The cosmological constant problem and topological gravity motivated by renormalization group (宇宙項問題とくりこみ群に動機付けられた位相的重力理論)

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Abstract

General relativity and quantum field theory have been successfully established by experiments and observations in the widespread scale from elementary particle phenomena to cosmological dynamics. However, there are unsolved problems in cosmological phenomena and various attempts have been made to solve them. One of the problems is the so-called cosmological constant problem which is a kind of hierarchy problem in physics, that is, if we apply the renormalization prescription in quantum field theory to the cosmological constant, fine-tuning of the parameters in gravity theory is required to match the experimental value of the cosmological constant. One of the approaches to solve this problem is called unimodular gravity.

Unimodular gravity was originally proposed by Einstein and it is obtained by imposing a constraint on general relativity. Although this constraint breaks diffeomorphism invariance in general relativity except the volume preserving diffeomorphism, the equation of motion in the unimodular gravity is not changed from that in the general relativity with the cosmological constant. The cosmological constant in unimodular gravity, however, appears as a constant of integration and therefore the cosmological constant is no longer a parameter in the theory but determined by an initial condition. Thus, there is a possibility to solve the cosmological constant problem by using the unimodular gravity.

We consider one of the extensions of the unimodular gravity, which can be regarded as a kind of topological field theory. In this model, we find that the cosmological constant and the gravitational coupling constant are not constants but behave as dynamical scalar fields. We also find that there is a solution where the scalar fields become constant and the values of the constants are determined by the initial conditions. In this way, the cosmological constant problem in quantum field theory is replaced by the problem of the initial condition in classical theory, and thus we may expect that the fine-tuning could be relaxed.

We first focus on the fact that the values of the scalar fields corresponding to the coupling constants in the gravity including the cosmological constant depend on the energy scale of the universe. When the coupling constants have the scale dependence, we may consider the analogue of the renormalization group equation (RGE) which determines the response with respect to the scale transformation of physical quantities. We propose a new model by deforming the above model of the topological field theory to have two fixed points at low energy scale and high energy scale by regarding the equation in the new model as RGE. When we consider the fixed point at high energy, we choose the parameters in the model so that the fixed point generates the inflation in the early universe. On the other hand, when we consider the fixed point at low energy, we choose the parameters so that the model reproduces the late-time acceleration of the current universe.

Furthermore, we find the potential where these fixed points are connected by the scale transformation and we also specify the region of the realistic parameters which realize the observational results. This is expected to be a clue for building a model which solves the cosmological constant problem while reproducing the inflation in the early universe and the current accelerating expansion by using topological field theory.

要旨

一般相対性理論と場の量子論は素粒子のように非常に小さなスケールの現象から宇宙のような広大なスケールでのダイナミクスに渡る様々な実験及び観測により、高い精度で検証され、成功を収めてきた。一方で宇宙論等には未解決の問題も依然として存在し、それを解決するための様々な試みも行われている。この問題の 1 つが階層性問題の類である宇宙項問題と呼ばれる問題である。すなわち、量子論における繰り込みの手法を宇宙項に用いた時、観測された値と合わせようとすると、理論のパラメータに不自然な微調整が必要となってしまう。この宇宙項問題に対するアプローチの 1 つとしてユニモジュラー重力と呼ばれる理論が知られている。

ユニモジュラー重力はもともとアインシュタインが提案した理論で、一般相対性理論に対して新たな拘束条件を課したものである。一般相対性理論は一般の座標変換の下での共変性を持つが、この拘束条件は体積要素の大きさを一定に保つものを除いてこの共変性を破る。それにも拘らず、この理論で得られる方程式は一般相対性理論のものと同等になる。ただし、この理論では宇宙項に含まれる宇宙定数が方程式を導出する際に積分定数として現れるため、宇宙定数は初期条件等によって決まるもので理論を特徴づけるパラメータではなくなる。この性質によりこの理論では宇宙項問題が解決される可能性がある。

申請者はこのユニモジュラー重力の拡張として提案された模型の一つで、位相的場の理論とみなせるものについて研究を行った。この模型では宇宙定数及び重力結合定数は定数ではなく、動的なスカラー場となる。この理論には、このスカラー場が低エネルギーで定数となる安定な解が存在し、これにより現在の宇宙の加速膨張解を再現することが出来るが、この定数は初期条件等により決まる。このようにこの理論では量子論としての宇宙項の問題が古典論としての初期条件の問題に置き換わるため、微調整の問題が緩和されることが期待される。

申請者は、この模型では宇宙定数を含む重力理論に現れる結合定数に対応するスカラー場が宇宙のエネルギースケールに依存していることに着目した。このような結合定数のスケール依存性は、スケール変換への応答を見るくりこみ群方程式と類似性がある。そこで申請者はこの模型を変形し、模型に現れる方程式を繰り込み群方程式に見立て、低エネルギーと高エネルギーの両方で固定点を持つような模型を構築した。この模型では、高エネルギーでの固定点が、宇宙初期のインフレーションを実現する一方で、低エネルギーの固定点が現在の宇宙の加速膨張を再現する。更に申請者はこの模型で、適切なポテンシャル項を仮定し、インフレーションと現在の宇宙の加速膨張を再現する固定点が現れる条件を求め、なおかつ、2 つの固定点を繋ぐような解が存在するパラメータ領域を特定した。このことは、位相的場の理論を使い、宇宙初期のインフレーションと現在の宇宙の加速膨張を再現するとともに宇宙項問題を解く模型を構築するための足掛かりとなると期待される。

Contents

Chapter 1

The cosmological constant problem

In the 20th century, Einstein's general relativity was established [1–3]. In general relativity, the dynamics of the space-time is described by the so-called Einstein field equation. When we apply the Einstein field equation to the universe, we use the approximation of the cosmological principle since it is very difficult to solve the equation without any approximation. The discovery that the Universe in the large-scale structure ($\simeq$ 100 Mpc) is isotropic means that it has no centre - there is no specified point in space. 'In a large scale, the space of the universe is a homogeneous and isotropic space'. This is the content of the so-called Cosmological Principle which is fundamental premise of cosmology [9]. Owing to this simple principle, we may assume a simple form of metric which is called Friedman-Lemaitre-Robertson-Walker (FLRW) metric. Assuming the FLRW metric, we can solve the Einstein field equation easily. This equation is called Friedmann equation which describes the dynamics of the universe. Based on these simple assumptions, we obtain so-called the ΛCDM model. The ΛCDM model can describe observational universe very accurately. The letter Λ is based on the existence of the cosmological constant, which is the simplest and strong candidate of the source of dark energy.

However, there are several problems about the cosmological constant which is called the cosmological constant problem. In this chapter, details of the cosmological constant problem [46–63] are reviewed. The history of the cosmological constant problem is not only long but also highly suggestive for the new physics.

In this chapter, the basics of the cosmological constant problem are given. In section 1, we briefly review the history of the cosmological constant. In section 2 the observational results of the cosmological constant are reviewed. In section 3, we explain the important issue of this thesis, that is, the fine-tuning of the cosmological constant and we provide additional explanation of this problem in sections 4, 5, 6 and 7.

1.1 The historical implication of the cosmological constant

In this section, a brief history of the cosmological constant is reviewed. The cosmological constant is originally introduced to counterbalance the effects of attractive force of the gravity and realize a static universe by Einstein [10]. After the discovery of the expansion of the universe by Hubble, Einstein abandoned to adopt the cosmological constant [11–13, 53]. However, from the recent observation, we find that the universe expands acceleratedly [17–33]. These imply that the existence of cosmological constant with positive value.

1.1.1 The Einstein equation and cosmology

In general relativity, the fundamental equation is given by the Einstein-Hilbert action [1] [4]. By virtue of Newtonian limit of the gravitational field, the general coordinate covariance and the Bianchi identities, the Einstein-Hilbert action is defined by,

$$S_{\mathrm{EH}} \equiv \int \sqrt{-g} d^4x \mathcal{L}_{\mathrm{EH}} = \frac{1}{16\pi G} \int \sqrt{-g} d^4x R \equiv \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x R \,, \tag{1.1.1}$$

where g is the determinant of the metric, and $\sqrt{-g}d^4x$ is the invariant volume element which follows from general coordinate covariance. G is the gravitational coupling constant which is determined by weak gravitational limit to recover the Newtonian gravity. Then, we may derive the gravitational field equation by varying the Einstein-Hilbert action and matter action S_{matter} with respect to $g_{\mu\nu}$,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu} \,. \tag{1.1.2}$$

Here, $G_{\mu\nu}$ is the Einstein tensor which satisfies the contracted Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ where ∇_μ is the covariant derivative. $T_{\mu\nu}$ is the energy-momentum tensor defined by,

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{matter}}}{\delta g_{\mu\nu}} \,. \tag{1.1.3}$$

The general coordinate covariance indicates that the conservation of the energy-momentum tensor $\nabla^\mu T_{\mu\nu} = 0$ which does not contradict with the Bianchi identity. This equation is known as the Einstein field equation which determines the relation of geometry of space-time and distribution of matter field. The Einstein equation follows the idea of Mach's principle [6, 8] . Mach's principle insists that the"local physical laws should be determined by the large-scale structure of the universe" [7]. In other words, the geometry of space-time tells how matters move, and conversely, the distribution of matter tells how space-time curve at the same time and finally it realizes our universe. When we apply this equation to the cosmology, there is no stable solution of the universe. However, if we introduce the cosmological constant, we may keep our universe static.

1.1.2 The original idea of the cosmological constant

The cosmological constant Λ was firstly introduced by Einstein in 1917 [10] as a repulsive force to keep the Universe static. In modern cosmology, the cosmological constant is the strongest candidate for so-called dark energy which causes accelerated expansion of the universe.

Although the Einstein field equation in Eq. (1.1.2) seems to be the simplest form, there remains the freedom of adding a constant term multiplied by metric, $g_{\mu\nu}\Lambda$. This is called "cosmological constant" which originally Einstein introduced in order to achieve a static universe keeping the Bianchi identity and covariance of the energy-momentum tensor. The Einstein field equation with the cosmological constant becomes,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu} \,. \tag{1.1.4}$$

There are many unsolved issues in the cosmological constant. We see these problems in this chapter.

1.2 The observational results of the existence of the cosmological constant

In this section, we consider the observational consequence of the non-zero cosmological constant. In cosmology, the homogeneous and isotropic metric is expressed as

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2\theta\phi^2\right)\right], \quad K = \text{const}\,. \tag{1.2.1}$$

This metric is called the Friedman-Lemaitre-Robertson-Walker (FLRW) metric. There are two important physical quantities in the FLRW metric. One is the scale factor $a(t)$ which describes the relative expansion of the universe. The other is the constant space curvature K. We can always choose the coordinate so that K takes the values $0, \pm 1$, expressing flat, positively (negatively) curved universe, respectively. In the FLRW universe, the energy is described as the perfect fluid. For a comoving observer, the energy-momentum tensor is expressed as

$$T_{\mu\nu} = \text{diag}(\rho, p, p, p)\,, \tag{1.2.2}$$

where ρ is an energy density and p is a pressure of the perfect fluid. When we apply the above assumptions to the Einstein equation with the cosmological constant, we obtain the two Friedmann equations. The (0,0)-component reads

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3}\,. \tag{1.2.3}$$

On the other hand, the spacial components reads

$$\frac{\ddot{a}}{a} = -\frac{1}{2}\left\{\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} - \Lambda\right\} - 4\pi G p\,. \tag{1.2.4}$$

Conservation of the energy-momentum tensor reads

$$\dot{\rho} = -3\frac{\dot{a}}{a}(p+\rho) = -3\frac{\dot{a}}{a}(w+1)\rho\,, \tag{1.2.5}$$

where we introduced the EoS parameter w as,

$$w = \frac{p}{\rho}\,. \tag{1.2.6}$$

From the above expressions, we can see the dynamics of the universe through the scale factor related to the property of matters. In general, an equation of state of the universe would be very complex since there co-exist several kinds of matters. However, it is possible to classify the following three typical matters which yield simple EoS parameter.

1. dust/non-relativistic matter
   The EoS parameter of an energy density for non-relativistic matter is expressed as $w = 0$ in Eq. (1.2.6). This matter describes a fluid of cold or heavy particles. In particular, this corresponds to so-called the cold dark matter (CDM) and baryons. For these matters, we can ignore the pressure.

2. Ultra-relativistic matter (radiation)
   This matter characterizes hot relativistic particles such as photons or all Standard Model particles in the early Universe where the temperature is very high. Radiation has an equation of state parameter with $w = 1/3$.

3. Cosmological constant
   The cosmological constant corresponds to a fluid with negative pressure $w = -1$. This pressure counteracts the attractive force of matters. Although the static solution of the universe is realized, it is unstable. This also leads to accelerated expansion of the universe. In fact, when we neglect a matter, radiation, and curvature in Eq. (1.2.3), we obtain the following equation

$$a \propto \exp\left(\sqrt{\frac{\Lambda}{3}}t\right)\,, \tag{1.2.7}$$

   which indicates the accelerated expansion of the universe for positive Λ.

Based on the FLRW universe, cosmological parameters are determined by the cosmic microwave background radiation (CMB) observed by WMAP [15], the number density of galaxy cluster, luminosity (cosmological) red-shift of supernova and so on. Therefore, WMAP can measure the fundamental parameters of the FLRW model, including the density and composition of the universe. WMAP also measures the ratio of the density of baryonic and non-baryonic matter. Furthermore, some of the properties of the non-baryonic matter can also be determined by WMAP. Planck satellite measurements play also an important role in the measurement of the cosmological parameters [16]. From these observations, we find the two important facts. One is that our current universe has a very small curvature. The other is that the universe mainly consists of the following three matters.

1. Baryon : About 5% of the total energy density in the universe. More than 95 % of the universe has never been directly observed in the laboratory.

2. Cold dark matter : 26% of the total energy density in the universe.
   Dark matter is likely to be composed of one or more species of sub-atomic particles that interact very weakly with baryons. Especially, cold dark matter is non-relativistic matters, and the kinetic energy can be neglected from the total energy. The cold dark matter is able to explain the galaxy or galaxy cluster formation. The cold dark matter may be regarded as the main component of the dark matters.

3. Dark Energy : 69% of the total energy density in the universe. By the observations of supernovae, it is found that the current universe is expanding. Combining Planck data with Pantheon supernovae and BAO data, the equation of state of dark energy is strictly constrained to $w = -1.03 \pm 0.03$, which is highly consistent with the cosmological constant.

This model of the universe is called ΛCDM model. However, there are some problems with the ΛCDM model. As we saw in the above discussion, 95% of the total density of the universe consists of the energy coming from the dark sector. There are many attempts to explain the origin of the dark matter and the dark energy. For the dark matter, there are many candidates, the massive compact halo objects (MACHOs) [34], the robust association of massive baryonic objects (RAMBOs) [35], the axion [36, 37] weakly interacting massive particles (WIMPs) [38] and so on. For the dark energy, there are also several candidates, phantom energy [39] with the range of the EoS parameter $w < -1$, Chaplygin gas [40], quintessence [41–43], K-essence [44, 45] with the range of EoS parameter $-1 < w < -\frac{1}{3}$ and so on. The simplest candidate is cosmological constant which is expressed as $w = -1$. As we saw in the previous section, the value of w is consistent with the cosmological constant. Furthermore, the cosmological constant emerges from the vacuum energy in the quantum field theory. Therefore, the cosmological constant is not only the simplest candidate, but also it seems that the strongest candidate for the dark energy.

However, there still remain many problems of the cosmological constant. Although we will discuss more details of the problems of the cosmological constant in the next section, we roughly present two problems of the cosmological constant briefly.

1. Old cosmological constant problem : The “old” cosmological constant problem emerges when we consider the general relativity and the quantum field theories. This is also called fine-tuning problem. The vacuum energy from the quantum field theory gravitates as the cosmological constant. However, the theoretical value of the vacuum energy exceeds the observed value of the cosmological constant by about 120 orders of magnitude. Therefore, we need tremendous fine-tuning of the counter term.

2. New cosmological constant problem : The “new” cosmological constant problem is called “coincidence” problem. This is the problem that there is no reason why the dark energy density is comparable to the dark matter density in the current universe.

1.3 Fine-tuning problem of the cosmological constant

As we discussed the previous section, the cosmological constant problem may be roughly divided into two types. One is the old cosmological constant problem. We discuss this problem more precisely in this section. This is a kind of the hierarchy problem which involves the fine-tuning of the parameters. Naively, the old cosmological constant problem is the question that “Why the value of the cosmological constant is extremely smaller than the prediction of the well-established theories?”. Actually, the contribution to the cosmological constant from quantum filed theory is much larger than the currently observed value $|\rho_\Lambda^{(\mathrm{obs})}| \lesssim \left(10^{-12}\ \mathrm{GeV}\right)^4$. To see the contributions from the field theory, we consider classical and quantum field theories. Because the source of the cosmological constant is divided into the classical contribution and quantum one respectively when we consider the field theory, we separately discuss the cosmological constant from the perspectives of the classical field theory and the quantum field theory.

1.3.1 The classical picture of the cosmological constant

The vacuum energy is interpreted as the cosmological constant in field theory. To see the relation between the vacuum energy and the cosmological constant, we consider the scalar field with potential which is minimally coupled to the Einstein gravity:

$$S_\phi = \int d^4x \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] . \tag{1.3.1}$$

By the variation with respect to $g_{\mu\nu}$, we obtain the following energy-momentum tensor:

$$T_{\mu\nu}^{(\phi)} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta \phi - V(\phi) \right) . \tag{1.3.2}$$

Since the vacuum state is the minimum energy state, i.e., $\phi = \phi_0 =$ constant and $V(\phi_0)$ is minimum of the energy, we obtain the energy-momentum tensor in the vacuum state as

$$\langle 0 | T_{\mu\nu} | 0 \rangle = -V(\phi_0) g_{\mu\nu} . \tag{1.3.3}$$

Since $V(\phi_0)$ is a constant, it is the same form as the cosmological constant. However, the standard model of the particle physics suggests that this value changes drastically at least twice. The standard model predicts that there are electroweak and QCD phase transition in the early universe [64–66]. For the electroweak phase transition, the difference of the value of the vacuum energy is given by, $\Delta V_{\mathrm{EW}} \sim (200\ \mathrm{GeV})^4 \sim 10^{56} \left| \rho_\Lambda^{(\mathrm{obs})} \right|$. On the other hand, for the QCD phase transition, this value is estimated by, $\Delta V_{\mathrm{QCD}} \sim (0.3\ \mathrm{GeV})^4 \sim 10^{44} \left| \rho_\Lambda^{(\mathrm{obs})} \right|$. Note that these values are much larger than the currently observed value of the cosmological constant. This huge discrepancy is the classical picture of the cosmological constant problem. To obtain the current value of the cosmological constant, we have to initially prepare the cosmological constant Λ_{ini}:

$$\Lambda_{\mathrm{ini}} - 10^{56} \left| \rho_\Lambda^{(\mathrm{obs})} \right| - 10^{44} \left| \rho_\Lambda^{(\mathrm{obs})} \right| \sim \left| \rho_\Lambda^{(\mathrm{obs})} \right| . \tag{1.3.4}$$

Therefore, we need the fine-tuning of the initial value of the cosmological constant.

1.3.2 The vacuum energy as the cosmological constant

Because of the locality in quantum field theory, the existence of the vacuum energy density is suggested. To see this, we consider a simple model which produces the vacuum energy. Let us consider the free field solution of the Klein-Gordon equation in the Minkowski space:

$$(\Box - m^2) \phi(x) = 0 . \tag{1.3.5}$$

The Fourier expansion of $\phi(x)$ reads

$$\phi(x) = \frac{1}{\left(\sqrt{2\pi}\right)^3} \int \frac{d^3 k}{\sqrt{2\omega}} \left(a_{\boldsymbol{k}} \mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} + a_{\boldsymbol{k}}^\dagger \mathrm{e}^{-i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \right) , \tag{1.3.6}$$

where $\omega = \sqrt{\boldsymbol{k}^2 + m^2}$. $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^\dagger$ are annihilation and creation operators which satisfy the following commutation relation:

$$[a_{\boldsymbol{k}}, a_{\boldsymbol{q}}^\dagger] = \delta^3(\boldsymbol{k} - \boldsymbol{q}) . \tag{1.3.7}$$

The (0,0) component of the energy-momentum tensor reads

$$T_{00} \equiv \mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\sum_{i=1}^{3}(\nabla_i\phi)^2 + \frac{1}{2}m^2\phi^2\,, \tag{1.3.8}$$

where $\mathcal{H}$ is the Hamiltonian density, and

$$\langle 0|T_{00}|0\rangle = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\omega = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\sqrt{\boldsymbol{k}^2+m^2}\,. \tag{1.3.9}$$

In general, the above quantity diverges. Since this type of energy can be interpreted as the ground-state energy, we may ignore this term within the scope of the conventional quantum field theory.

We can only detect the difference of the energy when we do not consider the global gravitational sector in experiments [67, 70, 71]. Preparing the cutoff scale $M_{\rm cutoff}$ to evaluate this divergence, the above integral can be written as

$$\begin{aligned}\langle 0|T_{00}|0\rangle &= \frac{1}{2}\int^{M_{\rm cutoff}}\frac{d^3k}{(2\pi)^3}\sqrt{\boldsymbol{k}^2+m^2}\\ &= \frac{1}{4\pi^2}\int^{M_{\rm cutoff}}\frac{d^3k}{(2\pi)^3}\sqrt{\boldsymbol{k}^2+m^2}\\ &= \frac{M_{\rm cutoff}^4}{16\pi^2}\left[1+\left(\frac{m}{M_{\rm cutoff}}\right)^2-\frac{1}{4}\frac{m^4}{M_{\rm cutoff}^4}\left[\log\left(\frac{M_{\rm cutoff}^2}{m^2}\right)+\frac{1}{8}-\frac{1}{2}\log 2\right]+\mathcal{O}\left(M_{\rm cutoff}^{-1}\right)\right]\,.\end{aligned} \tag{1.3.10}$$

Since the leading term is $\sim M_{\rm cutoff}^4$, this divergence is quartic. We may see that this term does not contribute in the quantum field theory under the Minkowski metric. Although this term diverges, it is a constant term in the action.

$$S_{\rm vacuum} = \int d^4x\,\langle 0|T_{00}|0\rangle = \int d^4x\Lambda_{\rm vacuum}\,, \tag{1.3.11}$$

where $\Lambda_{\rm vacuum}$ is a constant term. Apparently, this term does not contribute to any dynamics. Therefore, we can ignore this term. However, when we consider general relativity, this term emerges as the cosmological constant. The equivalence principle states that all energy affects the curvature of space-time. Therefore, the vacuum energy should also gravitates. Furthermore, the general coordinate covariance imposes how to gravitates this vacuum energy as

$$S_{\rm vacuum} = \int d^4x\Lambda_{\rm vacuum} \to \int d^4x\sqrt{-g}\Lambda_{\rm vacuum}\,. \tag{1.3.12}$$

This term is exactly the cosmological constant. Therefore, the vacuum energy contributes to the cosmological constant. However, when we regard $M_{\rm cutoff}$ as a Planck mass $M_{\rm Planck}$ to the value of the vacuum energy is estimated as,

$$\rho_{\rm vac} \equiv \langle 0|T_{00}|0\rangle \sim \Lambda_{\rm cutoff}^4 = M_{\rm pl}^4 \sim \left(10^{18}\ {\rm GeV}\right)^4 \sim 10^{121}\rho_\Lambda^{\rm (obs)} \,. \tag{1.3.13}$$

This is much larger than the observed value. To remove the large vacuum energy and to remain the extremely small value of the cosmological constant, we need tremendous fine-tuning of the counter term and it is extremely unnatural. This is the old cosmological constant (fine-tuning) problem from the perspective of the quantum field theory.

