mathrm{cross}(T,\delta_1) = x_1x_2^2x_3$, the corresponding cluster variable is

$$\begin{aligned} x_{\delta_1} &= \Phi\bigg(\frac{1}{x_2x_3}(x_2x_6 + x_4y_3 + x_3x_4y_2y_3 + x_3x_4y_1y_2y_3 + x_3^2x_4y_1y_2^2y_3)\bigg) \\ &= \frac{1}{x_1x_2x_3}(x_1x_2x_6 + x_4y_3 + x_3x_4y_2y_3 + x_3x_4y_1y_3 + x_3^2x_4y_1y_2y_3). \end{aligned}$$

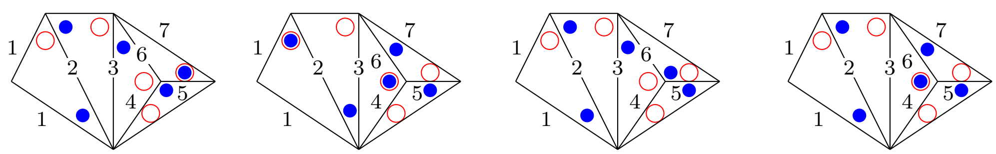
(2) δ_2: The minimal matching is

$A_-(T_{\delta_2}) =$  .

Then there are nine maximal independent sets of angles in T_{δ_2}, corresponding to nine monomials as follows:



$x_1x_2x_3x_5x_6$ $\quad$ $x_1x_2x_6^2y_4$ $\quad$ $x_1x_2x_4x_6x_7y_4y_5$ $\quad$ $x_1x_4x_6y_3y_4$ $\quad$ $x_1x_4^2x_7y_3y_4y_5$

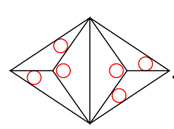


$x_1x_3x_4x_6y_2y_3y_4$ $\quad$ $x_1x_3x_4x_5y_3y_4y_5y_6$ $\quad$ $x_1x_3x_4^2x_7y_2y_3y_4y_5$ $\quad$ $x_1x_3^2x_4x_5y_2y_3y_4y_5y_6$

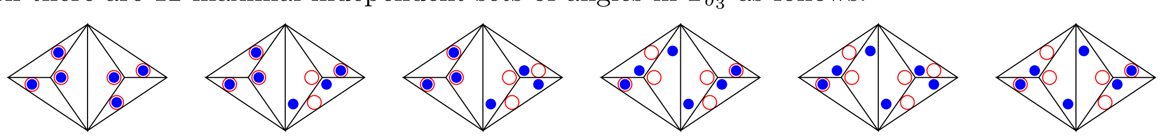
Remark that the three angles incident to the puncture of T_{δ_2} are not exterior angles and thus don't contribute to the coefficients. Since $\mathrm{cross}(T,\delta_2) = x_2x_3x_4x_5x_6$, the corresponding cluster variable is

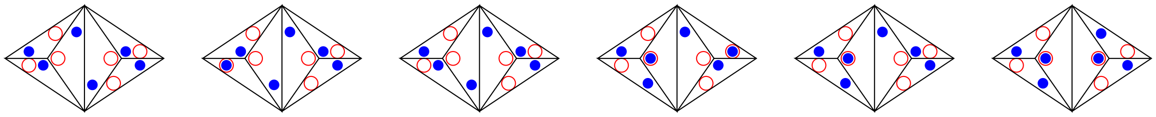
$$\begin{aligned} x_{\delta_2} &= \Phi\left(\frac{1}{x_2x_3x_4x_5x_6}\begin{pmatrix} x_1x_2x_3x_5x_6 + x_1x_2x_6^2y_4 + x_1x_2x_4x_6x_7y_4y_5 + x_1x_4x_6y_3y_4 + x_1x_4^2x_7y_3y_4y_5 \\ +x_1x_3x_4x_6y_2y_3y_4 + x_1x_3x_4x_5y_3y_4y_5y_6 + x_1x_3x_4^2x_7y_2y_3y_4y_5 + x_1x_3^2x_4x_5y_2y_3y_4y_5y_6 \end{pmatrix}\right) \\ &= \frac{1}{x_2x_3x_4x_5x_6}\begin{pmatrix} x_1x_2x_3x_5x_6 + x_1x_2x_6^2y_4 + x_1x_2x_4x_6x_7y_4y_5 + x_4x_6y_3y_4 + x_4^2x_7y_3y_4y_5 \\ +x_3x_4x_6y_2y_3y_4 + x_3x_4x_5y_3y_4y_5y_6 + x_3x_4^2x_7y_2y_3y_4y_5 + x_3^2x_4x_5y_2y_3y_4y_5y_6 \end{pmatrix}. \end{aligned}$$

(3) δ_3: The minimal matching is

$A_-(T_{\delta_3}) =$  .

Then there are 12 maximal independent sets of angles in T_{δ_3} as follows:





and 6 others obtained by rotation of angle π from the bottom row. Then the corresponding cluster variable is

$$x_{\delta_3} = \frac{1}{x_4x_5x_6x_7x_8x_9x_{10}}\begin{pmatrix} x_4x_5x_7^2x_9x_{10} + x_4x_5x_7x_{10}^2y_8 + x_3x_5^2x_7x_9x_{10}y_6 \\ +x_4x_5x_7x_8x_{10}y_8y_9 + x_5x_6x_9x_{10}y_4y_6 + x_3x_5^2x_{10}^2y_6y_8 \\ +x_3x_5^2x_8x_{10}y_6y_8y_9 + x_5x_6x_{10}^2y_4y_6y_8 + x_3x_5x_6x_8x_{10}y_6y_7y_8 \\ +x_5x_6x_8x_{10}y_4y_6y_8y_9 + x_3x_5x_6x_8^2y_6y_7y_8y_9 + x_6^2x_8x_{10}y_4y_6y_7y_8 \\ +x_6^2x_8^2y_4y_6y_7y_8y_9 + x_4x_6x_7x_8x_{10}y_4y_5y_6y_7y_8 \\ +x_3x_5x_6x_7x_8x_9y_6y_7y_8y_9y_{10} + x_4x_6x_7x_8^2y_4y_5y_6y_7y_8y_9 \\ +x_6^2x_7x_8x_9y_4y_6y_7y_8y_9y_{10} + x_4x_6x_7^2x_8x_9y_4y_5y_6y_7y_8y_9y_{10} \end{pmatrix},$$

which is not affected by Φ since x_2 and y_1 don't appear.

7.5. **f-vectors and intersection numbers.** We keep the notations of Subsection 7.3. We recall f-vectors of cluster variables [FuG, Definition 2.6]: For a cluster variable x of $\mathcal{A}(T)$, let $f_{x,1}, \dots, f_{x,N}$ be the maximal degrees of $y_1, \dots, y_N$ in the polynomial obtained from the Laurent expression of x by substituting 1 for each of $x_1, \dots, x_N$. The integer vector $f_x := (f_{x,1}, \dots, f_{x,N}) \in \mathbb{Z}_{\geq 0}^N$ is called the *f-vector of x*. For a tagged arc δ of (S, M) such that $\overline{\delta} \notin T$, by Theorem 7.5(1), the f-vector $(f_{x_\delta,1}, \dots, f_{x_\delta,N})$ of x_δ is given by

$$\prod_{i=1}^{N} y_i^{f_{x_\delta,i}} = \Phi\Bigg(\prod_{\tau \in T_\delta} y_\tau\Bigg). \tag{7.3}$$

On the other hand, for tagged arcs δ and ϵ of (S, M), Qiu and Zhou [QZ] defined the intersection number of δ and ϵ as follows: Assume that δ and ϵ intersect transversally in a minimum number of points in $S \setminus M$. Then we define the intersection number $\mathsf{Int}(\delta, \epsilon) = A + B + C$, where

- A is the number of intersection points of δ and ϵ in $S \setminus M$;
- B is the number of pairs of an end of δ and an end of ϵ that are incident to a common puncture such that their tags are different;
- $C = 0$ unless δ and ϵ form a pair of conjugate arcs, in which case $C = -1$.

Note that this definition is different from the "intersection number" $(\delta\,|\epsilon)$ defined in [FST, Definition 8.4]. We give the main result of this subsection.

Theorem 7.8. *For a tagged arc δ of (S, M), we have $f_{x_\delta,i} = \mathsf{Int}(\delta, i)$ for $i \in [1, N]$.*

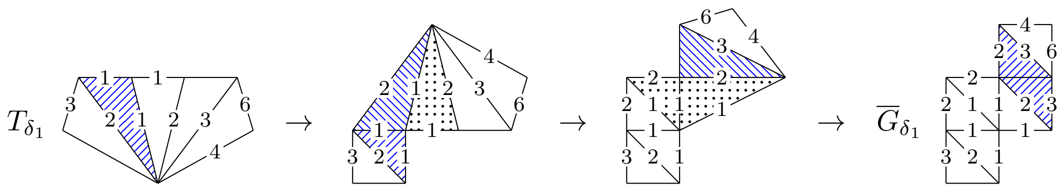
Proof. Considering in each case, it is easy to show that both $f_{x_\delta,i}$ and $\mathsf{Int}(\delta, i)$ are equal to $f \in \mathbb{Z}_{\geq 0}$ given as follows: If $\delta \in T$, then $f = 0$; Suppose that $\overline{\delta} \notin T$. If i is a plain arc of T, then f is the number of diagonals of T_δ corresponding to i. If i is a 1-notched arc of T, then f is the number of diagonals of T_δ corresponding to i minus the number of diagonals of T_δ corresponding to $\overline{i}$; Suppose that $\overline{\delta} \in T$ and $\delta \notin T$. We use the notations of Theorem 7.5(2). If $\delta = \overline{\delta}^{(s)}$, then $f = e_s(i) - \delta_{i\overline{\delta}}$, where $\delta_{i\overline{\delta}}$ is the Kronecker delta. If $\delta = \overline{\delta}^{(pq)}$, then $f = e_p(i) + e_q(i)$. $\square$

8. Musiker-Schiffler-Williams cluster expansion formulas

In this section, we recall the cluster expansion formula given by Musiker-Schiffler-Williams [MS, MSW11]. We call it the *MSW formula*. Fix a marked surface (S, M) and a tagged triangulation T of (S, M) satisfying Assumptions 7.1 and 7.2. Let γ be a plain arc of (S, M) such that $\gamma \notin T$. We use the notations of the introduction.

8.1. **Formula for plain arcs.** Recall the MSW formula for x_γ. In the triangulation T_γ constructed in the introduction, triangles have at most two sides that are non-boundary segments and at least one side that is a boundary segment. In the same way as Section 5, we can construct the snake graph $\overline{G}_\gamma := \overline{G}_{T_\gamma}$. In this part, we label all edges of $\overline{G}_\gamma$ by the corresponding tagged arcs of T.

Example 8.1. We construct the snake graph $\overline{G}_{\delta_1}$ for the tagged arc δ_1 given in Subsection 7.2 as follows:



Note that $\overline{G}_\gamma$ consists of n squares with diagonals τ_i for $1 \leq i \leq n$. We call these squares *tiles* of $\overline{G}_\gamma$. Let $G_\gamma := G_{T_\gamma}$ be the graph obtained from $\overline{G}_\gamma$ by removing the diagonal of each tile. It is easy to see that the following special perfect matching is uniquely determined.

Definition 8.2. [MSW11, Definition 4.7] Let e_0 be the edge of G_γ corresponding to the boundary segment of T_γ that follows τ_1 in the clockwise direction in the triangle T_0. The *minimal matching* $P_-(G_\gamma)$ is the perfect matching of G_γ containing e_0 and consisting only of boundary edges.

In Example 8.1, e_0 is the bottom edge of $\overline{G}_{\delta_1}$.

Theorem 8.3. [MS, Theorem 5.1] *For $P \in \mathbb{P}(G_\gamma)$, the set $P_-(G_\gamma) \triangle P$ consists of all boundary edges of some (possibly empty or disconnected) subgraph G_P of G_γ that is a union of tiles.*

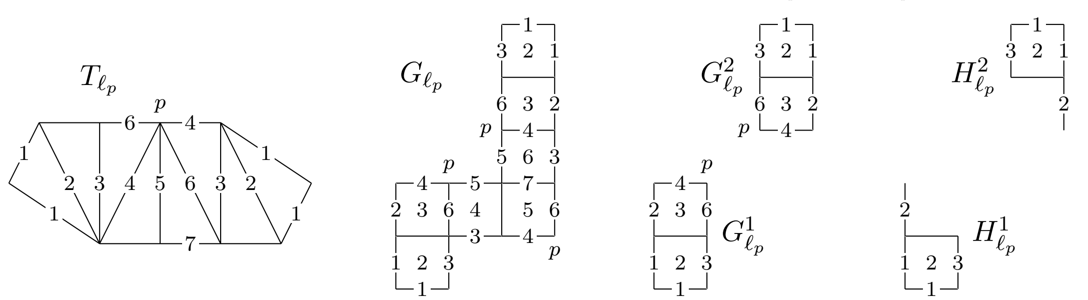
We denote by $J(P)$ the set of the diagonals of all tiles of $\overline{G}_\gamma$ that are contained in G_P. The following cluster expansion formula is obtained by using perfect matchings of G_γ.

Theorem 8.4. [MSW11, Theorem 4.10] *We have*

$$x_\gamma = \Phi\left(\frac{1}{\mathrm{cross}(T,\gamma)} \sum_{P \in \mathbb{P}(G_\gamma)} x(P)y(P)\right), \quad x(P) := \prod_{e \in P} x_e, \quad y(P) := \prod_{j \in J(P)} y_j.$$

8.2. **Formula for 1-notched arcs.** Recall the MSW formula for $x_{\gamma^{(p)}}$. Let $q \neq p$ be the other endpoint of $\gamma^{(p)}$. In the same way as above, for the loop ℓ_p defined in Theorem 7.3, we get the snake graph G_{ℓ_p} which is denoted by $G_{\gamma^{(p)}}$ in the introduction. By construction, G_{ℓ_p} contains two disjoint subgraphs $G^1_{\ell_p}$ and $G^2_{\ell_p}$ with same form as G_γ. Moreover, we consider the subgraph $H^i_{\ell_p}$ of $G^i_{\ell_p}$ obtained by removing the vertex p and the two edges ζ_1, ζ_m.

Example 8.5. Let ℓ_p be the loop such that $\iota(\ell_p) = \delta_2$ in given Subsection 7.2. We have the triangulated polygon T_{ℓ_p}, the snake graph G_{ℓ_p} and the subgraphs $G^i_{\ell_p}$ and $H^i_{\ell_p}$ of G_{ℓ_p} as follows:

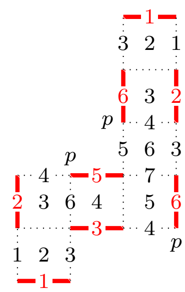
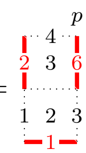


Definition 8.6. [MSW11, Definition 4.15] A perfect matching P of G_{ℓ_p} is γ-*symmetric* if $P|_{H^1_{\ell_p}} \simeq P|_{H^2_{\ell_p}}$. We denote by $\mathbb{P}(G_{\gamma^{(p)}})$ the set of γ-symmetric perfect matchings of G_{ℓ_p}. We also refer to elements of $\mathbb{P}(G_{\gamma^{(p)}})$ as perfect matchings of $G_{\gamma^{(p)}}$.

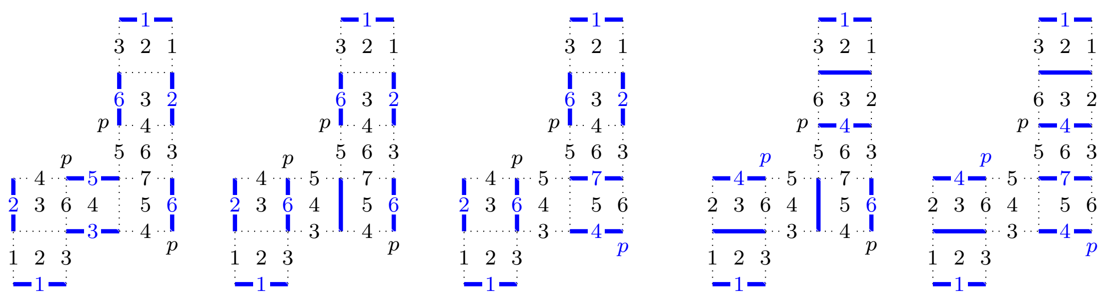
Theorem 8.7. [MSW11, Theorem 4.17, Lemma 12.4] *For $P \in \mathbb{P}(G_{\ell_p})$, let $\operatorname{res}(P)$ be a unique perfect matching of G_γ such that $\operatorname{res}(P) \setminus (\operatorname{res}(P) \cap \{\zeta_1, \zeta_m\}) = P|_{H^1_{\ell_p}}$. Then $P|_{G^i_{\ell_p}} \simeq \operatorname{res}(P)$ for some $i \in \{1, 2\}$. Moreover, we have*

$$x_{\gamma^{(p)}} = \Phi\left(\frac{1}{\operatorname{cross}(T, \gamma^{(p)})} \sum_{P \in \mathbb{P}(G_{\gamma^{(p)}})} \overline{x}(P)\,\overline{y}(P)\right), \quad \overline{x}(P) := \frac{x(P)}{x(\operatorname{res}(P))}, \quad \overline{y}(P) := \frac{y(P)}{y(\operatorname{res}(P))}.$$

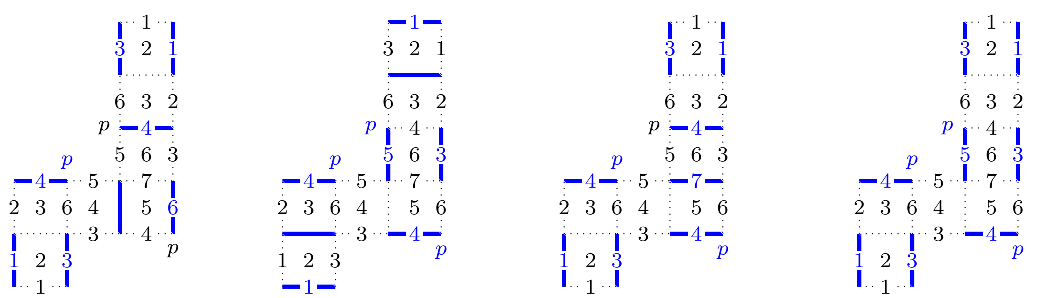
Example 8.8. For G_{ℓ_p} and $G^i_{\ell_p}$ in Example 8.5, their minimal matchings are

$P_-(G_{\ell_p}) =$  and $P_-(G^i_{\ell_p}) =$ 

Then there are nine γ-symmetric perfect matchings P of G_{ℓ_p}, corresponding to nine monomials $\overline{x}(P)\,\overline{y}(P)$ as follows:



$x_1x_2x_3x_5x_6$ $\quad x_1x_2x_6^2y_3$ $\quad x_1x_2x_4x_6x_7y_4y_5$ $\quad x_1x_4x_6y_3y_4$ $\quad x_1x_4^2x_7y_3y_4y_5$



$x_1x_3x_4x_6y_2y_3y_4$ $\quad x_1x_3x_4x_5y_3y_4y_5y_6$ $\quad x_1x_3x_4^2x_7y_2y_3y_4y_5$ $\quad x_1x_3^2x_4x_5y_2y_3y_4y_5y_6$

Since these are perfect matchings of G_{δ_2} for δ_2 given in Subsection 7.2, the corresponding cluster variable is

$$\begin{aligned} x_{\delta_2} &= \Phi\left(\frac{1}{x_2x_3x_4x_5x_6}\begin{pmatrix} x_1x_2x_3x_5x_6 + x_1x_2x_6^2y_4 + x_1x_2x_4x_6x_7y_4y_5 + x_1x_4x_6y_3y_4 \\ +x_1x_4^2x_7y_3y_4y_5 + x_1x_3x_4x_6y_2y_3y_4 + x_1x_3x_4x_5y_3y_4y_5y_6 \\ +x_1x_3x_4^2x_7y_2y_3y_4y_5 + x_1x_3^2x_4x_5y_2y_3y_4y_5y_6 \end{pmatrix}\right) \\ &= \frac{1}{x_2x_3x_4x_5x_6}\begin{pmatrix} x_1x_2x_3x_5x_6 + x_1x_2x_6^2y_4 + x_1x_2x_4x_6x_7y_4y_5 + x_4x_6y_3y_4 \\ +x_4^2x_7y_3y_4y_5 + x_3x_4x_6y_2y_3y_4 + x_3x_4x_5y_3y_4y_5y_6 \\ +x_3x_4^2x_7y_2y_3y_4y_5 + x_3^2x_4x_5y_2y_3y_4y_5y_6 \end{pmatrix}. \end{aligned}$$

8.3. **Formula for 2-notched arcs.** Recall the MSW formula for $x_{\gamma^{(pq)}}$. As above, we get loops ℓ_p and ℓ_q and the snake graphs G_{ℓ_p} and G_{ℓ_q}. Note that the pair (G_{ℓ_p}, G_{ℓ_q}) is denoted by $G_{\gamma^{(pq)}}$ in the introduction. Remark that γ may be a loop. Then we denote by ℓ_p and ℓ_q the loops as in Figure 8 although they are not loops.

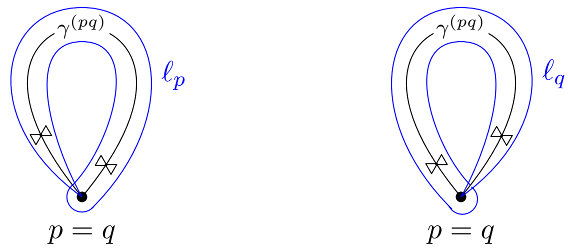


FIGURE 8. Analogues of ℓ_p and ℓ_q for a 2-notched loop

Definition 8.9. [MSW11, Definition 4.18] Let P_p and P_q be γ-symmetric perfect matchings of G_{ℓ_p} and G_{ℓ_q}, respectively. The pair (P_p, P_q) is γ-*compatible* if $\mathrm{res}(P_p) \simeq \mathrm{res}(P_q)$. We denote by $\mathbb{P}(G_{\gamma^{(pq)}})$ the set of γ-compatible pairs of $\mathbb{P}(G_{\gamma^{(p)}}) \times \mathbb{P}(G_{\gamma^{(q)}})$. We also refer to elements of $\mathbb{P}(G_{\gamma^{(pq)}})$ as perfect matchings of $G_{\gamma^{(pq)}}$.

Theorem 8.10. [MSW11, Theorem 4.20] *We have*

$$x_{\gamma^{(pq)}} = \Phi\left(\frac{1}{\mathrm{cross}(T,\gamma^{(pq)})} \sum_{(P_p,P_q)\in\mathbb{P}(G_{\gamma^{(pq)}})} \overline{\overline{x}}(P_p,P_q)\,\overline{\overline{y}}(P_p,P_q)\right),$$

where

$$\overline{\overline{x}}(P_p,P_q) := \frac{x(P_p)x(P_q)}{x(\mathrm{res}(P_p))^3}, \quad \overline{\overline{y}}(P_p,P_q) := \frac{y(P_p)y(P_q)}{y(\mathrm{res}(P_p))^3}.$$

9. Proof of Theorem 7.5

In this section, we keep the notations of the previous sections. We prove the bijection between (1) and (2) in Theorem 7.4 and Theorem 7.5 in the three cases of $\delta = \gamma$, $\gamma^{(p)}$ and $\gamma^{(pq)}$. Notice that the same notations Φ and $\mathrm{cross}(T,\delta)$ appear in Theorems 7.5, 8.4, 8.7 and 8.10. So we only need to consider $x(A)$ and $y(A)$ for $A \in \mathbb{A}(T_\delta)$. Let $A(T_\delta)$ be the set of angles incident to at least one diagonal of T_δ, and let $A_{\mathrm{ex}}(T_\delta)$ be the set of exterior angles of T_δ which are angles between boundary segments and diagonals of T_δ. In particular, $A_{\mathrm{ex}}(T_\delta)$ is contained in $A(T_\delta)$. For a set S, we denote by $\#S$ the cardinality of S.

9.1. **The case of plain arcs.** For a plain arc γ, we denote by $(G_\gamma)_1$ (resp., $(G_\gamma)_b$) the set of edges (resp., boundary edges) of G_γ. Let $A(\overline{G}_\gamma)$ be the set of angles between a diagonal τ_i and a side of the square with diagonal τ_i in $\overline{G}_\gamma$, and $\overline{\varphi} : A(\overline{G}_\gamma) \to (G_\gamma)_1$ the surjective map sending $a \in A(\overline{G}_\gamma)$ to the side that is opposite to a. By the unfolding process (see Section 8), there is a canonical surjection $\pi : A(\overline{G}_\gamma) \to A(T_\gamma)$ compatible with the construction of $\overline{G}_\gamma$.

Theorem 9.1. *[Lemma 6.2, Proposition 6.4] There exists a bijection $\varphi : A(T_\gamma) \to (G_\gamma)_1$ making the following diagram commutative:*

$$\begin{array}{ccc} & A(\overline{G}_\gamma) & \\ {}^{\pi}\swarrow & & \searrow^{\overline{\varphi}} \\ A(T_\gamma) & \xrightarrow[\varphi]{\sim} & (G_\gamma)_1 \end{array}$$

Moreover the map φ induces a bijection $\varphi : \mathbb{A}(T_\gamma) \to \mathbb{P}(G_\gamma)$ satisfying $x(A) = x(\varphi(A))$ for $A \in \mathbb{A}(T_\gamma)$.

Theorem 9.1 clearly gives the bijection between (1) and (2) in Theorem 7.4 for plain arcs. We only need to show that $y(A) = y(\varphi(A))$ for $A \in \mathbb{A}(T_\gamma)$ to prove Theorem 7.5 for plain arcs.

Lemma 9.2. *The restriction $\varphi|_{A_{\mathrm{ex}}(T_\gamma)}$ of φ deduces a bijection between $A_{\mathrm{ex}}(T_\gamma)$ and $(G_\gamma)_b$.*

Proof. The complement $A(T_\gamma) \setminus A_{\mathrm{ex}}(T_\gamma)$ consists of angles a_i between τ_i and τ_{i+1} for $i \in [1, n-1]$, in particular, $\#(A(T_\gamma) \setminus A_{\mathrm{ex}}(T_\gamma)) = n - 1$. It follows from the unfolding process that $\varphi(a_i) \in (G_\gamma)_1 \setminus (G_\gamma)_b$. Since $\#((G_\gamma)_1 \setminus (G_\gamma)_b) = n - 1$, the restriction $\varphi|_{A(T_\gamma)\setminus A_{\mathrm{ex}}(T_\gamma)}$ is bijective and so is $\varphi|_{A_{\mathrm{ex}}(T_\gamma)}$. □

Proposition 9.3. *For $A \in \mathbb{A}(T_\gamma)$, we have $Y(A) = J(\varphi(A))$, that is $y(A) = y(\varphi(A))$.*

Proof. By Theorem 9.1 and Lemma 9.2, $\varphi(A_-(T_\gamma))$ is a perfect matching of G_γ consisting only of boundary edges. In particular, since $e_0 \in \varphi(A_-(T_\gamma))$, where e_0 was defined in Definition 8.2, $\varphi(A_-(T_\gamma)) = P_-(G_\gamma)$ holds. Thus we have $\varphi(A_-(T_\gamma) \triangle A) = P_-(G_\gamma) \triangle \varphi(A)$. On the other hand, φ maps the four angles incident to τ_i in T_γ to sides of the square with diagonal τ_i in $\overline{G}_\gamma$. Therefore, $(A_-(T_\gamma) \triangle A) \cap A_{\mathrm{ex}}(T_\gamma)$ contains an angle incident to τ_i, which is equivalent to $\tau_i \in Y(A)$, if and only if $(P_-(G_\gamma) \triangle \varphi(A)) \cap (G_\gamma)_b$ contains an edge of the square with diagonal τ_i in $\overline{G}_\gamma$, which is equivalent to $\tau_i \in J(\varphi(A))$ by the definition. □

Proof of Theorem 7.5 for plain arcs. The assertion follows from Theorems 8.4 and 9.1 and Proposition 9.3. □

Finally, we prepare the following lemma to use later.

Lemma 9.4. *For $A \in \mathbb{A}(T_\gamma)$, if $A_-(T_\gamma) \triangle A$ contains an exterior angle incident to τ_i in T_γ, it contains all exterior angles incident to τ_i in T_γ.*

Proof. By Theorem 8.3, for $P \in \mathbb{P}(G_\gamma)$, if $P_-(T_\gamma) \triangle P$ contains a boundary sides of the square with diagonal τ_i in $\overline{G}_\gamma$, it contains all boundary sides of the square with diagonal τ_i in $\overline{G}_\gamma$. Since φ maps the four angles incident to τ_i in T_γ to sides of the square with diagonal τ_i in $\overline{G}_\gamma$, the assertion follows from Lemma 9.2. □

9.2. **The case of 1-notched arcs.** In this subsection, we show the following theorem.

Theorem 9.5. *There is a bijection $\varphi_p : \mathbb{A}(T_{\gamma^{(p)}}) \to \mathbb{P}(G_{\gamma^{(p)}})$ satisfying $x(A) = \overline{x}(\varphi_p(A))$ and $y(A) = \overline{y}(\varphi_p(A))$ for $A \in \mathbb{A}(T_{\gamma^{(p)}})$.*

Theorem 9.5 clearly gives the bijection between (1) and (2) in Theorem 7.4 for 1-notched arcs. To prove Theorem 9.5, we prepare the following notations as in Figure 9. By construction of the triangulated polygon T_{ℓ_p}, it contains two disjoint subgraphs $T^1_{\ell_p}$ and $T^2_{\ell_p}$ with same form as T_γ, where $T^1_{\ell_p}$ has the boundary segment ζ_m of T_{ℓ_p}. The subgraph $U^i_{\ell_p}$ of $T^i_{\ell_p}$ is obtained by removing the vertex p and the two sides ζ_1, ζ_m. For $i \in \{1, 2\}$, let v_i (resp., v'_i) be the common endpoint of τ_n and ζ_m (resp., ζ_1) in $T^i_{\ell_p}$. Let a_i (resp., a'_i) be the angle at v_i (resp., v'_i) that comes first in the counterclockwise (resp., clockwise) order around v_i (resp., v'_i). We denote by a°_i an angle between τ_n and the boundary segment of the triangle with sides τ_{n-1} and τ_n of $T^i_{\ell_p}$. If $n > 1$, it is uniquely determined, that is $a^\circ_i = a_i$ or $a^\circ_i = a'_i$.

By Theorem 9.1 and Proposition 9.3, there exists a bijection $\varphi^p : A(T_{\ell_p}) \to (G_{\ell_p})_1$ which induces a bijection $\varphi^p : \mathbb{A}(T_{\ell_p}) \to \mathbb{P}(G_{\ell_p})$ satisfying $x(A) = x(\varphi^p(A))$ and $y(A) = y(\varphi^p(A))$ for $A \in \mathbb{A}(T_{\ell_p})$.

Lemma 9.6. *The restrictions of φ^p induce bijections*

$$\varphi^p|_{A(U^i_{\ell_p}) \sqcup \{a^\circ_i\}} : A(U^i_{\ell_p}) \sqcup \{a^\circ_i\} \to (H^i_{\ell_p})_1, \quad \varphi^p|_{A(T^i_{\ell_p})} : A(T^i_{\ell_p}) \to (G^i_{\ell_p})_1$$

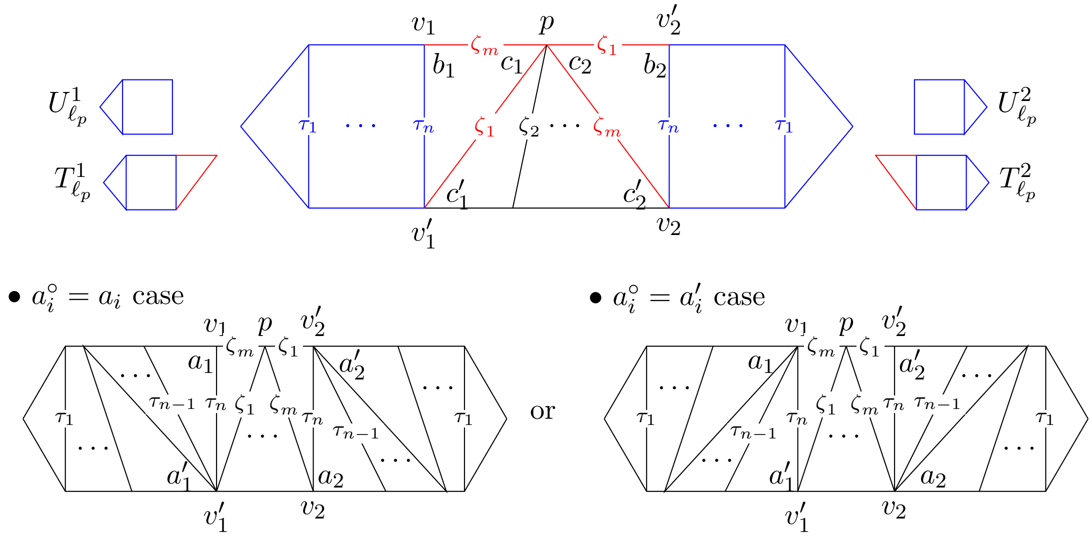


FIGURE 9. T_{ℓ_p} and subgraphs $T^i_{\ell_p}$ and $U^i_{\ell_p}$ of T_{ℓ_p}

for $i \in \{1,2\}$. Moreover, the map $\varphi^p|_{A(T^i_{\ell_p})}$ induces a bijection between $\mathbb{A}(T^i_{\ell_p})$ and $\mathbb{P}(G^i_{\ell_p})$.

Proof. The first assertion follows immediately from the unfolding process. The second assertion follows from $T^i_{\ell_p} \simeq T_\gamma$, $G^i_{\ell_p} \simeq G_\gamma$, and Theorem 9.1. □

Definition 9.7. We say that $A \in \mathbb{A}(T_{\ell_p})$ is γ-*symmetric* if the restrictions of A satisfies

$$A|_{A(U^1_{\ell_p}) \sqcup \{a_1^\circ\}} \simeq A|_{A(U^2_{\ell_p}) \sqcup \{a_2^\circ\}}.$$

We denote by $\mathbb{A}_{\mathrm{sym}}(T_{\ell_p})$ the set of γ-symmetric maximal independent sets of angles in T_{ℓ_p}.

Let $A \in \mathbb{A}(T_{\ell_p})$. It follows from Theorem 8.7 and Lemma 9.6 that $A|_{A(T^i_{\ell_p})} \in \mathbb{A}(T^i_{\ell_p})$ for some $i \in \{1,2\}$. Since it is uniquely determined up to isomorphism, we denote it by $\mathrm{res}(A)$.

Proposition 9.8. *The map φ^p induces a bijection $\varphi^p : \mathbb{A}_{\mathrm{sym}}(T_{\ell_p}) \to \mathbb{P}(G_{\gamma^{(p)}})$ satisfying $\overline{x}(A) = \overline{x}(\varphi^p(A))$ and $\overline{y}(A) = \overline{y}(\varphi^p(A))$ for $A \in \mathbb{A}_{\mathrm{sym}}(T_{\ell_p})$, where*

$$\overline{x}(A) := \frac{x(A)}{x(\mathrm{res}(A))}, \quad \overline{y}(A) := \frac{y(A)}{y(\mathrm{res}(A))}.$$

Proof. It follows from Lemma 9.6 that $A \in \mathbb{A}(T_{\ell_p})$ is γ-symmetric if and only if $\varphi^p(A) \in \mathbb{P}(G_{\gamma^{(p)}})$. Since φ^p is a bijection between $\mathbb{A}(T_{\ell_p})$ and $\mathbb{P}(G_{\ell_p})$, it induces a bijection between $\mathbb{A}_{\mathrm{sym}}(T_{\ell_p})$ and $\mathbb{P}(G_{\gamma^{(p)}})$. On the other hand, Theorem 9.1 and Proposition 9.3 imply that $x(A) = x(\varphi^p(A))$ and $y(A) = y(\varphi^p(A))$ for $A \in \mathbb{A}(T_{\ell_p})$, and also $x(\mathrm{res}(A)) = x(\varphi^p(\mathrm{res}(A)))$ and $y(\mathrm{res}(A)) = y(\varphi^p(\mathrm{res}(A)))$ for $A \in \mathbb{A}_{\mathrm{sym}}(T_{\ell_p})$ since $T^i_{\ell_p} \simeq T_\gamma$. Since φ^p is compatible with res, we have

$$\overline{x}(A) = \frac{x(\varphi^p(A))}{x(\varphi^p(\mathrm{res}(A)))} = \frac{x(\varphi^p(A))}{x(\mathrm{res}(\varphi^p(A)))} = \overline{x}(\varphi^p(A)),$$

similarly, $\overline{y}(A) = \overline{y}(\varphi^p(A))$ for $A \in \mathbb{A}_{\mathrm{sym}}(T_{\ell_p})$. □

All that is left is to give the following proposition for the proof of Theorem 9.5.

Proposition 9.9. *There is a bijection $\psi^p : \mathbb{A}_{\mathrm{sym}}(T_{\ell_p}) \to \mathbb{A}(T_{\gamma^{(p)}})$ satisfying $\overline{x}(A) = x(\psi^p(A))$ and $\overline{y}(A) = y(\psi^p(A))$ for $A \in \mathbb{A}_{\mathrm{sym}}(T_{\ell_p})$.*

To prove Proposition 9.9, we prepare some lemmas. We denote by $T_{\ell_p} \backslash T^2_{\ell_p}$ the subgraphs obtained from T_{ℓ_p} by removing $U^2_{\ell_p}$ and ζ_1 of $T^2_{\ell_p}$. Similarly, we define the notation $T_{\ell_p} \setminus T^1_{\ell_p}$. For $i \in \{1, 2\}$, let c_i and c'_i be the angles as in Figure 9.

Lemma 9.10. *For $A \in \mathbb{A}_{\mathrm{sym}}(T_{\ell_p})$ and $i \in \{1, 2\}$, $c_i \in A$ if and only if $c'_i \in A$.*

Proof. Suppose that $c_i \in A$. Since $T^i_{\ell_p}$ has $n+1$ triangles, it follows from $c_i \in A$ that $\#A|_{A(T^i_{\ell_p})} = n$. Thus $c'_i \in A$ since $T^i_{\ell_p}$ has $n + 1$ vertices incident to at least one diagonal in $T^i_{\ell_p}$. The proof of the converse assertion is similar. □

For $A \in \mathbb{A}_{\mathrm{sym}}(T_{\ell_p})$, the γ-symmetry implies that $a^\circ_1 \in A$ if and only if $a^\circ_2 \in A$. It is consistent to use the notations $a^\circ_i \in A$ and $a^\circ_i \notin A$. Let b_1 (resp., b_2) be the angles as in Figure 9.

Lemma 9.11. *For $A \in \mathbb{A}_{\mathrm{sym}}(T_{\ell_p})$,*
(1) if $a^\circ_i = a_i \in A$ or $a^\circ_i = a'_i \notin A$, then $c_2, c'_2 \notin A$,
(2) if $a^\circ_i = a_i \notin A$ or $a^\circ_i = a'_i \in A$, then $c_1, c'_1 \notin A$.
Moreover, $A = A|_{A(T_{\ell_p} \backslash T^j_{\ell_p})} \sqcup A|_{A(T^j_{\ell_p})}$ and $\mathrm{res}(A) = A|_{A(T^j_{\ell_p})}$ hold for $j \in \{1, 2\}$.

