EXTENDED MCKAY CORRESPONDENCE FOR QUOTIENT SURFACE SINGULARITIES

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ABSTRACT. Let G be a finite subgroup of GL(2) acting on $\mathbf{A}^2 \setminus \{0\}$ freely. The G-orbit Hilbert scheme G-Hilb$(\mathbf{A}^2)$ is a minimal resolution of the quotient $\mathbf{A}^2/G$ by [8]. We determine the generator sheaf of the ideal defining the universal G-cluster over G-Hilb$(\mathbf{A}^2)$, which somewhat strengthens the well-known McKay correspondence for a finite subgroup of SL(2).

1. Introduction.

For an algebraically closed field k, let G be any finite small subgroup of $\mathrm{GL}(2,k)$, that is, a finite subgroup with no pseudo-reflections, or equivalently, a finite subgroup of $\mathrm{GL}(2,k)$ acting on $\mathbf{A}_k^2$ with the unique fixed point at the origin. Throughout this article we assume that the characteristic of k is prime to the order $|G|$ of G.

The G-orbit Hilbert scheme G-Hilb$(\mathbf{A}_k^2)$ is the scheme parameterizing all the G-invariant zero-dimensional subschemes of $\mathbf{A}_k^2$ of length $|G|$, each with structure sheaf isomorphic to the group algebra $k[G]$ of G as a $k[G]$-module. It is by [8] the minimal resolution of the quotient $\mathbf{A}_k^2/G$.

For any geometric point y in the exceptional set of the resolution, let $\mathrm{Gen}(I_y)$ be the minimal $k[G]$-(sub)module generating the ideal $I_y \subset O_{\mathbf{A}_k^2}$ corresponding to y. The purpose of this article is to prove the following:

Theorem 1.1. *Let G be a finite small subgroup of* $\mathrm{GL}(2,k)$*, and E the exceptional set of the minimal resolution G-Hilb$(\mathbf{A}_k^2)$ of $\mathbf{A}_k^2/G$. Then the generator sheaf of the ideal defining the universal G-cluster, that is, the union of all* $\mathrm{Gen}(I_y)$ *over E is an O_F-module G-isomorphic to*

$$(\bigoplus_{\rho\neq\rho_0} O_{E(\rho)}(-1)\otimes_k \rho)\bigoplus O_F(-F)\otimes_k \rho_0$$

where ρ ranges over the set of all non-trivial irreducible representations of G, special in the sense of Definition 2.14, $E(\rho)$ is an irreducible component of E associated to ρ and F is the fundamental divisor of $E = F_{\mathrm{red}}$.

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See Theorem 3.13 and Corollary 3.14. Our main ingredient for the proof is the derived category method adopted by [3] and [8]. See also [11].

Here is an outline of the article. In Section 2, we recall reflexive modules and full modules. In Section 3, we review local McKay correspondence for two-dimensional quotient singularities from [8]. Then we formulate and prove its extended version, extended in the sense that it describes a family of the local versions over the exceptional set (Theorem 1.1).

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2. Reflexive modules and torsion free pullbacks

2.1. Derived category.

Definition 2.1. Let X be a scheme over a fixed field k. Let $Coh(X)$ be the category of coherent O_X-modules and $K(Coh(X))$ the category consisting of the bounded complexes of objects and morphisms in $Coh(X)$, where the morphisms in $K(Coh(X))$ are the homotopy equivalence classes of maps of complexes.

We define a morphism $f : P^\bullet \to Q^\bullet$ in $K(Coh(X))$ to be *a quasi-isomorphism* if f induces an isomorphism on cohomology.

The derived category $D(X)$ =$D(Coh(X))$ is the localization of $K(Coh(X))$ at Qis (the monoid of quasi-isomorphisms.), that is, it is the category $K(Coh(X))$ modulo equivalence defined by Qis. See [6, Def., pp. 49–50]. Similarly we define $Coh_c(X)$ to be the category of coherent O_X-modules with complete supports, $K(Coh_c(X))$ the category consisting of bounded complexes of objects and morphisms in $Coh_c(X)$, and $D_c(X)$ the derived category of $K(Coh_c(X))$.

2.2. **Reflexive O_X-modules and full O_Y-modules.** This subsection is taken from [1], [5], and [15]. Let Z be a scheme of finite type over k, $\mathcal{F}$ a coherent O_Z-module on Z. Then $\mathcal{F}$ is defined to be a *reflexive* O_Z-module iff $\mathcal{F}^{\vee\vee} \simeq \mathcal{F}$, where $\mathcal{F}^\vee = \mathcal{H}om_{O_Z}(\mathcal{F}, O_Z)$.

Lemma 2.2. *Let Z be a normal surface, $Z' = Z \setminus \operatorname{Sing}(Z)$ and $i : Z' \hookrightarrow Z$ the inclusion. The following is true.*

1. *any torsion free module over a discrete valuation ring is free, hence, any torsion free sheaf on a nonsingular curve is locally free.*
2. *if Z is nonsingular, any reflexive O_Z-module is locally free,*
3. *if $\mathcal{F}$ is a reflexive O_Z-module, then $i_*(i^*\mathcal{F}) = \mathcal{F}$, and it is uniquely determined by its restriction to Z',*
4. *if $\mathcal{F}$ is a locally free $O_{Z'}$-module, then $i_*(\mathcal{F})$ is a reflexive O_Z-module,*
5. *$\mathcal{G}^\vee$ is reflexive for any finite O_Z-module $\mathcal{G}$.*

Proof. See [7, Corollary 1.4] for the parts (1) and (2). See [7, Proposition 1.6] for the part (3). See [7, Corollary 1.2] for the part (5). The part (4) is proved as follows. Let $\mathcal{G} = i_*(\mathcal{F})$. Since $\mathcal{F} = i^*(\mathcal{G})$ is locally free on Z', we have $\mathcal{G} \simeq i_*(i^*\mathcal{G}) \simeq i_*(i^*(\mathcal{G}^{\vee\vee})) = \mathcal{G}^{\vee\vee}$, from which (4) follows. □

The following summarizes [4, Lemma 2,1, 2.2].

