Screened and unscreened solutions for relativistic star in de Rham-Gabadadze-Tolley massive gravity

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(Received 12 February 2019; published 25 October 2019)

We study the static and spherical symmetric configurations in the nonminimal model of the de Rham-Gabadadze-Tolley (dRGT) massive gravity with a flat reference metric. Considering the modified Tolman-Oppenheimer-Volkof equation, the Bianchi identity, and energy-momentum conservation, we find a new algebraic equation for the radial coordinate of the reference metric. We demonstrate that this equation suggests an absence of the Vainshtein mechanism in the minimal model of the dRGT massive gravity, while it has two branches of solutions where one connects with the Schwarzschild spacetime and another implies the significant deviation from the asymptotically flat spacetime in the nonminimal model. We also briefly discuss the boundary conditions for the relativistic stars in the dRGT massive gravity and a potential relation with the mass-radius relation of the stars.

DOI: 10.1103/PhysRevD.100.084060

I. INTRODUCTION

One of the aims of modified gravity theories is to explain phenomena that are hardly understood in the framework of the general relativity. Several modified gravity theories are motivated to study the dark energy related to the accelerated expansion of the Universe [1–3]. The cosmological constant Λ gives us a simple solution to the dark energy problem, where Λ may be interpreted as the vacuum energy induced from the quantum fluctuation of matter fields. However, it suffers from two theoretical problems: the fine-tuning problem and the coincidence problem (e.g., see [4,5]).

To explain the late-time acceleration of our Universe without invoking the cosmological constant, one needs to introduce the long-distance (IR) modifications of gravity theories so that the modification is responsible for the cosmic acceleration at present. On the other hand, such modifications often bring us the unsuitable feature that is to be excluded by the observations. It is well known that the Solar System observations are consistent with the prediction in the general relativity (see [6] for a review), and thus, the IR modification should be hidden in such a situation.

The modification for the dark energy is often regarded as the dynamical dark energy characterized by additional fields. Thus, if such dynamical fields induced from the IR modification are suppressed on local scales, one can safely avoid the constraint from the observations in the Solar System. The screening mechanisms [7–9] suggest the way for making the additional degrees of freedom ineffective in a short distance. Therefore, the viable modified gravity theories should possess the screening mechanism, and then, they do not conflict with Solar System constraints, keeping the validity to introduce the IR modification for the dark energy problem.

Although plenty of the previous research has verified the screening mechanisms in the static and spherical symmetric (SSS) configurations, the screening mechanisms are not well understood in the highly dense matter region to study the effect of the modification on the short-distance behavior. A typical situation can be found in the relativistic star. The hydrostatic equilibrium, which is maintained in the balance between the pressure of internal matters and its gravity, determines the inner and outer structures of relativistic stars. It means that the series of mass and radius of relativistic stars depends on the models of hadron

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physics in the high-density matter and gravitational theories in the strong-gravity region. From the perspective of hadron physics, various equations of state (EoS) have been investigated [10], corresponding to the inner structure of relativistic stars as in [11]. The EoS determines the mass-radius relation of relativistic stars and the maximum mass, and the existence of a massive neutron star with the mass larger than 2 $M_\odot$ is, at present, one of the criteria for the realistic model of EoS [12].

From another point of view, the gravity theories also determine the mass-radius relations, where the behavior in the nonperturbative or nonlinear region is of great importance to the inner structure of the relativistic stars. In the previous works [13–15], one of the authors has applied $F(R)$ gravity theories to the study on the relativistic stars. Because the curvature of spacetime, the Ricci scalar R, around relativistic stars is larger than that in the Solar System, some models of $F(R)$ gravity show significant differences from general relativity $[F(R) = R]$ around the relativistic stars. These notable results imply that the relativistic stars can be a useful tool for investigating the modifications of gravity. Furthermore, the study on the relativistic stars also provides us with a better understanding of screening mechanisms. Several works have attempted to study the Vainshtein mechanism [7], which is one of the screening mechanisms. By assuming a constant-density profile inside the star, the Vainshtein mechanism has been discussed [16] in the presence of matter fields.

In this work, we study the relativistic stars in the de Rham-Gabadadze-Tolley (dRGT) massive gravity [17], which is the theory of a ghost-free massive spin-2 particle. The theory has five ghost-free modes that are two tensor modes, two vector modes, and a scalar mode. The vector modes cannot couple with matters because of the energy-momentum conservation, while the additional scalar mode can produce different matter coupling with gravity from that in the general relativity. This additional scalar mode is considered to be suppressed by the Vainshtein mechanism; the nonlinear derivative couplings hide the scalar mode, and the gravitational coupling with matter becomes similar to that in the general relativity inside the so-called Vainshtein radius.

In our previous work [18], we studied the relativistic stars in the minimal model of dRGT massive gravity for the SSS configuration with a flat reference metric and found that the maximum mass of relativistic stars becomes smaller than that of general relativity. In the light of these results, we postulate that the lack of the Vainshtein mechanism results in the smaller maximal mass, due to the absence of nonlinear kinetic couplings in the relativistic star scales. A theoretical analysis for the minimal model shows that the minimal model does not have the Vainshtein mechanism [19].

The purpose of this article is to study the system of the relativistic stars in the nonminimal model, which is the broader framework of dRGT massive gravity, and to determine how the Vainshtein mechanism would affect the mass-radius relation of the relativistic star. We will derive the modified Tolman-Oppenheimer-Volkoff (TOV) equations to see how the modifications of gravity change the inside and outside structures of the relativistic star. After that, we will discuss the behavior of the solutions of modified TOV equations according to their mathematical structure, to find that the system of interest has a solution which is very similar to that in the general relativity thanks to the nonlinear kinetic terms. This study provides new insights into the nonperturbative aspects of dRGT massive gravity, and we argue that the Vainshtein mechanism potentially works around the relativistic star.

II. MODIFIED TOV EQUATION IN dRGT MASSIVE GRAVITY

A. dRGT massive gravity

In this section, we derive equations of motion of dRGT massive gravity in the SSS configuration and show the modified TOV equations. In the units of $c = \hbar = 1$, the action of the dRGT massive gravity [20] can be written as

$$S_{\mathrm{dRGT}} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-\det(g)} \times \left[R(g) - 2m_0^2\sum_{n=0}^{3}\beta_n e_n\left(\sqrt{g^{-1}f}\right)\right] + S_{\mathrm{mat}}, \quad (1)$$

where $\kappa^2 = 8\pi G$ is the gravitational coupling constant and S_{mat} is the matter action. We are using the units of $c = \hbar = 1$. The $e_k(\mathbb{X})$ are polynomials defined as the antisymmetric products of the components

$$e_k(\mathbf{X}) = \frac{1}{k!}\mathbf{X}^{I_1}{}_{[I_1}\cdots\mathbf{X}^{I_k}{}_{I_k]}. \quad (2)$$

The action (1) includes the two metric tensors $g_{\mu\nu}$ and $f_{\mu\nu}$; $g_{\mu\nu}$ denotes the dynamical variable in the dRGT massive gravity while $f_{\mu\nu}$ is fixed by hand and called the reference or fiducial metric. The $\sqrt{g^{-1}f}$ represents the matrix such that

$$(\sqrt{g^{-1}f})^{\mu}{}_{\rho}(\sqrt{g^{-1}f})^{\rho}{}_{\nu} = g^{\mu\rho}f_{\rho\nu}. \quad (3)$$