1.4 Radiative instability

In addition to the fine-tuning problem of the cosmological constant, another serious problem exists. The radiative instability [63] is one of the most serious problems of the cosmological constant. The important point is that the value of the cosmological constant is extremely sensitive to UV theory. We may see this problem by using the dimensional regularization [67–69] of the Eq. (1.3.9). Introducing the mass parameter μ, the vacuum energy density can be rewritten as

$$\langle 0|T_{00}|0\rangle = \frac{\mu^\epsilon}{2(2\pi)^{d-1}} \int^{M_{\rm cutoff}} d^{d-1}k k^{d-2}\omega d^{d-2}\Omega_d \,, \tag{1.4.1}$$

where we introduced the small parameter ϵ as $\epsilon = 4 - d$ and angular integration in d dimension as,

$$\int d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \,, \tag{1.4.2}$$

where $\Gamma(x)$ is the Gamma function. We obtain the expression of the vacuum energy in dimensional regularization as

$$\langle 0|T_{00}|0\rangle \simeq -\frac{m^4}{(8\pi)^2}\left[\frac{2}{\epsilon} + \ln\left(\frac{4\pi\mu^2}{m^2}\right) + \ {\rm finite}\ \right] . \tag{1.4.3}$$

We renormalize the above quantity by $\overline{\rm MS}$ scheme [67, 72]. In this scheme, we choose the counter term as

$$V_{\rm counter} \simeq \frac{m^4}{(8\pi)^2}\left[\frac{2}{\epsilon} + \ln\left(\frac{4\pi\mu^2}{k^2}\right) + \ {\rm finite}\ \right] . \tag{1.4.4}$$

Then the renormalized quantity is given by

$$V_{\rm vac}^{\rm (ren)} \equiv \Lambda_{\rm vac}^{\rm (bare)} + \langle 0|T_{00}|0\rangle + V_{\rm counter} \simeq \Lambda_{\rm vac}^{\rm (bare)} + \frac{m^4}{(8\pi)^2}\left[\log\left(\frac{m^2}{k^2}\right)\right] . \tag{1.4.5}$$

We observe the renormalized quantity at the scale of k. Note that the above quantity explicitly depends on m^4. In other words, the scalar field contribution to the cosmological constant is scaled as $\sim m^4$. This is called additive renormalization. When we rescale the scalar field mass as $m \to m + \delta m$, the cosmological constant is extremely sensitive to this change. The renormalized value of the cosmological constant is tremendously sensitive to the masses of all other fields of the quantum field theory. This is the essential point of the problem of the radiative instability. From the viewpoint of the effective field theory, we may not obtain the effective picture of the cosmological constant since it changes as we change the mass of the particle. This problem also emerges in the Higgs mass in the standard model.

1.5 Naturalness

Compared with the electroweak scale, the electron mass is very small ~ 1 [MeV]. However, we do not face the problem of the radiative instability as we saw in case of the cosmological constant in the electron mass. To see this, we consider the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \bar{\psi}\left(i\gamma_\mu\partial^\mu - m_{\rm e}\right)\psi + g\phi\bar{\psi}\psi\,, \tag{1.5.1}$$

where ψ is the Dirac field and g is the coupling of the scalar field and Dirac field. In the same way with the previous section, we obtain the correction to the electron mass as,

$$\delta m_{\rm e} \sim g^2 m_{\rm e} \ln\left(\frac{m_{\rm e}}{\mu}\right) . \tag{1.5.2}$$

Note that $\delta m_{\rm e}$ is proportional to m_e. This is called multiplicable renormalization in contrast to additive renormalization appeared in the cosmological constant. The smallness of the electron mass is preserved by *naturalness*. The "naturalness" insists that "a quantity in physics should be small only if the underlying theory enhances the new symmetry when that quantity sets to zero" [74,75]. In case of the electron mass, when we take the massless limit $m_{\rm e} \to 0$, all quantum correction to the mass vanishes. In this limit, the Chiral symmetry,

$$\psi \to \gamma_5\psi, \quad \phi \to -\phi\,, \tag{1.5.3}$$

appears in the Lagrangian in Eq. (1.5.1). This is the example of naturalness. However, there is no enhanced symmetry when we take the cosmological constant set to be zero. Therefore, the smallness of the cosmological constant is "unnatural" in the context of the naturalness.

1.6 Weinberg's no-go theorem

An adjustment mechanism of the cosmological constant has been proposed by Weinberg [46]. This mechanics can be applied when we consider that the cosmological constant may be "dynamically" relaxed to a small value or absorbing the huge vacuum energy by additional scalar field. The result states that it is impossible to construct the potential of the Lagrangian without fine-tuning or an unrealistic universe (scale invariant universe) under the following five assumptions [59, 83].

1. The theory has general covariance.
2. Four dimensional gravity is induced by massless spin-2 graviton.
3. A finite number of fields below the cut-off scale.
4. Theory has no ghost.
5. The fields are assumed to be space-time independent at late times.

We have to introduce additional scalar field or assume the energy-momentum tensor is not covariantly constant to make the cosmological constant dynamical. To verify this, we start from taking the covariant derivative for both sides of the Einstein equation,

$$\partial_\mu \Lambda = \kappa \nabla^\nu T_{\mu\nu} \,, \tag{1.6.1}$$

where we used contracted Bianchi identity. Therefore, there are two choices to make the cosmological constant dynamical quantity. One is to break general covariance of the energy-momentum tensor. The other is to introduce a new dynamical field ϕ which satisfies

$$\begin{aligned} &\Box\phi \propto T^\mu_\mu \propto R \\ &R = T^\mu_{\ \mu} = 0 \quad \text{at } \phi = \phi_0 \,, \end{aligned} \tag{1.6.2}$$

where we assumed the scalar field ϕ evolves until its equilibrium value ϕ_0. We will work on this case to make the cosmological constant dynamical. In order to consider the general case, we assume the multi scalar fields

$$\phi \to \phi_n \quad (n = 1, \cdots, N) \,. \tag{1.6.3}$$

We consider four dimensional quantum field theory which describes multi degrees of freedom below a certain UV cutoff $M_{\rm UV}$. We further assume the Poincaré invariance of the field theory which minimally couples to gravity via $g_{\mu\nu}$. We would like to find an equilibrium solution of the field equation at $g_{\mu\nu}, \phi =$constant on shell. In the above condition, this theory remains GL(4) symmetry, With all fields constant, the gravity and matter equation can be written as,

$$\left.\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}}\right|_{g,\phi_n=\mathrm{const}}=0, \quad \left.\frac{\partial \mathcal{L}}{\partial \phi_n}\right|_{g,\phi_n=\mathrm{const}}=0\,, \tag{1.6.4}$$

on shell. When we apply the contracted Bianchi identity, we have $N+6$ equations. Under the GL(4) symmetry, metric and scalar fields and the Lagrangian transform as

$$g_{\mu\nu} \to J^\alpha_\mu J^\beta_\nu g_{\alpha\beta} \quad \phi_n \to \mathcal{J}(J)\phi_n \quad \mathcal{L} \to \det(J)\mathcal{L} \tag{1.6.5}$$

where $\mathcal{J}(J)$ is a representation of GL(4). We can consider two distinct cases separately;

(i) The two relations in Eq. (1.6.4) hold independently.
(ii) They are not independent and related with each other.

First, we consider the case (i), from the matter equation and the GL(4) symmetry, the form of the Lagrangian is uniquely determined as

$$\mathcal{L}=\sqrt{-g}\Lambda\left(\phi_n\right), \quad \frac{\partial\Lambda\left(\phi_n\right)}{\partial\phi_i}=0\,. \tag{1.6.6}$$

However, the gravitational field equation requires

$$\Lambda\left(\phi_n\right)=0\,. \tag{1.6.7}$$

This means that we have to set the cosmological constant zero "by hand" and it corresponds to fine-tuning.
Next we consider the latter case (ii). We impose the following condition:

$$2g_{\mu\nu}\frac{\partial\mathcal{L}(g_{\mu\nu},\phi_n)}{\partial g_{\mu\nu}}=\sum_n f_n(\phi_m)\frac{\partial\mathcal{L}(g_{\mu\nu},\phi_n)}{\partial\phi_n}\,, \tag{1.6.8}$$

where $f_n(\phi_m)$ is a smooth function for ϕ_n. This requirement means that the trace of the matter fields is proportional to the cosmological constant for the equilibrium values of ϕ_n. The above equation implies that the Lagrangian $\mathcal{L}(g_{\mu\nu},\phi_n)$ describing the theory is invariant under a transformation generated by the following symmetry:

$$\delta g_{\mu\nu}=2\epsilon g_{\mu\nu}, \quad \delta\phi_n=-\epsilon f_n\left(\phi_m\right)\,. \tag{1.6.9}$$

Note that when we consider a matter field solution for the equilibrium value of $\phi_n^{(0)}$:

$$\frac{\partial\mathcal{L}}{\partial\phi_n}=0 \quad \text{at } \phi_n=\phi_n^{(0)}\,, \tag{1.6.10}$$

the trace of gravitational field equation

$$\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = 0\,, \tag{1.6.11}$$

is automatically satisfied. Since $f_n(\phi)$ is a smooth function, we may simplify the situation. We can split N fields ϕ_n as

$$\phi_n(n = 1, \cdots, N) = \left\{ \begin{array}{ll} \psi & \\ \sigma_a & (a = 1, \cdots, N-1) \end{array} \right. . \tag{1.6.12}$$

Under the above condition, we can modify the scale transformation in (1.6.9) as,

$$\delta g_{\mu\nu} = 2\epsilon g_{\mu\nu}, \quad \delta\psi = -\epsilon, \quad \delta\sigma_a = 0\,. \tag{1.6.13}$$

The above modification can be performed by a theorem of differential geometry [77] and the Poincaré symmetry. Firstly, we define the transverse surface $\mathcal{S}$ which satisfies $T(\phi_n) = 0$. Here $T(\phi_n)$ must be chosen not to be vanished on $\Sigma_n\,(\partial T/\partial\phi_n)\,f_n(\phi_m)$. We then choose σ_a as sets of $(\mathcal{N}-1)$ dimensional surface.
Here, we defined $\phi_n(\sigma_a, \psi)$ as the solution of the differential equations $\frac{\delta\phi_n}{\delta\psi} = f_n(\phi_m)$ under the condition where ϕ_n are at the point of $\mathcal{S}$ with smooth coordinates σ at $\psi = 0$. The symmetry in Eq. (1.6.13) implies

$$\mathcal{L} \propto \mathrm{e}^{2\psi} g_{\mu\nu}\,. \tag{1.6.14}$$

From the Poincaŕe-invariance of ground state, residual GL(4) symmetry, and field equation for σ_a, the form of the Lagrangian is determined as

$$\mathcal{L} = \mathrm{e}^{4\psi}\sqrt{-g}\mathcal{L}_0(\sigma), \quad \frac{\partial \mathcal{L}_0(\sigma)}{\partial\sigma_a} = 0\,. \tag{1.6.15}$$

When we differentiate $\mathcal{L}$ with respect to ψ, we obtain

$$\frac{\partial \mathcal{L}}{\partial\psi} = T^\mu_\mu\sqrt{-g}, \quad T^{\mu\nu} = g^{\mu\nu} e^{4\psi}\mathcal{L}_0(\sigma)\,. \tag{1.6.16}$$

Here, we used the relation (1.6.8). We can see that if $\partial\mathcal{L}/\partial\psi = 0$, the trace of the energy-momentum tensor vanishes. Therefore, when we find the equilibrium point $\psi = \psi_0$, the trace of the field equation is automatically satisfied. However, at that point, the Lagrangian is constrained as

$$\mathcal{L}_0(\sigma) = 0, \quad \text{or} \quad e^{2\psi} \to 0\,. \tag{1.6.17}$$

The former case corresponds to fine-tuning, again. On the other hand, the latter case corresponds to a scale invariant solution. Under the scale invariant theory, not only the

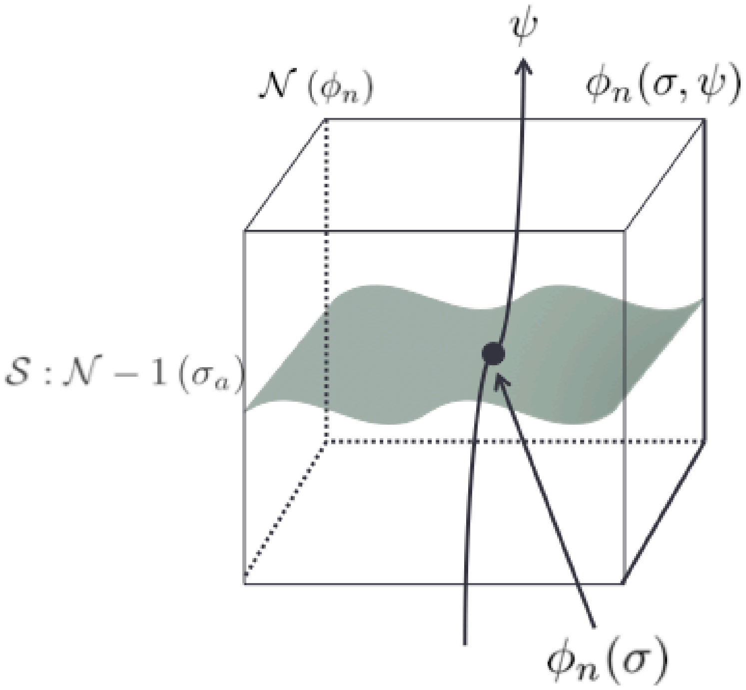

Fig. 1.1 Field splitting

cosmological constant vanishes, but also all the other dimensional parameters vanish. This is not our universe, so we cannot solve the essential problem.

We often see the scale symmetry in Eq. (1.6.13) is violated by conformal anomaly term which involves ${\Theta^{\mu}}_{\mu}$ [76]:

$$\mathcal{L}_{\text{eff}} = \sqrt{-g}\left[e^{4\phi}\mathcal{L}_0(\sigma) + \phi{\Theta^{\mu}}_{\mu}\right] . \tag{1.6.18}$$

The field equation for ϕ becomes

$$\frac{\partial\mathcal{L}_{\text{eff}}}{\partial\phi} = \sqrt{-g}\left(T^{\mu}_{\mu} + \Theta^{\mu}_{\mu}\right), \quad T^{\mu\nu} = g^{\mu\nu}e^{4\phi}\mathcal{L}_0(\sigma)\,. \tag{1.6.19}$$

Equilibrium solution for $\phi = \phi_0$ reads

$$4e^{4\phi_0}\mathcal{L}_0 + {\Theta^{\mu}}_{\mu} = 0\,. \tag{1.6.20}$$

However, it does not corresponds the gravitational field equation for constant metric which reads

$$0 = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial g_{\mu\nu}} \propto e^{4\phi} \mathcal{L}_0 + \phi \Theta^{\mu}_{\mu} \,. \tag{1.6.21}$$

This is simply because the symmetry in Eq. (1.6.13) is broken by the anomalous factor ${\Theta^{\mu}}_{\mu}$.

1.7 Anthropic consideration

The anthropic consideration is based on the anthropic principle [78–82]. The terminology of the anthropic principle is used in many kinds of the meanings. In this thesis, we define the anthropic principle in the following way. The physical parameters only have the physical meaning when the human observes them. The law of physics does not have predictability only by itself. Only after we define the physical quantity, we obtain the observational results. Therefore, what we observed is not directly included in the theory. Since the observer is human, theory must have correlation function between human and observational results. The anthropic principle means that we observe the physics through that correlation function. In other words, the human can see the physical parameters only from the stand point of human.

We apply the anthropic principle to the cosmological constant problem. When we consider the negative cosmological constant whose absolute value is not so small, our universe would suddenly collapse and no one observes our universe. On the other hand, if the cosmological constant is positive and the absolute value is larger than the current value to a certain degree, the universe expands infinitely before the formation of galaxies. In either case, no one can observe our universe. After all, the universe where our galaxy and we can exist must have the proper value of the cosmological constant and we observe it. This is not only related to the old cosmological constant problem but also deeply related to the new cosmological constant problem. It is the problem that we can exist in the proper physical value and proper era of the universe. We are currently living in a very special period of the history of the universe in which the ratio of the dark energy density and matter density is comparable. We cannot exist before the galaxy formation and this also means that we only observe the universe at the time we can exist.

However, even if we admit it, it seems that the question why the cosmological constant is so tremendously small still remains and it is difficult to say that the anthropic principle is a perfect solution of the cosmological constant problem. Therefore, it seems overhasty that we stop the consideration about the cosmological constant problem. In the next chapter, we discuss some attempts for the cosmological constant problem.

Chapter 2

Some attempts for the cosmological constant problem

2.1 Supersymmetry

Supersymmetry (SUSY) may be used for the cancellation of the vacuum energy density [84]. The vacuum energy contribution from the scalar field is denoted in Eq. (1.3.10). More precisely, the expression in Eq. (1.3.10) is generalized to an arbitrary spin field $j = 0, 1/2, 1, \cdots$:

$$\begin{aligned}\langle\rho_{\rm vac}\rangle &= \frac{1}{2}(-1)^{2j}(2j+1)\int_0^{M_{\rm cutoff}}\frac{d^3k}{(2\pi)^3}\sqrt{\mathbf{k}^2+m^2}\\ &= \frac{(-1)^{2j}(2j+1)}{16\pi^2}M_{\rm cutoff}^4\\ &\times\left(1+\left(\frac{m}{M_{\rm cutoff}}\right)^2-\frac{1}{4}\frac{m^4}{M_{\rm cutoff}^4}\left[\log\left(\frac{M_{\rm cutoff}^2}{m^2}\right)+\frac{1}{8}-\frac{1}{2}\log 2\right]+\mathcal{O}\left(M_{\rm cutoff}^{-1}\right)\right)\end{aligned} \tag{2.1.1}$$

Note that the fermion generates the negative energy density while the boson generates positive energy density. This implies that we may obtain the vanishing vacuum energy when SUSY is unbroken. To see this, we introduce supersymmetry generators to satisfy the following algebra:

$$\left\{Q_\alpha, \overline{Q}_{\dot\beta}\right\} = (\sigma_\mu)_{\alpha\dot\beta}\,P^\mu\,, \tag{2.1.2}$$

where the indices $(\alpha, \beta, \cdots, \dot\alpha, \dot\beta, \cdots)$ express the two component of Weyl spinor, $\sigma_i (i = 1, 2, 3)$ represents the Pauli matrices in addition to $\sigma_0 = I$, and P^μ is a four momenta.

When a theory is in a completely supersymmetric state, the vacuum state $|0\rangle$ satisfies,

$$Q_\alpha |0\rangle = \overline{Q}_{\dot\alpha} |0\rangle = 0 \quad \forall\ \alpha, \dot\alpha \,. \tag{2.1.3}$$

Therefore, for such a vacuum expectation value, the Hamiltonian vanishes as

$$\langle 0|H|0\rangle = \langle 0|P^0|0\rangle = 0\,. \tag{2.1.4}$$

This can be also applied to all the quantum correction since the SUSY enable us to cancel the contribution of boson loops by its super partner (fermion) loops. The above result can also be obtained in a different way. In the Wess-Zumino model which consists only of spin-0 and spin 1/2 particles have a special and simple form of the potential as,

$$V(\phi, \phi^*) = \sum_i \left| \frac{\partial W(\phi)}{\partial \phi_i} \right|^2 , \tag{2.1.5}$$

where ϕ is complex scalar field and W is so-called superpotential. In a completely supersymmetric condition, we impose

$$\frac{\partial W(\phi)}{\partial \phi_i} = 0\,. \tag{2.1.6}$$

From the above condition, we obtain $V(\phi, \phi^*) = 0$, which implies the cancellation of the vacuum energy. However, the SUSY is broken in our universe. If the SUSY is broken at some energy scale, the vacuum energy is given by

$$\langle \rho_{\rm vac} \rangle \sim \int_0^{M_{\rm cutoff}} \frac{d^3k}{(2\pi)^3} \left(\sqrt{\mathbf{k}^2 + m_{\rm boson}^2} - \sqrt{\mathbf{k}^2 + m_{\rm fermion}^2} \right) \sim g^2 M_{\rm cutoff}^2 \Lambda_{\rm SUSY}^2\,. \tag{2.1.7}$$

Here, we introduced SUSY breaking scale as $g^2\Lambda_{\rm SUSY}^2 = \left| m_{\rm boson}^2 - m_{\rm fermion}^2 \right|$, where g is the coupling of SUSY breaking. We assumed that the SUSY breaking scale is much smaller than the cutoff scale $\Lambda_{\rm SUSY} \ll M_{\rm cutoff}$. Although the quartic divergence is cancelled, the quadratic divergence remains in the above expression. From the fact that the absence of SUSY is below the TeV scale [85], we may put $g^2\Lambda_{\rm SUSY}^2 \geq ({\rm TeV})^2$. Therefore, Eq. (2.1.7) can be written as

$$\langle \rho_{\rm vac} \rangle \sim g^2 \Lambda_{\rm SUSY}^2 M_{\rm cutoff}^2 \sim g^2 \Lambda_{\rm SUSY}^4 \sim 10^{12} ({\rm GeV})^4 \sim 10^{60} \rho_\Lambda^{(\rm obs)} , \tag{2.1.8}$$

where we assumed $M_{\rm cutoff} \sim \Lambda_{\rm SUSY}$. This value is still much larger than the observed value of the cosmological constant. Therefore, we conclude that SUSY is not enough to solve the cosmological constant problem.

2.2 Scale invariance

As we discussed in section 1.6, our universe is not scale invariant. However, our universe is almost scale invariant except for the Higgs mass term and UV cutoff within the scope of the standard model. Therefore, if the origin of the Higgs mass is the spontaneous symmetry breaking of the scale invariance at the UV physics, one may allow the theory to be scale invariant at the high energy scale. It is easy to show that why the scale invariance may solve the cosmological constant problem. The scale invariance is the symmetry of the following scale transformation,

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu} \,. \tag{2.2.1}$$

The Einstein tensor $G_{\mu\nu}$ is invariant under the above transformation while the cosmological constant term is not invariant. Therefore, the cosmological constant does not allow to affect the Einstein equation when the scale invariance is realized. It seems that the mass terms in the energy-momentum tensor in the four dimensional space-time theory which includes the standard model particles are also forbidden. However, we may obtain non-zero massive particles when the scale invariance is spontaneously broken [86]. It is obtained if a scalar field ψ transforms as $\psi \to \psi + f \ln \Omega$, where f is a constant mass term. This leaves kinetic term invariant. The energy-momentum tensor with potential term is written by

$$V(\psi, \phi_i) g_{\mu\nu} = U(\phi_i) e^{-2\psi/f} g_{\mu\nu} \,, \tag{2.2.2}$$

where ϕ_i are other scalar fields which do not have the scale invariance. The above term can be surely scale invariant. When we take $U(\phi_i) = 1/2 m^2 \phi_i^2$, scalar fields ϕ_i obtain mass while ψ is massless. This is possible since the potential vanishes at this minima, regardless of the value of ψ in the vacuum state, and it is consistent with the coexistence of massive and massless fields. This is because that if we assume that the potential is minimized at $\phi_i = 0$ and $\psi = \psi_0$, the potential is minimized for an arbitrary ψ_0. To minimize the potential, however, we have to continue to take $\psi_0 \to \infty$. In this limit, all dimensional parameters involving mass term should vanish.

Finally, we obtain vacuum with one parameter minima $V(\psi, \phi_i)$ which does not correspond to our universe. When we assume that the scale invariance is preserved except for massive particles in the standard model during the two phase transition, we can keep the term for the cosmological constant prohibited and break the scale invariance in the standard model sector at the early time of the universe.