Proof. If $a^\circ_i = a_i \in A$, then $c'_2 \notin A$. If $a^\circ_i = a'_i \notin A$, then $b_2 \in A$, and $c_2 \notin A$. The assertion (1) follows from Lemma 9.10. Consequently, we have a decomposition $A = A|_{A(T_{\ell_p} \backslash T^2_{\ell_p})} \sqcup A|_{A(T^2_{\ell_p})}$. Since $\#A|_{A(T^2_{\ell_p})} = n+1$ and $T^2_{\ell_p}$ has $n+1$ triangles, then $A|_{A(T^2_{\ell_p})} \in \mathbb{A}(T^2_{\ell_p})$. Thus $\mathrm{res}(A) = A|_{A(T^2_{\ell_p})}$ holds. The proof of (2) is similar □

Next, we consider the triangulated polygon $T_{\gamma(p)}$ with one puncture p. We prepare the following notations as in Figure 10. Let v (resp., v') be the common endpoint of τ_n and ζ_m (resp., ζ_1) in $T_{\gamma(p)}$. Let d (resp., d') be the angle at v (resp., v') that comes first in the counterclockwise (resp., clockwise) order around v (resp., v'). We denote by d° an angle between τ_n and the boundary segment of the triangle with sides τ_{n-1} and τ_n of $T_{\gamma(p)}$. If $n > 1$, it is uniquely determined, that is $d^\circ = d$ or $d^\circ = d'$. Let e_1 (resp., e_2) be the angle between ζ_1 (resp., ζ_m) and a boundary segment of $T_{\gamma(p)}$.

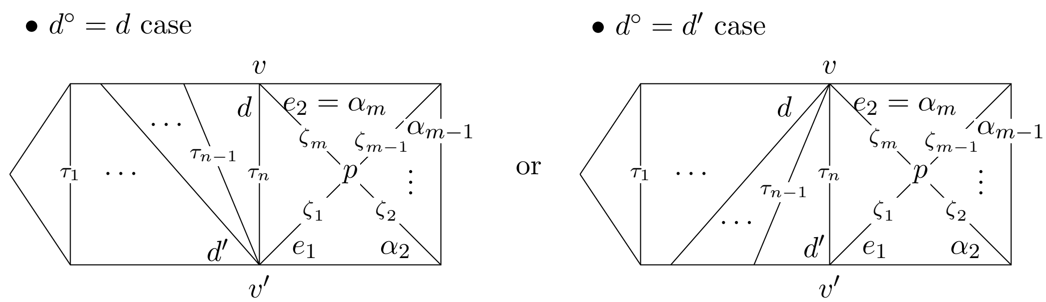


FIGURE 10. $T_{\gamma(p)}$

Lemma 9.12. *For $A \in \mathbb{A}(T_{\gamma(p)})$,*
(1) if $d^\circ = d \in A$ or $d^\circ = d' \notin A$, then $e_2 \notin A$,
(2) if $d^\circ = d \notin A$ or $d^\circ = d' \in A$, then $e_1 \notin A$.

Proof. We only prove (1) since the proof of (2) is similar. Suppose that $e_2 \in A$. For $k \in [2, m]$, we denote by α_k the angle between ζ_k and the boundary segment of the triangle with sides ζ_{k-1} and ζ_k. An easy induction shows that $\alpha_k \in A$ for all $k \in [2, m]$ since $\alpha_m = e_2 \in A$. Thus A has the angle between ζ_1 and ζ_m, and $d^\circ = d \notin A$ or $d^\circ = d' \in A$ follows easily. □

The graph $T_{\gamma^{(p)}}$ is obtained from $T_{\ell_p} \setminus T^2_{\ell_p}$ by identifying the two edges ζ_m along the direction from p to the other endpoint of ζ_m. Similarly, it is also obtained from $T_{\ell_p} \setminus T^1_{\ell_p}$ by identifying the two edges ζ_1 from p to the other endpoint of ζ_1. These constructions induce bijections

$$g_1 : A(T_{\ell_p} \setminus T^2_{\ell_p}) \to A(T_{\gamma^{(p)}}) \setminus \{e_2\} \ \text{ and } \ g_2 : A(T_{\ell_p} \setminus T^1_{\ell_p}) \to A(T_{\gamma^{(p)}}) \setminus \{e_1\}$$

such that $g_1(a_1^\circ) = d^\circ = g_2(a_2^\circ)$ holds. In particular, for $\{i,j\} = \{1,2\}$ and $A \in \mathbb{A}(T_{\ell_p} \setminus T^j_{\ell_p})$, we also have $g_i(A) \in \mathbb{A}(T_{\gamma^{(p)}})$ and $x(A) = x(g_i(A))$. Moreover, there are bijections

$$A_{\mathrm{ex}}(T_{\ell_p} \setminus T^2_{\ell_p}) \setminus \{b_1, c_1, \text{the angle between } \zeta_{m-1} \text{ and } \zeta_m\} \sqcup \{c_2'\} \xrightarrow{\sim} A_{\mathrm{ex}}(T_{\gamma^{(p)}}) \tag{9.1}$$

given by $a \mapsto g_1(a)$ if $a \neq c_2'$ and $c_2' \mapsto e_2$, and

$$A_{\mathrm{ex}}(T_{\ell_p} \setminus T^1_{\ell_p}) \setminus \{b_2, c_2, \text{the angle between } \zeta_1 \text{ and } \zeta_2\} \sqcup \{c_1'\} \xrightarrow{\sim} A_{\mathrm{ex}}(T_{\gamma^{(p)}}) \tag{9.2}$$

given by $a \mapsto g_2(a)$ if $a \neq c_1'$ and $c_1' \mapsto e_1$. Finally, we give one lemma for a general δ. For $k \in [1,n]$, let $T_\delta^{-;k}$ and $T_\delta^{+;k}$ be the two subpolygons of T_δ obtained by cutting T_δ along τ_k, where $T_\delta^{-;k}$ contains q. We denote by $A'(T_\delta^{\pm;k})$ the restriction $A(T_\delta)|_{T_\delta^{\pm;k}}$. We also define that $T_\delta^{-;n+1}$ (resp., $T_\delta^{+;0}$) is the subgraph obtained from $T_\delta^{-;n}$ (resp., $T_\delta^{+;1}$) by adding the triangle with sides τ_n, ζ_1 and ζ_m (resp., τ_1, ξ_1 and ξ_ℓ).

Lemma 9.13. *For $A \in \mathbb{A}(T_\delta)$, there is a unique completion $C_{\tau_k}(A|_{A'(T_\delta^{\pm;k})}) \in \mathbb{A}(T_\delta^{\pm;k\mp1})$ containing $A|_{A'(T_\delta^{\pm;k})}$.*

Proof. Since the equality

$$\#A|_{A'(T_\delta^{\pm;k})} = \#\{\text{triangles of } T_\delta^{\pm;k}\} = \#\{\text{vertices of } T_\delta^{\pm;k} \text{ incident to at least one diagonal}\}$$

holds, there is exactly one endpoint v of τ_k such that $A|_{A'(T_\delta^{\pm;k})}$ has no angle incident to v. Therefore, there is exactly one angle a_v of $A(T_\delta^{\pm;k\mp1}) \setminus A'(T_\delta^{\pm;k})$ incident to v, and we have a unique completion $C_{\tau_k}(A|_{A'(T_\delta^{\pm;k})}) = A|_{A'(T_\delta^{\pm;k})} \sqcup \{a_v\} \in \mathbb{A}(T_\delta^{\pm;k\mp1})$. □

For $\{i,j\} = \{1,2\}$ and $A \in \mathbb{A}(T_{\ell_p} \setminus T^i_{\ell_p})$, there exists a unique symmetric completion $\overline{A} \in \mathbb{A}_{\mathrm{sym}}(T_{\ell_p})$ of A, that is $\overline{A}|_{A(T_{\ell_p} \setminus T^i_{\ell_p}) \sqcup \{c_i'\}} = A$ and $\overline{A}|_{A(T^i_{\ell_p}) \sqcup \{c_i\}} \simeq C_{\tau_n}(A|_{A(U^j_{\ell_p}) \sqcup \{a_j^\circ\}})$. We are ready to prove Proposition 9.9.

Proof of Proposition 9.9. By Lemma 9.11, we can define the map $\psi^p : \mathbb{A}_{\mathrm{sym}}(T_{\ell_p}) \to \mathbb{A}(T_{\gamma^{(p)}})$ by

$$\mathbb{A}_{\mathrm{sym}}(T_{\ell_p}) \ni A \mapsto \begin{cases} g_1(A|_{A(T_{\ell_p} \setminus T^2_{\ell_p})}) & \text{if } a_i^\circ = a_i \in A \text{ or } a_i^\circ = a_i' \notin A, \\ g_2(A|_{A(T_{\ell_p} \setminus T^1_{\ell_p})}) & \text{if } a_i^\circ = a_i \notin A \text{ or } a_i^\circ = a_i' \in A. \end{cases} \tag{9.3}$$

We show that ψ^p is injective. Let $A, A' \in \mathbb{A}_{\mathrm{sym}}(T_{\ell_p})$ satisfying $A \neq A'$. In particular, the γ-symmetry implies that $A|_{A(T_{\ell_p} \setminus T^i_{\ell_p})} \neq A'|_{A(T_{\ell_p} \setminus T^i_{\ell_p})}$. If $a_i^\circ \in A \cap A'$ or $a_i^\circ \notin A \cup A'$, then $\psi^p(A) \neq \psi^p(A')$ follows from (9.3). Suppose that $a_i^\circ \in A$ and $a_i^\circ \notin A'$. Then $d^\circ = g_i(a_i^\circ) \in g_i(A) = \psi^p(A)$ and $d^\circ = g_j(a_j^\circ) \notin g_j(A') = \psi^p(A')$ for $j \in \{1,2\} \setminus \{i\}$. Thus $\psi^p(A) \neq \psi^p(A')$ holds, that is ψ^p is injective.

We show that ψ^p is surjective. Let $B \in \mathbb{A}(T_{\gamma^{(p)}})$. If $d^\circ = d \in A$ or $d^\circ = d' \notin A$, then $B \subseteq A(T_{\gamma^{(p)}}) \setminus \{e_2\}$ by Lemma 9.12(1). Thus $g_1^{-1}(B) \subseteq \mathbb{A}(T_{\ell_p} \setminus T^2_{\ell_p})$. There is the symmetric completion $\overline{g_1^{-1}(B)} \in \mathbb{A}_{\mathrm{sym}}(T_{\ell_p})$ such that $\psi^p\left(\overline{g_1^{-1}(B)}\right) = B$. If $d^\circ = d \notin A$ or $d^\circ = d' \in A$, then $e_1 \notin B$ by Lemma 9.12(2). In the same way as above, there is $\overline{g_2^{-1}(B)} \in \mathbb{A}_{\mathrm{sym}}(T_{\ell_p})$ such that $\psi^p\left(\overline{g_2^{-1}(B)}\right) = B$. Therefore, ψ^p is surjective.

Let $A \in \mathbb{A}_{\mathrm{sym}}(T_{\ell_p})$. Since there is at least one $i \in \{1,2\}$ such that $c_i, c_i' \notin A$ by Lemma 9.11, we have

$$\overline{x}(A) = \frac{x(A)}{x(\mathrm{res}(A))} = \frac{x(A)}{x(A|_{A(T^i_{\ell_p})})} = x(A|_{A(T_{\ell_p}\setminus T^i_{\ell_p})}) = x(\psi^p(A)).$$

We only need to prove $Y(A) \setminus Y(\mathrm{res}(A)) = Y(\psi^p(A))$ to give $\overline{y}(A) = y(\psi^p(A))$. Suppose that $a_i^\circ = a_i \in A$ or $a_i^\circ = a_i' \notin A$. From $A_-(T_{\ell_p}) = A_-(T_{\ell_p} \setminus T^2_{\ell_p}) \sqcup A_-(T^2_{\ell_p})$ and $c_2, c_2' \notin A$, we get a decomposition

$$\begin{aligned} A_-(T_{\ell_p}) \triangle A &= (A_-(T_{\ell_p})\triangle A)|_{A(T_{\ell_p}\setminus T^2_{\ell_p})} \sqcup (A_-(T_{\ell_p})\triangle A)|_{A(T^2_{\ell_p})} \\ &= (A_-(T_{\ell_p} \setminus T^2_{\ell_p})\triangle A|_{A(T_{\ell_p}\setminus T^2_{\ell_p})}) \sqcup (A_-(T^2_{\ell_p})\triangle A|_{A(T^2_{\ell_p})}). \end{aligned}$$

Thus we have

$$Y(A) = Y(A|_{A(T_{\ell_p}\setminus T^2_{\ell_p})}) \sqcup Y(A|_{A(T^2_{\ell_p})}) = Y(A|_{A(T_{\ell_p}\setminus T^2_{\ell_p})}) \sqcup Y(\mathrm{res}(A)), \tag{9.4}$$

where the second equality holds by Lemma 9.11(1). On the other hand, the equalities

$$\begin{aligned} g_1(A_-(T_{\ell_p} \setminus T^2_{\ell_p})\triangle A|_{A(T_{\ell_p}\setminus T^2_{\ell_p})}) &= g_1((A_-(T_{\ell_p})\triangle A)|_{A(T_{\ell_p}\setminus T^2_{\ell_p})}) \\ &= \psi^p(A_-(T_{\ell_p})\triangle A) = A_-(T_{\gamma(p)})\triangle \psi^p(A) \end{aligned} \tag{9.5}$$

hold by (9.3) and $\psi^p(A_-(T_{\ell_p})) = g_1(A_-(T_{\ell_p} \setminus T^2_{\ell_p})) = A_-(T_{\gamma(p)})$. Therefore, it follows from Lemma 9.4 that $A_-(T_{\ell_p} \setminus T^2_{\ell_p})\triangle A|_{A(T_{\ell_p}\setminus T^2_{\ell_p})}$ contains b_1 (resp., c_1, the angle between ζ_{m-1} and ζ_m) if and only if it contains a_1 (resp., c_1', the angles between ζ_{m-1} and boundary segments). Thus we have $Y(A|_{A(T_{\ell_p}\setminus T^2_{\ell_p})}) = Y(\psi^p(A))$ by (9.1) and (9.5). Consequently, we have

$$Y(A) \setminus Y(\mathrm{res}(A)) = Y(A|_{A(T_{\ell_p}\setminus T^2_{\ell_p})}) = Y(\psi^p(A))$$

by (9.4).

Suppose that $a_i^\circ = a_i \notin A$ or $a_i^\circ = a_i' \in A$. Since $c_1, c_1' \in A_-(T_{\ell_p})\triangle A$ by Lemma 9.11(2), then $\zeta_1 \in Y(A)$. We also have $\zeta_1 \in Y(\psi^p(A))$ since $e_1 \in A_-(T_{\gamma(p)})$ and $e_1 \notin \psi^p(A)$ by Lemma 9.12(2). Since we have the equalities

$$A_-(T^1_{\ell_p}) = A_-(T_{\ell_p})|_{A(T^1_{\ell_p})} \sqcup \{\text{the angle between } \tau_n \text{ and } \zeta_1\},$$

$$A_-(T_{\ell_p} \setminus T^1_{\ell_p}) = A_-(T_{\ell_p})|_{A(T_{\ell_p}\setminus T^1_{\ell_p})} \sqcup \{\text{the angle between } \zeta_1 \text{ and } \zeta_2\},$$

then the equalities

$$\begin{aligned} Y(A) &= Y(A)|_{T^1_{\ell_p}} \sqcup \{\zeta_1\} \sqcup Y(A)|_{T_{\ell_p}\setminus T^1_{\ell_p}} \\ &= Y(A|_{A(T^1_{\ell_p})}) \sqcup \{\zeta_1\} \sqcup Y(A|_{A(T_{\ell_p}\setminus T^1_{\ell_p})}) \\ &= Y(\mathrm{res}(A)) \sqcup \{\zeta_1\} \sqcup Y(A|_{A(T_{\ell_p}\setminus T^1_{\ell_p})}). \end{aligned}$$

hold by Lemma 9.4 and Lemma 9.11. In the same way as above proof, we have $Y(A|_{A(T_{\ell_p}\setminus T^1_{\ell_p})}) = Y(\psi^p(A)) \setminus \{\zeta_1\}$ by (9.2). Consequently, we have

$$Y(A) \setminus Y(\mathrm{res}(A)) = Y(A|_{A(T_{\ell_p}\setminus T^1_{\ell_p})}) \sqcup \{\zeta_1\} = Y(\psi^p(A)).$$

This finishes the proof. □

Proof of Theorem 9.5. By Propositions 9.8 and 9.9, there is a bijection $\varphi_p = \varphi^p(\psi^p)^{-1} : \mathbb{A}(T_{\gamma(p)}) \to \mathbb{P}(G_{\gamma(p)})$ satisfying

$$x(A) = \overline{x}((\psi^p)^{-1}(A)) = \overline{x}(\varphi^p(\psi^p)^{-1}(A)) \quad \text{and} \quad y(A) = \overline{y}((\psi^p)^{-1}(A)) = \overline{y}(\varphi^p(\psi^p)^{-1}(A))$$

for $A \in \mathbb{A}(T_{\gamma(p)})$. □

Proof of Theorem 7.5 for 1-notched arcs. The assertion follows immediately from Theorems 8.7 and 9.5. □

9.3. **The case of 2-notched arcs.** In this subsection, we show the following theorem.

Theorem 9.14. *There is a bijection $\varphi_{pq} : \mathbb{A}(T_{\gamma^{(pq)}}) \to \mathbb{P}(G_{\gamma^{(pq)}})$ satisfying $x(A) = \overline{\overline{x}}(\varphi_{pq}(A))$ and $y(A) = \overline{\overline{y}}(\varphi_{pq}(A))$ for $A \in \mathbb{A}(T_{\gamma^{(pq)}})$.*

Theorem 9.14 clearly gives the bijection between (1) and (2) in Theorem 7.4 for 2-notched arcs. To prove Theorem 9.14, we prepare the following notations as in Figure 11. For $\delta = \gamma$, $\gamma^{(p)}$, $\gamma^{(q)}$, or $\gamma^{(pq)}$, there are three subpolygons $T_\delta^q := T_\delta^{-;1}$, $T_\delta^c := T_\delta^{+;1} \cap T_\delta^{-;n}$ and $T_\delta^p := T_\delta^{+;n}$ of T_δ. We denote by $T_\delta^{*\star}$ the subpolygon $T_\delta^* \cup T_\delta^\star$ of T_δ for $*, \star \in \{q, c, p\}$. We have a decomposition $A(T_\delta) = A(T_\delta)^q \sqcup A(T_\delta)^c \sqcup A(T_\delta)^p$, where $A(T_\delta)^*$ consists of angles contained in T_δ^* for $* \in \{q, c, p\}$. For $A \in \mathbb{A}(T_\delta)$, we define a decomposition $A = A^q \sqcup A^c \sqcup A^p$, where $A^* \in A(T_\delta)^*$ for $* \in \{q, c, p\}$. For an arbitrary decomposition $S = S^q \sqcup S^c \sqcup S^p$ as above, we use the notations $S^{*\star} := S^* \sqcup S^\star$ for $*, \star \in \{q, c, p\}$.

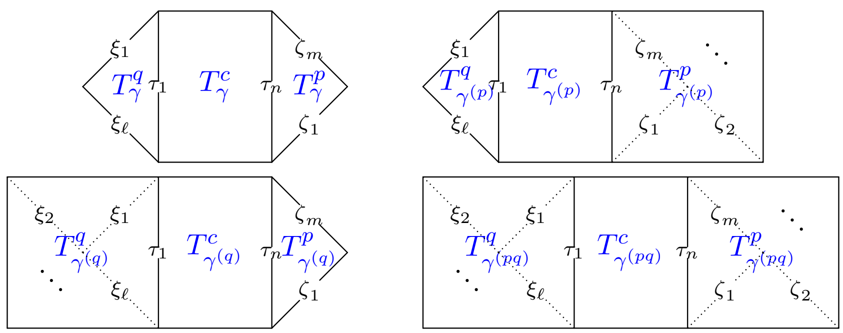


FIGURE 11. The decompositions of T_γ, $T_{\gamma^{(p)}}$, $T_{\gamma^{(q)}}$, and $T_{\gamma^{(pq)}}$

Since there is the natural inclusion from T_γ (resp., $T_{\gamma^{(p)}}$, $T_{\gamma^{(q)}}$) to $T_{\gamma^{(pq)}}$, we can view T_γ (resp., $T_{\gamma^{(p)}}$, $T_{\gamma^{(q)}}$) as a subpolygon of $T_{\gamma^{(pq)}}$, and $A(T_\gamma)$ (resp., $A(T_{\gamma^{(p)}})$, $A(T_{\gamma^{(q)}})$) as a subset of $A(T_{\gamma^{(pq)}})$.

Definition 9.15. The pair $(A_p, A_q) \in \mathbb{A}(T_{\gamma^{(p)}}) \times \mathbb{A}(T_{\gamma^{(q)}})$ is called *γ-compatible* if $A_p^c = A_q^c$ and $A_p^q \sqcup A_q^{cp} \in \mathbb{A}(T_\gamma)$, where we view $A_p^q \sqcup A_q^{cp}$ as a subset of $A(T_\gamma)$. We denote by $\mathbb{A}_{\mathrm{com}}(T_{\gamma^{(p)}}, T_{\gamma^{(q)}})$ the set of γ-compatible pairs of $\mathbb{A}(T_{\gamma^{(p)}}) \times \mathbb{A}(T_{\gamma^{(q)}})$.

Lemma 9.16. *If $n = 1$, $(A_p, A_q) \in \mathbb{A}_{\mathrm{com}}(T_{\gamma^{(p)}}, T_{\gamma^{(q)}})$ if and only if $A_p^q \sqcup A_q^p \in \mathbb{A}(T_\gamma)$. If $n > 1$, $(A_p, A_q) \in \mathbb{A}_{\mathrm{com}}(T_{\gamma^{(p)}}, T_{\gamma^{(q)}})$ if and only if $A_p^c = A_q^c$.*

Proof. If $n = 1$, the assertion follows from $A_p^c = \emptyset = A_q^c$. Suppose $n > 1$ and $A_p^c = A_q^c$. Since A_p and A_q have exactly one angle in each triangle, so does $A_p^q \sqcup A_q^{cp}$. Therefore, we only show that $A_p^q \sqcup A_q^{cp}$ has exactly one angle incident to each vertex which is incident to at least one diagonal of T_γ, which is equivalent that any two distinct angles a and b in $A_p^q \sqcup A_q^{cp}$ are not incident to a common vertex. If $a, b \in A_p^{qc}$ or $a, b \in A_q^{cp}$, the assertion holds since $A_p^{qc} \subset A_p$, $A_q^{cp} \subset A_q$, and A_p and A_q are so. Suppose that $a \in A_p^q$ and $b \in A_q^p$ are incident to a common vertex. Then $\tau_1, \dots, \tau_n$ must be incident to the vertex. Since $A_p \in \mathbb{A}(T_{\gamma^{(p)}})$, A_p^c contains the angle between τ_i and a boundary segment of the triangle with sides τ_i and τ_{i+1} for $i \in [1, n-1]$. Similarly, since $A_q \in \mathbb{A}(T_{\gamma^{(q)}})$, A_q^c contains the angle between τ_i and a boundary segment of the triangle with sides τ_{i-1} and τ_i for $i \in [2, n]$. It contradicts $A_p^c = A_q^c$. Thus the assertion holds. □

We define the maps $\mathsf{r} : \mathbb{A}_{\mathrm{com}}(T_{\gamma^{(p)}}, T_{\gamma^{(q)}}) \to \{\text{subsets of } A(T_\gamma)\}$ and $\mathsf{i} : \mathbb{A}_{\mathrm{com}}(T_{\gamma^{(p)}}, T_{\gamma^{(q)}}) \to \{\text{subsets of } A(T_{\gamma^{(pq)}})\}$ by

$$\mathsf{r}(A_p, A_q) = A_p^q \sqcup A_q^{cp}, \quad \mathsf{i}(A_p, A_q) = A_q^q \sqcup A_p^{cp}$$

for $(A_p, A_q) \in \mathbb{A}_{\mathrm{com}}(T_{\gamma^{(p)}}, T_{\gamma^{(q)}})$.

Lemma 9.17. *For* $(A_p, A_q) \in \mathbb{A}_{\mathrm{com}}(T_{\gamma^{(p)}}, T_{\gamma^{(q)}})$, $\mathsf{r}(A_p, A_q) \in \mathbb{A}(T_\gamma)$ *and* $\mathsf{i}(A_p, A_q) \in \mathbb{A}(T_{\gamma^{(pq)}})$ *hold.*

Proof. By the γ-compatibility, $\mathsf{r} = A_p^q \sqcup A_q^{cp} \in \mathbb{A}(T_\gamma)$. If $n > 1$, in the same as the proof of Lemma 9.16, $\mathsf{i}(A_p, A_q) \in \mathbb{A}(T_{\gamma^{(pq)}})$ holds. Suppose that $n = 1$. If $\mathsf{i}(A_p, A_q) \notin \mathbb{A}(T_{\gamma^{(pq)}})$, each of A_q^q and A_p^p has an angle incident to one endpoint of τ_1. Thus each of A_p^q and A_q^p must have an angle incident to the other endpoint of τ_1, so it contradicts $A_p^q \sqcup A_q^p \in \mathbb{A}(T_\delta)$. □

Lemma 9.18. *Let* $n = 1$ *and* $A = (A_p, A_q) \in \mathbb{A}_{\mathrm{com}}(T_{\gamma^{(p)}}, T_{\gamma^{(q)}})$. *Then the following conditions are equivalent:*

(1) $\tau_1 \in Y(A_p)$, *(2)* $\tau_1 \in Y(A_q)$, *(3)* $\tau_1 \in Y(\mathsf{r}(A))$, *(4)* $\tau_1 \in Y(\mathsf{i}(A))$.

Proof. In this case, $\mathsf{r}(A) = A_p^q \sqcup A_q^p$ has exactly two angles. Each of the conditions (1)-(3) is equivalent that the angle between τ_1 and ξ_ℓ is contained in A_p. Moreover, it is equivalent that A_p contains either the angle between τ_1 and ζ_m or the angle between ζ_m and a boundary segment of $T_{\gamma^{(pq)}}$, that is, the condition (4) holds. Therefore, the conditions (1)-(4) are equivalent. □

Proposition 9.19. *The map* i *is a bijection between* $\mathbb{A}_{\mathrm{com}}(T_{\gamma^{(p)}}, T_{\gamma^{(q)}})$ *and* $\mathbb{A}(T_{\gamma^{(pq)}})$ *satisfying* $\overline{\overline{x}}(A) = x(\mathsf{i}(A))$ *and* $\overline{\overline{y}}(A) = y(\mathsf{i}(A))$ *for* $A = (A_p, A_q) \in \mathbb{A}_{\mathrm{com}}(T_{\gamma^{(p)}}, T_{\gamma^{(q)}})$, *where*

$$\overline{\overline{x}}(A) := \frac{x(A_p)x(A_q)}{x(\mathsf{r}(A))}, \quad \overline{\overline{y}}(A) := \frac{y(A_p)y(A_q)}{y(\mathsf{r}(A))}.$$

Proof. First of all, we construct the inverse map of i. Let $B \in \mathbb{A}(T_{\gamma^{(pq)}})$. If $n > 1$, $C_{\tau_1}(B^{cp})^c = B^c = C_{\tau_n}(B^{qc})^c$ holds. If $n = 1$, then $C_{\tau_1}(B^p)^q \sqcup C_{\tau_n}(B^q)^p \in \mathbb{A}(T_\gamma)$ holds by the proof of Lemma 9.17. Thus $(C_{\tau_1}(B^{cp}), C_{\tau_n}(B^{qc})) \in \mathbb{A}_{\mathrm{com}}(T_{\gamma^{(p)}}, T_{\gamma^{(q)}})$ by Lemma 9.16. We define the map $\omega : \mathbb{A}(T_{\gamma^{(pq)}}) \to \mathbb{A}_{\mathrm{com}}(T_{\gamma^{(p)}}, T_{\gamma^{(q)}})$ by $\omega(B) = (C_{\tau_1}(B^{cp}), C_{\tau_n}(B^{qc}))$. Then it is easy to show that $\omega\,\mathsf{i}$ and $\mathsf{i}\,\omega$ are identities. Thus $\mathsf{i} : \mathbb{A}_{\mathrm{com}}(T_{\gamma^{(p)}}, T_{\gamma^{(q)}}) \to \mathbb{A}(T_{\gamma^{(pq)}})$ is a bijection.

We have

$$\overline{\overline{x}}(A) = \frac{x(A_p)x(A_q)}{x(A_p^q)x(A_q^{cp})} = x(A_p^{cp})x(A_q^q) = x(\mathsf{i}(A)).$$

We only need to prove $Y(A_p) \sqcup Y(A_q) = Y(\mathsf{i}(A)) \sqcup Y(\mathsf{r}(A))$, possibly with multiple elements, to give $\overline{\overline{y}}(A) = y(\mathsf{i}(A))$. Suppose that $n > 1$. By Lemma 9.4, $\tau_i \in Y(\mathsf{i}(A))$ (resp., $Y(A_p)$, $Y(A_q)$, $Y(\mathsf{r}(A))$) if and only if there is at least one exterior angle incident to τ_i in $(A_-(T_{\gamma^{(pq)}}) \triangle \mathsf{i}(A))^c$ (resp., $(A_-(T_{\gamma^{(p)}}) \triangle A_p)^c$, $(A_-(T_{\gamma^{(q)}}) \triangle A_q)^c$, $(A_-(T_\gamma) \triangle \mathsf{r}(A))^c$). On the other hand, we have the equalities

$$A_-(T_{\gamma^{(pq)}})^c = A_-(T_{\gamma^{(p)}})^c = A_-(T_{\gamma^{(q)}})^c = A_-(T_\gamma)^c \quad \text{and} \quad \mathsf{i}(A)^c = A_p^c = A_q^c = \mathsf{r}(A)^c.$$

Then $\tau_i \in Y(\mathsf{i}(A))$ (resp., $\tau_i \in Y(\mathsf{r}(A))$) if and only if $\tau_i \in Y(A_p)$ (resp., $\tau_i \in Y(A_q)$). Similarly, $\zeta_j \in Y(\mathsf{i}(A))$ (resp., $\xi_j \in Y(\mathsf{i}(A))$) if and only if $\zeta_j \in Y(A_p)$ (resp., $\xi_j \in Y(A_q)$). Thus we have $Y(A_p) \sqcup Y(A_q) = Y(\mathsf{i}(A)) \sqcup Y(\mathsf{r}(A))$.

Suppose that $n = 1$. As above, $\zeta_j \in Y(\mathsf{i}(A))$ (resp., $\xi_j \in Y(\mathsf{i}(A))$) if and only if $\zeta_j \in Y(A_p)$ (resp., $\xi_j \in Y(A_q)$). Therefore, Lemma 9.18 implies that $Y(A_p) \sqcup Y(A_q) = Y(\mathsf{i}(A)) \sqcup Y(\mathsf{r}(A))$. This finishes the proof. □

All that is left is to give the following proposition for the proof of Theorem 9.14.

Proposition 9.20. *There is a bijection* $\varphi^{pq} : \mathbb{A}_{\mathrm{com}}(T_{\gamma^{(p)}}, T_{\gamma^{(q)}}) \to \mathbb{P}(G_{\gamma^{(pq)}})$ *satisfying* $\overline{\overline{x}}(A) = \overline{\overline{x}}(\varphi^{pq}(A))$ *and* $\overline{\overline{y}}(A) = \overline{\overline{y}}(\varphi^{pq}(A))$ *for* $A = (A_p, A_q) \in \mathbb{A}_{\mathrm{com}}(T_{\gamma^{(p)}}, T_{\gamma^{(q)}})$.

Proof. By Propositions 9.8 and 9.9, there are bijections

$$\begin{array}{ccccc} \mathbb{A}(T_{\gamma^{(p)}}) \times \mathbb{A}(T_{\gamma^{(q)}}) & \xleftarrow[\psi^p \times \psi^q]{\sim} & \mathbb{A}_{\mathrm{sym}}(T_{\ell_p}) \times \mathbb{A}_{\mathrm{sym}}(T_{\ell_q}) & \xrightarrow[\varphi^p \times \varphi^q]{\sim} & \mathbb{P}(G_{\gamma^{(p)}}) \times \mathbb{P}(G_{\gamma^{(q)}}) \\ \cup & & \cup & & \cup \\ A = (A_p, A_q) & \longleftarrow & (S_p, S_q) & \longmapsto & (P_p, P_q) \end{array}$$

satisfying $x(A_*) = \overline{x}(P_*)$ and $y(A_*) = \overline{y}(P_*)$, where $A_* = \psi^*(S_*)$ $P_* = \varphi^*(S_*)$ for $* \in \{p,q\}$.

If $n > 1$, by construction of ψ^p and ψ^q, $A_p^c = A_q^c$ if and only if

$$C_{\tau_n}(S_p|_{A(U^1_{\ell_p})\sqcup\{a_1^\circ\}}) = C_{\tau_n}(S_p|_{A(U^2_{\ell_p})\sqcup\{a_2^\circ\}}) = C_{\tau_1}(S_q|_{A(U^1_{\ell_q})\sqcup\{a_1^\circ\}}) = C_{\tau_1}(S_q|_{A(U^2_{\ell_q})\sqcup\{a_2^\circ\}}).$$

Thus it is the same as $\mathrm{res}(S_p) = \mathrm{res}(S_q)$, that is $\mathrm{res}(P_p) = \mathrm{res}(P_q)$. By Lemma 9.16, $A \in \mathbb{A}_{\mathrm{com}}(T_{\gamma^{(p)}}, T_{\gamma^{(q)}})$ if and only if $(P_p, P_q) \in \mathbb{P}(G_{\gamma^{(pq)}})$.

If $n = 1$ and $\mathrm{res}(P_p) = \mathrm{res}(P_q)$, then $A_p^q \sqcup A_q^p$ corresponds to $\mathrm{res}(S_p) = \mathrm{res}(S_q)$. Thus $A_p^q \sqcup A_q^p \in \mathbb{A}(T_\gamma)$. Conversely, suppose that $A_p^q \sqcup A_q^p \in \mathbb{A}(T_\gamma)$. The each angle of S_p which is contained in the triangles $U^1_{\ell_p}$ and $U^2_{\ell_p}$ corresponds to the angle of A_p^q. Thus $A_p^q \sqcup A_q^p$ corresponds to $\mathrm{res}(S_p)$ since $A_p^q \sqcup A_q^p \in \mathbb{A}(T_\gamma)$. Similarly, $A_p^q \sqcup A_q^p$ corresponds to $\mathrm{res}(S_q)$. Therefore, we have $\mathrm{res}(S_p) = \mathrm{res}(S_q)$. So, by Lemma 9.16, $A \in \mathbb{A}_{\mathrm{com}}(T_{\gamma^{(p)}}, T_{\gamma^{(q)}})$ if and only if $(P_p, P_q) \in \mathbb{P}(G_{\gamma^{(pq)}})$, also in this case.

Consequently, we have a bijection

$$\varphi^{pq} := (\varphi^p \times \varphi^q)(\psi^p \times \psi^q)^{-1} : \mathbb{A}_{\mathrm{com}}(T_{\gamma^{(p)}}, T_{\gamma^{(q)}}) \to \mathbb{P}(G_{\gamma^{(pq)}}).$$

On the other hand, we have $\mathsf{r}(A) \simeq \mathrm{res}(S_p)$. As in the proof of Proposition 9.8, we also have $x(\mathrm{res}(S_p)) = x(\mathrm{res}(P_p))$ and $y(\mathrm{res}(S_p)) = y(\mathrm{res}(P_p))$. Therefore, we have

$$\overline{\overline{x}}(\varphi^{pq}(A)) = \frac{\overline{x}(P_p)\,\overline{x}(P_q)}{x(\mathrm{res}(P_p))} = \frac{x(A_p)x(A_q)}{x(\mathsf{r}(A))} = \overline{\overline{x}}(A)$$

and, similarly, $\overline{\overline{y}}(\varphi^{pq}(A)) = \overline{\overline{y}}(A)$. □

Proof of Theorem 9.14. By Propositions 9.19 and 9.20, there is a bijection

$$\varphi_{pq} = \varphi^{pq}\,\mathsf{i}^{-1} : \mathbb{A}(T_{\gamma^{(pq)}}) \to \mathbb{P}(G_{\gamma^{(pq)}})$$

satisfying

$$x(A) = \overline{\overline{x}}(\mathsf{i}^{-1}(A)) = \overline{\overline{x}}(\varphi^{pq}\,\mathsf{i}^{-1}(A)) \quad\text{and}\quad y(A) = \overline{\overline{y}}(\mathsf{i}^{-1}(A)) = \overline{\overline{y}}(\varphi^{pq}\,\mathsf{i}^{-1}(A))$$

for $A \in \mathbb{A}(T_{\gamma^{(pq)}})$. □

Proof of Theorem 7.5 for 2-notched arcs. The assertion follows immediately from Theorems 8.10 and 9.14. □

10. Proofs of our results for bipartite graphs

We refer the necessary notations in this section to the introduction. First, we prove the bijection between (1) and (3) in Theorem 7.4 and Proposition 7.6.

Proof of the bijection between (1) and (3) in Theorem 7.4. Angles incident to each vertex in $A(T_\delta)$ correspond bijectively with edges incident to the corresponding black vertex in B_δ. Angles in each triangle in $A(T_\delta)$ correspond bijectively with edges incident to the corresponding white vertex in B_δ. The assertion immediately follows from the definitions of maximal independent sets of angles and perfect matchings of graphs. □

Proof of Proposition 7.6. Let $E \in \mathbb{P}(B_\delta)$. For any vertex v of B_δ, v is incident to exactly zero or two edges in $E_-(B_\delta)\triangle E$. As a consequence, $E_-(B_\delta)\triangle E$ is a disjoint union of non-crossing cycles. Thus the assertion holds. □

Second, we have to be careful of the following special case to prove Proposition 7.7.

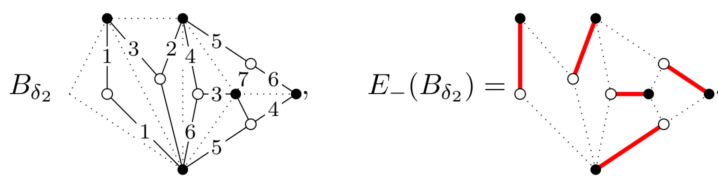
Lemma 10.1. *Suppose that $\delta = \gamma^{(pq)}$ and $n = 1$. For $A \in \mathbb{A}(T_{\gamma^{(pq)}})$, $\tau_1 \in Y(A)$ if and only if $\tau_1 \in I(\varpi(A))$.*

Proof. Since $A_-(T_{\gamma^{(pq)}})$ contains the angle between ξ_1 and a boundary segment of $T_{\gamma^{(pq)}}$, the assertion immediately follows from Proposition 7.6. □

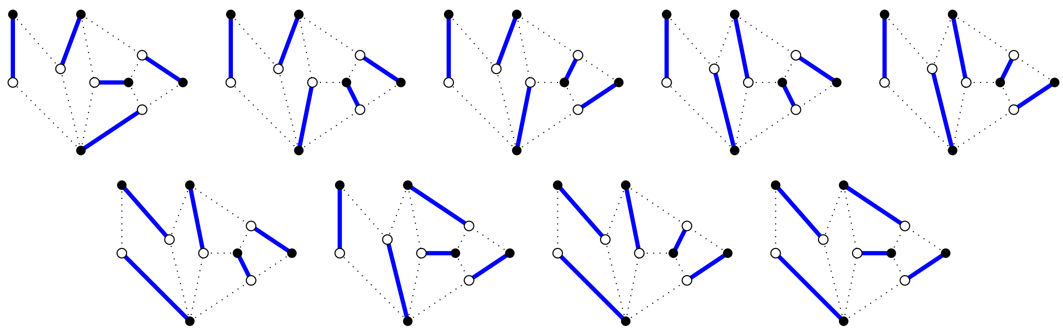
Finally, we prove Proposition 7.7 and give an example for the results of this section.