Here, m_0 is a parameter that defines the graviton mass, and in the following analysis, we set it as

$$m_0 \equiv 10^{-33}\ \mathrm{eV} \sim (10^{26}\ \mathrm{m})^{-1}. \quad (4)$$

This value is the same order of the cosmological constant, which represents the IR modification for the dark energy, and it is consistent with several observations (see for a review [21]). The parameters β_n's are free and expressed by only two parameters if we demand the flat Minkowski spacetime as a solution of the field equations and the appropriate coefficient of the graviton-mass term as in the Fierz-Pauli theory [22],

$$\beta_0 = 6 - 4\bar{\alpha}_3 + \bar{\alpha}_4, \qquad \beta_1 = -3 + 3\bar{\alpha}_3 - \bar{\alpha}_4,$$
$$\beta_2 = 1 - 2\bar{\alpha}_3 + \bar{\alpha}_4, \qquad \beta_3 = \bar{\alpha}_3 - \bar{\alpha}_4, \tag{5}$$

and they lead to the algebraic relations between parameters β_n,

$$\beta_2 = 1 - \beta_0 - 2\beta_1, \qquad \beta_3 = -3 + 2\beta_0 + 3\beta_1. \tag{6}$$

We call the case that

$$\beta_0 = 3, \qquad \beta_1 = -1, \qquad \beta_2 = \beta_3 = 0, \tag{7}$$

as the minimal model [22] by meaning the minimal nonlinear extension of the Fierz-Pauli theory [23]. In this work, we restrict our discussion for the case that $\beta_n \sim \mathcal{O}(1)$; otherwise, conditions for the UV completion are violated [24].

By the variation of action (1) with respect to the dynamical metric $g_{\mu\nu}$, we obtain the equations of motion in the dRGT massive gravity as follows (for the derivation, see [18]):

$$G_{\mu\nu} + m_0^2 I_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{8}$$

where

$$I_{\mu\nu} \equiv \sum_{n=0}^{3} (-1)^n \beta_n g_{\mu\lambda} Y^{\lambda}_{(n)\nu}(\sqrt{g^{-1}f}), \tag{9}$$

$$Y^{\lambda}_{(n)\nu}(\mathbf{X}) \equiv \sum_{r=0}^{n} (-1)^r (X^{n-r})^{\lambda}{}_{\nu} e_r(\mathbf{X}). \tag{10}$$

Here, the matrix $Y_{(n)}(\mathbf{X})$ is written in the following forms:

$$Y_0(\mathbf{X}) = \mathbf{1}, \qquad Y_1(\mathbf{X}) = \mathbf{X} - \mathbf{1}[\mathbf{X}],$$
$$Y_2(\mathbf{X}) = \mathbf{X}^2 - \mathbf{X}[\mathbf{X}] + \frac{1}{2}\mathbf{1}([\mathbf{X}]^2 - [\mathbf{X}^2]),$$
$$Y_3(\mathbf{X}) = \mathbf{X}^3 - \mathbf{X}^2[\mathbf{X}] + \frac{1}{2}\mathbf{X}([\mathbf{X}]^2 - [\mathbf{X}^2])$$
$$- \frac{1}{6}\mathbf{1}([\mathbf{X}]^3 - 3[\mathbf{X}][\mathbf{X}^2] + 2[\mathbf{X}^3]). \tag{11}$$

Now, we have to pay attention to the lack of diffeomorphism invariance because of the existence of graviton mass. While the diffeomorphism invariance guarantees the universal graviton-matter coupling in the general relativity, we should assume the universal couplings, which ensures the elimination of the ghost modes in the dRGT massive gravity.

B. Ansatz for SSS configuration

Considering a relativistic star, we impose the SSS configuration to the $g_{\mu\nu}$ metric. Then, the ansatz for $g_{\mu\nu}$ can be written as

$$g_{\mu\nu} dx^\mu dx^\nu = -\mathrm{e}^{2\nu(r)} dt^2 + \mathrm{e}^{2\lambda(r)} dr^2 + r^2 d\Omega^2. \tag{12}$$

$\nu(r)$ and $\lambda(r)$ are functions with respect to r, and $\mathrm{e}^{2\lambda(r)}$ is related to the mass function in the ordinary TOV equation:

$$\mathrm{e}^{-2\lambda(r)} \equiv 1 - \frac{2GM(r)}{r}, \tag{13}$$

where $M(r)$ is the mass parameter. The functions $\nu(r)$ and $\lambda(r)$ should satisfy boundary conditions that they vanish at the center of the star

$$\nu(r=0) = \lambda(r=0) = 0, \tag{14}$$

which suggests that the conical singularity should be removed [25]. The boundary conditions indicate that the mass parameter $M(r)$ should also vanish at the center,

$$M(r=0) = 0. \tag{15}$$

The equation of motion (8) determines the asymptotic behavior of these three functions. Note that the spacetime around the SSS configurations asymptotically matches with the Minkowski spacetime in the general relativity.

We assume the $f_{\mu\nu}$ metric as follows in our model:

$$f_{\mu\nu} dx^\mu dx^\nu = -dt^2 + d\chi(r)^2 + \chi(r)^2 d\Omega^2$$
$$= -dt^2 + \chi'(r)^2 dr^2 + \chi(r)^2 d\Omega^2, \tag{16}$$

where the prime denotes the derivative with respect to r. The reference metric $f_{\mu\nu}$ is chosen to represent the flat spacetime, while the radial coordinate is, in general, different from that of the physical metric $g_{\mu\nu}$. The relation of the radial coordinate between $g_{\mu\nu}$ and $f_{\mu\nu}$ spacetime is reflected to the new function $\chi(r)$.

We note that the function $\chi(r)$ plays a similar role to the Stukelberg field. The nondynamical $f_{\mu\nu}$ chosen by hand breaks the general coordinate transformation invariance, but it can be restored by changing the Stukelberg field as was discussed in our previous work [18]. If we use the gauge fixing condition $\chi(r) = r$, the $f_{\mu\nu}$ describes exactly the Minkowski spacetime from the observer in the coordinate system of $g_{\mu\nu}$. In our case, the radial coordinate can be chosen so that we can treat the equation of motion of the $g_{\mu\nu}$ in a similar way to the ordinary TOV equation thanks to the $\chi(r)$. The equations of motion of $g_{\mu\nu}$ contain a gauge fixing condition that determines $\chi(r)$. The solution of the gauge fixing condition gives the configuration of $\chi(r)$.

The calculation hereafter depends on the specific form of the reference metric. Although we conventionally choose the reference metric, we practically solve it as well as the dynamical metric according to the equation of motion. As we will see later, the function $\chi(r)$ is determined by the fourth-order algebraic equation derived from the divergence of equations of motion. Then, our choice of the reference metric can be justified when we find the solution with respect to $\chi(r)$ and $g_{\mu\nu}$.