From the perspective of Weinberg's no-go theorem, we may see how the initial scale invariance affects the no-go result. When we consider the Lagrangian $\mathcal{L}\left(g_{\mu\nu}, \psi_i, \varphi_n\right)$ with the scale invariant sector ψ_i and other sectors ϕ_n, the relation in Eq. (1.6.9) is modified as

$$\delta g_{\mu\nu} = 2\epsilon g_{\mu\nu}, \quad \delta\phi_n = -\epsilon f_n(\phi_m), \quad \delta\psi_i = -\epsilon\psi_i . \tag{2.2.3}$$

We may construct the scale invariant metric $\bar{g}_{\mu\nu}$ as

$$\bar{g}_{\mu\nu} = \left(\sum_{i,j} M_{ij}\psi_i\psi_j\right) g_{\mu\nu} \equiv \Omega^2 g_{\mu\nu}, \tag{2.2.4}$$

where we defined Ω as $\Omega^2 \equiv \sum_{i,j} M_{ij}\psi_i\psi_j$ with matrix M_{ij}. Then, the scale invariant Lagrangian with constant fields is written as

$$\mathcal{L}(g_{\mu\nu}, \psi_i, \phi_n) = \mathcal{L}(\bar{g}_{\mu\nu}, \phi_n) . \tag{2.2.5}$$

The relation in Eq. (1.6.8) becomes

$$2\bar{g}_{\mu\nu}\frac{\partial\mathcal{L}(\bar{g}_{\mu\nu}, \phi_n)}{\partial\bar{g}_{\mu\nu}} = \sum_n f_n(\phi_m)\frac{\partial\mathcal{L}(\bar{g}_{\mu\nu}, \phi_n)}{\partial\phi_n}, \tag{2.2.6}$$

Similarly, from the GL(4) symmetry, the Lagrangian is written as

$$\mathcal{L} = \sqrt{-\bar{g}}\bar{\mathcal{L}}_0(\phi_n) = \sqrt{-g}\Omega^4\bar{\mathcal{L}}_0(\phi_n) \equiv \sqrt{-g}\mathcal{L}_0(\psi_i, \phi_n) . \tag{2.2.7}$$

Here, we defined $\mathcal{L}_0(\psi_i, \phi_n) \equiv \Omega^4\bar{\mathcal{L}}_0(\phi_n)$. For simplicity, we assume a fully scale invariant theory. Then, we may take $\varphi_n = 0$ and the Lagrangian reduces to

$$\mathcal{L} = \sqrt{-g}\mathcal{L}_0(\psi_i, \phi_n = 0) \equiv \sqrt{-g}\mathcal{L}_0(\psi_i) . \tag{2.2.8}$$

When a kinetic term is the second order derivative, the potential term satisfies the following relation in four dimensional space-time classicaly:

$$\sum_i \psi_i\frac{\partial}{\partial\psi_i}V(\psi) = 4V(\psi) . \tag{2.2.9}$$

Since at the late time, matter field equations satisfy

$$\frac{\partial\mathcal{L}_0}{\partial\psi_i} = \frac{\partial V(\psi)}{\partial\psi_i} = 0, \tag{2.2.10}$$

where $V(\psi)$ is the scale invariant potential. Therefore, the gravitational field equation $\frac{\partial\mathcal{L}_0}{\partial g_{\mu\nu}} = 0$ requires

$$V(\psi) = 0 . \tag{2.2.11}$$

It seems to correspond to the fine-tuning as we have seen in the former case in Eq. (1.6.17). However, from Eq. (2.2.10) the stationary condition of the potential implies vanishing of the potential. Therefore, from the matter field equation, we obtain the condition $V(\psi) = 0$ automatically. In other words, we do not need to impose the condition in Eq. (2.2.11) by hand and the problem of the fine-tuning is solved when the theory is scale invariant classically. However, this vanishing property of the potential is spoiled by quantum corrections.

$$\begin{aligned} &V(\psi_0)|_{\text{tree}} = 0, \qquad \left.\frac{\partial V(\psi)}{\partial \psi}\right|_{\psi=\psi_0} = 0 \\ &V(\psi_0)|_{1-\text{loop}} \neq 0 \end{aligned} \tag{2.2.12}$$

2.3 Unimodular gravity

Unimodular gravity is firstly proposed by Einstein in 1919 [88] and it has been discussed for a long time [89–97]. Unimodular gravity is obtained from a restricted variation of the Einstein-Hilbert action, where the condition $\sqrt{-g} = 1$ is imposed in the action. Then, the field equations correspond to the traceless Einstein equations, and it can easily be shown to be equivalent to the gauged Einstein equations with a cosmological constant, Λ appearing as an integration constant. Unimodular gravity is described as the restricted Einstein gravity:

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2} R + \mathcal{L}_{\text{matter}} \right) , \tag{2.3.1}$$

where the metric is restricted by the unimodular condition,

$$\det g_{\mu\nu} = -1 \quad \text{or} \quad \sqrt{-g} = 1 \, . \tag{2.3.2}$$

Note that this condition breaks the diffeomorphism invariance. Generally, we may extend the unimodular condition by using a fixed density function ϵ_0 as

$$\sqrt{-g} = \epsilon_0 \, . \tag{2.3.3}$$

Then the action of unimodular gravity is written by

$$S_{\text{unimodular}} = \epsilon_0 \int d^4x \left(\frac{1}{2\kappa^2} R + \mathcal{L}_{\text{matter}} \right) . \tag{2.3.4}$$

When we divide the matter Lagrangian into vacuum energy coming from the matter and the other part as

$$\mathcal{L}_{\text{matter}} = \Lambda + \mathcal{L}^{(0)}_{\text{matter}} \, , \tag{2.3.5}$$

the vacuum energy does not affect the dynamics since it is just a constant term in the action. Therefore, the vacuum energy does not gravitate as the cosmological constant in the unimodular gravity. Neglecting the matter contribution, the variation of the action with respect to $g_{\mu\nu}$ yields

$$g^{\mu\nu}\delta R_{\mu\nu} + \delta g^{\mu\nu} R_{\mu\nu} = 0 \tag{2.3.6}$$

The first term corresponds to the total derivative term which we see in general relativity. Therefore, the equation of motion is obtained from the second term. However, we cannot put $R_{\mu\nu} = 0$ since $\delta g^{\mu\nu}$ is not fully independent. Because of the unimodular condition Eq. (2.3.2) or (2.3.3), the trace of $\delta g^{\mu\nu}$ vanishes. It seems that unimodular gravity is different from Einstein theory since the information of the trace part is lost. We will see, however, the equations in the unimodular gravity are not changed from those of the Einstein theory including the cosmological constant. We can only conclude that only traceless part of $R_{\mu\nu}\delta g^{\mu\nu} = 0$ vanishes. Therefore, we obtain

$$R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R = 0\,, \tag{2.3.7}$$

in four dimensional space-time. When we consider the matter sector, the energy-momentum tensor is modified as

$$T_{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_{\mathrm{matter}}\right)}{\delta g_{\mu\nu}} \to T_{\mu\nu} \equiv \frac{\delta \mathcal{L}_{\mathrm{matter}}}{\delta g_{\mu\nu}} \tag{2.3.8}$$

Similarly, the traceless part of the energy-momentum tensor vanishes by the variation of the metric under the unimodular condition. Thus, by variation of the action Eq. (2.3.4) with respect to $g_{\mu\nu}$, we obtain the equation of motion as,

$$R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R = 2\kappa^2 \left(T_{\mu\nu} - \frac{1}{4} g_{\mu\nu} T \right)\,. \tag{2.3.9}$$

From the Bianchi identity and conservation law of the energy-momentum tensor, the divergence of the above expression yields

$$\nabla_\nu \left(R + 2\kappa^2 T \right) = 0\,. \tag{2.3.10}$$

The above equation allows us to introduce the integration constant C as

$$R + 2\kappa^2 T = -4C\,. \tag{2.3.11}$$

Finally, we have the equation of motion as follows:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - C g_{\mu\nu} = \kappa^2 T_{\mu\nu}\,. \tag{2.3.12}$$

The third term is identical with that coming from the cosmological constant term. Traceless Einstein equation generates the cosmological constant. The reason why this strange correspondence occurs is that the Einstein gravity is originally gauge theory. Since the Einstein gravity is originally redundant, all the equations of motion are not independent. Note that the origin of the cosmological constant is completely different as we saw in the previous chapter. Although the vacuum energy does not gravitate as the cosmological constant in unimodular gravity, the integration constant appears as the cosmological constant.

Chapter 3

Generalization of unimodular gravity towards the cosmological constant problem

In this chapter, we discuss details of unimodular gravity and its first step of the application to the cosmological constant problem. In the first section, we generalize classical unimodular gravity to the fully diffeomorphism invariant theory. In the second section, we discuss the quantum unimodular gravity.

3.1 Reformulation of classical unimodular gravity

We see the unimodular condition in Eq. (2.3.2). In order to understand the gauge symmetry of unimodular gravity, we compare the diffeomorphism transformation of general relativity and unimodular gravity. In general relativity, theory is diffeomorphism invariant under the infinitesimal transformation

$$\delta g_{\mu\nu}(x) = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \tag{3.1.1}$$

On the other hand, in umimodular gravity, unimodular condition $\sqrt{-g} = \epsilon_0$ yields

$$\frac{\delta}{\delta g_{\mu\nu}} \sqrt{-g} = 0\,. \tag{3.1.2}$$

Therefore, we obtain

$$\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} = \nabla_\mu \xi^\mu = 0\,. \tag{3.1.3}$$

This is called transverse differomorphism or volume preserving diffeomorphism. At least locally, we can easily see that the classical equivalence of the classical general relativity and unimodular gravity since we can always choose the coordinate as unimodular in general relativity locally. In global cases, when the coordinate is fixed at the boundary or there are no boundary, global average

$$\langle\sqrt{-g}\rangle \equiv \frac{\int \sqrt{-g} d^4x}{\int d^4x} \tag{3.1.4}$$

is gauge invariant and it does not change $\langle\sqrt{-g}\rangle$. Thus, we may fix coordinate globally as $\langle\sqrt{-g}\rangle = 1$ or ϵ_0 .

By introducing the Lagrange multiplier field $\lambda(x)$, we may rewrite the action of unimodular gravity in Eq. (2.3.4) as [117]

$$S = \int d^4x \left[\sqrt{-g}\frac{R}{2\kappa^2} - \lambda(x)\left(\sqrt{-g} - \epsilon_0\right) + \sqrt{-g}\mathcal{L}_{\rm m}\right] . \tag{3.1.5}$$

It is useful to formulate unimodular gravity by using the Lagrange multiplier field. By the variation of the above action with respect to $g_{\mu\nu}$ and λ, we obtain the following equation as

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \kappa^2 T_{\mu\nu} - \frac{\lambda(x)}{2}g_{\mu\nu} \\ \sqrt{-g} &= \epsilon_0 \, , \end{aligned} \tag{3.1.6}$$

When we take the covariant derivative ∇^μ to the first equation, we obtain $\partial^\mu\lambda = 0$ by using the Bianchi identity and energy-momentum conservation. Therefore, we can conclude that the Lagrange multiplier is constant $\lambda(x) = \lambda_0$. We obtain Einstein equation with the cosmological constant from unimodular gravity again. Note that the equation of motion is unchanged by the shift of λ and matter Lagrangian $\lambda \to \lambda - \Lambda$ and $\mathcal{L}_{\rm m} \to \mathcal{L}_{\rm m}^{(0)} - \Lambda$. After this transformation, the action becomes

$$S' = \int d^4x \left[\sqrt{-g}\frac{R}{2\kappa^2} - \lambda(x)\left(\sqrt{-g} - \epsilon_0\right) + \sqrt{-g}\mathcal{L}_{\rm m}^{(0)}\right] - \epsilon_0\Lambda\int d^4x \, . \tag{3.1.7}$$

Since the last term including Λ remains just constant in the action, it does not affect to the dynamics. This means that we may cancel the huge vacuum energy contribution from matter by the redefinition of the Lagrange multiplier field. This is an important clue to tackle with the cosmological constant problem.

3.1.1 Fully difeomorphism invariant formulation of unimodular gravity

In this subsection, we discuss so-called Henneaux-Teitelboim formulation of unimodular gravity [94]. In this theory, the action has fully unconstrained gauge symmetry keeping fully diffeomorphism invariance and this is also equivalent to general relativity classically. Firstly, we use the Stückelberg trick to restore the diffeomorphism invariance. To do this, we introduce the Stückelberg fields $\phi^\alpha(x)$ and performing coordinate transformation $x^\alpha \to \phi^\alpha(x)$, the action in Eq. (3.1.5) except for the matter sector becomes

$$S_{\text{Stuck}} = \int d^4x \left[\frac{1}{2\kappa^2}\sqrt{-g}R - \lambda\left(\sqrt{-g} - \epsilon_0 |S^\alpha{}_\beta|\right)\right] , \tag{3.1.8}$$

where $S^\alpha{}_\beta$ is Jacobian matrix defined by

$$S^\alpha{}_\beta \equiv \frac{\partial \phi^\alpha(x)}{\partial x^\beta} . \tag{3.1.9}$$

The determinant of $S^\alpha{}_\beta$ is given by $|S^\alpha{}_\beta| = 4!\delta^{[\alpha}_\mu \delta^\beta_\nu \delta^\gamma_\rho \delta^{\delta]}_\sigma S^\mu{}_\alpha S^\nu{}_\beta S^\rho{}_\gamma S^\sigma{}_\delta$. This action is invariant under the diffeomorphism transformation $x^\mu \to x'^\mu(x^\nu)$ if the Stückelberg fields $\phi^\alpha(x)$ obey the law of scalar field transformation $\phi^\alpha(x) \to \Phi^\alpha(\phi(x'))$. In fact, it is easy to see that the determinant of the Jacobi matrix of that transformation is one:

$$\det S \equiv \det \begin{pmatrix} \frac{\partial\Phi^\alpha}{\partial\phi^\beta} & \frac{\partial\Phi^\alpha}{\partial x} \\ \frac{\partial x'^\alpha}{\partial\phi^\beta} & \frac{\partial x'^\alpha}{\partial x^\beta} \end{pmatrix} = 1 , \tag{3.1.10}$$

where we used $\det\left(\frac{\partial\Phi^\alpha}{\partial\phi^\beta}\right) = 1$. Note that $|S^\alpha{}_\beta|$ can also be rewritten by

$$|S^\alpha{}_\beta| = \partial_\alpha\left[4!\delta^{[\alpha}_\mu \delta^\beta_\nu \delta^\gamma_\rho \delta^{\delta]}_\sigma \phi^\mu S^\nu{}_\beta S^\rho{}_\gamma S^\sigma{}_\delta\right] . \tag{3.1.11}$$

When we chose $4!\delta^{[\alpha}_\mu \delta^\beta_\nu \delta^\gamma_\rho \delta^{\delta]}_\sigma \phi^\mu S^\nu{}_\beta S^\rho{}_\gamma S^\sigma{}_\delta$ as a vector density τ^α, the above action corresponds to the so-called Henneaux-Teitelboim action [94]:

$$S_{\text{H}-\text{T}} = \int d^4x \left[\sqrt{-g}\frac{R}{2\kappa^2} - \lambda\left(\sqrt{-g} - \partial_\mu \tau^\mu\right)\right] . \tag{3.1.12}$$

In other words, the action S_{Stuck} is the special case of $S_{\text{H}-\text{T}}$. By variation of $S_{\text{H}-\text{T}}$ with respect to $g_{\mu\nu}$, and taking the divergence of both sides, we obtain the following equation from Eq. (3.1.6):

$$\partial_\mu \lambda = 0 , \tag{3.1.13}$$

where we also used the Bianchi identity and energy-momentum conservation law. On the other hand, the equation of motion from the variation of λ yields

$$\sqrt{-g} = \partial_\mu \tau^\mu \,. \tag{3.1.14}$$

This is a fully diffeomorphism invariant expression of unimodular condition. Note that this model can also absorb the vacuum energy contribution from matter by the redefinition of the Lagrange multiplier field $\lambda \to \lambda - \Lambda$. Similarly, when we assume that $\mathcal{L}_{\rm m} \to \mathcal{L}_{\rm m}^{(0)} - \Lambda$, the action in Eq. (3.1.12) with matter becomes,

$$S'_{\rm H-T} = \int d^4x \left[\sqrt{-g}\frac{R}{2\kappa^2} - \lambda\left(\sqrt{-g} - \partial_\mu\tau^\mu\right) + \sqrt{-g}\mathcal{L}_{\rm m}^{(0)}\right] - \Lambda \int d^4x \partial_\mu \tau^\mu \,. \tag{3.1.15}$$

In the above expression, the last term only includes Λ. However, the last term is total derivative term and it does not affect the dynamics. Thus, we may cancel the vacuum energy contribution from matter by the redefinition of λ again.
Furthermore, we may generalize the above action [97]:

$$S_{\rm GH-T} = \int d^4x \sqrt{-g}\left[\frac{R}{2\kappa^2} - \lambda f\left(\phi\right) - g\left(\phi\right) + \mathcal{L}_{\rm m}\right] , \tag{3.1.16}$$

where we introduced ϕ as

$$\phi \equiv \frac{\partial_\mu \tau^\mu}{\sqrt{-g}} \,. \tag{3.1.17}$$

Equations of motion become

$$\begin{aligned} & G_{\mu\nu} = 8\pi G \left[T_{\mu\nu} + g_{\mu\nu}\left(\lambda F(\phi) + G(\phi)\right)\right] \\ & f(\phi) = 0 \\ & \partial_\alpha \left[\lambda f'(\phi) + g'(\phi)\right] = 0 \,, \end{aligned} \tag{3.1.18}$$

where $F \equiv \phi f'(\phi) - f(\phi)$ and $G \equiv \phi g'(\phi) - g(\phi)$. The latter two equations represent constraint equations. These imply there are constant solution for λ and ϕ as, $\lambda = \lambda_0$= constant and $\phi = \phi_0$= constant. Substituting these into the first equation, we find that there is a term $g_{\mu\nu}\left(\lambda F(\phi) + G(\phi)\right) \sim g_{\mu\nu} \times$ constant which plays a role of the cosmological constant as long as they do not be applied to the following properties.

1. f does not have real zeros.
2. f and f' have the same isolated zeros at the same time.
3. g is linear and f is identically zero.

Avoiding the first condition means that the second equation in Eq. (3.1.18) has a non-trivial solution for ϕ. For the second condition, if it is not valid, we obtain non-zero $\lambda F(\phi)$. If we avoid the third condition, we obtain non-trivial constraint equation from the third equation in Eq. (3.1.18). Therefore, avoiding the above condition enable us to identify the action in Eq. (3.1.21) is equivalent to general relativity with the cosmological constant. However, note that the equations of motion change if we split matter sector and redefinition of the Lagrange multiplier in general:

$$\mathcal{L}_{\mathrm{m}} \to \mathcal{L}_{\mathrm{m}}^{(0)} - \Lambda, \quad \lambda \to \lambda - \Lambda\,. \tag{3.1.19}$$

Contrary to this, we may extend the action $S_{\mathrm{H-T}}$ keeping the equation of motion unchanged by the above transformation in a simple way. When we replace partial derivative ∂_μ to covariant derivative ∇_μ and vector density τ^μ to vector field valuable $\sqrt{-g}V^\mu$ respectively, the action becomes

$$S_{\mathrm{GUMG}} = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \lambda\left(1 - \nabla_\mu V^\mu\right)\right]. \tag{3.1.20}$$

The constraint equation is changed as

$$\partial_\mu \tau^\mu = \sqrt{-g} \to \nabla_\mu V^\mu = 1\,. \tag{3.1.21}$$

Similarly, when we include the matter sector, the action is unchanged by Eq. (3.1.19) up to total derivative term. Therefore, the equation of motion is unchanged.

3.2 Quantum unimodular gravity

In this section, we discuss a possibility that classical unimodular gravity can be extended to quantum theory keeping the equivalence to general relativity. To consider quantum unimodular gravity in a method of path integral, we first consider the following generating function:

$$Z[J] = \int \mathcal{D}g_{\mu\nu}\mathcal{D}\lambda\mathcal{D}V^\mu \exp\left[iS_{\mathrm{GUMG}} + iS_{\mathrm{B}} + iS_{\mathrm{source}}[g, J]\right], \tag{3.2.1}$$

where $S_{\mathrm{source}}[g, J]$ denotes external source and S_{B} is boundary term expressed as

$$S_{\mathrm{B}} = \int d^3x\sqrt{-\gamma}\left[\frac{K}{\kappa^2} - \lambda n_\mu V^\mu\right]. \tag{3.2.2}$$

Here, γ is three dimensional induced metric on the boundary, n^μ is an outgoing normal vector orthogonal to the boundary and K is trace of the extrinsic curvature $K_{\mu\nu}$. We consider an effective theory in quantum unimodular gravity. In the above generating function, we impose the following three assumptions,

1. We only consider low energy contribution. In this effective theory, we only take the leading order contribution from the heavy particle modes to low energy where the diffeomorphism invariance is established.
2. Only metric couples to external source. The other quantities e.g., V^μ, λ do not couple to external source.
3. We assume that the generating function includes boundary term in Eq. (3.2.2).

Since a partial integration of V^μ yields the constraint $\delta_\mu \lambda = 0$, the generating function can be rewritten by

$$Z[J] = \int \mathcal{D}g_{\mu\nu}\mathcal{D}\lambda\delta[\delta_\mu\lambda]\exp\left[iS'_{\rm GUMG} + iS'_{\rm B} + iS_{\rm source}[g, J]\right] ,$$
$$S'_{\rm GUMG} + S'_{\rm B} = \int_V d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \lambda(x)\right] + \frac{1}{\kappa^2}\int_{\partial V} d^3x\sqrt{-\gamma}K\,. \tag{3.2.3}$$

When we impose the boundary condition $\delta_\mu\lambda|_{\partial V} = 0$, the integration of the generating function with respect to λ can be performed as

$$Z[J] = \int \mathcal{D}g_{\mu\nu}\exp\left[iS_{\rm GR}[J, \lambda] + iS_{\rm B} + iS_{\rm source}[g, J]\right] ,$$
$$S_{\rm GR} + S_{\rm B} = \int_V d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \lambda_0\right] + \frac{1}{\kappa^2}\int_{\partial V} d^3x\sqrt{-\gamma}K\,. \tag{3.2.4}$$

Here, we assume that λ_0 is the fixed value of λ at the boundary. The above action corresponds to general relativity with the cosmological constant and boundary term. In the above analysis, we only consider the fixed boundary. Thus, we consider the case where the λ is not fixed at the boundary from now on. Naively, if the boundary is not fixed, the integration about the delta function cannot be performed as we did in Eq. (3.2.3) to Eq. (3.2.4). Therefore, we alternatively choose that the λ is a constant $\lambda = \lambda_0$ in space-time integration. Then, the generating function becomes

$$Z[J] = \int \mathcal{D}g_{\mu\nu}d\lambda_0 M(\lambda_0)\exp\left[iS_{\rm GR}[J, \lambda] + iS_{\rm B} + iS_{\rm source}[g, J]\right] . \tag{3.2.5}$$

Here, we introduced $M(\lambda_0)$ as a contribution to measure. This is related to that how to impose the constraint of the unimodular gravity locally or globally in a classical analysis as we saw in the previous section. Locally, unimodular gravity is equivalent to general relativity with the cosmological constant. On the other hand, this cosmological constant is global parameter yielding global constraint. This implies that whether it is possible or not to impose the global gauge fixing in unimodular gravity in the previous section.

This equivalence is, however, break if we assume that the λ and V^μ are couple to the external source. We regard λ and V^μ are auxiliary fields which do not affect to the asymptotic states. Therefore, they only appear in internal lines in the Feynman diagrams. We assume that these fields can be integrated out in the path integral and it realize general relativity when we impose an appropriate boundary condition. On the other hand, if we allow them to couple with external source, they appear in the external lines in the Feynman diagrams. The coupling of an external line with λ and V^{μ} means, however, that one break the equivalence in quantum theory by hand.

