Multiple Zeta Values and Multi-Poly-Bernoulli Numbers in Positive Characteristic
(正標数における多重ゼータ値と多重ポリ Bernoulli 数)

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Abstract

It is said that there are various analogies between the arithmetic of number fields (characteristic 0 case) and function fields (positive characteristic case). Finding analogues is one of fundamental issues in positive characteristic number theory. In this thesis, we study positive characteristic analogue of multiple zeta values with their variants and positive characteristic analogue of multi-poly-Bernoulli numbers.

In Chapter 1, we discuss four formulae conjectured by J. A. Lara Rodríguez on certain power series in characteristic p, which yield sum-shuffle relations for positive characteristic analogue of double zeta values. We prove affirmatively the first two formulae. While we detect and correct errors in the last two formulae, and prove them. In Chapter 2, we introduce and study multi-poly-Bernoulli-Carlitz numbers (MPBCNs in short), a positive characteristic analogue of the multi-poly-Bernoulli numbers. In characteristic 0 case, the multi-poly-Bernoulli numbers (MPBNs in short) were introduced by Imatomi-Kaneko-Takeda in 2014. They found an explicit formula to express MPBNs in terms of factorials and Stirling numbers of the second kind. They also proved a certain relation between MPBNs and finite multiple zeta values. As a positive characteristic analogue of their result, we prove an explicit formula to express MPBCNs in terms of factorials and Stirling numbers of the second kind in positive characteristic. We also show a relation which connects MPBCNs with a positive characteristic analogue of finite multiple zeta values. In Chapter 3, we introduce positive characteristic alternating multiple zeta values. In characteristic 0 case, the alternating multiple zeta values are twisted generalization of multiple zeta values by units of integers. In positive characteristic case, we define their positive characteristic analogue by twisting them with units of 1-variable polynomial ring over finite fields. We explore their properties including non-vanishing property, sum-shuffle relation, period interpretation and linear independence. As a by-product, we settle a positive characteristic analogue of alternating multiple zeta values version of Goncharov's conjecture.

This docter thesis is based on three papers [**H18b, H18c, H19**].

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Contents

CHAPTER 0

Introduction

In §0.1, we recall multiple zeta values, the sum-shuffle relations, multi-poly-Bernoulli numbers, finite multiple zeta values and alternating multiple zeta values and their several properties. In §0.2, we explain their characteristic p counterparts. Further we briefly describe our main results in the thesis (cf. [**H18b, H18c, H19**]) which are characteristic p counterparts of these properties for the corresponding notions.

0.1. Characteristic 0 case

In §0.1, we recall some definitions whose characteristic p analogues are seen in §0.2. In §0.1.1, we recall the multiple zeta values and sum-shuffle relations. In §0.1.2, we recall Imatomi-Kaneko-Takeda's multi-poly-Bernoulli numbers. We introduce their result, that is, an explicit formula of multi-poly-Bernoulli numbers and a relation with Kaneko-Zagier's finite multiple zeta values. In §0.1.3, we recall alternating multiple zeta values which are certain generalization of multiple zeta values.

0.1.1. Multiple zeta values and sum-shuffle relations. We recall *multiple zeta values* in characteristic 0 which is a multi-variable generalization of *Riemann zeta values*,

$$\zeta(s) := \sum_{m>0} \frac{1}{m^s} \in \mathbb{R} \tag{0.1.1}$$

for $s \in \mathbb{Z}_{>1}$. It is said that the research of multiple zeta values is originated from Euler's work [**E**] in 1776.

Definition 0.1.1. For $\mathfrak{s} := (s_1, \ldots, s_r) \in \mathbb{N}^r$ and $s_1 > 1$ (the condition to be convergent),

$$\zeta(\mathfrak{s}) := \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}} \in \mathbb{R}. \tag{0.1.2}$$

We note r is called the depth and $\sum_{i=1}^{r} s_i$ is called the weight of the presentation of $\zeta(s_1, \ldots, s_r)$.

In his paper [**E**], Euler proposed three methods to calculate relations for double zeta values (depth 2 multiple zeta values) with non-mathematical proofs and unconventional notations (they were reformulated with mathematical proofs and conventional modern notations in [**H18a**]).