Proof of Proposition 7.7. It is trivial that ϖ induce a bijection between $A_{\rm ex}(T_\delta)$ and the set of boundary edges of B_δ. Therefore, for $A \in \mathbb{A}(T_\delta)$ and $\tau \in T_\delta$, $\tau \in Y'(A)$ if and only if $E_-(B_\delta) \triangle \varpi(A)$ contains at least one boundary edge of a square labeled by τ, thus $\tau \in I(\varpi(A))$. By Lemma 10.1, $\tau \in Y(A)$ if and only if $\tau \in I(\varpi(A))$. $\square$

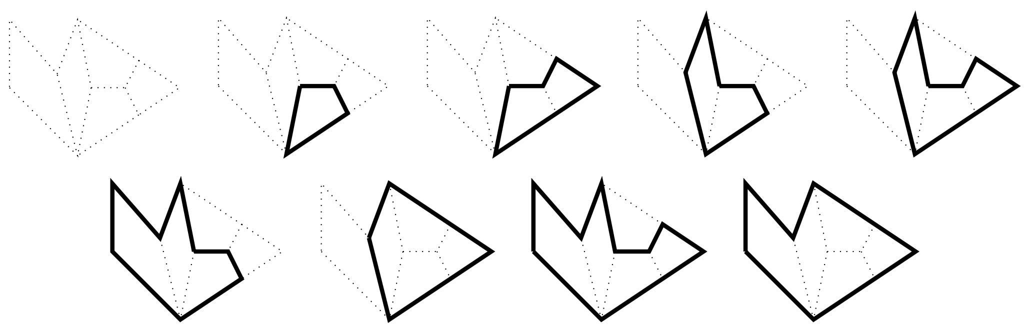
Example 10.2. For the tagged arc δ_2 given in Subsection 7.2(2), we have



Then there are nine perfect matchings of B_{δ_2} as follows:



It is easy to check that these correspond bijectively with maximal independent sets of angles in T_{δ_2} given in Subsection 7.2(2). Moreover, for each $E \in \mathbb{P}(B_{\delta_2})$, the subgraph B_E in Proposition 7.6 is given as follows:



By comparing with Subsection 7.4(2), we can check that Proposition 7.7 holds in this case.

11. Minimal cuts of quivers with potential

In this section, we show that maximal independent sets of angles in T_δ coincide with minimal cuts of quiver with potential obtained from T_δ, that is the bijection between (1) and (4) in Theorem 7.4.

11.1. **Quivers with potential and cuts.** We recall the definitions of quivers with potential [DWZ] and of their cuts [BFPPT, HI]. We denote by $\mathbb{Z}Q$ the path algebra of a quiver Q over the ring $\mathbb{Z}$ of integers.

Definition 11.1. (1) A *quiver with potential* (QP for short) is a pair (Q, W) of a quiver Q and an element $W \in \mathbb{Z}Q$ which is a linear combination of cyclic paths.

(2) A *cut* of a QP (Q, W) is a subset C of Q_1 such that any cyclic path appearing in W contains precisely one arrow in C.

We define a quiver Q_δ as follows: the set of vertices consists of diagonals and boundary segments of T_δ; the set of arrows consists of arrows from i to j, where i and j are in the common triangle of T_δ and j follows i in the counterclockwise order. We denote by $\overline{Q}_\delta$ the quiver obtained from Q_δ by adding arrows from i to j, where i and j are boundary segments which are not in the common triangle of T_δ and i is a predecessor of j with respect to clockwise order.

To define a potential $\overline{W}_\delta$ of $\overline{Q}_\delta$, we consider the following cycles of $\overline{Q}_\delta$. A *triangle cycle* is a cycle of length 3 inside a triangle of T_δ. An *exterior cycle* is a cycle winding around a vertex (possibly a puncture) of T_δ. We define

$$\overline{W}_\delta = \sum(\text{triangle cycles in } \overline{Q}_\delta) - \sum(\text{exterior cycles in } \overline{Q}_\delta).$$

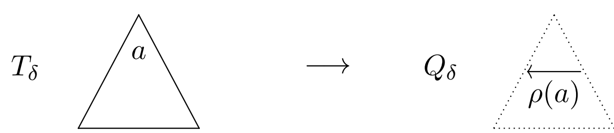
Note that this extends QPs for triangulated polygons without punctures defined in [DL] to QPs for triangulated polygons with punctures.

Lemma 11.2. *The number of triangle cycles in $\overline{Q}_\delta$ and the number of exterior cycles in $\overline{Q}_\delta$ coincide.*

Proof. By construction, the number of triangle cycles in $\overline{Q}_\delta$ and the number of triangles in T_δ coincide. Similarly, the number of exterior cycles in $\overline{Q}_\delta$ and the number of vertices incident to at least one diagonal in T_δ. So all these numbers coincide. □

We denote by $n(\delta)$ the number in *Lemma* 11.2.

11.2. **Minimal cuts of QPs and maximal independent sets of angles.** We have a natural injection $\rho : A(T_\delta) \to (Q_\delta)_1$ given by the following picture:



Cuts of $(\overline{Q}_\delta, \overline{W}_\delta)$ have the following property using the map ρ.

Lemma 11.3. *(a) Any cut C has the cardinality $|C| \geq n(\delta)$.*
(b) The equality in (a) holds if and only if C is contained in $\rho(A(T_\delta))$.

Proof. Since there are $n(\delta)$ triangle cycles (resp., $n(\delta)$ exterior cycles) not sharing arrows with each other, (a) holds. There is an exterior cycle sharing arrows with each triangle cycle. Since the shared arrows are contained in $\rho(A(T_\delta))$, the sufficiency of (b) holds. Since $\rho(A(T_\delta))$ is contained in the set of arrows appearing in a triangle cycle of $\overline{Q}_\delta$, then $|C| \leq n(\delta)$ for $C \subset \rho(A(T_\delta))$. Thus the necessity of (b) holds. □

Definition 11.4. A cut C of $(\overline{Q}_\delta, \overline{W}_\delta)$ is called *minimal* if $|C| = n(\delta)$.

By Theorem 7.4, $(\overline{Q}_\delta, \overline{W}_\delta)$ always has minimal cuts.

Proof of the bijection between (1) and (4) in Theorem 7.4. Let $A \subseteq A(T_\delta)$ and $C := \rho(A) \subseteq (Q_\delta)_1$. Then there is exactly one element a of A in any triangle of T_δ (resp., incident to any vertex of T_δ) if and only if the corresponding triangle cycle (resp., exterior cycle) contains precisely one arrow $\rho(a)$ in C. Thus $A \in \mathbb{A}(T_\delta)$ if and only if C is a cut. Since minimal cuts are precisely cuts contained in $\rho(A(T_\delta))$ by Lemma 11.3(b), the assertion follows. □

Consequently, we can give another cluster expansion formula in terms of minimal cuts.

Corollary 11.5. *We have*

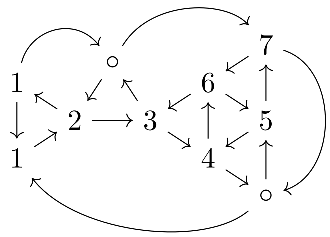
$$x_\delta = \Phi\left(\frac{1}{\operatorname{cross}(T,\delta)} \sum_C x(\rho^{-1}(C)) y(\rho^{-1}(C))\right),$$

where C runs over all minimal cuts of $(\overline{Q}_\delta, \overline{W}_\delta)$ and $\operatorname{cross}(T,\delta)$, *$x(\rho^{-1}(C))$ and $y(\rho^{-1}(C))$ are defined in Theorems 7.3 and 7.5.*

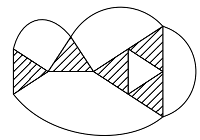
Proof. The assertion follows immediately from Theorems 7.4 and 7.5. $\square$

Example 11.6. For the tagged arc δ_2 given in Subsection 7.2(2), we have

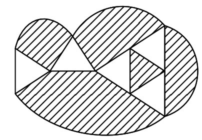
$(\overline{Q}_{\delta_2}, \overline{W}_{\delta_2}) = \Bigg($



$,\ \sum\Big($five triangle cycles

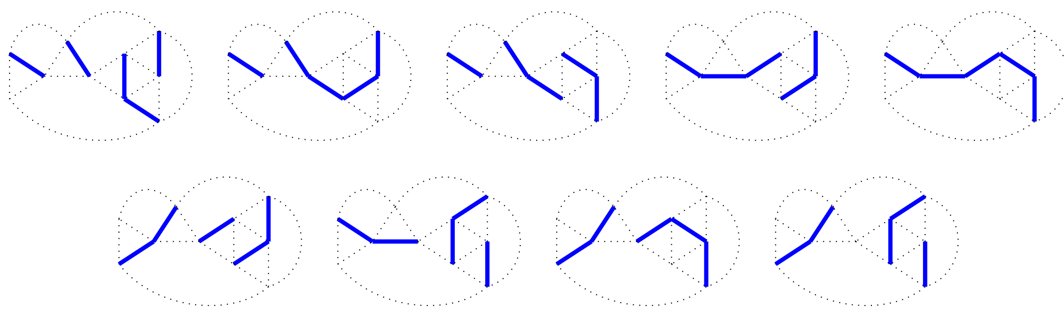


$\Big) - \sum\Big($five exterior cycles



$\Big)\Bigg).$

Then there are nine minimal cuts of $(\overline{Q}_{\delta_2}, \overline{W}_{\delta_2})$ as follows:



It is easy to check that these correspond bijectively with maximal independent sets of angles in T_{δ_2} given in Subsection 7.2(2).

12. Essential loops

Recall the definition of essential loops [MSW13]. Throughout this section, we suppose that a marked surface (S, M) has no punctures. An *essential loop* ζ in (S, M) is a closed curve in S, considered up to isotopy, such that: ζ is disjoint from M and the boundary of S; ζ does not intersect itself; ζ is not a contractible loop.

Choose a triangle $\triangle$ of T that ζ crosses. Let p be a point in the interior of $\triangle$ that lies on ζ. Let α and β be the two sides of $\triangle$ crossed ζ immediately before and following its travel through p, and let τ be the third side of $\triangle$. Let $\widetilde{\zeta}$ be the curve whose starting and ending points are p that exactly follows ζ. We can construct the triangulated polygon $T_{\widetilde{\zeta}}$ associated with $\widetilde{\zeta}$ in the same way as for plain arcs. Also, we obtain the snake graph $G_{\widetilde{\zeta}}$ from $T_{\widetilde{\zeta}}$. Let v (resp., w) be the endpoint of τ and α (resp., β) in the first triangle of $T_{\widetilde{\zeta}}$ or $G_{\widetilde{\zeta}}$, and let v' (resp., w') be the endpoint of τ and β (resp., α) in the last triangle of $T_{\widetilde{\zeta}}$ or $G_{\widetilde{\zeta}}$ (see Figure 12).

Definition 12.1. [MSW13, Definition 3.4, 3.8] The *band graph* $\widetilde{\mathrm{G}}_\zeta$ associated with the essential loop ζ is the graph obtained from $G_{\widetilde{\zeta}}$ by identifying the edges τ in the first and last squares such that v corresponds to v'. That is, the band graph lies on an annulus or a Möbius strip. A perfect matching P of $\widetilde{\mathrm{G}}_\zeta$ is called *good* either if $\tau \in P$ or if both edges incident to v and incident to w in P lie on the same square. We denote by $\mathbb{P}_g(\widetilde{\mathrm{G}}_\zeta)$ the set of good perfect matchings of $\widetilde{\mathrm{G}}_\zeta$.

Viewing $P \in \mathbb{P}_g(\widetilde{\mathrm{G}}_\zeta)$ as a subset of $(G_{\widetilde{\zeta}})_1$, we can obtain $\overline{\mathrm{P}} \in \mathbb{P}(G_{\widetilde{\zeta}})$ from P by adding either the edge τ in the first square or in the last square in $\widetilde{\mathrm{G}}_\zeta$. Then it is easy to show that there is a bijection $\mathbb{P}_g(\widetilde{\mathrm{G}}_\zeta)$ and the set

$$\mathbb{P}_g(G_{\widetilde{\zeta}}) := \{P \in \mathbb{P}(G_{\widetilde{\zeta}}) \mid P \text{ contains } \tau \text{ in the first or the last triangle of } G_{\widetilde{\zeta}}\}$$

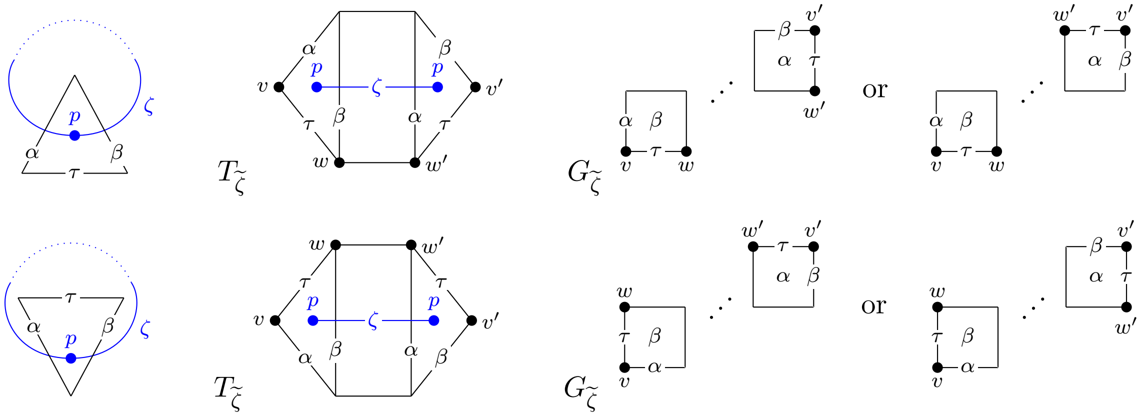


FIGURE 12. $T_{\widetilde{\zeta}}$ and $G_{\widetilde{\zeta}}$ associated with an essential loop ζ

given by sending P to $\overline{\mathrm{P}}$. In particular, there is a unique good perfect matching $P_-(\widetilde{\mathrm{G}}_\zeta)$ such that $\overline{P_-(\widetilde{\mathrm{G}}_\zeta)} = P_-(G_{\widetilde{\zeta}})$, called the *minimal matching* (see [MSW13, Remark 3.9]).

Definition 12.2. [MSW13, Definition 3.14] For an essential loop ζ in (S, M), we define a Laurent polynomial

$$x_\zeta := \frac{1}{\mathrm{cross}(T,\zeta)} \sum_{P \in \mathbb{P}_g(\widetilde{\mathrm{G}}_\zeta)} x(P)y(P).$$

One reason to consider x_ζ is that they give rise to a base for the cluster algebra with principal coefficients obtained from a triangulated surface without punctures. Let T be a triangulation of (S, M). A collection of arcs and essential loops in (S, M) is $\mathcal{C}^\circ$-*compatible* if they do not intersect each other.

Theorem 12.3. [MSW13, Theorem 1.1, 4.1] *Let (S, M) be a marked surface without punctures and T be a triangulation of (S, M). Then the set*

$$\Big\{\prod_{c \in C} x_c \mid C \textit{ is a } \mathcal{C}^\circ\textit{-compatible collection of } (S, M)\Big\}$$

is a base of $\mathcal{A}(T)$.

In this case, we study maximal independent sets of angles. For an essential loop ζ in (S, M), we can construct a triangulated polygon T_ζ in the same way as for plain arcs, that is, it is a triangulated annulus (see Figure 13). In particular, it is not twisted unlike band graphs. Since T_ζ has the same numbers of triangles and of vertices, then $\mathbb{A}(T_\zeta) \neq \emptyset$.

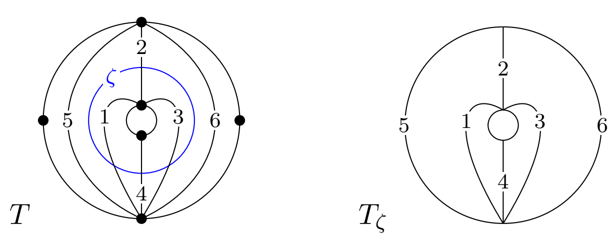


FIGURE 13. Example of T_ζ for an essential loop ζ

We define *max-condition* as the dual min-condition.

Definition 12.4. Let ζ be an essential loop in (S,M). We say that a maximal independent set of angles in T_ζ is *bad* if all angles incident to one boundary component satisfy min-condition and all angles incident to the other boundary component satisfy max-condition (see Figure 14). A non-bad maximal independent set of angles in T_ζ is called *good*. We denote by $\mathbb{A}_g(T_\zeta)$ the set of good maximal independent sets of angles in T_ζ.

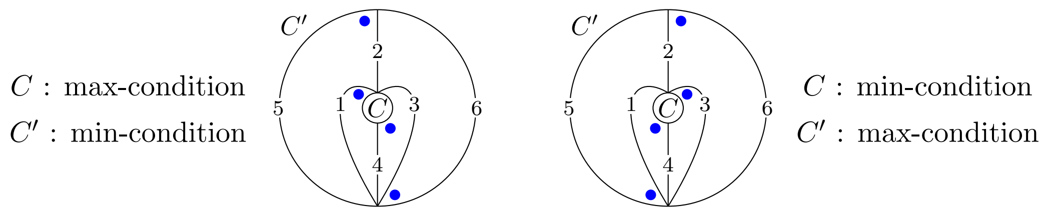


FIGURE 14. Bad maximal independent sets of angles in the above T_ζ with boundary components C and C'

Then we have the following result.

Theorem 12.5. *Let ζ be an essential loop in (S,M). There is a bijection $\psi_\zeta : \mathbb{P}_g(\widetilde{\mathrm{G}}_\zeta) \to \mathbb{A}_g(T_\zeta)$ satisfying $x(P) = x(\psi_\zeta(P))$ and $y(P) = y(\psi_\zeta(P))$ for $P \in \mathbb{P}_g(\widetilde{\mathrm{G}}_\zeta)$. In particular, we have the equation*

$$x_\zeta = \frac{1}{\mathrm{cross}(T,\zeta)} \sum_{A\in\mathbb{A}_g(T_\zeta)} x(A)y(A).$$

To prove Theorem 12.5, we need some preparations. By rotational symmetry of order two, we can assume that $T_{\widetilde{\zeta}}$ is the above case in Figure 12. Since there is a bijection between $\mathbb{P}_g(\widetilde{\mathrm{G}}_\zeta)$ and $\mathbb{P}_g(G_{\widetilde{\zeta}})$, Theorem 9.1 induces a bijection between $\mathbb{P}_g(\widetilde{\mathrm{G}}_\zeta)$ and the set

$$\mathbb{A}_g(T_{\widetilde{\zeta}}) := \{A \in \mathbb{A}(T_{\widetilde{\zeta}}) \mid A \text{ contains } c \text{ or } c'\},$$

where c (resp., c') is the angle between α and β in the first (resp., last) triangle of $T_{\widetilde{\zeta}}$ (see Figure 15). In particular, this bijection preserves the values of $x(-)$ and $y(-)$ by Theorem 9.1 and Proposition 9.3. We denote by c_A an angle c or c' contained in $A \in \mathbb{A}_g(T_{\widetilde{\zeta}})$. If both c and c' are contained in A, we define $c_A = c$. We only need to construct a bijection $\psi'_{\widetilde{\zeta}} : \mathbb{A}_g(T_{\widetilde{\zeta}}) \to \mathbb{A}_g(T_\zeta)$ satisfying $x(A \setminus \{c_A\}) = x(\psi'_{\widetilde{\zeta}}(A))$ and $y(A \setminus \{c_A\}) = y(\psi'_{\widetilde{\zeta}}(A))$ for $A \in \mathbb{A}_g(T_{\widetilde{\zeta}})$. Let a (resp., b) be the angle between α (resp., β) and τ in the first triangle of $T_{\widetilde{\zeta}}$ (see Figure 15). We denote by $\mathbb{A}_g(T_{\widetilde{\zeta}})_{\ni b}$ (resp., $\mathbb{A}_g(T_{\widetilde{\zeta}})_{\not\ni b}$) the subset of elements in $\mathbb{A}_g(T_{\widetilde{\zeta}})$ containing (resp., not containing) b, in particular, $\mathbb{A}_g(T_{\widetilde{\zeta}}) = \mathbb{A}_g(T_{\widetilde{\zeta}})_{\ni b} \sqcup \mathbb{A}_g(T_{\widetilde{\zeta}})_{\not\ni b}$.

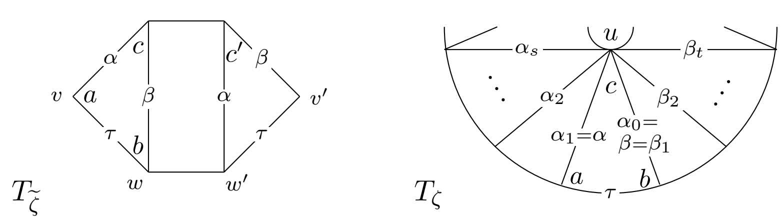


FIGURE 15. $T_{\widetilde{\zeta}}$ and T_ζ for an essential loop ζ

Let $A \in \mathbb{A}_g(T_{\widetilde{\zeta}})_{\ni b}$. Then $c' \in A$ follows from the definition of $\mathbb{A}_g(T_{\widetilde{\zeta}})$, that is $c_A = c'$. The triangulated annulus T_ζ is obtained from $T_{\widetilde{\zeta}}$ by removing the last triangle in $T_{\widetilde{\zeta}}$ and by identifying the edges α in the first triangle and in the last triangle. It is easy to show that this construction induces a natural map $\psi_{\zeta}^{\ni b} : \mathbb{A}_g(T_{\tilde{\zeta}})_{\ni b} \to \mathbb{A}(T_{\zeta})$. Abusing notation, let a (resp., b, c) be the angle between τ and α (resp., τ and β, α and β) in T_{ζ}. We denote by u the common endpoint of α and β in T_{ζ}. Let $\alpha_s, \ldots, \alpha_1 = \alpha, \alpha_0 = \beta = \beta_1, \ldots, \beta_t$ be all arcs incident to u winding counter-clockwisely around u (see Figure 15).

Lemma 12.6. *For $A \in \mathbb{A}_g(T_{\tilde{\zeta}})_{\ni b}$, then $\psi_{\zeta}^{\ni b}(A) \in \mathbb{A}_g(T_{\zeta})$. Moreover, the map $\psi_{\zeta}^{\ni b}$ induces a bijection between $\mathbb{A}_g(T_{\tilde{\zeta}})_{\ni b}$ and the set*

$$\mathbb{A}_g(T_{\zeta})_{\ni b} := \{A' \in \mathbb{A}_g(T_{\zeta}) \mid b \in A'\}.$$

Proof. Since $c' \in A$, then $\psi_{\zeta}^{\ni b}(A)$ does not contain the angle between α_s and a boundary segment incident to u. Thus the angle incident to u in $\psi_{\zeta}^{\ni b}(A)$ does not satisfy min-condition. Since $b \in \psi_{\zeta}^{\ni b}(A)$ satisfies max-condition, $\psi_{\zeta}^{\ni b}(A)$ is good.

By construction, there is a bijection between $\mathbb{A}_g(T_{\tilde{\zeta}})_{\ni b}$ and the set

$$\{A' \in \mathbb{A}_g(T_{\zeta})_{\ni b} \mid A' \text{ contains the angle between } \beta_i \text{ and } \beta_{i+1} \text{ for some } i \in [1, t-1]\}. \tag{12.1}$$

Let $A' \in \mathbb{A}_g(T_{\zeta})_{\ni b}$. If A' contains the angle between α_j and α_{j+1} for some $j \in [1, s-1]$, A' must contain the angle between α_j and the boundary segment of the triangle with sides α_j and α_{j-1}. Continuing this process, A' contains a, and it contradicts $b \in A'$. If A' contains the angle between α_s and a boundary segment incident to u, then A' must contain the angle between α_i and the boundary segment of the triangle with sides α_i and α_{i+1} for $[1, s-1]$. Then by the same argument for the other endpoint $u' \neq u$ of α_s, the angle of A' incident to u' satisfies max-condition. Continuing this process, A' consists only of exterior angles whose angles incident to the boundary with u satisfy min-condition and angles incident to the boundary with τ satisfy max-condition. It contradicts that A' is good. Therefore any $A' \in \mathbb{A}_g(T_{\zeta})_{\ni b}$ satisfies the condition of (12.1), thus the set (12.1) and $\mathbb{A}_g(T_{\zeta})_{\ni b}$ coincide. □

Let $A \in \mathbb{A}_g(T_{\tilde{\zeta}})_{\not\ni b}$. Then $c \in A$ follows from the definition of maximal independent sets of angles, that is $c_A = c$. The triangulated annulus T_{ζ} is obtained from $T_{\tilde{\zeta}}$ by removing the first triangle in $T_{\tilde{\zeta}}$ and by identifying the edges β in the first triangle and in the last triangle. In particular, c' in $T_{\tilde{\zeta}}$ corresponds to c in T_{ζ}. It is easy to show that this construction induces a natural map $\psi_{\zeta}^{\not\ni b} : \mathbb{A}_g(T_{\tilde{\zeta}})_{\not\ni b} \to \mathbb{A}(T_{\zeta})$.

Lemma 12.7. *For $A \in \mathbb{A}_g(T_{\tilde{\zeta}})_{\not\ni b}$, then $\psi_{\zeta}^{\not\ni b}(A) \in \mathbb{A}_g(T_{\zeta})$. In particular, the map $\psi_{\zeta}^{\not\ni b}$ induces a bijection between $\mathbb{A}_g(T_{\tilde{\zeta}})_{\not\ni b}$ and the set*

$$\mathbb{A}_g(T_{\zeta})_{\not\ni b} := \{A' \in \mathbb{A}_g(T_{\zeta}) \mid b \notin A'\}.$$

Proof. Since $c \in A$, then $\psi_{\zeta}^{\not\ni b}(A)$ does not contain the angle between β_t and a boundary segment incident to u. Thus $\psi_{\zeta}^{\not\ni b}(A)$ is good since $b \notin \psi_{\zeta}^{\not\ni b}(A)$.

By construction, there is a bijection between $\mathbb{A}_g(T_{\tilde{\zeta}})_{\not\ni b}$ and the set

$$\{A' \in \mathbb{A}_g(T_{\zeta})_{\not\ni b} \mid A' \text{ does not contain the angles between } \beta_i \text{ and } \beta_{i+1} \text{ for all } i \in [1, t-1]\}. \tag{12.2}$$

Let $A' \in \mathbb{A}_g(T_{\zeta})_{\not\ni b}$. If $c \in A'$, it satisfies the condition of (12.2). Suppose that $a \in A'$. Then, in the same way as the proof of Lemma 12.6, A' satisfies the condition of (12.2). Therefore, the set (12.2) and $\mathbb{A}_g(T_{\zeta})_{\not\ni b}$ coincide. Thus the assertion holds. □

Proof of Theorem 12.5. We have decompositions $\mathbb{A}_g(T_{\tilde{\zeta}}) = \mathbb{A}_g(T_{\tilde{\zeta}})_{\ni b} \sqcup \mathbb{A}_g(T_{\tilde{\zeta}})_{\not\ni b}$ and $\mathbb{A}_g(T_\zeta) = \mathbb{A}_g(T_\zeta)_{\ni b} \sqcup \mathbb{A}_g(T_\zeta)_{\not\ni b}$. We define the map $\psi'_\zeta : \mathbb{A}_g(T_{\tilde{\zeta}}) \to \mathbb{A}_g(T_\zeta)$ by

$$\psi'_\zeta(A) = \begin{cases} \psi_\zeta^{\ni b}(A) & \text{if } A \in \mathbb{A}_g(T_{\tilde{\zeta}})_{\ni b}, \\ \psi_\zeta^{\not\ni b}(A) & \text{if } A \in \mathbb{A}_g(T_{\tilde{\zeta}})_{\not\ni b}. \end{cases}$$

By Lemmas 12.1 and 12.2, ψ'_ζ is bijective. It satisfies $x(A \setminus \{c_A\}) = x(\psi'_\zeta(A))$ and $y(A \setminus \{c_A\}) = y(\psi'_\zeta(A))$ for $A \in \mathbb{A}_g(T_{\tilde{\zeta}})$ since $\psi_\zeta^{\ni b}$ and $\psi_\zeta^{\not\ni b}$ are natural maps. Therefore, we have a bijection

$$\begin{array}{cccccccc} \psi_\zeta : & \mathbb{P}_g(\widetilde{\mathrm{G}}_\zeta) & \longrightarrow & \mathbb{P}_g(G_{\tilde{\zeta}}) & \longrightarrow & \mathbb{A}_g(T_{\tilde{\zeta}}) & \longrightarrow & \mathbb{A}_g(T_\zeta) \\ & \cup\!\!\!\mid & & \cup\!\!\!\mid & & \cup\!\!\!\mid & & \cup\!\!\!\mid \\ & P & \longmapsto & \overline{\mathrm{P}} & \longmapsto & \varphi_{\tilde{\zeta}}(\overline{\mathrm{P}}) & \longmapsto & \psi'_\zeta \phi_{\tilde{\zeta}}(\overline{\mathrm{P}}), \end{array}$$

where $\varphi_{\tilde{\zeta}}$ is the bijection between $\mathbb{P}_g(G_{\tilde{\zeta}})$ and $\mathbb{A}_g(T_{\tilde{\zeta}})$ induced by Theorem 9.1, satisfying $x(P) = x(\psi_\zeta(P))$ and $y(P) = y(\psi_\zeta(P))$. $\square$

Part 4. F-matrices of cluster algebras from triangulated surfaces

13. Introduction

This part is based on the paper [GY]. The aim is to study tagged triangulations of (S, M) by using the intersection number of two tagged arcs was defined in [QZ, Definition 3.3] (see Subsection 7.5). For simplicity, in this part, we assume that if (S, M) is a closed surface with exactly one puncture, all tagged arcs are plain arcs. We denote by $\overline{\delta}$ the plain arc corresponding to a tagged arc δ of (S, M).

Fix a tagged triangulation T of (S, M) with n tagged arcs. For a tagged arc δ of (S, M), we define

$$\mathsf{Int}(T, \delta) := (\mathsf{Int}(t, \delta))_{t \in T} \in \mathbb{Z}^n_{\geq 0},$$

called an *intersection vector of* δ *with respect to* T. For a tagged triangulation $T' = \{\delta_1, \ldots, \delta_n\}$ of (S, M), we denote by $\mathsf{Int}(T, T')$ the non-negative integer matrix with columns

$$\mathsf{Int}(T, \delta_1), \ldots, \mathsf{Int}(T, \delta_n).$$

We are ready to state the main result of this part.

Theorem 13.1. *Let T be a tagged triangulation of (S, M). If tagged triangulations T' and T'' of (S, M) have $\mathsf{Int}(T, T') = \mathsf{Int}(T, T'')$ up to permutations of columns, then $T' = T''$.*

More generally, it is natural to consider whether a tagged arc δ of (S, M) is uniquely determined by $\mathsf{Int}(T, \delta)$. Clearly, if $\mathsf{Int}(T, \delta) = 0$, it is not true since $\mathsf{Int}(T, t) = 0$ for each $t \in T$. Thus we study the following property.

Definition 13.2. For a tagged triangulation T of (S, M), we say that T *detects tagged arcs* if it satisfies the following condition:

- If tagged arcs δ and ϵ of (S, M) have a common non-zero intersection vector $\mathsf{Int}(T, \delta) = \mathsf{Int}(T, \epsilon)$, then $\delta = \epsilon$.

We give a characterization of this property. In particular, a tagged triangulation does not detect tagged arcs generally.

Theorem 13.3. *Let T be a tagged triangulation of (S, M). Then T detects tagged arcs if and only if there are no tagged arcs δ and ϵ of T connecting two (possibly same) common punctures such that $\overline{\delta} \neq \overline{\epsilon}$.*

Next, we give a complete list of marked surfaces which have tagged triangulations detecting tagged arcs.

Theorem 13.4. (1) *If S is not closed, then there is at least one tagged triangulation of (S, M) detecting tagged arcs.*

(2) *If S is closed, then there is at least one tagged triangulation of (S, M) detecting tagged arcs if and only if the inequality*

$$p \geq \begin{cases} 10 & \text{if } g = 2, \\ \dfrac{7 + \sqrt{1 + 48g}}{2} & \text{if } g \neq 2, \end{cases} \tag{13.1}$$

holds, where p is the number of punctures of (S, M) and g is the genus of S.

(3) *All tagged triangulation of (S, M) detect tagged arcs if and only if (S, M) is one of the followings:*

- *a marked surface with no punctures;*

- *a marked surface of genus* 0 *with exactly* 1 *boundary component and at most* 2 *punctures;*
- *a marked surface of genus* 0 *with exactly* 2 *boundary components and a* 1 *puncture.*

Finally we apply our results to the cluster algebra $\mathcal{A}(T)$ with principal coefficients associated with a tagged triangulation T (see Section 3). Then each tagged arc δ of (S, M) gives rise to the cluster variables z_δ in $\mathcal{A}(T)$. It was shown in [Y19c] that the intersection vector $\mathsf{Int}(T, \delta)$ is equal to the f-vector of z_δ, that is, the maximal degree of F-polynomial of z_δ. As an application, we have the following result.

Corollary 13.5 (Corollary 16.3)**.** *Let T be a tagged triangulation of (S, M). For clusters $\mathbf{z}$ and $\mathbf{z}'$ of $\mathcal{A}(T)$, if the f-vectors of cluster variables in $\mathbf{z}$ coincide with ones in $\mathbf{z}'$, then $\mathbf{z} = \mathbf{z}'$.*

Remark 13.6. In cluster algebras, there are four families of integer vectors which are f-vectors, d-vectors, g-vectors and c-vectors (see e.g. [FZ02, FZ04, FZ07]). In cluster algebras defined from marked surfaces, they are given by $\mathsf{Int}(\cdot, \cdot)$, $(\cdot|\cdot)$ and shear coordinates [FoT, FeT].

Cluster algebras	f-vectors	d-vectors	g-vectors, c-vectors
Marked surfaces	$\mathsf{Int}(\cdot, \cdot)$	$(\cdot\|\cdot)$	shear coordinates

This part is organized as follows. In Section 14, we prove our main results Theorems 13.1 and 13.3 below. For that reason, we introduce modifications of tagged arcs. It plays a key role in our proofs that they are uniquely determined by their intersection vectors (Theorem 14.4). In Section 15, we study a more detailed result of Theorem 13.1. In Section 16, we recall the notions of f-vectors and F-matrices. Using the correspondence between f-vectors and intersection vectors in Theorem 7.8, we apply the results in the previous sections to study properties of f-vectors and F-matrices including Corollary 16.3. In Sections 17 and 18, we are devoted to prove Theorem 14.4. In Section 19, we give an example of our results.

14. Modifications of tagged arcs

For a tagged arc δ and a puncture p of (S, M), we define that $\delta^{(p)}$ is the tagged arc obtained from δ by changing its tags at p. If δ is not incident to p, then $\delta^{(p)} = \delta$. By definition, we have $\mathsf{Int}(\delta^{(p)}, \epsilon^{(p)}) = \mathsf{Int}(\delta, \epsilon)$ for any tagged arcs δ, ϵ and puncture p of (S, M). Therefore, to consider intersection vectors with respect to a tagged triangulation T of (S, M), we can assume that T satisfies the following condition as well as the previous part:

($\diamond$) The tagged triangulation T consists of plain arcs and 1-notched arcs, with at most one 1-notched arc incident to each puncture.

In this section, unless otherwise noted, let T be a tagged triangulation of (S, M) satisfying ($\diamond$). To prove Theorems 13.1 and 13.3, we first define modifications of tagged arcs with respect to T.

Let $\delta \notin T$ be a tagged arc of (S, M). We define the tagged arc $\hat{\delta}$ as follows: $\hat{\delta}$ is obtained from δ by changing tags at a puncture p if δ and a tagged arc of T are tagged notched at p; otherwise, $\hat{\delta} = \delta$ (see Figure 16). Note that a notched arc of T is a 1-notched arc inside a pair of conjugate arcs of T by ($\diamond$). We modify δ, denoted by $\mathsf{M}_T(\delta)$, without changing its intersection vector with respect to T as follows: We construct a deformed curve $\mathsf{M}'_T(\hat{\delta})$. First, we assume that $\overline{\delta} \notin T$.

- If $\hat{\delta}$ is a plain arc, $\mathsf{M}'_T(\hat{\delta}) = \hat{\delta}$.
- If $\hat{\delta}$ is a notched arc and is not a loop, $\mathsf{M}'_T(\hat{\delta})$ is obtained from $\hat{\delta}$ by replacing its ends tagged notched as in the left diagram of Figure 17.
- If $\hat{\delta}$ is a 2-notched loop and there are both sides of $\hat{\delta}$ in the same puzzle piece divided by T, $\mathsf{M}'_T(\hat{\delta})$ is obtained from $\hat{\delta}$ by replacing its ends as in the middle diagram of Figure 17.
- Otherwise, $\mathsf{M}'_T(\hat{\delta})$ is obtained from $\hat{\delta}$ by replacing its ends as in the right diagram of Figure 17.

Second, we assume that $\overline{\delta} \in T$. Since $\delta \notin T$, δ is a notched arc.

- If $\hat{\delta}$ is a 1-notched arc, $\mathsf{M}'_T(\hat{\delta})$ is a 1-punctured loop corresponding to $\hat{\delta}$.
- If $\hat{\delta}$ is a 2-notched arc, $\mathsf{M}'_T(\hat{\delta})$ is a pair of cycles which surround each endpoint of $\hat{\delta}$ and do not include any punctures in their curves (we call this circle a 1-*punctured cycle)*.

Finally, $\mathsf{M}_T(\delta)$ is obtained from $\mathsf{M}'_T(\hat{\delta})$ by changing tags at a puncture p if δ and a tagged arc of T are tagged notched at p; otherwise, $\mathsf{M}_T(\delta) = \mathsf{M}'_T(\hat{\delta})$ (see Figure 16). We call $\mathsf{M}_T(\delta)$ a *modified tagged arc of δ with respect to T*.

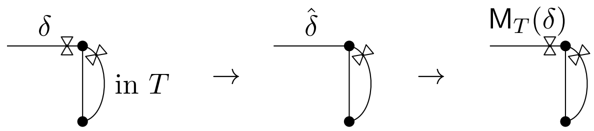


FIGURE 16. From δ to $\hat{\delta}$ and $\mathsf{M}_T(\delta)$

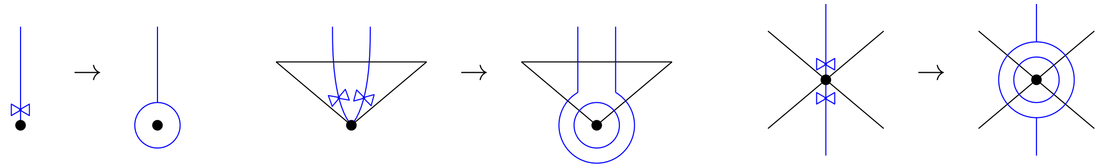
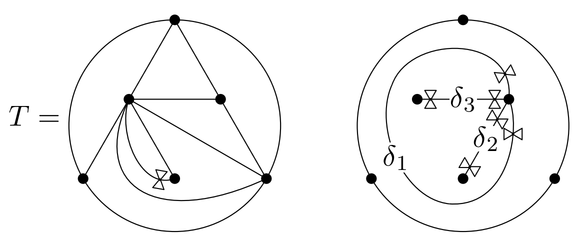
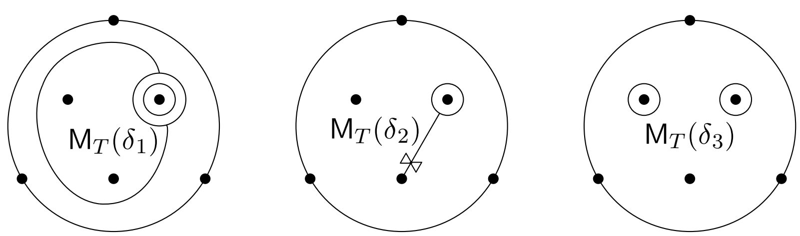


FIGURE 17. Modifications $\mathsf{M}'_T(\hat{\delta})$ of $\hat{\delta}$

Example 14.1. We consider the following tagged triangulation T and tagged arcs δ_1, δ_2 and δ_3:



Then the corresponding modified tagged arcs $\mathsf{M}_T(\delta_i)$ with respect to T are given as follows:



We can define the intersection number of a modified tagged arc m and a tagged arc δ in the same way as of tagged arcs, denote by $\mathsf{Int}(\mathsf{m}, \delta)$. Although the map M_T may seem strange, it is defined so as to satisfy the following properties.