Chapter 4

BRS structure of the simple model for the cosmological constant problem

In this chapter, we discuss a possibility that one of the model related to unimodular gravity may solve the cosmological constant problem. Inspired by unimodular gravity, a simple model has been proposed for solving the cosmological constant problem [104]. This model can be regarded as a kind of topological field theory [107]. In topological field theory, the Lagrangian is described as BRS exact and it has BRS symmetry. Especially, the model in [104] has an infinite number of BRS symmetry. The BRS symmetry in this model should be, however, broken spontaneously to obtain the non-zero cosmological constant. We investigate the details of this BRS symmetry and show that there remain one and only one BRS symmetry which survive keeping the cosmological constant non-zero. The cosmological constant problem arises from the quantum theory. In this analysis, the cosmological constant problem as a problem of a quantum theory is replaced to the initial condition of a dynamical cosmological constant as a problem of classical theory. We investigate the cosmological evolution based on this model and specify an appropriate region of the initial condition which reproduce the cosmic history. Furthermore, we will show that this model has a stable solution which describe the de-Sitter solution.

4.1 Construction of the model

As we see in the previous section, a redefinition of Lagrange multiplier in unimodular gravity enables us to absorb the huge quantum correction. The action is given by [104]

$$S = \int \sqrt{-g} d^4x \left[\mathcal{L}_{\text{gravity}} - \lambda \left(1 - \nabla_\mu J^\mu \right) \right] + \int d^4x \sqrt{-g} \mathcal{L}_{\text{matter}} \,, \tag{4.1.1}$$

where J^μ is a vector. Similarly, when we may split $\mathcal{L}_{\rm matter}$ into vacuum energy and other parts as $\mathcal{L}^{(0)}_{\rm matter} \to \mathcal{L}^{(0)}_{\rm matter} - \Lambda$. A redefinition of $\lambda \to \lambda - \Lambda$ changes the action as

$$S = \int d^4x \sqrt{-g}\left[\mathcal{L}_{\rm gravity} - \lambda\left(1 - \nabla_\mu J^\mu\right)\right] + \int d^4x \sqrt{-g}\mathcal{L}^{(0)}_{\rm matter} - \Lambda \int d^4x \sqrt{-g} \nabla_\mu J^\mu \,, \tag{4.1.2}$$

where we assumed that the $\mathcal{L}_{\rm gravity}$ does not include vacuum energy. The Λ are included only in the total derivative term. Thus, the quantum correction does not affect the dynamics and we can ignore it. We may choose the term $\nabla_\mu J^\mu$ in a various quantities. For example, we may choose this term as a topological invariant quantity like the Gauss-Bonnet invariant:

$$\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \tag{4.1.3}$$

or Instanton density

$$I \equiv \frac{1}{2}{\rm tr}F_{\mu\nu}{}^{*}F^{\mu\nu} \,, \tag{4.1.4}$$

where${}^{*}F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}$ is the dual field strength, and $\epsilon^{\mu\nu\rho\sigma}$ is Levi-Civita tensor. In addition to these topological invariant terms, we may choose that J^μ as a derivative of a scalar field $-\partial^\mu\varphi$. For simplicity, we especially consider a real scalar field. Then, the Lagrangian except for $\mathcal{L}_{\rm gravity}$ and $\mathcal{L}_{\rm matter}$ is written as

$$\mathcal{L}' \equiv -\lambda\left(1 + \nabla_\mu\partial^\mu\varphi\right) = \partial_\mu\lambda\partial^\mu\varphi - \lambda \,. \tag{4.1.5}$$

Note that the first term includes indefinite metric. An indefinite metric generates problems both classical and quantum theory. Classically, when we consider the Hamiltonian mechanics in this model, we see that the Hamiltonian is unbounded and the theory is unstable in general. On the other hand, when we consider quantum theory, this Lagrangian includes negative norm and unitarity is broken. To see this, we decompose λ and φ by using other real scalar fields ξ and η as

$$\lambda = \frac{1}{\sqrt{2}}(\eta + \xi), \quad \varphi = \frac{1}{\sqrt{2}}(\eta - \xi) \,, \tag{4.1.6}$$

the Lagrangian becomes

$$\mathcal{L}' = -\frac{1}{2}\partial_\mu\xi\partial^\mu\xi + \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{1}{\sqrt{2}}(\eta + \xi) \,. \tag{4.1.7}$$

Note that the kinetic term is different. This implies that negative kinetic term appears inevitably. A negative kinetic term implies that the Hamiltonian includes the term $H \propto -p^2$.

If we do not impose the upper bound of the momentum, the Hamiltonian is unbounded and it is unstable. Moreover, the negative kinetic term generates negative norm in quantum field theory. If a theory has a negative norm, the unitarity of S-matrix is broken [106]. We may remove this negative norm by introducing Faddeev-Popov (FP) ghost c and anti-ghost b [98–102, 105, 106]. Then, we may rewrite the Lagrangian as

$$\mathcal{L}'' = -\lambda\left(1 + \nabla_\mu \partial^\mu \varphi\right) - \partial_\mu b \partial^\mu c = \partial_\mu \lambda \partial^\mu \varphi - \lambda - \partial_\mu b \partial^\mu c \,. \tag{4.1.8}$$

When we introduce a fermionic parameter ϵ, the above Lagrangian is invariant under the following BRS transformation [102]

$$\delta_B \lambda = \delta_B c = 0, \quad \delta_B \varphi = \epsilon c, \quad \delta_B b = \epsilon \lambda \,. \tag{4.1.9}$$

Note that the BRS transformation satisfy the nilpotent condition $\delta_{\mathrm{B}}^2 = 0$. When we define the physical states as the states BRS invariant state, the negative norm states could be removed consistently by the following subsidiary condition [105, 106]:

$$Q_{\mathrm{B}} \left|\mathrm{phys}\right\rangle = 0 \,. \tag{4.1.10}$$

Here, $|\mathrm{phys}\rangle$ is physical subspace and Q_{B} is BRS charge expressed by

$$Q_B = \int d^4x \sqrt{-g}\left[(\partial^0 \lambda) c + \lambda (\partial^0 c)\right] \,, \tag{4.1.11}$$

in this model. Furthermore, when we assign the ghost numbers as

$$\begin{cases} c = 1 \\ b, \epsilon = -1, \end{cases}$$

the ghost number is conserved and we find that λ, φ, b and c are belong to the Kugo-Ojima's BRS-quartet. Then, λ and φ are identified as scalar and longitudinal mode respectively. Furthermore, this Lagrangian can be regarded as a kind of topological field theory [107]. In a topological field theory, the Lagrangian is described as BRS exact so that the Lagrangian is obtained by BRS transformation of a some quantity. Therefore, from the condition of nilpotent of the BRS transformation $\delta_{\mathrm{B}}^2 = 0$, BRS invariance is trivial in a topological field theory. Moreover, since the Lagrangian is BRS exact, the Lagrangian only consists of the BRS quartet so that the Lagrangian has only gauge degrees of freedom. Then, we consider the Lagrangian is described as the (gauge) scalar field φ and it is trivial $\mathcal{L}_\varphi = 0$. In this case, we impose the following gauge fixing condition as

$$F_{\mathrm{GF}} \equiv 1 + \nabla_\mu \partial^\mu \varphi = 0 \,. \tag{4.1.12}$$

When we perform the BRS transformation in Eq. (4.1.9) of the above quantity, we find that it corresponds to the Lagrangian $\mathcal{L}''$ in Eq. (4.1.8) up to total derivative term:

$$\begin{aligned}\delta_B\left(-bF_{\text{gauge fix}}\right) &= \epsilon\left(\lambda\left(1+\nabla_\mu\partial^\mu\varphi\right) - b\nabla_\mu\partial^\mu c\right) \\ &= \epsilon\left(\mathcal{L}'' + \text{Total derivative term}\right) . \end{aligned} \tag{4.1.13}$$

Therefore, this Lagrangian is surely BRS exact and it can be regarded as a topological field theory. From the BRS transformation Eq. (4.1.9), we identify the λ as Nakanishi-Lautrup field [131–133]. Note that the Nakanishi-Lautrup field is expressed by BRS exact so that the vacuum expectation value is zero $\langle 0|\,\lambda\,|0\rangle = 0$. If $\langle 0|\,\lambda\,|0\rangle \neq 0$, the BRS symmetry is spontaneously broken and unitarity is not guaranteed. However, as we will see in details later, the λ is able to play a role of the cosmological constant. Thus, we should consider an alternative way to construct the BRS transformation. Instead of the BRS transformation in Eq. (4.1.9), we may introduce the following BRS transformation:

$$\delta_B\lambda = \delta_B c = 0, \quad \delta_B\varphi = \epsilon c, \quad \delta_B b = \epsilon\left(\lambda - \lambda_0\right) . \tag{4.1.14}$$

Here, the λ_0 satisfies classical field equation of motion $\nabla_\mu\partial^\mu\lambda = 0$. The Lagrangian $\mathcal{L}''$ is also invariant under the above BRS transformation. Even if we take λ_0 to the cosmological constant Λ_{eff}, $\lambda - \Lambda_{\text{eff}}$ is BRS exact and it does not have non-zero expectation value. Therefore, $\lambda - \Lambda_{\text{eff}}$ does not affect the physical dynamics. If $\lambda - \lambda_0$ has non-zero expectation value, BRS symmetry is spontaneously broken and unitarity is not preserved. Thus, in the real world, we only choose λ as

$$\lambda = \lambda_0 . \tag{4.1.15}$$

Moreover, we should note that it is possible to include the classical fluctuation of λ_0 as long as λ_0 satisfies $\nabla_\mu\partial^\mu\lambda = 0$. On the other hand, the quantum fluctuation is prohibited by the BRS symmetry.

We should also note that the gauge fixing condition in Eq. (4.1.12) has a residual gauge symmetry. In fact the gauge fixing condition in Eq. (4.1.12) has the following gauge symmetry

$$\varphi \to \varphi + \delta\varphi , \tag{4.1.16}$$

where $\delta\varphi$ satisfies $\nabla^\mu\partial_\mu\delta\varphi = 0$. By using this residual gauge symmetry, we may choose the initial condition of φ as $\varphi = \text{constant}$.

4.2 Application to the cosmological evolution

In the previous section, we may remove the quantum correction Λ by the redefinition of the scalar field λ removing a ghost and the quantum aspect of the problem of huge vacuum energy is solved. That is, however, seems not to be solved from the perspective of the cosmological constant problem. We have no prediction about the value of the current value of the cosmological constant from λ (or $\lambda+\Lambda$). Since the λ is dynamical in general, that value can be determined by the initial conditions in classical theory. In other words, the problem of the vacuum energy in quantum theory is replaced to the problem of the identification of the initial condition in classical theory. In this section, we consider the cosmological evolution of this model to investigate the appropriate initial condition which reproduces the observed universe.

4.2.1 Assumptions for the application to the cosmological evolution

We consider the following three assumptions. Firstly, we assume that the gravitational sector is described by the Einstein gravity $\mathcal{L}_{\text{gravity}} = \frac{1}{2\kappa^2}R$. Secondly, we assume that the universe is flat and homogeneous isotropic. Then the metric is given by the FRW:

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3} \left(dx_i^2\right) . \tag{4.2.1}$$

Here, $a(t)$ is scale factor. Finally, we assume that the scalar fields only depend on time:

$$\lambda = \lambda(t), \quad \varphi = \varphi(t) . \tag{4.2.2}$$

In this analysis, we do not include the ghost c and anti-ghost b since they do not affect in the classical field theory by the super selection rule or ghost number conservation. Then, the action is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R + \Lambda - \lambda + \partial_\mu\lambda\partial^\mu\varphi\right] , \tag{4.2.3}$$

By the variation of the above action with respect to λ, we obtain

$$0 = 1 + \left(\frac{d^2\varphi}{dt^2} + 3H\frac{d\varphi}{dt}\right) , \tag{4.2.4}$$

where we defined Hubble parameter $H \equiv \dot{a}/a$. The general solution for φ is given by

$$\varphi(t) = -\int^t \frac{dt_1}{a(t_1)^3}\int^{t_1} dt_2 a(t_2)^3 + \varphi_1\int^t \frac{dt_1}{a(t_1)^3} + \varphi_2 . \tag{4.2.5}$$

Here, φ_1 and φ_2 are some constants. On the other hand, by the variation of φ yields

$$0 = \frac{d^2\lambda}{dt^2} + 3H\frac{d\lambda}{dt} . \tag{4.2.6}$$

The general solution of the above equation is given by

$$\lambda = \lambda_0 + \lambda_1 \int^t \frac{dt_1}{a(t_1)^3}\,, \tag{4.2.7}$$

where λ_0 and λ_1 are some constants. Furthermore, by the variation of the metric, we obtain the first and the second FRW equations as

$$\begin{aligned} \frac{3}{\kappa^2} H^2 &= \Lambda + \lambda - \frac{d\lambda}{dt}\frac{d\varphi}{dt} + \rho_{\rm mat} + \rho_{\rm rad} \\ -\frac{1}{\kappa^2}\left(3H^2 + 2\frac{dH}{dt}\right) &= -\Lambda - \lambda - \frac{d\lambda}{dt}\frac{d\varphi}{dt}\,, \end{aligned} \tag{4.2.8}$$

where $\rho_{\rm mat}$ and $+\rho_{\rm rad}$ are matter density and radiation density respectively. Deleting Λ from the above two equations, we find

$$\frac{1}{\kappa^2}\frac{dH}{dt} = \frac{d\lambda}{dt}\frac{d\varphi}{dt}\,. \tag{4.2.9}$$

Note that when $\lambda = \lambda_1$, H become constant and thus, we have the de-Sitter space-time solution. From Eq. (4.2.8), we obtain the relation between λ_1 and H_0 as

$$\lambda_1 = -\Lambda + \frac{3H_0^2}{\kappa^2}\,. \tag{4.2.10}$$

As we have mentioned, λ (or $\lambda + \Lambda$) can surely play a role of the cosmological constant when λ is constant. In a special case, we have a solution for $\varphi(t)$ from Eq. (4.2.4),

$$\varphi(t) = \frac{t}{3H_0}\,, \tag{4.2.11}$$

in the de-Sitter solution.

4.2.2 Stability analysis

In the previous subsection, we have seen the assumption to the application to the cosmology and equations of motion of the scalar fields which has de-Sitter solution. In this subsection, we investigate the stability of the de-Sitter space-time at linear level. If this model is stable, it surely reproduces the real universe. Furthermore, a stability of the de-Sitter solution implies that the de-Sitter space-time is a kind of attractor. Therefore, we may expect that it may relax the initial condition of the scalar field. We consider the perturbation of the solution of the de-Sitter space-time

$$H = H_0 + \delta H, \quad \lambda = -\Lambda + \frac{3H_0^2}{\kappa^2} + \delta\lambda, \quad \varphi = \frac{t}{3H_0} + \delta\varphi\,. \tag{4.2.12}$$

Substituting these into Eqs. (4.2.4), (4.2.6), and the first equation in Eq. (4.2.8), we obtain

$$\begin{aligned} 0 =&\delta\ddot{\varphi} + 3H_0\delta\dot{\varphi} - \frac{1}{3H_0}\delta H \,, \\ 0 =&\delta\ddot{\lambda} + 3H_0\delta\dot{\lambda} \,, \\ \frac{6}{\kappa^2}H_0\delta H =&\delta\lambda + \frac{1}{3H_0}\delta\dot{\lambda} \,. \end{aligned} \tag{4.2.13}$$

When we delete δH form the last two equations, we have

$$0 = \delta\ddot{\varphi} + 3H_0\delta\dot{\varphi} - \frac{\kappa^2}{18H_0^2}\left(\delta\lambda + \frac{1}{3H_0}\delta\eta\right) \,, \tag{4.2.14}$$

where we have introduced $\delta\eta$ as

$$\delta\eta \equiv \delta\dot{\lambda} \,. \tag{4.2.15}$$

Then, the second equation of (4.2.13) can be rewritten by

$$0 = \delta\dot{\eta} + 3H_0\delta\eta \,. \tag{4.2.16}$$

We express these equations in the following matrix form:

$$\begin{pmatrix} \delta\dot{\lambda} \\ \delta\dot{\eta} \\ \delta\ddot{\varphi} \end{pmatrix} = A \begin{pmatrix} \delta\lambda \\ \delta\eta \\ \delta\dot{\varphi} \end{pmatrix}$$

, where we have defined the matrix A as

$$A \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & \alpha & 0 \\ \beta & \gamma & \alpha \end{pmatrix} \qquad \alpha \equiv -3H_0, \quad \beta \equiv -\frac{\kappa^2}{18H_0^2}, \quad \gamma \equiv -\frac{1}{3}\beta$$

Then, we investigate the evolution of this perturbation. If there is a glowing mode of the perturbation, the system is unstable. We investigate the eigenvalue of the matrix A:

$$0 = \det\left(A - aI\right) \,, \tag{4.2.17}$$

where a is some constant. Then, we obtain

$$0 = \det\begin{pmatrix} -a & 1 & 0 \\ 0 & \alpha - a & 0 \\ \beta & \gamma & \alpha - a \end{pmatrix}$$

$$\iff 0 = -a(\alpha - a)^2$$
$$\therefore a = 0, \alpha \tag{4.2.18}$$

The condition of unstable and stable mode is given by

- Unstable mode : $\alpha > 0$
- Stable mode: $\alpha < 0$

respectively. Therefore, the condition of the stable mode is

$$\alpha = -3H_0 < 0 \Longrightarrow 3H_0 > 0\,, \tag{4.2.19}$$

which always holds in the current universe. Therefore, this model is stable under the linear perturbation and consistent with our universe.

4.2.3 Cosmology in the model

We investigate the time evolution of the scalar fields. After the inflation era, the universe has experienced the radiation-dominated era, matter-dominated era, and dark energy-dominated era. In the first two eras, we impose that the contribution of the scalar fields is neglected for simplicity. On the other hand, in the dark energy-dominated era, we ignore the matter contribution and the universe goes to the de-Sitter space time asymptotically at late time. At each era, the scale factor is expressed by

- Radiation-dominated era

$$a(t) = a_{\rm rad} t^{1/2}\,, \tag{4.2.20}$$

- Matter-dominated era

$$a(t) = a_{\rm mat} t^{2/3}\,, \tag{4.2.21}$$

- Dark energy-dominated era

$$a(t) = a_\Lambda \mathrm{e}^{H_c\sqrt{\Omega_\Lambda}t}\,, \tag{4.2.22}$$

where $a_{\rm rad}$, $a_{\rm mat}$ and a_Λ are constants which depend on the energy density of radiation, matter, and dark energy respectively. We express the current observed value of the Hubble constant as H_0 and dark energy density parameter as Ω_Λ. The scalar fields λ and φ at the radiation, matter, and dark energy dominated era are given by

$$\begin{aligned}
\varphi_{\rm rad}(t) &= \varphi_{\rm rad2} + \varphi_{\rm rad1} t - \frac{5}{4}t^2, \quad \lambda_{\rm rad}(t) = \lambda_{\rm rad\,1} - \frac{2\lambda_{\rm rad\,2}}{a_{\rm rad}^3} t^{-1/2} \\
\varphi_{\rm mat}(t) &= \varphi_{\rm mat2} + \varphi_{\rm mat1} t - \frac{1}{6}t^2, \quad \lambda_{\rm mat}(t) = \lambda_{\rm mat\,1} - \frac{\lambda_{\rm mat\,2}}{a_{\rm mat}^3} t^{-1} \\
\varphi_\Lambda(t) &= \varphi_{\Lambda 2} + \frac{\varphi_{\Lambda 1}}{3H_0\sqrt{\Omega_\Lambda}} e^{-3H_0\sqrt{\Omega_\Lambda}t} - \frac{1}{3H_0\sqrt{\Omega_\Lambda}} t, \quad \lambda_\Lambda(t) = \lambda_{\Lambda\,1} - \frac{\lambda_{\Lambda\,2}}{3H_0\sqrt{\Omega_\Lambda} a_\Lambda^3} e^{-3H_0\sqrt{\Omega_\Lambda}t} ,
\end{aligned} \tag{4.2.23}$$

respectively. Here, $\varphi_{\rm rad1}, \varphi_{\rm rad2}, \lambda_{\rm rad1}, \lambda_{\rm rad2}, \varphi_{\rm mat1}, \varphi_{\rm mat2}, \varphi_{\Lambda 1}, \varphi_{\Lambda 2}, \lambda_{\rm mat1}, \lambda_{\rm mat2}, \lambda_{\Lambda 1}$, and $\lambda_{\Lambda 2}$ are all constants. We now consider the approximation that the transition of the radiation-dominated universe to the matter-dominated universe occurs in a moment at $t = t_1$. Then, we connect the scalar fields as

$$\lambda_{\rm rad}(t_1) = \lambda_{\rm mat}(t_1), \quad \varphi_{\rm rad}(t_1) = \varphi_{\rm mat}(t_1) . \tag{4.2.24}$$

Moreover, we connect the time derivative of these scalar fields as

$$\left.\frac{d\varphi_{\rm rad}}{dt}\right|_{t=t_1} = \left.\frac{d\varphi_{\rm mat}}{dt}\right|_{t=t_1} , \quad \left.\frac{d\lambda_{\rm rad}}{dt}\right|_{t=t_1} = \left.\frac{d\lambda_{\rm mat}}{dt}\right|_{t=t_1} . \tag{4.2.25}$$

The above two equations yield

$$\begin{aligned}
&\varphi_{\rm mat\,1} = \frac{a_{\rm mat}^3}{a_{\rm rad}^3} \varphi_{\rm rad\,1} t_1^{1/2} - \frac{13}{6} a_{\rm mat}^3 t_1^3 , \quad \varphi_{\rm mat\,2} = \varphi_{\rm rad\,2} - \frac{\varphi_{\rm rad\,1}}{a_{\rm rad}^3} t_1^{-1/2} + \frac{13}{4} t_1^2 , \\
&\lambda_{\rm mat\,2} = \left(\frac{a_{\rm mat}}{a_{\rm rad}}\right)^3 t_1^{1/2} \lambda_{\rm rad\,2} , \quad \lambda_{\rm mat\,1} = \lambda_{\rm rad\,1} - \frac{\lambda_{\rm rad\,2}}{a_{\rm rad}^3} t_1^{-1/2} .
\end{aligned} \tag{4.2.26}$$