Lemma 2.3. *Let X be an affine normal surface with rational singularities and $f : Y \to X$ the minimal resolution of X.*

(1) *Let M be a reflexive O_X-module, $\mathcal{M}$ the torsion free pullback of M to Y, that is, $\mathcal{M} = f^*M/O_Y$-torsions. Then $f_*(\mathcal{M}) = M$, and*
  (i) *$\mathcal{M}$ is locally free,*
  (ii) *$\mathcal{M}$ is generated by global sections,*
  (iii) *$R^1 f_*(\mathcal{M}^\vee \otimes_{O_Y} \omega_Y) = 0$.*

(2) *Conversely if a coherent O_Y-module $\mathcal{M}$ satisfies (i) and (iii), then $M := f_*(\mathcal{M})$ is a reflexive O_X-module.*

Definition 2.4. We call an O_Y-module $\mathcal{M}$ a *full* O_Y-module (or simply a full sheaf following [4]) if the conditions (i)–(iii) are satisfied.

Notice that $\mathcal{M}$ is the torsion free pull-back of the direct image $f_*\mathcal{M}$ by the condition (ii) and hence is determined by $f_*\mathcal{M}$. The following is a corollary of Lemma 2.3 (and Lemma 2.2):

Corollary 2.5. *Under the same notation as in Lemma 2.3, there is a bijective correspondence between the following sets*
  (i) *the set of (indecomposable) reflexive O_X-modules M,*
  (ii) *the set of (indecomposable) full O_Y-modules $\mathcal{M}$.*

Corollary 2.6. *Under the same notation as in Lemma 2.3, every full O_Y-module $\mathcal{M}$ is determined by its restriction to $Y \setminus f^{-1}(\mathrm{Sing}\,(X))$.*

Proof. Clear from Lemma 2.2 (3). □

Corollary 2.7. *The invertible sheaf K_Y is a full sheaf.*

Proof. Clear from Lemma 2.3 (2) and Corollary 2.6. □

2.3. **The minimal resolution Y of the quotient $X = U/G$.** Let k be any algebraically closed field of any characteristic, and G a finite small subgroup of $\mathrm{GL}(2, k)$, which acts on $\mathbf{A}^2_k$ from the left. We assume throughout this article that the order $|G|$ of G and the characteristic of k are coprime.

Let $S = k[x, y]$ be the polynomial ring of two variables, and $R = S^G$ the subring of S consisting of all G-invariants. Let $U := \mathbf{A}^2_k = \mathrm{Spec}\ S$. $X = U/G = \mathrm{Spec}\ R$ and let $\pi : U \to X$ be the natural morphism. Since G is small, the surface X has a unique singular point 0, which is a rational singularity [14]. Let $f : Y \to X$ be the minimal resolution of $X = U/G$. Thus we have a commutative diagram:

(2.3.1)
$$\begin{array}{ccc} Y \times_k U & \xrightarrow{\pi_U} & U \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X = U/G \end{array}$$

Let F be *the fundamental divisor* of the singularity $(X, 0)$. That is, the minimal effective divisor D on Y such that $D \neq 0$, $\mathrm{Supp}\,(D) \subset f^{-1}(0)$, and $DE' \leq 0$ for any irreducible component E' of $f^{-1}(0)$.

The following is due to [15].

Theorem 2.8. *Let G be a finite small subgroup of* $\mathrm{GL}(2, k)$ *and $X = U/G$. Under the same notation as in Subsection 2.3,*

(1) *there is a bijective correspondence between the following sets*
   (i) *the set of irreducible components E_i of $E := f^{-1}(0)$,*
   (ii) *the set of non-trivial indecomposable full O_Y-modules $\mathcal{M}_i$, special in the sense that $H^1(Y, \mathcal{M}_i^\vee) = 0$,*

   where the correspondence $\mathcal{M}_i \mapsto E_i$ is given by

$$c_1(\mathcal{M}_i)E_j = \delta_{ij},$$

(2) *the rank of $\mathcal{M}_i$ is equal to $c_1(\mathcal{M}_i)F$, the multiplicity of E_i in F.*

2.4. **G-equivariant locally free O_U-modules and reflexive O_X-modules.** Let G be a small subgroup of $\mathrm{GL}(2, k)$. Any G-equivariant locally free O_U-module $\mathcal{F}$ of finite rank is of the form $\mathcal{F} = \widetilde{M}$ for a projective finite S-module M with G-action. Note that M is a free S-module [1] by [12] and [13].

Let M^G be the submodule consisting of all G-invariants of M. Then the natural exact sequence $0 \to M^G \to M$ of R-modules splits because the characteristic of k is prime to the order $|G|$. In other words, there is an R-submodule N of M such that $M \simeq M^G \oplus N$.

For a G-equivariant locally free O_U-module $\mathcal{F}$, we define a subsheaf $\mathcal{F}^G$ of $\pi_*(\mathcal{F})$ consisting of G-invariant sections by

$$\mathcal{F}^G(W) = \Gamma(\pi^{-1}(W), \mathcal{F})^G = \Gamma(W, \pi_*(\mathcal{F}))^G$$

for any open subset W of X. Since π is finite, $\mathcal{F}^G$ is a coherent O_X-submodule of $\pi_*\mathcal{F}$ associated to the R-module M^G. See [3, § 4] for more about G-sheaves. Let $U' = U \setminus \{0\}$, $X' = X \setminus \{\pi(0)\}$, $i : U' \hookrightarrow U$ and $j : X' \hookrightarrow X$ the natural inclusions.

