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Conditions for confluence of innermost terminating term rewriting systems

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Abstract This paper presents a counterexample for the open conjecture whether innermost joinability of all critical pairs ensures confluence of innermost terminating term rewriting systems. We then show that innermost joinability of all normalized instances of the critical pairs is a necessary and sufficient condition. Using this condition, we give a decidable sufficient condition for confluence of innermost terminating systems. Finally, we enrich the condition by introducing the notion of left-stable rules. As a corollary, confluence of innermost terminating left-weakly-shallow TRSs is shown to be decidable.

Keywords Confluence · Innermost termination · Term rewriting systems · Decidability

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1 Introduction

B. Gramlich [6] has shown that joinability of all critical pairs implies confluence for innermost terminating overlay systems, but does not always ensure confluence for non-overlay systems. E. Ohlebusch [11] has posed a conjecture that innermost joinability of all critical pairs ensures confluence for innermost terminating systems.

In this paper, we give a negative answer to this open conjecture and two characterizations for confluence of innermost terminating systems. We then

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show that innermost joinability of all normalized instances of all critical pairs is a necessary and sufficient condition. Since it might happen that this condition is undecidable, we give a decidable sufficient condition for confluence of innermost terminating systems, which is obtained by restricting rewriting allowed in join sequences of the critical pairs to ground innermost rewriting except for one-step parallel rewriting. We then introduce the notion of left-stable rules, where a rewrite rule is left-stable if no proper subterms of left-hand-sides of the rules instantiated by normalized substitution are reducible. Consequently we obtain the relaxed condition that for each critical pair the join sequence consists of ground innermost rewriting or consists of rewriting by left-stable rules. To prove the correctness, we use notions of basic rewrite sequences [12] and Priority Rewrite Systems [3]. We give some examples whose confluence is not proven by confluence provers available at this moment, but ensured by our sufficient condition. This means that the criterion is useful for tool implementation. As a corollary of the relaxed condition, confluence for innermost terminating left-weakly-shallow [13] TRSs is shown to be decidable.

2 Preliminaries

We follow [4] for fundamental notations and definitions. For a binary relation $\to$ on terms, we use ordinary notations $\leftarrow$, $\leftrightarrow$, $\xrightarrow{*}$, and so on. For terms s, s', we write $s \downarrow s'$ if $s \xrightarrow{*} t$ and $s' \xrightarrow{*} t$ for some term t. We say that $\to$ is *locally confluent* if $(\leftarrow \cdot \to) \subseteq \downarrow$, *confluent* if $(\xleftarrow{*} \cdot \xrightarrow{*}) \subseteq \downarrow$, and *Church-Rosser* if $\xleftrightarrow{*} \subseteq \downarrow$. We write $\mathrm{CR}(\to)$ if $\to$ has the Church-Rosser property. Confluence and Church-Rosser property are equivalent. We say that $\to$ is *terminating*, indicated as $\mathrm{SN}(\to)$, if it admits no infinite sequence $t_0 \to t_1 \to \cdots$. A term t is *reducible* with respect to $\to$ if there exists a term s such that $t \to s$, otherwise it is *irreducible* (or a *normal form*). We write $\mathrm{NF}_{\to}$ to denote the set of all the irreducible terms with respect to $\to$, where we often omit $\to$ if it is obviously identified.

We use $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to represent the set of terms over a finite set $\mathcal{F}$ of function symbols and a countable infinite set $\mathcal{V}$ of variables. The set of variables in a term t is denoted by $\mathrm{Var}(t)$. A term t is *ground* if $\mathrm{Var}(t) = \emptyset$. $\mathrm{Pos}(t)$ represents the set of *positions* of a term t and $\mathrm{Pos}_{\mathcal{F}}(t)$ represents the set of positions of function symbols of t. Let $\mathrm{Pos}_{\mathcal{V}}(t) = \mathrm{Pos}(t) \setminus \mathrm{Pos}_{\mathcal{F}}(t)$. For positions p and p', we write $p \geq p'$ when $p = p'.q$ with some position q, and they are in *parallel positions* (denoted by $p \parallel p'$) if neither $p \geq p'$ nor $p \leq p'$ holds. For a position p and a set Q of positions, we use $p.Q$ to represent $\{p.q \mid q \in Q\}$. We use $t|_p$ for the subterm of t at a position $p \in \mathrm{Pos}(t)$, and $t[s]_p$ for the term obtained from a term t by replacing its subterm at position $p \in \mathrm{Pos}(t)$ with a term s. We use $\trianglerighteq$ for the subterm relation, i.e., $t \trianglerighteq t|_p$.

We use $t\sigma$ for application of a *substitution* σ to a term t. The domain of a substitution σ is defined as $\mathrm{Dom}(\sigma) = \{x \in \mathcal{V} \mid x\sigma \neq x\}$. The composition of substitutions σ and σ' is defined as $t(\sigma\sigma') = (t\sigma)\sigma'$. $\sigma \leq \sigma'$ means that $\sigma\theta = \sigma'$ for some substitution θ. The union $\sigma \cup \sigma'$ is naturally defined when

$\mathrm{Dom}(\sigma) \cap \mathrm{Dom}(\sigma') = \emptyset$ as: $x(\sigma \cup \sigma')$ is $x\sigma'$ if $x \in \mathrm{Dom}(\sigma')$, and $x\sigma$ otherwise. $\sigma|_V$ represents the substitution obtained from σ by restricting its domain to a set V ($\subseteq \mathcal{V}$). A substitution σ is *normalized* if $x\sigma$ is irreducible for any variable x. A binary relation $\to$ on terms is *closed under substitution* if $s \to t$ implies $s\sigma \to t\sigma$ for any terms s, t and substitution σ.