Similarly, when we assume that the transition of the matter-dominated era to the dark energy-dominated era occurs at $t = t_2$, we impose $\varphi_{\rm mat}(t_2) = \varphi_\Lambda(t_2)$, $\lambda_{\rm mat}(t_2) = \lambda_\Lambda(t_2)$, $\left.\frac{d\varphi_{\rm mat}}{dt}\right|_{t=t_2} = \left.\frac{d\varphi_\Lambda}{dt}\right|_{t=t_2}$ $\left.\frac{d\lambda_{\rm mat}}{dt}\right|_{t=t_2} = \left.\frac{d\lambda_\Lambda}{dt}\right|_{t=t_2}$. Then, we find

$$\begin{aligned}
\varphi_{\rm mat\,2} - \frac{\varphi_{\rm mat\,1}}{a_{\rm mat}^3} t_2^{-1} - \frac{1}{6} t_2^2 &= \varphi_{\Lambda\,2} - \frac{\varphi_{\Lambda\,1}}{3H_0\sqrt{\Omega_\Lambda} a_\Lambda^3} \mathrm{e}^{-3H_0\sqrt{\Omega_\Lambda}t_2} - \frac{t_2}{3H_0\sqrt{\Omega_\Lambda}} , \\
\lambda_{\rm mat\,1} - \frac{\lambda_{\rm mat\,2}}{a_{\rm mat}^3} t_2^{-1} &= \lambda_{\Lambda\,1} - \frac{\lambda_{\Lambda\,2}}{3H_c\sqrt{\Omega_\Lambda} a_\Lambda^3} \mathrm{e}^{-3H_c\sqrt{\Omega_\Lambda}t_2} \\
\frac{\varphi_{\rm mat\,1}}{a_{\rm mat}^3} t_2^{-2} - \frac{1}{3} t_2 &= \frac{\varphi_{\Lambda\,1}}{a_\Lambda^3} \mathrm{e}^{-3H_0\sqrt{\Omega_\Lambda}t_2} - \frac{1}{3H_0\sqrt{\Omega_\Lambda}} , \quad \frac{\lambda_{\rm mat\,2}}{a_{\rm mat}^3} t_2^{-2} = \frac{\lambda_{\Lambda\,2}}{a_\Lambda^3} \mathrm{e}^{-3H_c\sqrt{\Omega_\Lambda}t_2} ,
\end{aligned} \tag{4.2.27}$$

and we have the following relations of constants:

$$\begin{aligned}
\varphi_{\Lambda\,1} =& \frac{a_\Lambda^3}{a_{\rm mat}^3} t_2^{-2} \mathrm{e}^{3H_0\sqrt{\Omega_\Lambda}t_2}\varphi_{\rm mat\,1} + \frac{a_\Lambda^3 \mathrm{e}^{3H_0\sqrt{\Omega_\Lambda}t_2}}{3H_0\sqrt{\Omega_\Lambda}} - \frac{1}{3}t_2 a_\Lambda^3 \mathrm{e}^{3H_0\sqrt{\Omega_\Lambda}t_2}\,, \\
\varphi_{\Lambda\,2} =& \varphi_{\rm mat\,2} - \left(1 - \frac{1}{3H_0 t_2\sqrt{\Omega_\Lambda}}\right)\frac{\varphi_{\rm mat\,1}}{t_2 a_{\rm mat}^3} + \frac{2t_2}{9H_0\sqrt{\Omega_\Lambda}} - \frac{1}{6}t_2^2 + \frac{1}{9H_0^2\Omega_\Lambda}\,, \\
\lambda_{\Lambda\,2} =& \left(\frac{a_\Lambda}{a_{\rm mat}}\right)^3 t_2^{-2}\mathrm{e}^{3H_0\sqrt{\Omega_\Lambda}t_2}\lambda_{\rm mat\,2}\,, \quad \lambda_{\Lambda\,1} = \lambda_{\rm mat\,1} - \left(1 - \frac{1}{3H_0 t_2\sqrt{\Omega_\Lambda}}\right)\frac{\lambda_{\rm mat\,2}}{t_2 a_{\rm mat}^3}\,.
\end{aligned} \tag{4.2.28}$$

Combining these, we may delete $\lambda_{\rm mat1}, \lambda_{\rm mat2}, \varphi_{\rm mat1}$, and $\varphi_{\rm mat2}$ as

$$\begin{aligned}
\lambda_0 + \Lambda =& \frac{3H_c^2}{\kappa^2} = \Lambda + \lambda_{\Lambda\,1} - \frac{\lambda_{\Lambda\,2}}{3H_0\sqrt{\Omega_\Lambda}a_\Lambda^3}\mathrm{e}^{-3H_0\sqrt{\Omega_\Lambda}t_2}\,, \\
\lambda_{\Lambda\,1} =& \lambda_{\rm rad1} - \frac{\lambda_{\rm rad2}}{a_{\rm rad}^3}t_1^{-1/2}\left[1 + t_1 t_2^{-1}\left(1 - \frac{t_2^{-1}}{3H_0\sqrt{\Omega_\Lambda}}\right)\right]\,, \\
\lambda_{\Lambda\,2} =& \lambda_{\rm rad\,2}\left(\frac{a_\Lambda}{a_{\rm rad}}\right)^3 \mathrm{e}^{3H_0\sqrt{\Omega_\Lambda}t_2}t_2^{-2}t_1^{1/2}\,, \\
\varphi_{\Lambda\,1} =& \varphi_{\rm rad\,1}\left(\frac{a_\Lambda}{a_{\rm rad}}\right)^3 t_2^{-2}t_1^{1/2}\mathrm{e}^{3H_0\sqrt{\Omega_\Lambda}t_2} \\
& - \frac{13}{6}a_\Lambda^3 t_1^3 t_2^{-2}\mathrm{e}^{3H_0\sqrt{\Omega_\Lambda}t_2} + \frac{a_\Lambda^3 \mathrm{e}^{3H_0\sqrt{\Omega_\Lambda}t_2}}{3H_0\sqrt{\Omega_\Lambda}} - \frac{1}{3}t_2 a_\Lambda^3 \mathrm{e}^{3H_0\sqrt{\Omega_\Lambda}t_2}\,, \\
\varphi_{\Lambda\,2} =& \varphi_{\rm rad\,2} - \left\{\frac{t_1^{-1/2}}{a_{\rm rad}^3} + \left(1 - \frac{1}{3H_0 t_2\sqrt{\Omega_\Lambda}}\right)\frac{t_1^{1/2}}{t_2 a_{\rm rad}^3}\right\}\varphi_{\rm rad\,1} + \frac{2t_2}{9H_0\sqrt{\Omega_\Lambda}} + \frac{1}{9H_0^2\Omega_\Lambda} \\
& + \frac{13}{6}\left(1 - \frac{1}{3H_0 t_2\sqrt{\Omega_\Lambda}}\right)\frac{t_1^3}{t_2} + \frac{13}{4}t_1^2 - \frac{1}{6}t_2^2\,.
\end{aligned} \tag{4.2.29}$$

Then, we may express $\lambda_{\rm mat}(t), \lambda_\Lambda(t), \varphi_{\rm mat}(t)$ and $\varphi_\Lambda(t)$ by using the parameters at the radiation-dominated era as

$$
\begin{aligned}
\lambda_{\rm mat}(t) &= \lambda_{\rm rad1} - \frac{\lambda_{\rm rad2}}{a_{\rm rad}^3}\left(t_1^{-1/2} + t_1^{1/2}t^{-1}\right) \\
\lambda_\Lambda(t) &= \lambda_{\rm rad1} - \frac{\lambda_{\rm rad2}}{a_{\rm rad}^3}t_1^{-1/2}\left[1 + t_1 t_2^{-1} - \frac{t_1 t_2^{-2}}{3H_0\sqrt{\Omega_\Lambda}}\left(1 - \mathrm{e}^{-3H_0\sqrt{\Omega_\Lambda}(t-t_2)}\right)\right] \\
\varphi_{\rm mat}(t) &= \varphi_{\rm rad2} + \varphi_{\rm rad1}t + \frac{13}{12}t_1^2 - \frac{13}{6}t_1 t - \frac{1}{6}t^2 \\
\varphi_\Lambda(t) &= \varphi_{\rm rad2} + \varphi_{\rm rad1}\left[(t_2 + \frac{1}{3H_0\sqrt{\Omega_\Lambda}}\left(1 - \mathrm{e}^{-3H_0\sqrt{\Omega_\Lambda}(t-t_2)}\right)\right] + \frac{13}{12}t_1^2 - \frac{13}{6}t_1 t_2 - \frac{1}{6}t_2^2 \\
&\quad - \frac{1}{3H_0\sqrt{\Omega_\Lambda}}\left(\frac{13}{6}t_1 + \frac{1}{3}t_2 - \frac{1}{3H_0\sqrt{\Omega_\Lambda}}\right)\left(1 - \mathrm{e}^{-3H_0\sqrt{\Omega_\Lambda}(t-t_2)}\right) - \frac{1}{3H_0\sqrt{\Omega_\Lambda}}(t - t_2)\,.
\end{aligned}
\tag{4.2.30}
$$

We then investigate the constraint of these constant parameter in the following three ways. The first constraint is obtained from the fact that $\Lambda + \lambda$ should be constant at the dark energy-dominated era. Then, we require

$$
\Lambda_0 \sim \Lambda + \lambda_{\Lambda_1} \gg \left| \frac{\lambda_{\Lambda\,2}}{3H_0\sqrt{\Omega_\Lambda}a_\Lambda^3}\mathrm{e}^{-3H_0\sqrt{\Omega_\Lambda}t_0} \right| , \tag{4.2.31}
$$

where Λ_0 and t_0 is the current observed value of the cosmological constant and cosmological time respectively. The second (third) constraints require us that the scalar fields in the matter (radiation)-dominated era are negligible compared with the matter (radiation) in $t_1 \ll t \ll t_2 (t \ll t_1)$. From the first constraint, we have

$$
\Lambda_0 \sim \Lambda + \lambda_{\Lambda\,1} \gg \left| \frac{\lambda_{\Lambda\,2}}{3H_0\sqrt{\Omega_\Lambda}a_\Lambda^3}e^{-3H_0\sqrt{\Omega_\Lambda}t_0} \right| \tag{4.2.32}
$$

By substituting Eq. (4.2.29), $\Lambda_0 \sim \Lambda + \lambda_{\Lambda\,1}$ becomes

$$
\Lambda_0 \sim \Lambda + \lambda_{\rm rad1} - \frac{\lambda_{\rm rad2}}{a_{\rm rad}^3}t_1^{-1/2}\left[1 + t_1 t_2^{-1}\left(1 - \frac{t_2^{-1}}{3H_0\sqrt{\Omega_\Lambda}}\right)\right] . \tag{4.2.33}
$$

On the other hand, $\Lambda_0 \gg \left| \frac{\lambda_{\Lambda\,2}}{3H_0\sqrt{\Omega_\Lambda}a_\Lambda^3}e^{-3H_0\sqrt{\Omega_\Lambda}t_0} \right|$ yields

$$
\Lambda_0 \gg \left| \frac{\lambda_{\rm rad2}}{3H_0\sqrt{\Omega_\Lambda}a_{\rm rad}^3}t_2^{-2}t_1^{1/2}\mathrm{e}^{-3H_0\sqrt{\Omega_\Lambda}(t_0-t_2)} \right| . \tag{4.2.34}
$$

From the second constraint, in the matter-dominated era $t_1 \ll t \ll t_2$, we obtain

$$\Lambda + \lambda_{\mathrm{rad}\,1} - \frac{\lambda_{\mathrm{rad}\,2}}{a_{\mathrm{rad}}^3} t_1^{-1/2} \left(1 + t_1 t^{-1}\right) - \frac{\lambda_{\mathrm{rad}\,2}}{a_{\mathrm{rad}}^3} t_1^{1/2} \left\{ \left(\frac{t_1^{1/2}}{a_{\mathrm{rad}}^3} \varphi_{\mathrm{rad}\,1} - \frac{13}{6} t_1^3 \right) t^{-4} - \frac{1}{3} t^{-1} \right\} \ll \rho = \Omega_{\mathrm{m}} \rho_0 a_{\mathrm{mat}}^{-3} t^{-2} \,, \qquad (4.2.35)$$

where Ω_{m} is the density parameter of the matter and ρ_0 is the critical density denoted as

$$\rho_0 = \frac{3H_0^2}{8\pi G} \qquad G : \text{Newton's gravitational constant} \,. \qquad (4.2.36)$$

Similarly, from the third constraint, in the radiation-dominated era $t \ll t_1$ we obtain

$$\Lambda + \lambda_{\mathrm{rad}\,1} - \frac{\lambda_{\mathrm{rad}\,2}}{a_{\mathrm{rad}}^3} \left(\frac{\varphi_{\mathrm{rad}\,1}}{a_{\mathrm{rad}}^3} t^{-3} - \frac{5}{2} t^{-1/2} \right) \ll \rho = \Omega_{\mathrm{r}} \rho_0 a_{\mathrm{rad}}^{-4} t^{-2} \,. \qquad (4.2.37)$$

Here, Ω_{m} is the density parameter of the radiation. Although we cannot solve the constraint in Eq. (4.2.35) and (4.2.37) straightforwardly, we continue to investigate these constraints in the following two assumptions. Firstly, when we consider the transition of matter to dark energy-dominated era at t_2, the right-hand side is almost equal to the left-hand side. When t goes to t_1, the term which is proportional to t^{-4} dominates. Thus, the constraint Eq. (4.2.35) becomes

$$\left| \lambda_{\mathrm{rad}\,2} \left(\varphi_{\mathrm{rad}\,1} - \frac{13}{6} a_{\mathrm{rad}}^3 t_1^{5/2} \right) \right| \ll \Omega_{\mathrm{m}} \rho_0 \frac{a_{\mathrm{rad}}^6}{a_{\mathrm{mat}}^3} t_1 \,. \qquad (4.2.38)$$

To consider the beginning of the radiation-dominated era, we define t_3 as the beginning of the radiation-dominated universe. Then, when we take t to t_3 in Eq. (4.2.37), the term including t^{-3} dominates and we obtain

$$\left| \lambda_{\mathrm{rad}\,2} \varphi_{\mathrm{rad}\,1} \right| \ll \Omega_{\mathrm{r}} \rho_0 a_{\mathrm{rad}}^2 t_3 \,. \qquad (4.2.39)$$

We then investigate details of the constraint Eq. (4.2.33), (4.2.34), (4.2.38), and (4.2.39). Therefore, we use the following values of the cosmological parameters in [108] to obtain the constraints of $\lambda_{\mathrm{rad1}}, \lambda_{\mathrm{rad2}}, \varphi_{\mathrm{rad1}}$ and λ_{rad2}.

- The scale factor and the cosmological time when the radiation density was equal to the matter density :
  $a_{\mathrm{rm}} = 2.8 \times 10^{-4}, t_1 = 4.7 \times 10^4$ yr$\sim 1.5 \times 10^{12}$ s$= 2.3 \times 10^{27}$ [eV^{-1}]

- The scale factor and the cosmological time when the radiation density was equal to the matter density :
  $a_{\mathrm{m}\Lambda} = 0.75, t_2 = 9.8 \times 10^9$ yr$\sim 3.1 \times 10^{17}$ s$= 4.7 \times 10^{32}$ [eV^{-1}]

- The scale factor and the cosmological time at the beginning of the radiation era: $t_3 = 1.0 \times 10^{-38}$ s$= 1.5 \times 10^{-20}$ $[\mathrm{eV}^{-1}]$

- The current value of the scale factor and the cosmological time : $a_0 = 1, t_0 = 13.5 \times 10^9$ yr$\sim 4.3 \times 10^{17}$ s$= 6.5 \times 10^{32}$ $[\mathrm{eV}^{-1}]$

- The current value of the Hubble parameter : $H_0 = 70$ kms^{-1}Mpc^{-1} $\sim 2.2 \times 10^{-18}$ s$^{-1} = 1.5 \times 10^{-33}$ [eV]

- The density parameter of the radiation : $\Omega_\mathrm{r} = 8.4 \times 10^{-5}$

- The density parameter of the matter: $\Omega_\mathrm{m} = 0.3$

- The density parameter of the dark energy : $\Omega_\Lambda \sim 0.7$

- $a_\mathrm{rad} \sim \left(2H_0\sqrt{\Omega_\mathrm{r}}\right)^{1/2} = 2.0 \times 10^{-10}$ s$^{-1/2} = 5.4 \times 10^{-18}$ $[\mathrm{eV}^{1/2}]$

- $a_\mathrm{mat} \sim \left(\frac{2}{3}H_0\sqrt{\Omega_\mathrm{m}}\right)^{2/3} \sim 5.7 \times 10^{-13}$ s$^{-2/3} = 6.7 \times 10^{-23}$ $[\mathrm{eV}^{2/3}]$

- $a_\Lambda \sim 0.2$ [0]

- The critical density : $\rho_o = \frac{3H_0^2}{8\pi G} = 5 \times 10^{-24}$ kgm$^{-3} = 4.2 \times 10^{-11}$ $[\mathrm{eV}^4]$

- Newton's gravitational constant : G $\sim 6.6 \times 10^{-11}$ m^{3}kg^{-1}s$^{-2} = 6.7 \times 10^{-57}$ $[\mathrm{eV}^{-2}]$

- The current value of the cosmological constant : $\Lambda_0 \sim 10^{-11}$ $[\mathrm{eV}^4]$

Then, the constraints in Eqs. (4.2.33), (4.2.34), (4.2.38), and (4.2.39) can be rewritten as,

$$\begin{cases} \Lambda + \lambda_\mathrm{rad1} - \lambda_\mathrm{rad2} \sim 10^{-11}\,[\mathrm{eV}^4]\,, \\ |\lambda_\mathrm{rad2}| \ll 10^{-47}\,[\mathrm{eV}^5]\,, \\ \left|\lambda_{\mathrm{rad\,2}}\left(\varphi_{\mathrm{rad\,1}} - 7.3 \times 10^{16}\ \left[\mathrm{eV}^{-1}\right]\right)\right| \ll 10^{-23}\ \left[\mathrm{eV}^4\right]\ , \\ |\lambda_{\mathrm{rad\,2}}\varphi_{\mathrm{rad\,1}}| \ll 10^{-62}\ \left[\mathrm{eV}^4\right]\ . \end{cases}$$

The first constraint of the above expression tells us that we may need fine-tuning of the initial condition of the scalar fields. We now investigate more about the initial condition of the scalar field λ. When we choose t as the present time $t = t_0$, we obtain

$$\lambda(t_0) \sim 10^{-11}[\mathrm{eV}^4] \sim \lambda_{\mathrm{rad\,1}} - \frac{\lambda_{\mathrm{rad\,2}}}{6.4 \times 10^{-39}[\mathrm{eV}]}\,. \tag{4.2.40}$$

Then, we have

$$\lambda_{\text{rad}\,1} \sim \left(10^{-3}\,[\text{eV}]\right)^4 , \quad \lambda_{\text{rad}\,2} \sim \left(10^{-10}\,[\text{eV}]\right)^5 . \tag{4.2.41}$$

This seems that the obtained value is very small. Although it might not be natural, we assume that $\lambda_{\text{rad}\,1} = 0$. We then find that the value of λ at the beginning of the radiation-dominated era becomes

$$\lambda = \lambda_{\text{rad}}(t_3) \sim (0.1\,[\text{keV}])^4 . \tag{4.2.42}$$

This implies that even if the current value of $\lambda(t_0)$ is $\lambda(t_0) \sim 10^{-11}[\text{eV}^4]$, at the beginning of the radiation-dominated universe, it becomes, $\lambda_{\text{rad}}(t_3) \sim (0.1\,[\text{keV}])^4$. Note that the converse is not possible since $\lambda_{\text{rad}\,1}$ is not zero in general. If we once require $\lambda_{\text{rad}}(t_3) \sim (0.1\,[\text{keV}])^4$ at $t = t_3$, we may find $\lambda \sim (0.1\,[\text{keV}])^4 \gg 10^{-11}[\text{eV}^4]$.

In this analysis, we have used the approximation that the scalar fields change in a moment at the transition of the each era. We now solve Eqs. (4.2.4), (4.2.6), and the first equation of (4.2.8) numerically. In.Fig 4.1, the time evolution of λ is shown. The λ at the beginning of the radiation-dominated universe is consistent with the analytical approximation in (4.2.41). In Fig. 4.2, the time evolution of ϕ is shown. Here, the value of ϕ is given by the scaling of φ as $\phi \equiv M_{\text{Pl}}^3\varphi$. The evolution of the energy density is shown in Fig. 4.3. We choose $\lambda_{\text{rad}\,1}$ and $\lambda_{\text{rad}\,2}$ to realize the current value of the dark energy density in our universe. Note that in the matter-dominated era or the radiation-dominated era, we see that one can surely ignore the dark energy density.

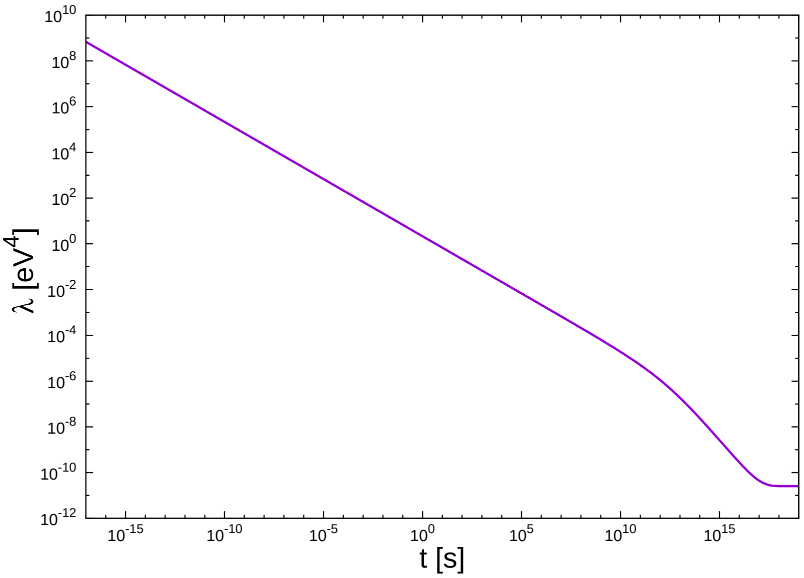

Fig. 4.1 The development of λ.

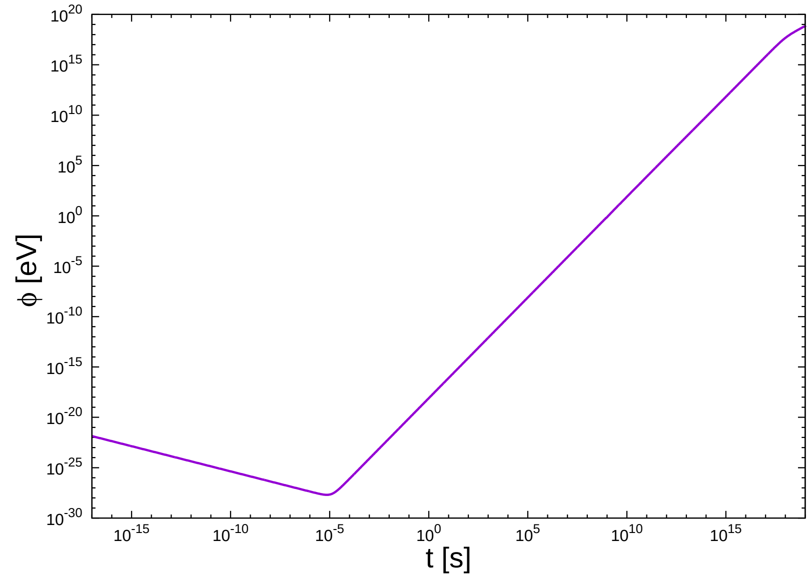

Fig. 4.2 The development of φ.

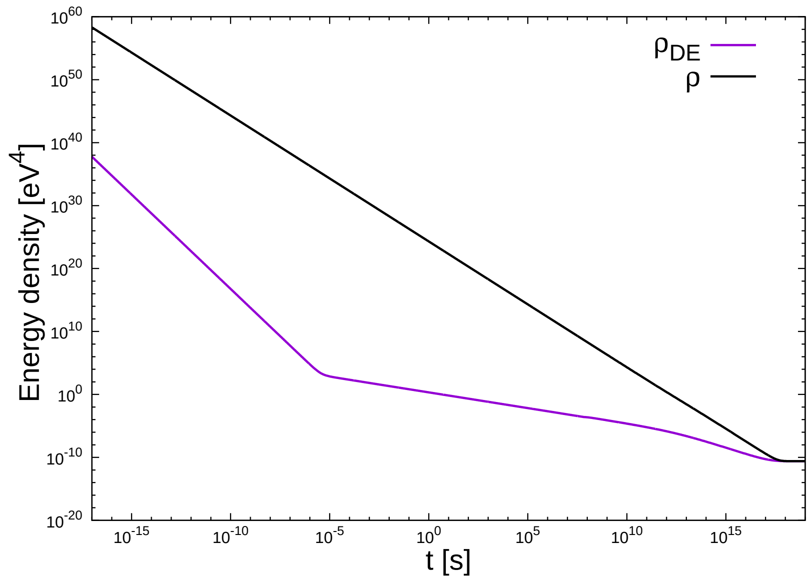

Fig. 4.3 The development of energy density.