C. Modified TOV equation

Substituting the ansatz with respect to the dynamical and reference metrics, $g_{\mu\nu}$ and $f_{\mu\nu}$, into the equation of motion (8), we obtain the modified TOV equations:

$$-\kappa^2\rho = -\frac{1}{r^2} + \frac{1}{r^2}(1 + r\partial_r)\mathrm{e}^{-2\lambda} + m_0^2 I^t{}_t, \tag{17}$$

$$\kappa^2 p = -\frac{1}{r^2} + \frac{1+2r\nu'}{r^2}\mathrm{e}^{-2\lambda} + m_0^2 I^r{}_r, \tag{18}$$

$$\kappa^2 p = \left(\nu'' + \nu'^2 + \frac{\nu' - \lambda'}{r} - \nu'\lambda'\right)\mathrm{e}^{-2\lambda} + m_0^2 I^\theta_\theta. \tag{19}$$

And the conservation law of the energy-momentum tensor leads to

$$-\frac{p'}{p+\rho} = \nu'. \tag{20}$$

Rewriting the above four equations, Eqs. (17), (18), (19), and (20), we obtain the following expressions:

$$GM' = 4\pi G\rho r^2 + \frac{1}{2}m_0^2 r^2 I^t{}_t, \tag{21}$$

$$-\frac{p'}{p+\rho} = \nu' = \frac{4\pi G p r^3 + GM - \frac{1}{2}m_0^2 r^3 I^r{}_r}{r(r-2GM)}, \tag{22}$$

$$\kappa^2 p = \left(\nu'' + \nu'^2 + \frac{\nu'}{r}\right)\left(1 - \frac{2GM}{r}\right) + \frac{1}{2}\left(\frac{1}{r} + \nu'\right)\left(1 - \frac{2GM}{r}\right)' + m_0^2 I^\theta{}_\theta. \tag{23}$$

One can find that the original TOV equations in the general relativity are modified by the interaction terms $m_0^2 I^i_i$, where $i = t, r, \theta$. We can compute the modification terms $I^t{}_t$, $I^r{}_r$, and $I^\theta{}_\theta$ with the ansatz for the physical and reference metrics as follows:

$$I^t{}_t \equiv \beta_0 + \beta_1\left(\frac{2\chi}{r} + \chi'\mathrm{e}^{-\lambda}\right) + \beta_2\left(\frac{\chi^2}{r^2} + \frac{2\chi\chi'}{r}\mathrm{e}^{-\lambda}\right) + \beta_3\frac{\chi^2\chi'}{r^2}\mathrm{e}^{-\lambda}, \tag{24}$$

$$I^r{}_r \equiv \beta_0 + \beta_1\left(\frac{2\chi}{r} + \mathrm{e}^{-\nu}\right) + \beta_2\left(\frac{\chi^2}{r^2} + \frac{2\chi}{r}\mathrm{e}^{-\nu}\right) + \beta_3\frac{\chi^2}{r^2}\mathrm{e}^{-\nu}, \tag{25}$$

$$I^\theta{}_\theta \equiv \beta_0 + \beta_1\left(\frac{\chi}{r} + \chi'\mathrm{e}^{-\lambda} + \mathrm{e}^{-\nu}\right) + \beta_2\left(\frac{1}{r}\chi\chi'\mathrm{e}^{-\lambda} + \frac{1}{r}\chi\mathrm{e}^{-\nu} + \chi'\mathrm{e}^{-\lambda-\nu}\right) + \beta_3\frac{\chi\chi'}{r}\mathrm{e}^{-\lambda-\nu}. \tag{26}$$

In addition to the equations of motion for the tt and rr components, we need to take into account the divergence of equations of motion,

$$\nabla_\mu(G^{\mu\nu} + m_0^2 I^{\mu\nu}) = \kappa^2\nabla_\mu T^{\mu\nu}. \tag{27}$$

If we assume the conservation of the energy-momentum tensor $\nabla_\mu T^{\mu\nu} = 0$, we obtain the new algebraic equations

$$\nabla_\mu I^{\mu\nu} = 0 \tag{28}$$

from the Bianchi identities $\nabla_\mu G^{\mu\nu} = 0$. Substituting Eqs. (24), (25), and (26) into (28), we find that t, θ, and ϕ components of Eq. (28) are identically satisfied, and that the nontrivial r component leads to the following equation:

$$0 = (\beta_1 r^2 + 2\beta_2 r\chi + \beta_3\chi^2)(\mathrm{e}^\nu)' + [2\beta_2(\mathrm{e}^\nu - \mathrm{e}^{\lambda+\nu}) + 2\beta_3(1 - \mathrm{e}^\lambda)]\chi + 2\beta_1 r(\mathrm{e}^\nu - \mathrm{e}^{\lambda+\nu}) + 2\beta_2 r(1 - \mathrm{e}^\lambda). \tag{29}$$

The ν' contains the modification term $I^r{}_r$ as in Eq. (22), and it can be written by up to second-order nonderivative terms for χ as given in Eq. (25). Therefore, the new constraint displays the fourth-order algebraic equation for χ.

Because the new constraint equation is the fourth-order algebraic equation, we can solve it analytically. For the convenience in the order estimation, we replace the variables to dimensionless ones as follows:

$$r_g \equiv GM_\odot \Leftrightarrow \kappa^2 = 8\pi\frac{r_g}{M_\odot}, \quad r \to r r_g, \quad \chi \to \chi r_g, \quad M(r) \to M(r)M_\odot, \quad \rho \to \rho(M_\odot/r_g^3), \quad p \to p(M_\odot/r_g^3), \quad m_0 \to \frac{m_0}{r_g}. \tag{30}$$

Here, we note that the dimension of graviton mass is $[L^{-1}]$ because of our units, and the magnitude of the dimensionless graviton mass is very tiny such as

$$m_0 \sim 10^{-23} \tag{31}$$

because we assume that the graviton mass is of order of the observed dark energy. Since we also demand that $\beta_n = \mathcal{O}(1)$, the modification of gravity seems to give the tiny effects to the observables. However, the additional algebraic equation (29), which cannot be found in the general relativity, changes the mathematical structures of the equations of motion.

III. ABSENCE OF THE VAINSHTEIN MECHANISM IN MINIMAL MODEL

To solve Eqs. (21), (22), and (29) and obtain the mass-radius relation of the relativistic star, we have to construct the solutions with a specific equation of state numerically. The typical way is imposing a boundary condition and solving it as a two-point boundary value problem. One of the two points in our case is the center of the relativistic star, and we impose Eq. (14). Another is the point far away from the star (analytically, at infinity), and thus, we need to check the asymptotic behaviors of the solutions. If the Vainshtein mechanism works outside the star, we can impose the boundary condition so that the Schwarzschild solution describes the spacetime outside the star.

When we solve the equation of motion, the fact that the relativistic star system in massive gravity has two scales makes the analysis complicated. These are the solar mass $M_{\odot}$, which characterizes the astrophysical scale, and the graviton mass m_0, which characterizes the cosmological scale. In this section, instead of solving the system numerically, we evaluate the behavior of $\chi(r)$ near and far from the star by approximations analytically. When we find the whole structure of the solution in the relativistic star system, it allows us to examine whether the screening mechanism can work or not, which determines the appropriate boundary condition outside the relativistic stars in the dRGT massive gravity.

A. $\chi(r)$ for Asymptotically flat spacetime

Before we study the asymptotic behavior of the spacetime, we think of a link between $g_{\mu\nu}$ and $\chi(r)$. First, when we make an assumption that $e^{2\nu} = e^{2\lambda} = 1$ in the physical metric $g_{\mu\nu}$ and $\chi(r) = r$ in the reference metric $f_{\mu\nu}$, one finds that it is consistent with the equation of motion because $g_{\mu\nu} = f_{\mu\nu} = \eta_{\mu\nu}$ is the solution with generic choices of the parameters β_n [26]. In the above case, one finds

$$I_{\mu\nu} = (\beta_0 + 3\beta_1 + 3\beta_2 + \beta_3)E_{(4)\mu\nu}, \tag{32}$$

where $E_{(4)}$ represents the 4×4 identity matrix. Equation (6) leads to

$$\beta_0 + 3\beta_1 + 3\beta_2 + \beta_3 = 0, \tag{33}$$

where the modification terms vanish. Because the energy-momentum tensor $T_{\mu\nu}$ also vanishes outside the star, the equation of motion (8) reduces to the Einstein equation in the vacuum.