A *term rewriting system* (TRS) $\mathcal{R}$ is a finite set of *rewrite rules*. A term s is rewritten to a term t if $s = s[l\sigma]_p$ and $t = s[r\sigma]_p$ for some rewrite rule $l \to r \in \mathcal{R}$, position p, and substitution σ. It is called a *rewrite step* of $\mathcal{R}$, denoted by $s \to^p_{\mathcal{R}} t$ (or simply $\to$). For a set B of positions, if the rewrite step $s \to^p t$ satisfies $q \leq p$ for some $q \in B$, we write this step as $s \to^{(B\leq)} t$

We say that a rewrite step $s \to^p_{\mathcal{R}} t$ is *innermost* if every proper subterm of $s|_p$ is irreducible, denoted by $s \mathrel{\underset{\mathrm{i}}{\to}}^p_{\mathcal{R}} t$. Trivially, $\underset{\mathrm{i}}{\to} \subseteq \to$, and hence $\mathrm{SN}(\to)$ implies $\mathrm{SN}(\underset{\mathrm{i}}{\to})$. It is also shown that $\mathrm{NF}_{\to} = \mathrm{NF}_{\underset{\mathrm{i}}{\to}}$. We use NF to denote either set of normal forms. As notational convention, we use x, y for variables, s, t, u, v, w for terms, p for a position, σ, θ, τ for substitutions, and $l \to r$ for rules of TRSs.

A substitution τ is a *unifier* of s and t if $s\tau = t\tau$. Let τ be a unifier of s and t. If $\tau \leq \tau'$ for any unifier τ' of s and t, then we say τ a *most general unifier (mgu)* of s and t. Let $l_1 \to r_1$ and $l_2 \to r_2$ be rules in a rewrite system whose variables have been renamed as $\mathrm{Var}(l_1) \cap \mathrm{Var}(l_2) = \emptyset$. Let $p \in \mathrm{Pos}_{\mathcal{F}}(l_1)$ be a position such that $p \neq \varepsilon$ if $l_1 \to r_1$ and $l_2 \to r_2$ are identical. If there exists an mgu τ of $l_1|_p$ and l_2, then $\langle l_1\tau[r_2\tau]_p, r_1\tau\rangle$ is a *critical pair of* l_1 *and* l_2 *at* p *with* τ*. We simply call it a critical pair.* If $p = \varepsilon$, the pair is *overlay.* A TRS is called *overlay* if every critical pair is overlay. We write $\mathrm{CP}_{\mathcal{R}}$, or CP by omitting $\mathcal{R}$, to indicate the set of critical pairs of TRS $\mathcal{R}$.

A term s is rewritten to t in a *parallel step* (resp. *bidirectional parallel step*) if $s = s[s_1, \ldots, s_n]_{p_1,\ldots,p_n}$, $t = s[t_1, \ldots, t_n]_{p_1,\ldots,p_n}$, and $s_i \to t_i$ (resp. $s_i \leftrightarrow t_i$) for a set of parallel positions $P = \{p_1, \ldots, p_n\} \subseteq \mathrm{Pos}(s)$ and $1 \leq i \leq n$. The step is denoted by $s \mathrel{+\!\!\!\!\!\!\to} t$ (resp. $s \mathrel{\leftarrow\!\!\!\!\!+\!\!\!\!\!\to} t$). Observe that P can be empty, that is $s \mathrel{\leftarrow\!\!\!\!\!+\!\!\!\!\!\to} t$ can be $s = t$.

Given a TRS, a function symbol is *defined* if it appears at position ε in the left-hand-side of some rewrite rule. A *weakly shallow* term is a term in which defined function symbols appear only either at the position ε or in the ground subterms. A TRS is *left-weakly-shallow* if all the rewrite rules have weakly-shallow left-hand-sides [13]. Note that the left-weak-shallow class includes constructor systems, whose left-hand sides contain no defined symbols below the root.

Example 1 The following innermost terminating and locally confluent TRS $\mathcal{R}_1$ with non-overlay critical pairs is neither confluent nor terminating.

$$\mathcal{R}_1 = \{f(c) \to g(c),\ g(c) \to f(c),\ c \to d\}$$
$$\mathrm{CP} = \{\langle f(d), g(c)\rangle,\ \langle g(d), f(c)\rangle\}$$

3 A counterexample to the conjecture

The following open conjecture is a variant of the famous result on the confluence for terminating TRSs by Knuth and Bendix [9].

Conjecture 1 ([11]) An innermost terminating TRS $\mathcal{R}$ is confluent, if $u \downarrow_{\mathrm{i}} v$ for every critical pair $\langle u, v \rangle$ of $\mathcal{R}$.

The following counterexample shows that this conjecture does not hold.

Example 2

$$\begin{aligned} \mathcal{R}_2 =& \{g(x) \to h(k(x)),\ g(x) \to x,\ h(k(x)) \to f(x), \\ & f(x) \to x,\ k(c) \to d,\ f(c) \to g(c)\} \\ \mathrm{CP} =& \{\langle x, h(k(x)) \rangle,\ \langle h(d), f(c) \rangle,\ \langle c, g(c) \rangle\} \end{aligned}$$

$\mathcal{R}_2$ is innermost terminating and every critical pair of $\mathcal{R}_2$ is innermost joinable. $\mathcal{R}_2$ is, however, not confluent, since $c \overset{*}{\leftarrow} h(k(c)) \to h(d)$ but c and $h(d)$ are not joinable.

4 Characterizations for confluence of innermost terminating TRSs

This section gives two characterizations for confluence of innermost terminating TRSs. By combining the first characterization and some lemmas, we obtain the second characterization, which is the main theorem in this section: *An innermost terminating TRS $\mathcal{R}$ is confluent if and only if every critical pair $\langle u, v \rangle$ of $\mathcal{R}$ is innermost joinable for normalized instances (IJN); $u\sigma \downarrow_{\mathrm{i}} v\sigma$ for any normalized substitution σ.*

The first characterization shows that confluence can be decomposed into two properties.

Lemma 1 *For innermost terminating TRSs,* $\mathrm{CR}(\to)$ *if and only if* $\mathrm{CR}(\underset{\mathrm{i}}{\to})$ *and* $\to \subseteq \underset{\mathrm{i}}{\overset{*}{\leftrightarrow}}$

Proof Only-if-part. Suppose $s \underset{\mathrm{i}}{\overset{*}{\leftrightarrow}} t$ or $s \to t$. Since $\mathrm{SN}(\underset{\mathrm{i}}{\to})$, we have $s \underset{\mathrm{i}}{\overset{*}{\to}} s'$ and $t \underset{\mathrm{i}}{\overset{*}{\to}} t'$ for some $s', t' \in \mathrm{NF}$. Thus, $s' \overset{*}{\leftrightarrow} t'$. From $\mathrm{CR}(\to)$, we obtain $s' = t'$. Therefore, $s \underset{\mathrm{i}}{\overset{*}{\to}} \cdot \underset{\mathrm{i}}{\overset{*}{\leftarrow}} t$.

