

# ULRICH MODULES OVER COHEN–MACAULAY LOCAL RINGS WITH MINIMAL MULTIPLICITY

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ABSTRACT. Let  $R$  be a Cohen–Macaulay local ring. In this paper we study the structure of Ulrich  $R$ -modules mainly in the case where  $R$  has minimal multiplicity. We explore generation of Ulrich  $R$ -modules, and clarify when the Ulrich  $R$ -modules are precisely the syzygies of maximal Cohen–Macaulay  $R$ -modules. We also investigate the structure of Ulrich  $R$ -modules as an exact category.

## INTRODUCTION

The notion of an *Ulrich module*, which is also called a *maximally generated (maximal) Cohen–Macaulay module*, has first been studied by Ulrich [29], and widely investigated in both commutative algebra and algebraic geometry; see [2, 4, 5, 9, 10, 12, 19, 22] for example. In [29] the natural question is posed to ask whether Ulrich modules exist over any Cohen–Macaulay local ring  $R$ . A lot of partial affirmative answers to this question have been obtained so far. One of them states that the conjecture holds whenever  $R$  has minimal multiplicity ([2]). Thus, in this paper, mainly assuming that  $R$  has minimal multiplicity, we are interested in what we can say about the structure of Ulrich  $R$ -modules.

We begin with exploring the number and generation of Ulrich modules. The following theorem is a special case of our main results in this direction ( $\Omega$  denotes the first syzygy).

**Theorem A.** *Let  $(R, \mathfrak{m}, k)$  be a  $d$ -dimensional complete Cohen–Macaulay local ring.*

- (1) *Assume that  $R$  is normal with  $d = 2$  and  $k = \mathbb{C}$  and has minimal multiplicity. If  $R$  does not have a rational singularity, then there exist infinitely many indecomposable Ulrich  $R$ -modules.*
- (2) *Let  $M, N$  be maximal Cohen–Macaulay  $R$ -modules with  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq d - 1$ . If either  $M$  or  $N$  is Ulrich, then so is  $\text{Hom}_R(M, N)$ .*
- (3) *Let  $\mathbf{x} = x_1, \dots, x_d$  be a system of parameters of  $R$  such that  $\mathfrak{m}^2 = \mathbf{x}\mathfrak{m}$ . If  $M$  is an Ulrich  $R$ -module, then so is  $\Omega(M/x_i M)$  for all  $1 \leq i \leq d$ . If one chooses  $M$  to be indecomposable and not to be a direct summand of  $\Omega^d k$ , then one finds an indecomposable Ulrich  $R$ -module not isomorphic to  $M$  among the direct summands of the modules  $\Omega(M/x_i M)$ .*

Next, we relate the Ulrich modules with the syzygies of maximal Cohen–Macaulay modules. To state our result, we fix some notation. Let  $R$  be a Cohen–Macaulay local ring with canonical module  $\omega$ . We denote by  $\text{mod } R$  the category of finitely generated  $R$ -modules, and by  $\text{Ul}(R)$  and  $\Omega\text{CM}^\times(R)$  the full subcategories of Ulrich modules and first syzygies of maximal Cohen–Macaulay modules without free summands, respectively. Denote by  $(-)^{\dagger}$  the canonical dual  $\text{Hom}_R(-, \omega)$ . Then  $\text{Ul}(R)$  is closed under  $(-)^{\dagger}$ , and contains  $\Omega\text{CM}^\times(R)$  if  $R$  has minimal multiplicity. The module  $\Omega^d k$  belongs to  $\Omega\text{CM}^\times(R)$ , and hence  $\Omega^d k, (\Omega^d k)^{\dagger}$  belong to  $\text{Ul}(R)$ . Thus it is natural to ask when the conditions in the theorem below hold, and we actually answer this.

**Theorem B.** *Let  $R$  be a  $d$ -dimensional singular Cohen–Macaulay local ring with residue field  $k$  and canonical module  $\omega$ , and assume that  $R$  has minimal multiplicity. Consider the following conditions.*

- (1) *The equality  $\text{Ul}(R) = \Omega\text{CM}^\times(R)$  holds.*
- (2) *The category  $\Omega\text{CM}^\times(R)$  is closed under  $(-)^{\dagger}$ .*
- (3) *The module  $(\Omega^d k)^{\dagger}$  belongs to  $\Omega\text{CM}^\times(R)$ .*
- (4) *One has  $\text{Tor}_1(\text{Tr}(\Omega^d k)^{\dagger}, \omega) = 0$ .*
- (5) *One has  $\text{Ext}_R^{d+1}(\text{Tr}(\Omega^d k)^{\dagger}, R) = 0$  and  $R$  is locally Gorenstein on the punctured spectrum.*

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- (6) There is an epimorphism  $\omega^{\oplus n} \rightarrow \Omega^d k$  for some  $n > 0$ .
- (7) There is an isomorphism  $\Omega^d k \cong (\Omega^d k)^\dagger$ .
- (8) The local ring  $R$  is almost Gorenstein (see Definition 3.8).

Then (1)–(6) are equivalent and (7) implies (1). If  $d > 0$  and  $k$  is infinite, then (1) implies (8). If  $d = 1$  and  $k$  is infinite, then (1)–(8) are equivalent. If  $R$  is complete normal with  $d = 2$  and  $k = \mathbb{C}$ , then (1)–(7) are equivalent unless  $R$  has a cyclic quotient singularity.

Finally, we study the structure of the category  $\text{Ul}(R)$  of Ulrich  $R$ -modules as an exact category in the sense of Quillen [24]. We prove that if  $R$  has minimal multiplicity, then  $\text{Ul}(R)$  admits an exact structure with enough projective/injective objects.

**Theorem C.** *Let  $R$  be a  $d$ -dimensional Cohen–Macaulay local ring with residue field  $k$  and canonical module, and assume that  $R$  has minimal multiplicity. Let  $\mathcal{S}$  be the class of short exact sequences  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  of  $R$ -modules with  $L, M, N$  Ulrich. Then  $(\text{Ul}(R), \mathcal{S})$  is an exact category having enough projective objects and enough injective objects with  $\text{proj Ul}(R) = \text{add } \Omega^d k$  and  $\text{inj Ul}(R) = \text{add}(\Omega^d k)^\dagger$ .*

The organization of this paper is as follows. In Section 1, we deal with a question of Cuong on the number of indecomposable Ulrich modules. We prove the first assertion of Theorem A to answer this question in the negative. In Section 2, we consider how to generate Ulrich modules from given ones, and prove the second and third assertions of Theorem A. In Section 3, we compare Ulrich modules with syzygies of maximal Cohen–Macaulay modules, and prove Theorem B; in fact, we obtain more equivalent and related conditions. The final Section 4 is devoted to giving applications of the results obtained in Section 3. In this section we study the cases of dimension one and two, and exact structures of Ulrich modules, and prove the rest assertions of Theorem B and Theorem C.

## CONVENTION

Throughout, let  $(R, \mathfrak{m}, k)$  be a Cohen–Macaulay local ring of Krull dimension  $d$ . We assume that all modules are finitely generated and all subcategories are full. A maximal Cohen–Macaulay module is simply called a Cohen–Macaulay module. For an  $R$ -module  $M$  we denote by  $\Omega M$  the first syzygy of  $M$ , that is, the kernel of the first differential map in the minimal free resolution of  $M$ . Whenever  $R$  admits a canonical module  $\omega$ , we denote by  $(-)^\dagger$  the canonical dual functor  $\text{Hom}_R(-, \omega)$ . For an  $R$ -module  $M$  we denote by  $e(M)$  and  $\mu(M)$  the multiplicity and the minimal number of generators of  $M$ , respectively.

## 1. A QUESTION OF CUONG

In this section, we consider a question raised by Cuong [6] on the number of Ulrich modules over Cohen–Macaulay local rings with minimal multiplicity. First of all, let us recall the definitions of an Ulrich module and minimal multiplicity.

**Definition 1.1.** (1) An  $R$ -module  $M$  is called *Ulrich* if  $M$  is Cohen–Macaulay with  $e(M) = \mu(M)$ .  
 (2) The ring  $R$  is said to have *minimal multiplicity* if  $e(R) = \text{edim } R - \dim R + 1$ .

An Ulrich module is also called a *maximally generated (maximal) Cohen–Macaulay* module. There is always an inequality  $e(R) \geq \text{edim } R - \dim R + 1$ , from which the name of minimal multiplicity comes. If  $k$  is infinite, then  $R$  has minimal multiplicity if and only if  $\mathfrak{m}^2 = Q\mathfrak{m}$  for some parameter ideal  $Q$  of  $R$ . See [3, Exercise 4.6.14] for details of minimal multiplicity.

The following question has been raised by Cuong [6].

**Question 1.2** (Cuong). If  $R$  is non-Gorenstein and has minimal multiplicity, then are there only finitely many indecomposable Ulrich  $R$ -modules?

To explore this question, we start by introducing notation, which is used throughout the paper.

**Notation 1.3.** We denote by  $\text{mod } R$  the category of finitely generated  $R$ -modules. We use the following subcategories of  $\text{mod } R$ :

$$\begin{aligned} \text{CM}(R) &= \{M \in \text{mod } R \mid M \text{ is Cohen–Macaulay}\}, \\ \text{Ul}(R) &= \{M \in \text{CM}(R) \mid M \text{ is Ulrich}\}, \end{aligned}$$

$$\Omega\text{CM}(R) = \left\{ M \in \text{CM}(R) \mid \begin{array}{l} M \text{ is the kernel of an epimorphism from a} \\ \text{free module to a Cohen–Macaulay module} \end{array} \right\},$$

$$\Omega\text{CM}^\times(R) = \{ M \in \Omega\text{CM}(R) \mid M \text{ does not have a (nonzero) free summand} \}.$$

**Remark 1.4.** (1) The subcategories  $\text{CM}(R), \text{Ul}(R), \Omega\text{CM}(R), \Omega\text{CM}^\times(R)$  of  $\text{mod } R$  are closed under finite direct sums and direct summands.  
(2) One has  $\Omega\text{CM}(R) \cup \text{Ul}(R) \subseteq \text{CM}(R) \subseteq \text{mod } R$ .

Here we make a remark to reduce to the case where the residue field is infinite.