4.3 Remarks

Note that this model realizes the de-Sitter space-time solution keeping one of the scalar fields φ to be dynamical. This implies that we may evade the Weinberg's no-go theorem.

Remember that the Weinberg' no-go theorem assumes that all fields will be constant at late time. As a result, they need fine-tuning of the flat potential of scalar fields. Contrary to this, we have obtained the dynamical solution of the scalar field at late-time. Therefore, we do not have to fine-tune the potential as we did in no-go results.

Chapter 5

Construction of renormalization group of the topological model

In the previous chapter, we discussed the topological model in order to solve the cosmological constant problem. A part of the Lagrangian is described by BRS exact so that it can be regarded as a kind of topological field theory. In this chapter, we will see that the coupling constants including the cosmological constant changes as the universe expands. This behavior is analogous to the renormalization group equation (RGE) since the couplings run in response to the change of energy scale of the universe. Motivated by these models, we propose new model which has two fixed points. One is an infrared (IR) fixed point may realize the late-time acceleration of the universe. The other is an ultraviolet (UV) fixed point which realize the inflation in the early universe. Especially, we construct a model which realize de-Sitter space-time solution at the UV and IR fixed points.

5.1 Generalization of the topological model

We have assumed that the vacuum energy gravitates only as the cosmological constant. In general, the huge quantum corrections from matter fields appear not only as the cosmological constant but also as other coupling constants. Although if we include the quantum corrections only from matter field, huge quantum corrections appear the following four coupling constants as

$$\mathcal{L}_{\mathrm{qc}} = \alpha R + \beta R^2 + \gamma R_{\mu\nu} R^{\mu\nu} + \delta R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} , \tag{5.1.1}$$

where α, β, γ and δ are coupling constants which diverge. Note that when we include the quantum correction, infinite number of counter term appear. This is one of the reasons that general relativity is not renormalizable theory. In [109], the generalized topological model has been proposed. The Lagrangian is given by

$$\mathcal{L} = -\Lambda + \mathcal{L}_{\mathrm{G}} + \mathcal{L}_{\mathrm{qc}} + \mathcal{L}_{\lambda,\varphi} + \mathcal{L}_{\mathrm{FP}} , \tag{5.1.2}$$

where

$$\begin{aligned}\mathcal{L}_{\rm G} &\equiv -\lambda_{(\Lambda)} + \lambda_{(\alpha)} R + \lambda_{(\beta)} R^2 + \lambda_{(\gamma)} R_{\mu\nu} R^{\mu\nu} + \lambda_{(\delta)} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\ \mathcal{L}_{\lambda,\varphi} &\equiv \partial_\mu \lambda_{(\Lambda)} \partial^\mu \varphi_{(\Lambda)} + \partial_\mu \lambda_{(\alpha)} \partial^\mu \varphi_{(\alpha)} + \partial_\mu \lambda_{(\beta)} \partial^\mu \varphi_{(\beta)} + \partial_\mu \lambda_{(\gamma)} \partial^\mu \varphi_{(\gamma)} + \partial_\mu \lambda_{(\delta)} \partial^\mu \varphi_{(\delta)} \\ \mathcal{L}_{\rm FP} &\equiv -\partial_\mu b_{(\Lambda)} \partial^\mu c_{(\Lambda)} - \partial_\mu b_{(\alpha)} \partial^\mu c_{(\alpha)} - \partial_\mu b_{(\beta)} \partial^\mu c_{(\beta)} - \partial_\mu b_{(\gamma)} \partial^\mu c_{(\gamma)} - \partial_\mu b_{(\delta)} \partial^\mu c_{(\delta)} \,.\end{aligned} \tag{5.1.3}$$

Here, i denotes $(i = \Lambda, \alpha, \beta, \gamma, \delta)$, λ_i and φ_i are scalar fields, b_i and c_i are FP ghosts and anti-ghosts respectively. Note that if we choose that $i = \Lambda$ and $\lambda_{(\alpha)} = \frac{1}{2\kappa^2}$ = constant, this Lagrangian corresponds to the Lagrangian in (4.1.8). When we redefine the λ_i as

$$\lambda_i \to \lambda_i - i \,, \tag{5.1.4}$$

the cosmological constant Λ and $\mathcal{L}_{\rm qc}$ term vanish and we obtain

$$\mathcal{L} = \mathcal{L}_{\rm G} + \mathcal{L}_{\lambda,\varphi} + \mathcal{L}_{\rm FP} \,. \tag{5.1.5}$$

We see that the quantum corrections do not affect to the dynamics by the shift of the scalar field λ again. The Lagrangian is invariant under the following BRS transformations:

$$\delta_{\rm B} \lambda_{(i)} = \delta_{\rm B} c_{(i)} = 0, \quad \delta_{\rm B} \varphi_{(i)} = \epsilon c, \quad \delta_{\rm B} b_{(i)} = \epsilon \left(\lambda_{(i)} - \lambda_{(i)0} \right) \,, \tag{5.1.6}$$

where $\lambda_{(i)0}$ satisfy the classical equations of motion:

$$0 = \nabla^\mu \partial_\mu \lambda_{(i)0} \,. \tag{5.1.7}$$

Furthermore, we may regard this Lagrangian as a kind of topological field theory including gravity if $\lambda_{(i)0} = 0$. In fact, the Lagrangian is obtained by the BRS transformations (5.1.6) as

$$\begin{aligned}\delta_{\rm B} \left(-b_i F_{\rm GF} \right) &= \epsilon \left(\mathcal{L} + \text{total derivative} \right) \\ F_{\rm GF} &\equiv \sum_{i=\Lambda,\alpha,\beta,\gamma,\delta} \left(\mathcal{O}_{(i)} + \nabla_\mu \partial^\mu \varphi_{(i)} \right) \,.\end{aligned} \tag{5.1.8}$$

Here, we introduced the possible gravitational operator $\mathcal{O}_{(i)}$ as

$$\mathcal{O}_{(1)} = 1, \quad \mathcal{O}_{(\alpha)} = R, \quad \mathcal{O}_{(\beta)} = R^2, \quad \mathcal{O}_{(\gamma)} = R_{\mu\nu} R^{\mu\nu}, \quad \mathcal{O}_{(\delta)} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \,. \tag{5.1.9}$$

We may further generalize the Lagrangian by introducing the possible gravitational operator $\mathcal{O}_{(i)}$ as

$$\mathcal{L} = \sum_i \left(\lambda_{(i)} \mathcal{O}_{(i)} + \partial_\mu \lambda_{(i)} \partial^\mu \varphi_{(i)} - \partial_\mu b_{(i)} \partial^\mu c_{(i)} \right) \,. \tag{5.1.10}$$

When we choose $\mathcal{O}_{(\Lambda)} = 1$ and $\mathcal{O}_{(\alpha)} = R$, it corresponds to the Lagrangian (4.1.8). The above model can be also regarded as a kind of topological field theory including gravity.

5.2 Model motivated by the RGE

In this section, we construct a kind of renormalization group equation (RGE) by using the special case of the Lagrangian in (5.1.10).

As well-known, higher derivative gravitational theory can be renormalizable but at the same time there includes the ghosts and thus the higher derivative gravity model losts unitary [110–113]. Since the divergences from the quantum correction can be removed, our model can be renormalizable. However, the unitarity is not guaranteed since the Lagrangian in (4.1.8) have the higher derivative in general. The predictability is an important criterion whether the theory can be renormalizable or not. When we consider the quantization of gravitational theory from general relativity, we need to introduce an infinite number of the counterterms to cancel the divergences. However, because of an infinite number of the counter terms, we have to abandon the predictability. The model In (5.1.10) seems that this model do not have the problem of these divergence. It is, however, we need an infinite number of initial condition alternatively since the scalar fields become dynamical and thus predictability is losts. However, note that if the scalar fields have a certain fixed point, we may recover the predictability. Thus, in this section, we try to construct the model which have the fixed point to recover the predictablility.

In this analysis, we consider the flat FRW metric:

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3} \left(dx^i\right)^2 \,. \tag{5.2.1}$$

Eq. (5.1.7) implies that the scalar fields $\lambda_{(i)}$ are characterized by the scale factor $a(t)$. Note that $\lambda_{(i)}$ are identified with the operators $\mathcal{O}_{(i)}$. The $a(t)$ dependence of these scalar fields is analogous to the renormalized coupling constant. Based on the above insight, we construct the model which has infrared (IR) fixed point regarded as the late-time acceleration and ultraviolet (UV) fixed point regarded as the early time inflation.

Instead of (5.1.6), we use the following BRS transformation:

$$\delta_{\mathrm{B}}\lambda_{(i)} = \delta_{\mathrm{B}}c_{(i)} = 0, \quad \delta_{\mathrm{B}}\varphi_{(i)} = \epsilon c_{(i)}, \quad \delta_{\mathrm{B}}b_{(i)} = \epsilon\lambda_{(i)} \,. \tag{5.2.2}$$

We then consider the BRS transformation of the some quantity as

$$\delta_{\mathrm{B}} \left(\sum_{i=\Lambda,\alpha} \left(b_{(i)} \left(\mathcal{O}_i + \nabla_\mu \partial^\mu \varphi_{(i)} + f_i \left(\lambda_{(j)} \right) \varphi_{(i)} \right) \right) \right) = \epsilon \mathcal{L} + (\text{total derivative}) \,, \tag{5.2.3}$$

where $f_i\left(\lambda_{(j)}\right)$ is a function of $\lambda_{(i)}$. Then the Lagrangian in Eq. (5.2.3) is written as

$$\mathcal{L}=\sum_i\left(\lambda_{(i)}\mathcal{O}_{(i)}+\partial_\mu\lambda_{(i)}\partial^\mu\varphi_{(i)}+\lambda_{(i)}f_i\left(\lambda_{(j)}\right)\varphi_{(i)}-\partial_\mu b_{(i)}\partial^\mu c_{(i)}-f_i\left(\lambda_{(j)}\right)b_{(i)}c_{(i)}\right)\,. \tag{5.2.4}$$

From Eq. (5.2.3), the gauge fixing condition is given by

$$F_{\rm GF}=\sum_{i=\Lambda,\alpha}\left(\mathcal{O}_i+\nabla_\mu\partial^\mu\varphi_{(i)}+f_i\left(\lambda_{(j)}\right)\varphi_{(i)}\right)\,. \tag{5.2.5}$$

Note that this gauge fixing condition is different from Eq. (5.1.8). Varying $\mathcal{L}$ with respect to φ_i, we obtain

$$\Box\lambda_{(i)}=-\lambda_{(i)}f_i\left(\lambda_{(j)}\right)\,. \tag{5.2.6}$$

In the FRW background in Eq. (5.2.1), the above equations become

$$\frac{d^2\lambda_{(i)}}{dt^2}+3H\frac{d\lambda_{(i)}}{dt}=\lambda_{(i)}f_i\left(\lambda_{(j)}\right)\,, \tag{5.2.7}$$

where $H\equiv\dot{a}/a$ is the Hubble parameter. When we define the parameter τ as, $\tau\equiv\ln a$, we then find

$$\frac{d}{dt}=H\frac{d}{d\tau},\,\frac{d^2}{dt^2}=H^2\frac{d^2}{d\tau^2}+\dot{H}\frac{d}{d\tau}\,. \tag{5.2.8}$$

From Eq. (5.2.8), we may rewrite the Eq. (5.2.7) as

$$H^2\left\{\frac{d^2\lambda_{(i)}}{d\tau^2}+\left(3+\frac{\dot{H}}{H^2}\right)\frac{d\lambda_{(i)}}{d\tau}\right\}=\lambda_{(i)}f_i\left(\lambda_{(j)}\right)\,. \tag{5.2.9}$$

Since the change of the scale factor $a(t)$ can be regarded as the scale transformation, we may consider the following RGE:

$$\frac{d\lambda_{(i)}}{d\tau}=g_i\left(\lambda_{(j)}\right)\,. \tag{5.2.10}$$

In the above expression, g_i can be regarded as the beta function in RGE which determine the response of the scale transformation. In cosmology, the Hubble parameter H is usually treated as energy scale. On the other hand, in an analogy with the RGE in the quantum field theory, suggest that we can alternatively use the scale factor $a(t)$ as the energy scale. Then we obtain

$$\frac{d^2\lambda_{(i)}}{d\tau^2}=\sum_k\frac{\partial g_i\left(\lambda_{(j)}\right)}{\partial\lambda_{(k)}}g_k\left(\lambda_{(j)}\right)\,. \tag{5.2.11}$$

Therefore, by using g_i, Eq. (5.2.9) can be rewritten by

$$f_i\left(\lambda_{(j)}\right)=\frac{H^2}{\lambda_{(i)}}\left\{\sum_k \frac{\partial g_i\left(\lambda_{(j)}\right)}{\partial \lambda_{(k)}} g_k\left(\lambda_{(j)}\right)+\left(3+\frac{\dot{H}}{H^2}\right) g_i\left(\lambda_{(j)}\right)\right\}. \tag{5.2.12}$$

Note that RGE in Eq. (5.2.7) requires $f_i\left(\lambda_{(j)}\right)$ not to be time dependent. Thus, from the above expression in Eq. (5.2.12), it seems that we can only have any physical meaning when the Hubble parameter is a constant at least in the neighborhood of the fixed point. In other words, the space-time have to be described by the de Sitter space-time solution at least asymptotically. As we will see later, we construct the model with the renormalization flow which connect the two fixed points corresponding to UV and IR fixed point. Since the value of H is different between inflation and late-time acceleration, H have to change during the cosmological evolution. However, as we will see later, we can impose the scale dependence of H into $f_i\left(\lambda_{(j)}\right)$ or $g_i\left(\lambda_{(j)}\right)$. In this analysis, we assume that there is a UV or IR fixed point in the RGE in Eq. (5.2.10). In the early time or late time of the universe, we suppose that the space-time asymptotically goes to the de Sitter solution. When we properly choose $f_i\left(\lambda_{(j)}\right)$ by Eq. (5.2.12), the early universe is identified with the ultraviolet (UV) fixed point and the late time universe is identified with the IR fixed point. Because the change of τ is identified with the change of the scale and therefore τ is described by using the scale factor as $a=e^{\tau}$. We define the UV limit and IR limit by using the relation of τ and a. For the UV limit, we take $\tau\to-\infty$ and at the same time, we obtain $a\to 0$. On the other hand, for the IR limit, we take $\tau\to\infty$ and similarly we have, $a\to\infty$. Firstly, we consider the UV fixed point $\lambda\to\lambda^*_{\rm UV}$ and its neighborhood. We assume the following condition in the neighborhood of the UV fixed point $\lambda^*_{\rm UV}$:

$$\frac{dg_{(i)}\left(\lambda_{(j)}\right)}{d\lambda_{(i)}}>0\,. \tag{5.2.13}$$

Then we obtain

$$g_{(i)}\left(\lambda_{(j)}\right)\approx r_{(i)\rm UV}\left(\lambda_{(j)}\right)\left(\lambda_{(i)}-\lambda_{(i)\rm UV}\right)\,, \tag{5.2.14}$$

in the neighborhood of the UV fixed point. Here, we denote $r_{(i)\rm UV}$ as a function of $\lambda_{(j)}$ which satisfy $r_{(i)\rm UV}\left(\lambda_{(j)}\right)>0$. If $r_{(i)\rm UV}\left(\lambda_{(j)}\right)$ is possible to be approximately constant when $\lambda_{(i)}\approx\lambda^*_{\rm UV}$, we obtain

$$r_{(i)\rm UV}\left(\lambda_{(j)}\right)\approx r_{(i)\rm UV}\left(\lambda_{(j)\rm UV}\right)\,. \tag{5.2.15}$$

Then, Eq. (5.2.10) can be solved by

$$\lambda_{(i)}\approx\lambda_{(i)\rm UV}+\lambda_{(i)\rm UV0}\,a(t)^{r_{(i)\rm UV}\left(\lambda_{(j)\rm UV}\right)}\,, \tag{5.2.16}$$

where $\lambda_{(i)\mathrm{UV}0}$ is an integration constant. Secondly, we consider the IR fixed point. In the UV fixed point case, we assumed that the condition in Eq. (5.2.13). On the other hand, in the IR fixed point case, we assume that

$$\frac{dg_{(i)}\left(\lambda_{(j)}\right)}{d\lambda_{(i)}} < 0 \,, \tag{5.2.17}$$

in the neighborhood of the IR fixed point. Therefore, the solution can be obtained from the replacement of $r_{(i)\mathrm{UV}} \to -r_{(i)\mathrm{IR}}$ and $\lambda_{(i)\mathrm{UV}} \to \lambda_{(i)\mathrm{IR}}$. Then we have

$$g_{(i)}\left(\lambda_{(j)}\right) \approx -r_{(i)\,\mathrm{IR}}\left(\lambda_{(j)}\right)\left(\lambda_{(i)} - \lambda_{(i)\mathrm{R}}\right) \,, \tag{5.2.18}$$

$$\lambda_{(i)} \approx \lambda_{(i)\mathrm{IR}} + \lambda_{(i)\mathrm{IR}0}\left(\frac{1}{a(t)}\right)^{r_{(i)\mathrm{IR}}\left(\left(\lambda_{(j)\mathrm{IR}}\right)\right)} \,, \tag{5.2.19}$$

where $\lambda_{(i)\mathrm{IR}0}$ is also an integration constant. Note that if $a(t) \to 0$ in Eq. (5.2.16) or $a(t) \to \infty$ in Eq. (5.2.19), they go to $\lambda_{(i)\mathrm{UV}}$ and $\lambda_{(i)\mathrm{UV}}$ respectively. Therefore, as long as we require that Eqs. (5.2.16) and (5.2.19) in the neighborhood of the UV and IR fixed point respectively, we may surely obtain the UV and IR fixed point respectively. Therefore, in the neighborhood of the UV fixed point, $f_i\left(\lambda_{(j)}\right)$ is written as

$$f_i\left(\lambda_{(j)}\right) = \frac{H^2}{\lambda_{(i)\mathrm{UV}}}\left(r_{(i)\mathrm{UV}}\left(\lambda_{(j)\mathrm{UV}}\right) + 3\right) r_{(i)\mathrm{UV}}\left(\lambda_{(j)\mathrm{UV}}\right)\left(\lambda_{(i)} - \lambda_{(i)\mathrm{UV}}\right) + \mathcal{O}\left(\left(\lambda_{(i)} - \lambda_{(i)\mathrm{UV}}\right)^2\right) \,. \tag{5.2.20}$$

Similarly, in the neighborhood of the IR fixed point, we obtain

$$f_i\left(\lambda_{(j)}\right) = \frac{H^2}{\lambda_{(i)\mathrm{IR}}}\left(r_{(i)\mathrm{IR}}\left(\lambda_{(j)\mathrm{IR}}\right) - 3\right) r_{(i)\mathrm{IR}}\left(\lambda_{(j)\mathrm{IR}}\right)\left(\lambda_{(i)} - \lambda_{(i)\mathrm{IR}}\right) + \mathcal{O}\left(\left(\lambda_{(i)} - \lambda_{(i)\,\mathrm{IR}}\right)^2\right) \,. \tag{5.2.21}$$

When we only consider the Einstein gravity with cosmological constant, the action is given as

$$S = \int d^4x\sqrt{-g}\left[\lambda_{(\alpha)}R - \lambda_{(\Lambda)} + \sum_{i=\Lambda,\alpha}\left(\partial_\mu\lambda_{(i)}\partial^\mu\varphi_{(i)} - \partial_\mu b_{(i)}\partial^\mu c_{(i)} + \lambda_{(i)}f_{(i)}\left(\lambda_{(j)}\right)\varphi_{(i)}\right)\right] + S_{\mathrm{matter}} \,. \tag{5.2.22}$$

By varying the above action with respect to metric $g^{\mu\nu}$, we obtain

$$\lambda_{(\alpha)}G_{\mu\nu} + \frac{1}{2}\lambda_{(\Lambda)}g_{\mu\nu} + \sum_{i=\Lambda,\alpha}\left[\frac{1}{2}g_{\mu\nu}\left(\partial_\rho\lambda_{(i)}\partial^\rho\varphi_{(i)} + \lambda_{(i)}f_{(i)}\left(\lambda_{(j)}\right)\varphi_{(i)}\right) + \partial_\mu\lambda_{(i)}\partial_\nu\varphi_{(i)}\right] = T_{\mu\nu} \,, \tag{5.2.23}$$

where we ignored the FP ghost terms since we only consider the classical dynamics. As we did in the previous chapter, we assume that the scalar fields depend only on time:

$$\lambda_{(i)} = \lambda_{(i)}(t), \quad \varphi_{(i)} = \varphi_{(i)}(t) \tag{5.2.24}$$

Then, under the flat FRW background, Eq. (5.2.23) becomes,

$$H^2 = \frac{1}{6\lambda_{(\alpha)}} \left\{ \lambda_{(\Lambda)} - 3H\dot{\lambda}_{(\alpha)} - \sum_{i=\Lambda,\alpha} \left(\dot{\lambda}_{(i)}\dot{\varphi}_{(i)} - \lambda_{(i)} f_i\left(\lambda_j\right) \varphi_{(i)} \right) \right\} . \tag{5.2.25}$$

When we consider the neighbourhood of the UV fixed point, the above equation becomes

$$\begin{aligned} H^2 \approx & \frac{1}{6\lambda_{(\alpha)}} \left(\lambda_{(\Lambda)\mathrm{UV}} + \lambda_{(\Lambda)\mathrm{UV}0} a(t)^{r_{(\Lambda)\mathrm{UV}}\left(\lambda_{(j)\mathrm{UV}}\right)} - 3H\dot{\lambda}_{(\alpha)} \right. \\ & \left. + \sum_{i=\Lambda,\alpha} r_{(i)}\left(\lambda_{(j)}\right) H a(t)^{r_{(i)}\left(\lambda_{(j)}\right)} \lambda_{(i)\mathrm{UV}0}\dot{\varphi}_{(i)} \right) \\ & + \frac{H^2}{6\lambda_{(\alpha)}} \sum_{i=\Lambda,\alpha} \left\{ \sum_k \frac{\partial g_i\left(\lambda_{(j)}\right)}{\partial \lambda_{(k)}} g_i\left(\lambda_{(j)}\right) + \left(3 + \frac{\dot{H}}{H^2}\right) g_i\left(\lambda_{(j)}\right) \right\} \varphi_{(i)} . \end{aligned} \tag{5.2.26}$$

When we take the UV limit:

$$a(t) \to 0, \quad g_{(i)} \to 0, \quad \lambda_{(i)} \to \lambda_{(i)\mathrm{UV}} , \tag{5.2.27}$$

we surely obtain the de-Sitter spacetime solution as

$$H = H_{\mathrm{UV}} = \sqrt{\frac{\lambda_{(\Lambda)\mathrm{UV}}}{6\lambda_{(\alpha)\mathrm{UV}}}} = \text{ const} . \tag{5.2.28}$$