If-part. From $\to \subseteq \underset{\mathrm{i}}{\overset{*}{\leftrightarrow}}$ and $\mathrm{CR}(\underset{\mathrm{i}}{\to})$, we obtain $\overset{*}{\leftrightarrow} \subseteq \underset{\mathrm{i}}{\overset{*}{\leftrightarrow}} \subseteq \downarrow_{\mathrm{i}} \subseteq \downarrow$. Therefore $\mathrm{CR}(\to)$ holds. □

We define a property *IJN* as follows.

Definition 1 A critical pair $\langle u, v \rangle$ is *innermost-joinable for normalized instances (IJN)* if $u\sigma \underset{\mathrm{i}}{\overset{*}{\to}} \cdot \underset{\mathrm{i}}{\overset{*}{\leftarrow}} v\sigma$ for any normalized substitution σ. A TRS $\mathcal{R}$ enjoys the property *IJN* if all the critical pairs of $\mathcal{R}$ are IJN.

Now we give some lemmas to obtain the second characterization (Theorem 1) for confluence of innermost terminating TRSs.

Lemma 2 *Let $l \to r$ be a rule in $\mathcal{R}$, and σ be a normalized substitution. If $l\sigma \to r\sigma$ is a non-innermost step, then there exists a non-overlay critical pair $\langle u, v\rangle$ with a substitution τ such that*

1. *$l\sigma \mathrel{\underset{\mathrm{i}}{\to}} u\theta$ and $r\sigma = v\theta$ for some normalized substitution θ, and*
2. *τ is normalized.*

Proof Since $l\sigma \to r\sigma$ is a non-innermost step and σ is a normalized substitution, there exists an innermost step $l\sigma = l\sigma[l'\sigma']_p \mathrel{\underset{\mathrm{i}}{\to}} l\sigma[r'\sigma']_p$ for some rule $l' \to r' \in \mathcal{R}$, substitution σ', and rewriting position $p > \varepsilon$ which is in $\mathrm{Pos}_{\mathcal{F}}(l)$. Here, we can choose as σ' a normalized substitution so that $\mathrm{Dom}(\sigma') = \mathrm{Var}(l')$, because $l'\sigma'$ is an innermost redex. We can also assume $\mathrm{Var}(l) \cap \mathrm{Var}(l') = \emptyset$ and $\mathrm{Dom}(\sigma) \cap \mathrm{Dom}(\sigma') = \emptyset$ without loss of generality. This means that $\langle l\tau[r'\tau]_p, r\tau\rangle$ is a critical pair $\langle u, v\rangle$ with an mgu τ of $l|_p$ and l', where $u \leftarrow l\tau[l'\tau]_p = l\tau \to v$. Since τ is an mgu and $\sigma'' = \sigma \cup \sigma'$ is a unifier of $l|_p$ and l', there exists a substitution θ such that $\sigma'' = \tau\theta$. Here, σ'' is normalized, hence τ and θ are normalized. We will show that $l\sigma \mathrel{\underset{\mathrm{i}}{\to}} u\theta$ and $r\sigma = v\theta$ to complete the proof. We have

$$\begin{aligned} &l\sigma = l\sigma[l'\sigma']_p = l\tau\theta[l'\tau\theta]_p = (l\tau[l'\tau]_p)\theta, \quad \text{and} \\ &r\sigma = r\tau\theta. \end{aligned}$$

Therefore,

$$\begin{aligned} &l\sigma = (l\tau[l'\tau]_p)\theta \mathrel{\underset{\mathrm{i}}{\to}} (l\tau[r'\tau]_p)\theta = u\theta, \quad \text{and} \\ &r\sigma = (r\tau)\theta = v\theta. \end{aligned}$$

□

This lemma is used also in Sections 5 and 6.

Lemma 3 *If a TRS enjoys $\mathrm{SN}(\mathrel{\underset{\mathrm{i}}{\to}})$ and IJN then $\to \subseteq \downarrow_{\mathrm{i}}$.*

Proof We show that if $s \to t$ then $s \downarrow_{\mathrm{i}} t$, by Noetherian induction on $\{s, t\}$ with respect to the multiset extension of $\mathrel{\underset{\mathrm{i}}{\overset{+}{\to}}}$, where we use $>^{\mathrm{i}}_{\mathrm{mul}}$ to represent the multiset extension.

Since $s \downarrow_{\mathrm{i}} t$ is trivial if $s \to t$ is an innermost step, we assume that $s \to t$ is not innermost, and terms s and t are respectively represented by $s[l\sigma]_p$ and $s[r\sigma]_p$ for a substitution σ and a rule $l \to r$.

If $x\sigma$ is reducible for some $x \in \mathrm{Var}(l)$, there exists an innermost rewrite sequence $s = s[l\sigma]_p \mathrel{\underset{\mathrm{i}}{\overset{+}{\to}}} s[l\sigma']_p = s'$ for some substitution σ', hence the rewrite sequence $t = s[r\sigma]_p \mathrel{\underset{\mathrm{i}}{\overset{*}{\to}}} s[r\sigma']_p = t'$ is also possible. Since $s' \to t'$ and $\{s, t\} >^{\mathrm{i}}_{\mathrm{mul}} \{s', t'\}$, we have $s' \downarrow_{\mathrm{i}} t'$ by induction hypothesis. Thus $s \downarrow_{\mathrm{i}} t$.

Otherwise, by Lemma 2, there exist a critical pair $\langle u, v\rangle$ and a normalized substitution θ such that $s \mathrel{\underset{\mathrm{i}}{\to}} s[u\theta]_p$ and $t = s[v\theta]_p$. Since θ is a normalized substitution, $s[u\theta]_p \mathrel{\underset{\mathrm{i}}{\overset{*}{\to}}} \cdot \mathrel{\underset{\mathrm{i}}{\overset{*}{\leftarrow}}} s[v\theta]_p$ holds from the IJN property. Therefore $s \downarrow_{\mathrm{i}} t$.