**Remark 1.5.** Consider the faithfully flat extension  $S := R[t]_{\mathfrak{m}_R[t]}$  of  $R$ . Then we observe that:

- (1) If  $X$  is a module in  $\Omega\text{CM}^\times(R)$ , then  $X \otimes_R S$  is in  $\Omega\text{CM}^\times(S)$ .
- (2) A module  $Y$  is in  $\text{Ul}(R)$  if and only if  $Y \otimes_R S$  is in  $\text{Ul}(S)$  (see [13, Lemma 6.4.2]).

The converse of (1) also holds true; we prove this in Corollary 3.4.

If  $R$  has minimal multiplicity, then all syzygies of Cohen–Macaulay modules are Ulrich:

**Proposition 1.6.** *Suppose that  $R$  has minimal multiplicity. Then  $\Omega\text{CM}^\times(R)$  is contained in  $\text{Ul}(R)$ .*

*Proof.* By Remark 1.5 we may assume that  $k$  is infinite. Since  $R$  has minimal multiplicity, we have  $\mathfrak{m}^2 = Q\mathfrak{m}$  for some parameter ideal  $Q$  of  $R$ . Let  $M$  be a Cohen–Macaulay  $R$ -module. There is a short exact sequence  $0 \rightarrow \Omega M \rightarrow R^{\oplus n} \rightarrow M \rightarrow 0$ , where  $n$  is the minimal number of generators of  $M$ . Since  $M$  is Cohen–Macaulay, taking the functor  $R/Q \otimes_R -$  preserves the exactness; we get a short exact sequence

$$0 \rightarrow \Omega M/Q\Omega M \xrightarrow{f} (R/Q)^{\oplus n} \rightarrow M/QM \rightarrow 0.$$

The map  $f$  factors through the inclusion map  $X := \mathfrak{m}(R/Q)^{\oplus n} \rightarrow (R/Q)^{\oplus n}$ , and hence there is an injection  $\Omega M/Q\Omega M \rightarrow X$ . As  $X$  is annihilated by  $\mathfrak{m}$ , so is  $\Omega M/Q\Omega M$ . Therefore  $\mathfrak{m}\Omega M = Q\Omega M$ , which implies that  $\Omega M$  is Ulrich. ■

As a direct consequence of [7, Corollary 3.3], we obtain the following proposition.

**Proposition 1.7.** *Let  $R$  be a 2-dimensional normal excellent henselian local ring with algebraically closed residue field of characteristic 0. Then there exist only finitely many indecomposable modules in  $\Omega\text{CM}(R)$  if and only if  $R$  has a rational singularity.*

Combining the above propositions yields the following result.

**Corollary 1.8.** *Let  $R$  be a 2-dimensional normal excellent henselian local ring with algebraically closed residue field of characteristic 0. Suppose that  $R$  has minimal multiplicity and does not have a rational singularity. Then there exist infinitely many indecomposable Ulrich  $R$ -modules. In particular, Question 1.2 has a negative answer.*

*Proof.* Proposition 1.7 implies that  $\Omega\text{CM}(R)$  contains infinitely many indecomposable modules, and so does  $\text{Ul}(R)$  by Proposition 1.6. ■

Here is an example of a non-Gorenstein ring satisfying the assumption of Corollary 1.8, which concludes that the question of Cuong is negative.

**Example 1.9.** Let  $B = \mathbb{C}[x, y, z, t]$  be a polynomial ring with  $\deg x = \deg t = 3$ ,  $\deg y = 5$  and  $\deg z = 7$ . Consider the  $2 \times 3$ -matrix  $M = \begin{pmatrix} x & y & z \\ y & z & x^3 - t^3 \end{pmatrix}$  over  $B$ , and let  $I$  be the ideal of  $B$  generated by  $2 \times 2$ -minors of  $M$ . Set  $A = B/I$ . Then  $A$  is a nonnegatively graded  $\mathbb{C}$ -algebra as  $I$  is homogeneous. By virtue of the Hilbert–Burch theorem ([3, Theorem 1.4.17]),  $A$  is a 2-dimensional Cohen–Macaulay ring, and  $x, t$  is a homogeneous system of parameters of  $A$ . Directly calculating the Jacobian ideal  $J$  of  $A$ , we can verify that  $A/J$  is Artinian. The Jacobian criterion implies that  $A$  is a normal domain. The quotient ring  $A/tA$  is isomorphic to the numerical semigroup ring  $\mathbb{C}[H]$  with  $H = \langle 3, 5, 7 \rangle$ . Since this ring is not Gorenstein (as  $H$  is not symmetric), neither is  $A$ . Let  $a(A)$  and  $F(H)$  stand for the  $a$ -invariant of  $A$  and the Frobenius number of  $H$ , respectively. Then

$$a(A) + 3 = a(A) + \deg(t) = a(A/tA) = F(H) = 4,$$

where the third equality follows from [25, Theorem 3.1]. Therefore we get  $a(A) = 1 \not\leq 0$ , and  $A$  does not have a rational singularity by the Flenner–Watanabe criterion (see [20, Page 98]).

Let  $A'$  be the localization of  $A$  at  $A_+$ , and let  $R$  be the completion of the local ring  $A'$ . Then  $R$  is a 2-dimensional complete (hence excellent and henselian) normal non-Gorenstein local domain with residue field  $\mathbb{C}$ . The maximal ideal  $\mathfrak{m}$  of  $R$  satisfies  $\mathfrak{m}^2 = (x, t)\mathfrak{m}$ , and thus  $R$  has minimal multiplicity. Having a rational singularity is preserved by localization since  $A$  has an isolated singularity, while it is also preserved by completion. Therefore  $R$  does not have a rational singularity.

We have seen that Question 1.2 is not true in general. However, in view of Corollary 1.8, we wonder if having a rational singularity is essential. Thus, we pose a modified question.

**Question 1.10.** Let  $R$  be a 2-dimensional normal local ring with a rational singularity. Then does  $R$  have only finitely many indecomposable Ulrich modules?

Proposition 1.7 leads us to an even stronger question:

**Question 1.11.** If  $\Omega\text{CM}(R)$  contains only finitely many indecomposable modules, then does  $\text{Ul}(R)$  so?

## 2. GENERATING ULRICH MODULES

In this section, we study how to generate Ulrich modules from given ones. First of all, we consider using the Hom functor to do it.

**Proposition 2.1.** *Let  $M, N$  be Cohen–Macaulay  $R$ -modules such that  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq d - 1$ . If either  $M$  or  $N$  is Ulrich, then so is  $\text{Hom}_R(M, N)$ .*

*Proof.* Take a free resolution

$$\cdots \rightarrow F_{d+1} \rightarrow F_d \rightarrow F_{d-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

of the  $R$ -module  $M$ . Dualizing this by  $N$  and using the assumption on Ext, we get an exact sequence

$$0 \rightarrow \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(F_0, N) \rightarrow \text{Hom}_R(F_1, N) \rightarrow \cdots \rightarrow \text{Hom}_R(F_{d-1}, N) \rightarrow \text{Hom}_R(F_d, N).$$

The depth lemma implies that  $\text{Hom}_R(M, N)$  is a Cohen–Macaulay  $R$ -module.

We may assume that  $k$  is infinite by Remark 1.5(2), so that we can find a reduction  $Q$  of  $\mathfrak{m}$  which is a parameter ideal of  $R$ . Write  $Q = (x_1, \dots, x_d)$ .

We show the assertion by induction on  $d$ . Let us consider the case  $d = 1$ . There are exact sequences

$$0 \rightarrow N \xrightarrow{x_1} N \rightarrow N/QN \rightarrow 0, \quad 0 \rightarrow M \xrightarrow{x_1} M \rightarrow M/QM \rightarrow 0,$$

which induce injections

$$\begin{aligned} \text{Hom}_R(M, N)/Q \text{Hom}_R(M, N) &\hookrightarrow \text{Hom}_R(M, N/QN), \\ \text{Hom}_R(M, N)/Q \text{Hom}_R(M, N) &\hookrightarrow \text{Ext}_R^1(M/QM, N). \end{aligned}$$

If  $N$  (resp.  $M$ ) is Ulrich, then  $N/QN$  (resp.  $M/QM$ ) is annihilated by  $\mathfrak{m}$ , and so is  $\text{Hom}_R(M, N/QN)$  (resp.  $\text{Ext}_R^1(M/QM, N)$ ). In either case, the quotient module  $\text{Hom}_R(M, N)/Q \text{Hom}_R(M, N)$  is annihilated by  $\mathfrak{m}$ , which shows that  $\text{Hom}_R(M, N)$  is an Ulrich  $R$ -module.

Next we consider the case  $d \geq 2$ . Clearly,  $M/x_1M$  and  $N/x_1N$  are Cohen–Macaulay  $R/(x_1)$ -modules. There are isomorphisms

$$\begin{aligned} \text{Ext}_{R/(x_1)}^i(M/x_1M, N/x_1N) &\cong \text{Ext}_R^{i+1}(M/x_1M, N) \\ &\cong \begin{cases} 0 & (1 \leq i \leq d - 2), \\ \text{Hom}_R(M, N)/x_1 \text{Hom}_R(M, N) & (i = 0), \end{cases} \end{aligned}$$

where the first isomorphism follows from [3, Lemma 3.1.16], and the second isomorphism is shown by using the exact sequence  $0 \rightarrow M \xrightarrow{x_1} M \rightarrow M/x_1M \rightarrow 0$  and our assumption on Ext. Furthermore, it is easily observed that if  $M$  (resp.  $N$ ) is an Ulrich  $R$ -module, then  $M/x_1M$  (resp.  $N/x_1N$ ) is an Ulrich  $R/(x_1)$ -module. The induction hypothesis implies that  $\text{Hom}_{R/(x_1)}(M/x_1M, N/x_1N)$  is an Ulrich  $R/(x_1)$ -module, which is isomorphic to  $\text{Hom}_R(M, N)/x_1 \text{Hom}_R(M, N)$ . Now it is deduced that  $\text{Hom}_R(M, N)$  is an Ulrich  $R$ -module.  $\blacksquare$

As an immediate consequence of Proposition 2.1, we obtain the following corollary, which is a special case of [9, Theorem 5.1].

**Corollary 2.2.** *Suppose that  $R$  admits a canonical module. If  $M \in \text{Ul}(R)$ , then  $M^\dagger \in \text{Ul}(R)$ .*

Next, we consider taking extensions of given Ulrich modules to obtain a new one.