The multiple zeta values are special values of the *multiple polylogarithms* (MPLs in short). For $\mathfrak{s} := (s_1, \ldots, s_r) \in \mathbb{Z}^r$ and r-variables $z_1, \ldots, z_r$, MPLs are defined by

$$\mathrm{Li}_{\mathfrak{s}}(z_1, \ldots, z_r) := \sum_{m_1 > \cdots > m_r > 0} \frac{z_1^{m_1} \cdots z_r^{m_r}}{m_1^{s_1} \cdots m_r^{s_r}} \in \mathbb{Q}[[z_1, \ldots, z_r]]. \tag{0.1.3}$$

Clearly, for $\mathfrak{s} := (s_1, \ldots, s_r) \in \mathbb{N}^r$ with $s_1 > 1$, MPLs recover multiple zeta values as the $(z_1, \ldots, z_r) = (1, \ldots, 1)$ case.

It is known that the multiple zeta values satisfy the *sum-shuffle relations* (also known as stuffle relations, harmonic product relations) which are derived from series expression of the multiple zeta values. The easiest one is described as follows:

$$\zeta(m)\zeta(n) = \zeta(m, n) + \zeta(n, m) + \zeta(m + n)$$

for $m, n \geq 2$. We can extend this relation to the higher depth case by using induction on the depth.

Remark 0.1.2. It is known that the multiple zeta values also satisfy the *integral shuffle relations* which are derived from integral expression of the multiple zeta values. The easiest one is described as follows:

$$\zeta(m)\zeta(n) = \sum_{i=0}^{n-1} \binom{m-1+i}{i} \zeta(m+i, n-i) + \sum_{j=0}^{m-1} \binom{n-1+j}{j} \zeta(n+j, m-j)$$

for $m, n \geq 2$.

By sum-shuffle relations, we see that a $\mathbb{Q}$-vector space generated by 1 and all multiple zeta values forms a $\mathbb{Q}$-algebra. On the $\mathbb{Q}$-algebra we have the following folklore conjecture:

Conjecture 0.1.3. Let $\overline{\mathfrak{Z}}$ be the $\overline{\mathbb{Q}}$-algebra generated by multiple zeta values and $\overline{\mathfrak{Z}}_w$ be the $\overline{\mathbb{Q}}$-vector space spanned by the weight w multiple zeta values for $w \geq 2$. Then one has the following:

(i) $\overline{\mathfrak{Z}}$ forms an weight-graded algebra, that is, $\overline{\mathfrak{Z}} = \overline{\mathbb{Q}} \oplus_{w \geq 2} \overline{\mathfrak{Z}}_w$;
(ii) $\overline{\mathfrak{Z}}$ is defined over $\mathbb{Q}$, i.e. the canonical map $\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathfrak{Z} \to \overline{\mathfrak{Z}}$ is bijective.

We note that this conjecture is a stronger version of Goncharov's conjecture in [**Go98**, Conjecture 4.2] where he stated a similar conjecture over $\mathbb{Q}$ instead of $\overline{\mathbb{Q}}$.

0.1.2. The multi-poly-Bernoulli numbers and Kaneko-Zagier's finite multiple zeta values. In 1997, M. Kaneko introduced and investigated generalizations of the Bernoulli numbers, the poly-Bernoulli numbers in [**K97**]. He obtained explicit formulae for the poly-Bernoulli numbers which includes the Stirling numbers of the second kind. He and T. Arakawa found that poly-Bernoulli numbers express the special values of Arakawa-Kaneko zeta functions at negative integers in [**AK99**]. After 2000, the several multi-poly-Bernoulli numbers, generalizations of the poly-Bernoulli numbers, were posted by several researchers, Hamahata-Masubuchi [**HM07**], Imatomi-Kaneko-Takeda [**IKT14**] and M.-S. Kim-T. Kim [**KiK**] in different ways from each other.

The *Bernoulli numbers* B_n $(n = 0, 1, \ldots)$ are rational numbers defined by the following generating function

$$\sum_{n=0}^{\infty} B_n \frac{z^n}{n!} := \frac{ze^z}{e^z - 1}. \tag{0.1.4}$$

It is known that we have the following equation

$$B_n = 0 \quad (\text{for } n \geq 3 \text{ so that } 2 \nmid n). \tag{0.1.5}$$

Moreover, the Bernoulli numbers are expressed as follows [**Kr**]:

$$B_n = (-1)^n \sum_{m=1}^{n+1} \frac{(-1)^{m-1}(m-1)!}{m} \left\{ {n \atop m-1} \right\}, \tag{0.1.6}$$

where $\left\{ {n \atop m} \right\} \in \mathbb{Z}$ are the *Stirling numbers of the second kind* defined by

$$\frac{(e^z-1)^m}{m!} = \sum_{n=0}^{\infty} \left\{ {n \atop m} \right\} \frac{z^n}{n!}. \tag{0.1.7}$$

Imatomi-Kaneko-Takeda concerned the following two types of the multi-poly-Bernoulli numbers which generalize the Bernoulli numbers.