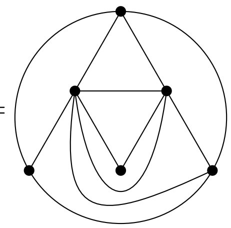
Proposition 14.2. (1) *For a tagged arc δ of (S, M), we have $\mathsf{Int}(T, \delta) = \mathsf{Int}(T, \mathsf{M}_T(\delta))$.*
(2) *The map M_T restricting to the set*

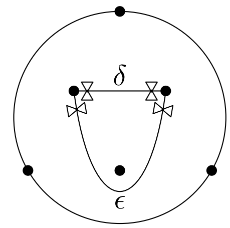
$$A := \{\textit{tagged arcs } \delta \textit{ of } (S, M) \mid \delta \notin T \textit{ and } \mathsf{M}_T(\delta) \textit{ is not a pair of } 1\textit{-punctured cycles}\}$$

is injective. Moreover, if $\mathsf{M}_T(\delta) = \mathsf{M}_T(\epsilon)$ for $\delta \in A$ and any tagged arc $\epsilon \notin T$, then $\delta = \epsilon$ holds.

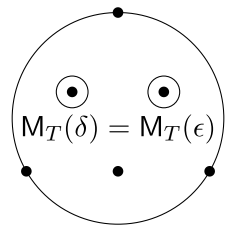
Proof. The assertions follow from the definition of intersection numbers and the map M_T. □

Remark 14.3. For a tagged arc $\delta \notin T \cup A$ of (S, M), $\mathsf{M}(\delta)$ does not always correspond to δ bijectively. Indeed, we consider the following tagged triangulation T and tagged arcs δ, ϵ:

$T =$ 



Then the corresponding modified tagged arcs $\mathsf{M}_T(\delta)$ and $\mathsf{M}_T(\epsilon)$ with respect to T are given as follows:



The following theorem is a key of the proofs of Theorems 13.1 and 13.3.

Theorem 14.4. *If modified tagged arcs* m *and* m' *with respect to* T *have* $\mathsf{Int}(T, \mathsf{m}) = \mathsf{Int}(T, \mathsf{m}')$, *then* $\mathsf{m} = \mathsf{m}'$.

We will prove Theorem 14.4 in Section 18.

Corollary 14.5. *If tagged arcs* δ *and* ϵ *in* A *have* $\mathsf{Int}(T, \delta) = \mathsf{Int}(T, \epsilon)$, *then* $\delta = \epsilon$.

Proof. Proposition 14.2(1) implies that $\mathsf{Int}(T, \mathsf{M}_T(\delta)) = \mathsf{Int}(T, \mathsf{M}_T(\epsilon))$. By Theorem 14.4 and Proposition 14.2(2), we have $\delta = \epsilon$. □

These results provide the proofs of Theorems 13.1 and 13.3.

Proof of Theorem 13.1. By changing tags, we can assume that T satisfies $(\diamondsuit)$. Let $T' = \{\delta_1, \ldots, \delta_n\}$ and $T'' = \{\epsilon_1, \ldots, \epsilon_n\}$ be tagged triangulations of (S, M) such that $\mathsf{Int}(T, \delta_i) = \mathsf{Int}(T, \epsilon_i)$ for any i. We set $V = (v_1 \cdots v_n) = \mathsf{Int}(T, T')$, where $v_i = \mathsf{Int}(T, \delta_i) \in \mathbb{Z}^n_{\geq 0}$. Without loss of generality, we assume that $\delta_i \in A$ for $i \in \{1, \ldots, k\}$ and $\delta_j \notin A$ for $j \in \{k+1, \ldots, n\}$, that is, either $\delta_j, \epsilon_j \in T$ or $\mathsf{M}_T(\delta_j) = \mathsf{M}_T(\epsilon_j)$ is a pair of 1-punctured cycles by Theorem 14.4. Corollary 14.5 implies that $\delta_i = \epsilon_i$ for $i \in \{1, \ldots, k\}$.

If $T' \neq T''$, then there exist $f, g \in \{k+1, \ldots, n\}$ such that $\mathsf{Int}(\delta_f, \epsilon_g) \neq 0$. Otherwise, it conflicts with the maximality of T'. Since $\overline{\delta_f}$ and $\overline{\epsilon_g}$ are contained in T, δ_f and ϵ_g must have different tags at the common endpoint. Without loss of generality, we assume that δ_f is contained in T and $\mathsf{M}_T(\delta_g) = \mathsf{M}_T(\epsilon_g)$ is a pair of 1-punctured cycles. Since δ_f and δ_g have the common endpoint and $\mathsf{Int}(\delta_f, \delta_g) = 0$, δ_f is a 1-notched arc of T by $(\diamondsuit)$. Then $\hat{\delta_g}$ is not a 2-notched arc, thus it is contradictory to the fact that $\mathsf{M}_T(\epsilon_g)$ is a pair of 1-punctured cycles. This finishes the proof. □

Proof of Theorem 13.3. By changing tags, we can assume that T satisfies $(\diamondsuit)$. First, we prove "if" part. Let δ and ϵ be tagged arcs with a common non-zero intersection vector $\mathsf{Int}(T, \delta) = \mathsf{Int}(T, \epsilon)$ with respect to T. Then δ and ϵ are not contained in T by definition of intersection vectors. By Corollary 14.5, it suffice to show that if $\mathsf{M}_T(\delta)$ is a pair of 1-punctured cycles, then $\delta = \epsilon$. In this case, δ and ϵ are 2-notched arcs such that $\overline{\delta}$ and $\overline{\epsilon}$ are plain arcs of T such that both endpoints of $\overline{\delta}$ correspond to ones of $\overline{\epsilon}$ since $\mathsf{M}_T(\delta) = \mathsf{M}_T(\epsilon)$ by Theorem 14.4. Therefore, we have $\delta = \epsilon$ by the assumption.

Second, we prove "only if" part. Suppose that T has a pair of different plain arcs γ and γ' such that both endpoints of γ correspond to ones of γ' which are punctures. Let δ and ϵ be 2-notched arcs such that $\overline{\delta} = \gamma$ and $\overline{\epsilon} = \gamma'$. Then we have $\delta \neq \epsilon$ and $\mathsf{Int}(T, \delta) = \mathsf{Int}(T, \epsilon)$ which is not zero, that is, T does not detect tagged arcs. □

15. Proof of Theorem 13.4

First of all, we prove Theorem 13.4(3).

Proof of Theorem 13.4(3). It is easy to show that for (S, M) as in Theorem 13.4(3), any tagged triangulation of (S, M) detects tagged arcs by Theorem 13.3. Conversely, if (S, M) is not one of the above cases, a part of (S, M) must have one of the pairs of plain arcs δ and ϵ as in Table 2. Then a tagged triangulation T of (S, M) including δ and ϵ does not detect tagged arcs by Theorem 13.3. □

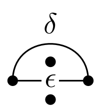
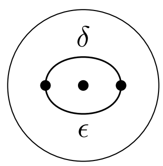
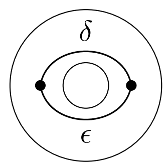
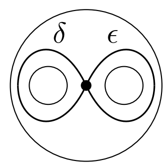
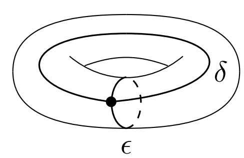
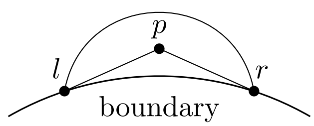
g	0				≥ 1
b	0	1	2	≥ 3	any
p	≥ 4	≥ 3	≥ 2	≥ 1	≥ 1
δ, ϵ					

Table 2. Tagged arcs δ and ϵ connecting two (possibly same) common punctures such that $\overline{\delta} \neq \overline{\epsilon}$, where g is the genus, b is the number of components of the boundary and p is the number of punctures in (S, M)

We consider the case that S is not closed. The following lemma is basic.

Lemma 15.1. *If S is not closed, then there is a tagged triangulation of (S, M) whose any tagged arc is a plain arc with at least one marked point on the boundary of S as its endpoints.*

Proof. For a puncture p of (S, M), we can construct triangles with p and two marked points l and r (possibly $l = r$) on the boundary of S as follows:



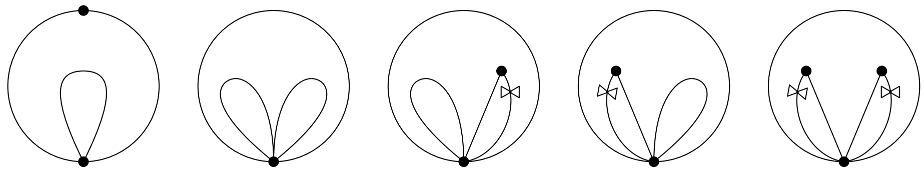
Then, for another puncture q of (S, M), it is easy to construct triangles with q, l and r in the same way. We have the set of triangles containing all punctures of (S, M) by the inductive construction. There is a tagged triangulation of (S, M) containing these triangles, thus it is what is desired. □

Proof of Theorem 13.4(1). The assertion follows from Theorem 13.3 and Lemma 15.1. □

Next, we consider the case that S is closed. In the rest of this section, let g be the genus of S and p be the number of punctures of (S, M). To prove Theorem 13.4(2), we need some preparations.

Lemma 15.2. *We assume that S is closed and $g > 0$. If a tagged triangulation T of (S, M) has loops, then T does not detect tagged arcs.*

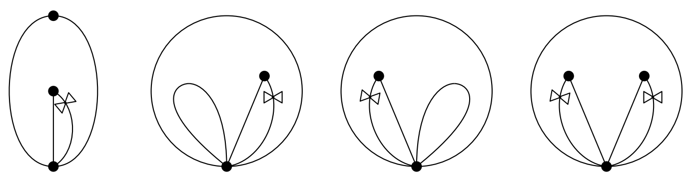
Proof. A puzzle piece with loops is one of the followings:



In these puzzle pieces, only the 2-punctured piece does not have a pairs of different plain arcs connecting two (possibly same) common punctures. Therefore, by Theorem 13.3, if a tagged triangulation T with loops of (S, M) detects tagged arcs, then T is obtained by gluing two 2-punctured pieces and by changing tags if necessary. This is in conflict with $g > 0$. $\square$

Lemma 15.3. *We assume that S is closed and $g > 0$. If a tagged triangulation T of (S, M) satisfies* ($\diamondsuit$) *and has 1-notched arcs, then T does not detect tagged arcs.*

Proof. A puzzle piece with 1-notched arcs is one of the followings:



In these puzzle pieces, only the 2-punctured piece does not have a pairs of different plain arcs connecting two (possibly same) common punctures. Therefore, the assertion follows in the same way as Lemma 15.2. $\square$

Theorem 15.4. [JR, Theorem 1.1] *We assume that S is closed. If p is the minimal integer to satisfy* (13.1)*, then there is a tagged triangulation T of (S, M) satisfying the following conditions:*

- (T1) *any tagged arc of T is a plain arc;*
- (T2) *any triangle of T has three distinct vertices;*
- (T3) *the intersection of two distinct triangles of T is either empty, a single vertex, or a single edge.*

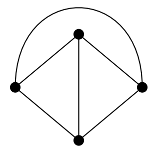
Conversely, if there is a tagged triangulation of (S, M) satisfying (T1)-(T3), then (13.1) *holds.*

Proposition 15.5. *We assume that S is closed and $g > 0$. Then a tagged triangulation T of (S, M) satisfies (T1)-(T3) if and only if T detects tagged arcs.*

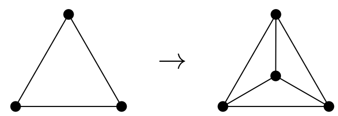
Proof. We assume that T satisfies (T1)-(T3) and does not detect tagged arcs. By Theorem 13.3, there are tagged arcs δ and ϵ of T connecting two common punctures such that $\overline{\delta} \neq \overline{\epsilon}$. Then they are not contained in a single triangle of T by (T2). The intersection of a triangle with δ and a triangle with ϵ has two vertices and does not have an edge connecting them. It conflicts with (T3).

Conversely, we assume that T detects tagged arcs. By Lemma 15.3, we can also assume that T satisfies (T1). By Lemma 15.2, T satisfies (T2). It is easy to show that if the intersection of two distinct triangles of T is either two vertices, three vertices, or two edges, then there are tagged arcs δ and ϵ of T connecting two common punctures such that $\overline{\delta} \neq \overline{\epsilon}$. Thus it is a contradiction by Theorem 13.3. If the intersection of two distinct triangles of T is three edges, then (S, M) must be a sphere with exactly three punctures, thus it conflicts with our assumption. Therefore, T satisfies (T3). $\square$

Proof of Theorem 13.4(2). When $g = 0$, we have $p \geq 4$ by our assumption, in which case (13.1) holds. We consider the tagged triangulation

$T =$ 

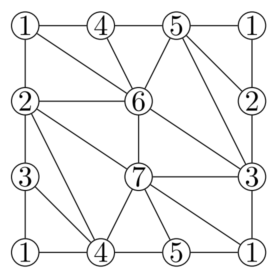
on the 2-dimensional sphere S. The tagged triangulation T does not have different plain arcs connecting two common punctures. We add a puncture and arcs to a triangle of T as follows:



Then we have inductively a tagged triangulation without different plain arcs connecting two common punctures for any p. By Theorem 13.3, it detects tagged arcs.

We assume that $g > 0$. By Theorem 15.4 and Proposition 15.5, if there is a tagged triangulation of (S, M) detecting tagged arcs, then (13.1) holds. Conversely, if p is the minimal integer to satisfy (13.1), then there is a tagged triangulation T of (S, M) detecting tagged arcs. In the same way as the case of $g = 0$, we have inductively a tagged triangulation without different plain arcs connecting two common punctures for any p satisfying (13.1). By Theorem 13.3, it detects tagged arcs. □

Example 15.6. When $g = 1$, (13.1) means that $p \geq 7$. We consider the tagged triangulation

$T =$ 

on the torus S with 7 punctures, where we identify each of two vertical lines and two horizontal lines. Then T does not have different plain arcs connecting two common punctures. Thus T detects tagged arcs by Theorem 13.3.

16. f-vectors in cluster algebras

In this section, we apply our results to the theory of cluster algebras.

16.1. **Cluster algebras and f-vectors.** One of the remarkable properties of cluster algebras is the *strongly Laurent phenomenon* [FZ07, Proposition 3.6]: Every element of the cluster algebra $\mathcal{A}(\hat{Q})$ with principal coefficients is a Laurent polynomial over $\mathbb{Z}[y_1, \ldots, y_n]$ in the initial cluster variables, that is, $\mathcal{A}(\hat{Q}) \subseteq \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_n]$. Then we denote the Laurent expression of a cluster variable z of $\mathcal{A}(\hat{Q})$ by $z(x_1, \ldots, x_n, y_1, \ldots, y_n)$. The *$F$-polynomial* of z is the rational function $z(1, \ldots, 1, y_1, \ldots, y_n)$, which is a polynomial by the strongly Laurent phenomenon. Let $f_{z,1}, \ldots, f_{z,n}$ be the maximal degrees of $y_1, \ldots, y_n$ in $z(1, \ldots, 1, y_1, \ldots, y_n)$, respectively. The *$f$-vector* of z is the integer vector $f_z := (f_{z,1}, \ldots, f_{z,n}) \in \mathbb{Z}^n_{\geq 0}$. For a cluster $\mathbf{z} = (z_1, \ldots, z_n)$ of $\mathcal{A}(\hat{Q})$, the *F-matrix* of $\mathbf{z}$ is defined by the non-negative integer $n \times n$-matrix $F_{\mathbf{z}}$ with columns $f_{z_1}, \ldots, f_{z_n}$ [FuG, Definition 2.6].

Example 16.1. Let Q be a quiver $1 \leftarrow 2 \leftarrow 3$ of type A_3. We can compute the mutation of the initial seed $((x_1, x_2, x_3, y_1, y_2, y_3), \hat{Q})$ of $\mathcal{A}(\hat{Q})$ in direction 1 as follows:

$$\mu_1\left((x_1, x_2, x_3, y_1, y_2, y_3), \begin{array}{ccccc} 1' & & 2' & & 3' \\ \uparrow & & \uparrow & & \uparrow \\ 1 & \longleftarrow & 2 & \longleftarrow & 3 \end{array}\right) = \left(\left(\frac{y_1 + x_2}{x_1}, x_2, x_3, y_1, y_2, y_3\right), \begin{array}{ccccc} 1' & & 2' & & 3' \\ \downarrow & \nwarrow & \uparrow & & \uparrow \\ 1 & \longrightarrow & 2 & \longleftarrow & 3 \end{array}\right)$$

Repeating mutations, we get all the cluster variables as in Table 3. Therefore, the cluster algebra is

$$\mathcal{A}(\hat{Q}) = \mathbb{Z}\Bigg[x_1, x_2, x_3, \frac{y_1 + x_2}{x_1}, \frac{y_2 x_1 + x_3}{x_2}, \frac{1 + y_3 x_2}{x_3}, \frac{y_1 y_2 x_1 + y_1 x_3 + x_2 x_3}{x_1 x_2}, \frac{y_2 x_1 + x_3 + y_2 y_3 x_1 x_2}{x_2 x_3}, \frac{y_1 y_2 x_1 + y_1 x_3 + y_1 y_2 y_3 x_1 x_2 + x_2 x_3}{x_1 x_2 x_3}\Bigg].$$

The F -polynomial of a cluster variable

$$z = \frac{y_1 y_2 x_1 + y_1 x_3 + y_1 y_2 y_3 x_1 x_2 + x_2 x_3}{x_1 x_2 x_3}$$

is $y_1 y_2 + y_1 + y_1 y_2 y_3 + 1$, thus we have $f_z = (1, 1, 1)$. All f-vectors appear in Table 3.

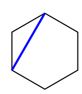
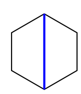
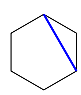
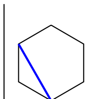
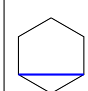
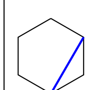
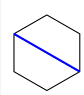
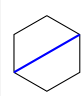
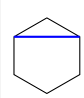
Cluster variable z	$f_z = (f_{z,1}, f_{z,2}, f_{z,3})$	Tagged arcs δ such that $z_\delta = z$
x_1	$(0,0,0)$	
x_2	$(0,0,0)$	
x_3	$(0,0,0)$	
$\frac{y_1 + x_2}{x_1}$	$(1,0,0)$	
$\frac{y_2 x_1 + x_3}{x_2}$	$(0,1,0)$	
$\frac{1 + y_3 x_2}{x_3}$	$(0,0,1)$	
$\frac{y_1 y_2 x_1 + y_1 x_3 + x_2 x_3}{x_1 x_2}$	$(1,1,0)$	
$\frac{y_2 x_1 + x_3 + y_2 y_3 x_1 x_2}{x_2 x_3}$	$(0,1,1)$	
$\frac{y_1 y_2 x_1 + y_1 x_3 + y_1 y_2 y_3 x_1 x_2 + x_2 x_3}{x_1 x_2 x_3}$	$(1,1,1)$	

TABLE 3. In $\mathcal{A}(\hat{Q})$ for a quiver Q of type A_3, all the 9 cluster variables, the corresponding f-vectors and tagged arcs

In Table 3, different cluster variables have different f-vectors. In general, it is not true (see Proposition 16.5). However, we conjecture that different clusters have different F-matrices.

Conjecture 16.2. *Let $\mathcal{A}$ be an arbitrary cluster algebra with principal coefficients. If clusters $\mathbf{z}$ and $\mathbf{z}'$ in $\mathcal{A}$ have $F_{\mathbf{z}} = F_{\mathbf{z}'}$, then $\mathbf{z} = \mathbf{z}'$.*

In the next subsection, we prove Conjecture 16.2 for the cluster algebra with principal coefficients defined from each tagged triangulation of (S, M).

16.2. **Applications for cluster algebras defined from tagged triangulations.** Thanks to Theorem 7.8, we can apply the results in the previous sections to the theory of cluster algebras.

Corollary 16.3. *Let T be a tagged triangulation of (S, M). If tagged triangulations T' and T'' of (S, M) satisfy $F_{\mathbf{z}_{T'}} = F_{\mathbf{z}_{T''}}$, then $\mathbf{z}_{T'} = \mathbf{z}_{T''}$.*

Proof. The assertion follows immediately from Theorems 13.1 and 7.8. □

Definition 16.4. For a cluster algebra $\mathcal{A}$, we say that $\mathcal{A}$ *detects cluster variables by f-vectors* if it satisfies the following condition:

- For non-initial cluster variables z and z' of $\mathcal{A}(T)$, if $f_z = f_{z'}$, then $z = z'$.

Proposition 16.5. *Let T be a tagged triangulation of (S, M). Then T detects cluster variables by f-vectors if and only if either of the following conditions holds:*

- *(S, M) is a closed surface with exactly one puncture;*
- *there are no tagged arcs δ and ϵ of T connecting two (possibly same) common punctures such that $\overline{\delta} \neq \overline{\epsilon}$.*

Proof. If (S, M) is not a closed surface with exactly one puncture, the assertion follows from Theorems 13.3, 3.1 and 7.8. If (S, M) is a closed surface with exactly one puncture, there are no 2-notched arcs corresponding to cluster variables by Theorem 3.1(2). Therefore, the assertion follows from Corollary 14.5 and Theorem 7.8. □

Corollary 16.6. (1) *If S is not closed, then there is at least one tagged triangulation of (S, M) detecting cluster variables by f-vectors.*

(2) *If S is closed, then there is at least one tagged triangulation of (S, M) detecting cluster variables by f-vectors if and only if the inequality* (13.1) *holds.*

(3) *All tagged triangulation of (S, M) detect cluster variables by f-vectors if and only if (S, M) is one of the followings:*

- *a closed surface with exactly one puncture;*
- *a marked surface with no punctures;*
- *a marked surface of genus 0 with exactly 1 boundary component and at most 2 punctures;*
- *a marked surface of genus 0 with exactly 2 boundary components and a 1 puncture.*

Proof. The assertion follows immediately from Theorem 13.4 and Proposition 16.5. □

17. List of segments in each puzzle piece

In this section, we prepare some tables to show Theorem 14.4.

We fix a tagged triangulation T of (S, M) satisfying $(\diamond)$ which is not T_3. Let $\square$ be a puzzle piece of T. If $\square$ is a triangle piece or a 1-puncture piece, we say that $\square$ is Case($-$) (resp., Case(τ_i)) if its edges are not loops (resp., its edge τ_i is an only loop in $\square$). Similarly, we can define Case(τ_i,τ_{i+1}) if $\square$ is a triangle piece (see Figures 18 and 19). Let δ be a tagged arc of (S, M) which is not contained in T. We have the set of curves $\delta \cap \square$ and call its each curve a *segment (of δ) in* $\square$. It is easy to show that Table 4 (resp., Table 6, Table 7) gives a complete list of segments of δ in a triangle piece (resp., a 1-puncture piece, a 2-puncture piece), where a_i is the intersection number of each segment and τ_i. Moreover, we have the set of 'curves' $\mathsf{M}_T(\delta) \cap \square$ and call its each curve a *modified segment (of δ) in* $\square$. Let m be a modified segment in $\square$ which is not a segment. If there are two distinct segments s and s' in $\square$ such that $\mathsf{M}_T(s) = \mathsf{M}_T(s') = m$, then m is one as in Figure 20. Otherwise,

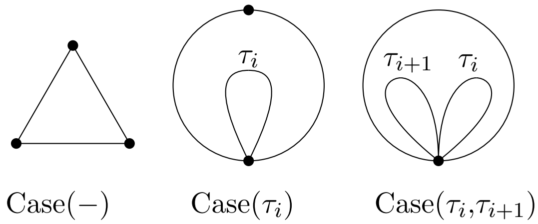


FIGURE 18. Cases of a triangle piece

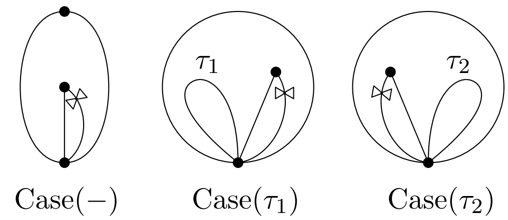


FIGURE 19. Cases of a 1-puncture piece

there is exactly one segment s in $\square$ such that $\mathsf{M}_T(s) = m$. In this case, abusing notation, we denote $\mathsf{M}_T(s)$ by s. In particular, Table 5 gives all segments s in $\square$ such that $s \neq \mathsf{M}_T(s)$.

On the other hand, it is also easy to show that Table 8 (resp., Table 9, Table 10) gives a complete list of $\delta \cap \square$ and $\mathsf{M}_T(\delta) \cap \square$, where $\square$ is a triangle piece (resp., a 1-puncture piece, a 2-puncture piece). Note that if an end which is not in $\square$ is tagged notched at a vertex of $\square$, it does not appear in $\delta \cap \square$, but appear in $\mathsf{M}_T(\delta) \cap \square$. So we identify its end to the corresponding modified segment as in Figure 21 in Tables 8, 9 and 10. For example, in the five line from the top of Table 8 Case $(-)$, the segment e_1 is given by this identification.

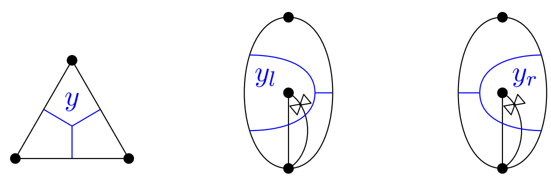


FIGURE 20. Exceptional segments of modified tagged arcs

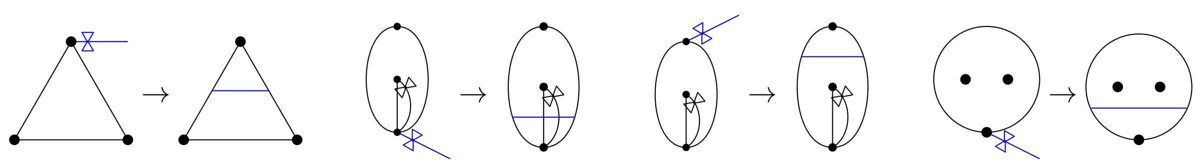
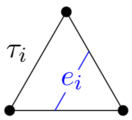
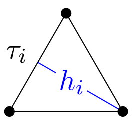


FIGURE 21. Identifications in $\square$

Segments		(a_1, a_2, a_3)
	e_1	$(0,1,1)$
	e_2	$(1,0,1)$
	e_3	$(1,1,0)$
	h_1	$(1,0,0)$
	h_2	$(0,1,0)$
	h_3	$(0,0,1)$

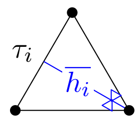
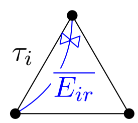
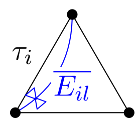
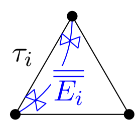
Segments		(a_1, a_2, a_3)		
		Case$(-)$	Case(τ_1)	Case(τ_1,τ_2)
	$\overline{h_1}$	$(1,1,1)$	$(1,1,1)$	$(3,2,2)$
	$\overline{h_2}$		$(2,2,1)$	$(2,3,2)$
	$\overline{h_3}$		$(2,1,2)$	$(2,2,3)$
	$\overline{E_{1r}}$	$(0,1,0)$	$\times$	$\times$
	$\overline{E_{2r}}$	$(0,0,1)$	$(0,0,1)$	$\times$
	$\overline{E_{3r}}$	$(1,0,0)$	$(2,1,0)$	$\times$
	$\overline{E_{1l}}$	$(0,0,1)$	$\times$	$\times$
	$\overline{E_{2l}}$	$(1,0,0)$	$(2,0,1)$	$\times$
	$\overline{E_{3l}}$	$(0,1,0)$	$(0,1,0)$	$\times$
	$\overline{\overline{E_1}}$	$(2,1,1)$	$(4,2,2)$	$(4,4,4)$
	$\overline{\overline{E_2}}$	$(1,2,1)$	$(2,2,2)$	
	$\overline{\overline{E_3}}$	$(1,1,2)$		

Table 4. Segments of a tagged arc in triangle pieces and the corresponding intersection sub-vectors (a_1, a_2, a_3)

Triangle piece case

Segments	Modified segments
$\overline{h_i}$	y
$\overline{2h_i}$	$\{e_1, e_2, e_3\}$
$\overline{E_{ir}}$	h_{i+1}
$\overline{E_{il}}$	h_{i-1}
$\overline{\overline{E_i}}$	$\{2e_{i-1}, 2e_{i+1}\}$

2-puncture piece case

Q	$2h$
$\overline{R_0}$	$\{r_0, r_1\}$
$\overline{L_0}$	$\{l_0, l_{-1}\}$
$2\overline{c_n}$	$\{s_{n-1}, s_n, h\}$

1-puncture piece case

Segments	Modified segments
$\overline{l_\pm}$	y_l
$\overline{r_\pm}$	y_r
$\underline{L}$	l_+
$\underline{R}$	r_+
$\overline{L}$	l_-
$\overline{R}$	r_-
$\underline{\overline{L}}$, $\underline{\overline{R}}$	$\{u, d\}$
$\overline{P_-}$	$\{r_p, l_p\}$

TABLE 5. Segments s and the corresponding modified segments $\mathsf{M}_T(s)$ in $\square$ such that $s \neq \mathsf{M}_T(s)$, where $h_{3k+j} = h_j$ and $e_{3k+j} = e_j$ for any $k, j \in \mathbb{Z}$

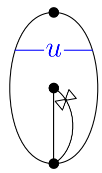
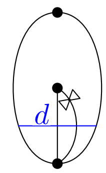
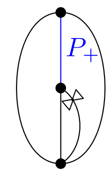
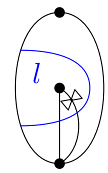
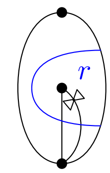
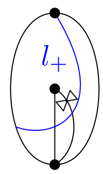
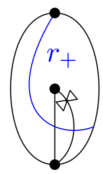
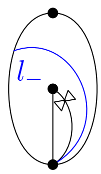
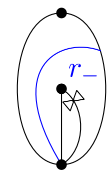
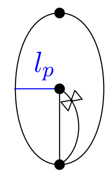
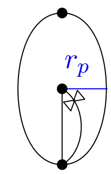
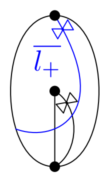
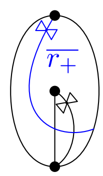
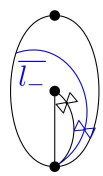
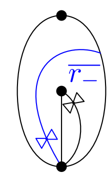
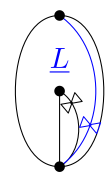
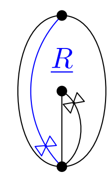
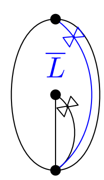
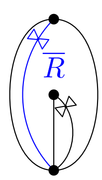
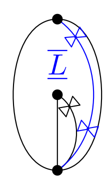
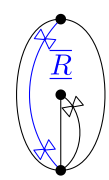
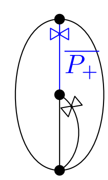
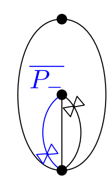
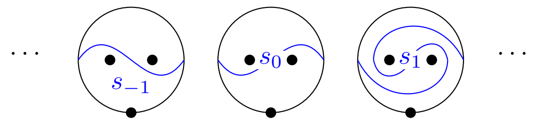
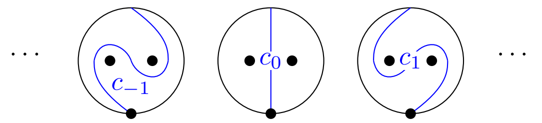
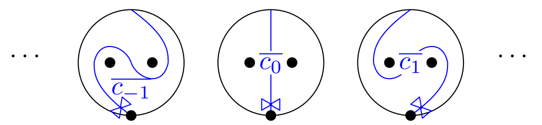
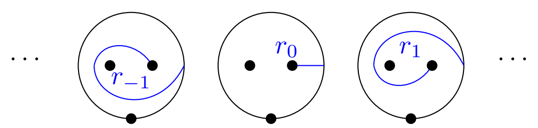
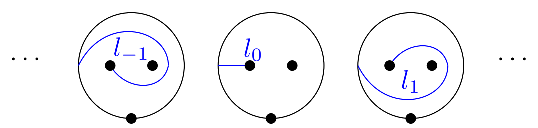
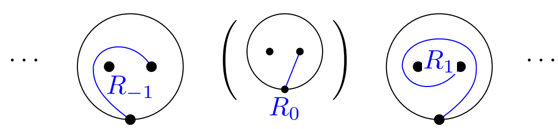
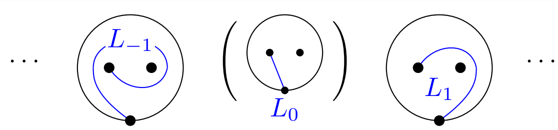
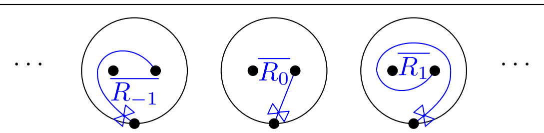
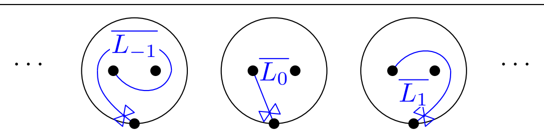
Segments					
(a_1, a_2, a_3, a_4)	(1,1,0,0)	(1,1,1,1)	(0,0,0,1)	(2,0,1,1)	(0,2,1,1)
					
(1,0,1,1)	(0,1,1,1)	(1,0,0,0)	(0,1,0,0)	(1,0,0,1)	(0,1,0,1)
					
(2,1,1,1)	(1,2,1,1)	(2,1,1,1)	(1,2,1,1)	(1,0,1,1)	(0,1,1,1)
(3,2,2,2)	(2,3,2,2)	(3,2,1,1)	(2,3,1,1)	×	×
					
(1,0,0,0)	(0,1,0,0)	(2,2,1,1)	(2,2,1,1)	(1,1,0,1)	(1,1,0,2)
×	×	(4,4,2,2)	(4,4,2,2)	(2,2,1,2)	(2,2,0,2)

Table 6. Segments of a tagged arc in 1-puncture pieces and the corresponding intersection sub-vectors (a_1, a_2, a_3, a_4) that are values of Case($-$) (above) and of Case(τ_1) (below)

Segments		$(a_1, a_2, a_3, a_4, a_5)$	
	s_n	$(2, \|n+1\|, \|n+1\|, \|n\|, \|n\|)$	
	c_n	$(1,0,0,0,0)$ $(1,n,n,n-1,n-1)$ $(1,-n-1,-n-1,-n,-n)$	$(n=0)$ $(n>0)$ $(n<0)$
	$\overline{c_n}$	$(3,1,1,1,1)$ $(3,n+1,n+1,n,n)$ $(3,-n,-n,-n+1,-n+1)$	$(n=0)$ $(n>0)$ $(n<0)$
	r_n	$(1,n,n,n-1,n)$ $(1,-n,-n,-n,-n+1)$	$(n>0)$ $(n\leq 0)$
	l_n	$(1,n,n+1,n,n)$ $(1,-n-1,-n,-n,-n)$	$(n\geq 0)$ $(n<0)$
	R_n	$(0,n,n,n-1,n)$ $(0,-n-1,-n-1,-n-1,-n)$	$(n>0)$ $(n<0)$
	L_n	$(0,n-1,n,n-1,n-1)$ $(0,-n-1,-n,-n,-n)$	$(n>0)$ $(n<0)$
	$\overline{R_n}$	$(2,1,1,0,2)$ $(2,n+1,n+1,n,n+1)$ $(2,-n,-n,-n,-n+1)$	$(n=0)$ $(n>0)$ $(n<0)$
	$\overline{L_n}$	$(2,0,2,1,1)$ $(2,n,n+1,n,n)$ $(2,-n,-n+1,-n+1,-n+1)$	$(n=0)$ $(n>0)$ $(n<0)$

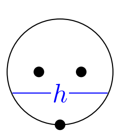
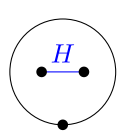
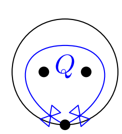
 $(2,1,1,1,1)$  $(0,0,1,0,1)$  $(4,2,2,2,2)$

TABLE 7. Segments of a tagged arc in 2-puncture pieces and the corresponding intersection sub-vectors $(a_1, a_2, a_3, a_4, a_5)$, where s_n (resp., c_n, $\overline{c_n}$) is obtained from s_0 (resp., c_0, $\overline{c_0}$) by moving its endpoints along the boundary clockwise in angle π and the other cases are in angle 2π

Case(−)

Sets of segments	Sets of modified segments	(a_1, a_2, a_3)
$\{m_1e_1, m_2e_2, m_3e_3\}$		$(m_2+m_3, m_3+m_1, m_1+m_2)$
$\{h_1, m_2e_2, m_3e_3\}$		$(1+m_2+m_3, m_3, m_2)$
$\{2h_1, m_2e_2, m_3e_3\}$		$(2+m_2+m_3, m_3, m_2)$
$\{\overline{h_1}, m_2e_2, m_3e_3\}$	$\{y, m_2e_2, m_3e_3\}$	$(1+m_2+m_3, 1+m_3, 1+m_2)$
$\{\overline{h_1}, e_1, m_2e_2, m_3e_3\}$	$\{y, e_1, m_2e_2, m_3e_3\}$	$(1+m_2+m_3, 2+m_3, 2+m_2)$
$\{2\overline{h_1}, m_2e_2, m_3e_3\}$	$\{e_1, (m_2+1)e_2, (m_3+1)e_3\}$	appear in above
$\{\overline{h_2}, k_1e_1, k_2e_2, k_3e_3\}$	$\{y, k_1e_1, k_2e_2, k_3e_3\}$	
$\{\overline{h_3}, k_1e_1, k_2e_2, k_3e_3\}$		
$\{2\overline{h_2}, m_3e_3\}$	$\{e_1, e_2, (1+m_3)e_3\}$	
$\{2\overline{h_3}, m_2e_2\}$	$\{e_1, (1+m_2)e_2, e_3\}$	
$\{\overline{E_{3r}}\}$, $\{\overline{E_{2l}}\}$	$\{h_1\}$	
$\{\overline{\overline{E_1}}\}$	$\{e_2, e_3\}$	

Case(τ_i) and Case(τ_i,τ_j) come down to Case(−) as follows:

Case(τ_i) and not appear in Case(−)

i	Sets of segments	Sets of modified segments	(a_1, a_2, a_3)
1	$\{\overline{E_{2l}}, e_2\}$	$\{h_1, e_2\}$	appear in above
1	$\{\overline{E_{3r}}, e_3\}$	$\{h_1, e_3\}$	
1	$\{\overline{\overline{E_1}}, e_2, e_3\}$	$\{2e_2, 2e_3\}$	
2, 3	$\{\overline{\overline{E_1}}, e_1\}$	$\{e_1, e_2, e_3\}$	
1, 3	$\{\overline{\overline{E_2}}, e_2\}$		
1, 2	$\{\overline{\overline{E_3}}, e_3\}$		