However, it is not trivial that the asymptotic flatness in $g_{\mu\nu}$ is identical to the condition $\chi(r) = r$. Next, we only assume that the physical spacetime shows the asymptotic flatness outside the star, $g_{\mu\nu} = \eta_{\mu\nu}$. Because the Einstein tensor $G_{\mu\nu}$ and energy-momentum tensor $T_{\mu\nu}$ vanish, the modification terms should vanish, $I^{\mu\nu} = 0$, which suggests that Eq. (28) also satisfies. Thus, we find Eqs. (24), (25), and (26) lead to

$$0 = \beta_0 + \beta_1\left(\frac{2\chi}{r} + \chi'\right) + \beta_2\left(\frac{\chi^2}{r^2} + \frac{2\chi\chi'}{r}\right) + \beta_3\frac{\chi^2\chi'}{r^2}, \tag{34}$$

$$0 = (\beta_0 + \beta_1) + 2(\beta_1 + \beta_2)\frac{\chi}{r} + (\beta_2 + \beta_3)\left(\frac{\chi}{r}\right)^2, \tag{35}$$

$$0 = (\beta_0 + \beta_1) + (\beta_1 + \beta_2)\left(\frac{\chi}{r} + \chi'\right) + (\beta_2 + \beta_3)\frac{\chi\chi'}{r}. \tag{36}$$

Using Eq. (6) for Eq. (35), we obtain

$$0 = \left(\frac{\chi}{r} - 1\right)\left[(\beta_0 + \beta_1 - 2)\frac{\chi}{r} - (\beta_0 + \beta_1)\right], \tag{37}$$

and the solutions are

$$\frac{\chi}{r} = 1, \qquad \frac{\beta_0 + \beta_1}{\beta_0 + \beta_1 - 2}. \tag{38}$$

Note that we have only the first solution in the minimal model because the second one diverges. Moreover, we always have $\chi/r = 1$ in the case $\beta_0 + \beta_1 = 2$. The second solution does not give $\chi/r = 1$ when $\beta_0 + \beta_1$ takes a finite value. In any models, one can find that the χ should take the form of $\chi(r) = Ar$, where A is a constant.

When we substitute this linear solution to Eq. (36), we find

$$0 = (A - 1)[(\beta_0 + \beta_1 - 2)A - (\beta_0 + \beta_1)], \tag{39}$$

and thus, we obtain the results identical with Eq. (38),

$$A = 1, \qquad \frac{\beta_0 + \beta_1}{\beta_0 + \beta_1 - 2}. \tag{40}$$

By substituting $\chi(r) = Ar$ into Eq. (34), we find

$$\begin{aligned} 0 &= \beta_0 + 3\beta_1 A + 3\beta_2 A^2 + \beta_3 A^3 \\ &= (A - 1)[(2\beta_0 + 3\beta_1 - 3)A^2 - (\beta_0 + 3\beta_1)A - \beta_0]. \end{aligned} \tag{41}$$

One solution is $A = 1$ and the others satisfy the following equation:

$$(2\beta_0 + 3\beta_1 - 3)A^2 - (\beta_0 + 3\beta_1)A - \beta_0 = 0. \tag{42}$$

Note that $A = 1$ does not satisfy Eq. (42) in any choice of β_0 and β_1. When we substitute the second solution of Eqs. (35) or (36) into Eq. (42), we obtain the consistent solution $\chi(r) = Ar$ for the specific choices of β_0 and β_1.

We have found that $\chi(r) = r$ with the generic parameters and that $\chi(r) = Ar$ with $A \neq 1$ for the specific parameters when we require the asymptotic flatness for $g_{\mu\nu}$. If we substitute $\chi(r) = Ar$ in Eq. (16), the reference metric takes the following form:

$$f_{\mu\nu}dx^\mu dx^\nu = -dt^2 + A^2 dr^2 + A^2 r^2 d\Omega^2. \tag{43}$$

If one redefines the radial coordinate $Ar \to r$, we can remove the factor A in the reference metric. We can absorb the scaling by A into the scaling ambiguity of the definition of $\chi(r)$; therefore, $\chi(r) = Ar$ also represents the Minkowski spacetime in the reference metric $f_{\mu\nu}$. We also note that the scaling factor A, which is determined by β_n, is of the order of unity when we assume $\beta_n = \mathcal{O}(1)$ and do not use the specified choice so that $\beta_0 + \beta_1 \approx 2$. In the following, we calculate the case of $A = 1$, $\chi(r) = r$ for simplicity.

Finally, we consider the inverse problem and only assume that $\chi(r) = r$ outside the star. Because p and M' vanish, the equations of motion (21)–(23) give

$$0 = \beta_0 + \beta_1(2 + \mathrm{e}^{-\lambda}) + \beta_2(1 + 2\mathrm{e}^{-\lambda}) + \beta_3 \mathrm{e}^{-\lambda}, \tag{44}$$

$$\begin{aligned} 2r\mathrm{e}^{2\lambda}\nu' = (1 - \mathrm{e}^{-2\lambda}) - m_0^2 r^2[\beta_0 + \beta_1(2 + \mathrm{e}^{-\nu}) \\ + \beta_2(1 + 2\mathrm{e}^{-\nu}) + \beta_3 \mathrm{e}^{-\nu}], \end{aligned} \tag{45}$$

$$\begin{aligned} 0 = \left(\nu'' + \nu'^2 + \frac{\nu'}{r}\right)\mathrm{e}^{-2\lambda} + \frac{1}{2}\left(\frac{1}{r} + \nu'\right)(\mathrm{e}^{-2\lambda})' \\ + m_0^2[\beta_0 + \beta_1(1 + \mathrm{e}^{-\lambda} + \mathrm{e}^{-\nu}) \\ + \beta_2(\mathrm{e}^{-\lambda} + \mathrm{e}^{-\nu} + \mathrm{e}^{-\lambda-\nu}) + \beta_3 \mathrm{e}^{-\lambda-\nu}]. \end{aligned} \tag{46}$$

Here, we have used $\mathrm{e}^{-2\lambda} \equiv 1–2GM/r$. When we substitute Eq. (6) into Eq. (44), we find

$$\mathrm{e}^{-\lambda} = 1, \tag{47}$$

and furthermore, Eqs. (45) and (46) are given by

$$0 = \frac{2\nu'}{r} + m_0^2(1 - \mathrm{e}^{-\nu}), \tag{48}$$

$$0 = \left(\nu'' + \nu'^2 + \frac{\nu'}{r}\right) + m_0^2(1 - \mathrm{e}^{-\nu}). \tag{49}$$

For the general m_0^2, we only find the trivial solution

$$\mathrm{e}^{\nu} = 1. \tag{50}$$

Based on the discussion in this subsection, we have found that asymptotic flatness in $g_{\mu\nu}$ is equivalent to $\chi(r) = r$, which allows us to study the behavior of the physical spacetime in terms of the $\chi(r)$. If $\chi(r)$ shows the asymptotically flat feature, we can infer and conclude that the physical spacetime is also asymptotically flat.