Definition 0.1.4 ([**IKT14**, (1)] and [**AK99**, (8)]). For $\mathfrak{s} := (s_1, \ldots, s_r) \in \mathbb{Z}^r$, the *multi-poly-Bernoulli numbers* (MPBNs for short) *of B-type, C-type* are the rational numbers which are defined by the following generating functions respectively (for $Li_{\mathfrak{s}}(z_1, \ldots, z_r)$, see (0.1.3)):

$$\sum_{n=0}^{\infty} B_n^{\mathfrak{s}} \frac{z^n}{n!} := \frac{\mathrm{Li}_{\mathfrak{s}}(1-e^{-z}, \overbrace{1, \ldots, 1}^{r-1})}{1-e^{-z}},$$

$$\sum_{n=0}^{\infty} C_n^{\mathfrak{s}} \frac{z^n}{n!} := \frac{\mathrm{Li}_{\mathfrak{s}}(1-e^{-z}, \overbrace{1, \ldots, 1}^{r-1})}{e^{z}-1}.$$

Remark 0.1.5. When $r = 1$, B_n^s and C_n^s are the *poly-Bernoulli numbers of B-type, C-type* (cf. [**AK99, K97**]), respectively. When $r = 1$ and $s_1 = 1$, $\mathrm{Li}_{\mathfrak{s}}(z_1, \ldots, z_r) = -\log(1-z)$ and then $B_n^{(1)}$ agrees with (0.1.4) of the Bernoulli numbers. We note that $B_1^{(1)} = 1/2$ and $C_1^{(1)} = -1/2$ and $B_n^{(1)} = C_n^{(1)} = B_n$ for $n \neq 1$.

In [**IKT14**], they obtained the explicit formulae for MPBNs which are the following finite sums involving the Stirling numbers of the second kind.

Proposition 0.1.6 ([**IKT14**, Theorem 3]). *For $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{Z}^r$ and $n \geq 0$, we have*

$$B_n^{\mathfrak{s}} = (-1)^n \sum_{n+1 \geq m_1 > m_2 > \cdots > m_r > 0} (-1)^{m_1-1}(m_1-1)! \left\{ {n \atop m_1-1} \right\} \frac{1}{m_1^{s_1} \cdots m_r^{s_r}}$$

and

$$C_n^{\mathfrak{s}} = (-1)^n \sum_{n+1 \geq m_1 > m_2 > \cdots > m_r > 0} (-1)^{m_1-1}(m_1-1)! \left\{ {n+1 \atop m_1} \right\} \frac{1}{m_1^{s_1} \cdots m_r^{s_r}}.$$

Imatomi-Kaneko-Takeda also showed that the following relations hold between the MPBNs and the Bernoulli numbers for the special case $(s_1, \ldots, s_r) = (1, \ldots, 1)$.

Proposition 0.1.7 ([**IKT14**, Proposition 4]). *For $r \geq 1$ and $n \geq r-1$, we have*

$$B_n^{(\overbrace{1,\ldots,1}^{r})} = \frac{1}{n+1} \binom{n+1}{r} B_{n-r+1}^{(1)},$$

$$C_n^{(\overbrace{1,\ldots,1}^{r})} = \frac{1}{n+1} \binom{n+1}{r} C_{n-r+1}^{(1)}.$$

Further, they obtained relations which connect the MPBNs and the finite multiple zeta values. The finite multiple zeta values are variants of the multiple zeta values which are introduced by Kaneko and Zagier in [**KZ**].

Definition 0.1.8 ([**KZ**]). We set a $\mathbb{Q}$-algebra $\mathscr{A}$ by

$$\mathscr{A} := \prod_p \mathbb{Z}/p\mathbb{Z} \Big/ \bigoplus_p \mathbb{Z}/p\mathbb{Z}$$

where p runs over all prime numbers. For $\mathfrak{s} := (s_1, \dots, s_r) \in \mathbb{Z}^r$, the *finite multiple zeta values* are defined by

$$\zeta_{\mathscr{A}}(\mathfrak{s}) := (\zeta_{\mathscr{A}}(\mathfrak{s})_{(p)}) \in \mathscr{A}$$

where

$$\zeta_{\mathscr{A}}(\mathfrak{s})_{(p)} := \sum_{p > m_1 > \cdots > m_r > 0} \frac{1}{m_1^{s_1} \cdots m_r^{s_r}} \in \mathbb{Z}/p\mathbb{Z}.$$

By the work of [**IKT14**], the finite multiple zeta values are related to multi-poly-Bernoulli numbers.