□

Lemma 4 *For a TRS satisfying IJN, it holds that $\mathrel{\underset{\mathrm{i}}{\leftarrow}} \cdot \mathrel{\underset{\mathrm{i}}{\to}} \subseteq \downarrow_{\mathrm{i}}$.*

Proof Suppose $s \xleftarrow[\mathrm{i}]{} \cdot \xrightarrow[\mathrm{i}]{} t$. If the rewriting steps to s and t occur at the same position by different rules, s and t can be represented by $s[u\theta]_p$ and $s[v\theta]_p$ respectively for some critical pair $\langle u, v\rangle$ and substitution θ. Since s and t are obtained by innermost rewriting, θ is normalized as an immediate consequence from the definition of innermost rewriting. Hence, we have $s = s[u\theta]_p \xrightarrow[\mathrm{i}]{*} \cdot \xleftarrow[\mathrm{i}]{*} s[v\theta]_p = t$ by IJN property, so that $s \downarrow_{\mathrm{i}} t$ holds.

Otherwise, the rewriting steps to s and t occur at parallel positions. Therefore, there exists a term w such that $s \xrightarrow[\mathrm{i}]{} w \xleftarrow[\mathrm{i}]{} t$. □

Lemma 5 *If a TRS enjoys* $\mathrm{SN}(\xrightarrow[\mathrm{i}]{})$ *and IJN then* $\mathrm{CR}(\xrightarrow[\mathrm{i}]{})$.

Proof Since $\xrightarrow[\mathrm{i}]{}$ is terminating and also locally confluent by Lemma 4, it is confluent by Newman's lemma. Hence $\mathrm{CR}(\xrightarrow[\mathrm{i}]{})$. □

Combining Lemmas 1, 3, and 5, we obtain the following characterization.

Theorem 1 *Let a TRS $\mathcal{R}$ be innermost terminating. Then, $\mathcal{R}$ is confluent if and only if $\mathcal{R}$ is IJN, i.e., all the critical pairs are* innermost-joinable for normalized instances.

Theorem 1 gives a necessary and sufficient condition for confluence of innermost terminating TRSs. Unfortunately, it remains open whether the condition is decidable.

Open Problem. *Is confluence of innermost terminating TRSs decidable?*

5 A decidable sufficient condition

We have shown a necessary and sufficient condition for confluence of innermost terminating TRSs in Theorem 1 in the previous section. We, however, do not know whether the condition is decidable or not. Thus, we propose decidable sufficient conditions for confluence of innermost terminating TRSs.

We weaken the condition IJN by allowing a bidirectional parallel step.

Definition 2 A pair $\langle u, v\rangle$ of terms is *pseudo-innermost-joinable for normalized instances (PIJN)* if and only if $u\sigma \xrightarrow[\mathrm{i}]{*} \cdot \leftrightarrow\!\!\!\!\Vdash \cdot \xleftarrow[\mathrm{i}]{*} v\sigma$ for every normalized substitution σ. If all the critical pairs of $\mathcal{R}$ are PIJN, we say that $\mathcal{R}$ is PIJN.

The following lemma gives that PIJN and IJN properties are equivalent for innermost terminating systems.

Lemma 6 *For a TRS satisfying* $\mathrm{SN}(\xrightarrow[\mathrm{i}]{})$ *and PIJN, it holds that* $\leftrightarrow\!\!\!\!\Vdash \subseteq \downarrow_{\mathrm{i}}$.

Proof We show that if $s \leftrightarrow\!\!\!\!\Vdash t$ then $s \downarrow_{\mathrm{i}} t$ by Noetherian induction on $\{s, t\}$ with respect to multiset extension of $(\xrightarrow[\mathrm{i}]{} \cup \rhd)$, where we write $>^{\mathrm{i},\rhd}_{\mathrm{mul}}$ to represent the multiset extension. Note that $(\xrightarrow[\mathrm{i}]{} \cup \rhd)$ is terminating since $\rhd \cdot \xrightarrow[\mathrm{i}]{} \subseteq \xrightarrow[\mathrm{i}]{} \cdot \rhd$. Terms s and t can be represented as $s = s[s_1, \ldots, s_n]_{p_1,\ldots,p_n}$ and $t =$

$s[t_1,\ldots,t_n]_{p_1,\ldots,p_n}$ where $s_i \leftrightarrow t_i$ for $1 \leq i \leq n$ and set of parallel positions $P = \{p_1,\ldots,p_n\}$. We assume $P \neq \emptyset$, since the lemma is obvious if $P = \emptyset$.

In the case that $\varepsilon \notin P$, we obtain for each i that $\{s,t\} >^{\mathrm{i},\rhd}_{\mathrm{mul}} \{s_i,t_i\}$ and $s_i \leftrightarrow t_i$, and hence $s_i \xrightarrow[\mathrm{i}]{*} w_i \xleftarrow[\mathrm{i}]{*} t_i$ for some w_i by induction hypothesis. Therefore $s = s[s_1,\ldots,s_n]_{p_1,\ldots,p_n} \xrightarrow[\mathrm{i}]{*} s[w_1,\ldots,w_n]_{p_1,\ldots,p_n} \xleftarrow[\mathrm{i}]{*} s[t_1,\ldots,t_n]_{p_1,\ldots,p_n} = t$.

Otherwise $P = \{\varepsilon\}$, and hence we can assume $s \to t$ without loss of generality. If the step is innermost, it is trivial. Thus we suppose that it is a non-innermost step. We can write $s = l\sigma \to r\sigma = t$ for some rule $l \to r$ and substitution σ. If $\sigma|_{\mathrm{Var}(l)}$ is not normalized, then $s = l\sigma \xrightarrow[\mathrm{i}]{+} l\sigma'$ and $t = r\sigma \xrightarrow[\mathrm{i}]{*} r\sigma'$ for some substitution σ'. From $\{s,t\} >^{\mathrm{i},\rhd}_{\mathrm{mul}} \{l\sigma', r\sigma'\}$ and $l\sigma' \to r\sigma'$, by induction hypothesis $l\sigma' \downarrow_{\mathrm{i}} r\sigma'$. Thus $s \downarrow_{\mathrm{i}} t$. If $\sigma|_{\mathrm{Var}(l)}$ is normalized, then by Lemma 2, there exist a critical pair $\langle u,v\rangle$ and a normalized substitution θ such that $s \xrightarrow[\mathrm{i}]{} u\theta$ and $t = v\theta$. Since θ is a normalized substitution, $u\theta \xrightarrow[\mathrm{i}]{*} s' \leftrightarrow\!\!\!\!\!\!\!\!\;\;\Vert\;\; t' \xleftarrow[\mathrm{i}]{*} v\theta$ holds for some terms s' and t' from PIJN property. By induction hypothesis, we have $s' \downarrow_{\mathrm{i}} t'$ since $\{s,t\} >^{\mathrm{i},\rhd}_{\mathrm{mul}} \{s',t'\}$. Therefore $s \downarrow_{\mathrm{i}} t$. ◻