Case(τ_i,τ_j) and not appear in others

Sets of segments	Sets of modified segments	(a_1, a_2, a_3)
$\{\overline{\overline{E_k}}, e_1, e_2, e_3\}$	$\{2e_1, 2e_2, 2e_3\}$	appear in above

Table 8. Sets of segments and the corresponding sets of modified segments in a triangle piece for the case of $a_1 \geq a_2, a_3$, where $m_i \in \mathbb{Z}_{\geq 0}$ and $k_i \in \{0, 1\}$ such that $k_1 \leq k_2, k_3$

Case$(-)$

Sets of segments	Sets of modified segments	(a_1, a_2, a_3, a_4)
$\{m_1r, m_2u, m_3d\}$		$(m_2+m_3, 2m_1+m_2+m_3, m_1+m_3, m_1+m_3)$
$\{r_+, m_1r, m_3d\}$		$(m_3, 1+2m_1+m_3, 1+m_1+m_3, 1+m_1+m_3)$
$\{2r_+, m_1r, m_3d\}$		$(m_3, 2+2m_1+m_3, 2+m_1+m_3, 2+m_1+m_3)$
$\{r_-, m_1r, m_2u\}$		$(m_2, 1+2m_1+m_2, m_1, m_1)$
$\{2r_-, m_1r, m_2u\}$		$(m_2, 2+2m_1+m_2, m_1, m_1)$
$\{\overline{r_+}, m_1r, m_3d\}$	$\{y_r, m_1r, m_3d\}$	$(1+m_3, 2+2m_1+m_3, 1+m_1+m_3, 1+m_1+m_3)$
$\{\overline{r_+}, u, m_1r, m_3d\}$	$\{y_r, u, m_1r, m_3d\}$	$(2+m_3, 3+2m_1+m_3, 1+m_1+m_3, 1+m_1+m_3)$
$\{\overline{r_-}, m_1r, m_2u\}$	$\{y_r, m_1r, m_2u\}$	$(1+m_2, 2+2m_1+m_2, 1+m_1, 1+m_1)$
$\{\overline{r_-}, d, m_1r, m_2u\}$	$\{y_r, d, m_1r, m_2u\}$	$(2+m_2, 3+2m_1+m_2, 2+m_1, 2+m_1)$
$\{r_p, m_1r, m_2u, m_3d\}$		$(m_2+m_3, 1+2m_1+m_2+m_3, m_1+m_3, 1+m_1+m_3)$
$\{r_p, r_+, m_1r, m_3d\}$		$(m_3, 2+2m_1+m_3, 1+m_1+m_3, 2+m_1+m_3)$
$\{r_p, r_-, m_1r, m_2u\}$		$(m_2, 2+2m_1+m_2, m_1, 1+m_1)$
$\{r_p, \overline{r_+}, m_1r, m_3d\}$	$\{r_p, y_r, m_1r, m_3d\}$	$(1+m_3, 3+2m_1+m_3, 1+m_1+m_3, 2+m_1+m_3)$
$\{r_p, \overline{r_+}, u, m_1r, m_3d\}$	$\{r_p, y_r, u, m_1r, m_3d\}$	$(2+m_3, 4+2m_1+m_3, 1+m_1+m_3, 2+m_1+m_3)$
$\{r_p, \overline{r_-}, m_1r, m_2u\}$	$\{r_p, y_r, m_1r, m_2u\}$	$(1+m_2, 3+2m_1+m_2, 1+m_1, 2+m_1)$
$\{r_p, \overline{r_-}, d, m_1r, m_2u\}$	$\{r_p, y_r, d, m_1r, m_2u\}$	$(2+m_2, 4+2m_1+m_2, 2+m_1, 3+m_1)$
$\{P_+\}$		$(0,0,0,1)$
$\{\overline{P_+}\}$		$(1,1,0,1)$
$\{2r_p, m_1r, m_2u, m_3d\}$		$(m_2+m_3, 2+2m_1+m_2+m_3, m_1+m_3, 2+m_1+m_3)$
$\{r_p, l_p, m_2u, m_3d\}$		$(1+m_2+m_3, 1+m_2+m_3, m_3, 2+m_3)$
$\{2\overline{r_+}, m_1r, m_3d\}$	$\{(m_1+2)r, 2u, (m_3+2)d\}$	appear in above
$\{2\overline{r_-}, m_1r, m_2u\}$	$\{(m_1+2)r, (m_2+2)u, 2d\}$	
$\{\overline{P_-}\}$	$\{r_p, l_p\}$	
$\{\underline{R}\}$	$\{r_+\}$	
$\{\overline{R}\}$	$\{r_-\}$	
$\{\overline{\overline{R}}\}$, $\{\overline{\overline{L}}\}$	$\{u, d\}$	

Case(τ_i) and not appear in Case$(-)$

$\{\overline{P_+}, d\}$		$(2,2,1,2)$
$\{\overline{P_-}, u\}$	$\{r_p, l_p, u\}$	appear in above
$\{\overline{\overline{R}}, u, d\}$, $\{\overline{\overline{L}}, u, d\}$	$\{2u, 2d\}$	

Table 9. Sets of segments and the corresponding sets of modified segments in a 1-puncture piece for the case of $a_1 \leq a_2$, where $m_i \in \mathbb{Z}_{\geq 0}$

TABLE 10. Sets of segments and the corresponding sets of modified segments in a 2-puncture piece for the case of $a_3 - a_2 \le a_5 - a_4$, where $m_i \in \mathbb{Z}_{\ge 0}$ and the notation $(\uparrow)$ (resp., $(\leftarrow)$) means that it is equal to the polynomial just above (resp., left)

Sets of segments = Sets of modified segments	
$(a_1, a_2, a_3, a_4, a_5)$	
$\{c_n, m_1 s_{n-1}, m_2 s_n\}$	
$(1+2(m_1+m_2), m_2, m_2, m_1, m_1)$	$(n=0)$
$((\uparrow), n+m_1(n)+m_2(n+1), (\leftarrow), n-1+m_1(n-1)+m_2(n), (\leftarrow))$	$(n>0)$
$((\uparrow), -n-1+m_1(-n)+m_2(-n-1), (\leftarrow), -n+m_1(-n+1)+m_2(-n), (\leftarrow))$	$(n<0)$
$\{\overline{c_n}, m_1 s_{n-1}, m_2 s_n\}$	
$(3+2(m_1+m_2), 1+m_2, 1+m_2, 1+m_1, 1+m_1)$	$(n=0)$
$((\uparrow), n+1+m_1(n)+m_2(n+1), (\leftarrow), n+m_1(n-1)+m_2(n), (\leftarrow))$	$(n>0)$
$((\uparrow), -n+m_1(-n)+m_2(-n-1), (\leftarrow), -n+1+m_1(-n+1)+m_2(-n), (\leftarrow))$	$(n<0)$
$\{\overline{c_n}, m_1 s_{n-1}, m_2 s_n, h\}$	
$(5+2(m_1+m_2), 2+m_2, (\leftarrow), 2+m_1, (\leftarrow))$	$(n=0)$
$((\uparrow), n+2+m_1(n)+m_2(n+1), (\leftarrow), n+1+m_1(n-1)+m_2(n), (\leftarrow))$	$(n>0)$
$((\uparrow), -n+1+m_1(-n)+m_2(-n-1), (\leftarrow), -n+2+m_1(-n+1)+m_2(-n), (\leftarrow))$	$(n<0)$
$\{r_n, m_1 s_{2n-2}, m_2 s_{2n-1}, m_3 h\}$ $(m_1 \neq 0)$	
$(1+2(m_1+m_2+m_3), m_1+m_3, (\leftarrow), 2m_1+m_2+m_3, 1+2m_1+m_2+m_3)$	$(n=0)$
$((\uparrow), n+m_1(2n-1)+m_2(2n)+m_3, (\leftarrow), n-1+m_1(2n-2)+m_2(2n-1)+m_3, n+(\leftarrow))$	$(n>0)$
$((\uparrow), -n+m_1(-2n+1)+m_2(-2n)+m_3, (\leftarrow), -n+m_1(-2n+2)+m_2(-2n+1)+m_3, -n+1+(\leftarrow))$	$(n<0)$
$\{r_n, m_1 s_{2n-1}, m_2 s_{2n}, m_3 h\}$	
$(1+2(m_1+m_2+m_3), m_2+m_3, (\leftarrow), m_1+m_3, 1+m_1+m_3)$	$(n=0)$
$((\uparrow), n+m_1(2n)+m_2(2n+1)+m_3, (\leftarrow), n-1+m_1(2n-1)+m_2(2n)+m_3, n+(\leftarrow))$	$(n>0)$
$((\uparrow), -n+m_1(-2n)+m_2(-2n-1)+m_3, (\leftarrow), -n+m_1(-2n+1)+m_2(-2n)+m_3, -n+1+(\leftarrow))$	$(n<0)$
$\{2r_n, m_1 s_{2n-2}, m_2 s_{2n-1}, m_3 h\}$ $(m_1 \neq 0)$	
$(2+2(m_1+m_2+m_3), m_1+m_3, (\leftarrow), 2m_1+m_2+m_3, 2+2m_1+m_2+m_3)$	$(n=0)$
$((\uparrow), 2n+m_1(2n-1)+m_2(2n)+m_3, (\leftarrow), 2n-2+m_1(2n-2)+m_2(2n-1)+m_3, 2n+(\leftarrow))$	$(n>0)$
$((\uparrow), -2n+m_1(-2n+1)+m_2(-2n)+m_3, (\leftarrow), -2n+m_1(-2n+2)+m_2(-2n+1)+m_3, -2n+2+(\leftarrow))$	$(n<0)$
$\{2r_n, m_1 s_{2n-1}, m_2 s_{2n}, m_3 h\}$	
$(2+2(m_1+m_2+m_3), m_2+m_3, (\leftarrow), m_1+m_3, 2+m_1+m_3)$	$(n=0)$
$((\uparrow), 2n+m_1(2n)+m_2(2n+1)+m_3, (\leftarrow), 2n-2+m_1(2n-1)+m_2(2n)+m_3, 2n+(\leftarrow))$	$(n>0)$
$((\uparrow), -2n+m_1(-2n)+m_2(-2n-1)+m_3, (\leftarrow), -2n+m_1(-2n+1)+m_2(-2n)+m_3, -2n+2+(\leftarrow))$	$(n<0)$
$\{r_n, r_{n+1}, m_1 s_{2n}, m_3 h\}$	
$(2+2(m_1+m_3), 1+m_1+m_3, (\leftarrow), m_3, 2+m_3)$	$(n=0)$
$((\uparrow), 1+m_1+m_3, (\leftarrow), 1+2m_1+m_3, 3+2m_1+m_3)$	$(n=-1)$
$((\uparrow), 2n+1+m_1(2n+1)+m_3, (\leftarrow), 2n-1+m_1(2n)+m_3, 2n+1+(\leftarrow))$	$(n>0)$
$((\uparrow), -2n-1+m_1(-2n-1)+m_3, (\leftarrow), -2n-1+m_1(-2n)+m_3, -2n+1+(\leftarrow))$	$(n<-1)$
$\{\overline{R_n}\}$ $(n \neq 0)$	
$(2, n+1, n+1, n, n+1)$	$(n>0)$
$(2, -n, -n, -n, -n+1)$	$(n<0)$

$\{r_n, c_{2n-1}, m_1 s_{2n-2}, m_2 s_{2n-1}\}$	
$(2+2(m_1+m_2), m_1, (\leftarrow), 1+2m_1+m_2, 2+2m_1+m_2)$	$(n=0)$
$((\uparrow), 3n-1+m_1(2n-1)+m_2(2n), (\leftarrow), 3n-3+m_1(2n-2)+m_2(2n-1), 3n-2+(\leftarrow))$	$(n>0)$
$((\uparrow), -3n+m_1(-2n+1)+m_2(-2n), (\leftarrow), -3n+1+m_1(-2n+2)+m_2(-2n+1), -3n+2+(\leftarrow))$	$(n<0)$
$\{r_n, c_{2n}, m_1 s_{2n-1}, m_2 s_{2n}\}$	
$(2+2(m_1+m_2), m_2, (\leftarrow), m_1, 1+m_1)$	$(n=0)$
$((\uparrow), 3n+m_1(2n)+m_2(2n+1), (\leftarrow), 3n-2+m_1(2n-1)+m_2(2n), 3n-1+(\leftarrow))$	$(n>0)$
$((\uparrow), -3n-1+m_1(-2n)+m_2(-2n-1), (\leftarrow), -3n+m_1(-2n+1)+m_2(-2n), -3n+1+(\leftarrow))$	$(n<0)$
$\{r_n, \overline{c_{2n-1}}, m_1 s_{2n-2}, m_2 s_{2n-1}\}$	
$(4+2(m_1+m_2), 1+m_1, (\leftarrow), 2+2m_1+m_2, 3+2m_1+m_2)$	$(n=0)$
$((\uparrow), 3n+m_1(2n-1)+m_2(2n), (\leftarrow), 3n-2+m_1(2n-2)+m_2(2n-1), 3n-1+(\leftarrow))$	$(n>0)$
$((\uparrow), -3n+1+m_1(-2n+1)+m_2(-2n), (\leftarrow), -3n+2+m_1(-2n+2)+m_2(-2n+1), -3n+3+(\leftarrow))$	$(n<0)$
$\{r_n, \overline{c_{2n}}, m_1 s_{2n-1}, m_2 s_{2n}\}$	
$(4+2(m_1+m_2), 1+m_2, (\leftarrow), 1+m_1, 2+m_1)$	$(n=0)$
$((\uparrow), 3n+1+m_1(2n)+m_2(2n+1), (\leftarrow), 3n-1+m_1(2n-1)+m_2(2n), 3n+(\leftarrow))$	$(n>0)$
$((\uparrow), -3n+m_1(-2n)+m_2(-2n-1), (\leftarrow), -3n+1+m_1(-2n+1)+m_2(-2n), -3n+2+(\leftarrow))$	$(n<0)$
$\{r_n, l_{n-1}, m_1 s_{2n-2}, m_2 s_{2n-1}, m_3 h\}$	
$(2+2(m_1+m_2+m_3), m_1+m_3, 1+m_1+m_3, 1+2m_1+m_2+m_3, 2+2m_1+m_2+m_3)$	$(n=0)$
$((\uparrow), 2n-1+m_1(2n-1)+m_2(2n)+m_3, 2n+(\leftarrow), 2n-2+m_1(2n-2)+m_2(2n-1)+m_3, 2n-1+(\leftarrow))$	$(n>0)$
$((\uparrow), -2n+m_1(-2n+1)+m_2(-2n)+m_3, -2n+1+(\leftarrow), -2n+1+m_1(-2n+2)+m_2(-2n+1)+m_3, -2n+2+(\leftarrow))$	$(n<0)$
$\{r_n, l_n, m_1 s_{2n-1}, m_2 s_{2n}, m_3 h\}$	
$(2+2(m_1+m_2+m_3), m_2+m_3, 1+m_2+m_3, m_1+m_3, 1+m_1+m_3)$	$(n=0)$
$((\uparrow), 2n+m_1(2n)+m_2(2n+1)+m_3, 2n+1+(\leftarrow), 2n-1+m_1(2n-1)+m_2(2n)+m_3, 2n+(\leftarrow))$	$(n>0)$
$((\uparrow), -2n-1+m_1(-2n)+m_2(-2n-1)+m_3, -2n+(\leftarrow), -2n+m_1(-2n+1)+m_2(-2n)+m_3, -2n+1+(\leftarrow))$	$(n<0)$
$\{2c_n, m_1 s_{n-1}, m_2 s_n\}$	
$(2+2(m_1+m_2), m_2, m_2, m_1, m_1)$	$(n=0)$
$((\uparrow), 2n+m_1(n)+m_2(n+1), (\leftarrow), 2n-2+m_1(n-1)+m_2(n), (\leftarrow))$	$(n>0)$
$((\uparrow), -2n-2+m_1(-n)+m_2(-n-1), (\leftarrow), -2n+m_1(-n+1)+m_2(-n), (\leftarrow))$	$(n<0)$
$\{m_1 s_{n-1}, m_2 s_n, m_3 h\}$	
$(2(m_1+m_2+m_3), m_2+m_3, (\leftarrow), m_1+m_3, (\leftarrow))$	$(n=0)$
$((\uparrow), m_1(n)+m_2(n+1)+m_3, (\leftarrow), m_1(n-1)+m_2(n)+m_3, (\leftarrow))$	$(n>0)$
$((\uparrow), m_1(-n)+m_2(-n-1)+m_3, (\leftarrow), m_1(-n+1)+m_2(-n)+m_3, (\leftarrow))$	$(n<0)$
$\{H\}$	
$(0,0,1,0,1)$	
$\{R_n\}$	
$(0,n,n,n-1,n)$	$(n>0)$
$(0,-n-1,-n-1,-n-1,-n)$	$(n<0)$

The other cases are modified to the above cases as follows:

Sets of segments	$\{2\overline{c_n}, m_1 s_{n-1}, m_2 s_n\}$	$\{Q\}$	$\{\overline{R_0}\}$
Sets of modified segments	$\{(m_1+1)s_{n-1}, (m_2+1)s_n, h\}$	$\{2h\}$	$\{r_0, r_1\}$

18. Proof of Theorem 14.4

Let T be a tagged triangulation of (S, M). To prove Theorem 14.4, we can assume that T satisfies $(\diamond)$. Let $v \in \mathbb{Z}_{\geq 0}^n$ be an intersection vector with respect to T. We show that there is a unique modified tagged arc m such that $\mathsf{Int}(T, \mathsf{m}) = v$.

First of all, we assume that T is not T_3. We only need to show that, for any puzzle piece $\Box$, there is a unique set of modified segments $S = S_\Box$ in $\Box$ such that $\sum_{s \in S} \mathsf{Int}(s, \Box) = v|_\Box$. Indeed, gluing puzzle pieces of T, their segments are glued simultaneously. Then we can obtain m.

First, we consider the case that $\Box$ is a triangle piece. That is, $v|_\Box = (a_1, a_2, a_3)$. By symmetry, we can assume that $a_1 \geq a_2, a_3$. We consider the simultaneous equations

$$\begin{cases} m_2 + m_3 = a_1 \\ m_3 + m_1 = a_2 \\ m_1 + m_2 = a_3. \end{cases}$$

If $(m_1, m_2, m_3) \in \mathbb{Z}_{\geq 0}^3$, then we have $S = \{m_1 e_1, m_2 e_2, m_3 e_3\}$. Now, we assume that $(m_1, m_2, m_3) \notin \mathbb{Z}_{\geq 0}^3$.

- If $a_2 + a_3 = a_1 - 1$, then we have $S = \{h_1, a_3 e_2, a_2 e_3\}$.
- If $a_2 + a_3 = a_1 - 2$, then we have $S = \{2h_1, a_3 e_2, a_2 e_3\}$.
- If $a_2 + a_3 = a_1 + 1$, then we have $S = \{y, (a_3 - 1)e_2, (a_2 - 1)e_3\}$.
- If $a_2 + a_3 = a_1 + 2$, then we have $S = \{y, e_1, (a_3 - 2)e_2, (a_2 - 2)e_3\}$.

By Table 8, these cover all cases of (a_1, a_2, a_3). Therefore, $v|_\Box$ gives the unique set of modified segments $S_\Box$.

Second, we consider the case that $\Box$ is a 1-puncture piece. That is, $v|_\Box = (a_1, a_2, a_3, a_4)$. By symmetry, we can assume that $a_1 \leq a_2$.

a) Suppose that $a_3 = a_4$. In this case, S is one as in Table 11.

S	$a_1 - a_2 + 2a_3$	$a_1 + a_2 - 2a_3$
$\{m_1 r, m_2 u, m_3 d\}$	$2m_3$	$2m_2$
$\{r_+, m_1 r, m_3 d\}$	$1 + 2m_3$	-1
$\{2r_+, m_1 r, m_3 d\}$	$2 + 2m_3$	-2
$\{r_-, m_1 r, m_2 u\}$	-1	$1 + 2m_2$
$\{2r_-, m_1 r, m_2 u\}$	-2	$2 + 2m_2$
$\{y_r, m_1 r, m_3 d\}$	$1 + 2m_3$	1
$\{y_r, u, m_1 r, m_3 d\}$	$1 + 2m_3$	3
$\{y_r, m_1 r, m_2 u\}$	1	$1 + 2m_2$
$\{y_r, d, m_1 r, m_2 u\}$	3	$1 + 2m_2$

Table 11. All cases of S for a 1-puncture piece and $a_3 = a_4$

a1) If $a_1 - a_2 + 2a_3 = -2$, then $S = \{2r_-, a_3 r, a_1 u\}$.
a2) If $a_1 - a_2 + 2a_3 = -1$, then $S = \{r_-, a_3 r, a_1 u\}$.
a3) If $a_1 - a_2 + 2a_3 = 0$, then $S = \{a_3 r, a_1 u\}$.
a4) If $a_1 - a_2 + 2a_3 = 1$, then S is either $\{r_+, m_1 r\}$ or $\{y_r, m_1 r, m_2 u\}$.
  * If $a_1 + a_2 - 2a_3 = -1$, then $S = \{r_+, (a_3 - 1)r\}$.

- $*$ If $a_1 + a_2 - 2a_3 \neq -1$, then $S = \{y_r, (a_3 - 1)r, (a_1 - 1)u\}$.

a5) If $a_1 - a_2 + 2a_3 = 2$, then S is either $\{m_1 r, m_2 u, d\}$ or $\{2r_+, m_1 r\}$.
- $*$ If $a_1 + a_2 - 2a_3 = -2$, then $S = \{2r_+, (a_3 - 2)r\}$.
- $*$ If $a_1 + a_2 - 2a_3 \neq -2$, then $S = \{(a_3 - 1)r, (a_1 - 1)u, d\}$.

a6) If $a_1 - a_2 + 2a_3 = 3$, then S is either $\{r_+, m_1 r, d\}$ or $\{y_r, d, m_1 r, m_2 u\}$.
- $*$ If $a_1 + a_2 - 2a_3 = -1$, then $S = \{r_+, (a_3 - 2)r, d\}$.
- $*$ If $a_1 + a_2 - 2a_3 \neq -1$, then $S = \{y_r, d, (a_3 - 2)r, (a_1 - 2)u\}$.

a7) If $a_1 - a_2 + 2a_3 \in 2\,\mathbb{Z}_{\geq 2}$, then S is either $\{m_1 r, m_2 u, m_3 d\}$ or $\{2r_+, m_1 r, m_3 d\}$.
- $*$ If $a_1 + a_2 - 2a_3 = -2$, then $S = \{2r_+, (a_3 - a_1 - 2)r, a_1 d\}$.
- $*$ If $a_1 + a_2 - 2a_3 \neq -2$, then $S = \left\{\frac{1}{2}(a_2 - a_1)r, \left(\frac{1}{2}(a_1 + a_2) - a_3\right)u, \left(\frac{1}{2}(a_1 - a_2) + a_3\right)d\right\}$.

a8) If $a_1 - a_2 + 2a_3 \in 2\,\mathbb{Z}_{\geq 2} + 1$, then S is one of $\{r_+, m_1 r, m_3 d\}$, $\{y_r, m_1 r, m_3 d\}$ and $\{y_r, u, m_1 r, m_3 d\}$.
- $*$ If $a_1 + a_2 - 2a_3 = -1$, then $S = \{r_+, (a_3 - a_1 - 1)r, a_1 d\}$.
- $*$ If $a_1 + a_2 - 2a_3 = 1$, then $S = \{y_r, (a_3 - a_1)r, (a_1 - 1)d\}$.
- $*$ If $a_1 + a_2 - 2a_3 = 3$, then $S = \{y_r, u, (a_3 - a_1 + 1)r, (a_1 - 2)d\}$.

b) Suppose that $a_4 - a_3 = 1$. In this case, S is one as in Table 12.

S	$a_1 - a_2 + 2a_3$	$a_1 + a_2 - 2a_3$
$\{r_p, m_1 r, m_2 u, m_3 d\}$	$-1 + 2m_3$	$1 + 2m_2$
$\{r_p, r_+, m_1 r, m_3 d\}$	$2m_3$	0
$\{r_p, r_-, m_1 r, m_2 u\}$	-2	$2 + 2m_2$
$\{r_p, y_r, m_1 r, m_3 d\}$	$2m_3$	2
$\{r_p, y_r, u, m_1 r, m_3 d\}$	$2m_3$	4
$\{r_p, y_r, m_1 r, m_2 u\}$	0	$2 + 2m_2$
$\{r_p, y_r, d, m_1 r, m_2 u\}$	2	$4 + 2m_2$
$\{P_+\}$	0	0
$\{\overline{P_+}\}$	0	2
$\{\overline{P_+}, d\}$	2	2

TABLE 12. All cases of S for a 1-puncture piece and $a_4 - a_3 = 1$

b1) If $a_1 - a_2 + 2a_3 \in 2\,\mathbb{Z} + 1$, then $S = \left\{r_p, \frac{1}{2}(a_2 - a_1 - 1)r, \left(\frac{1}{2}(a_1 + a_2 - 1) - a_3\right)u, \left(\frac{1}{2}(a_1 - a_2 + 1) + a_3\right)d\right\}$.

b2) If $a_1 - a_2 + 2a_3 = -2$, then $S = \{r_p, r_-, a_3 r, a_1 u\}$.

b3) If $a_1 - a_2 + 2a_3 = 0$ and $a_1 + a_2 - 2a_3 = 0$, then S is either $\{r_p, r_+, m_1 r\}$ or $\{P_+\}$.
- $*$ If $a_3 = 0$, then $S = \{P_+\}$.
- $*$ If $a_3 \neq 0$, then $S = \{r_p, r_+, (a_3 - 1)r\}$.

b4) If $a_1 - a_2 + 2a_3 = 0$ and $a_1 + a_2 - 2a_3 = 2$, then S is either $\{r_p, y_r, m_1 r\}$ or $\{\overline{P_+}\}$.
- $*$ If $a_3 = 0$, then $S = \{\overline{P_+}\}$.
- $*$ If $a_3 \neq 0$, then $S = \{r_p, y_r, (a_3 - 1)r\}$.

b5) If $a_1 - a_2 + 2a_3 = 0$ and $a_1 + a_2 - 2a_3 \geq 4$, then $S = \{r_p, y_r, (a_3 - 1)r, (a_1 - 1)u\}$.

b6) If $a_1 - a_2 + 2a_3 = 2$ and $a_1 + a_2 - 2a_3 = 0$, then $S = \{r_p, r_+, (a_3 - 2)r, d\}$.

b7) If $a_1 - a_2 + 2a_3 = 2$ and $a_1 + a_2 - 2a_3 = 2$, then S is either $\{r_p, y_r, m_1 r, d\}$ or $\{\overline{P_+}, d\}$.

- $*$ If $a_3 = 1$, then $S = \{\overline{P_+}, d\}$.
- $*$ If $a_3 \neq 1$, then $S = \{r_p, y_r, (a_3 - 2)r, d\}$.

b8) If $a_1 - a_2 + 2a_3 = 2$ and $a_1 + a_2 - 2a_3 \geq 4$, then $S = \{r_p, y_r, d, (a_3 - 2)r, (a_1 - 2)u\}$.
b9) If $a_1 - a_2 + 2a_3 \in 2\,\mathbb{Z}_{\geq 2}$ and $a_1 + a_2 - 2a_3 = 0$, then $S = \{r_p, r_+, (a_3 - a_1 - 1)r, a_1 d\}$.
b10) If $a_1 - a_2 + 2a_3 \in 2\,\mathbb{Z}_{\geq 2}$ and $a_1 + a_2 - 2a_3 = 2$, then $S = \{r_p, y_r, (a_3 - a_1)r, (a_1 - 1)d\}$.
b11) If $a_1 - a_2 + 2a_3 \in 2\,\mathbb{Z}_{\geq 2}$ and $a_1 + a_2 - 2a_3 = 4$, then $S = \{r_p, y_r, u, (a_3 - a_1 + 1)r, (a_1 - 2)d\}$.

c) Suppose that $a_4 - a_3 = 2$. In this case, S is either $\{2r_p, m_1 r, m_2 u, m_3 d\}$ or $\{r_p, l_p, m_2 u, m_3 d\}$.
c1) If $a_1 = a_2$, then $S = \{r_p, l_p, (a_1 - a_3 - 1)u, a_3 d\}$.
c2) If $a_1 \neq a_2$, then $S = \left\{2r_p, \frac{1}{2}(a_2 - a_1 - 2)r, \left(\frac{1}{2}(a_1 + a_2) - a_3 - 1\right)u, \left(\frac{1}{2}(a_1 - a_2) + a_3 + 1\right)d\right\}$.

By Table 9, these cover all cases of (a_1, a_2, a_3, a_4). Therefore, $v|_\square$ gives the unique set of modified segments $S_\square$.

Finally, we consider the case that $\square$ is a 2-puncture piece. That is, $v|_\square = (a_1, a_2, a_3, a_4, a_5)$. By symmetry, we can assume that $a_3 - a_2 \leq a_5 - a_4$. We consider the division into cases as in Table 13.

$a_1 = 0$				a)
$a_1 \neq 0$	a_1 : odd	$a_5 - a_4 = 0$		b)
		$a_5 - a_4 \neq 0$		c)
	a_1 : even	$a_5 - a_4 = 2$		d)
		$a_5 - a_4 = 1$	$a_3 - a_2 = 0$	e)
			$a_3 - a_2 = 1$	f)
		$a_5 - a_4 = 0$		g)

a) $\{H\}$, $\{R_n\}$
b) $\{c_n, m_1 s_{n-1}, m_2 s_n\}$, $\{\overline{c_n}, m_1 s_{n-1}, m_2 s_n\}$, $\{\overline{c_n}, m_1 s_{n-1}, m_2 s_n, h\}$
c) $\{r_n, m_1 s_{2n-2}, m_2 s_{2n-1}, m_3 h\}$ $(m_1 \neq 0)$, $\{r_n, m_1 s_{2n-1}, m_2 s_{2n}, m_3 h\}$
d) $\{2r_n, m_1 s_{2n-2}, m_2 s_{2n-1}, m_3 h\}$ $(m_1 \neq 0)$,
$\{2r_n, m_1 s_{2n-1}, m_2 s_{2n}, m_3 h\}$, $\{r_n, r_{n+1}, m_1 s_{2n}, m_3 h\}$
e) $\{\overline{R_n}\}$ $(n \neq 0)$, $\{r_n, c_{i+1}, m_1 s_i, m_2 s_{i+1}\}$, $\{r_n, \overline{c_{i+1}}, m_1 s_i, m_2 s_{i+1}\}$ $(i = 2n - 2$ or $2n - 1)$
f) $\{r_n, l_{n-1}, m_1 s_{2n-2}, m_2 s_{2n-1}, m_3 h\}$, $\{r_n, l_n, m_1 s_{2n-1}, m_2 s_{2n}, m_3 h\}$
g) $\{2c_n, m_1 s_{n-1}, m_2 s_n\}$, $\{m_1 s_{n-1}, m_2 s_n, m_3 h\}$ (If $n > 0$, $m_2 \neq 0$. If $n < 0$, $m_1 \neq 0$.)

Table 13. Division into cases

a) Suppose that $a_1 = 0$. If $a_2 = 0$, then $S = \{H\}$. If $a_2 \neq 0$, then $S = \{R_{a_2}\}$.

b) Suppose that $a_1 \neq 0$ is odd and $a_5 - a_4 = 0$. In this case, S is one of the followings:
$$\{c_n, m_1 s_{n-1}, m_2 s_n\}, \{\overline{c_n}, m_1 s_{n-1}, m_2 s_n\}, \{\overline{c_n}, m_1 s_{n-1}, m_2 s_n, h\}.$$
Set
$$m = \frac{a_1 - 1}{2}.$$
b1) If $|a_2 - a_4| > m$, then $S = \{c_n, m_1 s_{n-1}, m_2 s_n\}$ for $n \neq 0$.
b1i) If $a_2 - a_4 > m$, then $n > 0$. In this case,
$$n + mn \leq a_2 \leq n + m(n + 1),$$
thus
$$\frac{a_2 - m}{m + 1} \leq n \leq \frac{a_2}{m + 1}.$$

Since
$$\frac{a_2}{m+1} - \frac{a_2 - m}{m+1} = \frac{m}{m+1} < 1,$$
then n is uniquely given as
$$\left\lfloor \frac{a_2}{m+1} \right\rfloor,$$
where $\lfloor x \rfloor := \max\{n \in \mathbb{Z} \mid n \leq x\}$. We have $m_2 = a_2 - (m+1)n$ and $m_1 = m - m_2$, that is,

$$S = \left\{ c_n, \left(\frac{n+1}{2} a_1 - a_2 + \frac{n-1}{2} \right) s_{n-1}, \left(-\frac{n}{2} a_1 + a_2 - \frac{n}{2} \right) s_n \right\}, n = \left\lfloor \frac{2a_2}{a_1+1} \right\rfloor.$$

b1ii) If $a_2 - a_4 < -m$, then $n < 0$. In the same way as b1i), we obtain
$$n = \left\lfloor \frac{-a_2 - 1}{m+1} \right\rfloor, m_1 = a_2 + (m+1)n + m + 1 \text{ and } m_2 = m - m_1,$$
that is,
$$S = \left\{ c_n, \left(\frac{n+1}{2} a_1 + a_2 + \frac{n+1}{2} \right) s_{n-1}, \left(-\frac{n}{2} a_1 - a_2 - \frac{n+2}{2} \right) s_n \right\}, n = \left\lfloor \frac{-2(a_2+1)}{a_1+1} \right\rfloor.$$

b2) If $|a_2 - a_4| \leq m$, then $S = \{c_0, m_1 s_{-1}, m_2 s_0\}$, $\{\overline{c_n}, m_1 s_{n-1}, m_2 s_n\}$ or $\{\overline{c_n}, m_1 s_{n-1}, m_2 s_n, h\}$.

b2i) If $a_2 + a_4 \leq m$, then $S = \{c_0, a_4 s_{-1}, a_2 s_0\}$.

b2ii) If $a_2 + a_4 = m + 1$, then
$$S = \{\overline{c_0}, (m - a_2)s_{-1}, (m - a_4)s_0\} = \{\overline{c_0}, (\tfrac{1}{2}a_1 - a_2 - \tfrac{1}{2})s_{-1}, (\tfrac{1}{2}a_1 - a_4 - \tfrac{1}{2})s_0\}.$$

b2iii) If $a_2 + a_4 = m + 2$, then S is one of the followings:
$$\{\overline{c_1}, (a_2 - 2)s_0\}, \{\overline{c_{-1}}, (a_4 - 2)s_{-1}\}, \{\overline{c_0}, (m - a_2)s_{-1}, (m - a_4)s_0, h\}.$$
If $a_4 = 1$, then $S = \{\overline{c_1}, (a_2 - 2)s_0\}$. If $a_2 = 1$, then $S = \{\overline{c_{-1}}, (a_4 - 2)s_{-1}\}$. Otherwise,
$$S = \{\overline{c_0}, (m - a_2)s_{-1}, (m - a_4)s_0, h\} = \{\overline{c_0}, (\tfrac{1}{2}a_1 - a_2 - \tfrac{1}{2})s_{-1}, (\tfrac{1}{2}a_1 - a_4 - \tfrac{1}{2})s_0, h\}.$$

b2iv) If $a_2 + a_4 > m + 2$ and $|a_2 - a_4| = m$, then $S = \{\overline{c_n}, m_1 s_{n-1}, m_2 s_n\}$, $n \neq 0$ and it is not as in (b2iii). In the same way as b1i), if $a_2 - a_4 = m$, then we have
$$n = \left\lfloor \frac{2(a_2 - 1)}{a_1 - 1} \right\rfloor, m_2 = a_2 - mn - 1 \text{ and } m_1 = m - 1 - m_2,$$
that is,
$$S = \left\{ \overline{c_n}, \left(\frac{n+1}{2} a_1 - a_2 - \frac{n+1}{2} \right) s_{n-1}, \left(-\frac{n}{2} a_1 + a_2 + \frac{n-2}{2} \right) s_n \right\}, n = \left\lfloor \frac{2(a_2-1)}{a_1-1} \right\rfloor.$$
If $a_2 - a_4 = -m$, then we have
$$n = \left\lfloor -\frac{2a_2}{a_1 - 1} \right\rfloor, m_1 = a_2 + mn + m - 1 \text{ and } m_2 = m - 1 - m_1,$$
that is,
$$S = \left\{ \overline{c_n}, \left(\frac{n+1}{2} a_1 + a_2 - \frac{n+3}{2} \right) s_{n-1}, \left(-\frac{n}{2} a_1 - a_2 + \frac{n}{2} \right) s_n \right\}, n = \left\lfloor -\frac{2a_2}{a_1-1} \right\rfloor.$$

b2v) If $a_2 + a_4 > m + 2$ and $|a_2 - a_4| = m - 1$, then $S = \{\overline{c_n}, m_1 s_{n-1}, m_2 s_n, h\}$, $n \neq 0$ and it is not as in (b2iii). In the same way as b1i), if $a_2 - a_4 = m - 1$, then we have
$$n = \left\lfloor \frac{2(a_2 - 2)}{a_1 - 3} \right\rfloor, m_2 = a_2 - (m-1)n - 2 \text{ and } m_1 = m - 2 - m_2,$$
that is,
$$S = \left\{ \overline{c_n}, \left(\frac{n+1}{2} a_1 - a_2 - \frac{3n+1}{2} \right) s_{n-1}, \left(-\frac{n}{2} a_1 + a_2 + \frac{3n-4}{2} \right) s_n, h \right\}, n = \left\lfloor \frac{2(a_2-2)}{a_1-3} \right\rfloor.$$

If $a_2 - a_4 = -m+1$, then we have

$$n = \left\lfloor \frac{-2a_2+2}{a_1-3} \right\rfloor, m_1 = a_2 + (m-1)n + m - 3 \text{ and } m_2 = m-2-m_1,$$

that is,

$$S = \left\{ \overline{c_n}, \left(\frac{n+1}{2}a_1 + a_2 - \frac{3n+7}{2} \right) s_{n-1}, \left(-\frac{n}{2}a_1 - a_2 + \frac{3n+2}{2} \right) s_n, h \right\}, n = \left\lfloor \frac{-2a_2+2}{a_1-3} \right\rfloor.$$

c) Suppose that $a_1 \neq 0$ is odd and $a_5 - a_4 \neq 0$. In this case, S is one of the followings:

$$\{r_n, m_1 s_{2n-2}, m_2 s_{2n-1}, m_3 h\}, \{r_n, m_1 s_{2n-1}, m_2 s_{2n}, m_3 h\}.$$

Set

$$m = \frac{a_1 - 1}{2}.$$

c1) If $a_2 \leq m$, then $S = \{r_0, m_1 s_i, m_2 s_{i+1}, m_3 h\}$ for $i = 2n-2$ or $2n-1$. If $a_4 > m$, then

$$S = \left\{ r_0, \left(-\frac{1}{2}a_1 + a_4 + \frac{1}{2} \right) s_{-2}, \left(\frac{1}{2}a_1 - a_2 - \frac{1}{2} \right) s_{-1}, \left(\frac{1}{2}a_1 + a_2 - a_4 - \frac{1}{2} \right) h \right\}.$$

If $a_4 \leq m$, then

$$S = \left\{ r_0, \left(\frac{1}{2}a_1 - a_2 - \frac{1}{2} \right) s_{-2}, \left(\frac{1}{2}a_1 - a_4 - \frac{1}{2} \right) s_{-1}, \left(-\frac{1}{2}a_1 + a_2 + a_4 + \frac{1}{2} \right) h \right\}.$$

c2) If $a_2 > m$, then $S = \{r_n, m_1 s_i, m_2 s_{i+1}, m_3 h\}$ for $n \neq 0$ and $i = 2n-2$ or $2n-1$.