Remark 2.9. If $\mathcal{F}$ is a G-equivariant locally free O_U-module,[2] $\mathcal{F}^G$ is a reflexive O_X-module as in [2, Proposition 2.2]. Conversely if $\mathcal{F} = (\pi^*(\mathcal{G}))^{\vee\vee}$ for a reflexive O_X-module $\mathcal{G}$, then $\mathcal{F}$ is a G-equivariant locally free O_U-module such that $\mathcal{F}^G \simeq \mathcal{G}$.

Definition 2.10. For a normal surface Z over k, we denote by $\mathrm{Coh}(Z)$ (resp. $\mathrm{LF}(Z)$, $\mathrm{Rfl}(Z)$) the category of coherent O_Z-modules, (resp. the category of locally free O_Z-modules of finite rank, the category of coherent reflexive O_Z-modules). If Z has a G-action, then we denote by $\mathrm{Coh}^G(Z)$ (resp. $\mathrm{LF}^G(Z)$, $\mathrm{Rfl}^G(Z)$) the category of G-equivariant coherent O_Z-modules, (resp. the category of G-equivariant locally free O_Z-modules of finite rank, the category of G-equivariant coherent reflexive O_Z-modules). Let $G\text{-}\mathrm{Hom}_{O_Z}(A, B)$

[1] M is a projective S-module if and only if M is a free S-module.

[2] $\mathcal{F}$ is a free O_U-module by [12] and [13].

(resp. $\mathrm{Hom}_{O_Z}(A, B)$) be the set of all G-homomorphisms (resp. all homomorphisms) from A to B for $A, B \in \mathrm{Coh}^G(Z)$ (resp. $A, B \in \mathrm{Coh}(Z)$).

Then we have the following equivalence of categories. (See [2, Proposition 2.2].)

Lemma 2.11. *The functor G-inv : $\mathcal{F} \mapsto \mathcal{F}^G$ is an equivalence of the categories $\mathrm{LF}^G(U)$ and $\mathrm{Rfl}(X)$.*

Definition 2.12. For an irreducible representation ρ of G, we define

$$\begin{aligned} M_\rho &: = (O_U \otimes_k \rho^*)^G := [\pi_*(O_U) \otimes_k \rho^*]^G, \\ \mathcal{M}_\rho &:= f^*M/O_Y\text{-torsions}. \end{aligned}$$

Corollary 2.13. *Let ρ, σ be irreducible representations of G, and ρ^* the dual representation of ρ, Then M_ρ is a reflexive O_X-module and*

$$\mathcal{H}om_{O_X}(M_\rho, M_\sigma) \simeq M_{\rho^* \otimes_k \sigma}.$$

In particular, the dual O_X-module $M_\rho^\vee$ of M_ρ is isomorphic to M_{ρ^*}.

Definition 2.14. An irreducible representation ρ of G is said to be *special* if its corresponding full O_Y-module $\mathcal{M}_\rho$ is special, that is, $H^1(Y, \mathcal{M}_\rho^\vee) = 0$. A $k[G]$-module N is said to be *special* if N is an irreducible $k[G]$-module isomorphic to a special representation.

We show the existence of non-special representations when $G \not\subset \mathrm{SL}(2, k)$.

Lemma 2.15. *For a full sheaf $\mathcal{M}$ on Y, $\mathcal{M}$ is special if and only if $\mathcal{M} \otimes_{O_Y} K_Y$ is a full sheaf.*

Proof. Since $\mathcal{M}$ and K_Y are generated by global sections by Corollary 2.7, so is $\mathcal{M} \otimes_{O_Y} K_Y$. Lemma follows from $H^1((\mathcal{M} \otimes_{O_Y} K_Y)^\vee \otimes K_Y) = H^1(\mathcal{M}^\vee)$. □

2.5. **Universal G-cluster and full sheaves.** The following is due to [8].

Theorem 2.16. *Let $U = \mathbf{A}_k^2$, G a finite subgroup of $\mathrm{GL}(2, k)$ such that the order $|G|$ of G is prime to the characteristic of k. Then $Y = G$-$\mathrm{Hilb}(U)$ is connected and it is a minimal resolution of $X := U/G$.*

Definition 2.17. Let $\mathcal{Z}$ be the universal cluster over $Y = G$-$\mathrm{Hilb}(U)$. Consider the commutative diagram:

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{q} & U \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X = U/G \end{array} \tag{2.5.1}$$

where $q := (\pi_U)_{\mathcal{Z}}$ and $p := (\pi_Y)_{\mathcal{Z}}$. Note that

$$\mathcal{Z} \times_Y (Y \setminus F) \simeq (U \times_X (X \setminus \{O\}).$$

Let y be a closed point of Y. Let Z_y be the fiber of $\mathcal{Z}$ over y, that is, the G-cluster corresponding to y.

Lemma 2.18. *The following is true:*

(1) $q_*(O_{\mathcal{Z}}) = O_U$ *and* $f_*p_*(O_{\mathcal{Z}}) \simeq \pi_*q_*(O_{\mathcal{Z}}) = \pi_*(O_U)$,
(2) $\pi_*(O_U)$ *is a reflexive* O_X*-module,*
(3) $p_*(O_{\mathcal{Z}})$ *is the torsion free pullback of* $\pi_*(O_U)$ *by* f,
(4) *for any irreducible representation* ρ *of* G,

$$\mathcal{M}_\rho \cong [p_*q^*(O_U) \otimes_k \rho^*]^G = [p_*(O_{\mathcal{Z}}) \otimes_k \rho^*]^G,$$

(5) $p_*(O_{\mathcal{Z}}) = \bigoplus_{\rho\in \mathrm{Irr}(G)} \mathcal{M}_\rho \otimes_k \rho$ *and* $\pi_*(O_U) = \bigoplus_{\rho\in \mathrm{Irr}(G)} M_\rho \otimes_k \rho$.