Lemma 7 *For a TRS $\mathcal{R}$ satisfying* $\mathrm{SN}(\xrightarrow[\mathrm{i}]{})$*, the property PIJN coincides with IJN.*

Proof Only-if-part. We show that $u\theta \xrightarrow[\mathrm{i}]{*} \cdot \xleftarrow[\mathrm{i}]{*} v\theta$ holds for any $\langle u,v\rangle \in \mathrm{CP}_{\mathcal{R}}$ and any normalized substitution θ. Since $\mathcal{R}$ is PIJN, $u\theta \xrightarrow[\mathrm{i}]{*} u' \leftrightarrow\!\!\!\!\!\!\!\!\;\;\Vert\;\; v' \xleftarrow[\mathrm{i}]{*} v\theta$ for some terms u' and v'. By Lemma 6 $u' \downarrow_{\mathrm{i}} v'$. Therefore $u\theta \xrightarrow[\mathrm{i}]{*} \cdot \xleftarrow[\mathrm{i}]{*} v\theta$.

If-part. Obviously *PIJN* holds for $\mathcal{R}$. ◻

If we succeed in finding a decidable sub-relation $\to'$ of $\xrightarrow[\mathrm{i}]{}$ such that $s \xrightarrow{*}' t$ implies $s\theta \xrightarrow{*}' t\theta$ for any normalized substitution, Theorem 1 and Lemma 7 induce a decidable sufficient condition for confluence of innermost terminating TRSs. We say that $s \xrightarrow[\mathrm{gi}]{} t$ is a *ground innermost rewrite step* if $s \xrightarrow[\mathrm{i}]{}^{p} t$ and $s|_p$ is ground. Since $\xrightarrow[\mathrm{gi}]{} \subseteq \xrightarrow[\mathrm{i}]{}$ and $\xrightarrow[\mathrm{gi}]{*}$ is closed under substitution, we obtain the following lemma.

Lemma 8 *A pair $\langle u,v\rangle$ satisfying $u \xrightarrow[\mathrm{gi}]{*} \cdot \leftrightarrow\!\!\!\!\!\!\!\!\;\;\Vert\;\; \cdot \xleftarrow[\mathrm{gi}]{*} v$ is PIJN.*

By Lemmas 8 and 7, and Theorem 1, we obtain the following sufficient condition for confluence.

Corollary 1 *Let a TRS $\mathcal{R}$ be innermost terminating. Then, $\mathcal{R}$ is confluent if all the critical pairs $\langle u,v\rangle$ enjoy $u \xrightarrow[\mathrm{gi}]{*} \cdot \leftrightarrow\!\!\!\!\!\!\!\!\;\;\Vert\;\; \cdot \xleftarrow[\mathrm{gi}]{*} v$.*

6 Extending the decidable sufficient condition

In this section, we extend the sufficient condition obtained in the previous section.

Definition 3 A term t is *stable* if $t\sigma$ is irreducible for any normalized substitution σ. A rule $l \to r$ is *left-stable* if every proper subterm of l is stable. We present a rewrite step by a left-stable rule as $\to_{\mathrm{ls}}$.

In other words, a rule $l \to r$ is left-stable if and only if the rewrite step $l\sigma \to r\sigma$ is innermost for any normalized substitution σ. Left-stability is decidable by the following lemma.

Lemma 9 *$l \to r \in \mathcal{R}$ is not left-stable if and only if there exist $l' \to r' \in \mathcal{R}$, a position p ($\neq \varepsilon$), and a critical pair of l and l' at a position p with a substitution τ such that $\tau|_{\mathrm{Var}(l)}$ is normalized.*

Proof If-part. It is trivial that $l\tau \to r\tau$ is non-innermost since $l\tau = l\tau[l'\tau]_p$. Only-if-part. Directly shown by Lemma 2. $\square$

Note that rewrite sequences $\xrightarrow[\mathrm{i}]{*}_{\mathrm{ls}}$ are not closed under normalized substitutions as shown by the following example.

Example 3 Consider $\mathcal{R} = \{f(x) \to g(h(x)),\ h(a) \to b\}$, whose rules are left-stable.

$$f(f(x)) \xrightarrow[\mathrm{i}]{}_{\mathrm{ls}} f(g(h(x))) \xrightarrow[\mathrm{i}]{}_{\mathrm{ls}} g(h(g(h(x))))$$
$$f(f(a)) \xrightarrow[\mathrm{i}]{}_{\mathrm{ls}} f(g(h(a))) \to_{\mathrm{ls}} g(h(g(h(a)))),$$

where the last step is not innermost.

Nevertheless, a TRS is confluent if every critical pair is innermost joinable with left-stable rules (Theorem 2). To show this, we use the notion of basic rewrite sequence [12], which is a rewriting version of basic narrowing [8]. Intuitively, a basic rewrite sequence admits no rewrite steps at positions in terms introduced by substitutions in previous rewrite steps.

We now give a formal definition. For a set B of parallel positions and a position p, we introduce notations: $B_{p<} = \{q \in B \mid p < q\}$, $B_{p\|} = \{q \in B \mid p \parallel q\}$. Note that for a position p such that $p \not\geq q$ for any $q \in B$, B is divided into $B_{p\|}$ and $B_{p<}$.

Definition 4 1. For a term s and a set B of parallel positions in $\mathrm{Pos}(s)$, the pair of s and B, represented by s^B, is an *annotated term*.

2. An annotated term s^B is rewritten to an annotated term $t^{B'}$ if $s \to^p t$, $p \not\geq q$ for any $q \in B$, and $B' = B_{p\|} \cup p.\mathrm{Pos}_{\mathcal{V}}(r)$. This is called a *basic rewrite step on annotated terms*, denoted by $s^B \xrightarrow[\mathrm{b}]{} t^{B'}$.
3. A rewrite sequence $s_1 \to s_2 \to \cdots \to s_n$ is *basic*, if there exist sets $B_1, \ldots, B_n$ of parallel positions such that $s_i^{B_i} \xrightarrow[\mathrm{b}]{} s_{i+1}^{B_{i+1}}$ for each i $(1 \leq i \leq n-1)$.