Proposition 0.1.9 ([**IKT14**, Theorem 8]). *For $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{Z}^r$, we have*

$$\zeta_{\mathscr{A}}(\mathfrak{s})_{(p)} = -C_{p-2}^{(s_1-1, s_2, \dots, s_r)} \mod p$$

and for $d \geq 0$,

$$\zeta_{\mathscr{A}}(\underbrace{1, \dots, 1}_{d}, s_1, \dots, s_r)_{(p)} = -C_{p-d-2}^{(s_1-1, s_2, \dots, s_r)} \mod p.$$

Here we note that the second relation generalizes the first relation.

0.1.3. The alternating multiple zeta values. The *alternating multiple zeta values*[1] are twisted generalizations of multiple zeta values by units of integers and defined as the following series:

Definition 0.1.10. For $\mathfrak{s} := (s_1, \dots, s_r) \in \mathbb{N}^r$, $\boldsymbol{\epsilon} := (\epsilon_1, \dots, \epsilon_r) \in \{\pm 1\}^r$ with $(s_1, \epsilon_1) \neq (1, 1)$,

$$\zeta(\mathfrak{s}; \boldsymbol{\epsilon}) := \sum_{n_1 > \cdots > n_r > 0} \frac{\epsilon_1^{n_1} \cdots \epsilon_r^{n_r}}{n_1^{s_1} \cdots n_r^{s_r}} \in \mathbb{R}.$$

In these two decades, they were also studied by Broadhurst, Deligne-Goncharov, Glanois, Hoffman, Ihara-Kaneko-Zagier and Kaneko-Tsumura. Due to their contributions, now it is known that alternating multiple zeta values have connections to the studies of knots [**Br96a**], Feynman diagrams [**Br96b**], modular forms [**KT13**], mixed Tate motives over $\mathbb{Z}[1/2]$ [**DG05**]. The alternating multiple zeta values also have integral expressions and series expressions and they enjoy sum-shuffle relations and integral shuffle relations [**Go98, Ra02**]. Thus we see that a $\mathbb{Q}$-vector space generated by 1 and all alternating multiple zeta values forms a $\mathbb{Q}$-algebra.

[1] They are also known as Euler sums [**Br96a**] or level 2 colored multiple zeta values [**BJOP02**].

0.2. Characteristic p case

Next we turn into the positive characteristic case. In 1919, Kornblum proved a positive characteristic analogue of Dirichlet prime number theorem in his PhD thesis [**KL**]. It is said that the study of arithmetic in characteristic p has been started after that. Since then, many analogues have been detected by Anderson, Carlitz, Drinfeld, Goss, Thakur, etc (for more detail, see [**G96, T04**]). Those analogies enable us to consider and develop the arithmetic of characteristic p. Hence finding positive characteristic analogues is one of fundamental issues.

In §0.2.1, we describe Lara Rodríguez' full conjecture which gives the sum-shuffle relations for double zeta values in characteristic p with specific indices. We settle the conjecture (partially affirmatively and partially negatively). In §0.2.2, we introduce a characteristic p analogue of multi-poly-Bernoulli numbers and characteristic p counterparts of results by Imatomi-Kaneko-Takeda [**IKT14**]. In §0.2.3, we introduce characteristic p analogue of alternating multiple zeta values. We recall their definition and briefly describe their four fundamental properties which consist of non-vanishing properties, sum-shuffle relations, period interpretations and linear independence.

We recall the following notations which are used in the rest of this thesis.

q	a power of a prime number p.
$\mathbb{F}_q$	a finite field with q elements.
θ, t	independent variables.
A	the polynomial ring $\mathbb{F}_q[\theta]$.
A_+	the set of monic polynomials in A.
A_{d+}	the set of elements of A_+ of degree d.
k	the rational function field $\mathbb{F}_q(\theta)$.
k_∞	the completion of k at the infinite place ∞, $\mathbb{F}_q((\frac{1}{\theta}))$.
$\overline{k_\infty}$	a fixed algebraic closure of k_∞.
$\mathbb{C}_\infty$	the completion of $\overline{k_\infty}$ at the infinite place ∞.
$\overline{k}$	a fixed algebraic closure of k in $\mathbb{C}_\infty$.
$\lvert\cdot\rvert_\infty$	a fixed absolute value for the completed field $\mathbb{C}_\infty$ so that $\lvert\theta\rvert_\infty = q$.
$\mathbb{T}$	the Tate algebra over $\mathbb{C}_\infty$, the subring of $\mathbb{C}_\infty[[t]]$ consisting of power series convergent on the closed unit disc $\lvert t\rvert_\infty \le 1$.
D_i	$\prod_{j=0}^{i-1}(\theta^{q^i} - \theta^{q^j}) \in A_+$ where $D_0 := 1$.
L_i	$\prod_{j=1}^{i}(\theta - \theta^{q^j}) \in A_+$ where $L_0 := 1$.
Γ_{n+1}	the Carlitz gamma, $\prod_i D_i^{n_i}$ $(n = \sum_i n_i q^i \in \mathbb{Z}_{\ge 0}$ $(0 \le n_i \le q-1))$.
$\Pi(n)$	the Carlitz factorial, Γ_{n+1}
$\mathrm{Int}(x)$	$= \begin{cases} 0 & \text{if } x \text{ is not an integer,} \\ 1 & \text{if } x \text{ is an integer.} \end{cases}$