B. Asymptotic behavior near and away from stars

To study the asymptotic behavior of $\chi(r)$, we introduce a mass scale M_s, to denote the dimensionless mass of the relativistic star. Since we formulated the equations in terms of dimensionless quantities normalized by the solar scale, and we expect M_s is also at the solar scale, $M_s = \mathcal{O}(1)$. Around the object with a particular mass scale, we can introduce the significant scale, the Vainshtein radius. In the dRGT massive gravity, the Vainshtein radius is defined by

$$r_V = \left(\frac{M_s}{M_{\mathrm{Pl}}}\right)^{1/3}\frac{1}{\Lambda_3}, \tag{51}$$

where M_{Pl} is the Planck mass and Λ_3 is the cutoff scale in the dRGT massive gravity, defined as

$$\Lambda_3 = (M_{\mathrm{Pl}} m_0^2)^{1/3}. \tag{52}$$

In our normalization, rescaled by the solar-mass scale, we find

$$r_V = \left(\frac{M_s}{m_0^2}\right)^{1/3}. \tag{53}$$

Assuming $M_s = \mathcal{O}(1)$, the Vainshtein radius is $r_V = m_0^{-2/3} \sim 10^{15}$.

As we have mentioned, the possible difficulty is that the Vainshtein radius is the product of astrophysical M_s and cosmological scales m_0. In order to deal with the important intermediate scale r_V, we focus on the scale $M_s \ll r \ll r_V$ to address the spacetime outside but not far away from the star. If the Vainshtein mechanism works, the dRGT massive gravity restores the results in the general relativity inside the Vainshtein radius, and the physical spacetime should be the Schwarzschild spacetime. Thus, in the region $M_s \ll r \ll r_V$, we assume that the physical metric is given by the Schwarzschild spacetime,

$$\mathrm{e}^{2\nu(r)} = 1 - \frac{2M_s}{r}, \qquad \mathrm{e}^{-2\lambda(r)} = 1 - \frac{2M_s}{r}. \tag{54}$$

$M_s/r \ll 1$ in $M_s \ll r \ll r_V$, and we can treat M_s/r as the perturbation from the Minkowski spacetime.

Furthermore, the discussion in the previous subsection implies $\chi(r)$ should take the following form:

$$\frac{\chi(r)}{r} = 1 + \mathcal{O}\left(\frac{M_s}{r}\right) \tag{55}$$

to balance the order of the perturbations in both sides of the equations of motion. Note that we can rescale $\chi(r)$ with the arbitrary factor to express the above form if it is necessary. In other words, when we find the above $\chi(r)$ as a solution to the equation of motion in $M_s \ll r \ll r_V$, we have the Schwarzschild spacetime outside the star, which suggests the Vainshtein mechanism works properly. If $\chi(r)$ shows the large deviation from the asymptotic form $\chi(r) = r$, it implies that the Vainshtein mechanism does not work.

C. Asymptotic flatness around and away from the stars

Before we discuss the relativistic star in the general case of the dRGT massive gravity, we consider the minimal model in which the parameters β_n are chosen as in Eq. (7). In our previous work, we directly derived the mass-radius relation in the minimal model by the numerical simulation and discussed the effect of the modification on the maximal mass of the stars. Here, we refine our previous result from the viewpoints of the boundary conditions and the Vainshtein mechanism.

Substituting β_n in the minimal model (7) into Eq. (29), we obtain the following algebraic equation:

$$\nu' = \frac{2}{r}(\mathrm{e}^{\lambda} - 1). \tag{56}$$

And Eq. (22) is given by

$$\begin{aligned} \nu' &= \frac{1}{2}\left(\kappa^2 pr + \frac{1}{r} - m_0^2 r I^r{}_r\right)\mathrm{e}^{2\lambda} - \frac{1}{2r}, \\ I^r{}_r &= 3 - \left(\frac{2\chi}{r} + \mathrm{e}^{-\nu}\right). \end{aligned} \tag{57}$$

By eliminating ν', we find the first-order algebraic equation for χ, whose solution is

$$\chi = \frac{r}{2}\left\{3 - \mathrm{e}^{-\nu} + \frac{1}{m_0^2 r^2}[(4\mathrm{e}^{\lambda} - 3)\mathrm{e}^{-2\lambda} - 1 - \kappa^2 pr^2]\right\}. \tag{58}$$

This result shows that the minimal model does not have the Vainshtein mechanism because the additional terms proportional to $1/(m_0^2 r^2)$ become relevant at the small scale $r \ll 1/m_0$, including the interior region of the star.

As an illustration, we assume the Schwarzschild spacetime outside the star. If this assumption is appropriate, we get Eq. (55) from Eq. (58). Substituting Eq. (54) and $p = 0$, we obtain

$$\begin{aligned} \frac{\chi}{r} &= 1 - \frac{M_s}{2r} + \mathcal{O}\left(\frac{M_s^2}{r^2}\right) + \frac{1}{m_0^2 r^2}\left[\frac{M_s}{r} + \mathcal{O}\left(\frac{M_s^2}{r^2}\right)\right] \\ &= \left(\frac{r_V}{r}\right)^3\left[1 + \mathcal{O}\left(\frac{M_s}{r}\right)\right] + 1 - \frac{M_s}{2r} + \mathcal{O}\left(\frac{M_s^2}{r^2}\right). \end{aligned} \tag{59}$$

Equation (59) displays the significant deviations from Eq. (55) because the first term becomes dominant inside the Vainshtein radius, $r_V/r \gg 1$. It suggests that the physical metric $g_{\mu\nu}$ does not describe the Schwarzschild spacetime outside the star. Therefore, we can conclude that the Vainshtein mechanism does not work in the minimal model with the flat reference metric.

IV. SCREENED AND UNSCREENED SOLUTIONS IN NONMINIMAL MODEL

In the previous section, we have discussed the physical spacetime around the star in the context of $\chi(r)$. We have found that the Vainshtein mechanism does not always work around the star in the minimal model, where β_n's are specially chosen. In this section, we consider the general case, the nonminimal model of the dRGT massive gravity and check the asymptotic behavior of $\chi(r)$ and examine the Vainshtein mechanism.

A. Fourth-order equation for $\chi(r)$

To solve the new algebraic equation, we eliminate the ν' from Eq. (22) with Eq. (29) as we performed in the case of the minimal model. For the convention, we express Eq. (22) in the following form written in the new variables:

$$\nu' \equiv n_0 - m_0^2 n_1 I^r{}_r \quad (n_0, n_1 > 0), \tag{60}$$

where we define

$$n_0 \equiv \frac{1}{2}\kappa^2 pr\mathrm{e}^{2\lambda} + \frac{1}{2r}(\mathrm{e}^{2\lambda} - 1), \qquad n_1 \equiv \frac{1}{2}r\mathrm{e}^{2\lambda}. \tag{61}$$