In figures of this section, we abuse annotated terms. For example, we use $u \xleftarrow[\mathrm{i}]{} s^B \xrightarrow[\mathrm{b}]{} v^{B'}$ to represent that $u \xleftarrow[\mathrm{i}]{} s$ and $s^B \xrightarrow[\mathrm{b}]{} v^{B'}$.

Example 4 Consider the innermost terminating TRS $\mathcal{R}_1$ in Example 1. It has an infinite basic rewrite sequence $f(c) \to g(c) \to f(c) \to \cdots$.

The following properties hold on basic rewrite sequences.

Proposition 1 *1. Basic rewrite steps are closed under substitution, i.e., $s^B \xrightarrow[\mathrm{b}]{} t^{B'}$ implies $(s\theta)^B \xrightarrow[\mathrm{b}]{} (t\theta)^{B'}$ for any substitution θ.*

2. All rewrite sequences $s \xrightarrow[\mathrm{i}]{*} t \twoheadrightarrow u$ are basic, i.e., $s^{\emptyset} \xrightarrow[\mathrm{b}]{*} t^{B} \xrightarrow[\mathrm{b}]{*} u^{B'}$ for some sets B and B' of parallel positions.

We next introduce a rewrite relation $\underset{\mathrm{i}}{\Rrightarrow}$ that satisfies all of the following properties.

1. $\underset{\mathrm{i}}{\Rrightarrow} \subseteq \xrightarrow[\mathrm{i}]{}$,
2. $\mathrm{NF}_{\underset{\mathrm{i}}{\Rrightarrow}} = \mathrm{NF}_{\xrightarrow[\mathrm{i}]{}}$, and
3. $\mathrm{CR}(\underset{\mathrm{i}}{\Rrightarrow})$.

This rewrite relation always exists, which is shown in the appendix by using the notion of priority rewrite systems (PRS) [3]. We write NF for the set of irreducible terms with respect to $\to$, $\xrightarrow[\mathrm{i}]{}$ and $\underset{\mathrm{i}}{\Rrightarrow}$ since all of them are equal by the property 2.

We prepare some lemmas.

Lemma 10 *For rewrite steps $s^{B} \xrightarrow[\mathrm{b}]{}_{\mathrm{ls}} t^{B'}$ and $s \underset{\mathrm{i}}{\overset{*}{\Rrightarrow}}{}^{(B\leq)} u$, there exists a term v such that $u \xrightarrow[\mathrm{i}]{*} v$ and $t \underset{\mathrm{i}}{\overset{*}{\Rrightarrow}}{}^{(B'\leq)} v$.*

Proof The outline is shown in Figure 1. There exist a left-stable rule $l \to r$, substitution σ, and a position $p \in \mathrm{Pos}(s)$ such that $s = s[l\sigma]_p$ and $t = s[r\sigma]_p$. Moreover, $p \not\geq q$ for any $q \in B$ by the definition of the rewrite relation. The sequence $s = s[l\sigma]_p \underset{\mathrm{i}}{\overset{*}{\Rrightarrow}}{}^{(B\leq)} u$ can be reformulated as $s = s[l\sigma]_p \underset{\mathrm{i}}{\overset{*}{\Rrightarrow}}{}^{((B_{p<})\leq)} s[u']_p \underset{\mathrm{i}}{\overset{*}{\Rrightarrow}}{}^{((B_{p\|})\leq)} u[u']_p = u$.

Let σ' be a substitution such that $x\sigma \underset{\mathrm{i}}{\overset{*}{\Rrightarrow}} x\sigma' \in \mathrm{NF}$ for each $x \in \mathrm{Var}(l)$. Then $l\sigma \underset{\mathrm{i}}{\overset{*}{\Rrightarrow}} l\sigma'$. Moreover $l\sigma \underset{\mathrm{i}}{\overset{*}{\Rrightarrow}}{}^{((B_{p<})\leq)} u'$ has no rewrite step at positions in $\mathrm{Pos}_{\mathcal{F}}(l)$ since $l \to r$ is left-stable. Hence by $\mathrm{CR}(\underset{\mathrm{i}}{\Rrightarrow})$ there exists a rewrite sequence $u' \underset{\mathrm{i}}{\overset{*}{\Rrightarrow}} l\sigma'$. Thus, we obtain a rewrite sequence $u = u[u']_p \underset{\mathrm{i}}{\overset{*}{\Rrightarrow}} u[l\sigma']_p \xrightarrow[\mathrm{i}]{} u[r\sigma']_p$, since the last innermost step is justified by the left-stable rule $l \to r$ and the normalized substitution σ'. Now, we take $u[r\sigma']_p$ as v.

On the other hand, we have $t = s[r\sigma]_p \underset{\mathrm{i}}{\overset{*}{\Rrightarrow}}{}^{((p.\mathrm{Pos}_{\mathcal{V}}(r))\leq)} s[r\sigma']_p \underset{\mathrm{i}}{\overset{*}{\Rrightarrow}}{}^{((B_{p\|})\leq)} u[r\sigma']_p$. Since $B' = B_{p\|} \cup p.\mathrm{Pos}_{\mathcal{V}}(r)$ where $p.\mathrm{Pos}_{\mathcal{V}}(r)$ and $B_{p\|}$ are parallel, it holds that $t = s[r\sigma]_p \underset{\mathrm{i}}{\overset{*}{\Rrightarrow}}{}^{(B'\leq)} u[r\sigma']_p = v$. Therefore we have completed the proof. □

Lemma 11 *If $u^{\emptyset} \xrightarrow[\mathrm{b}]{*}_{\mathrm{ls}} w^{B}$ and $v^{\emptyset} \xrightarrow[\mathrm{b}]{*}_{\mathrm{ls}} w^{B'}$, then $u \downarrow_{\mathrm{i}} v$.*

Proof By repeatedly applying Lemma 10 to the sequences, we obtain $u \xrightarrow[\mathrm{i}]{*} w' \underset{\mathrm{i}}{\overset{*}{\Lleftarrow}} w$ and $v \xrightarrow[\mathrm{i}]{*} w'' \underset{\mathrm{i}}{\overset{*}{\Lleftarrow}} w$. It follows from $\mathrm{CR}(\underset{\mathrm{i}}{\Rrightarrow})$ that $w' \Downarrow_{\mathrm{i}} w''$. Thus $u \downarrow_{\mathrm{i}} v$. □

Now we obtain the main lemma of this section.

