三角関数の抽象化

高　須　照　夫

数Ⅱの三角関数では，数少ないいくつかの公式からたくさんの公式がつぎつぎにみちびかれてゆく様子がよくわかる。この様子をできるだけ簡潔に整理してみた。数学の抽象化の一面をのぞかせるのにてごろな教材であると考えられる。以下，わくの中は公理（約束），それ以外は公理をもとにしてみちびかれることがらである。

> 関数 f(x)，g(x) がつぎの関係をみたすものとする。
> ① $f(-x)=-f(x)$　　② $g(-x)=g(x)$
> ③ $f\left(\frac{\pi}{2}-x\right)=g(x)$

(1) $g\left(\frac{\pi}{2}-x\right)=f\left\{\frac{\pi}{2}-\left(\frac{\pi}{2}-x\right)\right\}=f(x)$　③

(2) $f\left(x+\frac{\pi}{2}\right)=f\left\{\frac{\pi}{2}-(-x)\right\}=g(-x)=g(x)$　③，②

(3) $g\left(x+\frac{\pi}{2}\right)=g\left\{\frac{\pi}{2}-(-x)\right\}=f(-x)=-f(x)$

(4) $f(\pi-x)=f\left(\frac{\pi}{2}+\frac{\pi}{2}-x\right)=g\left(\frac{\pi}{2}-x\right)=f(x)$　(2)，(1)

(5) $g(\pi-x)=g\left(\frac{\pi}{2}+\frac{\pi}{2}-x\right)=-f\left(\frac{\pi}{2}-x\right)=-g(x)$　(3)，③

(6) $f(x+\pi)=f(-x)=-f(x)$　(4)，①

(7) $g(x+\pi)=-g(-x)=-g(x)$　(5)，②

(8) $f(x+2\pi)=f(\pi+x+\pi)=-f(x+\pi)=f(x)$　(6)

(9) $g(x+2\pi)=g(\pi+x+\pi)=-g(x+\pi)=g(x)$　(7)

> 更　に
> ④ $f(x+y)=f(x)\cdot g(y)+g(x)\cdot f(y)$　をみたすとき

(10) $f(x-y)=f(x)g(-y)+g(x)f(-y)=f(x)g(y)-g(x)f(y)$　④，②，①

(11) $g(x+y)=f\left(\frac{\pi}{2}-x-y\right)$　③

$=f\left(\frac{\pi}{2}-x\right)g(y)-g\left(\frac{\pi}{2}-x\right)f(y)$　(10)

$=g(x)g(y)-f(x)f(y)$　③，(1)

(12) $g(x-y)=g(x)g(-y)-f(x)f(-y)$　(11)

$=g(x)g(y)+f(x)f(y)$　②，①

(13) $f(2x)=2f(x)g(x)$　④

(14) $g(2x)=g^2(x)-f^2(x)$　(11)

(15) $f(x)g(y)=\frac{1}{2}\{f(x+y)+f(x-y)\}$　④，(10)

(16) $g(x)f(y)=\frac{1}{2}\{f(x+y)-f(x-y)\}$　④，(10)

(17) $g(x)g(y)=\frac{1}{2}\{g(x+y)+g(x-y)\}$　(11)，(12)

(18) $f(x)f(y)=-\frac{1}{2}\{g(x+y)-g(x-y)\}$　(11)，(12)

(19) $f(x)+f(y)=2f\left(\frac{x+y}{2}\right)g\left(\frac{x-y}{2}\right)$　(15)

(20) $f(x)-f(y)=2g\left(\frac{x+y}{2}\right)f\left(\frac{x-y}{2}\right)$ (16)

(21) $g(x)+g(y)=2g\left(\frac{x+y}{2}\right)g\left(\frac{x-y}{2}\right)$ (17)

(22) $g(x)-g(y)=-f\left(\frac{x+y}{2}\right)f\left(\frac{x-y}{2}\right)$ (18)

更　に
⑤ $f^2(x)+g^2(x)=1$ をみたすとき

(23) $g(2x)=2g^2(x)-1=1-2f^2(x)$ (14), ⑤

(24) $f^2\left(\frac{x}{2}\right)=\frac{1-g(x)}{2}$ (23)

(25) $g^2\left(\frac{x}{2}\right)=\frac{1+g(x)}{2}$ (23)