Similarly, in the neighborhood of the IR fixed point, we obtain

$$\begin{aligned} H^2 \approx & \frac{1}{6\lambda_{(\alpha)}} \left(\lambda_{(\Lambda)\mathrm{IR}} + \lambda_{(\Lambda)\mathrm{IR}0} a(t)^{-r_{(\Lambda)\mathrm{IR}}\left(\lambda_{(j)\mathrm{IR}}\right)} - 3H\dot{\lambda}_{(\alpha)} \right. \\ & \left. - \sum_{i=\Lambda,\alpha} r_{(i)}\left(\lambda_{(j)}\right) H a(t)^{-r_{(i)}\left(\lambda_{(j)}\right)} \lambda_{(i)\,\mathrm{IR}\,0}\dot{\varphi}_{(i)} \right) \\ & + \frac{H^2}{6\lambda_{(\alpha)}} \sum_{i=\Lambda,\alpha} \left\{ \sum_k \frac{\partial g_i\left(\lambda_{(j)}\right)}{\partial \lambda_{(k)}} g_i\left(\lambda_{(j)}\right) + \left(3 + \frac{\dot{H}}{H^2}\right) g_i\left(\lambda_{(j)}\right) \right\} \varphi_{(i)} . \end{aligned} \tag{5.2.29}$$

When we take the IR limit,

$$a(t) \to \infty, \quad g_{(i)} \to 0, \quad \lambda_{(i)} \to \lambda_{(i)\mathrm{IR}}\,, \tag{5.2.30}$$

we have the de-Sitter solution again:

$$H = H_{\mathrm{IR}} = \sqrt{\frac{\lambda_{(\Lambda)\mathrm{IR}}}{6\lambda_{(\alpha)\mathrm{IR}}}} = \text{const.} \tag{5.2.31}$$

We then try to construct the specific model motivated by RGE. We require that we connect the UV fixed point and IR fixed point by using renormalization flow. We then consider the following simple form:

$$f_{(i)}\left(\lambda_{(j)}\right) = A_{(i)}\left(\lambda_{(j)}\right)\left(\lambda_{(i)} - \lambda_{(i)\mathrm{UV}}\right)\left(\lambda_{(i)} - \lambda_{(i)\mathrm{IR}}\right)\,, \tag{5.2.32}$$

where $A_{(i)}\left(\lambda_{(j)}\right)$ is a positive function of λ. Comparing the above expression and (5.2.20), we obtain

$$\frac{\lambda_{(\Lambda)\mathrm{UV}}}{6\lambda_{(\alpha)\mathrm{UV}}\lambda_{(i)\mathrm{UV}}}\left(r_{(i)\mathrm{UV}}\left(\lambda_{(j)\mathrm{UV}}\right)+3\right) r_{(i)\mathrm{UV}}\left(\lambda_{(j)\mathrm{UV}}\right) = A_{(i)}\left(\lambda_{(j)\mathrm{UV}}\right)\left(\lambda_{(i)\mathrm{UV}} - \lambda_{(i)\mathrm{IR}}\right)\,, \tag{5.2.33}$$

in the neighborhood of UV fixed point. We may solve this equation with respect to $r_{(i)\mathrm{UV}}$ as

$$r_{(i)\mathrm{UV}} = -\frac{3}{2} + \frac{1}{2}\sqrt{9 + \frac{24\lambda_{(\alpha)\mathrm{UV}}\lambda_{(i)\mathrm{UV}}A_{(i)}\left(\lambda_{(j)\mathrm{UV}}\right)}{\lambda_{(\Lambda)\mathrm{UV}}}\left(\lambda_{(i)\mathrm{UV}} - \lambda_{(i)\mathrm{IR}}\right)}\,. \tag{5.2.34}$$

When we assume that $\lambda_{(i)\mathrm{UV}} > \lambda_{(i)\mathrm{IR}}$, the requirement of positive $\lambda_{(i)\mathrm{UV}}$ is automatically satisfied. On the other hand, when we consider in the neighborhood of the IR fixed point, we obtain

$$\frac{\lambda_{(\Lambda)\mathrm{IR}}}{6\lambda_{(\alpha)\mathrm{IR}}\lambda_{(i)\mathrm{IR}}}\left(r_{(i)\mathrm{IR}}\left(\lambda_{(j)\mathrm{IR}}\right)-3\right) r_{(i)\mathrm{IR}}\left(\lambda_{(j)\mathrm{IR}}\right) - A_{(i)}\left(\lambda_{(j)\mathrm{IR}}\right)\left(\lambda_{(i)\mathrm{UV}} - \lambda_{(i)\mathrm{IR}}\right)\,, \tag{5.2.35}$$

We may also solve this equation with respect to $r_{(i)\mathrm{IR}}$ as

$$r_{(i)\mathrm{IR}} = \frac{3}{2} \pm \frac{1}{2}\sqrt{9 - \frac{24\lambda_{(\alpha)\mathrm{IR}}\lambda_{(i)\mathrm{IR}}A_{(i)}\left(\lambda_{(j)\mathrm{IR}}\right)}{\lambda_{(\Lambda)\mathrm{IR}}}\left(\lambda_{(i)\mathrm{UV}} - \lambda_{(i)\mathrm{IR}}\right)}\,. \tag{5.2.36}$$

For a positive $r_{(i)\mathrm{IR}}$ and , we impose that

$$9 \geq \frac{24\lambda_{(\alpha)\mathrm{IR}}\lambda_{(i)\mathrm{IR}}A_{(i)}\left(\lambda_{(j)\mathrm{IR}}\right)}{\lambda_{(\Lambda)\mathrm{IR}}}\left(\lambda_{(i)\mathrm{UV}} - \lambda_{(i)\mathrm{IR}}\right), \quad \lambda_{(i)\mathrm{UV}} > \lambda_{(i)\mathrm{IR}}\,. \tag{5.2.37}$$

Thus, as long as $A_{(i)}\left(\lambda_{(j)\mathrm{IR}}\right)$ satisfies (5.2.37), we can surely connect the UV fixed point and IR fixed point by the renormalization flow $f_{(i)}\left(\lambda_{(j)}\right)$ in Eq. (5.2.32).

5.3 Summary

In this chapter, we have constructed models of topological field theory which includes gravity based on the model in [104], [103], and [109]. In these models, the coupling constants are described by scalar fields $\lambda_{(i)}(x)$. This parameter runs as the scale of the universe changes. Therefore, we can analogously identify $\lambda_{(i)}(x)$ as running coupling constants in RGE. As an example, we have proposed a model which connects the inflation in the early universe and the accelerating expansion of the present universe. The inflation and the late the acceleration are generated by the de-Sitter space-time solution with UV and IR fixed point, respectively. However, we note that several problems which ruin the useful characteristic in the original models still remain.

1. Since the Lagrangian has the non-linear term of $\lambda_{(i)}$, we cannot absorb the huge vacuum energy contribution as we did in [104], [103], and [109].

2. Although we need to have non-zero $\lambda_{(i)}$ in the real universe, there is no physical degrees of freedom of $\lambda_{(i)}$ in the BRS transformation in Eq. (5.2.2). It implies that this BRS transformation is spontaneously broken.

3. Although the original model in [103,104,109] has no physical parameters if $\lambda_{(i)0} = 0$, the models proposed in this paper should have several parameters.

Therefore, it is interesting if we construct any model which solve some of the above problems by keeping the structure similar to the RGE. In this chapter, we have constructed models where the scalar fields $\lambda_{(i)}$'s play the role of the running coupling constants as in the RGE. We have treated the scalar fields as classical fields although the RGE in quantum field theory, origins from the quantum effect. Therefore, the models proposed in this chapter might be realized by the effective field theory which connect the low energy region with the high energy regions. If these models are really given as a kind of effective field theories, we do not have to impose that the models satisfy the unitarity and evading the ghosts. We have succeeded to construct such models and we have shown that we can construct the model with two fixed points with proper flow.

The models have, however, arbitrariness, which could be removed by the constraints from the observations and/or the consistencies of the models.

We further mention that there is possibility to approach the cosmological constant problem in this model. In chapter 3, we saw that in the model in Eq. (4.1.8), we need the fine-tuning of the initial condition of the dynamical scalar field. This imply that the dynamics of the scalar field which realize our current universe is sensitive to the change of the initial condition and therefore it seems that we cannot solve the cosmological constant problem consistently. On the other hand, in this chapter the model in Eq. (5.2.22) has the non-dynamical initial condition. Therefore, even the model in Eq. (5.2.22) in this chapter does not have the full property in Eq. (4.1.8), there is a possibility that we can replace the problem of the fine-tuning of the initial condition of the dynamical scalar field with the choice of the two fixed points. Thus, we can approach the cosmological constant problem in this manner.

Chapter 6

Conclusions

6.1 Summary and discussion

In chapter 1, we discussed the cosmological constant problem in details. The cosmological constant problem emerges from two well-established theory, general relativity and quantum field theory. We do not consider the ground state vacuum energy in quantum field theory because it does not affect the experiments. This effect is, however, appear explicitly when we consider general relativity based on the equivalence principle and the general coordinate covariance. Although the cosmological constant is regarded as the vacuum energy in this manner, the naturally expected value of the quantum field theory is much too larger than the observed one. This implies that we need tremendous fine-tuning of the counter term. In addition to this, a further complication appears when we consider the phase transition. This involves change of the vacuum energy which is also much larger than the observed cosmological constant. Then, we may consider screening of the cosmological constant by the dynamical scalar fields. However, we again stress that if we work on the cosmological constant problem by using additional scalar fields, we need to overcome Weinberg's no-go theorem.

In chapter 2, we therefore discussed some attempts which might work the solution of the cosmological constant problem. There are many attempts to solve the fine-tuning of the cosmological constant. We have proposed the three attempts for solving the cosmological constant problem. They do not seem to work well to solve the cosmological constant problem completely.

In chapter 3, we reviewed the unimodular gravity in details. We only consider the original model of unimodular gravity in chapter 2. We generalized the original model and discussed the quantum theory of unimodular gravity.

In chapter 4, we proposed the model for solving the cosmological constant problem motivated by the extension of the unimodular gravity. In this model, one of the Lagrangian is regarded as a kind of topological field theory since the Lagrangian is described as BRS exact. The important point was that we mainly considered the cancellation of the huge vacuum energy by the redefinition of the dynamical scalar field. Therefore, the fine-tuning problem in quantum theory is replaced by the problem of initial condition in classical theory. This system is stable under the de-Sitter space-time which describes the dynamics of the late time universe. Therefore, we may expect that the initial condition of the scalar field will be relaxed. As a result, it seems that we need fine-tuning of one of the parameters of the scalar fields. However, there is a possibility that this model may relax the original degree of the fine-tuning. Since one of the scalar fields remains dynamical at the late-time, this model evades Weinberg's no-go theorem which assume that all fields become constant at the late-time.

In chapter 5, we further generalized the model in chapter 4. We extended the vacuum energy cancellation mechanism described by a kind of topological field theory to the higher curvature gravitational theory. From the perspective of the renormalizability and predictability, we considered a topological field theory including gravity motivated by RGE. Since the scalar fields are dynamical, we need an infinite number of counter term and we lose predictability. These scalar fields can be regarded as coupling constants depending on the scale factor. Therefore, the scale factor is identified with the scale transformation and then we may construct like RGE. We then assumed the existence of the UV and IR fixed points which correspond to the inflation and late-time acceleration respectively and connected them by RG flow. Then, we expected that the predictability could be recovered since the fixed points are not dynamical. Finally, we proposed the model with a simple form of the RG flow and investigated the region which connect the IR and UV fixed point properly. Furthermore, we mentioned the possibility that this model can approach the cosmological constant problem.

6.2 Future directions

6.2.1 Generalization of the no-go evading Lagrangian

We proposed the Lagrangian which evades Weinberg's no-go theorem. It seems that we may generalize this type of Lagrangian. The indefinite metric term plays an important role for evading no-go result keeping the system has the de-sitter space-time solution. The fields are assumed to be space-time "independent" at late times in the Weinberg's assumption. Weinberg assumed that all fields are Poincaré invariant. On the other hand, we allowed it to be broken in the only scalar fields sector. In other words, we preserve Poincaré invariance at the level of curvature, but not at the level of the scalar fields. We have considered that there is no equilibrium solution for one of the scalar fields in the chapter 3. In chapter 3, we considered only one scalar field. We may add extra scalar fields to obtain a general Lagrangian which evades the no-go theorem. It seems, however, that we need to introduce an indefinite metric to reproduce such a Lagrangian in general. If we introduce an indefinite metric, a ghost problem reappears as we saw in chapter 3.

To avoid ghosts, we may have two choices. One is to consider only an effective theory. The indefinite metric generates negative norm and unitarity is not guaranteed. On the other hand, we do not have to consider that unitarity when we consider only an effective theory, . We may need to modify the problem of the divergence of the vacuum energy since the vacuum energy depends on the cutoff scale. Furthermore, as we discussed in chapter 2, the cosmological constant is extremely sensitive to UV physics. Therefore, it is doubtful to consider the effective theory which involves the cosmological constant in this manner. The other is that we may extend the BRS transformation to the multi-fields. In chapter 3, we considered the BRS transformation which makes one of the scalar fields to be a physical quantity. By introducing multi-scalar fields, and considering the generalized BRS transformation, we may keep some of scalar fields to be physical and the other fields to be gauge degrees of freedom avoiding the problem of ghosts.

Appendix A

Other attempts for solving the cosmological constant problem

In chapter 2, we reviewed representative attempts for approaching the cosmological constant problem. There are, however, many other attempts still remain, In this Appendix, other approaches to the cosmological constant problem are given.

A.1 Vacuum energy sequester

The global vacuum energy sequester model has ben proposed in [118, 119]. In this model, the vacuum energy coming from matter can be consistently removed by introducing the global variables to general relativity. Furthermore, we can evade Weinberg's no-go theorem and radiative instability. In addition to this, we can approach the coincidence problem in this model.

A.1.1 The model

In the global vacuum energy sequester model, the action with the classical cosmological constant is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - \Lambda\right] - \int d^4x\sqrt{-\tilde{g}}\mathcal{L}\left(\tilde{g}^{\mu\nu}, \Phi\right) + F\left(\frac{\Lambda}{\lambda^4\mu^4}\right), \tag{A.1.1}$$

where μ expresses scale parameter and λ is chosen as the hierarchy between matter scales $m_{\rm phys}$ and Planck mass $M_{\rm pl}$:

$$\frac{m_{\rm phys}}{M_{\rm pl}} \propto \lambda\,. \tag{A.1.2}$$

We assume that the matter Φ minimally couples to the rescaled metric $\tilde{g}_{\mu\nu} = \lambda^2 g_{\mu\nu}$. The important point of this model is that we regard the Λ and λ as dynamical valuables and we define F as a function of Λ, λ and μ. By the variation of the action in Eq. (A.1.1) with respect to $g^{\mu\nu}$, Λ and λ, we obtain the following equation of motion:

$$M_{\rm pl}^2 G_{\mu\nu} = -\Lambda g_{\mu\nu} + \lambda^4 \tilde{T}_{\mu\nu} \,, \tag{A.1.3}$$

$$\frac{F'}{\lambda^4 \mu^4} = \int d^4x \sqrt{-g} \,, \tag{A.1.4}$$

$$4\Lambda \frac{F'}{\lambda^4 \mu^4} = \int d^4x \sqrt{-g} \lambda^4 \tilde{T} \,, \tag{A.1.5}$$

respectively. Here, we define $F' = \frac{dF(x)}{dx}$ and $\tilde{T}$ is the trace of the energy-momentum tensor $\tilde{T}_{\mu\nu}$ defined by

$$\tilde{T}_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta \tilde{g}^{\mu\nu}} \int d^4x \sqrt{-\tilde{g}} \mathcal{L} \left(\tilde{g}^{\mu\nu}, \Psi \right) \,. \tag{A.1.6}$$

We assume that the space-time volume of the universe is finite. To realize this, we need to impose the following two properties to the universe.

1. The universe is closed, i.e., the curvature is positive $K > 0$.
2. The universe finally goes to Big crunch.

Then, we can consistently define the space-time average quantity as

$$\langle A \rangle = \frac{\int d^4x \sqrt{-g} A}{\int d^4x \sqrt{-g}} \,, \tag{A.1.7}$$

where A is an arbitrary quantity. A quantity $\langle A \rangle$ is well defined if the above two properties are satisfied. Therefore, the Eqs. (A.1.4) and (A.1.5) become

$$\Lambda = \frac{1}{4} \langle T \rangle \,, \quad \lambda^{-4} T_{\mu\nu} \equiv \tilde{T}_{\mu\nu} \,. \tag{A.1.8}$$

Then, Eq. (A.1.3) can be rewritten as

$$M_{\rm Pl}^2 G_{\mu\nu} = T_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \langle T \rangle \,. \tag{A.1.9}$$

Note that the vacuum energy contribution Λ to the equation is irrelevant. When we split the $T_{\mu\nu}$ into the vacuum energy contribution $V_{\rm vac}$ and other parts $T_{\mu\nu}^{(0)}$ as

$$T_{\mu\nu} = g_{\mu\nu} V_{\rm vac} + T_{\mu\nu}^{(0)} \,, \tag{A.1.10}$$

Eq. (A.1.9) becomes

$$M_{\rm Pl}^2 G_{\mu\nu} = T_{\mu\nu}^{(0)} - \frac{1}{4} g_{\mu\nu} \left\langle T^{(0)} \right\rangle \,. \tag{A.1.11}$$

Therefore, the vacuum energy is consistently removed in the equation of motion in Eq. (A.1.12) and we find that the effective cosmological constant in the sequester model $\Lambda_{\rm seq}$ is expressed as

$$\Lambda_{\rm seq} \equiv \left\langle T^{(0)} \right\rangle . \tag{A.1.12}$$

Note that this quantity only depends on the space-time average of the matter fields. In other words, the effective cosmological constant is not sensitive to the radiative correction and we can avoid radiative instability. We also note that we can evade Weinberg's no-go theorem in this manner. Finally, we mention about the relation of coincidence problem between the sequester model. Since the effective cosmological constant is expressed by the space-time average of the trace of energy-momentum tensor of matter, it can be the clue of solving the coincidence problem. If we take the proper boundary of space-time, the ratio of the cosmological constant and matter can be comparable.

A.1.2 Relation with naturalness

We review the sequester model in the perspective of naturalness. In this model, there are two types of symmetries. One is the scale symmetry which is described as

$$\Lambda \Rightarrow A^4\Lambda, \quad \lambda \Rightarrow A\lambda, \quad g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{\rm Pl}} \to \frac{\eta_{\mu\nu}}{A^2} + \frac{h_{\mu\nu}}{M_{\rm Pl}A} . \tag{A.1.13}$$

Then, the action becomes

$$S \to S + \mathcal{O}\left(\frac{1}{M_{\rm Pl}}\right) . \tag{A.1.14}$$

When we take $M_{\rm Pl} \to \infty$, the action is invariant under the scale transformation in Eq. (A.1.13). The other is the shift symmetry written as

$$\mathcal{L} \Rightarrow \mathcal{L} + B, \quad \Lambda \Rightarrow \Lambda + B\lambda^4 . \tag{A.1.15}$$

Then the action is also invariant under the above transformation as long as $\mu \to \infty$:

$$S \to S + \mathcal{O}\left(\frac{1}{\mu^4}\right) . \tag{A.1.16}$$

Therefore, since the symmetries are enhanced if we take the inverse of the parameters set to zero, the smallness of the effective cosmological constant $\Lambda_{\rm seq}/M_{\rm Pl}^2$ is guaranteed by naturalness.

A.1.3 Phase transition is irrelevant

We show that the phase transitions of the universe are negligible for the effective cosmological constant. For simplicity, we assume that the potential are constant before and after the phase transition $V_{\rm b}$ and $V_{\rm a}$. Then, the equations of motion before and after phase transition become

$$M_{\rm Pl}^2 G_{\mu\nu} = T_{\mu\nu}^{(0)} - \frac{1}{4} g_{\mu\nu} \left\langle T^{(0)} \right\rangle = \left\{ \begin{array}{ll} -g_{\mu\nu} \left\langle V_{\rm b} - V \right\rangle & t < t_* \\ -g_{\mu\nu} \left\langle V_{\rm a} - V \right\rangle & t > t_* \end{array} \right. , \tag{A.1.17}$$

where t_* is the time of the phase transition. If we define the ΔV as $\Delta V \equiv V_{\rm b} - V_{\rm a}$, for an early phase transition, we obtain

$$\begin{aligned} \left\langle V_{\rm a} - V \right\rangle &= -\Delta V \frac{\int_{t_{\rm i}}^{t_*} dt a^3}{\int_{t_{\rm i}}^{t_{\rm f}} dt a^3} \ll \Delta V \,, \\ \left\langle V_{\rm b} - V \right\rangle &\sim \mathcal{O}(1) \Delta V \,, \end{aligned} \tag{A.1.18}$$

where a is scale factor, $t_{\rm i}$ and $t_{\rm f}$ is the starting and the ending time of the universe respectively. Therefore, after the phase transition, the difference of the value of the vacuum energy is irrelevant if the phase transitions occur early time. This model have good properties to approach the cosmological constant problem. It seems, however, remains the important problem. Remember that to define the space-time average in Eq. (A.1.7), the space-time volume of the universe should be finite. It may cause the problem of causality. We have to impose that our universe goes to Big crunch in advance and it may break the causality.

A.2 Quantum cosmology

One of the approaches of quantizing the whole universe is called quantum cosmology. Under the several assumptions, we can calculate the probability for $\Lambda = 0$ in the method of quantum cosmology. The wave function can be calculated by the Wheeler-DeWitt equation which derives from Dirac's quantization [120] .

A.2.1 ADM decomposition

We use the Hamiltonian formulation for the canonical quantization. When we apply the Hamiltonian formulation to the relativistic theory, we decompose the space-time into time and space, called Arnowitt, Deser, Misner (ADM) decomposition [121]. Then, the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ becomes

$$ds^2 = N^2dt^2 + h_{ab}\left(N^a dt + dx^a\right)\left(N^b dt + dx^b\right) , \tag{A.2.1}$$

where N is a lapse function, N_a is a shift vector, and h_{ab} is a three-dimensional spacial metric. We decompose the space-time into the time direction and three-dimensional hypersurface Σ_t which is orthogonal to the time direction. We can express the four-dimensional Ricci scalar ${}^{(4)}R$ by using N, N_a, h_{ab} and three-dimensional Ricci scalar ${}^{(3)}R$ as

$${}^{(4)}R ={}^{(3)} R + K_{ab}K^{ab} - K^2 - 2\nabla^b(n_a\nabla^a n_b - n_b\nabla^a n_a)\,, \tag{A.2.2}$$

where n^a is a normal unit vector on Σ_t, K_{ab} is an extrinsic curvature defined by

$$K_{ab} \equiv \frac{1}{2N}\left(D_aN_b + D_bN_a - \frac{\partial h_{ab}}{\partial t}\right), \quad D_a : \text{covariant derivative on } \Sigma_t\,, \tag{A.2.3}$$

and K is its trace. Then, the Einstein-Hilbert action $S_{\rm EH}$ becomes

$$\begin{aligned} S_{\rm EH} &= \frac{1}{2\kappa^2}\int_V d^4x\sqrt{-g}R \\ &= \frac{1}{2\kappa^2}\int_V \left({}^{(3)}R + K_{ab}K^{ab} - K^2\right) N\sqrt{h} - \frac{1}{\kappa^2}\int_{\partial V}(n_a\nabla^a n_b - n_b\nabla^a n_a)d\Sigma^b\,. \end{aligned} \tag{A.2.4}$$

When the boundary is fixed, we obtain

$$S_{\rm EH} = \frac{1}{2\kappa^2}\int_V N\left({}^{(3)}R + K_{ab}K^{ab} - K^2\right)\sqrt{h}dtd^3x \equiv \frac{1}{2\kappa^2}\int_V Ndtd^3x\mathcal{L}_{\rm EH}\,. \tag{A.2.5}$$

The conjugate momenta for N and N_i is zero, and thus we obtain the primary constraints as

$$p_N \equiv \frac{\partial\mathcal{L}}{\partial\dot{N}} = 0, \quad p_a \equiv \frac{\partial\mathcal{L}}{\partial\dot{N}^a} = 0\,. \tag{A.2.6}$$

A.2.2 Minisuperspace

The is often used for the canonical quantization of gravity. We have to consider an infinite number degrees of phase space in quantum gravity. In the minisuperspace approximation, we treat space of all three-dimensional metric and matter in Riemanian manifold are on the hypersurface Σ_t:

$$\text{Riem}(\Sigma_t) \equiv \{h_{ab}(x), \Phi(x)|x \in \Sigma_t\}\ . \tag{A.2.7}$$

Therefore, we impose an infinite degrees of freedom on the $(h_{ab}(x), \Phi(x))$ with a finite degrees of freedom in this approximation.