Using Eqs. (22) and (25), we find that Eq. (29) in the generic case of the parameters β_n leads to

$$\begin{aligned} 0 &= (\beta_1 r^2 + 2\beta_2 r\chi + \beta_3\chi^2)\mathrm{e}^{\nu}(n_0 - m_0^2 n_1 I^r{}_r) + [2\beta_2(\mathrm{e}^{\nu} - \mathrm{e}^{\lambda+\nu}) + 2\beta_3(1 - \mathrm{e}^{\lambda})]\chi + 2\beta_1 r(\mathrm{e}^{\nu} - \mathrm{e}^{\lambda+\nu}) + 2\beta_2 r(1 - \mathrm{e}^{\lambda}) \\ &= -m_0^2 n_1 \mathrm{e}^{\nu}(\beta_3\chi^2 + 2\beta_2 r\chi + \beta_1 r^2)\left[\frac{1}{r^2}(\beta_2 + \beta_3\mathrm{e}^{-\nu})\chi^2 + \frac{2}{r}(\beta_1 + \beta_2\mathrm{e}^{-\nu})\chi + (\beta_0 + \beta_1\mathrm{e}^{-\nu})\right] \\ &\quad + \beta_3 n_0\mathrm{e}^{\nu}\chi^2 + [2\beta_2 r n_0\mathrm{e}^{\nu} + 2\beta_2(\mathrm{e}^{\nu} - \mathrm{e}^{\lambda+\nu}) + 2\beta_3(1 - \mathrm{e}^{\lambda})]\chi + 2\beta_1 r(\mathrm{e}^{\nu} - \mathrm{e}^{\lambda+\nu}) + 2\beta_2 r(1 - \mathrm{e}^{\lambda}) + \beta_1 r^2 n_0\mathrm{e}^{\nu}. \end{aligned} \tag{62}$$

Expanding the above expression as the polynomial with respect to χ, we obtain the following fourth-order algebraic equation:

$$\begin{aligned}
0 = &-\frac{m_0^2 n_1 \mathrm{e}^{\nu}}{r^2}\beta_3(\beta_2+\beta_3\mathrm{e}^{-\nu})\chi^4 - \frac{2m_0^2 n_1 \mathrm{e}^{\nu}}{r}[\beta_2(\beta_2+\beta_3\mathrm{e}^{-\nu})+\beta_3(\beta_1+\beta_2\mathrm{e}^{-\nu})]\chi^3 \\
&+\{\beta_3 n_0 \mathrm{e}^{\nu} - m_0^2 n_1 \mathrm{e}^{\nu}[\beta_1(\beta_2+\beta_3\mathrm{e}^{-\nu}) - 4\beta_2(\beta_1+\beta_2\mathrm{e}^{-\nu}) - \beta_3(\beta_0+\beta_1\mathrm{e}^{-\nu})]\}\chi^2 \\
&+\{2[\beta_2 n_0 r\mathrm{e}^{\nu} + \beta_2(\mathrm{e}^{\nu}-\mathrm{e}^{\lambda+\nu}) + \beta_3(1-\mathrm{e}^{\lambda})] - 2m_0^2 n_1 r\mathrm{e}^{\nu}[\beta_1(\beta_1+\beta_2\mathrm{e}^{-\nu}) + \beta_2(\beta_0+\beta_1\mathrm{e}^{-\nu})]\}\chi \\
&+ r[2\beta_1(\mathrm{e}^{\nu}-\mathrm{e}^{\lambda+\nu}) + 2\beta_2(1-\mathrm{e}^{\lambda}) + \beta_1 n_0 r\mathrm{e}^{\nu}] - m_0^2 n_1 r^2 \mathrm{e}^{\nu}\beta_1(\beta_0+\beta_1\mathrm{e}^{-\nu}). \qquad (63)
\end{aligned}$$

As we mentioned below Eq. (7), we need to choose β_2, $\beta_3 \neq 0$ to realize the nonminimal model of the dRGT massive gravity. For this restriction of β_2 and β_3, we find that Eq. (63) is the fourth order with respect to χ. Note that in the minimal model, one can confirm that Eq. (63) is indeed reduced to the first-order equation, which restores Eq. (58).

For the further convenience in the later calculation, we rewrite Eq. (63) with normalizing the coefficient of the χ^4 term,

$$\begin{aligned}
&\chi^4 + a\chi^3 - \frac{1}{m_0^2}[(b_0+m_0^2 b_1)\chi^2 + (c_0+m_0^2 c_1)\chi \\
&\quad + (d_0 + m_0^2 d_1)] = 0, \qquad (64)
\end{aligned}$$

where we define the coefficients as follows:

$$\begin{aligned}
a &= \frac{2r[\beta_3(\beta_1+\beta_2\mathrm{e}^{-\nu}) + \beta_2(\beta_2+\beta_3\mathrm{e}^{-\nu})]}{\beta_3(\beta_2+\beta_3\mathrm{e}^{-\nu})}, \\
b_0 &= \frac{n_0 r^2}{n_1(\beta_2+\beta_3\mathrm{e}^{-\nu})}, \\
b_1 &= -\frac{r^2[4\beta_2^2\mathrm{e}^{-\nu} + \beta_0\beta_3 + \beta_1(5\beta_2+2\beta_3\mathrm{e}^{-\nu})]}{\beta_3(\beta_2+\beta_3\mathrm{e}^{-\nu})}, \\
c_0 &= \frac{2r^2[\beta_2 r n_0 + \beta_2(1-\mathrm{e}^{\lambda}) + \beta_3(\mathrm{e}^{-\nu}-\mathrm{e}^{\lambda-\nu})]}{n_1\beta_3(\beta_2+\beta_3\mathrm{e}^{-\nu})}, \\
c_1 &= -\frac{2r^3[\beta_2(\beta_0+\beta_1\mathrm{e}^{-\nu}) + \beta_1(\beta_1+\beta_2\mathrm{e}^{-\nu})]}{\beta_3(\beta_2+\beta_3\mathrm{e}^{-\nu})}, \\
d_0 &= \frac{r^3[\beta_1 r n_0 + 2\beta_1(1-\mathrm{e}^{\lambda}) + 2\beta_2(\mathrm{e}^{-\nu}-\mathrm{e}^{\lambda-\nu})]}{n_1\beta_3(\beta_2+\beta_3\mathrm{e}^{-\nu})}, \\
d_1 &= -\frac{r^4\beta_1(\beta_0+\beta_1\mathrm{e}^{-\nu})}{\beta_3(\beta_2+\beta_3\mathrm{e}^{-\nu})}. \qquad (65)
\end{aligned}$$

When we obtain the real solutions of Eq. (64) and study their asymptotic behavior away from the star, we can discuss the physical spacetime to connect the Schwarzschild spacetime as we have done in the case of the minimal model of the dRGT massive gravity.

B. Branch analysis for χ around star

As we have performed in the minimal model, we assume the Schwarzschild spacetime in the region $M_s \ll r \ll r_V$ and study $\chi(r)$ outside the star in the nonminimal model. If $\chi(r)$ shows the asymptotic behavior as expected in Eq. (55), we can conclude that the Vainshtein mechanism works in the nonminimal model; otherwise, the screening mechanism does not work in the general model of the dRGT massive gravity.