(26) $f(3x)=3f(x)-4f^3(x)$ ④, (13), (23), ⑤

(27) $g(3x)=4g^3(x)-3g(x)$ (11), (13), (23), ⑤

⑥ $t(x)=\frac{f(x)}{g(x)}$ なる関数 $t(x)$ をつくると

(28) $t(-x)=-t(x)$ ⑥, ①, ②, ⑥

(29) $t\left(\frac{\pi}{2}-x\right)=\frac{1}{t(x)}$ ⑥, ③, (1), ⑥

(30) $t\left(x+\frac{\pi}{2}\right)=\frac{1}{t(-x)}=-\frac{1}{t(x)}$ (29), (28)

(31) $t(\pi-x)=t\left(\frac{\pi}{2}+\frac{\pi}{2}-x\right)=-\frac{1}{t\left(\frac{\pi}{2}-x\right)}=-t(x)$ (30), (29)

(32) $t(x+\pi)=-t(-x)=t(x)$ (31), (28)

(33) $t(x+y)=\frac{f(x+y)}{g(x+y)}$ ⑥

$=\frac{f(x)g(y)+g(x)f(y)}{g(x)g(y)-f(x)f(y)}$ ④, (11)

$=\frac{t(x)+t(y)}{1-t(x)t(y)}$ ⑥

(34) $t(x-y)=\frac{t(x)-t(y)}{1+t(x)t(y)}$ (33), (28)

(35) $t(2x)=\frac{2t(x)}{1-t^2(x)}$ (33)

(36) $t^2\left(\frac{x}{2}\right)=\frac{1-g(x)}{1+g(x)}$ (24), (25)

つぎに $0<x<\frac{\pi}{2}$ のとき $f(x)>0$, $g(x)>0$ とすると ⑦

(37) $g(0)=g^2(x)+f^2(x)=1$ (12), ⑤

(38) $f^2(0)=1-g^2(0)=1-1=0$ ⑤, (37)

$\therefore\ f(0)=0$

(39) $f\left(\frac{\pi}{2}\right)=g(0)=1$ ③, (37)

(40) $g\left(\frac{\pi}{2}\right)=f(0)=0$ ③, (38)

(41) $f^2\left(\frac{\pi}{4}\right)=\frac{1-g\left(\frac{\pi}{2}\right)}{2}=\frac{1}{2}$ (24), (40)

$$\therefore\ f\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$ ⑦

(42) $g\left(\frac{\pi}{4}\right)=f\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$ ③, (41)

(43) $\alpha=\frac{\pi}{6}$ とすると $\alpha+2\alpha=\frac{\pi}{2}$

$$\therefore\ f(2\alpha)=g(\alpha)$$ ③

$2f(\alpha)g(\alpha)=g(\alpha)$ (13)

$$\therefore\ f(\alpha)=\frac{1}{2}$$ ⑦

すなわち $f\left(\frac{\pi}{6}\right)=\frac{1}{2}$

(44) $g^2\left(\frac{\pi}{6}\right)=1-f^2\left(\frac{\pi}{6}\right)=\frac{3}{4}$ ⑤

$$\therefore\ g\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$ ⑦

(45) $f\left(\frac{\pi}{3}\right)=g\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$ ③, (44)

(46) $g\left(\frac{\pi}{3}\right)=f\left(\frac{\pi}{6}\right)=\frac{1}{2}$ ③, (43)

(47) $f^2\left(\frac{\pi}{12}\right)=\frac{1-g\left(\frac{\pi}{6}\right)}{2}=\frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right)=\frac{2-\sqrt{3}}{4}$ (24), (44)

$$\therefore\ f\left(\frac{\pi}{12}\right)=\frac{\sqrt{6}-\sqrt{2}}{4}$$ ⑦

(48) $g\left(\frac{\pi}{12}\right)=\frac{\sqrt{6}+\sqrt{2}}{4}$ (25), (44), ⑦

(49) $f\left(\frac{5}{12}\pi\right)=g\left(\frac{\pi}{12}\right)=\frac{\sqrt{6}+\sqrt{2}}{4}$ ③, (48)

(50) $g\left(\frac{5}{12}\pi\right)=\frac{\sqrt{6}-\sqrt{2}}{4}$ ③, (47)

(51) $f^2\left(\frac{\pi}{8}\right)=\frac{1-g\left(\frac{\pi}{4}\right)}{2}=\frac{1-\frac{\sqrt{2}}{2}}{2}=\frac{2-\sqrt{2}}{4}$ (24), (42)

$$\therefore\ f\left(\frac{\pi}{8}\right)=\frac{\sqrt{2-\sqrt{2}}}{2}$$ ⑦

(52) $g\left(\frac{\pi}{8}\right)=\frac{\sqrt{2+\sqrt{2}}}{2}$ (25), (42), ⑦

(53) $f\left(\frac{3}{8}\pi\right)=g\left(\frac{\pi}{8}\right)=\frac{\sqrt{2+\sqrt{2}}}{2}$ ③, (52)