0.2.1. The multiple zeta values and sum shuffle relations in characteristic p. We recall the characteristic p analogues of multiple zeta values which were invented by Thakur [**T04**]. Firstly, we recall the power sums. For $s \in \mathbb{Z}$ and $d \in \mathbb{Z}_{\ge 0}$, power sums are defined by

$$S_d(s) := \sum_{a \in A_{d+}} \frac{1}{a^s} \in k.$$

For $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ and $d \in \mathbb{Z}_{\geq 0}$, we define

$$S_d(\mathfrak{s}) := S_d(s_1) \sum_{d > d_2 > \cdots > d_r \geq 0} S_{d_2}(s_2) \cdots S_{d_r}(s_r) \in k.$$

For $s \in \mathbb{Z}_{\geq 0}$, the *Carlitz zeta values* [**Ca35**] are defined by

$$\zeta_A(s) := \sum_{a \in A_+} \frac{1}{a^s} \in k_\infty.$$

These values are characteristic p analogues of (0.1.1). Thakur generalized this definition to that of *multiple zeta values in characteristic* p.

Definition 0.2.1. For $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$,

$$\zeta_A(\mathfrak{s}) := \sum_{d_1 > \cdots > d_r \geq 0} S_{d_1}(s_1) \cdots S_{d_r}(s_r) = \sum_{\substack{a_1, \ldots, a_r \in A_+ \\ \deg a_1 > \cdots > \deg a_r \geq 0}} \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in k_\infty.$$

These values are characteristic p analogues of (0.1.2). For $a, b \in \mathbb{Z}_{>0}$, we define

$$\Delta_d(a, b) := S_d(a) S_d(b) - S_d(a + b).$$

The following relation holds by summing $\Delta_d(a, b)$ over d.

$$\zeta_A(a)\zeta_A(b) = \zeta_A(a, b) + \zeta_A(b, a) + \zeta_A(a + b) + \sum_{d=0}^{\infty} \Delta_d(a, b). \tag{0.2.1}$$

Remark 0.2.2. By the lack of integral expression of multiple zeta values in characteristic p, it is not known if we have the integral shuffle relations for multiple zeta values in characteristic p.

In [**La10**], J. A. Lara Rodríguez conjectured precise formulation of the relations in the case of depth 2 with special weights. This conjecture contained five formulae and the first formula was proved by himself in [**La12**]. The rests are stated as follows:

Conjecture 0.2.3 (Conjecture 1.2.1, [**La10**, Conjecture 2.8]). For $n, d \in \mathbb{Z}_{\geq 1}$ and general q, we have

(0.2.2)

$$\Delta_d(q^n + 1, q^n) = \mathrm{Int}\left(\frac{2}{q}\right) S_d(2, 2q^n - 1) - \sum_{j=1}^{\frac{q^n - 1}{q-1}} S_d\Big(3 + (j-1)(q-1), 2q^n - 2 - (j-1)(q-1)\Big).$$

(0.2.3)

$$\Delta_d(q^n - 1, q^n + 1) = -\sum_{j=1}^{\frac{q^n + q - 2}{q-1}} S_d\Big(2 + (j-1)(q-1), 2q^n - 2 - (j-1)(q-1)\Big).$$

(0.2.4)

$$\Delta_d(q^{n-1}, q^n + 1) = \mathrm{Int}\left(\frac{2}{q}\right) S_d\Big(2, q^n + q^{n-1} - 1\Big) - \sum_{j=1}^{\frac{q^{n-1} - 1}{q-1}} S_d\Big(3 + (j-1)(q-1), q^n + q^{n-1} - 2 + (j-1)(q-1)\Big).$$