**Proposition 2.3.** *Let  $Q$  be a parameter ideal of  $R$  which is a reduction of  $\mathfrak{m}$ . Let  $M, N$  be Ulrich  $R$ -modules, and take any element  $a \in Q$ . Let  $\sigma : 0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0$  be an exact sequence, and consider the multiplication  $a\sigma : 0 \rightarrow M \rightarrow X \rightarrow N \rightarrow 0$  as an element of the  $R$ -module  $\text{Ext}_R^1(N, M)$ . Then  $X$  is an Ulrich  $R$ -module.*

*Proof.* It follows from [26, Theorem 1.1] that the exact sequence

$$a\sigma \otimes_R R/aR : 0 \rightarrow M/aM \rightarrow X/aX \rightarrow N/aN \rightarrow 0$$

splits; we have an isomorphism  $X/aX \cong M/aM \oplus N/aN$ . Applying the functor  $-\otimes_{R/aR} R/Q$ , we get an isomorphism  $X/QX \cong M/QM \oplus N/QN$ . Since  $M, N$  are Ulrich, the modules  $M/QM, N/QN$  are  $k$ -vector spaces, and so is  $X/QX$ . Hence  $X$  is also Ulrich.  $\blacksquare$

As an application of the above proposition, we give a way to make an Ulrich module over a Cohen–Macaulay local ring with minimal multiplicity.

**Corollary 2.4.** *Let  $Q$  be a parameter ideal of  $R$  such that  $\mathfrak{m}^2 = Q\mathfrak{m}$ . Let  $M$  be an Ulrich  $R$ -module. Then for each  $R$ -regular element  $a \in Q$ , the syzygy  $\Omega(M/aM)$  is also an Ulrich  $R$ -module.*

*Proof.* There is an exact sequence  $\sigma : 0 \rightarrow \Omega M \rightarrow R^{\oplus n} \rightarrow M \rightarrow 0$ , where  $n$  is a minimal number of generators of  $M$ . We have a commutative diagram

$$\begin{array}{ccccccc} & & & & 0 & & 0 \\ & & & & \uparrow & & \uparrow \\ & & & & M/aM & \equiv & M/aM \\ & & & & \uparrow & & \uparrow \\ \sigma : & 0 & \longrightarrow & \Omega M & \longrightarrow & R^{\oplus n} & \longrightarrow & M & \longrightarrow & 0 \\ & & & \parallel & & \uparrow & & a\uparrow & & \\ a\sigma : & 0 & \longrightarrow & \Omega M & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & & \\ & & & & 0 & & 0 & & & \end{array}$$

with exact rows and columns. Since the minimal number of generators of  $M/aM$  is equal to  $n$ , the middle column shows  $X \cong \Omega(M/aM)$ . Propositions 1.2 and 2.3 show that  $X$  is Ulrich, and we are done.  $\blacksquare$

**Remark 2.5.** In Corollary 2.4, if the parameter ideal  $Q$  annihilates the  $R$ -module  $\text{Ext}_R^1(M, \Omega M)$ , then we have  $a\sigma = 0$ , and  $\Omega(M/aM) \cong M \oplus \Omega M$ . Hence, in this case, the operation  $M \mapsto \Omega(M/aM)$  does not produce an essentially new Ulrich module.

Next, we investigate the annihilators of Tor and Ext modules.

**Proposition 2.6.** *For an  $R$ -module  $M$  one has*

$$\begin{aligned} \text{ann}_R \text{Ext}_R^1(M, \Omega M) &= \bigcap_{i>0, N \in \text{mod } R} \text{ann}_R \text{Ext}_R^i(M, N) \\ &= \text{ann}_R \text{Tor}_1^R(M, \text{Tr}M) = \bigcap_{i>0, N \in \text{mod } R} \text{ann}_R \text{Tor}_i^R(M, N). \end{aligned}$$

*Proof.* It is clear that

$$\begin{aligned} I &:= \bigcap_{i>0, N \in \text{mod } R} \text{ann}_R \text{Ext}_R^i(M, N) \subseteq \text{ann}_R \text{Ext}_R^1(M, \Omega M) \\ J &:= \bigcap_{i>0, N \in \text{mod } R} \text{ann}_R \text{Tor}_i^R(M, N) \subseteq \text{ann}_R \text{Tor}_1^R(M, \text{Tr}M). \end{aligned}$$

It is enough to show that  $\text{ann} \text{Ext}^1(M, \Omega M) \cup \text{ann} \text{Tor}_1(M, \text{Tr}M)$  is contained in  $I \cap J$ .

(1) Take any element  $a \in \text{ann}_R \text{Ext}_R^1(M, \Omega M)$ . The proof of [15, Lemma 2.14] shows that the multiplication map  $(M \xrightarrow{a} M)$  factors through a free module, that is,  $(M \xrightarrow{a} M) = (M \xrightarrow{f} F \xrightarrow{\pi} M)$  with  $F$  free. Hence, for all  $i > 0$  and  $N \in \text{mod } R$  we have commutative diagrams:

$$\begin{array}{ccc} \text{Tor}_i(M, N) & \xrightarrow{a} & \text{Tor}_i(M, N) \\ \text{Tor}_i(f, N) \searrow & & \nearrow \text{Tor}_i(\pi, N) \\ & \text{Tor}_i(F, N) & \end{array} \quad \begin{array}{ccc} \text{Ext}^i(M, N) & \xrightarrow{a} & \text{Ext}^i(M, N) \\ \text{Ext}^i(\pi, N) \searrow & & \nearrow \text{Ext}^i(f, N) \\ & \text{Ext}^i(F, N) & \end{array}$$

As  $\text{Tor}_i(F, N) = \text{Ext}^i(F, N) = 0$ , the element  $a$  is in  $I \cap J$ .

(2) Let  $b \in \text{ann}_R \text{Tor}_1^R(M, \text{Tr}M)$ . By [31, Lemma (3.9)], the element  $b$  annihilates  $\underline{\text{Hom}}_R(M, M)$ . Hence the map  $b \cdot \text{id}_M$ , which is nothing but the multiplication map  $(M \xrightarrow{b} M)$ , factors through a free  $R$ -module. Similarly to (1), we get  $b$  is in  $I \cap J$ .  $\blacksquare$

**Definition 2.7.** We denote by  $\text{ann}^h M$  the ideal in the above proposition.

Note that  $\text{ann}^h M = R$  if and only if  $M$  is a free  $R$ -module.

For an  $R$ -module  $M$  we denote by  $\text{add } M$  the subcategory of  $\text{mod } R$  consisting of direct summands of finite direct sums of copies of  $M$ .

With the notation of Remark 2.5, we are interested in when the operation  $M \mapsto \Omega(M/aM)$  actually gives rise to an essentially new Ulrich module. The following result presents a possible way: if we choose an indecomposable Ulrich module  $M$  that is not a direct summand of  $\Omega^d k$ , then we find an indecomposable Ulrich module not isomorphic to  $M$  among the direct summands of the modules  $\Omega(M/x_i M)$ .

**Proposition 2.8.** *Suppose that  $R$  is henselian. Let  $Q = (x_1, \dots, x_d)$  be a parameter ideal of  $R$  which is a reduction of  $\mathfrak{m}$ . Let  $M$  be an indecomposable Ulrich  $R$ -module. If  $M$  is a direct summand of  $\Omega(M/x_i M)$  for all  $1 \leq i \leq d$ , then  $M$  is a direct summand of  $\Omega^d k$ .*

*Proof.* For all integer  $1 \leq i \leq d$  the module  $\text{Ext}_R^1(M, \Omega M)$  is a direct summand of  $\text{Ext}_R^1(\Omega(M/x_i M), \Omega M)$ . The latter module is annihilated by  $x_i$  since it is isomorphic to  $\text{Ext}_R^2(M/x_i M, \Omega M)$ . Hence  $Q$  is contained in  $\text{ann}_R \text{Ext}_R^1(M, \Omega M) = \text{ann}^h M$ , and therefore  $Q \text{Ext}_R^{\geq 0}(M, N) = 0$  for all  $N \in \text{mod } R$ . It follows from [28, Corollary 3.2(1)] that  $M$  is a direct summand of  $\Omega^d(M/QM)$ . As  $M$  is Ulrich, the module  $M/QM$  is a  $k$ -vector space, and  $\Omega^d(M/QM)$  belongs to  $\text{add}(\Omega^d k)$ , whence so does  $M$ . Since  $R$  is henselian and  $M$  is indecomposable, the Krull–Schmidt theorem implies that  $M$  is a direct summand of  $\Omega^d k$ .  $\blacksquare$

### 3. COMPARISON OF $\text{Ul}(R)$ WITH $\Omega\text{CM}^\times(R)$

In this section, we study the relationship of the Ulrich  $R$ -modules with the syzygies of Cohen–Macaulay  $R$ -modules. We begin with giving equivalent conditions for a given Cohen–Macaulay module to be a syzygy of a Cohen–Macaulay module, after stating an elementary lemma.

**Lemma 3.1.** *Let  $M, N$  be  $R$ -modules. The evaluation map  $\text{ev} : M \otimes_R \text{Hom}_R(M, N) \rightarrow N$  is surjective if and only if there exists an epimorphism  $(f_1, \dots, f_n) : M^{\oplus n} \rightarrow N$ .*

*Proof.* The “only if” part follows by taking an epimorphism  $R^{\oplus n} \rightarrow \text{Hom}_R(M, N)$  and tensoring  $M$ . To show the “if” part, pick any element  $y \in N$ . Then we have  $y = f_1(x_1) + \dots + f_n(x_n)$  for some  $x_1, \dots, x_n \in M$ . Therefore  $y = \text{ev}(\sum_{i=1}^n x_i \otimes f_i)$ , and we are done.  $\blacksquare$

**Proposition 3.2.** *Let  $R$  be a Cohen–Macaulay local ring with canonical module  $\omega$ . Then the following are equivalent for a Cohen–Macaulay  $R$ -module  $M$ .*

- (1)  $M \in \Omega\text{CM}(R)$ .
- (2)  $\underline{\text{Hom}}_R(M, \omega) = 0$ .
- (3) There exists a surjective homomorphism  $\omega^{\oplus n} \rightarrow \text{Hom}_R(M, \omega)$ .
- (4) The natural homomorphism  $\Phi : \omega \otimes_R \text{Hom}_R(\omega, \text{Hom}_R(M, \omega)) \rightarrow \text{Hom}_R(M, \omega)$  is surjective.
- (5)  $M$  is torsionless and  $\text{Tr}\Omega\text{Tr}M$  is Cohen–Macaulay.
- (6)  $\text{Ext}_R^1(\text{Tr}M, R) = \text{Ext}_R^1(\text{Tr}\Omega\text{Tr}M, \omega) = 0$ .
- (7)  $\text{Tor}_1^R(\text{Tr}M, \omega) = 0$ .