c2i) If $a_2 - a_4 > 0$, then $n > 0$ and $m_3 = m - (a_2 - a_4 - 1)$. In this case,

$$n + (m - m_3)(2n-1) + m_3 \leq a_2 \leq n + (m - m_3)(2n+1) + m_3,$$

thus

$$\frac{a_2 - m}{2m - 2m_3 + 1} \leq n \leq \frac{a_2 + m - 2m_3}{2m - 2m_3 + 1}.$$

Since

$$\frac{a_2 + m - 2m_3}{2m - 2m_3 + 1} - \frac{a_2 - m}{2m - 2m_3 + 1} = \frac{2m - 2m_3}{2m - 2m_3 + 1} < 1,$$

then n is uniquely given as

$$\left\lfloor \frac{a_2 + m - 2m_3}{2m - 2m_3 + 1} \right\rfloor.$$

Let $f = a_2 - (n + (m - m_3)(2n-1) + m_3)$. If $0 \leq f < m - m_3$, then

$$\begin{aligned} S &= \{r_n, (m - m_3 - f)s_{2n-2}, f s_{2n-1}, m_3 h\} \\ &= \Big\{ r_n, \left(\tfrac{1}{2}a_1 + (2n-2)a_2 - (2n-1)a_4 - \tfrac{2n-1}{2} \right) s_{2n-2}, \\ &\quad \left(-\tfrac{1}{2}a_1 + (-2n+3)a_2 + (2n-2)a_4 + \tfrac{2n-3}{2} \right) s_{2n-1}, \left(\tfrac{1}{2}a_1 - a_2 + a_4 + \tfrac{1}{2} \right) h \Big\}, \end{aligned}$$

if $m - m_3 \leq f \leq 2(m - m_3)$, then

$$\begin{aligned} S &= \{r_n, (2(m - m_3) - f)s_{2n-1}, (f - (m - m_3))s_{2n}, m_3 h\} \\ &= \Big\{ r_n, \left(\tfrac{1}{2}a_1 + (2n-1)a_2 - 2na_4 - \tfrac{2n+1}{2} \right) s_{2n-1}, \\ &\quad \left(-\tfrac{1}{2}a_1 - (2n-2)a_2 + (2n-1)a_4 + \tfrac{2n-1}{2} \right) s_{2n}, \left(\tfrac{1}{2}a_1 - a_2 + a_4 + \tfrac{1}{2} \right) h \Big\}, \end{aligned}$$

where

$$n = \left\lfloor \frac{-a_1 + 6a_2 - 4a_4 - 3}{2(2a_2 - 2a_4 - 1)} \right\rfloor.$$

c2ii) If $a_2 - a_4 \leq 0$, then $n < 0$ and $m_3 = m - (a_4 - a_2)$. In the same way as c2i), n is uniquely given as

$$\left\lfloor \frac{m - a_2}{2m - 2m_3 + 1} \right\rfloor.$$

Let $f = a_2 - (-n - (m - m_3)(2n+1) + m_3)$. If $0 \leq f \leq m - m_3$, then

$$\begin{aligned}S &= \{r_n, fs_{2n-1}, (m-m_3-f)s_{2n}, m_3h\}\\ &= \Big\{r_n, \Big(-\tfrac{1}{2}a_1 - (2n+1)a_2 + (2n+2)a_4 + \tfrac{2n+1}{2}\Big)s_{2n-1},\\ &\qquad \Big(\tfrac{1}{2}a_1 + 2na_2 - (2n+1)a_4 - \tfrac{2n+1}{2}\Big)s_{2n}, \Big(\tfrac{1}{2}a_1 + a_2 - a_4 - \tfrac{1}{2}\Big)h\Big\},\end{aligned}$$

if $m - m_3 < f \leq 2(m-m_3)$, then

$$\begin{aligned}S &= \{r_n, (f-(m-m_3))s_{2n-2}, (2(m-m_3)-f)s_{2n-1}, m_3h\}\\ &= \Big\{r_n, \Big(-\tfrac{1}{2}a_1 - 2na_2 + (2n+1)a_4 + \tfrac{2n+1}{2}\Big)s_{2n-2},\\ &\qquad \Big(\tfrac{1}{2}a_1 + (2n-1)a_2 - 2na_4 - \tfrac{2n+1}{2}\Big)s_{2n-1}, \Big(\tfrac{1}{2}a_1 + a_2 - a_4 - \tfrac{1}{2}\Big)h\Big\},\end{aligned}$$

where

$$n = \left\lfloor \frac{m-a_2}{2m-2m_3+1} \right\rfloor.$$

d) Suppose that $a_1 \neq 0$ is even and $a_5 - a_4 = 2$. In this case, S is one of the followings:

$$\{2r_n, m_1s_{2n-2}, m_2s_{2n-1}, m_3h\}, \{2r_n, m_1s_{2n-1}, m_2s_{2n}, m_3h\}, \{r_n, r_{n+1}, m_1s_{2n}, m_3h\}.$$

Set

$$m = \frac{a_1-2}{2}.$$

d1) Suppose that $a_2 = m+1$. If $a_2 \leq a_4$, then $S = \{r_{-1}, r_0, (a_4-a_2)s_{-2}, (m+a_2-a_4)h\}$. If $a_2 > a_4$, then $S = \{r_0, r_1, (a_2-a_4-1)s_0, a_4h\}$.

d2) Suppose that $a_2 < m+1$. If $a_4 > m$, then

$$\begin{aligned}S &= \{2r_0, (a_4-m)s_{-2}, (m-a_2)s_{-1}, (m+a_2-a_4)h\}\\ &= \{2r_0, (-\tfrac{1}{2}a_1 + a_4 + 1)s_{-2}, (\tfrac{1}{2}a_1 - a_2 - 1)s_{-1}, (\tfrac{1}{2}a_1 + a_2 - a_4 - 1)h\}.\end{aligned}$$

If $a_4 \leq m$, then

$$\begin{aligned}S &= \{2r_0, (m-a_2)s_{-1}, (m-a_4)s_0, (a_2+a_4-m)h\}\\ &= \{2r_0, (\tfrac{1}{2}a_1 - a_2 - 1)s_{-1}, (\tfrac{1}{2}a_1 - a_4 - 1)s_0, (-\tfrac{1}{2}a_1 + a_2 + a_4 + 1)h\}.\end{aligned}$$

d3) Suppose that $a_2 > m+1$.

d3i) If $a_2 - a_4 > 0$, then $n > 0$ and $m_3 = m - a_2 + a_4 + 2$. If $S = \{2r_n, m_1s_{2n-2}, m_2s_{2n-1}, m_3h\}$ or $\{2r_n, m_1s_{2n-1}, m_2s_{2n}, m_3h\}$, a_2 satisfy

$$2n + (m-m_3)(2n-1) + m_3 \leq a_2 \leq 2n + (m-m_3)(2n+1) + m_3$$

In particular, there is no $n' \in \mathbb{Z}$ such that $a_2 = 2n' + 1 + (m-m_3)(2n'+1) + m_3$. If there is such $n' \in \mathbb{Z}$, then

$$S = \{r_n, r_{n+1}, (a_2-a_4-2)s_{2n}, (m-a_2+a_4+2)h\},$$

where

$$n = n' = \frac{a_2-m-1}{2(m-m_3)+2}.$$

If not, in the same way as c2i), we have

$$n = \left\lfloor \frac{a_2+m-2m_3}{2(m-m_3)+2} \right\rfloor.$$

Set $f = a_2 - (2n + (m-m_3)(2n-1) + m_3)$. If $0 \leq f \leq m - m_3$, then

$$\begin{aligned}S &= \{2r_n, (m-m_3-f)s_{2n-2}, fs_{2n-1}, m_3h\}\\ &= \big\{2r_n, \big(\tfrac{1}{2}a_1 + (2n-2)a_2 - (2n-1)a_4 - 2n + 1\big)s_{2n-2},\\ &\qquad \big(-\tfrac{1}{2}a_1 - (2n-3)a_2 + (2n-2)a_4 + 2n - 3\big)s_{2n-1}, \big(\tfrac{1}{2}a_1 - a_2 + a_4 + 1\big)h\big\},\end{aligned}$$

if $m - m_3 < f \leq 2(m-m_3)$, then

$$\begin{aligned}S &= \{2r_n, (2(a_2-a_4-2)-f)s_{2n-1}, (f+a_2-a_4-2)s_{2n}, m_3h\}\\ &= \big\{2r_n, \big(\tfrac{1}{2}a_1+(2n-1)a_2-2na_4-2n-1\big)s_{2n-1},\\ &\quad \big(-\tfrac{1}{2}a_1-(2n-2)a_2+(2n-1)a_4+2n-1\big)s_{2n}, \big(\tfrac{1}{2}a_1-a_2+a_4+1\big)h\big\},\end{aligned}$$

where

$$n = \left\lfloor \frac{a_2+m-2m_3}{2(m-m_3)+2} \right\rfloor.$$

d3ii) If $a_2-a_4<0$, then $n<0$ and $m_3=m+a_2-a_4$. In the same way as d3i), if there is $n' \in \mathbb{Z}$ such that $a_2=-2n'-1+(m-m_3)(-2n'-1)+m_3$, then

$$S = \{r_n, r_{n+1}, (-a_2+a_4)s_{2n}, (m+a_2-a_4)h\},$$

where

$$n = n' = \left\lceil \frac{-m+2m_3-a_2+1}{2(m-m_3)+2} \right\rceil.$$

If not, we have

$$n = \left\lfloor \frac{m-a_2}{2(m-m_3)+2} \right\rfloor$$

and S is obtained as follows: Set $f=a_2-(-2n+(m-m_3)(-2n-1)+m_3)$. If $0 \le f \le m-m_3$, then

$$\begin{aligned}S &= \{2r_n, fs_{2n-1}, (m-m_3-f)s_{2n}, m_3h\}\\ &= \big\{2r_n, \big(-\tfrac{1}{2}a_1-(2n+1)a_2+(2n+2)a_4+2n+1\big)s_{2n-1},\\ &\quad \big(\tfrac{1}{2}a_1+2na_2-(2n+1)a_4-2n-1\big)s_{2n}, \big(\tfrac{1}{2}a_1+a_2-a_4-1\big)h\big\},\end{aligned}$$

if $m-m_3<f\le 2(m-m_3)$, then

$$\begin{aligned}S &= \{2r_n, (f-m+m_3)s_{2n-2}, (2(m-m_3)-f)s_{2n-1}, m_3h\}\\ &= \big\{2r_n, \big(-\tfrac{1}{2}a_1-2na_2+(2n+1)a_4+2n+1\big)s_{2n-2},\\ &\quad \big(\tfrac{1}{2}a_1+(2n-1)a_2-2na_4-2n-1\big)s_{2n-1}, \big(\tfrac{1}{2}a_1+a_2-a_4-1\big)h\big\},\end{aligned}$$

where

$$n = \left\lfloor \frac{m-a_2}{2(m-m_3)+2} \right\rfloor.$$

e) Suppose that $a_1 \neq 0$ is even, $a_5-a_4=1$ and $a_3-a_2=0$. In this case, S is one of the followings:

$$\{\overline{R_n}\}(n\neq 0), \{r_n, c_{i+1}, m_1s_i, m_2s_{i+1}\}, \{r_n, \overline{c_{i+1}}, m_1s_i, m_2s_{i+1}\}$$

for $i=2n-2$ or $2n-1$. Suppose that $a_1=2$, $a_2\neq 0$, $|a_2-a_4|\le 1$ and $|a_2-a_5|\le 1$. If $|a_2-a_4|=1$, then $S=\{\overline{R_{a_4}}\}$. If $|a_2-a_4|=0$, then $S=\{\overline{R_{-a_4}}\}$. Otherwise, $S=\{r_n, c_{i+1}, m_1s_i, m_2s_{i+1}\}$ or $\{r_n, \overline{c_{i+1}}, m_1s_i, m_2s_{i+1}\}$. Set

$$m = \frac{a_1-2}{2}.$$

e1) If $a_2 \le m$, then $n=0$.

e1i) Suppose that $a_4>m$. If $a_4-a_2=m+1$, then

$$S = \{r_0, c_{-1}, a_2s_{-2}, (\tfrac{1}{2}a_1-a_2-1)s_{-1}\}.$$

If not, then

$$S = \{r_0, \overline{c_{-1}}, (a_2-1)s_{-2}, (\tfrac{1}{2}a_1-a_2-1)s_{-1}\}.$$

e1ii) Suppose that $a_4 \le m$. If $a_2+a_4=m$, then

$$S = \{r_0, c_0, a_4s_{-1}, a_2s_0\}.$$

If not, then

$$S = \{r_0, \overline{c_0}, (a_4-1)s_{-1}, (a_2-1)s_0\}.$$

e2) Suppose that $a_2 > m$, then $n \neq 0$.

e2i) If $a_2 - a_4 > 0$, then $n > 0$. Suppose that $a_2 - a_4 = m + 2$. In the same way as b1i), then n is uniquely given as
$$\left\lfloor \frac{a_2 + m + 1}{2m + 3} \right\rfloor.$$
Set $f = a_2 - (3n - 1 + m(2n - 1))$. If $0 \leq f \leq m$, then
$$\begin{aligned} S &= \{r_n, c_{2n-1}, (m - f)s_{2n-2}, fs_{2n-1}\} \\ &= \big\{r_n, c_{2n-1}, (na_1 - a_2 + n - 1)s_{2n-2}, \big((-n + \tfrac{1}{2})a_1 + a_2 - n\big)s_{2n-1}\big\}, \end{aligned}$$
if $m + 1 \leq f \leq 2m + 1$, then
$$\begin{aligned} S &= \{r_n, c_{2n}, (2m + 1 - f)s_{2n-1}, (f - m - 1)s_{2n}\} \\ &= \big\{r_n, c_{2n}, \big((n + \tfrac{1}{2})a_1 - a_2 + n - 1\big)s_{2n-1}, (-na_1 + a_2 - n)s_{2n}\big\}, \end{aligned}$$
where
$$n = \left\lfloor \frac{a_1 + 2a_2}{2(a_1 + 1)} \right\rfloor.$$
Suppose that $a_2 - a_4 \neq m + 2$. Then n is uniquely given as
$$\left\lfloor \frac{a_2 + m - 1}{2m + 1} \right\rfloor.$$
Set $f = a_2 - (3n + (m - 1)(2n - 1))$. If $0 \leq f \leq m - 1$, then
$$\begin{aligned} S &= \{r_n, \overline{c_{2n-1}}, (m - 1 - f)s_{2n-2}, fs_{2n-1}\} \\ &= \big\{r_n, \overline{c_{2n-1}}, (na_1 - a_2 - n)s_{2n-2}, \big((-n + \tfrac{1}{2})a_1 + a_2 + n - 2\big)s_{2n-1}\big\}, \end{aligned}$$
if $m \leq f \leq 2m - 1$, then
$$\begin{aligned} S &= \{r_n, \overline{c_{2n}}, (2m - 1 - f)s_{2n-1}, (f - m)s_{2n}\} \\ &= \big\{r_n, \overline{c_{2n}}, \big((n + \tfrac{1}{2})a_1 - a_2 - n - 1\big)s_{2n-1}, (-na_1 + a_2 + n - 1)s_{2n}\big\}, \end{aligned}$$
where
$$n = \left\lfloor \frac{a_1 + 2a_2 - 4}{2(a_1 - 1)} \right\rfloor.$$

e2ii) If $a_2 - a_4 \leq 0$, then $n < 0$. Suppose that $a_2 - a_4 = -m - 1$. In the same way as above, then n is uniquely given as
$$\left\lfloor \frac{-a_2 + m}{2m + 3} \right\rfloor.$$
Set $f = a_2 - (-3n - 1 + m(-2n - 1))$. If $0 \leq f \leq m$, then
$$\begin{aligned} S &= \{r_n, c_{2n}, fs_{2n-1}, (m - f)s_{2n}\} \\ &= \big\{r_n, c_{2n}, \big((n + \tfrac{1}{2})a_1 + a_2 + n\big)s_{2n-1}, (-na_1 - a_2 - n - 1)s_{2n}\big\}, \end{aligned}$$
if $m + 1 \leq f \leq 2m + 1$, then
$$\begin{aligned} S &= \{r_n, c_{2n-1}, (f - m - 1)s_{2n-2}, (2m + 1 - f)s_{2n-1}\} \\ &= \big\{r_n, c_{2n-1}, (na_1 + a_2 + n)s_{2n-2}, \big(-(n + \tfrac{1}{2})a_1 - a_2 - n - 1\big)s_{2n-1}\big\}, \end{aligned}$$
where
$$n = \left\lfloor \frac{a_1 - 2a_2 - 2}{2(a_1 + 1)} \right\rfloor.$$
Suppose that $a_2 - a_4 \neq -m - 1$. Then n is uniquely given as
$$\left\lfloor \frac{-a_2 + m}{2m + 1} \right\rfloor.$$
Set $f = a_2 - (-3n + (m - 1)(-2n - 1))$. If $0 \leq f \leq m - 1$, then
$$\begin{aligned} S &= \{r_n, \overline{c_{2n}}, fs_{2n-1}, (m - 1 - f)s_{2n}\} \\ &= \big\{r_n, \overline{c_{2n}}, \big((n + \tfrac{1}{2})a_1 + a_2 - n - 2\big)s_{2n-1}, (-na_1 - a_2 + n)s_{2n}\big\}, \end{aligned}$$

if $m \le f \le 2m-1$, then

$$S = \{r_n, \overline{c_{2n-1}}, (f-m)s_{2n-2}, (2m-1-f)s_{2n-1}\}$$
$$= \big\{r_n, \overline{c_{2n-1}}, (na_1+a_2-n-1)s_{2n-1}, \big((-n+\tfrac{1}{2})a_1 - a_2 + n - 1\big)s_{2n}\big\},$$

where

$$n = \left\lfloor \frac{a_1 - 2a_2 - 2}{2(a_1-1)} \right\rfloor.$$

f) Suppose that $a_1 \neq 0$ is even, $a_5 - a_4 = 1$ and $a_3 - a_2 = 1$. In this case, S is one of the followings:

$$\{r_n, l_{n-1}, m_1 s_{2n-2}, m_2 s_{2n-1}, m_3 h\}, \{r_n, l_n, m_1 s_{2n-1}, m_2 s_{2n}, m_3 h\}.$$

Set

$$m = \frac{a_1 - 2}{2}.$$

In the same way as d), we construct S.

f1) If $a_2 \le m$, then $n = 0$. If $a_4 > m$, then

$$S = \{r_0, l_{-1}, (a_4 - m - 1)s_{-2}, (m - a_2)s_{-1}, (m + a_2 - a_4 + 1)h\}$$
$$= \{r_0, l_{-1}, (-\tfrac{1}{2}a_1 + a_4)s_{-2}, (\tfrac{1}{2}a_1 - a_2 - 1)s_{-1}, (\tfrac{1}{2}a_1 + a_2 - a_4)h\}.$$

If not, then

$$S = \{r_0, l_0, (m-a_2)s_{-1}, (m-a_4)s_0, (a_2 + a_4 - m)h\}$$
$$= \{r_0, l_0, (\tfrac{1}{2}a_1 - a_2 - 1)s_{-1}, (\tfrac{1}{2}a_1 - a_4 - 1)s_0, (-\tfrac{1}{2}a_1 + a_2 + a_4 + 1)h\}.$$

f2) If $a_2 > m$, then $n \neq 0$.

f2i) If $a_2 - a_4 > 0$, then $n > 0$ and $m_3 = m - (a_2 - a_4 - 1)$. Moreover, n is uniquely given as

$$\left\lfloor \frac{a_2 + m - 2m_3 + 1}{2(m - m_3 + 1)} \right\rfloor.$$

Let $f = a_2 - (2n - 1 + (m - m_3)(2n-1) + m_3)$. If $0 \le f \le m - m_3$, then

$$S = \{r_n, l_{n-1}, (m - m_3 - f)s_{2n-2}, f s_{2n-1}, m_3 h\}$$
$$= \big\{r_n, l_{n-1}, \big(\tfrac{1}{2}a_1 + (2n-2)a_2 - (2n-1)a_4 - 1\big)s_{2n-2},$$
$$\big(-\tfrac{1}{2}a_1 + (-2n+3)a_2 + (2n-2)a_4\big)s_{2n-1}, \big(\tfrac{1}{2}a_1 - a_2 + a_4\big)h\big\},$$

if $m - m_3 + 1 \le f \le 2(m - m_3) + 1$, then

$$S = \{r_n, l_n, (2(m-m_3) + 1 - f)s_{2n-1}, (f - m + m_3 - 1)s_{2n}, m_3 h\}$$
$$= \big\{r_n, l_n, \big(\tfrac{1}{2}a_1 + (2n-1)a_2 - 2na_4 - 1\big)s_{2n-1},$$
$$\big(-\tfrac{1}{2}a_1 + (-2n+2)a_2 + (2n-1)a_4\big)s_{2n}, \big(\tfrac{1}{2}a_1 - a_2 + a_4\big)h\big\},$$

where

$$n = \left\lfloor \frac{-a_1 + 6a_2 - 4a_4}{4(a_2 - a_4)} \right\rfloor.$$

f2ii) If $a_2 - a_4 \le 0$, then $n < 0$ and $m_3 = m + a_2 - a_4 + 1$. Moreover, n is uniquely given as

$$\left\lfloor \frac{m - a_2}{2(m - m_3 + 1)} \right\rfloor.$$

Let $f = a_2 - (-2n - 1 + (m - m_3)(-2n-1) + m_3)$. If $0 \le f \le m - m_3$, then

$$S = \{r_n, l_n, f s_{2n-1}, (m - m_3 - f)s_{2n}, m_3 h\}$$
$$= \big\{r_n, l_n, \big(-\tfrac{1}{2}a_1 - (2n+1)a_2 + (2n+2)a_4\big)s_{2n-1},$$
$$\big(\tfrac{1}{2}a_1 + 2na_2 - (2n+1)a_4 - 1\big)s_{2n}, \big(\tfrac{1}{2}a_1 + a_2 - a_4\big)h\big\},$$

if $m-m_3+1 \le f \le 2(m-m_3)+1$, then

$$\begin{aligned} S &= \{r_n, l_{n-1}, (f-m+m_3-1)s_{2n-2}, (2(m-m_3)+1-f)s_{2n-1}, m_3h\} \\ &= \big\{r_n, l_n, \big(-\tfrac{1}{2}a_1 - 2na_2 + (2n+1)a_4\big)s_{2n-2}, \\ &\quad \big(\tfrac{1}{2}a_1 + (2n-1)a_2 - 2na_4 - 1\big)s_{2n-1}, \big(\tfrac{1}{2}a_1 + a_2 - a_4\big)h\big\}, \end{aligned}$$

where

$$n = \left\lfloor \frac{a_1 - 2a_2 - 2}{4(a_2 - a_4)} \right\rfloor.$$

g) Suppose that $a_1 \neq 0$ is even, $a_5 - a_4 = 0$. In this case, S is one of the followings:

$$\{2c_n, m_1s_{n-1}, m_2s_n\}, \{m_1s_{n-1}, m_2s_n, m_3h\},$$

where for $S = \{m_1s_{n-1}, m_2s_n, m_3h\}$ we assume the following conditions: If $n > 0$, $m_2 \neq 0$; If $n < 0$, $m_1 \neq 0$. Note that if $S = \{m_3h\}$, we have $n = 0$. Set $m = a_1/2$.

g1) Suppose that $|a_2 - a_4| > m$. Then $S = \{2c_n, m_1s_{n-1}, m_2s_n\}$ and $n \neq 0$. In the same way as b1i), if $a_2 - a_4 > m$, then

$$\begin{aligned} S &= \{2c_n, ((m+1)n + m - a_2 - 1)s_{n-1}, (a_2 - (m+1)n)s_n\} \\ &= \left\{2c_n, \left(\tfrac{n+1}{2}a_1 - a_2 + n - 1\right)s_{n-1}, \left(-\tfrac{n}{2}a_1 + a_2 - n\right)s_n\right\}, \end{aligned}$$

where

$$n = \left\lfloor \frac{a_2}{m+1} \right\rfloor = \left\lfloor \frac{2a_2}{a_1+2} \right\rfloor.$$

If $a_2 - a_4 < -m$, then

$$\begin{aligned} S &= \{2c_n, (a_2 + (m+1)(n+1))s_{n-1}, (-a_2 - (m+1)n - 2)s_n\} \\ &= \left\{2c_n, \left(\tfrac{n+1}{2}a_1 + a_2 + n + 1\right)s_{n-1}, \left(-\tfrac{n}{2}a_1 - a_2 - n - 2\right)s_n\right\}, \end{aligned}$$

where

$$n = \left\lfloor \frac{-a_2 - 2}{m+1} \right\rfloor = \left\lfloor \frac{-2(a_2+2)}{a_1+2} \right\rfloor.$$

g2) Suppose that $|a_2 - a_4| \le m$. If $a_2 + a_4 < m$, then

$$S = \{2c_0, a_4s_{-1}, a_2s_0\}.$$

Suppose that $a_2 + a_4 \ge m$.

g2i) If $a_2, a_4 \le m$, then

$$S = \{(\tfrac{1}{2}a_1 - a_2)s_{-1}, (\tfrac{1}{2}a_1 - a_4)s_0, (-\tfrac{1}{2}a_1 + a_2 + a_4)h\}.$$

g2ii) If either $a_2 > m$ or $a_4 > m$ holds, then $S = \{m_1s_{n-1}, m_2s_n, m_3h\}$ for $n \neq 0$. Since S is not $\{m_3h\}$ by our assumptions, $a_2 \neq a_4$. In the same way as b1i), if $a_2 - a_4 > 0$, then

$$\begin{aligned} S &= \left\{ \begin{matrix} ((a_2 - a_4)(n+1) + m_3 - a_2)s_{n-1}, (a_2 - ((a_2 - a_4)n + m_3))s_n, \\ (m - a_2 + a_4)h \end{matrix} \right\} \\ &= \left\{ \begin{matrix} (\tfrac{1}{2}a_1 + (n+1)a_2 - na_4)s_{n-1}, (-\tfrac{1}{2}a_1 + (-n+2)a_2 + (n-1)a_4)s_n, \\ (\tfrac{1}{2}a_1 - a_2 + a_4)h \end{matrix} \right\} \end{aligned}$$

where

$$n = \left\lfloor \frac{2a_2 - a_4 - m}{a_2 - a_4} \right\rfloor = \left\lfloor \frac{-a_1 + 4a_2 - 2a_4}{2(a_2 - a_4)} \right\rfloor.$$

If $a_2 - a_4 < 0$, then

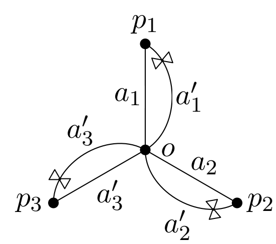
$$S = \left\{ \begin{array}{c} (a_2 - ((-a_2 + a_4)(-n-1) + m_3))s_{n-1}, ((-a_2 + a_4)(-n) + m_3 - a_2)s_n, \\ (m + a_2 - a_4)h \end{array} \right\}$$
$$= \left\{ \begin{array}{c} (-\frac{1}{2}a_1 - (n+1)a_2 + (n+2)a_4)s_{n-1}, (\frac{1}{2}a_1 + na_2 - (n+1)a_4)s_n, \\ (\frac{1}{2}a_1 + a_2 - a_4)h \end{array} \right\}$$

where

$$n = \left\lfloor \frac{m - a_4}{-a_2 + a_4} \right\rfloor = \left\lfloor \frac{a_1 - 2a_4}{2(-a_2 + a_4)} \right\rfloor.$$

By Table 10, these cover all cases of $(a_1, a_2, a_3, a_4, a_5)$. Therefore, $v|_\square$ gives the unique set of modified segments $S_\square$. This finishes the proof for $T \neq T_3$.

To finish the proof of Theorem 14.4, we assume that $T = T_3$ (which consists of three pairs (τ_i, τ_i') of conjugate arcs for $i \in \{1, 2, 3\}$ as in Figure 1). For a modified tagged arc m, we set $\mathsf{Int}(T_3, \mathsf{m}) = (a_1, a_1', a_2, a_2', a_3, a_3') \in \mathbb{Z}^n_{\geq 0}$, where a_i (resp., a_i') is the intersection number of τ_i (resp., τ_i') and m as follows:



We show that m is uniquely determined by $\mathsf{Int}(T_3, \mathsf{m})$. By symmetry, we can assume that $a_i' - a_i \geq 0$ for $i \in \{1, 2, 3\}$. Let ℓ_i be a loop at o cutting out a monogon with exactly one puncture p_i as in Figure 22. Note that ℓ_i is not a tagged arc, but we can define the intersection number $\mathsf{Int}(\ell_i, \mathsf{m})$

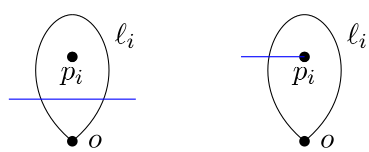
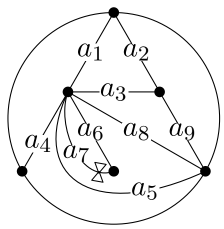


FIGURE 22. The loop ℓ_i corresponding to a_i' and two kinds of segments of m intersecting with ℓ_i

of ℓ_i and m. It is clear that the number of intersection points of m and τ_i coincides with one of m and τ_i' except at p_i. Thus $a_i' - a_i$ is the number of the end points of m at p_i. Since m only intersects with ℓ_i in two ways as in Figure 22, the set of segments of m in the monogon enclosed by ℓ_i consists of a_i segments in the left diagram of Figure 22 and $a_i' - a_i$ segments in the right diagram of Figure 22, in particular, is uniquely determined. Furthermore, by this observation, we have $\mathsf{Int}(\ell_i, \mathsf{m}) = 2a_i + (a_i' - a_i)$. The set of segments of m in the triangle consisting of ℓ_1, ℓ_2 and ℓ_3 is unique determined in the same way as a triangle piece which is Case(τ_i,τ_{i+1}). Gluing their segments simultaneously, we can obtain m. This finishes the proof of Theorem 14.4.

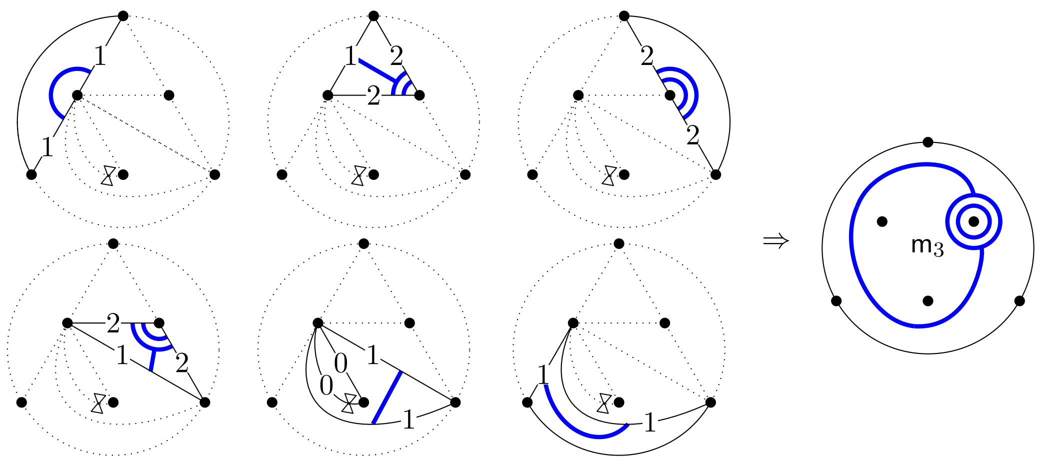
19. Example

Let T be the tagged triangulation in Example 14.1. We consider the following intersection vectors:

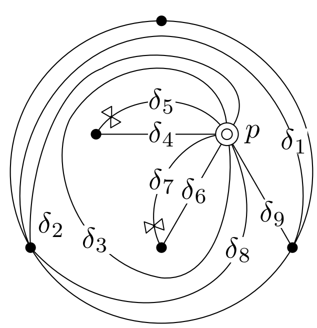


$$v_1 = (1,1,0,0,0,0,0,0,0)$$
$$v_2 = (1,1,1,0,0,0,0,0,1)$$
$$v_3 = (1,2,2,1,1,0,0,1,2)$$
$$v_4 = (0,1,0,0,0,0,0,0,1)$$
$$v_5 = (1,1,2,1,1,1,1,1,1)$$
$$v_6 = (0,1,1,0,0,0,1,1,1)$$
$$v_7 = (0,1,1,0,0,1,0,1,1)$$
$$v_8 = (0,1,1,0,1,0,0,1,1)$$
$$v_9 = (0,1,1,0,0,0,0,0,0)$$

By Theorem 14.4, each v_i give a unique modified tagged arc m_i with respect to T. In fact, for instance, m_3 is given in the way of the previous section as follows:



Similarly, we can obtain all m_i. Then there is a unique tagged arc δ_i such that $\mathsf{M}_T(\delta_i) = \mathsf{m}_i$ for $i \in \{1, \dots, 4, 6, \dots, 9\}$. Finally, there is a unique tagged arc δ_5 such that $\mathsf{M}_T(\delta_5) = \mathsf{m}_5$ and $\{\delta_1, \dots, \delta_9\}$ is a tagged triangulation as follows:



where all ends around the puncture p are tagged notched.

Part 5. Density of g-vector cones from triangulated surfaces

20. Introduction

This part is based on the paper [Y19b]. Let Q be a quiver without loops and 2-cycles, and let $\mathcal{A}(\hat{Q})$ be the associated cluster algebra with principal coefficients (see Section 1). We denote by $\operatorname{cluster} Q$ the set of clusters in $\mathcal{A}(\hat{Q})$. Each cluster variable x in $\mathcal{A}(\hat{Q})$ has a numerical invariant $g_Q(x)$, called the g-vector of x [FZ07]. For each $\mathbf{x} \in \operatorname{cluster} Q$, one can define a cone

$$C_Q(\mathbf{x}) := \left\{ \sum_{x \in \mathbf{x}} a_x g_Q(x) \mid a_x \in \mathbb{R}_{\geq 0} \right\}$$

in $\mathbb{R}^n$, called the g-vector cone of $\mathbf{x}$. Note that these cones and their faces form a fan [R14a, Theorem 8.7]. We say that Q is finite type if $\operatorname{cluster} Q < \infty$. The following result is well-known.

Theorem 20.1. [R14a, Theorem 10.6] *If a quiver Q is finite type, then we have*

$$\bigcup_{\mathbf{x} \in \operatorname{cluster} Q} C_Q(\mathbf{x}) = \mathbb{R}^n .$$

In this part, we study an analogue of Theorem 20.1 for cluster algebras defined from marked surfaces. Let (S, M) be a marked surface and T a tagged triangulation of (S, M) (see Subsection 2). We denote by $|T|$ the number of tagged arcs of T. In $\mathcal{A}(T)$, cluster variables correspond to tagged arcs, and clusters correspond to tagged triangulations (Theorem 3.1). Our first aim is to give the following analogue of Theorem 20.1.

Theorem 20.2. *If (S, M) is a closed surface with exactly one puncture, then we have*

$$\overline{\bigcup_{\mathbf{x} \in \operatorname{cluster} Q_T} C_{Q_T}(\mathbf{x})} = \overline{\bigcup_{\mathbf{x} \in \operatorname{cluster} Q_T^{\mathrm{op}}} C_{Q_T^{\mathrm{op}}}(\mathbf{x})} = \Big\{ (a_\delta)_{\delta \in T} \in \mathbb{R}^{|T|} \mid \sum_{\delta \in T} a_\delta \geq 0 \Big\},$$

where $\overline{(-)}$ is the closure with respect to the natural topology on $\mathbb{R}^{|T|}$. Otherwise, we have

$$\overline{\bigcup_{\mathbf{x} \in \operatorname{cluster} Q_T} C_{Q_T}(\mathbf{x})} = \mathbb{R}^{|T|} .$$

The second aim of this part is to apply Theorem 20.2 to representation theory. Let W be a non-degenerate potential of Q_T such that the associated Jacobian algebra $J = J(Q_T, W)$ is finite dimensional [DWZ]. The existence of such a potential W follows from [Lab09, Lab16, Lad, TV]. Using the Ginzburg differential graded algebra $\Gamma = \Gamma_{Q_T, W}$ associated with (Q_T, W) [G], Amiot [A] constructed a generalized cluster category $\mathcal{C} = \mathcal{C}_{Q_T, W}$ with cluster tilting object Γ. The g-vector of each rigid object (resp., τ-rigid pair) in $\mathcal{C}$ (resp., $\mathsf{mod}\, J$) is a certain element in the Grothendieck group $K_0(\mathsf{add}\, \Gamma)$ (resp., $K_0(\mathsf{proj}\, J)$). The g-vectors of indecomposable direct summands of a cluster tilting object X (resp., a τ-tilting pair (M, P)) form a cone $C_\Gamma(X)$ in $K_0(\mathsf{add}\, \Gamma) \otimes_{\mathbb{Z}} \mathbb{R}$ (resp., $C_J(M, P)$ in $K_0(\mathsf{proj}\, J) \otimes_{\mathbb{Z}} \mathbb{R}$), called the g-vector cone of X (resp., (M, P)). Note that these g-vector cones and their faces form a fan [DIJ]. Such a fan plays an important role in the study of scattering diagrams and their wall-chamber structures (see e.g. [B, BST, GHKK, GS, KS, Y18]).

We denote by $\text{c-tilt}\, \mathcal{C}$ (resp., $\text{s}\tau\text{-tilt}\, J$) the set of isomorphism classes of basic cluster tilting objects in $\mathcal{C}$ (resp., τ-tilting pairs in $\mathsf{mod}\, J$). We also denote by $\text{c-tilt}^+ \mathcal{C}$ (resp., $\text{c-tilt}^- \mathcal{C}$, $\text{s}\tau\text{-tilt}^+ J$, $\text{s}\tau\text{-tilt}^- J$) the subset of $\text{c-tilt}\, \mathcal{C}$ (resp., $\text{c-tilt}\, \mathcal{C}$, $\text{s}\tau\text{-tilt}\, J$, $\text{s}\tau\text{-tilt}\, J$) consisting of mutation equivalence classes containing Γ (resp., $\Gamma[1]$, $(J, 0)$, $(0, J)$). We set

$$\text{c-tilt}^\pm \mathcal{C} := \text{c-tilt}^+ \mathcal{C} \cup \text{c-tilt}^- \mathcal{C} \ \text{ and } \ \text{s}\tau\text{-tilt}^\pm J := \text{s}\tau\text{-tilt}^+ J \cup \text{s}\tau\text{-tilt}^- J.$$

The following analogues of Theorem 20.2 hold.

Theorem 20.3. *Let T be a tagged triangulation of a marked surface (S,M). For a non-degenerate potential W of Q_T such that $J = J(Q_T,W)$ is finite dimensional, let $\mathcal{C} = \mathcal{C}_{Q_T,W}$ and $\Gamma = \Gamma_{Q_T,W}$. Then we have the equalities*

$$\overline{\bigcup_{U\in \text{c-tilt}^{\pm}\mathcal{C}} C_{\Gamma}(U)} = K_0(\mathsf{add}\,\Gamma)\otimes_{\mathbb{Z}}\mathbb{R} \quad \textit{and} \quad \overline{\bigcup_{(M,P)\in \text{s}\tau\text{-tilt}^{\pm} J} C_J(M,P)} = K_0(\mathsf{proj}\,J)\otimes_{\mathbb{Z}}\mathbb{R}\,.$$

This theorem means that g-vector cones are dense in the scattering diagram of J. It gives the following application.