Proof. Let $q' : \mathcal{Z}' := \mathcal{Z} \setminus q^{-1}(0) \to U'$ (resp. $p' : \mathcal{Z}' \to Y' = Y \setminus F$) be the restriction of q (resp. of p). First we prove $q_*(O_{\mathcal{Z}}) = O_U$. Since q' is an isomorphism, $(q')_*(O_{\mathcal{Z}'}) \simeq O_{U'}$. It follows that $O_U \subset q_*(O_{\mathcal{Z}}) \subset i_*((q')_*(O_{\mathcal{Z}'})) = i_*(O_{U'}) = O_U$ since $\mathrm{codim}(U \setminus U') = 2$. Hence $q_*(O_{\mathcal{Z}}) = O_U$. This proves (1).

Since O_U is Cohen-Macaulay and is finite as an O_X-module, it is also Cohen-Macaulay over O_X. This proves (2).

Next we prove (3). See also the proof of [8, Theorem 3.1]. Since $\mathcal{Z}$ is finite and flat over Y, $p_*(O_{\mathcal{Z}})$ is locally free. Moreover, it is generated by global sections because it is a quotient of the quasi-coherent O_Y-module $O_Y \otimes_k k[x, y]$. So it follows from $f_*p_*(O_{\mathcal{Z}}) \cong \pi_*O_U$ that $p_*(O_{\mathcal{Z}})$ is the torsion free pullback of $\pi_*(O_U)$ by f.

Next we prove (4). It is proved in [8, Corollary 3.2] that $[p_*(O_{\mathcal{Z}}) \otimes_k \rho^*]^G$ is a full O_Y-module. We have

$$\begin{aligned} f_*([p_*(O_{\mathcal{Z}}) \otimes_k \rho^*]^G) &= [f_*p_*(O_{\mathcal{Z}}) \otimes_k \rho^*]^G = [\pi_*q_*(O_{\mathcal{Z}}) \otimes_k \rho^*]^G \\ &= [\pi_*(O_U) \otimes_k \rho^*]^G = M_\rho. \end{aligned}$$

Taking the torsion-free pullbacks of the both sides, we obtain (4).

Next we prove (5). By [3, Subsection 4.2] and since the characteristic of k is prime to $|G|$, there is a decomposition $p_*(O_{\mathcal{Z}}) = \bigoplus_{\rho\in \mathrm{Irr}(G)} \mathcal{N}_\rho \otimes_k \rho$ for some O_Y-module $\mathcal{N}_\rho$. For any irreducible representation γ of G, we have

$$\mathcal{M}_\gamma := [p_*(O_{\mathcal{Z}}) \otimes_k \gamma^*]^G \simeq \bigoplus_{\rho\in \mathrm{Irr}(G)} \mathcal{N}_\rho \otimes_k (\mathrm{Hom}_k(\gamma, \rho))^G \simeq \mathcal{N}_\gamma.$$

The rest of (5) follows from (1) and (4). □

3. Extended McKay correspondence

The purpose of this section is to prove Theorem 3.13 together with Corollary 3.14, which reformulate Theorem 1.1 in a more precise way.

3.1. **Local McKay correspondence.** We first recall results from [8].

Definition 3.1. We define two functors

$$\Psi : D^G(U) \to D(Y), \quad \Phi : D(Y) \to D^G(U)$$

by

$$\Psi(A) = [p_*\mathbf{L}q^*(A)]^G = [(\pi_Y)_*(O_{\mathcal{Z}} \overset{\mathbf{L}}{\otimes}_{O_{Y\times_k U}} \pi_U^*(A)]^G,$$

$$\Phi(B) = \mathbf{R}(\pi_U)_*(O_{\mathcal{Z}}^\vee \overset{\mathbf{L}}{\otimes}_{O_{Y\times_k U}} \pi_Y^*(B \otimes_k \rho_0) \otimes_{O_{Y\times_k U}} \pi_U^* K_U[2]).$$

where $O_{\mathcal{Z}}^\vee$ is the derived dual of $O_{\mathcal{Z}}$, that is, $\mathbf{R}\mathcal{H}om_{O_{Y\times_k U}}(O_{\mathcal{Z}}, O_{Y\times_k U})$.

Note $K_U \cong O_U \otimes_k \det(\rho_U^*)$ where ρ_U denotes the given representation $G \subset \mathrm{GL}(U)$.

Theorem 3.2. *Φ is fully faithful and Ψ is a left adjoint of Φ.*

See [8, § 6] and [9, Proposition 1.1, Lemma 2.9].

Definition 3.3. We number all irreducible representations ρ in the following manner: ρ_0 is trivial, ρ_i $(1 \le i \le m)$ is non-trivial special and ρ_j $(j \ge m+1)$ is non-trivial nonspecial.

We denote $\mathcal{M}_{\rho_i}$ by $\mathcal{M}_i$.

Lemma 3.4. *Let ρ_i be an irreducible special representation of G, $\mathcal{M}_i = \mathcal{M}_{\rho_i}$, and E_i the irreducible component of F with $c_1(\mathcal{M}_i)E_j = \delta_{ij}$ as in Theorem 2.8. Then*

$$\Psi(O_0 \otimes_k \rho^*) = \begin{cases} O_{E_i}(-1)[1] & \text{if } \rho = \rho_i \text{ is non-trivial special} \\ O_F & \text{if } i = 0, \text{ that is, } \rho = \rho_0 \text{ is trivial,} \\ 0 & \text{if } \rho \text{ is non-trivial, nonspecial.} \end{cases}$$

See [8, § 5] for Lemma 3.4.