*Proof.* (1)  $\Rightarrow$  (2): By the assumption, there is an exact sequence  $0 \rightarrow M \rightarrow F \rightarrow N \rightarrow 0$  such that  $N$  is Cohen–Macaulay and  $F$  is free. Take  $f \in \text{Hom}_R(M, \omega)$ . There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & \omega & \longrightarrow & W & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

with exact rows. Since  $N$  is Cohen–Macaulay, we have  $\text{Ext}_R^1(N, \omega) = 0$ . Hence the second row splits, and  $f$  factors through  $F$ . This shows  $\underline{\text{Hom}}_R(M, \omega) = 0$ .

(2)  $\Rightarrow$  (1): There is an exact sequence  $0 \rightarrow M \xrightarrow{f} \omega^{\oplus m} \rightarrow N \rightarrow 0$  such that  $N$  is Cohen–Macaulay. Since  $\underline{\text{Hom}}_R(M, \omega^{\oplus m}) = \underline{\text{Hom}}_R(M, \omega)^{\oplus m} = 0$ , there are a free  $R$ -module  $F$ , homomorphisms  $g : M \rightarrow F$  and  $h : F \rightarrow \omega^{\oplus m}$  such that  $f = hg$ . We get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{g} & F & \longrightarrow & L & \longrightarrow & 0 \\ & & \parallel & & \downarrow h & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{f} & \omega^{\oplus m} & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

with exact rows. The second square is a pullback-pushout diagram, which gives an exact sequence  $0 \rightarrow F \rightarrow L \oplus \omega^{\oplus m} \rightarrow N \rightarrow 0$ . This shows that  $L$  is Cohen–Macaulay, and hence  $M \in \Omega\text{CM}(R)$ .

(2)  $\Leftrightarrow$  (7): This equivalence follows from [31, Lemma (3.9)].

(1)  $\Rightarrow$  (3): Let  $0 \rightarrow M \rightarrow R^{\oplus n} \rightarrow N \rightarrow 0$  be an exact sequence with  $F$  free. Applying  $(-)^{\dagger}$ , we have an exact sequence  $0 \rightarrow N^{\dagger} \rightarrow \omega^{\oplus n} \rightarrow M^{\dagger} \rightarrow 0$ .

(3)  $\Rightarrow$  (1): There is an exact sequence  $0 \rightarrow K \rightarrow \omega^{\oplus n} \rightarrow M^{\dagger} \rightarrow 0$ . It is seen that  $K$  is Cohen–Macaulay. Taking  $(-)^{\dagger}$  gives an exact sequence  $0 \rightarrow M \rightarrow R^{\oplus n} \rightarrow K^{\dagger} \rightarrow 0$ , which shows  $M \in \Omega\text{CM}(R)$ .

(3)  $\Leftrightarrow$  (4): This follows from Lemma 3.1.

(5)  $\Leftrightarrow$  (6): The module  $\text{Tr}\Omega\text{Tr}M$  is Cohen–Macaulay if and only if  $\text{Ext}_R^i(\text{Tr}\Omega\text{Tr}M, \omega) = 0$  for all  $i > 0$ . One has  $\text{Ext}_R^1(\text{Tr}M, R) = 0$  if and only if  $M$  is torsionless, if and only if  $M \cong \Omega\text{Tr}\Omega\text{Tr}M$  up to free summands; see [1, Theorem (2.17)]. Hence  $\text{Ext}_R^i(\text{Tr}\Omega\text{Tr}M, \omega) = \text{Ext}_R^{i-1}(M, \omega) = 0$  for all  $i > 1$ .

(1)  $\Leftrightarrow$  (5): This equivalence follows from [17, Lemma 2.5] and its proof.  $\blacksquare$

**Remark 3.3.** The equivalence (1)  $\Leftrightarrow$  (5) in Proposition 3.2 holds without the assumption that  $R$  admits a canonical module. Indeed, its proof does not use the existence of a canonical module.

The property of being a syzygy of a Cohen–Macaulay module (without free summand) is preserved under faithfully flat extension.

**Corollary 3.4.** *Let  $R \rightarrow S$  be a faithfully flat homomorphism of Cohen–Macaulay local rings. Let  $M$  be a Cohen–Macaulay  $R$ -module. Then  $M \in \Omega\text{CM}^{\times}(R)$  if and only if  $M \otimes_R S \in \Omega\text{CM}^{\times}(S)$ .*

*Proof.* Using Remark 3.3, we see that  $M \in \Omega\text{CM}(R)$  if and only if  $\text{Ext}_R^1(\text{Tr}_R M, R) = 0$  and  $\text{Tr}_R \Omega_R \text{Tr}_R M$  is Cohen–Macaulay. Also,  $M$  has a nonzero  $R$ -free summand if and only if the evaluation map  $M \otimes_R \text{Hom}_R(M, R) \rightarrow R$  is surjective by Lemma 3.1. Since the latter conditions are both preserved under faithfully flat extension, they are equivalent to saying that  $M \otimes_R S \in \Omega\text{CM}(S)$  and that  $M \otimes_R S$  has a nonzero  $S$ -free summand, respectively. Now the assertion follows.  $\blacksquare$

Next we state and prove a couple of lemmas. The first one concerns Ulrich modules and syzygies of Cohen–Macaulay modules with respect to short exact sequences.

**Lemma 3.5.** *Let  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  be an exact sequence of  $R$ -modules.*

(1) *If  $L, M, N$  are in  $\text{Ul}(R)$ , then the equality  $\mu(M) = \mu(L) + \mu(N)$  holds.*

(2) *Suppose that  $L, M, N$  are in  $\text{CM}(R)$ . Then:*

(a) *If  $M$  is in  $\text{Ul}(R)$ , then so are  $L$  and  $N$ .*      (b) *If  $M$  is in  $\Omega\text{CM}^{\times}(R)$ , then so is  $L$ .*

*Proof.* (1) We have  $\mu(M) = e(M) = e(L) + e(N) = \mu(L) + \mu(N)$ .

(2) Assertion (a) follows by [2, Proposition (1.4)]. Let us show (b). As  $M$  is in  $\Omega\text{CM}^{\times}(R)$ , there is an exact sequence  $0 \rightarrow M \xrightarrow{\beta} R^{\oplus a} \xrightarrow{\gamma} C \rightarrow 0$  with  $C$  Cohen–Macaulay. As  $M$  has no free summand,  $\gamma$  is a

minimal homomorphism. In particular,  $\mu(C) = a$ . The pushout of  $\beta$  and  $\gamma$  gives a commutative diagram

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \downarrow & & \downarrow & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \\
& & \parallel & & \downarrow \beta & & \downarrow & \\
0 & \longrightarrow & L & \longrightarrow & R^{\oplus a} & \xrightarrow{\delta} & D \longrightarrow 0 \\
& & & & \downarrow \gamma & & \downarrow & \\
& & & & C & \xlongequal{\quad} & C & \\
& & & & \downarrow & & \downarrow & \\
& & & & 0 & & 0 & 
\end{array}$$

with exact rows and columns. We see that  $a = \mu(C) \leq \mu(D) \leq a$ , which implies that  $\delta$  is a minimal homomorphism. Hence  $L = \Omega D \in \Omega \text{CM}^\times(R)$ .  $\blacksquare$

The following lemma is used to reduce to the case of a lower dimensional ring.

**Lemma 3.6.** *Let  $Q = (x_1, \dots, x_d)$  be a parameter ideal of  $R$  that is a reduction of  $\mathfrak{m}$ . Let  $M$  be a Cohen–Macaulay  $R$ -module. Then  $M$  is an Ulrich  $R$ -module if and only if  $M/x_i M$  is an Ulrich  $R/x_i R$ -module.*

*Proof.* Note that  $Q/x_i R$  is a reduction of  $\mathfrak{m}/x_i R$ . We see that  $(\mathfrak{m}/x_i R)(M/x_i M) = (Q/x_i R)(M/x_i M)$  if and only if  $\mathfrak{m}M = QM$ . Thus the assertion holds.  $\blacksquare$

Now we explore syzygies of the residue field of a Cohen–Macaulay local ring with minimal multiplicity.

**Lemma 3.7.** *Assume that  $R$  is singular and has minimal multiplicity.*

- (1) *One has  $\Omega_R^d k \in \Omega \text{CM}^\times(R)$ . In particular,  $\Omega_R^d k$  is an Ulrich  $R$ -module.*
- (2) *There is an isomorphism  $\Omega_R^{d+1} k \cong (\Omega_R^d k)^{\oplus n}$  for some  $n \geq 0$ .*
- (3) *Let  $Q = (x_1, \dots, x_d)$  be a parameter ideal of  $R$  with  $\mathfrak{m}^2 = Q\mathfrak{m}$ , and suppose that  $d \geq 1$ . Then  $\Omega_R^1(\Omega_{R/(x_1)}^i k) \cong \Omega_R^{i+1} k$  for all  $i \geq 0$ . In particular,  $\Omega_R^1(\Omega_{R/(x_1)}^{d-1} k) \cong \Omega_R^d k$ .*
- (4) *For each  $M \in \text{Ul}(R)$  there exists a surjective homomorphism  $(\Omega_R^d k)^{\oplus n} \rightarrow M$  for some  $n \geq 0$ .*

*Proof.* (1)(2) We may assume that  $k$  is infinite; see Remark 1.5. So we find a parameter ideal  $Q = (x_1, \dots, x_d)$  of  $R$  with  $\mathfrak{m}^2 = Q\mathfrak{m}$ . The module  $\mathfrak{m}/Q$  is a  $k$ -vector space, and there is an exact sequence  $0 \rightarrow k^{\oplus n} \rightarrow R/Q \rightarrow k \rightarrow 0$ . Taking the  $d$ th syzygies gives an exact sequence

$$0 \rightarrow (\Omega^d k)^{\oplus n} \rightarrow R^{\oplus t} \rightarrow \Omega^d k \rightarrow 0.$$

Since  $\Omega^d k$  has no free summand by [27, Theorem 1.1], we obtain  $\Omega^d k \in \Omega \text{CM}^\times(R)$  and  $(\Omega^d k)^{\oplus n} \cong \Omega^{d+1} k$ . The last assertion of (1) follows from this and Proposition 1.6.