Lemma 12 *If a critical pair of a TRS $\mathcal{R}$ is joinable by basic rewrite sequences with left-stable rules of $\mathcal{R}$, then the pair is IJN, and hence PIJN.*

Proof Let $\langle u, v\rangle$ be a critical pair and θ be a normalized substitution. Then, $u^{\emptyset} \xrightarrow[\mathrm{b}]{*}_{\mathrm{ls}} t^{B}$ and $v^{\emptyset} \xrightarrow[\mathrm{b}]{*}_{\mathrm{ls}} t^{B'}$ for some annotated terms t^{B} and $t^{B'}$. By part 1 of Proposition 1, properties $(u\theta)^{\emptyset} \xrightarrow[\mathrm{b}]{*}_{\mathrm{ls}} (t\theta)^{B}$ and $(v\theta)^{\emptyset} \xrightarrow[\mathrm{b}]{*}_{\mathrm{ls}} (t\theta)^{B'}$ hold. Therefore $u\theta \downarrow_{\mathrm{i}} v\theta$ follows from Lemma 11. □

$$\begin{tikzcd}[column sep=large, row sep=large]
 & s = (s[l\sigma]_p)^B \arrow[r, "\mathrm{b}"', "\mathrm{ls}"'] \arrow[dl, Rightarrow, "*_{((B_p<)\leq)}"', "\mathrm{i}"] \arrow[d, Rightarrow, dashed, "*_{((B_p<)\leq)}", "\mathrm{i}"'] \arrow[dl, phantom, "\text{(CR)}" near end, shift left=6ex] & (s[r\sigma]_p)^{B'} = t \arrow[d, Rightarrow, dashed, "*_{((p.\mathrm{Pos}_{\mathcal{V}}(r))\leq)}", "\mathrm{i}"'] \\
s[u']_p \arrow[r, Rightarrow, dashed, "*", "\mathrm{i}"'] \arrow[d, Rightarrow, "*_{((B_p\parallel)\leq)}", "\mathrm{i}"'] & s[l\sigma']_p \arrow[r, dashed, "\mathrm{i}"'] \arrow[d, Rightarrow, dashed, "*_{((B_p\parallel)\leq)}", "\mathrm{i}"'] & s[r\sigma']_p \arrow[d, Rightarrow, dashed, "*_{((B_p\parallel)\leq)}", "\mathrm{i}"'] \\
u = u[u']_p \arrow[r, Rightarrow, dashed, "*", "\mathrm{i}"'] & u[l\sigma']_p \arrow[r, dashed, "\mathrm{i}"'] & u[r\sigma']_p = v
\end{tikzcd}$$

Fig. 1 Proof diagram of Lemma 10

Theorem 2 *Let a TRS $\mathcal{R}$ be innermost terminating. Then, $\mathcal{R}$ is confluent if each critical pair $\langle u, v\rangle$ enjoys either*

1. $u \xrightarrow[\mathrm{gi}]{*} \cdot \leftarrow\!\!\!\Vert\!\!\!\rightarrow \cdot \xleftarrow[\mathrm{gi}]{*} v$, *or*
2. $u \xrightarrow[\mathrm{i}]{*}_{\mathrm{ls}} \cdot \rightarrow\!\!\!\!\!\Vert_{\mathrm{ls}} \cdot \Vert\!\!\!\!\!\leftarrow_{\mathrm{ls}} \cdot \xleftarrow[\mathrm{i}]{*}_{\mathrm{ls}} v$.

Proof Critical pairs satisfying the condition 1 are PIJN by Lemma 8. Critical pairs satisfying the condition 2 are also PIJN by 2 of Proposition 1 and Lemma 12. By combining Lemma 7 and Theorem 1, the theorem follows. □

Example 5 Consider the following innermost terminating $\mathcal{R}_3$.

$$\begin{aligned}
\mathcal{R}_3 = &\{d(c(x,x)) \to d(c(f(x), f(x))),\ f(x) \to g(x),\ c(g(x), g(x)) \to e,\\
&\ f(d(e)) \to g(d(e))\}\\
\mathrm{CP} = &\{\langle d(e), d(c(f(g(x)), f(g(x))))\rangle,\ \langle g(d(e)), g(d(e))\rangle\}
\end{aligned}$$

The first critical pair is innermost joinable by left-stable rules $f(x) \to g(x)$ and $c(g(x), g(x)) \to e$. Thus, it is confluent by Theorem 2.

Example 6 Consider the following innermost terminating $\mathcal{R}_4$.

$$\begin{aligned}
\mathcal{R}_4 = &\{h(k(x,x,e,e)) \to h(k(l(x), l(x), k(a,a,e,e), k(a,a,e,e))),\\
&\ k(a,a,y,y) \to g(l(a), y),\ g(a,e) \to e,\ g(l(x), e) \to g(x,e),\\
&\ k(l(l(x)), l(l(x)), y, y) \to g(x,y),\ k(x,x,h(e), g(e,e)) \to e\}\\
\mathrm{CP} = &\{\langle h(g(l(a), e)),\ h(k(l(a), l(a), k(a,a,e,e), k(a,a,e,e)))\rangle,\\
&\ \langle h(g(x,e)),\ h(k(l(l(l(x))), l(l(l(x))), k(a,a,e,e), k(a,a,e,e)))\rangle\}
\end{aligned}$$

The first critical pair is innermost joinable by ground innermost rewrite sequence, and the second critical pair is innermost joinable by left-stable rules. Thus, it is confluent by Theorem 2.

Innermost termination of these systems has been checked by AProVE [5]. We have tried to check their confluence by confluence checkers ACP [1,2], Saigawa [7], and CSI [14], but none of them reached any conclusion. Thus, this theorem seems beneficial for checking confluence of innermost terminating TRSs.

Theorem 2 induces some corollaries.

Corollary 2 *An innermost terminating left-weakly shallow TRS is confluent if and only if all the critical pairs are innermost joinable. Hence, confluence of such TRSs is decidable.*

Proof This corollary is shown by combining Theorem 2 and the fact that rules usable in innermost rewrite steps are left-stable for a left-weakly shallow TRS. □

This corollary can be compared with the undecidability of confluence for weakly-shallow (or flat) TRSs [10].