A.2.3 Canonical quantization

The momentum conjugate to h_{ab} is given by

$$p_{ab} \equiv \frac{\partial \mathcal{L}}{\partial \dot{h}_{ab}} = \frac{\sqrt{h}}{2\kappa^2}\left(K^{ab} - K h^{ab}\right) . \tag{A.2.8}$$

We then have the canonical commutation relations:

$$\left\{h_{ab}(\boldsymbol{k}), p^{cd}(\boldsymbol{q})\right\} = \delta_a^{(c}\delta_b^{d)}\delta^3\left((\boldsymbol{k}) - (\boldsymbol{q})\right) . \tag{A.2.9}$$

We obtain the Hamiltonian

$$\begin{aligned} H &= \int d^3x \left(p_N \dot{N} + p^a \dot{N}_a + p_{ab}\dot{h}_{ab} - \mathcal{L}_{\mathrm{EH}}\right) \\ &= \int d^3x \left(N\mathcal{H}^g + N_a \mathcal{H}^a\right) , \end{aligned} \tag{A.2.10}$$

Here, $\mathcal{H}^g$ is

$$\begin{aligned} \mathcal{H}^g &\equiv \sqrt{h}\left({}^{(3)}R + K_{ab}K^{ab} - K^2\right) \\ &= 2\kappa^2 G_{abcd}p^{ab}p^{cd} - \frac{\sqrt{h}^{(3)}R}{2\kappa^2} , \end{aligned} \tag{A.2.11}$$

where G_{abcd} is defied by

$$G_{abcd} \equiv \frac{\sqrt{h}}{2}\left(h_{ac}h_{bd} + h_{ad}h_{bc} - h_{ab}h_{cd}\right) . \tag{A.2.12}$$

From the Dirac quantization, we obtain the zero energy Schrödinger equation by using Hamiltonian constraint:

$$\mathcal{H}^g(h_{ab}, p^{ab})\Psi(h_{ab}) = 0 . \tag{A.2.13}$$

Here, $\Psi(h_{ab})$ is called "wave function of the universe". Furthermore, in the canonical quantization, we replace the p^{ab} as

$$p^{ab} \to \frac{1}{i}\frac{\delta}{\delta h_{ab}} . \tag{A.2.14}$$

Then, we obtain the so-called Wheeler-DeWitt equation:

$$\left(2\kappa^2 G_{abcd}\frac{\delta}{\delta h_{ab}}\frac{\delta}{\delta h_{cd}} + \frac{\sqrt{h}^{(3)}R}{2\kappa^2}\right)\Psi(h_{ab}) = 0 . \tag{A.2.15}$$

Hartle and Hawking have proposed no-boundary condition, which leads to

$$\Psi(h_{ab}) \sim \int \mathcal{D}g_{\mu\nu} \exp\left[-S_E\right] , \tag{A.2.16}$$

where S_E is the Euclidean action. This no-boundary condition require also the closed universe and finite volume of the universe as we saw in the previous chapter.

A.2.4 Hawking's approach

Inspired by eleven-dimensional supergravity, Hawking has introduced the three gauge field into the action [122–126]. The action is given by

$$S = -\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R + 2\Lambda_B\right) - \frac{1}{48}\int d^4x\sqrt{-g}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} , \tag{A.2.17}$$

where $F_{\mu\nu\rho\sigma} \equiv \nabla_{[\mu}A_{\nu\rho\sigma]}$ is the field strength of the three form field $A_{\mu\nu\rho}$ which is included in eleven-dimensional super gravity [127]. Then, in the four dimensional Euclidean space-time, the trace of the Einstein equation of motion becomes

$$R \equiv -4\Lambda_{\text{eff}} = -4\left(\Lambda - \kappa^2\omega\right) , \tag{A.2.18}$$

where ω is unknown parameter which is independent of the dimension of space-time. Then, under the no-boundary condition, S_E becomes

$$S_E = -3\pi\frac{M_{\text{Pl}}}{\Lambda_{\text{eff}}} , \tag{A.2.19}$$

and therefore we obtain

$$\Psi \propto \exp\left[3\pi\frac{M_{\text{Pl}}}{\Lambda_{\text{eff}}}\right] . \tag{A.2.20}$$

Note that this wave function of the universe has the peak at $\Lambda_{\text{eff}} = 0$.

A.2.5 Coleman's approach

Coleman has proposed the model which has topological effect of wormholes [129]. This model realize the similar solution in Eq. (A.2.20) without using the three-form gauge fields. Alternatively, we introduce the following action,

$$S_{\text{WH}} = \sum_i \left(a_i + a_i^\dagger\right)\int d^4x\mathcal{O}_i(x) \equiv \sum_i A_i \int d^4x\mathcal{O}_i(x) , \tag{A.2.21}$$

where a_i and $a_i^\dagger$ are the annihilation and creation operator of the early universe, and $\mathcal{O}_i$ is the local operator corresponding to i. The path integral over 4-manifold with a certain boundary condition is given by

$$\int \mathcal{D}g_{\mu\nu}\mathcal{D}\Phi \mathrm{e}^{-S} = \int_* \mathcal{D}g_{\mu\nu}\mathcal{D}\Phi \left\langle E \left| \mathrm{e}^{-(S+S_{\mathrm{wormhole}})} \right| E \right\rangle , \tag{A.2.22}$$

where $*$ denotes that wormholes and early universe are excluded, and $|E\rangle$ is a normalized early universe which depends on the boundary condition. When we impose the no-boundary condition, we obtain

$$a_i \,|E\rangle = 0\,. \tag{A.2.23}$$

$|E\rangle$ can be expanded by using the eigenstates of the A_i as

$$|E\rangle = \int f_E(\alpha)\prod_i d\alpha_i|\alpha\rangle, \quad |\alpha\rangle : \text{Eigenvalue of } A_i \tag{A.2.24}$$

Note that if the scale is sufficiently large, the manifolds are only connected by wormholes. Thus, they can be integrated out over. For the no-boundary condition, the wave function is expressed as

$$\Psi_\alpha^{\mathrm{NB}}(E,\alpha) = \sum_{\text{all manifolds}} e^{-S_{\mathrm{T}}(\alpha)} = e^{-\alpha^2/2}\psi_\alpha^{\mathrm{NB}}(E)Z(\alpha)\,, \tag{A.2.25}$$

where we denote $S_{\mathrm{T}}(\alpha)$ as $S_{\mathrm{T}}(\alpha) \equiv S + S_{\mathrm{WH}}(\alpha)$. Then, we sum over the vacuum to another vacuum which are closely connected each other and obtain

$$\Psi_\alpha^{\mathrm{NB}} \propto \exp\left[\sum_{\text{connected manifold}} e^{-S_{\mathrm{T}}(\alpha)}\right]. \tag{A.2.26}$$

The power of the above quantity can be expressed by using the background gravitational effective action I_{BG} as

$$\sum_{\text{connected manifold}} e^{-S_{\mathrm{T}}(\alpha)} = \sum_{\mathrm{T}} \exp\left[-I_{\mathrm{BG}}\right] , \tag{A.2.27}$$

where subscript T under the Σ means that we sum over the topologies. The leading contribution from I_{BG} for sufficiently large universe is

$$I_{\mathrm{BG}} \sim \int d^4x\sqrt{-g}\Lambda_{\mathrm{re}}(\alpha)\,. \tag{A.2.28}$$

Here, $\Lambda_{\rm re}(\alpha)$ denotes the renormalized cosmological constant. Substituting the above expression into Eq. (A.2.26), we obtain

$$\Psi_\alpha^{\rm NB} \propto \exp\left[\exp\left(3\pi\frac{M_P^2}{\Lambda_{\rm re}}\right)\right] . \tag{A.2.29}$$

Note that the above expression is very similar to Eq. (A.2.20). The limit $\Lambda_{\rm re} \to 0$ give the highest probability of the universe. This approach is, however, violated by including the additional term in Eq. (A.2.28) [130]. If we include the following term in Eq. (A.2.28)

$$\int d^4x\sqrt{-g}\mathcal{O}(1)\eta(8\pi G\Lambda)^2 , \tag{A.2.30}$$

Eq. (A.2.29) becomes

$$\Psi_\alpha^{\rm NB} \propto \exp\left[\exp\left(3\pi\frac{M_P^2}{\Lambda_{\rm re}} + \eta\mathcal{O}(1)\right)\right] . \tag{A.2.31}$$

Note that we can also obtain $|\eta| \to \infty$ for realizing the maximum of the probability. However, it means that the unitarity is violated. This implies that the number of large wormholes is large and an universe is covered by wormholes [130, 134–136].

Furthermore, this approach and Hawking's approach are used only in the method of Euclidean path integral based on no-boundary condition. However, whether it can be applied to quantum gravity or not is not clear in these models.

A.3 Modifying the evaluation of the global Hubble parameter

Recently, Q.Wang, Z. Zhu, and W. G. Unruh have proposed the new method of the evaluation of the vacuum energy [137]. In this method, the vacuum energy does not gravitate as the cosmological constant. This method is motivated by the Wheeler' s foam of space-time [138, 139]. It suggest that the structure of the vacuum is foamy and wee see it as the cosmological constant. Conventionally, we treat the vacuum energy as a constant all over the space-time, and therefore the Hubble parameter is evaluated as $H \propto \Lambda^2$. On the other hand, when we include the fluctuation of the vacuum itself, the correlation of the vacuum strongly changes at the very small region of the each space-time. Then, we see that the global Hubble parameter becomes $H \propto \Lambda e^{-\alpha\Lambda}$. If we take Λ to very large, the Hubble parameter goes to asymptotically, but do not exactly zero. Therefore, we can expect that we obtain the very tiny Hubble parameter which we currently observed.

A.3.1 The new evaluation of the fluctuation of the vacuum energy

For evaluating the fluctuation of the vacuum energy, we consider the following massless scalar field:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} \left(a_k \mathrm{e}^{-ikx} + a_k^\dagger \mathrm{e}^{ikx} \right), \quad \boldsymbol{k}^2 = \omega^2 . \tag{A.3.1}$$

Then, we define the vacuum state by

$$a_{\boldsymbol{k}} |0\rangle = 0 . \tag{A.3.2}$$

We see that the total vacuum energy is constant all over the space-time:

$$H = \int d^3x T_{00} = \int d^3x \frac{1}{2} \left(\dot{\phi}^2 + (\nabla\phi)^2 \right) \sim \int d^3x \left\langle T_{00} \right\rangle \sim \int d^3x \Lambda^4 . \tag{A.3.3}$$

Here,

$$\begin{aligned} T_{00} \equiv \frac{1}{2} \int \frac{d^3k d^3q}{2(2\pi)^3} \left(k^0 q^0 + \boldsymbol{k} \cdot \boldsymbol{q} \right) \Big[& a_{\boldsymbol{k}} a_{\boldsymbol{q}}^\dagger \mathrm{e}^{-i(k-q)x} \\ & + a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{q}} \mathrm{e}^{i(k-q)x} - a_{\boldsymbol{k}} a_{\boldsymbol{q}} \mathrm{e}^{-i(k+q)x} - a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{q}}^\dagger \mathrm{e}^{i(k+q)x} \Big] . \end{aligned} \tag{A.3.4}$$

Note that the magnitude of the fluctuation of the above quantity T_{00} also diverges in the order of Λ^4:

$$\sqrt{\left\langle \left(T_{00} - \left\langle T_{00} \right\rangle \right)^2 \right\rangle} \sim \sqrt{\left\langle T_{00} \right\rangle^2} \sim \Lambda^4 . \tag{A.3.5}$$

In other words, the correlation of the vacuum is extremely fluctuating in each space-time and it is comparable to the magnitude of the total vacuum energy density.

A.3.2 Minimal extension of the FLRW metric

Since the vacuum energy is lo longer constant in space-time, we need to modify the evaluation of that quantity. Therefore, we introduce the extension of the flat FLRW metric by replacing the scale factor $a(t)$ to $a(t, \boldsymbol{x})$ as

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1}^{3} dx^i \to ds^2 = -dt^2 + a^2(t, \boldsymbol{x}) \sum_{i=1}^{3} dx^i . \tag{A.3.6}$$

Then the Einstein equation can be written as

$$\begin{aligned} G_{00} &= 3H^2 + \frac{\partial_i a \partial^i a}{a^4} - \frac{2\partial^2 a}{a^3} = \kappa^2 T_{00}, \\ G_{ij} &= -\left(2a\ddot{a} + \dot{a}^2\right)\delta_{ij} - \frac{\partial_i a \partial_j a}{a^2} + 2\frac{\partial_i a}{a}\frac{\partial_j a}{a} - \frac{\partial_i \partial_j a}{a} = \kappa^2 T_{ij}, \\ G_{0i} &= -2\partial_i H = \kappa^2 T_{0i}\,. \end{aligned} \tag{A.3.7}$$

Here, i and j are spacial indices $i, j = 1, 2, 3$, and we define $\partial^2 \equiv \partial_i \partial^i$ and the local Hubble parameter $H(t, \boldsymbol{x}) \equiv \dot{a}(t, \boldsymbol{x})/a(t, \boldsymbol{x})$. Note that if we take $a(t, \boldsymbol{x}) \to a(t)$, we recover the conventional Friedmann equations. If $i \neq j$, the vacuum expectation value of T_{ij} and T_{0i} are zero:

$$\langle T_{ij} \rangle = \langle T_{0i} \rangle = 0, \quad \text{for } i \neq j \tag{A.3.8}$$

On the other hand, the vacuum expectation value of its squared quantity is large:

$$\sqrt{\langle T_{ij}^2 \rangle} \sim \sqrt{\langle T_{0i} \rangle^2} \sim \sqrt{\langle T_{00} \rangle^2} \sim \sqrt{G}\Lambda^4\,. \tag{A.3.9}$$

As we have seen so far, the space-time fluctuate itself and therefore the quantum vacuum is tremendously inhomogeneous. It implies that the local Hubble parameter is no longer available. From Eq. (A.3.7), we note that the local Hubble parameter H

1. becomes a huge quantity,

$$\sqrt{\langle H^2(t, \boldsymbol{x}) \rangle} \sim \sqrt{G}\sqrt{\langle T_{00} \rangle} \sim \sqrt{G}\Lambda^2 \tag{A.3.10}$$

2. changes drastically at each spatially different point

$$|H(t, \boldsymbol{x}) - H(t, \boldsymbol{y})| \sim \sqrt{\langle T_{0i} \rangle^2}\, |\boldsymbol{x} - \boldsymbol{y}| \sim \sqrt{G}\Lambda^4\, |\boldsymbol{x} - \boldsymbol{y}| \tag{A.3.11}$$

Furthermore, we can see that the local Hubble parameter periodically and strongly changes. From Eq. (A.3.7), we obtain

$$G_{00} + \frac{1}{a^2}\sum_i G_{ii} = -\frac{6\ddot{a}}{a} \tag{A.3.12}$$

This yields time-dependent harmonic oscillator:

$$\ddot{a} + \Pi^2(t, \mathbf{x})a = 0, \quad \Pi^2 = \frac{\kappa^2}{6}\left(T_{00} + \frac{1}{a^2}\sum_{i=1}^{3} T_{ii}\right) = \frac{\kappa^2}{3}\dot{\phi}^2\,. \tag{A.3.13}$$

Note that the frequency is very large because

$$\left\langle \Pi^2 \right\rangle \sim \sqrt{G}\Lambda^4\,. \tag{A.3.14}$$

Therefore, the local Hubble parameter changes periodically and largely between positive and negative value.

$$H(t,\boldsymbol{x}) \sim P(\Pi t), \quad P(\Pi t): \text{ periodic function.} \tag{A.3.15}$$

This period is very short because the frequency is very high $T = 1/\Pi \sim 1/\Lambda^2$. Thus, the local Hubble parameter changes drastically at the spatially and temporary different point. This implies huge global cancellation of the Hubble parameter in a certain large scale of the universe and it may realize the slowly expansion of the current universe. The image is shown in Fig A.1

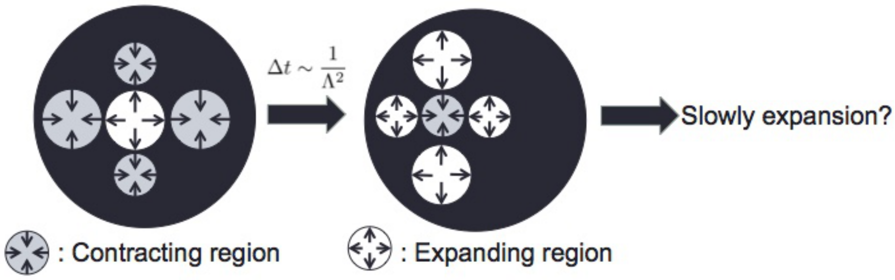

Fig. A.1 Slow expansion in global universe

A.3.3 Global Hubble parameter

We then define the global Hubble parameter by

$$H_{\mathrm{G}}(t) \equiv \frac{\dot{D}(t)}{D(t)}, \quad D(t) = \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} \sqrt{a^2(t,\boldsymbol{x}')}d\boldsymbol{x}'\,. \tag{A.3.16}$$

Here, $D(t)$ express a physical distance which is expressed $D(t) = a(t)\,|\boldsymbol{x}_2 - \boldsymbol{x}_1|$ when $a(t,\boldsymbol{x}) = a(t)$. Although $\Pi(t,\boldsymbol{x})$ is not exactly periodic, it is al least quasi periodic of the order Λ^{-1}. Then, Eq. (A.3.13) can be written as,

$$a(t,\mathbf{x}) \propto \exp\left[\int^t h(t',\boldsymbol{x})dt'\right] F_{\mathrm{QP}}(t,\boldsymbol{x}), \quad h(t,\boldsymbol{x}) > 0\,. \tag{A.3.17}$$

Here, $F_{\mathrm{QP}}(t,\boldsymbol{x})$ is quasi periodic function of order Λ^{-1}. Substituting the above quantity into Eq. (A.3.16), we obtain $D(t)$ and the relation between the global Hubble parameter H_{G} and $h(t,\boldsymbol{x})$ as

$$D(t) = D_0 \mathrm{e}^{H_{\mathrm{G}}t}, \quad D_0 \equiv \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} \sqrt{F_{\mathrm{QP}}^2(t,x)}d\boldsymbol{x}, \quad H_{\mathrm{G}} = \frac{1}{t}\int^t h(t',\boldsymbol{x})dt'\,. \tag{A.3.18}$$

We now investigate a solution for F_{QP}. We assume that we can regard $a(t,\boldsymbol{x})$ as classical quantity as a kind of semiclassical approximation and only quantize the fields propagating on $a(t,\boldsymbol{x})$. It is difficult of fully solve the quasi periodic harmonic oscillator and obtain the global Hubble parameter H_{G}. It is, however, can be solved if the following slowly varying condition is satisfied:

$$T\frac{d\Pi}{dt} \ll \Pi\,. \tag{A.3.19}$$

Actually the above condition is satisfied in this model when $\Lambda \to \infty$. Because

$$\sqrt{\langle \Pi^2 \rangle} \sim G^{1/4}\Lambda^2, \quad T \sim \frac{1}{\Pi} \sim \frac{1}{\sqrt{G}\Lambda^2}, \quad \sqrt{\left\langle \left(\frac{d\Pi}{dt}\right)^2 \right\rangle} \sim \sqrt{G}\Lambda^3\,, \tag{A.3.20}$$

we have

$$T\frac{d\Pi}{dt} \sim \Lambda, \quad \Pi \sim \Lambda^2\,, \tag{A.3.21}$$

and therefore, we surely have

$$T\frac{d\Pi}{dt} \ll \Pi \quad \text{for} \quad \Lambda \to \infty\,. \tag{A.3.22}$$

Then, we can solve the H_{G} by using adiabatic approximation [140]:

$$H_{\mathrm{G}} \sim \Lambda e^{-\alpha\sqrt{G}\Lambda}, \quad \alpha \ : \ \text{dimensionless parameter}\,. \tag{A.3.23}$$

Note that if we take $\Lambda \to \infty$, the global Hubble parameter behave $H_{\mathrm{G}} \to 0$ asymptotically. Therefore, we can easily obtain the small but non-zero global Hubble parameter.

A.3.4 Back reaction is irrrelevant

For $\Lambda \to \infty$, the back reaction can be ignored since the vacuum play a role of huge energy pool. While the effect of the back reaction only occurs at the length $\sim \Lambda^{-1}$, the fluctuating space-time occurs at the length of $\sim \frac{1}{\sqrt{G}\Lambda^2}$. Therefore, the back reaction effect is much smaller than the fluctuating effect of space-time if we take $\Lambda \to \infty$ as shown in Fig A.2. This model is inspired by the foamy structure of the vacuum by Wheeler and the resulting value of the global Hubble parameter is reasonable to reproduce our universe. It has, however, a weak point when we take the cutoff scale. From Eq. (A.3.23), we need to take Λ larger than the Planck mass to obtain a small value of H_{G} since $\sqrt{G} \sim M_{\mathrm{Pl}}^{-1}$. When we consider the energy scale beyound the Planck scale, we have to take the effect of quantum gravity into consideration and the approximation that we treat the $a(t,\boldsymbol{x})$ as a classical quantity could be violated.

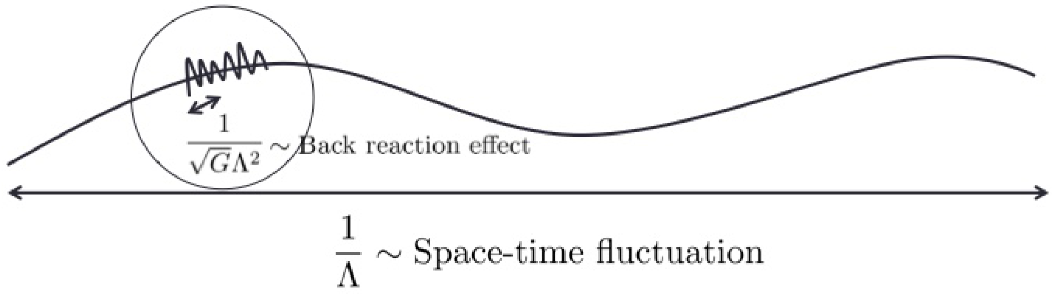

Fig. A.2 The scale of the fluctuating space-time and the back reaction

A.3.5 Other attempts

We reviewed the several attempts for the cosmological constant problem. There still remains many approaches. Universe multiplication by Linde assume the mirror universe with negative energy and the vacuum energy contributions are cancelled each other [141–143]. The super symmetric large extra dimension scenario [144–147] use the six-dimensional supergravity [148, 149]. In this scenario, the universe is described by one of two three-branes which is not supersymmetric and four dimensional vacuum energy only curves the two extra dimension. The idea of fat graviton has proposed by Sundrum [150]. This is a kind of effective field theories with graviton length $\sim$ 1meV. As a result, the contribution of tadpole diagram is suppressed and vacuum energy does not contribute to the cosmological constant. We leave off other attempts in this chapter, but we emphasize that there still exist several approaches to the cosmological constant problem.

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