(3) Set  $x = x_1$ . We show that  $\Omega(\Omega_{R/xR}^i k) \cong \Omega^{i+1} k$  for all  $i \geq 0$ . We may assume  $i \geq 1$ ; note then that  $x$  is  $\Omega^i k$ -regular. By [27, Corollary 5.3] we have an isomorphism  $\Omega^i k/x\Omega^i k \cong \Omega_{R/xR}^i k \oplus \Omega_{R/xR}^{i-1} k$ . Hence

$$(3.7.1) \quad \Omega^i k \oplus \Omega^{i+1} k \cong \Omega(\Omega^i k/x\Omega^i k) \cong \Omega(\Omega_{R/xR}^i k) \oplus \Omega(\Omega_{R/xR}^{i-1} k),$$

where the first isomorphism follows from the proof of Corollary 2.4. There is an exact sequence  $0 \rightarrow \Omega_{R/xR}^i k \rightarrow (R/xR)^{\oplus a_{i-1}} \rightarrow \dots \rightarrow (R/xR)^{\oplus a_0} \rightarrow k \rightarrow 0$  of  $R/xR$ -modules, which gives an exact sequence

$$0 \rightarrow \Omega(\Omega_{R/xR}^i k) \rightarrow R^{\oplus b_{i-1}} \rightarrow \dots \rightarrow R^{\oplus b_0} \rightarrow \Omega k \rightarrow 0$$

of  $R$ -modules. This shows  $\Omega(\Omega_{R/xR}^i k) \cong \Omega^{i+1} k \oplus R^{\oplus u}$  for some  $u \geq 0$ , and similarly we have an isomorphism  $\Omega(\Omega_{R/xR}^{i-1} k) \cong \Omega^i k \oplus R^{\oplus v}$  for some  $v \geq 0$ . Substituting these in (3.7.1), we see  $u = v = 0$  and obtain an isomorphism  $\Omega(\Omega_{R/xR}^i k) \cong \Omega^{i+1} k$ .

(4) According to Lemma 3.1 and Remark 1.5, we may assume that  $k$  is infinite. Take a parameter ideal  $Q = (x_1, \dots, x_d)$  of  $R$  with  $\mathfrak{m}^2 = Q\mathfrak{m}$ . We prove this by induction on  $d$ . If  $d = 0$ , then  $M$  is a  $k$ -vector space, and there is nothing to show. Assume  $d \geq 1$  and set  $x = x_1$ . Clearly,  $R/xR$  has minimal multiplicity. By Lemma 3.6,  $M/xM$  is an Ulrich  $R/xR$ -module. The induction hypothesis gives an exact sequence  $0 \rightarrow L \rightarrow (\Omega_{R/xR}^{d-1} k)^{\oplus n} \rightarrow M/xM \rightarrow 0$  of  $R/xR$ -modules. Lemma 3.5(2) shows that  $L$  is also an Ulrich  $R/xR$ -module, while Lemma 3.5(1) implies

$$\mu_{R/xR}(L) + \mu_{R/xR}(M/xM) = \mu_{R/xR}((\Omega_{R/xR}^{d-1} k)^{\oplus n}).$$



Note that  $\mu_R(X) = \mu_{R/xR}(X)$  for an  $R/xR$ -module  $X$ . Thus, taking the first syzygies over  $R$ , we get an exact sequence of  $R$ -modules:

$$0 \rightarrow \Omega L \rightarrow \Omega(\Omega_{R/xR}^{d-1}k)^{\oplus n} \rightarrow \Omega(M/xM) \rightarrow 0.$$

From the proof of Corollary 2.4 we see that there is an exact sequence  $0 \rightarrow \Omega M \rightarrow \Omega(M/xM) \rightarrow M \rightarrow 0$ , while  $\Omega(\Omega_{R/xR}^{d-1}k)$  is isomorphic to  $\Omega^d k$  by (3). Consequently, we obtain a surjection  $(\Omega^d k)^{\oplus n} \rightarrow M$ . ■

Here we recall the definition of an almost Gorenstein local ring, which is introduced in [11].

**Definition 3.8.** Let  $R$  be a Cohen–Macaulay local ring of dimension  $d$  with canonical module  $\omega$ . Then  $R$  is called *almost Gorenstein* if there exists an exact sequence

$$0 \rightarrow R \rightarrow \omega \rightarrow C \rightarrow 0$$

of finitely generated  $R$ -modules such that  $e(C) = \mu(C)$ .

We have reached the stage to state and prove the main result of this section.

**Theorem 3.9.** *Let  $R$  be a  $d$ -dimensional Cohen–Macaulay local ring with residue field  $k$  and canonical module  $\omega$ . Suppose that  $R$  has minimal multiplicity. Then the following are equivalent.*

- (1) *The equality  $\Omega\text{CM}^\times(R) = \text{Ul}(R)$  holds.*
- (2) *For an exact sequence  $M \rightarrow N \rightarrow 0$  in  $\text{CM}(R)$ , if  $M \in \Omega\text{CM}^\times(R)$ , then  $N \in \Omega\text{CM}^\times(R)$ .*
- (3) *The category  $\Omega\text{CM}^\times(R)$  is closed under  $(-)^{\dagger}$ .*
- (4) *The module  $(\Omega^d k)^{\dagger}$  belongs to  $\Omega\text{CM}^\times(R)$ . (4') *The module  $(\Omega^d k)^{\dagger}$  belongs to  $\Omega\text{CM}(R)$ .**
- (5) *One has  $\underline{\text{Hom}}_R((\Omega^d k)^{\dagger}, \omega) = 0$ .*
- (6) *One has  $\text{Tor}_1^R(\text{Tr}((\Omega^d k)^{\dagger}), \omega) = 0$ .*
- (7) *One has  $\text{Ext}_R^{d+1}(\text{Tr}((\Omega^d k)^{\dagger}), R) = 0$  and  $R$  is locally Gorenstein on the punctured spectrum.*
- (8) *The natural homomorphism  $\omega \otimes_R \text{Hom}_R(\omega, \Omega^d k) \rightarrow \Omega^d k$  is surjective.*
- (9) *There exists a surjective homomorphism  $\omega^{\oplus n} \rightarrow \Omega^d k$ .*

*If  $d$  is positive,  $k$  is infinite and one of the above nine conditions holds, then  $R$  is almost Gorenstein.*

*Proof.* (1)  $\Rightarrow$  (2): This follows from Lemma 3.5(2).

(2)  $\Rightarrow$  (3): Let  $M$  be an  $R$ -module in  $\Omega\text{CM}^\times(R)$ . Then  $M \in \text{Ul}(R)$  by Proposition 1.6, and hence  $M^{\dagger} \in \text{Ul}(R)$  by Corollary 2.2. It follows from Lemma 3.7(4) that there is a surjection  $(\Omega^d k)^{\oplus n} \rightarrow M^{\dagger}$ . Since  $(\Omega^d k)^{\oplus n}$  is in  $\Omega\text{CM}^\times(R)$  by Lemma 3.7(1), the module  $M^{\dagger}$  is also in  $\Omega\text{CM}^\times(R)$ .

(3)  $\Rightarrow$  (4): Lemma 3.7(1) says that  $\Omega^d k$  is in  $\Omega\text{CM}^\times(R)$ , and so is  $(\Omega^d k)^{\dagger}$  by assumption.

(4)  $\Rightarrow$  (1): The inclusion  $\Omega\text{CM}^\times(R) \subseteq \text{Ul}(R)$  follows from Proposition 1.6. Take any module  $M$  in  $\text{Ul}(R)$ . Then  $M^{\dagger}$  is also in  $\text{Ul}(R)$  by Corollary 2.2. Using Lemma 3.7(4), we get an exact sequence  $0 \rightarrow X \rightarrow (\Omega^d k)^{\oplus n} \rightarrow M^{\dagger} \rightarrow 0$  of Cohen–Macaulay modules, which induces an exact sequence  $0 \rightarrow M \rightarrow (\Omega^d k)^{\dagger \oplus n} \rightarrow X^{\dagger} \rightarrow 0$ . The assumption and Lemma 3.5(2) imply that  $M$  is in  $\Omega\text{CM}^\times(R)$ .

(4)  $\Leftrightarrow$  (4'): As  $R$  is singular, by [27, Corollary 4.4] the module  $(\Omega^d k)^{\dagger}$  does not have a free summand.

(4')  $\Leftrightarrow$  (5)  $\Leftrightarrow$  (6)  $\Leftrightarrow$  (8)  $\Leftrightarrow$  (9): These equivalences follow from Proposition 3.2.

(4')  $\Leftrightarrow$  (7): We claim that, under the assumption that  $R$  is locally Gorenstein on the punctured spectrum,  $(\Omega^d k)^{\dagger} \in \Omega\text{CM}(R)$  if and only if  $\text{Ext}_R^{d+1}(\text{Tr}((\Omega^d k)^{\dagger}), R) = 0$ . In fact, since  $(\Omega^d k)^{\dagger}$  is Cohen–Macaulay, it satisfies Serre’s condition  $(S_d)$ . Therefore it is  $d$ -torsionfree, that is,  $\text{Ext}_R^i(\text{Tr}((\Omega^d k)^{\dagger}), R) = 0$  for all  $1 \leq i \leq d$ ; see [21, Theorem 2.3]. Hence,  $\text{Ext}_R^{d+1}(\text{Tr}((\Omega^d k)^{\dagger}), R) = 0$  if and only if  $(\Omega^d k)^{\dagger}$  is  $(d+1)$ -torsionfree, if and only if it belongs to  $\Omega\text{CM}(R)$  by [21, Theorem 2.3] again. Thus the claim follows.