Compared with the minimal model, we have a remarkable difficulty to obtain $\chi(r)$ in the nonminimal model because of the higher-order algebraic equation, Eq. (64). To make it manageable in an analytical manner, we begin the analysis with the assumption that the physical spacetime is described by the Schwarzschild solution outside the star, instead of looking for the exact solutions. Then, we examine the asymptotic behavior of $\chi(r)$ and check whether it is consistent with the assumption. When we use Eq. (54) with the condition $p = 0$, we find

$$n_0 = \frac{1}{r}\left[\frac{M_s}{r} + \mathcal{O}\left(\frac{M_s^2}{r^2}\right)\right], \qquad \frac{1}{n_1} = \frac{2}{r}\left[1 - \frac{2M_s}{r}\right]. \qquad (66)$$

Furthermore, when we use $\beta_n = \mathcal{O}(1)$, we can expand the coefficients of the fourth-order equation, Eq. (65), in terms of $\mathcal{O}(M_s/r)$ as follows:

$$\begin{aligned}
a &= 2r\left[A + \tilde{A}\frac{M_s}{r} + \mathcal{O}\left(\frac{M_s^2}{r^2}\right)\right], \\
b_0 &= B_0\frac{M_s}{r} + \mathcal{O}\left(\frac{M_s^2}{r^2}\right), \\
b_1 &= -r^2\left[B_1 + \tilde{B}_1\frac{M_s}{r} + \mathcal{O}\left(\frac{M_s^2}{r^2}\right)\right], \\
c_0 &= rC_0\left[\frac{M_s}{r} + \mathcal{O}\left(\frac{M_s^2}{r^2}\right)\right], \\
c_1 &= -2r^3\left[C_1 + \tilde{C}_1\frac{M_s}{r} + \mathcal{O}\left(\frac{M_s^2}{r^2}\right)\right], \\
d_0 &= r^2 D_0\left[\frac{M_s}{r} + \mathcal{O}\left(\frac{M_s^2}{r^2}\right)\right], \\
d_1 &= -r^4\left[D_1 + \tilde{D}_1\frac{M_s}{r} + \mathcal{O}\left(\frac{M_s^2}{r^2}\right)\right], \qquad (67)
\end{aligned}$$

where

$$\begin{aligned}
&A=\frac{\beta_3(\beta_1+\beta_2)+\beta_2(\beta_2+\beta_3)}{\beta_3(\beta_2+\beta_3)}, \qquad \tilde{A}=\frac{2\beta_2\beta_3-\beta_3^2A}{\beta_3(\beta_2+\beta_3)},\\
&B_0=\frac{2\beta_3}{\beta_3(\beta_2+\beta_3)}, \qquad B_1=\frac{4\beta_2^2+\beta_0\beta_3+\beta_1(5\beta_2+2\beta_3)}{\beta_3(\beta_2+\beta_3)}, \qquad \tilde{B}_1=\frac{(4\beta_2^2+2\beta_1\beta_3)-\beta_3^2B_1}{\beta_3(\beta_2+\beta_3)},\\
&C_0=\frac{-4\beta_3}{\beta_3(\beta_2+\beta_3)}, \qquad C_1=\frac{\beta_2(\beta_0+\beta_1)+\beta_1(\beta_1+\beta_2)}{\beta_3(\beta_2+\beta_3)}, \qquad \tilde{C}_1=\frac{2\beta_1\beta_2-\beta_3^2C_1}{\beta_3(\beta_2+\beta_3)},\\
&D_0=\frac{-2\beta_1-4\beta_2}{\beta_3(\beta_2+\beta_3)}, \qquad D_1=\frac{\beta_1(\beta_0+\beta_1)}{\beta_3(\beta_2+\beta_3)}, \qquad \tilde{D}_1=\frac{\beta_1^2-\beta_3^2D_1}{\beta_3(\beta_2+\beta_3)}.
\end{aligned} \tag{68}$$

Therefore, when we assume the Schwarzschild spacetime for the physical metric $g_{\mu\nu}$, the fourth-order equation takes the following form:

$$\begin{aligned}
0=&\left(\frac{\chi}{r}\right)^4+\left[2A+2\tilde{A}\frac{M_s}{r}+\mathcal{O}\left(\frac{M_s^2}{r^2}\right)\right]\left(\frac{\chi}{r}\right)^3-\frac{1}{m_0^2r^2}\left\{\left[B_0\frac{M_s}{r}+\mathcal{O}\left(\frac{M_s^2}{r^2}\right)\right]-m_0^2r^2\left[B_1+\tilde{B}_1\frac{M_s}{r}+\mathcal{O}\left(\frac{M_s^2}{r^2}\right)\right]\right\}\left(\frac{\chi}{r}\right)^2\\
&-\frac{1}{m_0^2r^2}\left\{\left[C_0\frac{M_s}{r}+\mathcal{O}\left(\frac{M_s^2}{r^2}\right)\right]-m_0^2r^2\left[2C_1+2\tilde{C}_1\frac{M_s}{r}+\mathcal{O}\left(\frac{M_s^2}{r^2}\right)\right]\right\}\left(\frac{\chi}{r}\right)\\
&-\frac{1}{m_0^2r^2}\left\{\left[D_0\frac{M_s}{r}+\mathcal{O}\left(\frac{M_s^2}{r^2}\right)\right]-m_0^2r^2\left[D_1+\tilde{D}_1\frac{M_s}{r}+\mathcal{O}\left(\frac{M_s^2}{r^2}\right)\right]\right\}.
\end{aligned} \tag{69}$$

We find that $1/(m_0^2r^2)$ corrections show up as in Eq. (58) for the minimal model, which would bring the origin of the large deviation from the asymptotic flatness. Noting $m_0^2r^2 \ll M_s/r \ll 1$ in the region $M_s \ll r \ll r_V$ because

$$\frac{M_s/r}{m_0^2r^2}=\left(\frac{r_V}{r}\right)^3 \gg 1, \tag{70}$$

the fourth-order equation Eq. (69) can be further approximated and given by

$$\begin{aligned}
0=&\left(\frac{\chi}{r}\right)^4+\left[2A+2\tilde{A}\frac{M_s}{r}+\mathcal{O}\left(\frac{M_s^2}{r^2}\right)\right]\left(\frac{\chi}{r}\right)^3\\
&-\left(\frac{r_V}{r}\right)^3\left\{\left[B_0+\mathcal{O}\left(\frac{M_s}{r}\right)\right]\left(\frac{\chi}{r}\right)^2+\left[C_0+\mathcal{O}\left(\frac{M_s}{r}\right)\right]\left(\frac{\chi}{r}\right)+\left[D_0+\mathcal{O}\left(\frac{M_s}{r}\right)\right]\right\}.
\end{aligned} \tag{71}$$

Finally, we solve the above algebraic equation Eq. (71) to $\chi(r)$. If we assume the asymptotically flat solution for the reference metric $f_{\mu\nu}$, $\chi/r=\mathcal{O}(1)+\mathcal{O}(M_s/r)$ up to the scaling, the first line of Eq. (71) is of order of $\mathcal{O}(1)$, while the second line is of order $\mathcal{O}((r_V/r)^3)\gg 1$. Thus, the first line is negligible, the second line is dominant, we find

$$\begin{aligned}
0=&\left[B_0+\mathcal{O}\left(\frac{M_s}{r}\right)\right]\left(\frac{\chi}{r}\right)^2+\left[C_0+\mathcal{O}\left(\frac{M_s}{r}\right)\right]\left(\frac{\chi}{r}\right)\\
&+\left[D_0+\mathcal{O}\left(\frac{M_s}{r}\right)\right],
\end{aligned} \tag{72}$$

and the solution is given by

$$\frac{\chi}{r}=\frac{-C_0\pm\sqrt{C_0^2-4B_0D_0}}{2B_0}+\mathcal{O}\left(\frac{M_s}{r}\right). \tag{73}$$

Actually, the above solution is consistent with the assumption $\chi/r=\mathcal{O}(1)+\mathcal{O}(M_s/r)$, and after rescaling the solution, we find

$$\frac{\chi}{r}=1+\mathcal{O}\left(\frac{M_s}{r}\right). \tag{74}$$

This solution suggests that the nonminimal model of the dRGT massive gravity possesses the Vainshtein mechanism around the relativistic star. Therefore, one can study the relativistic star with the particular equation of state with the boundary condition to connect to the Schwarzschild spacetime outside the star, as in the general relativity. We emphasize that the existence of the real solutions depends on the parameter β_n's whose region is evaluated with the condition that the determinant $D=C_0^2-4B_0D_0\geq 0$. Moreover, we need to require that, at least, one of the two solutions is positive definite to express the radial coordinate.