Corollary 20.4. *Any basic cluster tilting object in $\mathcal{C}$ (resp., τ-tilting pair in $\mathsf{mod}\,J$) is contained in* c-tilt$^{\pm}\,\mathcal{C}$ *(resp.,* sτ-tilt$^{\pm}\,J$*). In particular, if (S,M) is a closed surface with exactly one puncture, then the exchange graph of* c-tilt $\mathcal{C}$ *(resp.,* sτ-tilt J*) has precisely two connected components* c-tilt$^{+}\,\mathcal{C}$ *and* c-tilt$^{-}\,\mathcal{C}$ *(resp.,* sτ-tilt$^{+}\,J$ *and* sτ-tilt$^{-}\,J$*). Otherwise, it is connected, thus* c-tilt $\mathcal{C}$ = c-tilt$^{+}\,\mathcal{C}$ = c-tilt$^{-}\,\mathcal{C}$ *(resp.,* sτ-tilt J = sτ-tilt$^{+}\,J$ = sτ-tilt$^{-}\,J$*).*

Note that Corollary 20.4 was given by Qiu and Zhou [QZ] for marked surfaces with non-empty boundary. But our proof is entirely different from theirs since a key point of their proof is to show that $J(Q_T,W)$ is skewed-gentle for an admissible triangulation T of (S,M) and a certain potential W of Q_T.

To prove Theorem 20.2, our main ingredient is shear coordinates on (S,M). To study coefficients in cluster algebras defined from T, Fomin and Thurston [FoT] used a certain class of curves in S, called laminates, and finite multi-sets of pairwise non-intersecting laminates, called laminations (see also [FoG07, T]). To a laminate ℓ of (S,M), they associated an integer vector $b_T(\ell)\in\mathbb{Z}^{|T|}$ whose entries are shear coordinates of ℓ and defined $b_T(L) := \sum_{\ell\in L} b_T(\ell)\in\mathbb{Z}^{|T|}$ for a lamination L on (S,M). They showed that the map $L\mapsto b_T(L)$ induces a bijection between the set of laminations on (S,M) and $\mathbb{Z}^{|T|}$.

For a multi-set L of laminates of (S,M), in the same way as g-vector cones, we can define a cone $C_T(L)$ in $\mathbb{R}^{|T|}$, called the shear coordinate cone of L with respect to T. Recall that there is a natural injective map e from the set of tagged arcs of (S,M) to the set of laminates of (S,M) (see Subsection 21.1). We denote by $\mathbb{T}$ the set of tagged triangulations of (S,M). The following result plays an important role to prove Theorem 20.2.

Theorem 20.5. *Let T be a tagged triangulation of a marked surface (S,M). Then we have*

$$\overline{\bigcup_{T'\in\mathbb{T}} C_T(\mathsf{e}(T'))} = \mathbb{R}^{|T|}\,.$$

If (S,M) is a closed surface with exactly one puncture p, then we have

$$\overline{\bigcup_{T'\in\mathbb{T}^+} C_T(\mathsf{e}(T'))} = \overline{\bigcup_{T'\in\mathbb{T}^-} (-C_T(\mathsf{e}(T')))} = \Big\{(a_\delta)_{\delta\in T}\in\mathbb{R}^{|T|} \mid \sum_{\delta\in T} a_\delta \le 0\Big\},$$

where $\mathbb{T}^+$ (resp., $\mathbb{T}^-$) is the set of tagged triangulations of (S,M) tagged at p in the same (resp., different) way as T.

This part is organized as follows. In Section 21, we recall the notions of laminations, and their shear coordinates. We study shear coordinates of laminates and their asymptotic behavior under Dehn twists, and prove Theorem 20.5. In Section 22, we show that the shear coordinate of a laminate with respect to T corresponds with the g-vector of a cluster variable in $\mathcal{A}(\hat{Q})$ or $\mathcal{A}(\hat{Q^{\rm op}})$. Consequently, Theorem 20.2 follows from Theorem 20.5. In Section 23, we recall τ-tilting theory, cluster tilting theory, and the relationships between them and cluster algebras. Finally, we prove Theorem 20.3 and Corollary 20.4.

21. Density of shear coordinate cones from triangulated surfaces

21.1. Laminations on marked surfaces. We recall the notions of [FoT]. A *laminate* of (S, M) is a non-self-intersecting curve in S, considered up to isotopy relative to M, which is either

- a closed curve, or
- a curve whose ends are unmarked points on ∂S or spirals around punctures (either clockwise or counterclockwise),

and the following curves are not allowed (see Figure 23):

- a curve cutting out a disk with at most one puncture;
- a curve with two endpoints on ∂S such that it is isotopic to a piece of ∂S containing at most one marked point;
- a curve whose both ends are spirals around a common puncture in the same direction such that it does not enclose anything else.

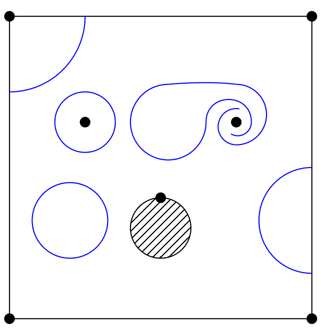


Figure 23. Curves which are not laminates

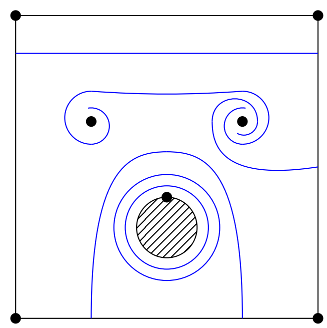


Figure 24. A lamination on an annulus with 2 punctures

Definition 21.1. We say that two laminates of (S, M) are *compatible* if they don't intersect. A finite multi-set of pairwise compatible laminates of (S, M) is called a *lamination* on (S, M) (see Figures 24).

Let ℓ be a laminate of (S, M). For an ideal/tagged triangulation T of (S, M), we define the *shear coordinate* $b_{\gamma,T}(\ell)$ *of* ℓ with respect to $\gamma \in T$ (see [FoT, Definition 12.2, 13.1]).

First, we assume that T is an ideal triangulation. If $\gamma \in T$ is not inside a self-folded triangle of T, then $b_{\gamma,T}(\ell)$ is defined by a sum of contributions from all intersections of γ and ℓ as follows: Such an intersection contributes $+1$ (resp., -1) to $b_{\gamma,T}(\ell)$ if a segment of ℓ cuts through the quadrilateral surrounding γ as in the left (resp., right) diagram of Figure 25. Suppose that $\gamma \in T$ is inside a

Figure 25. The contribution from a segment of the laminate ℓ on the left (resp., right) is $+1$ (resp., -1)

self-folded triangle $\{\gamma, \gamma'\}$ of T, where γ' is a loop enclosing exactly one puncture p. Then we define $b_{\gamma,T}(\ell) = b_{\gamma',T}(\ell^{(p)})$, where $\ell^{(p)}$ is a laminate obtained from ℓ by changing the directions of its spirals at p if they exist.

Next, we assume that T is a tagged triangulation. If there is an ideal triangulation T^0 satisfying $T = \iota(T^0)$, then we define $b_{\gamma,T}(\ell) = b_{\gamma^0,T^0}(\ell)$, where $\gamma = \iota(\gamma^0)$. For an arbitrary T, we can obtain a tagged triangulation $T^{(p_1 \cdots p_m)}$ from T by simultaneous changing all tags at punctures $p_1, \dots, p_m$ (possibly $m = 0$), in such a way that there is a unique ideal triangulation T^0 satisfying $T^{(p_1 \cdots p_m)} = \iota(T^0)$ (see [MSW11, Remark 3.11]). Then we define

$$b_{\gamma,T}(\ell) = b_{\gamma^{(p_1 \cdots p_m)},T^{(p_1 \cdots p_m)}}\big((\cdots((\ell^{(p_1)})^{(p_2)})\cdots)^{(p_m)}\big),$$

where $\gamma^{(p_1 \cdots p_m)}$ corresponds to γ.

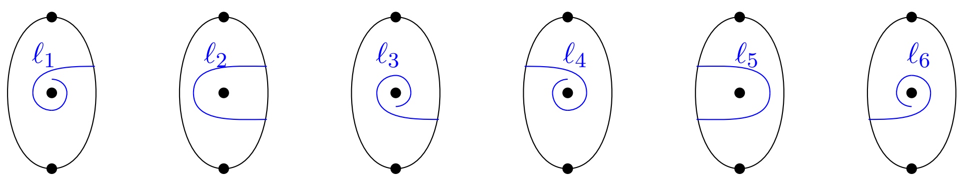
For a multi-set $L = L' \sqcup \{\ell\}$ of laminates of (S, M), the *shear coordinate* $b_{\gamma,T}(L)$ *of* L with respect to $\gamma \in T$ is inductively defined by

$$b_{\gamma,T}(L) = b_{\gamma,T}(L') + b_{\gamma,T}(\ell).$$

We denote by $b_T(L)$ a vector $(b_{\gamma,T}(L))_{\gamma \in T} \in \mathbb{Z}^{|T|}$. Note that the shear coordinate cone $C_T(L)$ is a cone spanned by $b_T(\ell)$ for $\ell \in L$. These vectors have the following property.

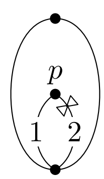
Theorem 21.2. [FoT, Theorems 12.3, 13.6] *Let T be a tagged triangulation of (S, M). The map sending laminations L to $b_T(L)$ induces a bijection between the set of laminations on (S, M) and $\mathbb{Z}^{|T|}$.*

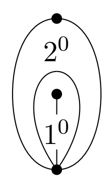
Example 21.3. For a digon (S, M) with exactly one puncture, all laminates are given as follows:



.

We consider the following tagged triangulation T:

$T =$ 

, where $T^0 =$ 

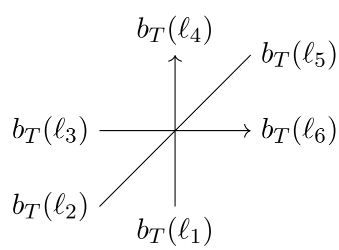
.

The shear coordinate $b_{2,T}(\ell_1)$ is given by $b_{2^0,T^0}(\ell_1) = -1$. Since $\ell_3^{(p)} = \ell_1$, we have the equalities

$$b_{1,T}(\ell_3) = b_{1^0,T^0}(\ell_3) = b_{2^0,T^0}(\ell_3^{(p)}) = b_{2^0,T^0}(\ell_1) = -1.$$

Similarly, for $i \in \{1, 2\}$ and $j \in \{1, \dots, 6\}$, the shear coordinates $b_{i,T}(\ell_j)$ and $b_T(\ell_j)$ are given as follows:

i \ j	1	2	3	4	5	6
1	0	-1	-1	0	1	1
2	-1	-1	0	1	1	0



In particular, we have

$$\bigcup_{j=1}^{6} C_T(\{\ell_j, \ell_{j+1}\}) = \mathbb{R}^2,$$

where $\ell_7 = \ell_1$. On the other hand, all laminations on (S, M) are given by $\{m\ell_j, n\ell_{j+1}\}$ for $j \in \{1, \dots, 6\}$ and $m, n \in \mathbb{Z}_{\geq 0}$. Since $C_T(\{\ell_{j-1}, \ell_j\}) \cap C_T(\{\ell_j, \ell_{j+1}\}) = C_T(\{\ell_j\})$ and b_T induces a bijection

$$b_T : \{\{m\ell_j, n\ell_{j+1}\} \mid m, n \in \mathbb{Z}_{\geq 0}\} \longleftrightarrow C_T(\{\ell_j, \ell_{j+1}\}) \cap \mathbb{Z}^2,$$

there is a bijection between the set of laminations on (S, M) and $\mathbb{Z}^2$.

21.2. **Elementary and exceptional laminates.** Non-closed laminates of (S, M) are divided into two types, elementary and exceptional. For a tagged arc δ of (S, M), we define an *elementary laminate* $\mathsf{e}(\delta)$ as follows:

- $\mathsf{e}(\delta)$ is a laminate running along δ in a small neighborhood of it;
- If δ has an endpoint o on a component C of ∂S, then the corresponding endpoint of $\mathsf{e}(\delta)$ is located near o on C in the clockwise direction as in the left diagram of Figure 26;
- If δ has an endpoint at a puncture p, then the corresponding end of $\mathsf{e}(\delta)$ is a spiral around p clockwise (resp., counterclockwise) if δ is tagged plain (resp., notched) at p as in the right diagram of Figure 26.

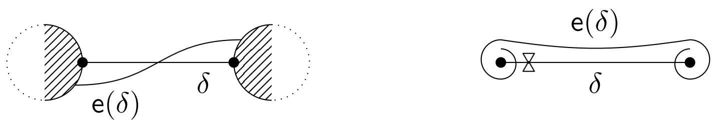


FIGURE 26. Elementary laminates of tagged arcs

It follows from the construction that the map e from the set of tagged arcs of (S, M) to the set of laminates is injective. For an elementary laminate ℓ, we denote by $\mathsf{e}^{-1}(\ell)$ a unique tagged arc δ such that $\mathsf{e}(\delta) = \ell$. Note that, for a tagged arc δ, a lamination $\{\mathsf{e}(\delta)\}$ is a reflection of the elementary lamination of δ defined in [FoT, Definition 17.2]. Our convention is more convenient for our aim.

Elementary laminates have the following properties.

Proposition 21.4. *(1) Let δ and δ' be tagged arcs such that $\delta^\circ \neq \delta'^\circ$. Then δ and δ' are compatible if and only if $\mathsf{e}(\delta)$ and $\mathsf{e}(\delta')$ are compatible.*

(2) The map e induces a bijection between the set of partial tagged triangulations of (S, M) without pairs of conjugate arcs and the set of laminations of (S, M) consisting only of distinct elementary laminates.

Proof. (1) Since e transforms δ and δ' just around the marked points of (S, M), it is enough to consider neighborhoods of their endpoints. In particular, if δ and δ' have no common endpoints, the assertion holds. Suppose that δ and δ' have at least one common endpoint. Since $\delta^\circ \neq \delta'^\circ$, (δ, δ') is not a pair of conjugate arcs. Thus δ and δ' are compatible if and only if the ends of δ and δ' at each common endpoint are tagged in the same way. By the definition of e, it is equivalent that $\mathsf{e}(\delta)$ and $\mathsf{e}(\delta')$ are compatible.

(2) If two distinct tagged arcs δ and δ' satisfying $\delta^\circ = \delta'^\circ$ are compatible, then (δ, δ') is a pair of conjugate arcs, in which case $\mathsf{e}(\delta)$ and $\mathsf{e}(\delta')$ are not compatible. Therefore, the assertion follows from (1). □

Laminates which are neither closed nor elementary are called *exceptional*. They are characterized as follows.

Proposition 21.5. *A laminate is exceptional if and only if it is one of the following curves (Figure 27):*

- *a curve enclosing exactly one puncture whose both endpoints lie on a common boundary segment;*

- *a curve enclosing exactly one puncture whose both ends are spirals around a common puncture in the same direction.*

Proof. Applying the same transformation as e^{-1} to a non-closed laminate ℓ and forgetting its tags, we obtain a unique ideal arc. In general, an ideal arc γ is not obtained from a tagged arc by forgetting its tags if and only if γ is a loop cutting out a monogon with exactly one puncture. Therefore, ℓ is not elementary if and only if it is one of the desired cases. □

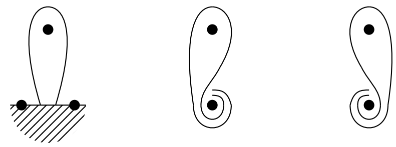
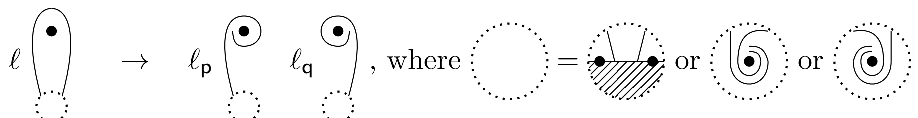


FIGURE 27. Exceptional laminates

Note that exceptional laminates coincide with excluded curves for quasi-laminations in [R14b]. To interpret shear coordinates of exceptional laminates as ones of elementary laminates, we introduce the following notations. For an exceptional laminate ℓ of (S, M), elementary laminates ℓ_{p} and ℓ_{q} are given by



In particular, $(\mathsf{e}^{-1}(\ell_{\mathsf{p}}), \mathsf{e}^{-1}(\ell_{\mathsf{q}}))$ is a pair of conjugate arcs. For a lamination L on (S, M), we denote by L_{pq} the multi-set of elementary laminates obtained from L by replacing exceptional laminates $\ell \in L$ with ℓ_{p} and ℓ_{q}.

Example 21.6. In Example 21.3, ℓ_2 and ℓ_5 are exceptional, and $(\ell_2)_{\mathsf{p}} = \ell_3$, $(\ell_2)_{\mathsf{q}} = \ell_1$, $(\ell_5)_{\mathsf{p}} = \ell_4$ and $(\ell_5)_{\mathsf{q}} = \ell_6$. Thus we have the equalities

$$b_T(\ell_2) = b_T(\{(\ell_2)_{\mathsf{p}}, (\ell_2)_{\mathsf{q}}\}) \text{ and } b_T(\ell_5) = b_T(\{(\ell_5)_{\mathsf{p}}, (\ell_5)_{\mathsf{q}}\}).$$

In general, the same property as Example 21.6 holds for arbitrary exceptional laminates. We denote by $\gamma \cap \delta$ the set of their intersection points.

Lemma 21.7. *Let T be a tagged triangulation of (S, M). For an exceptional laminate ℓ of (S, M), we have*

$$b_T(\ell) = b_T(\{\ell_{\mathsf{p}}, \ell_{\mathsf{q}}\}).$$

Proof. By Proposition 21.5, there is a unique puncture p enclosed by ℓ. We only need to prove

$$b_{\delta,T}(\ell) = b_{\delta,T}(\{\ell_{\mathsf{p}}, \ell_{\mathsf{q}}\}) \tag{21.1}$$

for any $\delta \in T$. If $\delta \in T$ is not incident to p, then (21.1) is clear. We assume that δ is incident to p. Let $\delta_1, \dots, \delta_m$ be tagged arcs of T incident to p winding clockwisely around p such that the following conditions are satisfied (see Figure 28):

- ℓ crosses them at points $p_1, \dots, p_m$ in this order;
- The segment of δ_i from p to p_i, that of δ_{i+1} from p to p_{i+1}, and that of ℓ from p_i to p_{i+1} form a contractible triangle.

Note that if these arcs contains a pair (δ, ϵ) of conjugate arcs, then we can choice the order of δ and ϵ.
Moreover, δ_1 is different from δ_m. Indeed, if $\delta_1 = \delta_m$, considering triangles with a side δ_1, there is a tagged arc of T incident to p such that ℓ crosses it before p_1 or after p_m, a contradiction.

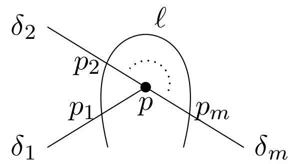


FIGURE 28. Local configuration around a puncture p

The contributions to $b_{\delta_i,T}(\ell)$ at $\delta_i \cap \ell$ except at p_i coincide with the contributions to $b_{\delta_i,T}(\ell_{\mathsf{p}})$ and $b_{\delta_i,T}(\ell_{\mathsf{q}})$ at them. We denote by $c_i(\ell)$ (resp, $c_i(\ell_{\mathsf{p}})$, $c_i(\ell_{\mathsf{q}})$) the contribution to $b_{\delta_i,T}(\ell)$ (resp., $b_{\delta_i,T}(\ell_{\mathsf{p}})$, $b_{\delta_i,T}(\ell_{\mathsf{q}})$) at p_i for $i \in \{1,\dots,m\}$. To prove (21.1), we only need to show

$$c_i(\ell) = c_i(\ell_{\mathsf{p}}) + c_i(\ell_{\mathsf{q}}). \tag{21.2}$$

First, we assume that neither (δ_1, δ_2) nor (δ_{m-1}, δ_m) form a pair of conjugate arcs. Then it is easy to give the following values:

$$c_i(\ell) = \begin{cases} -1 & \text{if } i = 1, \\ 1 & \text{if } i = m, \\ 0 & \text{if } i \neq 1, m, \end{cases} \qquad c_i(\ell_{\mathsf{p}}) = \begin{cases} -1 & \text{if } i = 1, \\ 0 & \text{if } i \neq 1, \end{cases} \qquad c_i(\ell_{\mathsf{q}}) = \begin{cases} 1 & \text{if } i = m, \\ 0 & \text{if } i \neq m. \end{cases}$$

Therefore, (21.2) holds.

Second, we assume that (δ_1, δ_2) is a pair of conjugate arcs tagged in the different ways at p, in which case $m = 2$. Then by exchanging δ_1 and δ_2 if necessary (see Figure 29), we have

$$c_i(\ell_{\mathsf{p}}) = \begin{cases} c_1(\ell) & \text{if } i = 1, \\ 0 & \text{if } i = 2, \end{cases} \qquad c_i(\ell_{\mathsf{q}}) = \begin{cases} 0 & \text{if } i = 1, \\ c_2(\ell) & \text{if } i = 2. \end{cases} \tag{21.3}$$

Therefore, (21.2) holds.

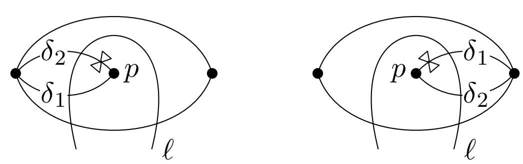


FIGURE 29. Pairs (δ_1, δ_2) of conjugate arcs tagged in the different ways at p such that (21.3) holds

Finally, we assume that (δ_1, δ_2) is a pair of conjugate arcs tagged in the same way at p. We define a set $\mathbb{M}$ as follows: If (δ_{m-1}, δ_m) is a pair of conjugate arcs, then $\mathbb{M} = \{m-1, m\}$; Otherwise, $\mathbb{M} = \{m\}$. Then we have $c_i(\ell_{\mathsf{p}}) = 0 = c_j(\ell_{\mathsf{q}})$ for $i \notin \{1,2\}$ and $j \notin \mathbb{M}$, and

$$c_i(\ell) = \begin{cases} c_i(\ell_{\mathsf{p}}) & \text{if } i \in \{1,2\}, \\ c_i(\ell_{\mathsf{q}}) & \text{if } i \in \mathbb{M}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, (21.2) holds. Moreover, it follows from the symmetry for the case that (δ_{m-1}, δ_m) is a pair of conjugate arcs tagged in the same way at p. Consequently, (21.1) holds for any $\delta \in T$. $\square$

For a lamination L on (S, M), we have decompositions

$$L = L_{\mathrm{el}} \sqcup L_{\mathrm{ex}} \sqcup L_{\mathrm{cl}} = L_{\mathrm{nc}} \sqcup L_{\mathrm{cl}}, \tag{21.4}$$

where L_{el} (resp., L_{ex}, L_{cl}) consists of all elementary (resp., exceptional, closed) laminates in L. For multi-sets L and L' of laminates of (S,M), we define non-multi-sets

$$\mathsf{e}^{-1}(L) := \{\delta\ :\ \text{a tagged arc} \mid \mathsf{e}(\delta) \in L\} \quad \text{and} \quad L \setminus L' := \{\ell \in L \mid \ell \notin L'\}.$$

The following properties are used to prove Theorem 20.5 in Subsection 21.4.

Proposition 21.8. *Let L be a lamination on (S,M) with $L_{\mathrm{cl}} = \emptyset$. Then the following properties hold:*

(1) *$C_T(L) \subseteq C_T(L_{\mathsf{pq}})$.*
(2) *$\mathsf{e}^{-1}(L_{\mathsf{pq}})$ is a partial tagged triangulation of (S,M).*

Moreover, we take a set U of tagged arcs of (S,M) such that $T' = \mathsf{e}^{-1}(L_{\mathsf{pq}}) \sqcup U$ is a tagged triangulation. Then we have the equality

(3) *$C_T(\mathsf{e}\,T') = C_T(L_{\mathsf{pq}} \sqcup \mathsf{e}\,U)$.*

Proof. (1) The assertion immediately follows from Lemma 21.7.

(2) By Proposition 21.4(2), $\mathsf{e}^{-1}(L_{\mathrm{el}})$ is a partial tagged triangulation of (S,M). Since L is a lamination, any laminate in $(L_{\mathrm{ex}})_{\mathsf{pq}}$ is compatible with all laminates in $L_{\mathrm{el}} \setminus (L_{\mathrm{ex}})_{\mathsf{pq}}$. Then, by Proposition 21.4(1), any tagged arc of $\mathsf{e}^{-1}((L_{\mathrm{ex}})_{\mathsf{pq}})$ is compatible with all tagged arcs of $\mathsf{e}^{-1}(L_{\mathrm{el}} \setminus (L_{\mathrm{ex}})_{\mathsf{pq}})$. Moreover, $\mathsf{e}^{-1}((L_{\mathrm{ex}})_{\mathsf{pq}})$ is a partial tagged triangulation since $(\mathsf{e}^{-1}(\ell_{\mathsf{p}}), \mathsf{e}^{-1}(\ell_{\mathsf{q}}))$ is a pair of conjugate arcs for $\ell \in L_{\mathrm{ex}}$. Therefore,

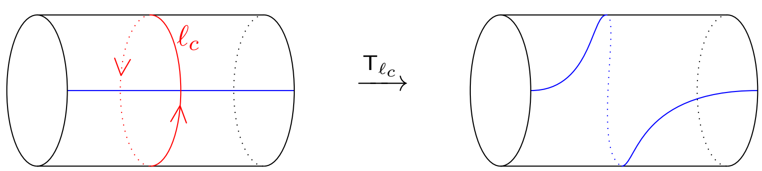
$$\mathsf{e}^{-1}(L_{\mathsf{pq}}) = \mathsf{e}^{-1}(L_{\mathrm{el}} \setminus (L_{\mathrm{ex}})_{\mathsf{pq}}) \sqcup \mathsf{e}^{-1}((L_{\mathrm{ex}})_{\mathsf{pq}})$$

is a partial tagged triangulation of (S,M).

(3) Since L_{pq} coincides with the multiplicity of $\mathsf{e}\,\mathsf{e}^{-1}(L_{\mathsf{pq}})$, we have the equalities

$$C_T(L_{\mathsf{pq}} \sqcup \mathsf{e}\,U) = C_T(\mathsf{e}\,\mathsf{e}^{-1}(L_{\mathsf{pq}}) \sqcup \mathsf{e}\,U) = C_T(\mathsf{e}\,T'). \qquad \square$$

21.3. **Shear coordinates and Dehn twists.** We consider the Dehn twist along a closed laminate and its effect on shear coordinates. In this subsection, we fix an ideal or tagged triangulation T, a closed laminate ℓ_c of (S,M) and its direction. We denote by T_{ℓ_c} the Dehn twist of (S,M) along ℓ_c defined from the direction of ℓ_c as follows:

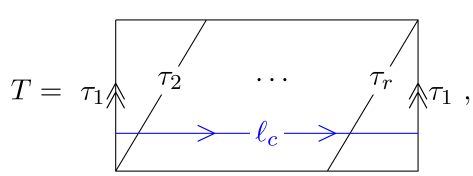


The aim of this subsection is to prove the following.

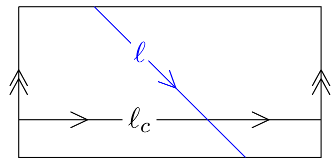
Theorem 21.9. *Let ℓ_c be a closed laminate and ℓ a laminate of (S,M) intersecting with ℓ_c, and let $\delta \in T$. Then there is $m' \in \mathbb{Z}_{\geq 0}$ such that for any $m \geq m'$, we have*

$$b_{\delta,T}(\mathsf{T}_{\ell_c}^{m}(\ell)) = b_{\delta,T}(\mathsf{T}_{\ell_c}^{m'}(\ell)) + (m - m')\#(\ell \cap \ell_c) b_{\delta,T}(\ell_c).$$

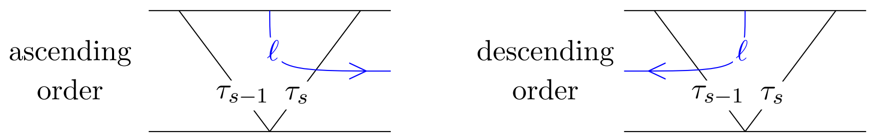
First, we assume that (S,M) is an annulus without punctures and T is its ideal triangulation consisting of arcs $\tau_1, \ldots, \tau_r$ crossing ℓ_c in order of occurrence along ℓ_c (we can have $\tau_i = \tau_j$ even if $i \neq j$), that is,



where two vertical lines τ_1 are identified. Any elementary laminate ℓ of (S,M) intersects with ℓ_c at most once since they intersect in a minimal number of points. We assume that ℓ intersects with ℓ_c. We define the direction of ℓ as crossing ℓ_c from left to right:

$T =$ 

Let $s \in \{1,\dots,r\}$ such that the starting point of ℓ is on the triangle of T with sides τ_{s-1} and τ_s, where $\tau_{r+i} = \tau_i$. In particular, ℓ intersects at least one of τ_{s-1} and τ_s. Thus ℓ intersects with the t_ℓ $(\in \mathbb{Z}_{\geq 1})$ diagonals either $\tau_s, \tau_{s+1}, \dots, \tau_{s+t_\ell-1}$ or $\tau_{s-1}, \tau_{s-2}, \dots, \tau_{s-t_\ell}$ of T in order. In the former (resp., latter) case, we say that ℓ *intersects with T in ascending* (resp., *descending*) *order*:



Proposition 21.10. *Let (S,M) be an annulus without punctures and ℓ an elementary laminate of (S,M) intersecting with ℓ_c.*

(1) *If ℓ intersects with T in ascending order, then so is $\mathsf{T}_{\ell_c}(\ell)$ and for $\delta \in T$, we have*

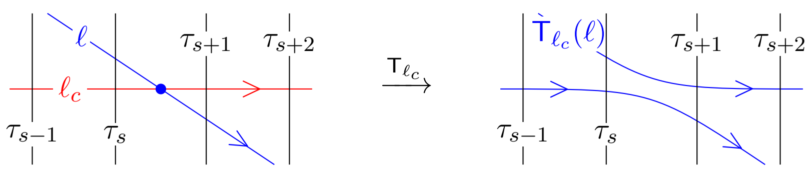
$$b_{\delta,T}(\mathsf{T}_{\ell_c}(\ell)) = b_{\delta,T}(\ell) + b_{\delta,T}(\ell_c).$$

If ℓ and $\mathsf{T}_{\ell_c}(\ell)$ intersect with T in descending order, then for $\delta \in T$, we have

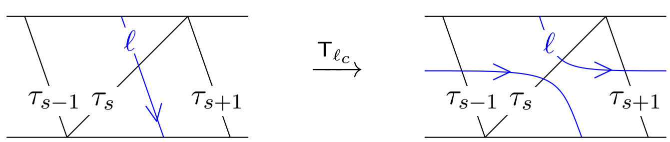
$$b_{\delta,T}(\mathsf{T}_{\ell_c}(\ell)) = b_{\delta,T}(\ell) - b_{\delta,T}(\ell_c).$$

(2) *There is $m \gg 0$ such that $\mathsf{T}^m_{\ell_c}(\ell)$ intersects with T in ascending order.*

Proof. (1) We only prove the first assertion since the proof of the second assertion is similar. Suppose that ℓ intersects with T in ascending order. If $t_\ell > 1$, then the assertion holds since T_{ℓ_c} only transforms ℓ around an intersection point of ℓ and ℓ_c as follows:



If $t_\ell = 1$, then ℓ and $\mathsf{T}_{\ell_c}(\ell)$ are given as follows:

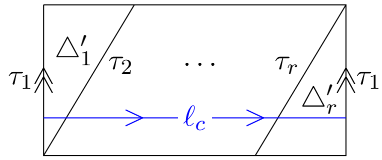


Then the assertion is directly given by enumerating their shear coordinates.

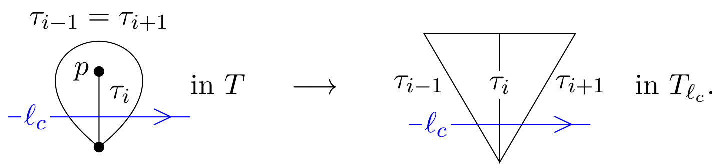
(2) Suppose that ℓ intersects with T in descending order. If $t_\ell < r$, then $\mathsf{T}_{\ell_c}(\ell)$ intersects with T in ascending order. If $t_\ell \geq r$, then $\mathsf{T}_{\ell_c}(\ell)$ intersects with T in descending order and $t_{\mathsf{T}_{\ell_c}(\ell)} = t_\ell - r$. By the induction, the assertion holds. $\square$

Next, we consider an arbitrary marked surface (S,M) and its ideal triangulation T. For ℓ_c in Theorem 21.9, we construct an annulus (S_{ℓ_c}, M_{ℓ_c}) and its triangulation T_{ℓ_c} as follows: Let $\tau_1, \dots, \tau_r$ be the arcs of T crossing ℓ_c in order of occurrence along ℓ_c (τ_i and τ_j can be the same even if $i \neq j$). Hence ℓ_c crosses r triangles $\triangle_1, \dots, \triangle_r$ in this order. For $i \in \{1,\dots,r\}$, let $\triangle'_i$ be a copy of the triangle $\triangle_i$, hence $\triangle'_i$ has the sides τ_i and τ_{i+1}. Then an annulus (S_{ℓ_c}, M_{ℓ_c}) and its triangulation T_{ℓ_c} are obtained by gluing $\triangle'_1, \dots, \triangle'_r$ along the edges τ_i, that is,

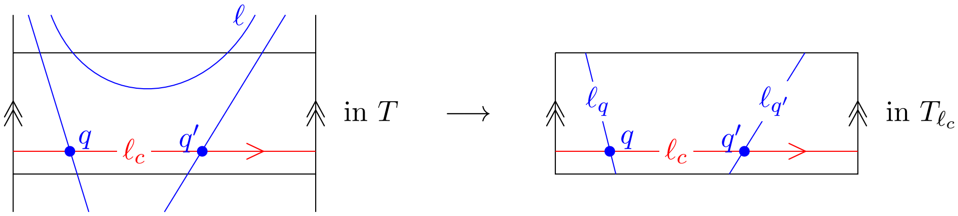
(21.5) $T_{\ell_c} =$



in (S_{ℓ_c}, M_{ℓ_c}), where two vertical lines τ_1 are identified. In particular, if τ_i is inside a self-folded triangle of T, then the corresponding triangles are given by



For a laminate ℓ of (S, M) intersecting with ℓ_c, let $q \in \ell \cap \ell_c$ and ℓ_q a laminate of (S_{ℓ_c}, M_{ℓ_c}) corresponding to the connected segment of ℓ in T_{ℓ_c} containing q as follows:



Proposition 21.11. *Let T be an ideal triangulation of (S, M), $\gamma \in T$, and ℓ a laminate of (S, M) intersecting with ℓ_c. We assume that for all $q \in \ell \cap \ell_c$, ℓ_q intersects with T_{ℓ_c} in ascending order. Then we have*

$$b_{\gamma,T}(\mathsf{T}_{\ell_c}(\ell)) = b_{\gamma,T}(\ell) + \#(\ell \cap \ell_c) b_{\gamma,T}(\ell_c).$$

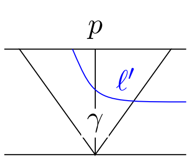
Proof. The proof is divided into the following three cases (1)–(3).

(1) If γ is not a side of triangles of T, then the assertion is clear.

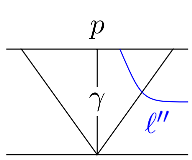
(2) We assume that γ is a side of some triangle of T and $\gamma \cap \ell_c \neq \emptyset$. If γ is not inside a self-folded triangle of T, then the construction of T_{ℓ_c} preserves the quadrilateral surrounding γ. Therefore, we have

$$\begin{aligned} b_{\gamma,T}(\mathsf{T}_{\ell_c}(\ell)) - b_{\gamma,T}(\ell) &= \sum_{q \in \ell \cap \ell_c} \sum_{1 \le i \le r, \tau_i = \gamma} \Big(b_{\tau_i, T_{\ell_c}}(\mathsf{T}_{\ell_c}(\ell_q)) - b_{\tau_i, T_{\ell_c}}(\ell_q) \Big) \\ &\overset{21.10(1)}{=} \sum_{q \in \ell \cap \ell_c} \sum_{1 \le i \le r, \tau_i = \gamma} b_{\tau_i, T_{\ell_c}}(\ell_c) = \#(\ell \cap \ell_c) b_{\gamma,T}(\ell_c). \end{aligned}$$

Suppose that γ is inside a self-folded triangle $\{\gamma, \gamma'\}$ of T enclosing a puncture p. Recall that $b_{\gamma,T}(\ell) = b_{\gamma',T}(\ell^{(p)})$, where $\ell^{(p)}$ is a laminate obtained from ℓ by changing the directions of its spirals at p if they exist (see Subsection 21.1). Thus the assertion follows from the previous case if for all $q \in \ell^{(p)} \cap \ell_c$, $(\ell^{(p)})_q$ intersects with T_{ℓ_c} in ascending order. This is checked as follows: If no ends of ℓ are spirals at p, then $\ell^{(p)} = \ell$, hence it is clear. Otherwise, $\ell \neq \ell^{(p)}$, and $\ell \cap \ell_c$ and $\ell^{(p)} \cap \ell_c$ are identified in the natural way. Then ℓ_q and $(\ell^{(p)})_q$ are only different in that their ends around p are given by



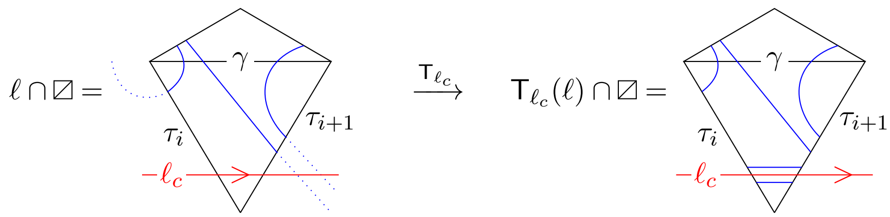
and



in T_{ℓ_c}, where $\{\ell', \ell''\} = \{\ell_q, (\ell^{(p)})_q\}$.

Therefore, $(\ell^{(p)})_q$ also intersects with T_{ℓ_c} in ascending order.

(3) We assume that γ is a side of some triangle $\triangle_i$ of T and $\gamma \cap \ell_c = \emptyset$. Then we prove $b_{\gamma,T}(\mathsf{T}_{\ell_c}(\ell)) = b_{\gamma,T}(\ell)$. In this case, γ is not inside a self-folded triangle of T. Indeed, if γ is inside a self-folded triangle of T, then γ is either τ_i or τ_{i+1}, hence it is a contradiction. Therefore, there is the quadrilateral $\boxslash$ surrounding γ of T. Since for all $q \in \ell \cap \ell_c$, ℓ_q intersects with T_{ℓ_c} in ascending order, the Dehn twist T_{ℓ_c} affects $\ell \cap \boxslash$ as follows:



Therefore, it gives $b_{\gamma,T}(\mathsf{T}_{\ell_c}(\ell)) = b_{\gamma,T}(\ell)$. □

We are ready to prove Theorem 21.9.