According to this claim, it suffices to prove that if (4') holds, then  $R$  is locally Gorenstein on the punctured spectrum. For this, pick any nonmaximal prime ideal  $\mathfrak{p}$  of  $R$ . There are exact sequences

$$0 \rightarrow \Omega^d k \rightarrow R^{\oplus a_{d-1}} \rightarrow \dots \rightarrow R^{\oplus a_0} \rightarrow k \rightarrow 0, \quad 0 \rightarrow (\Omega^d k)_{\mathfrak{p}} \rightarrow R_{\mathfrak{p}}^{\oplus a_{d-1}} \rightarrow \dots \rightarrow R_{\mathfrak{p}}^{\oplus a_0} \rightarrow 0.$$

We observe that  $(\Omega^d k)_{\mathfrak{p}}$  is a free  $R_{\mathfrak{p}}$ -module with  $\text{rank}_{R_{\mathfrak{p}}}((\Omega^d k)_{\mathfrak{p}}) = \sum_{i=0}^{d-1} (-1)^i a_{d-1-i} = \text{rank}_R(\Omega^d k)$ . The module  $\Omega^d k$  has positive rank as it is torsionfree, and we see that  $(\Omega^d k)_{\mathfrak{p}}$  is a nonzero free  $R_{\mathfrak{p}}$ -module. Since we have already shown that (4') implies (9), there is a surjection  $\omega^{\oplus n} \rightarrow \Omega^d k$ . Localizing this at  $\mathfrak{p}$ , we see that  $\omega_{\mathfrak{p}}^{\oplus n}$  has an  $R_{\mathfrak{p}}$ -free summand, which implies that the  $R_{\mathfrak{p}}$ -module  $R_{\mathfrak{p}}$  has finite injective dimension. Thus  $R_{\mathfrak{p}}$  is Gorenstein.

So far we have proved the equivalence of the conditions (1)–(9). It remains to prove that  $R$  is almost Gorenstein under the assumption that  $d$  is positive,  $k$  is infinite and (1)–(9) all hold. We use induction on  $d$ .

Let  $d = 1$ . Let  $Q$  be the total quotient ring of  $R$ , and set  $E = \text{End}_R(\mathfrak{m})$ . Let  $K$  be an  $R$ -module with  $K \cong \omega$  and  $R \subseteq K \subseteq \bar{R}$  in  $Q$ , where  $\bar{R}$  is the integral closure of  $R$ . Using [23, Proposition 2.5], we have:

$$(3.9.1) \quad \mathfrak{m} \cong \text{Hom}_R(\mathfrak{m}, R) = E \quad \text{and} \quad \mathfrak{m}^\dagger \cong \text{Hom}_R(\mathfrak{m}, K) \cong (K :_Q \mathfrak{m}).$$

By (4) the module  $\mathfrak{m}^\dagger$  belongs to  $\Omega\text{CM}^\times(R)$ . It follows from [18, Theorem 2.14] that  $R$  is almost Gorenstein; note that the completion of  $R$  also has Gorenstein punctured spectrum by (4').

Let  $d > 1$ . Since  $(\Omega^d k)^\dagger \in \Omega\text{CM}(R)$ , there is an exact sequence  $0 \rightarrow (\Omega^d k)^\dagger \rightarrow R^{\oplus m} \rightarrow N \rightarrow 0$  for some  $m \geq 0$  and  $N \in \text{CM}(R)$ . Choose a parameter ideal  $Q = (x_1, \dots, x_d)$  of  $R$  satisfying the equality  $\mathfrak{m}^2 = Q\mathfrak{m}$ , and set  $\overline{(-)} = (-) \otimes_R R/(x_1)$ . An exact sequence

$$0 \rightarrow \overline{(\Omega^d k)^\dagger} \rightarrow \overline{R}^{\oplus m} \rightarrow \overline{N} \rightarrow 0$$

is induced, which shows that  $\overline{(\Omega^d k)^\dagger}$  is in  $\Omega\text{CM}(\overline{R})$ . Applying  $(-)^{\dagger}$  to the exact sequence  $0 \rightarrow \Omega^d k \xrightarrow{x} \Omega^d k \rightarrow \overline{\Omega^d k} \rightarrow 0$  and using [3, Lemma 3.1.16], we obtain isomorphisms

$$\overline{(\Omega^d k)^\dagger} \cong \text{Ext}_R^1(\overline{\Omega^d k}, \omega) \cong \text{Hom}_{\overline{R}}(\overline{\Omega^d k}, \overline{\omega}).$$

The module  $\Omega_{\overline{R}}^{d-1} k$  is a direct summand of  $\overline{\Omega^d k}$  by [27, Corollary 5.3], and hence  $\text{Hom}_{\overline{R}}(\Omega_{\overline{R}}^{d-1} k, \overline{\omega})$  is a direct summand of  $\text{Hom}_{\overline{R}}(\overline{\Omega^d k}, \overline{\omega})$ . Summarizing these, we observe that  $\text{Hom}_{\overline{R}}(\Omega_{\overline{R}}^{d-1} k, \overline{\omega})$  belongs to  $\Omega\text{CM}(\overline{R})$ . Since  $\overline{R}$  has minimal multiplicity, we can apply the induction hypothesis to  $\overline{R}$  to conclude that  $\overline{R}$  is almost Gorenstein, and so is  $R$  by [11, Theorem 3.7].  $\blacksquare$

**Remark 3.10.** When  $d \geq 2$ , it holds that

$$\text{Ext}_R^{d+1}(\text{Tr}((\Omega^d k)^\dagger), R) \cong \text{Ext}_R^{d-1}(\text{Hom}_R(\omega, \Omega^d k), R).$$

Thus Theorem 3.9(7) can be replaced with the condition that  $\text{Ext}_R^{d-1}(\text{Hom}_R(\omega, \Omega^d k), R) = 0$ .

Indeed, using the Hom- $\otimes$  adjointness twice, we get isomorphisms

$$\text{Hom}_R(\omega, \Omega^d k) \cong \text{Hom}_R(\omega, (\Omega^d k)^{\dagger\dagger}) \cong \text{Hom}_R((\Omega^d k)^\dagger \otimes_R \omega, \omega) \cong \text{Hom}_R((\Omega^d k)^\dagger, \omega^\dagger) \cong (\Omega^d k)^{\dagger*},$$

and  $(\Omega^d k)^{\dagger*}$  is isomorphic to  $\Omega^2 \text{Tr}((\Omega^d k)^\dagger)$  up to free summand.

We have several more conditions related to the equality  $\Omega\text{CM}^\times(R) = \text{Ul}(R)$ .

**Corollary 3.11.** *Let  $R$  be as in Theorem 3.9. Consider the following conditions:*

- (1)  $(\Omega^d k)^\dagger \cong \Omega^d k$ , (2)  $(\Omega^d k)^\dagger \in \text{add}(\Omega^d k)$ , (3)  $\text{ann}^h(\Omega^d k)^\dagger = \mathfrak{m}$ , (4)  $\Omega\text{CM}^\times(R) = \text{Ul}(R)$ .

*It then holds that (1)  $\implies$  (2)  $\iff$  (3)  $\implies$  (4).*

*Proof.* The implications (1)  $\implies$  (2)  $\implies$  (3) are obvious. The proof of Proposition 2.8 shows that if an Ulrich  $R$ -module  $M$  satisfies  $\text{ann}^h M = \mathfrak{m}$ , then  $M$  is in  $\text{add}(\Omega^d k)$ . This shows (3)  $\implies$  (2). Proposition 3.7(1) says that  $\Omega^d k$  is in  $\Omega\text{CM}^\times(R)$ , and so is  $(\Omega^d k)^\dagger$  by assumption. Theorem 3.9 shows (2)  $\implies$  (4).  $\blacksquare$

We close this section by constructing an example by applying the above corollary.

**Example 3.12.** Let  $S = \mathbb{C}[[x, y, z]]$  be a formal power series ring. Let  $G$  be the cyclic group  $\frac{1}{2}(1, 1, 1)$ , and let  $R = S^G$  be the invariant (i.e. the second Veronese) subring of  $S$ . Then  $\Omega\text{CM}^\times(R) = \text{Ul}(R)$ . In fact, by [31, Proposition (16.10)], the modules  $R$ ,  $\omega$ ,  $\Omega\omega$  are the nonisomorphic indecomposable Cohen–Macaulay  $R$ -modules and  $(\Omega\omega)^\dagger \cong \Omega\omega$ . By [27, Theorem 4.3] the module  $\Omega^2\mathbb{C}$  does not have a nonzero free or canonical summand. Hence  $\Omega^2\mathbb{C}$  is a direct sum of copies of  $\Omega\omega$ , and thus  $(\Omega^2\mathbb{C})^\dagger \cong \Omega^2\mathbb{C}$ . The equality  $\Omega\text{CM}^\times(R) = \text{Ul}(R)$  follows from Corollary 3.11.

#### 4. APPLICATIONS

This section is devoted to stating applications of our main theorems obtained in the previous section.

**4.1. The case of dimension one.** We begin with studying the case where  $R$  has dimension 1.

**Theorem 4.1.** *Let  $(R, \mathfrak{m}, k)$  be a 1-dimensional Cohen–Macaulay local ring with  $k$  infinite and canonical module  $\omega$ . Suppose that  $R$  has minimal multiplicity, and set  $(-)^{\dagger} = \text{Hom}_R(-, \omega)$ . Then*

$$\Omega\text{CM}^{\times}(R) = \text{Ul}(R) \iff \mathfrak{m}^{\dagger} \in \Omega\text{CM}^{\times}(R) \iff \mathfrak{m}^{\dagger} \cong \mathfrak{m} \iff R \text{ is almost Gorenstein.}$$

*Proof.* Call the four conditions (i)–(iv) from left to right. The implications (i)  $\Leftrightarrow$  (ii)  $\Rightarrow$  (iv) are shown by Theorem 3.9, while (iii)  $\Leftrightarrow$  (iv) by [18, Theorem 2.14] and (3.9.1). Lemma 3.7(1) shows (iii)  $\Rightarrow$  (ii). ■

Now we pose a question related to Question 1.2.

**Question 4.2.** Can we classify 1-dimensional Cohen–Macaulay local rings  $R$  with minimal multiplicity (and infinite residue field) satisfying the condition  $\#\text{ind Ul}(R) < \infty$ ?