Lemma 3.5. *Under the same notation as above,*

$$\begin{aligned} &G\text{-}\mathrm{Ext}^p_{O_U}(O_{Z_y}, (O_0 \otimes_k \rho_i)) := \mathrm{Hom}^p_{D_c^G(O_U)}(O_{Z_y}, (O_0 \otimes_k \rho_i)) \\ &= \begin{cases} \mathrm{Ext}^p_{O_Y}(O_F, O_y) & (i = 0) \\ \mathrm{Ext}^{p-1}_{O_Y}(O_{E_i}(-1), O_y) & (1 \le i \le m) \\ 0 & (i \ge m+1) \end{cases} \\ &= \begin{cases} k & (i = 0, p = 0, 1, y \in F), \\ k & (1 \le i \le m, p = 1, 2, y \in E_i), \\ 0 & (\textit{otherwise}). \end{cases} \end{aligned}$$

See [8, § 7] for the above lemma. See also Definition 3.3.

Theorem 3.6. *(Local McKay correspondence) Let $\mathfrak{m}$ be the maximal ideal of O_U at the origin, $y \in F$ and Z_y the G-invariant cluster of U corresponding to y and I_{Z_y} the ideal of O_U defining Z_y. Then the $k[G]$-module* $\mathrm{Gen}(I_y) := I_{Z_y}/\mathfrak{m}I_{Z_y}$ *is given by*

$$\begin{cases} \rho_i \oplus \rho_0 & \text{if } y \in E_i \setminus \cup_{j\ne i} E_j \\ \rho_i \oplus \rho_j \oplus \rho_0 & \text{if } y \in E_i \cap E_j,\ i \ne j. \end{cases}$$

See [8, Theorem 7.1].

3.2. **The ideals n_Y and I_Y.** Let $U = \mathbf{A}^2_k$, $X = U/G$, $Y = G$-Hilb(U) and $f : Y \to X$ the natural morphism, which is the minimal resolution of X. Let $\pi_Y : Y \times_k U \to Y$ and $\pi_U : Y \times_k U \to U$ be the first and the second projection. Since $Y \times_X X \simeq Y$, we see that Y is a closed subscheme of $Y \times_k X$. Let I_Y be the ideal of $O_{Y\times_k X}$ defining Y, and $n_Y = (1_Y \times \pi)^{-1}(I_Y) \cdot O_{Y\times_k U}$.

Let $\mathcal{Z}$ be the universal subscheme of $Y \times_k U$, and $\mathcal{I}$ the ideal sheaf of $O_{Y\times_k U}$ defining $\mathcal{Z}$. We have a commutative diagram of structure sheaves:

$$\begin{array}{ccc} O_{\mathcal{Z}} & \xleftarrow{q^*} & O_U \\ {\scriptstyle p^*}\uparrow & & \uparrow{\scriptstyle \pi^*} \\ O_Y & \xleftarrow{f^*} & O_X. \end{array}$$

where

$$O_Y \simeq O_Y \otimes_{O_X} O_X \simeq O_Y \otimes_k O_X/I_Y,$$

$$O_Y \otimes_k O_X \overset{\mathrm{id}_Y^* \otimes \pi^*}{\to} O_Y \otimes_k O_U \overset{p^*\otimes q^*}{\to} O_{\mathcal{Z}},$$

$$O_Y \otimes_k O_X/I_Y \simeq O_Y \overset{p^*}{\to} O_{\mathcal{Z}} = O_Y \otimes_k O_U/\mathcal{I}.$$

3.3. **The fundamental divisor F.** Let F be the fundamental divisor of the singularity $(X, 0)$ (see Subsection 2.3).

Lemma 3.7. *Let $\mathfrak{m}$ be the (maximal) ideal of O_U defining the origin 0, and I_F the ideal of O_Y defining the fundamental divisor F. Then*

(1) $O_{Y\times_X U} \simeq O_Y \otimes_k O_U/\mathfrak{n}_Y$, *that is, $\mathfrak{n}_Y$ is the ideal of $O_Y \otimes_k O_U$ defining the fiber product $Y \times_X U$.*
(2) $\mathcal{Z} \subset Y \times_X U$, *that is,* $\mathfrak{n}_Y \subset \mathcal{I}$,
(3) $\pi_U^*\mathfrak{m} + \mathfrak{n}_Y = \pi_U^*\mathfrak{m} + I_F O_{Y\times_k U} = \pi_U^*\mathfrak{m} + \mathcal{I}$,
(4) $\pi_U^*\mathfrak{m}\mathcal{I} + \mathfrak{n}_Y\mathcal{I} = \pi_U^*\mathfrak{m}\mathcal{I} + \mathcal{I}^2 = \pi_U^*\mathfrak{m}\mathcal{I} + I_F\mathcal{I}$.

Proof. Let $\mathfrak{m}_X$ be the (maximal) ideal of O_X defining the singular point 0. The ideal I_Y of $O_Y \otimes_k O_X$ is generated by $(f^*a) \otimes 1 - 1 \otimes a$ for $a \in \mathfrak{m}_X$. Therefore $\mathfrak{n}_Y$ is generated by $(f^*a) \otimes 1 - 1 \otimes (\pi^*a)$ for $a \in \mathfrak{m}_X$. Hence $O_Y \otimes_k O_U/\mathfrak{n}_Y$ is the structure sheaf of the fiber product $Y \times_X U$. The commutative diagram [8, (3.1)] (which follows from the proof of [8, Theorem 3.1]) implies $\mathcal{Z} \subset Y \times_X U$. Therefore, we have $(1 \otimes \pi^*)I_Y \subset \mathcal{I}$, so that $\mathfrak{n}_Y \subset \mathcal{I}$. This proves parts (1) and (2).