(54) $g\left(\frac{3}{8}\pi\right)=f\left(\frac{\pi}{8}\right)=\frac{\sqrt{2-\sqrt{2}}}{2}$ ③, (51)

(55) $\alpha=\frac{\pi}{10}$ とおくと $2\alpha+3\alpha=\frac{\pi}{2}$ $\therefore\ f(2\alpha)=g(3\alpha)$ ③

$2f(\alpha)g(\alpha)=4g^3(\alpha)-3g(\alpha)$ ④, (27)

$2f(\alpha)=4g^2(\alpha)-3$ ⑦

$2f(\alpha)=4\{1-f^2(\alpha)\}-3$ ⑤

$4f^2(\alpha)+2f(\alpha)-1=0$

$f(\alpha)=\frac{-1+\sqrt{5}}{4}$ ⑦

$$\therefore\ f\left(\frac{\pi}{10}\right)=\frac{-1+\sqrt{5}}{4}$$

(56) $g^2\left(\frac{\pi}{10}\right)=1-f^2\left(\frac{\pi}{10}\right)$ ⑤

$$=1-\frac{6-2\sqrt{5}}{16}=\frac{10+2\sqrt{5}}{16}$$ (55)

$$\therefore\ g\left(\frac{\pi}{10}\right)=\frac{\sqrt{10+2\sqrt{5}}}{4}$$ ⑦

(57) $f\left(\frac{\pi}{5}\right)=2f\left(\frac{\pi}{10}\right)g\left(\frac{\pi}{10}\right)=\frac{1}{4}\sqrt{10-2\sqrt{5}}$ (13), (55), (56)

(58) $g\left(\frac{\pi}{5}\right)=1-2f^2\left(\frac{\pi}{10}\right)=1-2\cdot\frac{6-2\sqrt{5}}{16}=1-\frac{3-\sqrt{5}}{4}=\frac{1+\sqrt{5}}{4}$ (23), (55)

(59) $f\left(\frac{2}{5}\pi\right)=g\left(\frac{\pi}{10}\right)=\frac{\sqrt{10+2\sqrt{5}}}{4}$ ③, (56)

(60) $g\left(\frac{2}{5}\pi\right)=f\left(\frac{\pi}{10}\right)=\frac{-1+\sqrt{5}}{4}$ ③, (55)

(61) $f\left(\frac{3}{10}\pi\right)=g\left(\frac{\pi}{5}\right)=\frac{\sqrt{10-2\sqrt{5}}}{4}$ ③, (57)

(62) $g\left(\frac{3}{10}\pi\right)=f\left(\frac{\pi}{5}\right)=\frac{1+\sqrt{5}}{4}$ ③, (58)

(63) $t(0)=\frac{f(0)}{g(0)}=0$ ⑥, (37), (38)

(64) $t\left(\frac{\pi}{2}\right)$　なし ⑥, (39), (40)

(65) $t\left(\frac{\pi}{4}\right)=1$ ⑥, (43), (44)

(66) $t\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}$ ⑥, (43), (44)

(67) $t\left(\frac{\pi}{3}\right)=\sqrt{3}$ ⑥, (45), (46)

(68) $t\left(\frac{\pi}{12}\right)=2-\sqrt{3}$ ⑥, (47), (48)

(69) $t\left(\frac{5}{12}\pi\right)=2+\sqrt{3}$ ⑥, (49), (50)

(70) $t\left(\frac{\pi}{8}\right)=\sqrt{2}-1$ ⑥, (51), (52)

(71) $t\left(\frac{3}{8}\pi\right)=\sqrt{2}+1$ ⑥, (53), (54)

(72) $t\left(\frac{\pi}{10}\right)=\sqrt{\frac{5-2\sqrt{5}}{5}}$ ⑥, (55), (56)

(73) $t\left(\frac{\pi}{5}\right)=\sqrt{5-2\sqrt{5}}$ ⑥, (57), (58)

(74) $t\left(\frac{2}{5}\pi\right)=\sqrt{5+3\sqrt{5}}$ ⑥, (59), (60)