According to Proposition 1.6, over such a ring  $R$  we have the property that  $\#\text{ind } \Omega\text{CM}(R) < \infty$ , which is studied in [17]. If  $R$  has finite Cohen–Macaulay representation type (that is, if  $\#\text{ind CM}(R) < \infty$ ), then of course this question is affirmative. However, we do not have any partial answer other than this. The reader may wonder if the condition  $\#\text{ind Ul}(R) < \infty$  implies the equality  $\Omega\text{CM}^{\times}(R) = \text{Ul}(R)$ . Using the above theorem, we observe that this does not necessarily hold:

**Example 4.3.** Let  $R = k[[t^3, t^7, t^8]]$  be (the completion of) a numerical semigroup ring, where  $k$  is an algebraically closed field of characteristic zero. Then  $R$  is a Cohen–Macaulay local ring of dimension 1 with minimal multiplicity. It follows from [12, Theorem A.3] that  $\#\text{ind Ul}(R) < \infty$ . On the other hand,  $R$  is not almost-Gorenstein by [8, Example 4.3], so  $\Omega\text{CM}^{\times}(R) \neq \text{Ul}(R)$  by Theorem 4.1.

**4.2. The case of dimension two.** From now on, we consider the case where  $R$  has dimension 2. We recall the definition of a Cohen–Macaulay approximation. Let  $R$  be a Cohen–Macaulay local ring with canonical module. A homomorphism  $f : X \rightarrow M$  of  $R$ -modules is called a *Cohen–Macaulay approximation* (of  $M$ ) if  $X$  is Cohen–Macaulay and any homomorphism  $f' : X' \rightarrow M$  with  $X'$  being Cohen–Macaulay factors through  $f$ . It is known that  $f$  is a (resp. minimal) Cohen–Macaulay approximation if and only if there exists an exact sequence

$$0 \rightarrow Y \xrightarrow{g} X \xrightarrow{f} M \rightarrow 0$$

of  $R$ -modules such that  $X$  is Cohen–Macaulay and  $Y$  has finite injective dimension (resp. and that  $X, Y$  have no common direct summand along  $g$ ). For details of Cohen–Macaulay approximations, we refer the reader to [20, Chapter 11].

The module  $E$  appearing in the following remark is called the *fundamental module* of  $R$ .

**Remark 4.4.** Let  $(R, \mathfrak{m}, k)$  be a 2-dimensional Cohen–Macaulay local ring with canonical module  $\omega$ .

(1) There exists a nonsplit exact sequence

$$(4.4.1) \quad 0 \rightarrow \omega \rightarrow E \rightarrow \mathfrak{m} \rightarrow 0$$

which is unique up to isomorphism. This is because  $\text{Ext}_R^1(\mathfrak{m}, \omega) \cong \text{Ext}_R^2(k, \omega) \cong k$ .

(2) The module  $E$  is Cohen–Macaulay and uniquely determined up to isomorphism.

(3) The sequence (4.4.1) gives a minimal Cohen–Macaulay approximation of  $\mathfrak{m}$ .

(4) There is an isomorphism  $E \cong E^{\dagger}$ . In fact, applying  $(-)^{\dagger}$  to (4.4.1) induces an exact sequence

$$0 \rightarrow \mathfrak{m}^{\dagger} \rightarrow E^{\dagger} \rightarrow R \rightarrow \text{Ext}_R^1(\mathfrak{m}, \omega) \rightarrow \text{Ext}_R^1(E, \omega) = 0.$$

Applying  $(-)^{\dagger}$  to the natural exact sequence  $0 \rightarrow \mathfrak{m} \rightarrow R \rightarrow k \rightarrow 0$  yields  $\mathfrak{m}^{\dagger} \cong \omega$ , while  $\text{Ext}_R^1(\mathfrak{m}, \omega) \cong k$ . We get an exact sequence  $0 \rightarrow \omega \rightarrow E^{\dagger} \rightarrow \mathfrak{m} \rightarrow 0$ , and the uniqueness of (4.4.1) shows  $E^{\dagger} \cong E$ .

To prove the main result of this section, we prepare two lemmas. The first one relates the fundamental module of a 2-dimensional Cohen–Macaulay local ring  $R$  with  $\text{Ul}(R)$  and  $\Omega\text{CM}^{\times}(R)$ .

**Lemma 4.5.** *Let  $(R, \mathfrak{m}, k)$  be a 2-dimensional Cohen–Macaulay local ring with canonical module  $\omega$  and fundamental module  $E$ .*

(1) *Assume that  $R$  has minimal multiplicity. Then  $E$  is an Ulrich  $R$ -module.*

(2) *For each module  $M \in \Omega\text{CM}^{\times}(R)$  there exists an exact sequence  $0 \rightarrow M \rightarrow E^{\oplus n} \rightarrow N \rightarrow 0$  of  $R$ -modules such that  $N$  is Cohen–Macaulay.*

*Proof.* (1) We may assume that  $k$  is infinite by Remark 1.5(2). Let  $Q = (x, y)$  be a parameter ideal of  $R$  with  $\mathfrak{m}^2 = Q\mathfrak{m}$ . We have  $\mathfrak{m}/x\mathfrak{m} \cong \mathfrak{m}/(x) \oplus k$ ; see [27, Corollary 5.3]. Note that  $(\mathfrak{m}/(x))^2 = y(\mathfrak{m}/(x))$ . By [30, Corollary 2.5] the minimal Cohen–Macaulay approximation of  $\mathfrak{m}/x\mathfrak{m}$  as an  $R/(x)$ -module is  $E/xE$ . In view of the proof of [20, Proposition 11.15], the minimal Cohen–Macaulay approximations of  $\mathfrak{m}/(x)$  and  $k$  as  $R/(x)$ -modules are  $\mathfrak{m}/(x)$  and  $\mathrm{Hom}_{R/(x)}(\mathfrak{m}/(x), \omega/x\omega)$ , respectively. Thus we get an isomorphism

$$E/xE \cong \mathfrak{m}/(x) \oplus \mathrm{Hom}_{R/(x)}(\mathfrak{m}/(x), \omega/x\omega).$$

In particular,  $E/xE$  is an Ulrich  $R/(x)$ -module by Lemma 3.7(1) and Corollary 2.2. It follows from Lemma 3.6 that  $E$  is an Ulrich  $R$ -module.

(2) Take an exact sequence  $0 \rightarrow M \xrightarrow{f} R^{\oplus n} \xrightarrow{e} L \rightarrow 0$  such that  $L$  is Cohen–Macaulay. As  $M$  has no free summand, the homomorphism  $e$  is minimal. This means that  $f$  factors through the natural inclusion  $i : \mathfrak{m}^{\oplus n} \rightarrow R^{\oplus n}$ , that is,  $f = ig$  for some  $g \in \mathrm{Hom}_R(M, \mathfrak{m}^{\oplus n})$ . The direct sum  $p : E^{\oplus n} \rightarrow \mathfrak{m}^{\oplus n}$  of copies of the surjection  $E \rightarrow \mathfrak{m}$  (given by (4.4.1)) is a Cohen–Macaulay approximation. Hence there is a homomorphism  $h : M \rightarrow E^{\oplus n}$  such that  $g = ph$ . We get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{f} & R^{\oplus n} & \longrightarrow & L \longrightarrow 0 \\ & & \parallel & & \uparrow ip & & \uparrow \\ 0 & \longrightarrow & M & \xrightarrow{h} & E^{\oplus n} & \longrightarrow & N \longrightarrow 0 \end{array}$$

with exact rows. This induces an exact sequence  $0 \rightarrow E^{\oplus n} \rightarrow R^{\oplus n} \oplus N \rightarrow L \rightarrow 0$ , and therefore  $N$  is a Cohen–Macaulay  $R$ -module.  $\blacksquare$

A short exact sequence of Ulrich modules is preserved by certain functors:

**Lemma 4.6.** *Let  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  be an exact sequence of modules in  $\mathrm{Ul}(R)$ . Then it induces exact sequences of  $R$ -modules*

- (a)  $0 \rightarrow X \otimes_R k \rightarrow Y \otimes_R k \rightarrow Z \otimes_R k \rightarrow 0$ ,
- (b)  $0 \rightarrow \mathrm{Hom}_R(Z, k) \rightarrow \mathrm{Hom}_R(Y, k) \rightarrow \mathrm{Hom}_R(X, k) \rightarrow 0$ , and
- (c)  $0 \rightarrow \mathrm{Hom}_R(Z, (\Omega^d k)^\dagger) \rightarrow \mathrm{Hom}_R(Y, (\Omega^d k)^\dagger) \rightarrow \mathrm{Hom}_R(X, (\Omega^d k)^\dagger) \rightarrow 0$ .

*Proof.* The sequence  $X \otimes_R k \rightarrow Y \otimes_R k \rightarrow Z \otimes_R k \rightarrow 0$  is exact and the first map is injective by Lemma 3.5(1). Hence (a) is exact, and so is (b) by a dual argument. In what follows, we show that (c) is exact. We first note that  $(\Omega^d k)^\dagger$  is a minimal Cohen–Macaulay approximation of  $k$ ; see the proof of [20, Proposition 11.15]. Thus there is an exact sequence  $0 \rightarrow I \rightarrow (\Omega^d k)^\dagger \rightarrow k \rightarrow 0$  such that  $I$  has finite injective dimension. As  $\mathrm{Ul}(R) \subseteq \mathrm{CM}(R)$ , we have  $\mathrm{Ext}_R^1(M, I) = 0$  for all  $M \in \{X, Y, Z\}$ . We obtain a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(Y, I) & \longrightarrow & \mathrm{Hom}_R(Y, (\Omega^d k)^\dagger) & \longrightarrow & \mathrm{Hom}_R(Y, k) \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & \mathrm{Hom}_R(X, I) & \longrightarrow & \mathrm{Hom}_R(X, (\Omega^d k)^\dagger) & \longrightarrow & \mathrm{Hom}_R(X, k) \longrightarrow 0 \end{array}$$

with exact rows, where  $\alpha$  is surjective. The exactness of (b) implies that  $\gamma$  is surjective. By the snake lemma  $\beta$  is also surjective, and therefore (c) is exact.  $\blacksquare$

Now we can state and show our main result in this section.