The inclusion $\pi_U^*\mathfrak{m} + \mathfrak{n}_Y \subset \pi_U^*\mathfrak{m} + \mathcal{I}$ is clear. We shall prove the converse. Since $\mathfrak{n}_Y$ is generated by $(f^*a)\otimes 1 - 1 \otimes (\pi^*a)$ for $a \in \mathfrak{m}_X$, we have $(f^*a)\otimes 1 \in \pi_U^*\mathfrak{m} + \mathfrak{n}_Y$ because $\pi^*a \in \mathfrak{m}$. Thus $\pi_U^*\mathfrak{m} + \mathfrak{n}_Y = \pi_U^*\mathfrak{m} + I_F O_{Y\times U}$. We see

$$\begin{aligned} O_{Y\times_k U}/(\pi_U^*\mathfrak{m} + I_F O_{Y\times_k U}) &\simeq (O_Y/I_F) \otimes_{O_{Y\times_k U}} (O_U/\mathfrak{m}) = O_F, \\ O_{Y\times_k U}/(\pi_U^*\mathfrak{m} + \mathcal{I}) &\simeq (O_{Y\times_k U}/\mathcal{I}) \otimes_{O_{Y\times_k U}} (O_U/\mathfrak{m}) \\ &\simeq O_{\mathcal{Z}} \otimes_{O_{Y\times_k U}} (O_U/\mathfrak{m}). \end{aligned}$$

Now Lemma 3.4 implies that

$$(\pi_Y)_*(\mathcal{O}_{\mathcal{Z}} \otimes \mathcal{O}_0) = \bigoplus_\rho \Psi^0(O_0 \otimes \rho^*) \otimes \rho = O_F$$

where $\Psi^0(-)$ is the 0-th cohomology of $\Psi(-)$. It follows that

$$\pi_U^*\mathfrak{m} + I_F O_{Y\times U} = \pi_U^*\mathfrak{m} + \mathcal{I}.$$

This proves part (3). Part (4) follows from part (3). □

Lemma 3.8. *Let $\mathfrak{n} := (\mathfrak{m} \cap k[x,y]^G)O_U$. Then the following is true:*

1. $I_{Z_y} \simeq \mathcal{I}/\mathfrak{m}_y\mathcal{I} \simeq (\mathcal{I}/\mathcal{I} \cap (\mathfrak{m}_y \otimes_k O_U))$,
2. $\mathcal{I} \cap (\mathfrak{m}_y \otimes_k O_U) = \mathfrak{m}_y\mathcal{I}$,
3. $\mathfrak{n}_Y + \mathfrak{m}_y \otimes_k O_U = O_Y \otimes_k \mathfrak{n} + \mathfrak{m}_y \otimes_k O_U$ *if* $y \in F$,
4. $(\mathfrak{n}_Y + \mathfrak{m}_y\mathcal{I})/\mathfrak{m}_y\mathcal{I} \simeq \mathfrak{n}$ *if* $y \in F$.

Proof. Since $O_{\mathcal{Z}}$ is O_Y-flat,

$$\begin{aligned} O_{Z_y} &\simeq O_{\mathcal{Z}} \otimes_{O_Y} O_y = (O_{Y\times_k U}/\mathcal{I}) \otimes_{O_Y} O_y \\ &\simeq (O_Y \times_k O_U)/(\mathcal{I} + \mathfrak{m}_y \otimes_k O_U) \\ &\simeq (O_y \times_k O_U)/\left(\mathcal{I}/\mathcal{I} \cap (\mathfrak{m}_y \otimes_k O_U)\right), \end{aligned}$$

whence $I_{Z_y} \simeq \mathcal{I}/\mathcal{I} \cap (\mathfrak{m}_y \otimes_k O_U)$. This proves the part (1).

Since $O_{\mathcal{Z}}$ is O_Y-flat again, the following is an exact sequence:

$$0 \to \mathcal{I} \otimes_{O_{Y\times_k U}} O_y \to O_y \otimes_k O_U \ (\simeq O_U) \to O_{Z_y}(\simeq O_{\mathcal{Z}} \otimes_{O_{Y\times_k U}} O_y) \to 0,$$

so that we have $I_{Z_y} = \mathcal{I}/\mathfrak{m}_y\mathcal{I}$. Hence we have

$$\mathcal{I} \cap (\mathfrak{m}_y \otimes_k O_U) = \mathfrak{m}_y\mathcal{I},$$

which proves the part (2). Since $\mathfrak{n}_Y$ is generated by $f^*a \otimes 1 - 1 \otimes \pi^*a$ for $a \in \mathfrak{m}_X$ and $\pi^*\mathfrak{m}_X = \mathfrak{n}$, we have $\mathfrak{n}_Y + \mathfrak{m}_y \otimes_k O_U = O_Y \otimes_k \mathfrak{n} + \mathfrak{m}_y \otimes_k O_U$ for $y \in F$, which is the part (3). It follows that

$$\begin{aligned} (\mathfrak{n}_Y + \mathfrak{m}_y\mathcal{I})/\mathfrak{m}_y\mathcal{I} &\simeq \mathfrak{n}_Y/\mathfrak{n}_Y \cap \mathcal{I} \cap (\mathfrak{m}_y \otimes_k O_U) \\ &= \mathfrak{n}_Y/\mathfrak{n}_Y \cap (\mathfrak{m}_y \otimes_k O_U) \quad \text{by Lemma 3.7 (2)} \\ &\simeq \mathfrak{n}_Y + \mathfrak{m}_y \otimes_k O_U/(\mathfrak{m}_y \otimes_k O_U) \\ &\simeq (O_Y \otimes_k \mathfrak{n} + \mathfrak{m}_y \otimes_k O_U)/(\mathfrak{m}_y \otimes_k O_U) \simeq \mathfrak{n}. \end{aligned}$$