On the other hand, in general, we have four solutions for Eq. (71). Since we have two of the four, which are of the order of unity at the leading order, we can analyze the leading order of the other two according to the coefficients of Eq. (71). When we express the four solutions as α, β, γ, and δ, they satisfy

$$\alpha+\beta+\gamma+\delta=-\left[2A+\mathcal{O}\left(\frac{M_s}{r}\right)\right], \tag{75}$$

$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=-\left(\frac{r_V}{r}\right)^3\left[B_0+\mathcal{O}\left(\frac{M_s}{r}\right)\right], \tag{76}$$

$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=\left(\frac{r_V}{r}\right)^3\left[C_0+\mathcal{O}\left(\frac{M_s}{r}\right)\right], \tag{77}$$

$$\alpha\beta\gamma\delta=-\left(\frac{r_V}{r}\right)^3\left[D_0+\mathcal{O}\left(\frac{M_s}{r}\right)\right]. \tag{78}$$

Furthermore, if we assume α and β approximately obey Eq. (72), we find

$$\alpha+\beta=-\frac{C_0}{B_0}+\mathcal{O}\left(\frac{M_s}{r}\right), \quad \alpha\beta=\frac{D_0}{B_0}+\mathcal{O}\left(\frac{M_s}{r}\right). \tag{79}$$

Thus, the sum and product of the other two solutions are given by

$$\gamma+\delta=\left(\frac{C_0}{B_0}-2A\right)+\mathcal{O}\left(\frac{M_s}{r}\right), \quad \gamma\delta=-\left(\frac{r_V}{r}\right)^3\frac{B_0}{D_0}. \tag{80}$$

In order to satisfy the above relation, we can deduce the relevant expressions of the two solutions, γ and δ, as follows:

$$\gamma=\sqrt{\frac{B_0}{D_0}}\left(\frac{r_V}{r}\right)^{3/2}+\mathcal{O}(1), \tag{81}$$

$$\delta=-\sqrt{\frac{B_0}{D_0}}R\left(\frac{r_V}{r}\right)^{3/2}+\mathcal{O}(1). \tag{82}$$

We find that these two solutions include the significant deviation from the asymptotically flat reference metric. As in the minimal model, we can understand that the nonminimal model includes the asymptotically nonflat solutions, Eq. (81), although it potentially possesses the asymptotically flat solutions, Eq. (74). We note again that the existence of the real solutions depends on the parameter choice β_n; for instance, we would find the proper parameter regions so that $B_0/D_0>0$.

From Eq. (71), we have found the two different branches of solutions for Eq. (71),

$$\chi=\left(\frac{r_V}{r}\right)^{3/2}+\mathcal{O}(1), \qquad 1+\mathcal{O}\left(\frac{M_s}{r}\right). \tag{83}$$

The former has the large correction $\mathcal{O}((r_V/r)^{3/2})\gg 1$ in the region of our interest $M_s\ll r\ll r_V$ although the latter is of the order of unity. Only one branch exists in the minimal model, which does not admit the Vainshtein mechanism, while the new branch appears in the nonminimal model. It is notable that the minimal model predicts $\mathcal{O}((r_V/r)^3)$, while the nonminimal model predicts $\mathcal{O}((r_V/r)^{3/2})$.

V. SUMMARY AND DISCUSSION

We have studied the asymptotic behavior of the spacetime around the relativistic star in the dRGT massive gravity with the flat reference metric. We have explicitly shown that the Vainshtein mechanism does not work in the minimal model, which is consistent with the previous theoretical analysis [19]. Remarkably, we have found that the modification terms become relevant even inside the relativistic star, and thus, that the modification of gravity becomes reasonable not only outside the star but also inside the star. Using the same analysis method, we have considered the nonminimal model of the dRGT massive gravity. We have derived the fourth-order algebraic equation based on several approximations and demonstrated the solutions that suggest the nonminimal model has the relativistic star solutions with and without the Vainshtein mechanism.

The condition that the Vainshtein mechanism works or not gives a definite difference in the equation of motion. The modification terms to the Einstein equation are integrated into $m_0^2 I_{\mu\nu}$, and $\chi(r)$ characterizes $I_{\mu\nu}$. Because $I_{\mu\nu}$ contains the third order terms of $\chi(r)$ in Eq. (11), the condition $\chi\sim\mathcal{O}(1)$ implies that the modification term is of $\mathcal{O}(m_0^2)$, and, on the other hand, the condition $\chi\sim\mathcal{O}(1/m_0)$ predicts that the modification includes the term of $\mathcal{O}(1/m_0)$ in general. The former case shows that the modifications to the equation of motion can be ignored, and the latter case shows that the essential contributions from the modifications arise in the modified TOV equation. Therefore, the absence of the Vainshtein mechanism drastically changes the mass-radius relation of the relativistic star. In our previous work [18], we have obtained the mass-radius relations for the neutron star and quark star in the minimal model of the dRGT massive gravity, which displays significant differences from those in the general relativity. From the above discussion, we can understand that the lack of the Vainshtein mechanism in the minimal model has produced the differences because the TOV equations receive the non-negligible modifications.

A couple of comments and discussion on prospects regarding what we have elucidated in the present paper are as follows: The branch including the Vainshtein mechanism allows us to impose the ordinary boundary condition, where we connect the external solution with the Schwarzschild spacetime, around the relativistic star. Therefore, we can compute the mass-radius relation even in the nonminimal model of the dRGT massive gravity based on the techniques that had been established in our previous work. Although we might face another difficulty to solve the fourth-order equation of $\chi(r)$, we can solve the modified TOV equation with the arbitrary EoS. However, the Vainshtein mechanism may result in almost the same mass-radius relation as that in the general relativity.

Regarding the two branches in the nonminimal model, we have not constrained the parameter regions to obtain the realistic solution of $\chi(r)$ although we have discussed the leading order and deviation from the Minkowski spacetime. Concerning the relativistic star solution, we should evaluate the parameters as well as the mass-radius relation. Because we have derived the fundamental equations, we could discuss the particular combination of the parameters to simplify the equation but obtain the physical solutions, which we will address in our future works.

ACKNOWLEDGMENTS

This work is supported by the Nagoya University Program for Leading Graduate Schools funded by the Ministry of Education of the Japanese Government under Program No. N01 (M. Y.) and by International Postdoctoral Exchange Fellowship Program at Central China Normal University and Project funded by China Postdoctoral Science Foundation 2018M632895 (T. K.). This work is also supported by MINECO (Spain), FIS2016-76363-P, by Project No. 2017 SGR247 (AGAUR, Catalonia) (S. D. O.), by MEXT KAKENHI Grant-in-Aid for Scientific Research on Innovative Areas "Cosmic Acceleration" No. 15H05890 (S. N.), and by the JSPS Grant-in-Aid for Scientific Research (C) No. 18K03615 (S. N.).

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