**Theorem 4.7.** *Let  $R$  be a 2-dimensional complete singular normal local ring with residue field  $\mathbb{C}$  and having minimal multiplicity. Suppose that  $R$  does not have a cyclic quotient singularity. Then:*

$$(\Omega^d k)^\dagger \cong \Omega^d k \iff (\Omega^d k)^\dagger \in \mathrm{add}(\Omega^d k) \iff \mathrm{ann}^h(\Omega^d k)^\dagger = \mathfrak{m} \iff \Omega\mathrm{CM}^\times(R) = \mathrm{Ul}(R).$$

*Proof.* In view of Corollary 3.11, it suffices to show that if  $R$  does not have a cyclic quotient singularity, then the fourth condition implies the first one. By virtue of [31, Theorem 11.12] the fundamental module  $E$  is indecomposable. Applying Lemma 4.5(2) to  $(\Omega^d k)^\dagger$ , we have an exact sequence  $0 \rightarrow (\Omega^d k)^\dagger \xrightarrow{\alpha} E^{\oplus n} \rightarrow N \rightarrow 0$  such that  $N$  is Cohen–Macaulay. Since  $E$  is Ulrich by Lemma 4.5(1), so are all the three modules in this sequence by Lemma 3.5(2). Thus we can apply Lemma 4.6 to see that the induced map

$$\mathrm{Hom}_R(\alpha, (\Omega^d k)^\dagger) : \mathrm{Hom}_R(E^{\oplus n}, (\Omega^d k)^\dagger) \rightarrow \mathrm{Hom}_R((\Omega^d k)^\dagger, (\Omega^d k)^\dagger)$$

is surjective. This implies that  $\alpha$  is a split monomorphism, and  $(\Omega^d k)^\dagger$  is isomorphic to a direct summand of  $E^{\oplus n}$ . Since  $E$  is indecomposable, it follows that  $(\Omega^d k)^\dagger$  is isomorphic to  $E^{\oplus m}$  for some  $m$ . We obtain

$$(\Omega^d k)^\dagger \cong E^{\oplus m} \cong (E^\dagger)^{\oplus m} \cong (\Omega^d k)^{\dagger\dagger} \cong \Omega^d k,$$

where the second isomorphism follows by Remark 4.4(4).  $\blacksquare$

**Remark 4.8.** Let  $R$  be a cyclic quotient surface singularity over  $\mathbb{C}$ . Nakajima and Yoshida [22, Theorem 4.5] give a necessary and sufficient condition for  $R$  to be such that the number of nonisomorphic indecomposable Ulrich  $R$ -modules is equal to the number of nonisomorphic nonfree indecomposable special Cohen–Macaulay  $R$ -modules. By [14, Corollary 2.9], the latter is equal to the number of isomorphism classes of indecomposable modules in  $\Omega\text{CM}^\times(R)$ . Therefore, they actually gives a necessary and sufficient condition for  $R$  to satisfy  $\Omega\text{CM}^\times(R) = \text{Ul}(R)$ .

Using our Theorem 4.7, we give some examples of a quotient surface singularity over  $\mathbb{C}$  to consider Ulrich modules over them.

**Example 4.9.** (1) Let  $S = \mathbb{C}[[x, y]]$  be a formal power series ring. Let  $G$  be the cyclic group  $\frac{1}{3}(1, 1)$ , and let  $R = S^G$  be the invariant (i.e. the third Veronese) subring of  $S$ . Then  $\Omega\text{CM}^\times(R) = \text{Ul}(R)$ . This follows from [22, Theorem 4.5] and Remark 4.8, but we can also show it by direct calculation: we have

$$\text{Cl}(R) = \{[R], [\omega], [\mathfrak{p}]\} \cong \mathbb{Z}/3\mathbb{Z},$$

where  $\omega = (x^3, x^2y)R$  is a canonical ideal of  $R$ , and  $\mathfrak{p} = (x^3, x^2y, xy^2)R$  is a prime ideal of height 1 with  $[\omega] = 2[\mathfrak{p}]$ . Since the second Betti number of  $\mathbb{C}$  over  $R$  is 9, we see  $\Omega^2\mathbb{C} \cong \mathfrak{p}^{\oplus 3}$ . As  $[\mathfrak{p}^\dagger] = [\omega] - [\mathfrak{p}] = [\mathfrak{p}]$ , we have  $\mathfrak{p}^\dagger \cong \mathfrak{p}$  and  $(\Omega^2\mathbb{C})^\dagger \cong \Omega^2\mathbb{C}$ . Theorem 4.7 shows  $\Omega\text{CM}^\times(R) = \text{Ul}(R)$ .

(2) Let  $S = \mathbb{C}[[x, y]]$  be a formal power series ring. Let  $G$  be the cyclic group  $\frac{1}{8}(1, 5)$ , and let  $R = S^G$  be the invariant subring of  $S$ . With the notation of [22], the Hirzebruch–Jung continued fraction of this group is  $[2, 3, 2]$ . It follows from [22, Theorem 4.5] and Remark 4.8 that  $\Omega\text{CM}^\times(R) \neq \text{Ul}(R)$ .

**4.3. An exact structure of the category of Ulrich modules.** Finally, we consider realization of the additive category  $\text{Ul}(R)$  as an exact category in the sense of Quillen [24]. We begin with recalling the definition of an exact category given in [16, Appendix A].

**Definition 4.10.** Let  $\mathcal{A}$  be an additive category. A pair  $(i, d)$  of composable morphisms

$$X \xrightarrow{i} Y \xrightarrow{d} Z$$

is *exact* if  $i$  is the kernel of  $d$  and  $d$  is the cokernel of  $i$ . Let  $\mathcal{E}$  be a class of exact pairs closed under isomorphism. The pair  $(\mathcal{A}, \mathcal{E})$  is called an *exact category* if the following axioms hold. Here, for each  $(i, d) \in \mathcal{E}$  the morphisms  $i$  and  $d$  are called an *inflation* and a *deflation*, respectively.

- (Ex0)  $1 : 0 \rightarrow 0$  is a deflation.
- (Ex1) The composition of deflations is a deflation.
- (Ex2) For each morphism  $f : Z' \rightarrow Z$  and each deflation  $d : Y \rightarrow Z$ , there is a pullback diagram as in the left below, where  $d'$  is a deflation.
- (Ex2<sup>op</sup>) For each morphism  $f : X \rightarrow X'$  and each inflation  $i : X \rightarrow Y$ , there is a pushout diagram as in the right below, where  $i'$  is an inflation.

$$\begin{array}{ccc} Y' & \xrightarrow{d'} & Z' \\ \downarrow & & \downarrow f \\ Y & \xrightarrow{d} & Z \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{i} & Y \\ f \downarrow & & \downarrow \\ X' & \xrightarrow{i'} & Y' \end{array}$$

We can equip a structure of an exact category with our  $\text{Ul}(R)$  as follows.

**Theorem 4.11.** *Let  $R$  be a  $d$ -dimensional Cohen–Macaulay local ring with residue field  $k$  and canonical module, and assume that  $R$  has minimal multiplicity. Let  $\mathcal{S}$  be the class of exact sequences  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  of  $R$ -modules with  $L, M, N$  Ulrich. Then  $\text{Ul}(R) = (\text{Ul}(R), \mathcal{S})$  is an exact category having enough projective objects and enough injective objects with  $\text{proj Ul}(R) = \text{add}(\Omega^d k)$  and  $\text{inj Ul}(R) = \text{add}((\Omega^d k)^\dagger)$ .*

*Proof.* We verify the axioms in Definition 4.10.

(Ex0): This is clear.

(Ex1): Let  $d : Y \rightarrow Z$  and  $d' : Z \rightarrow W$  be deflations. Then there is an exact sequence  $0 \rightarrow X \rightarrow Y \xrightarrow{d} W \rightarrow 0$  of  $R$ -modules. Since  $Y$  is in  $\text{Ul}(R)$  and  $X, W \in \text{CM}(R)$ , it follows from that  $X \in \text{Ul}(R)$ . Thus this sequence belongs to  $\mathcal{S}$ , and  $d'd$  is a deflation.

(Ex2): Let  $f : Z' \rightarrow Z$  be a homomorphism in  $\text{Ul}(R)$  and  $d : Y \rightarrow Z$  a deflation in  $\mathcal{S}$ . Then we get an exact sequence  $0 \rightarrow Y' \rightarrow Y \oplus Z' \xrightarrow{(d, f)} Z \rightarrow 0$ . Since  $Y \oplus Z' \in \text{Ul}(R)$  and  $Y', Z \in \text{CM}(R)$ , Lemma 3.5(2) implies  $Y' \in \text{Ul}(R)$ . Make an exact sequence  $0 \rightarrow X' \rightarrow Y' \xrightarrow{d'} Z' \rightarrow 0$ . As  $Y' \in \text{Ul}(R)$  and  $X', Z' \in \text{CM}(R)$ , the module  $Z'$  is in  $\text{Ul}(R)$  by Lemma 3.5(2) again. Thus  $d'$  is a deflation.

(Ex2<sup>op</sup>): We can check this axiom by the opposite argument to (Ex2).

Now we conclude that  $(\text{Ul}(R), \mathcal{S})$  is an exact category. Let us prove the remaining assertions. Lemma 4.6(c) yields the injectivity of  $(\Omega^d k)^\dagger$ . Since  $(-)^{\dagger}$  gives an exact duality of  $(\text{Ul}(R), \mathcal{S})$ , the module  $\Omega^d k$  is a projective object. We also observe from Lemma 3.7 and Corollary 2.2 that  $(\text{Ul}(R), \mathcal{S})$  has enough projective objects with  $\text{proj Ul}(R) = \text{add}(\Omega^d k)$ , and has enough injective objects with  $\text{inj Ul}(R) = \text{add}((\Omega^d k)^\dagger)$  by the duality  $(-)^{\dagger}$ .  $\blacksquare$

**Remark 4.12.** Let  $(R, \mathfrak{m})$  be 1-dimensional Cohen–Macaulay local ring with infinite residue field. Let  $(t)$  be a minimal reduction of  $\mathfrak{m}$ . Then  $\text{Ul}(R) = \text{CM}(R[\frac{\mathfrak{m}}{t}])$  by [12, Proposition A.1]. This equality actually gives an equivalence  $\text{Ul}(R) \cong \text{CM}(R[\frac{\mathfrak{m}}{t}])$  of categories, since Hom-sets do not change; see [20, Proposition 4.14]. Thus the usual exact structure on  $\text{CM}(R[\frac{\mathfrak{m}}{t}])$  coincides with the exact structure on  $\text{Ul}(R)$  given above via this equivalence.

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