For $0 \leq i \leq n$, we have

(0.2.5)

$$\begin{aligned}\Delta_d(q^n+1, q^n+1-q^i) = \mathrm{Int}\left(\frac{2}{q}\right) & S_d(2, 2q^n-q^i) \\ & - \sum_{j=1}^{\frac{q^n-q^i}{q-1}} S_d\Big(3+(j-1)(q-1), 2q^n-q^i-1-(j-1)(q-1)\Big) \\ & + \sum_{j=\frac{q^n-q^i}{q-1}+1}^{\frac{q^n-q^i}{q-1}} S_d\Big(3+(j-1)(q-1), 2q^n-q^i-1-(j-1)(q-1)\Big).\end{aligned}$$

By using H. J. Chen's result in [**Chen15**], we will prove affirmatively the second and third formulae in Theorem 1.3.1 and 1.3.2. Whereas we detect and correct errors in the fourth and fifth formulae, and prove corrected ones in Theorem 1.3.3 and 1.3.4. These results are stated as follows:

THEOREM 0.2.4 ([**H18b**, Theorem 3-6]). *In the Lara Rodríguez' full conjecture, the two relations* (0.2.2) *and* (0.2.3) *hold while the other two corrected* (0.2.4) *and* (0.2.5) *hold.*

By this theorem, we obtain sum-shuffle relations for characteristic p double zeta values with certain indices. Those relations are stated in Corollary 1.3.5.

0.2.2. The multi-poly-Bernoulli numbers and finite multiple zeta values in characteristic p. In 1935, L. Carlitz [**Ca35**] introduced and investigated characteristic p analogues of the Bernoulli numbers, the Bernoulli-Carlitz numbers BC_n. By using them, he obtained an analogue of Euler's famous formula $\zeta(m) = -\frac{(2\pi i)^m}{2(m!)}B_m$ (for even m) in [**Ca35**] and the von Staudt-Clausen theorem in [**Ca37, Ca40**].[2] E. Gekeler proved several identities for the Bernoulli-Carlitz numbers in [**Ge89**]. Furthermore, H. Kaneko and T. Komatsu obtained explicit formulae for Bernoulli-Carlitz numbers by using characteristic p analogues of the Stirling numbers in [**KK16**].

In Chapter 2, we introduce and study characteristic p analogues of the multi-poly-Bernoulli numbers which are related to characteristic p analogue of finite multiple zeta values. The results in Chapter 2 are characteristic p analogues of the results in [**IKT14**].

A characteristic p analogues of multi-poly-Bernoulli numbers are defined by the following generating function:

DEFINITION 0.2.5 (Definition 2.2.7, [**H18c**, Definition 21]). For $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, $\mathbf{j} = (j_1, \ldots, j_r) \in J_{\mathfrak{s}}$ (for $J_{\mathfrak{s}}$, see Notation 2.1.8), we define *multi-poly-Bernoulli-Carlitz numbers* (MPBCNs for short) $BC_n^{\mathfrak{s},\mathbf{j}}$ to be elements of k as follows:

$$\sum_{n\geq 0} BC_n^{\mathfrak{s},\mathbf{j}} \frac{z^n}{\Pi(n)} := \frac{Li_{\mathfrak{s}}(e_C(z)u_{1j_1}, u_{2j_2}, \ldots, u_{rj_r})}{e_C(z)}.$$

Here definition of u_{ij_i} $(1 \leq i \leq r)$ is described in Notation 2.1.8.

[2] An analogue of von Staudt-Clausen theorem stated in [**Ca37, Ca40**] was corrected by L. Carlitz [**Ca41**] for $q = 2$.

We also discuss generalizations of the vanishing condition (a characteristic p analogue of (0.1.5))

$$BC_n = 0 \quad (q-1 \nmid n) \tag{0.2.6}$$

shown in [**Ca37**] and explicit formulae which (a characteristic p analogue of (0.1.6))

$$BC_n = \sum_{j=0}^{\infty} \frac{(-1)^j D_j}{L_j^2} \left\{ \begin{matrix} n \\ q^j - 1 \end{matrix} \right\}_C \tag{0.2.7}$$

shown in [**KK16**]. A generalization of (0.2.6) is obtained as the following proposition.

Proposition 0.2.6 (Proposition 2.2.9, [**H18c**, Remark 23]). *For $r \in \mathbb{N}$, $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, $\mathbf{j} = (j_1, \ldots, j_r) \in J_{\mathfrak{s}}$ and $n \in \mathbb{Z}_{\geq 0}$ with $(q-1) \nmid n$,*

$$BC_n^{\mathfrak{s},\mathbf{j}} = 0.$$

A generalization of (0.2.7) is obtained as the following theorem.