This completes the proof. □

Definition 3.9. We define

$$\begin{aligned} \mathcal{V} &:= \mathcal{I}/(\pi_U^*\mathfrak{m}\mathcal{I} + \mathfrak{n}_Y), \\ \mathcal{V}^\dagger &: = \mathcal{I}/(\pi_U^*\mathfrak{m}\mathcal{I} + I_F\mathcal{I}) \simeq (\mathcal{I}/\pi_U^*\mathfrak{m}\mathcal{I}) \otimes_{O_Y} O_F. \end{aligned}$$

These are G-equivariant coherent sheaves on $Y \times \{0\} \cong Y$ and we consider them as objects of $\mathrm{Coh}^G(Y)$. Notice by Lemma 3.7 (4) we have an inclusion $\pi_U^*\mathfrak{m}\mathcal{I} + I_F\mathcal{I} \subset \pi_U^*\mathfrak{m}\mathcal{I} + \mathfrak{n}_Y$, which induces a natural surjection $\mathcal{V}^\dagger \twoheadrightarrow \mathcal{V}$.

Lemma 3.10. *For $y \in F$,*

$$\mathcal{V} \otimes_{O_Y} O_y \simeq I_{Z_y}/(\mathfrak{m} I_{Z_y} + \mathfrak{n});\ \ \mathcal{V}^\dagger \otimes_{O_Y} O_y \simeq I_{Z_y}/\mathfrak{m} I_{Z_y}.$$

Proof. Let $y \in Y$ and $\mathfrak{m}_y$ the maximal ideal of $O_{Y,y}$. Hence $O_y = O_{Y,y}/\mathfrak{m}_y$. Since $I_{Z_y} = \mathcal{I}/\mathfrak{m}_y \mathcal{I}$, we have

$$\begin{aligned}
\mathcal{V} \otimes_{O_Y} O_y &\simeq (\mathcal{I}/(\pi_U^* \mathfrak{m} \mathcal{I} + \mathfrak{n}_Y)) \otimes_{O_Y} O_y = \mathcal{I}/(\pi_U^* \mathfrak{m} \mathcal{I} + \mathfrak{n}_Y + \mathfrak{m}_y \mathcal{I}) \\
&\simeq (\mathcal{I}/\mathfrak{m}_y \mathcal{I})/\left(\mathfrak{m}(\mathcal{I}/\mathfrak{m}_y \mathcal{I}) + (\mathfrak{n}_Y + \mathfrak{m}_y \mathcal{I})/\mathfrak{m}_y \mathcal{I}\right) \\
&\simeq I_{Z_y}/(\mathfrak{m} I_{Z_y} + \mathfrak{n}), \\
\mathcal{V}^\dagger \otimes_{O_Y} O_y &= \mathcal{I}/(\pi_U^* \mathfrak{m} \mathcal{I} + I_F \mathcal{I} + \mathfrak{m}_y \mathcal{I}) = \mathcal{I}/(\pi_U^* \mathfrak{m} \mathcal{I} + \mathfrak{m}_y \mathcal{I}) \\
&\simeq (\mathcal{I}/\mathfrak{m}_y \mathcal{I})/(\pi_U^* \mathfrak{m} \mathcal{I} + \mathfrak{m}_y \mathcal{I})/\mathfrak{m}_y \mathcal{I}) \\
&\simeq (\mathcal{I}/\mathfrak{m}_y \mathcal{I})/\mathfrak{m}(\mathcal{I}/\mathfrak{m}_y \mathcal{I}) \simeq I_{Z_y}/\mathfrak{m} I_{Z_y}.
\end{aligned}$$

□

Definition 3.11. For a coherent $O_{Y \times U}$-module J, we define a functor

$$\Psi_J : D_c^G(U) \to D_c(Y)$$

by

$$\Psi_J(A) = [p_*(\mathbf{L}\pi_U^*(A) \overset{\mathbf{L}}{\otimes}_{O_{Y \times_k U}} J)]^G = \mathbf{R}(\pi_Y)_*(\mathbf{L}\pi_U^*(A) \overset{\mathbf{L}}{\otimes}_{O_{Y \times_k U}} J)]^G$$

where $A \in D_c^G(U)$. Note that $\Psi = \Psi_{O_\mathcal{Z}}$.

Lemma 3.12. *The following is true:*

(1) $\Psi(O_0 \otimes_k \rho_i^*) = \begin{cases} O_F & (i = 0), \\ O_{E_i}(-1)[1] & (\rho_i\ :\ \textit{non-trivial special}), \end{cases}$

(2) $\Psi_{O_{Y \times_k U}}(O_0 \otimes_k \rho_i^*) = \begin{cases} O_Y & (i = 0), \\ 0 & (i \neq 0), \end{cases}$

(3) $\Psi_\mathcal{I}(O_0 \otimes_k \rho_i^*) = \begin{cases} O_Y(-F) & (i = 0), \\ O_{E_i}(-1) & (i \neq 0), \\ 0 & \textit{(otherwise)}. \end{cases}$

Proof. The part (1) follows from Lemma 3.4. Since $O_{Y \times_k U}$ is $O_{Y \times_k U}$-flat, by definition,

$$\begin{aligned}
\Psi_{O_{Y \times_k U}}(O_0 \otimes_k \rho_i^*) &= [p_*(O_Y \otimes_k (O_0 \otimes_k \rho_i^*))]^G \\
&= [O_Y \otimes_k \rho_i^*]^G = \begin{cases} O_Y & (i = 0) \\ 0 & (i \neq 0) \end{cases}
\end{aligned}$$