(75) $t\left(\frac{3}{10}\pi\right)=\sqrt{3\sqrt{5}-5}$ ⑥, (61), (62)

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⑧ $A+B+C=\pi$,　$A, B, C>0$

$f(A):f(B):f(C)=a:b:c$,　$a>0$

なる a, b, c をとり，$\frac{a}{f(A)}=2R$ とする。

(Ⅰ)　$a\cdot g(B)+b\cdot g(A)=2R\{f(A)g(B)+g(A)f(B)\}$ ⑧

$=2Rf(A+B)$ ④

$=2Rf(\pi-C)$ ⑧

$=2Rf(C)=c$ (4)

故に $\begin{cases} (\mathrm{I}_1) & a=b\cdot g(C)+c\cdot g(B) \\ (\mathrm{I}_2) & b=c\cdot g(A)+a\cdot g(C) \\ (\mathrm{I}_3) & c=a\cdot g(B)+b\cdot g(A) \end{cases}$

(Ⅱ) $(\mathrm{I}_1)\times a+(\mathrm{I}_2)\times b-(\mathrm{I}_3)\times c$

$$a^2+b^2-c^2=2ab\cdot g(C)$$

$$\begin{cases} (\mathrm{II}_1) & a^2=b^2+c^2-2bc\cdot g(A) \\ (\mathrm{II}_2) & b^2=c^2+a^2-2ca\cdot g(B) \\ (\mathrm{II}_3) & c^2=a^2+b^2-2ab\cdot g(C) \end{cases}$$

(Ⅲ) $f^2\left(\frac{A}{2}\right)=\frac{1-g(A)}{2}=\frac{1}{2}\left\{1-\frac{b^2+c^2-a^2}{2bc}\right\}$ ㉔, (Ⅱ)

$$=\frac{2bc-(b^2+c^2-a^2)}{4bc}=\frac{1}{4bc}(a-b+c)(a+b-c)$$

$$=\frac{1}{bc}(s-b)(s-c) \quad \text{ここに } s=\frac{1}{2}(a+b+c)$$

$b,\ c>0,\quad s-a=\frac{1}{2}(-a+b+c)$

$$b+c-a=2R\{f(B)+f(C)-f(A)\}$$

$$f(B)+f(C)-f(A)=2f\left(\frac{B+C}{2}\right)g\left(\frac{B-C}{2}\right)-2f\left(\frac{A}{2}\right)g\left(\frac{A}{2}\right)$$

$$=2g\left(\frac{A}{2}\right)\left\{g\left(\frac{B-C}{2}\right)-g\left(\frac{B+C}{2}\right)\right\}$$

$$=4g\left(\frac{A}{2}\right)f\left(\frac{B}{2}\right)f\left(\frac{C}{2}\right)>0$$

$\therefore\ s-a>0$ 他も同様

$$\therefore\ f\left(\frac{A}{2}\right)=\sqrt{\frac{(s-b)(s-c)}{bc}}$$

(Ⅳ) $g\left(\frac{A}{2}\right)=\sqrt{\frac{s(s-a)}{bc}}$ ㉕, (Ⅱ)

(Ⅴ) $t\left(\frac{A}{2}\right)=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ ⑥, (Ⅲ), (Ⅳ)

⑨ $S=\frac{1}{2}bc\cdot f(A)$ なるSを定めると

(Ⅵ) $S=\frac{1}{2}2Rf(B)\cdot c\cdot\frac{a}{2R}=\frac{1}{2}ca\cdot f(B)$ ⑨

$S=\frac{1}{2}b\cdot 2Rf(C)\cdot\frac{a}{2R}=\frac{1}{2}ab\cdot f(C)$ ⑨

(Ⅶ) $S=\frac{1}{2}bc\cdot 2f\left(\frac{A}{2}\right)\cdot g\left(\frac{A}{2}\right)$ ⒀

$=\frac{1}{2}bc\cdot 2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)s}{bc}}$ (Ⅲ), (Ⅵ)

$$=\sqrt{s(s-a)(s-b)(s-c)}$$

(Ⅷ) $S=\frac{1}{2}bc\cdot\frac{a}{2R}=\frac{abc}{4R}$ ⑨, ⑧

(Ⅸ) $S=\frac{1}{2}ab\cdot f(C)=\frac{1}{2}a\cdot\frac{a\cdot f(B)}{f(A)}\cdot f(C)$ ⑧

$=\frac{1}{2}a^2\cdot\frac{f(B)\cdot f(C)}{f(B+C)}$ (4)