Proof of Theorem 21.9. First of all, we prove Theorem 21.9 for an ideal triangulation T. For $q \in \ell \cap \ell_c$, by Proposition 21.10(2) there exists $m_q \in \mathbb{Z}_{\geq 0}$ such that $\mathsf{T}_{\ell_c}^{m_q}(\ell_q)$ intersects with T_{ℓ_c} in ascending order. Thus for

$$m \geq m' := \max_{q \in \ell \cap \ell_c} \{m_q\},$$

$\mathsf{T}_{\ell_c}^{m}(\ell_q)$ intersects with T_{ℓ_c} in ascending order for each $q \in \ell \cap \ell_c$. Therefore, by Theorem 21.11, we have

$$\begin{aligned} b_{\gamma,T}(\mathsf{T}_{\ell_c}^{m+1}(\ell)) &= b_{\gamma,T}(\mathsf{T}_{\ell_c}^{m}(\ell)) + \#(\ell \cap \ell_c) b_{\gamma,T}(\ell_c) = \cdots \\ &= b_{\gamma,T}(\mathsf{T}_{\ell_c}^{m'}(\ell)) + (m+1-m')\#(\ell \cap \ell_c) b_{\gamma,T}(\ell_c). \end{aligned}$$

For an arbitrary tagged triangulation T, we recall that there is a unique ideal triangulation T^0 satisfying $T' = \iota(T^0)$, where T' is obtained from T by simultaneous changing all tags at some punctures if necessary (see Subsection 21.1 for details). Then the shear coordinate $b_{\gamma,T}(\ell)$ of a laminate ℓ is equal to $b_{\gamma^0,T^0}(\ell')$, where ℓ' is a laminate obtained from ℓ by changing the directions of its spirals at these punctures if they exist. Since the change of directions of its spirals and the Dehn twist T_{ℓ_c} are compatible, the proof of Theorem 21.9 comes down to the case of ideal triangulations. □

21.4. **Proof of Theorem 20.5.** In this subsection, we fix a tagged triangulation T of (S, M). By Theorem 21.2, to prove the first assertion of Theorem 20.5, we only need to show that for each lamination L on (S, M),

$$b_T(L) \in \overline{\bigcup_{T' \in \mathbb{T}} C_T(\mathsf{e}(T'))}. \tag{21.6}$$

To prove (21.6), we need some preparation. We have decompositions (21.4) of L. By Proposition 21.8(2), $\mathsf{e}^{-1}((L_{\mathrm{nc}})_{\mathsf{pq}})$ is a partial tagged triangulation of (S, M). Then we take a set U of tagged arcs of (S, M) such that $T_L := \mathsf{e}^{-1}((L_{\mathrm{nc}})_{\mathsf{pq}}) \sqcup U$ is a tagged triangulation.

Lemma 21.12. *Let L be a lamination on (S, M).*

1. *Any closed laminate ℓ in L_{cl} does not intersect with tagged arcs of $\mathsf{e}^{-1}((L_{\mathrm{nc}})_{\mathsf{pq}})$, but it intersects with at least one tagged arc of U.*
2. $C_T(\mathsf{e}\,\mathsf{T}_\ell(T_L)) = C_T((L_{\mathrm{nc}})_{\mathsf{pq}} \sqcup \mathsf{e}\,\mathsf{T}_\ell(U))$.

Proof. (1) Since L is a lamination, any closed laminate ℓ in L_{cl} does not intersect with all laminates in L_{nc}. Thus ℓ does not intersect with all tagged arcs of $\mathsf{e}^{-1}((L_{\mathrm{nc}})_{\mathsf{pq}})$ since e^{-1} and $(-)_{\mathsf{pq}}$ transform laminates just around the marked points of (S,M). Moreover, since T_L is a tagged triangulation and ℓ is not contractible, ℓ intersects with at least one tagged arc of T_L, hence it is of U.

(2) By (1), we have $\mathsf{T}_\ell(T_L)=\mathsf{e}^{-1}((L_{\mathrm{nc}})_{\mathsf{pq}})\sqcup\mathsf{T}_\ell(U)$ and it is a tagged triangulation. The desired equality is given by Proposition 21.8(3). □

Let $\ell_1,\ldots,\ell_t$ be all distinct closed laminates in L_{cl} and n_i the multiplicity of ℓ_i in L_{cl} for $i\in\{1,\ldots,t\}$. By Lemma 21.12(1), $N_i:=\sum_{\epsilon\in U}\#(\ell_i\cap\epsilon)$ is not zero. In particular, N_i is equal to $\sum_{\epsilon\in U}\#(\ell_i\cap\mathsf{e}(\epsilon))$. We fix the direction of ℓ_i for $1\le i\le t$ and consider the Dehn twists T_{ℓ_i}. Since ℓ_i are not intersect, T_{ℓ_i} are commutative. We set

$$\mathsf{T}:=\prod_{i=1}^{t}\mathsf{T}_{\ell_i}^{\frac{N_1\cdots N_t}{N_i}n_i}.$$

Proposition 21.13. *Let L be a lamination on (S,M). Then we have*

$$b_T(L)\in\overline{\bigcup_{m\ge0}C_T(\mathsf{e}\,\mathsf{T}^m(T_L))}.$$

Proof. By Theorem 21.9, for $m_T\gg0$ and $m\ge m_T$, we have

$$\begin{aligned}b_T(\mathsf{e}\,\mathsf{T}^m(U))&=b_T(\mathsf{e}\,\mathsf{T}^{m_T}(U))+\sum_{i=1}^{t}\Big(\frac{N_1\cdots N_t}{N_i}n_i\Big)(m-m_T)N_ib_T(\ell_i)\\&=b_T(\mathsf{e}\,\mathsf{T}^{m_T}(U))+(m-m_T)N_1\cdots N_t\sum_{i=1}^{t}n_ib_T(\ell_i)\\&=b_T(\mathsf{e}\,\mathsf{T}^{m_T}(U))+(m-m_T)N_1\cdots N_tb_T(L_{\mathrm{cl}}).\end{aligned}$$

This equality gives

$$\lim_{m\to\infty}\frac{b_T(\mathsf{e}\,\mathsf{T}^m(U))}{m-m_T}=N_1\cdots N_tb_T(L_{\mathrm{cl}}),$$

thus

$$b_T(L_{\mathrm{cl}})\in\overline{\bigcup_{m\ge0}C_T(\mathsf{e}\,\mathsf{T}^m(U))}.$$

Since $C_T(L_{\mathrm{nc}})\subseteq C_T((L_{\mathrm{nc}})_{\mathsf{pq}})$ by Proposition 21.8(1), we have

$$\begin{aligned}b_T(L)=b_T(L_{\mathrm{nc}})+b_T(L_{\mathrm{cl}})&\in C_T(L_{\mathrm{nc}})+\overline{\bigcup_{m\ge0}C_T(\mathsf{e}\,\mathsf{T}^m(U))}\\&\subseteq\overline{\bigcup_{m\ge0}C_T((L_{\mathrm{nc}})_{\mathsf{pq}}\sqcup\mathsf{e}\,\mathsf{T}^m(U))}=\overline{\bigcup_{m\ge0}C_T(\mathsf{e}\,\mathsf{T}^m(T_L))},\end{aligned}$$

where the last equality is given by Lemma 21.12(2). □

Proof of the first assertion of Theorem 20.5. Since $\mathsf{T}^m(T_L)$ is a tagged triangulation of (S,M) for any $m\in\mathbb{Z}_{\ge0}$, Proposition 21.13 finishes the proof of (21.6). Hence the assertion holds. □

To prove the second assertion of Theorem 20.5, we give the following results in a more general setting.

Proposition 21.14. *Let T be an ideal triangulation of (S,M) without self-folded triangles.*

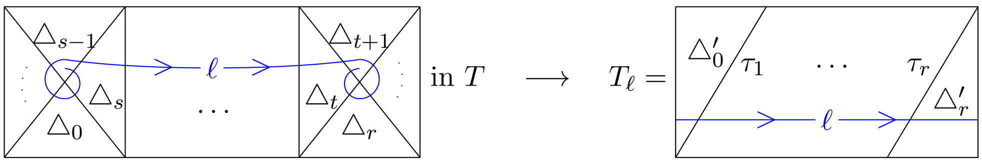
(1) *For a laminate ℓ, we have*

$$\sum_{\gamma\in T}b_{\gamma,T}(\ell)\in\{0,\pm1\}.$$

(2) *For a tagged arc ϵ of (S,M) whose both endpoints are punctures, we have*

$$\sum_{\gamma\in T} b_{\gamma,T}(\mathsf{e}(\epsilon)) = \begin{cases} -1 & \text{if both ends of } \epsilon \text{ are tagged plain,} \\ 1 & \text{if both ends of } \epsilon \text{ are tagged notched,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (1) Fix a direction of ℓ. For a closed laminate ℓ, let T_ℓ be in (21.5). For a non-closed laminate ℓ, we define a polygon (S_ℓ, M_ℓ) and its ideal triangulation T_ℓ consisting of triangles $\triangle'_0, \triangle'_1, \dots, \triangle'_r$ and $\tau_i = \triangle'_{i-1} \cap \triangle'_i$ in the same way, where we need a slight modification at the spirals. If the starting (resp., ending) end of ℓ is a spiral around a puncture p, then the triangles of T incident to p are $\triangle_0, \dots, \triangle_s$ for $1 < s < r$ (resp., $\triangle_t, \dots, \triangle_r$ for $1 < t < r$) as follows:



Let ℓ be an arbitrary laminate of (S,M) and we consider the ideal triangulation T_ℓ consisting of triangles $\triangle'_0, \dots, \triangle'_r$ ($\triangle'_0 = \triangle'_r$ if ℓ is closed). We call $\triangle'_i$ a *left* (resp., *right*) *triangle* if a side of $\triangle'_i$ is a boundary segment of T_ℓ on the left (resp., right) side of ℓ. Then we have

$$b_{\tau_i,T_\ell}(\ell) = \begin{cases} 1 & \text{if } \triangle'_{i-1} \text{ is a left triangle and } \triangle'_i \text{ is a right triangle,} \\ -1 & \text{if } \triangle'_{i-1} \text{ is a right triangle and } \triangle'_i \text{ is a left triangle,} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\sum_{k=1}^{r} b_{\tau_k,T_\ell}(\ell) = \begin{cases} 1 & \text{if } \triangle'_0 \text{ is a left triangle and } \triangle'_r \text{ is a right triangle,} \\ -1 & \text{if } \triangle'_0 \text{ is a right triangle and } \triangle'_r \text{ is a left triangle,} \\ 0 & \text{otherwise.} \end{cases} \tag{21.7}$$

On the other hand, since T has no self-folded triangles, we have

$$\sum_{\delta\in T} b_{\delta,T}(\ell) = \sum_{k=1}^{r} b_{\tau_k,T_\ell}(\ell).$$

Thus the assertion follows from (21.7).

(2) We consider $T_{\mathsf{e}(\epsilon)}$ as above and define the direction of ϵ from ℓ by the obvious way. If the starting point of ϵ is tagged plain (resp., notched), then $\triangle'_0$ is a right (resp., left) triangle. If the ending point of ϵ is tagged plain (resp., notched), then $\triangle'_0$ is a left (resp., right) triangle. Thus the assertion also follows from (21.7). □

Proof of the second assertion of Theorem 20.5. Let (S,M) be a closed surface with exactly one puncture p and T its tagged triangulation. In this case, all ends of tagged arcs of T are tagged plain or they are tagged notched. Thus we can assume that T is an ideal triangulation without self-folded triangles. Let ϵ and ϵ' be tagged arcs of (S,M) tagged plain and notched, respectively. We only need to show that

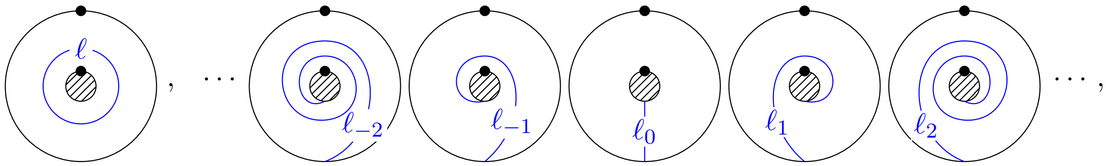
$$b_T(\mathsf{e}(\epsilon)) \in \Big\{(a_\gamma)_{\gamma\in T} \in \mathbb{R}^{|T|} \mid \sum_{\gamma\in T} a_\gamma \le 0\Big\} \ \text{ and } \ b_T(\mathsf{e}(\epsilon')) \in \Big\{(a_\gamma)_{\gamma\in T} \in \mathbb{R}^{|T|} \mid \sum_{\gamma\in T} a_\gamma \ge 0\Big\},$$

that is,

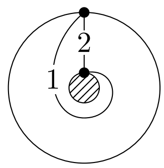
$$\sum_{\gamma\in T} b_{\gamma,T}(\mathsf{e}(\epsilon)) \le 0 \ \text{ and } \ \sum_{\gamma\in T} b_{\gamma,T}(\mathsf{e}(\epsilon')) \ge 0.$$

It immediately follows from Proposition 21.14(2). □

21.5. **Example of Proposition 21.13.** For an annulus (S, M) with exactly two marked points, all laminates are given as follows:



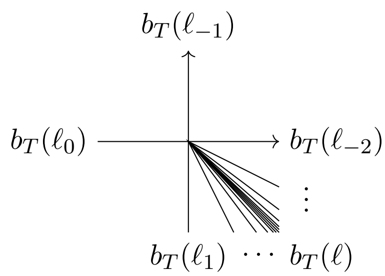
where ℓ is closed and $\ell_m = \mathsf{T}_\ell^m(\ell_0)$ is elementary for $m \in \mathbb{Z}$. Their shear coordinates with respect to

$T =$ 

are given by

$$b_T(\ell) = (b_{1,T}(\ell), b_{2,T}(\ell)) = (1, -1),$$

$$b_T(\ell_m) = \begin{cases} (m-1, -m) & \text{if } m \geq 0, \\ (-m-1, m+2) & \text{if } m < 0. \end{cases}$$



For a tagged triangulation T' of (S, M), the shear coordinate cone $C_T(\mathsf{e}(T'))$ is given by $\{\alpha b_T(\ell_j) + \beta b_T(\ell_{j+1}) \mid \alpha, \beta \in \mathbb{Z}_{\geq 0}\}$ for some $j \in \mathbb{Z}$. Then the set of integer vectors which are not contained in these shear coordinate cones is $\{b_T(\{k\ell\}) \mid k \in \mathbb{Z}_{>0}\}$. Taking $L = \{k\ell\}$ and $T_L = T$, Proposition 21.13 means that

$$b_T(k\ell) = (k, -k) \in \overline{\bigcup_{m \geq 0} C_T(\mathsf{e}\,\mathsf{T}^m(T))} = \overline{\bigcup_{m \geq 0} C_T(\{\ell_m, \ell_{m+1}\})}.$$

It is described in the above picture.

22. Cluster algebras

Let Q be a quiver without loops and 2-cycles. Then the associated cluster algebra $\mathcal{A}(\hat{Q})$ is contained in $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1, \dots, y_n]$. We consider the $\mathbb{Z}^n$-grading in $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1, \dots, y_n]$ given by

$$\deg(x_i) = e_i, \quad \deg(y_j) = (\#\{i \to j \text{ in } Q\} - \#\{i \leftarrow j \text{ in } Q\})_{1 \leq i \leq n},$$

where $e_1, \dots, e_n$ are the standard basis vectors in $\mathbb{Z}^n$. Every cluster variable x of $\mathcal{A}(\hat{Q})$ is homogeneous with respect to the $\mathbb{Z}^n$-grading, and its degree $g_Q(x)$ is called *g-vector* of x [FZ07, Proposition 6.1].

We denote by $\operatorname{cluster} Q$ the set of clusters in $\mathcal{A}(\hat{Q})$ and by $\operatorname{cl-var} Q$ the set of cluster variables in $\mathcal{A}(\hat{Q})$. We denote by $\mathbb{T}_T$ the set of tagged triangulations of (S, M) obtained from T by sequences of flips, and by $\mathbb{A}_T$ the set of tagged arcs of each tagged triangulation contained in $\mathbb{T}_T$. By Theorem 3.1, there is a bijection

$$x_{(-)} : \mathbb{A}_T \longleftrightarrow \operatorname{cl-var} Q_T.$$

Moreover, it induces a bijection

$$x_{(-)} : \mathbb{T}_T \longleftrightarrow \operatorname{cluster} Q_T$$

which sends T to the initial cluster in $\mathcal{A}(T)$ and commutes with flips and mutations. This bijection also satisfies the following property.

Theorem 22.1. [R14b, Proposition 5.2] *Let T be a tagged triangulation of (S,M) and $\delta \in \mathbb{A}_T$. Then we have*

$$-b_T(\mathsf{e}(\delta)) = g_{Q_T}(x_\delta).$$

Note that, in another way, Theorem 22.1 can be directly given by the cluster expansion formula in Part 3. Moreover, it was proved in [FeT, Theorem 8.6] for orbifolds in the same way as [R14b, Proposition 5.2].

22.1. **Proof of Theorem 20.2.** We recall the following notion to prove Theorem 20.2.

Definition 22.2. [BQ] Let (S,M) be an arbitrary marked surface. The *tagged rotation* of a tagged arc δ of (S,M) is the tagged arc $\rho(\delta)$ defined as follows:

- If δ has an endpoint o on a component C of ∂S, then $\rho(\delta)$ is obtained from δ by moving o to the next marked point on C in the counterclockwise direction;
- If δ has an endpoint at a puncture p, then $\rho(\delta)$ is obtained from δ by changing its tags at p.

By Theorem 2.1, we have

$$\mathbb{T} = \begin{cases} \mathbb{T}_T \sqcup \mathbb{T}_{\rho T} & \text{if } (S,M) \text{ is a closed surface with exactly one puncture,} \\ \mathbb{T}_T & \text{otherwise.} \end{cases} \tag{22.1}$$

Let S^* be the same surface as S oriented in the opposite direction and $M^* = M$. For a tagged arc or laminate γ of (S,M), we denote by γ^* the corresponding one of (S^*,M^*). In particular, the tagged triangulation T^* of (S^*,M^*) is naturally induced by $T \in \mathbb{T}$ and we have $Q_{T^*} = Q_T^{\mathrm{op}}$. By Theorem 3.1, the composition of maps $\rho^{-1}(-)$, $(-)^*$ and $x_{(-)}$ gives a bijection

$$x_{(\rho^{-1}(-))^*} : \mathbb{A}_{\rho T} \longleftrightarrow \text{cl-var}\, Q_T^{\mathrm{op}}.$$

Moreover, it induces a bijection

$$x_{(\rho^{-1}(-))^*} : \mathbb{T}_{\rho T} \longleftrightarrow \text{cluster}\, Q_T^{\mathrm{op}}$$

which sends ρT to the initial cluster in $\mathcal{A}(\hat{Q}_T^{\mathrm{op}})$ and commutes with flips and mutations.

Theorem 22.3. *Let T be a tagged triangulation of (S,M). Then for each $\delta \in \mathbb{A}_{\rho T}$, we have*

$$b_T(\mathsf{e}(\delta)) = g_{Q_T^{\mathrm{op}}}(x_{(\rho^{-1}(\delta))^*}).$$

Proof. For a tagged arc δ of (S,M), the equalities

$$b_T(\mathsf{e}(\rho(\delta))) = b_T((\mathsf{e}(\delta^*))^*) = -b_{T^*}(\mathsf{e}(\delta^*))$$

hold. Since $Q_{T^*} = Q_T^{\mathrm{op}}$, Theorem 22.1 gives

$$-b_{T^*}(\mathsf{e}(\delta^*)) = g_{Q_{T^*}}(x_{\delta^*}) = g_{Q_T^{\mathrm{op}}}(x_{\delta^*})$$

for $\delta \in \mathbb{A}_T$, hence $b_T(\mathsf{e}(\rho(\delta))) = g_{Q_T^{\mathrm{op}}}(x_{\delta^*})$. □

Proof of Theorem 20.2. By Theorems 22.1 and 22.3, we have

$$\bigcup_{T' \in \mathbb{T}_T} C_T(\mathsf{e}(T')) = \bigcup_{\mathbf{x} \in \text{cluster}\, Q_T} \big(-C_{Q_T}(\mathbf{x})\big) \quad \text{and} \quad \bigcup_{T' \in \mathbb{T}_{\rho T}} C_T(\mathsf{e}(T')) = \bigcup_{\mathbf{x} \in \text{cluster}\, Q_T^{\mathrm{op}}} C_{Q_T^{\mathrm{op}}}(\mathbf{x}).$$

If (S,M) is a closed surface with exactly one puncture, then $\mathbb{T}_T$ and $\mathbb{T}_{\rho T}$ coincide with $\mathbb{T}^+$ and $\mathbb{T}^-$ in Theorem 20.5, respectively. Therefore, the assertion follows from Theorem 20.5 and (22.1). □

22.2. **Example for a cluster algebra.** For the tagged triangulation T in Subsection 21.5, the quiver Q_T is the Kronecker quiver $1 \leftleftarrows 2$. The set $\operatorname{cluster} Q_T$ is described by

$$\begin{array}{ccccccc} (x_1, x_2) & \text{———} & \left(\frac{x_2^2+y_1}{x_1}, x_2\right) & \text{———} & \left(\frac{x_2^2+y_1}{x_1}, \frac{x_2^4+y_1^2y_2x_1^2+2y_1x_2^2+y_1^2}{x_1^2x_2}\right) & \text{———} & \cdots \\ | & & & & & & \\ \left(x_1, \frac{y_2x_1^2+1}{x_2}\right) & \text{———} & \left(x', \frac{y_2x_1^2+1}{x_2}\right) & \text{———} & (x', x'') & \text{———} & \cdots \end{array}$$

where

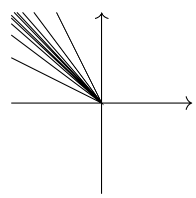
$$x' = \frac{y_1y_2^2x_1^4 + 2y_1y_2x_1^2 + x_2^2 + y_1}{x_1x_2^2},$$

$$x'' = \frac{y_1^2y_2^3x_1^6 + 3y_1^2y_2^2x_1^4 + 2y_1y_2x_1^2x_2^2 + x_2^4 + 3y_1^2y_2x_1^2 + 2y_1x_2^2 + y_1^2}{x_1^2x_2^3}.$$

The corresponding g-vectors are as follows:

$$\begin{array}{ccccccc} (1,0),(0,1) & \text{———} & (-1,2),(0,1) & \text{———} & (-1,2),(-2,3) & \text{———} & \cdots \\ | & & & & & & \\ (1,0),(0,-1) & \text{———} & (-1,0),(0,-1) & \text{———} & (-1,0),(-2,1) & \text{———} & \cdots \end{array}$$

The g-vector cones of clusters are reflections of the corresponding shear coordinate cones in Subsection 21.5 as follows:



23. Representation theory

23.1. **τ-tilting theory and cluster tilting theory.** In this subsection, we recall τ-tilting and cluster tilting theory to prepare for the proofs of Theorem 20.3 and Corollary 20.4.

First, we recall τ-tilting theory [AIR]. Let Λ be a finite dimensional algebra over a field. We denote by $\mathsf{mod}\,\Lambda$ (resp., $\mathsf{proj}\,\Lambda$) the category of finitely generated (resp., finitely generated projective) left Λ-modules. We denote by τ the Auslander-Reiten translation of $\mathsf{mod}\,\Lambda$ and by $|M|$ is the number of non-isomorphic indecomposable direct summands of $M \in \mathsf{mod}\,\Lambda$. Let $M \in \mathsf{mod}\,\Lambda$ and $P \in \mathsf{proj}\,\Lambda$. We say that a pair (M, P) is

- *τ-rigid* if $\operatorname{Hom}_\Lambda(M, \tau M) = 0 = \operatorname{Hom}_\Lambda(P, M)$;
- *τ-tilting* if (M, P) is τ-rigid and $|\Lambda| = |M| + |P|$
- *basic* if M and P are basic;
- a *direct summand* of $(M', P') \in \mathsf{mod}\,\Lambda \times \mathsf{proj}\,\Lambda$ if M is a direct summand of M' and P is a direct summand of P';
- *indecomposable* if (M, P) is basic and $|M| + |P| = 1$.

Recall that we denote by $\mathrm{s}\tau\text{-tilt}\,\Lambda$ the set of isomorphism classes of basic τ-tilting pairs in $\mathsf{mod}\,\Lambda$. For $N \in \mathrm{s}\tau\text{-tilt}\,\Lambda$ and an indecomposable direct summand N' of N, there is a unique indecomposable τ-rigid pair N'' such that $N/N' \oplus N'' \in \mathrm{s}\tau\text{-tilt}\,\Lambda$ [AIR, Theorem 0.4]. Therefore, one can define mutations in $\mathrm{s}\tau\text{-tilt}\,\Lambda$.

Let $\Lambda = \bigoplus_{i=1}^n P_i$ be a decomposition of Λ, where P_i is an indecomposable projective Λ-module. Then $[P_1], \ldots, [P_n]$ form a basis for $K_0(\mathsf{proj}\,\Lambda)$, thus there is a natural bijection between $K_0(\mathsf{proj}\,\Lambda)$

and $\mathbb{Z}^n$. Let $M \in \mathsf{mod}\,\Lambda$. There is a minimal projective presentation of M

$$P^1 \to P^0 \to M \to 0.$$

We set

$$g_\Lambda(M) := [P^0] - [P^1] \in K_0(\mathsf{proj}\,\Lambda) \simeq \mathbb{Z}^n,$$

called the *g-vector of* M. We denote by $\mathrm{i}\tau\text{-rigid}\,\Lambda$ the set of isomorphism classes of indecomposable τ-rigid pairs in $\mathsf{mod}\,\Lambda$. The *g-vector of* $(M,P) \in \mathrm{i}\tau\text{-rigid}\,\Lambda$ is $g_\Lambda(M,P) := g_\Lambda(M) - g_\Lambda(P)$.

For our aim, we also need to consider the opposite algebra Λ^{op} of Λ. For $M \in \mathsf{mod}\,\Lambda$, the notation $\mathrm{Tr}\,M$ denotes the transpose of M. We define $(-)^* := \mathrm{Hom}_\Lambda(-,\Lambda) : \mathsf{proj}\,\Lambda \longleftrightarrow \mathsf{proj}\,\Lambda^{\mathrm{op}}$. Then $(-)^*$ gives $K_0(\mathsf{proj}\,\Lambda) \simeq K_0(\mathsf{proj}\,\Lambda^{\mathrm{op}})$.

Theorem 23.1. [AIR, Theorem 2.14][F, Subsection 3.4] *There is a bijection*

$$\varphi : \mathrm{i}\tau\text{-rigid}\,\Lambda \longleftrightarrow \mathrm{i}\tau\text{-rigid}\,\Lambda^{\mathrm{op}}$$

given by $(M,P) \mapsto (\mathrm{Tr}\,M \oplus P^*, M_{\mathrm{pr}})$ *such that*

$$g_\Lambda(M,P) = -g_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\,M \oplus P^*, M_{\mathrm{pr}}),$$

where M_{pr} *is a maximal projective direct summand of* M*. The map* φ *induces a bijection*

$$\varphi : \mathrm{s}\tau\text{-tilt}\,\Lambda \longleftrightarrow \mathrm{s}\tau\text{-tilt}\,\Lambda^{\mathrm{op}}$$

which sends $(\Lambda,0)$ *to* $(0,\Lambda)$ *and commutes with mutations.*

Next, we recall cluster tilting theory in 2-Calabi-Yau triangulated categories. Let $\mathcal{C}$ be a Hom-finite Krull-Schmidt 2-Calabi-Yau triangulated category. We call $X \in \mathcal{C}$ *rigid* if $\mathrm{Hom}_\mathcal{C}(X, X[1]) = 0$. We denote by $\mathsf{add}\,U$ the category of all direct summands of finite direct sums of copies of U. We call $U \in \mathcal{C}$ *cluster tilting* if $\mathsf{add}\,U = \{X \in \mathcal{C} \mid \mathrm{Hom}_\mathcal{C}(U, X[1]) = 0\}$. We denote by $\mathrm{irigid}\,\mathcal{C}$ the set of isomorphism classes of indecomposable rigid objects in $\mathcal{C}$. Recall that we denote by $\text{c-tilt}\,\mathcal{C}$ the set of isomorphism classes of basic cluster tilting objects in $\mathcal{C}$. We assume that $\mathcal{C}$ has cluster tilting objects, that is, $\text{c-tilt}\,\mathcal{C} \neq \emptyset$. In this case, any maximal rigid object in $\mathcal{C}$ is cluster tilting [ZZ, Theorem 2.6]. Iyama and Yoshino [IY] gave mutations in $\text{c-tilt}\,\mathcal{C}$ (see also [BMRRT]).

Let $U = \bigoplus_{i=1}^n U_i$ be a decomposition of U, where U_i is indecomposable. Then $[U_1], \dots, [U_n]$ form a basis for $K_0(\mathsf{add}\,U)$, thus there is a natural bijection between $K_0(\mathsf{add}\,U)$ and $\mathbb{Z}^n$. For $U \in \text{c-tilt}\,\mathcal{C}$ and $X \in \mathcal{C}$, there is a triangle

$$U^1 \to U^0 \to X \to U_1[1],$$

where $U^1, U^0 \in \mathsf{add}\,U$. We define

$$g_U(X) := [U^0] - [U^1] \in K_0(\mathsf{add}\,U) \simeq \mathbb{Z}^n,$$

called the *g-vector of* X *with respect to* U.

There is a close relationship between cluster tilting theory and τ-tilting theory as follows.

Theorem 23.2. [AIR, Theorem 4.1] *Let* $U \in \text{c-tilt}\,\mathcal{C}$ *and* $\Lambda = \mathrm{End}_\mathcal{C}(U)^{\mathrm{op}}$*. Then there is a bijection*

$$\mathsf{H} := \mathrm{Hom}_\mathcal{C}(U,-) : \mathrm{irigid}\,\mathcal{C} \longleftrightarrow \mathrm{i}\tau\text{-rigid}\,\Lambda$$

such that

$$g_U(X) = g_\Lambda(\mathsf{H}(X))$$

for $X \in \mathrm{irigid}\,\mathcal{C}$*. Moreover, it induces a bijection*

$$\mathsf{H} : \text{c-tilt}\,\mathcal{C} \longleftrightarrow \mathrm{s}\tau\text{-tilt}\,\Lambda$$

which sends U *to* $(\Lambda,0)$ *and commutes with mutations.*

For $\bullet \in \{+,-\}$, we denote by $\mathrm{irigid}^\bullet\,\mathcal{C}$ (resp., $\mathrm{i}\tau\text{-rigid}^\bullet\,\Lambda$) the set of indecomposable direct summands of an object in $\text{c-tilt}^\bullet\,\mathcal{C}$ (resp., $\mathrm{s}\tau\text{-tilt}^\bullet\,\Lambda$). Clearly, the map H in Theorem 23.2 gives bijections

$$\mathrm{irigid}^\bullet\,\mathcal{C} \longleftrightarrow \mathrm{i}\tau\text{-rigid}^\bullet\,\Lambda \quad \text{and} \quad \text{c-tilt}^\bullet\,\mathcal{C} \longleftrightarrow \mathrm{s}\tau\text{-tilt}^\bullet\,\Lambda.$$

23.2. **Representation theory and cluster algebras.** We consider a relationship between representation theory and cluster algebras to prove Theorem 20.3 and Corollary 20.4. For a quiver with potential (Q, W), we have the associated Jacobian algebra $J(Q, W)$, Ginzburg differential graded algebra $\Gamma_{Q,W}$, and generalized cluster category $\mathcal{C}_{Q,W}$ (see e.g. [A, DWZ, G, K08, K11] for details). The following is the main result in the additive categorification of cluster algebras.

Theorem 23.3. *Let Q be a quiver without loops and 2-cycles and W a non-degenerate potential of Q such that $J(Q, W)$ is finite dimensional.*

(1) [A, Theorem 2.1] *The category $\mathcal{C}_{Q,W}$ is a Hom-finite Krull-Schmidt 2-Calabi-Yau triangulated category with a cluster tilting object $\Gamma_{Q,W}$.*

(2) [FK, Theorem 6.3][CKLP, Corollary 3.5] *There is a bijection*
$$\mathsf{X} : \mathrm{irigid}^+ \mathcal{C}_{Q,W} \longleftrightarrow \text{cl-var}\, Q$$
such that
$$g_{\Gamma_{Q,W}}(X) = g_Q(\mathsf{X}(X))$$
for $X \in \mathrm{irigid}^+ \mathcal{C}_{Q,W}$. Moreover, it induces a bijection
$$\mathsf{X} : \text{c-tilt}^+ \mathcal{C}_{Q,W} \longleftrightarrow \text{cluster}\, Q$$
which sends $\Gamma_{Q,W}$ to the initial cluster in $\mathcal{A}(\hat{Q})$ and commutes with mutations.

Note that the map X in Theorem 23.3(2) is called the *cluster character* associated with (Q, W) (see e.g. [BY, CC, Pa, Pl11a, Pl11b]).

We also study $\mathrm{irigid}^- \mathcal{C}_{Q,W}$ and $\text{c-tilt}^- \mathcal{C}_{Q,W}$. We have
$$\mathrm{End}_{\mathcal{C}_{Q,W}}(\Gamma_{Q,W})^{\mathrm{op}} \simeq J(Q, W) \text{ and } J(Q, W)^{\mathrm{op}} \simeq J(Q^{\mathrm{op}}, W^{\mathrm{op}}),$$
where Q^{op} is the opposite quiver of Q and W^{op} is a non-degenerate potential of Q^{op} corresponding to W.

Corollary 23.4. *Let Q be a quiver without loops and 2-cycles and W a non-degenerate potential of Q such that $J(Q, W)$ is finite dimensional. Then there is a bijection*
$$\mathsf{X}' : \mathrm{irigid}^- \mathcal{C}_{Q,W} \longleftrightarrow \text{cl-var}\, Q^{\mathrm{op}}$$
such that
$$g_{\Gamma_{Q,W}}(X) = -g_{Q^{\mathrm{op}}}(\mathsf{X}'(X))$$
for $X \in \mathrm{irigid}^- \mathcal{C}_{Q,W}$. Moreover, it induces a bijection
$$\mathsf{X}' : \text{c-tilt}^- \mathcal{C}_{Q,W} \longleftrightarrow \text{cluster}\, Q^{\mathrm{op}}$$
which sends $\Gamma_{Q,W}[1]$ to the initial cluster in $\mathcal{A}(\hat{Q^{\mathrm{op}}})$ and commutes with mutations.

Proof. Let X' be the following composition:
$$\begin{aligned}\mathrm{irigid}^- \mathcal{C}_{Q,W} &\xrightarrow{\mathsf{H}} \mathrm{i}\tau\text{-rigid}^- \mathrm{End}_{\mathcal{C}_{Q,W}}(\Gamma_{Q,W})^{\mathrm{op}} \longleftrightarrow \mathrm{i}\tau\text{-rigid}^- J(Q, W)\\ &\xrightarrow{\varphi} \mathrm{i}\tau\text{-rigid}^+ J(Q, W)^{\mathrm{op}} \longleftrightarrow \mathrm{i}\tau\text{-rigid}^+ J(Q^{\mathrm{op}}, W^{\mathrm{op}})\\ &\xrightarrow{\mathsf{H}^{-1}} \mathrm{irigid}^+ \mathcal{C}_{Q^{\mathrm{op}},W^{\mathrm{op}}}\\ &\xrightarrow{\mathsf{X}} \text{cl-var}\, Q^{\mathrm{op}}.\end{aligned}$$
By Theorems 23.1, 23.2 and 23.3, it induces a bijection between $\text{c-tilt}^- \mathcal{C}_{Q,W}$ and $\text{cluster}\, Q^{\mathrm{op}}$ which sends $\Gamma_{Q,W}[1]$ to the initial cluster in $\mathcal{A}(\hat{Q^{\mathrm{op}}})$ and commutes with mutations. Moreover, we have the equalities
$$g_{\Gamma_{Q,W}}(X) = g_{J(Q,W)}(\mathsf{H}(X)) = -g_{J(Q,W)^{\mathrm{op}}}(\varphi\, \mathsf{H}(X)) = -g_{\Gamma_{Q^{\mathrm{op}},W^{\mathrm{op}}}}(\mathsf{H}^{-1} \varphi\, \mathsf{H}(X)) = -g_{Q^{\mathrm{op}}}(\mathsf{X}'(X))$$
for $X \in \mathrm{irigid}^- \mathcal{C}_{Q,W}$. □

23.3. **Proofs of Theorem 20.3 and Corollary 20.4.** We keep the notations in the previous subsection. Let $\Gamma, U = \bigoplus_{i=1}^{n} U_i \in \text{c-tilt}\,\mathcal{C}$ and $N = \bigoplus_{i=1}^{n} N_i \in \text{s}\tau\text{-tilt}\,\Lambda$, where U_i and N_i are indecomposable. We define *g-vector cones*

$$C_\Gamma(U) := \left\{ \sum_{i=1}^{n} a_i g_\Gamma(U_i) \mid a_i \in \mathbb{R}_{\geq 0} \right\} \ \text{ and } \ C_\Lambda(N) := \left\{ \sum_{i=1}^{n} a_i g_\Lambda(N_i) \mid a_i \in \mathbb{R}_{\geq 0} \right\}.$$

Proof of Theorem 20.3. Let T be a tagged triangulation of (S, M) and W a non-degenerate potential of $Q = Q_T$ such that $J(Q, W)$ is finite dimensional. By Theorem 23.2, we have

$$C_{\Gamma_{Q,W}}(U) = C_{J(Q,W)}(\mathsf{H}(U))$$

for $U \in \text{c-tilt}\,\mathcal{C}_{Q,W}$. Therefore, we only need to prove the assertion for $\mathcal{C}_{Q,W}$. By Theorem 23.3(2) and Corollary 23.4, the equalities

$$\bigcup_{U \in \text{c-tilt}^+ \mathcal{C}_{Q,W}} C_{\Gamma_{Q,W}}(U) = \bigcup_{\mathbf{x} \in \text{cluster}\, Q_T} C_{Q_T}(\mathbf{x})$$

and

$$\bigcup_{U \in \text{c-tilt}^- \mathcal{C}_{Q,W}} C_{\Gamma_{Q,W}}(U) = \bigcup_{\mathbf{x} \in \text{cluster}\, Q_T^{\text{op}}} \left(-C_{Q_T^{\text{op}}}(\mathbf{x})\right).$$

hold. Thus the assertion follows from Theorem 20.2. □

Proof of Corollary 20.4. A g-vector cone $C_{\Gamma_{Q,W}}(U)$ has dimension n for any $U \in \text{c-tilt}\,\mathcal{C}_{Q,W}$ [DK, Theorem 2.4]. For $U \not\cong V \in \text{c-tilt}\,\mathcal{C}_{Q,W}$, $C_{\Gamma_{Q,W}}(U)$ and $C_{\Gamma_{Q,W}}(V)$ have no intersections except for their boundaries [DIJ, Corollary 6.7]. Thus there are no cluster tilting objects in $\text{c-tilt}\,\mathcal{C}_{Q,W} \setminus \text{c-tilt}^{\pm}\,\mathcal{C}_{Q,W}$ by Theorem 20.3. The assertion follows from Theorem 2.1. □

23.4. **Example for representation theory.** For the tagged triangulation T in Subsection 21.5, the quiver Q_T is the Kronecker quiver $1 \leftleftarrows 2$. The set $\text{s}\tau\text{-tilt}\,J(Q_T, 0)$ is as follows:

$$\begin{array}{ccccccc}
\left(1 \oplus \begin{smallmatrix}2\\1\,1\end{smallmatrix}, 0\right) & \text{---} & \left(\begin{smallmatrix}2\,2\\1\,1\,1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\,1\end{smallmatrix}, 0\right) & \text{---} & \left(\begin{smallmatrix}2\,2\\1\,1\,1\end{smallmatrix} \oplus \begin{smallmatrix}2\,2\,2\\1\,1\,1\,1\end{smallmatrix}, 0\right) & \text{---} & \cdots \\
| & & & & & & \\
\left(1, \begin{smallmatrix}2\\1\,1\end{smallmatrix}\right) & \text{---} & \left(0, 1 \oplus \begin{smallmatrix}2\\1\,1\end{smallmatrix}\right) & \text{---} & (2, 1) & \text{---} & \cdots
\end{array}$$

The corresponding g-vectors

$$\begin{array}{ccccccc}
(1,0),(0,1) & \text{---} & (-1,2),(0,1) & \text{---} & (-1,2),(-2,3) & \text{---} & \cdots \\
| & & & & & & \\
(1,0),(0,-1) & \text{---} & (-1,0),(0,-1) & \text{---} & (-1,0),(-2,1) & \text{---} & \cdots
\end{array}$$

coincide with the g-vectors of the corresponding cluster variables as in Subsection 22.2.

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