The part (3) follows from the parts (1) and (2) and the exact sequence:

$$\longrightarrow \Psi_\mathcal{I}^{-2}(O_0 \otimes_k \rho_0^*) \to \Psi_{O_{Y \times_k U}}^{-2}(O_0 \otimes_k \rho_0^*) \to \Psi_{O_\mathcal{Z}}^{-2}(O_0 \otimes_k \rho_0^*)$$

$$\longrightarrow \Psi_\mathcal{I}^{-1}(O_0 \otimes_k \rho_0^*) \to \Psi_{O_{Y \times_k U}}^{-1}(O_0 \otimes_k \rho_0^*) \to \Psi_{O_\mathcal{Z}}^{-1}(O_0 \otimes_k \rho_0^*)$$

$$\longrightarrow \Psi_\mathcal{I}^{0}(O_0 \otimes_k \rho_0^*) \longrightarrow \Psi_{O_{Y \times_k U}}^{0}(O_0 \otimes_k \rho_0^*) \to \Psi_{O_\mathcal{Z}}^{0}(O_0 \otimes_k \rho_0^*) \longrightarrow 0.$$

This proves the part (3). □

Theorem 3.13. *There are isomorphisms*

$$\mathcal{V} \simeq \sum_{\rho_i \neq \rho_0} O_{E_i}(-1) \otimes_k \rho_i, \quad \mathcal{V}^\dagger \simeq \mathcal{V} \oplus O_F(-F) \otimes_k \rho_0,$$

where ρ_i ranges over all non-trivial special irreducible representations of G.

Corollary 3.14. *Let $\mathfrak{n} = (\mathfrak{m} \cap k[x,y]^G)O_U$. Then the fibers of $\mathcal{V}$ and $\mathcal{V}^\dagger$ over $y \in F$ are*

$$\mathcal{V} \otimes_{O_Y} O_y = \begin{cases} \rho_i & (y \in E_i \setminus \cup_{j \neq i} E_j) \\ \rho_i \oplus \rho_j & (y \in E_i \cap E_j,\ i \neq j), \end{cases}$$

$$\mathcal{V}^\dagger \otimes_{O_Y} O_y = \begin{cases} \rho_i \oplus \rho_0 & (y \in E_i \setminus \cup_{j \neq i} E_j) \\ \rho_i \oplus \rho_j \oplus \rho_0 & (y \in E_i \cap E_j,\ i \neq j). \end{cases}$$

We note $E_i = C(\rho_i)$ and $\mathrm{Gen}(I_{Z_y}) = I_{Z_y}/\mathfrak{m}I_{Z_y}$ in Theorem 1.1.

Proof of Theorem 3.13 and Corollary 3.14. By Lemma 3.12, we have

$$\begin{aligned} \mathcal{I}/\pi_U^*\mathfrak{m}\mathcal{I} = p_*(\mathbf{L}\pi_U^*(O_0) \overset{\mathbf{L}}{\otimes}_{O_{Y \times_k U}} \mathcal{I}) &\simeq \sum_{\rho:\text{irred.}} \Psi_{\mathcal{I}}(O_0 \otimes_k \rho^*) \otimes \rho \\ &\simeq \bigoplus_{i=1}^n O_{E_i}(-1) \otimes_k \rho_i \bigoplus O_Y(-F) \otimes_k \rho_0, \end{aligned}$$

which is an isomorphism in $D_c(Y)$. However since the rhs is concentrated to degree zero only, it is an isomorphism of O_Y-modules. It follows

$$\begin{aligned} \mathcal{V}^\dagger = \mathcal{I}/(\pi_U^*\mathfrak{m}\mathcal{I} + I_F\mathcal{I}) &\simeq (\mathcal{I}/\pi_U^*\mathfrak{m}\mathcal{I}) \otimes_{O_Y} O_F \\ &\simeq \bigoplus_{i=1}^n O_{E_i}(-1) \otimes_k \rho_i \bigoplus O_F(-F) \otimes_k \rho_0. \end{aligned}$$

It remains to compute $\mathcal{V}$. By Lemma 3.10,

$$\begin{aligned} I_{Z_y}/\mathfrak{m}I_{Z_y} &= \bigoplus_{i=1}^n O_{E_i}(-1) \otimes_{O_Y} O_y \otimes_k \rho_i \bigoplus O_y \otimes_k \rho_0, \\ I_{Z_y}/(\mathfrak{m}I_{Z_y} + \mathfrak{n}) &\simeq \bigoplus_{i=1}^n O_{E_i}(-1) \otimes_{O_Y} O_y \otimes_k \rho_i \end{aligned}$$

where every generator of $O_F(-F) \otimes_k \rho_0 \subset \mathcal{V}^\dagger$ is the image of a G-invariant polynomial in $\mathcal{I}$, which reduces to zero in $\mathcal{V}$ because $\mathfrak{m} \cap k[x,y]^G \subset \mathfrak{n}$. Hence the surjection $\mathcal{V}^\dagger \twoheadrightarrow \mathcal{V}$ restricts to a surjection:

$$\alpha : \bigoplus_{i=1}^n O_{E_i}(-1) \otimes_k \rho_i \twoheadrightarrow \mathcal{V}$$

and the induced map $\alpha \otimes O_y$ is an isomorphism for every $y \in F$. Since E_i's are reduced, α itself is an isomorphism. This proves Theorem 3.13 and Corollary 3.14. $\square$

Remark 3.15. Whether or not one can find an isomorphism similar to Theorem 3.13 is of some interest when $U = \mathbf{C}^n$ and G is a very natural subgroup of $\mathrm{GL}(U)$. For example, U is the Griess algebra or the vertex operator algebra for the big Monster $\mathbf{M}$. For $G = M_{24}$, the Mathieu group of degree 24, U is the Leech lattice.

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