Theorem 0.2.7 (Theorem 2.3.1, [**H18c**, Theorem 27]). *For $r \in \mathbb{N}$, $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, $\mathbf{j} = (j_1, \ldots, j_r) \in J_{\mathfrak{s}}$ and $n \in \mathbb{Z}_{\geq 0}$,*

$$BC_n^{\mathfrak{s},\mathbf{j}} = \sum_{\log_q(n+1) \geq i_1 > \cdots > i_r \geq 0} \Pi(q^{i_1} - 1) \left\{ \begin{matrix} n \\ q^{i_1} - 1 \end{matrix} \right\}_C \frac{u_{1j_1}^{q^{i_1}} \cdots u_{rj_r}^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}}. \tag{0.2.8}$$

Theorem 0.2.7 is characteristic p analogue of Proposition 0.1.6 while Proposition 0.2.6 has no counterpart in characteristic 0. In §2.3 we show that multi-poly-Bernoulli-Carlitz numbers with special indices are expressed by Bernoulli-Carlitz numbers as follows:

Theorem 0.2.8 (Theorem 2.3.4, [**H18c**, Theorem 30]). *For $r \geq 1$ and $n \geq q^{r-1} - 1$, we have*

(0.2.9)

$$BC_n^{\overbrace{(1,\ldots,1)}^{r}, \overbrace{(0,\ldots,0)}^{r}} = \sum_{\log_q(n+1) \geq i_1 > \cdots > i_r \geq 0} \left\{ \begin{matrix} n \\ q^{i_1} - 1 \end{matrix} \right\}_C BC_{q^{i_1}-1} \frac{BC_{q^{i_2}-1}}{\Pi(q^{i_2} - 1)} \cdots \frac{BC_{q^{i_r}-1}}{\Pi(q^{i_r} - 1)}.$$

This theorem is characteristic p analogue of Proposition 0.1.7.

We show that multi-poly-Bernoulli-Carlitz numbers are connected to a characteristic p analogue of finite multiple zeta values. The finite multiple zeta values in characteristic p are characteristic p analogue of the finite multiple zeta values (Definition 0.1.8) which were introduced by Chang and Mishiba [**CM17**].

Definition 0.2.9 (Definition 2.1.1, [**CM17**, (2.1)]). We set a k-algebra as follows:

$$\mathscr{A}_k := \prod_{\wp} A/\wp A \Big/ \bigoplus_{\wp} A/\wp A$$

where $\wp$ runs over all monic irreducible polynomials in A. For $\mathfrak{s} := (s_1, \ldots, s_r) \in \mathbb{N}^r$ and a monic irreducible polynomial $\wp \in A$, *finite multiple zeta values* are defined as follows:

$$\zeta_{\mathscr{A}_k}(\mathfrak{s}) := (\zeta_{\mathscr{A}_k}(\mathfrak{s})_{\wp}) \in \mathscr{A}_k$$

where

$$\zeta_{\mathscr{A}_k}(\mathfrak{s})_{\wp} := \sum_{\substack{\deg \wp > \deg a_1 > \cdots > \deg a_r \geq 0 \\ a_1, \ldots, a_r \in A_+}} \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in A/\wp A.$$

THEOREM 0.2.10 (Theorem 2.3.6, [**H18c**, Theorem 32]). *For $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ and a monic irreducible polynomial $\wp \in A$ so that $\wp \nmid \Gamma_{s_1} \cdots \Gamma_{s_r}$, we have the following:*

$$\zeta_{\mathscr{A}_k}(\mathfrak{s})_\wp = \frac{1}{\Gamma_{s_1} \cdots \Gamma_{s_r}} \sum_{\mathbf{j} \in J_{\mathfrak{s}}} a_{\mathbf{j}}(\theta) \sum_{i=r-1}^{\deg \wp - 1} \frac{1}{L_i} \frac{BC^{\mathfrak{s},\mathbf{j}}_{q^i - 1}}{BC_{q^i - 1}} \mod \wp. \tag{0.2.10}$$

For $\mathfrak{s} = (\underbrace{1, \dots, 1}_{d}, s_1, \dots, s_r) \in \mathbb{N}^{r+d}$ $(d \geq 0)$ and a monic irreducible polynomial $\wp \in A$ so that $\wp \nmid \Gamma_{s_1} \cdots \Gamma_{s_r}$, we have the following (for the definition of $a_{\mathbf{j}}(\theta)$ and $a_{\mathbf{j}'}(\theta)$, see Notation 2.1.8):

(0.2.11)

$$\zeta_{\mathscr{A}_k}(\mathfrak{s})_\wp = \frac{1}{\Gamma_{s_1} \cdots \Gamma_{s_r}} \sum_{\mathbf{j}' \in J_{\mathfrak{s}'}} a_{\mathbf{j}'}(\theta) \sum_{\deg \wp > i_0 > \cdots > i_d \geq r-1} \frac{1}{L_{i_0} \cdots L_{i_d}} \frac{BC^{\mathfrak{s}',\mathbf{j}'}_{q^{i_d} - 1}}{BC_{q^{i_d} - 1}} \mod \wp.$$

Here we put $\mathfrak{s}' = (s_1, \dots, s_r)$.