For overlay system, since every rule in an overlay TRS is left-stable the following corollary is immediately obtained from Theorem 2.

Corollary 3 *An innermost terminating overlay TRS is confluent if and only if all the critical pairs are innermost joinable.*

This corollary may be a bit weaker than the result of [6], in which the condition is that all its critical pairs are joinable, i.e., in the sense that it does not require innermost joinability. On the other hand, the latter result is shown via termination property of the TRS.

7 Conclusion

We have given a negative answer to the open conjecture posed by E. Ohlebusch [11], and shown some conditions necessary and sufficient for confluence of innermost terminating TRSs. Using one of the conditions, we have given decidable sufficient conditions for the confluence. It has also been shown that confluence of innermost terminating left-weakly shallow TRSs is decidable. At present, the problem of deciding whether an innermost terminating TRS is confluent, remains open.

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A Priority rewriting system

We show that there exists a rewrite relation $\underset{\mathrm{i}}{\Rightarrow}$ satisfying that $\underset{\mathrm{i}}{\Rightarrow} \subseteq \underset{\mathrm{i}}{\to}$, $\mathrm{NF}_{\underset{\mathrm{i}}{\Rightarrow}} = \mathrm{NF}_{\underset{\mathrm{i}}{\to}}$, and $\mathrm{CR}(\underset{\mathrm{i}}{\Rightarrow})$ by using priority term rewriting systems (PRS) [3].

Definition 5 A *priority term rewriting system (PRS)* [3] is a pair $(\mathcal{R}, \blacktriangleright)$ of a TRS $\mathcal{R}$ and a partial order $\blacktriangleright$ on the rules of $\mathcal{R}$. A rule $l_1 \to r_1$ has a higher priority than a rule $l_2 \to r_2$ iff $l_1 \to r_1 \blacktriangleright l_2 \to r_2$. A *priority rewrite step* is defined as: $s \xrightarrow[\mathrm{i},\blacktriangleright]{} t$ iff for a substitution σ, a position $p \in \mathrm{Pos}(s)$ and a rule $l \to r \in \mathcal{R}$,

- $s = s[l\sigma]_p \underset{\mathrm{i}}{\to} s[r\sigma]_p = t$ and
- $l \to r$ is maximal with respect to $\blacktriangleright$ among rules that reduce $l\sigma$, i.e. $l' \to r' \blacktriangleright l \to r$ for any different rule $l' \to r' \in \mathcal{R}$ such that $l\sigma = l'\sigma'$ for some σ'.

It is clear that $\xrightarrow[\mathrm{i},\blacktriangleright]{} \subseteq \underset{\mathrm{i}}{\to}$.

Example 7 For a PRS

$$\begin{array}{ll} f(g(x)) \to b & (1) \\ g(a) \to c & (2) \text{ and } (1) \blacktriangleright (2) \blacktriangleright (3), \\ g(x) \to x & (3) \end{array}$$

$f(g(a)) \xrightarrow[\mathrm{i},\blacktriangleright]{} f(c)$, but $f(g(a)) \underset{\mathrm{i}}{\to} f(a)$ is not a priority rewrite step.

Lemma 13 $\mathrm{NF}_{\to} = \mathrm{NF}_{\underset{\mathrm{i},\blacktriangleright}{\to}}$

Proof Since $\mathrm{NF}_{\to} = \mathrm{NF}_{\underset{\mathrm{i}}{\to}}$, we will show $\mathrm{NF}_{\underset{\mathrm{i}}{\to}} = \mathrm{NF}_{\underset{\mathrm{i},\blacktriangleright}{\to}}$. For $\mathrm{NF}_{\underset{\mathrm{i}}{\to}} \subseteq \mathrm{NF}_{\underset{\mathrm{i},\blacktriangleright}{\to}}$, it is obvious from $\underset{\mathrm{i},\blacktriangleright}{\to} \subseteq \underset{\mathrm{i}}{\to}$.

Now, we prove $\mathrm{NF}_{\underset{\mathrm{i},\blacktriangleright}{\to}} \subseteq \mathrm{NF}_{\underset{\mathrm{i}}{\to}}$ by contradiction. We assume that $t \underset{\mathrm{i}}{\to} t'$ for $t \in \mathrm{NF}_{\underset{\mathrm{i},\blacktriangleright}{\to}}$. Then there exist a rule $l \to r \in \mathcal{R}$, normalized substitution σ and position $p \in \mathrm{Pos}(t)$ such that $t = t[l\sigma]_p$ and $t' = t[r\sigma]_p$. Since $t \in \mathrm{NF}_{\underset{\mathrm{i},\blacktriangleright}{\to}}$, $l\sigma \underset{\mathrm{i},\blacktriangleright}{\to} r\sigma$ is not possible. This means that there must exist $l' \to r' \in \mathcal{R}$ such that $l' \to r' \blacktriangleright l \to r$ and $l\sigma = l'\sigma'$ for some substitution σ'. Note that $l'\sigma' \underset{\mathrm{i}}{\to} r'\sigma'$ since every proper subterm of $l\sigma(= l'\sigma')$ is irreducible. This means that t can be reduced by $\underset{\mathrm{i},\blacktriangleright}{\to}$. This is a contradiction. $\square$

Lemma 14 *If* $\blacktriangleright$ *is total then* $\mathrm{CR}(\underset{\mathrm{i},\blacktriangleright}{\to})$.

Proof We first show that $u \underset{\mathrm{i},\blacktriangleright}{\leftarrow} \cdot \underset{\mathrm{i},\blacktriangleright}{\to} v$ implies $u = v$ or $u \underset{\mathrm{i},\blacktriangleright}{\to} \cdot \underset{\mathrm{i},\blacktriangleright}{\leftarrow} v$. If the rewriting steps to u and v occur at the same position, $u = v$ since the same rule is used for the rewriting from the totality of $\blacktriangleright$. Otherwise, the rewriting steps to u and v occur at parallel positions, and hence there exists a term t such that $u \underset{\mathrm{i},\blacktriangleright}{\to} t \underset{\mathrm{i},\blacktriangleright}{\leftarrow} v$. Therefore the lemma is easily obtained. $\square$