This theorem is a characteristic p analogue of Proposition 0.1.9.

0.2.3. The alternating multiple zeta values in characteristic p. In Chapter 3, we introduce the *alternating multiple zeta values in characteristic p* which are defined as the following infinite sums.

DEFINITION 0.2.11 (Definition 3.1.1, [**H19**, (1)]). For $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ and $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$,

$$\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon}) := \sum_{\substack{a_1, \dots, a_r \in A_+ \\ \deg a_1 > \cdots > \deg a_r \geq 0}} \frac{\epsilon_1^{\deg a_1} \cdots \epsilon_r^{\deg a_r}}{a_1^{s_1} \cdots a_r^{s_r}} \in k_\infty. \tag{0.2.12}$$

We call $\mathrm{wt}(\mathfrak{s}) := \sum_{i=1}^r s_i$ the weight and $\mathrm{dep}(\mathfrak{s}) := r$ the depth of the presentation of $\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon})$. We also note that $\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon})$ is generalization of Thakur's multiple zeta values, characteristic p analogue of multiple zeta values.

Thakur multiple zeta values were studied by G. W. Anderson, Thakur and Chang. They found the following properties:

(a). Non-vanishing property ([**T09a**, Theorem 4])
(b). Sum-shuffle relations ([**T10**, Theorem 3])
(c). Period interpretation ([**AT09**, Theorem 1])
(d). Linear independence ([**C14**, Theorem 3.4.5])

A characteristic p analogue of Conjecture 0.1.3 was settled affirmatively by (d) in [**C14**, Theorem 2.2.1].

We investigate properties of AMZVs corresponding to (a)-(d) as following:

(A). Non-vanishing property (Theorem 3.2.1)
(B). Sum-shuffle relations (Theorem 3.2.8)
(C). Period interpretation (Theorem 3.3.4)
(D). Linear independence (Theorem 3.4.7)

For the property (A), it is proven by an inequality of the absolute values of power sums proved by Thakur [**T09a**]. For the property (B), it is proven by Chen's formula [**Chen15**] and induction method by Thakur [**T10**]. For the property (C), it is proven by construction of a certain suitable pre-t-motives. The construction is based on the result in [**AT09**] which enables us to give a period interpretation of multiple zeta values in characteristic p. For the property (D), it is proven by the use of the Anderson-Brownawell-Papanikolas criterion [**ABP04**] and the alternating analogue of MZ property. This method is based on the [**C14**]. By the linear independence result (D), we have the following theorem.

THEOREM 0.2.12 (Theorem 3.4.10, [**H19**, Theorem 4.10]).

(i) $\overline{\mathcal{AZ}}$ *forms an weight-graded algebra, that is,* $\overline{\mathcal{AZ}} = \overline{k} \bigoplus_{w \in \mathbb{N}} \overline{\mathcal{AZ}}_w$*,*

(ii) $\overline{\mathcal{AZ}}$ *is defined over* k*, that is, we have the canonical map* $\overline{k} \otimes_k \mathcal{AZ} \to \overline{\mathcal{AZ}}$ *which is bijective.*

This settles characteristic p analogue of alternating version of Conjecture 0.1.3.

CHAPTER 1

On Lara Rodriguez' full conjecture for double zeta values in function fields

In characteristic 0 case, it is known that multiple zeta values satisfy sum-shuffle relations (also known as stuffle relations or harmonic product relation).

In characteristic p case, Thakur showed that characteristic p multiple zeta values satisfy sum-shuffle relations in 2010. He showed the product of two characteristic p multiple zeta values is expressed by a linear combination of characteristic p multiple zeta values although he did not present their coefficients explicitly. In 2010, Lara Rodríguez posed a conjecture which is called the full conjecture. It gives an explicit form of sum-shuffle relations for certain weights. In this chapter, we solve the conjecture. The full conjecture contains four formulae on certain power series in characteristic p, which yield sum-shuffle relations for positive characteristic analogue of double zeta values. In §1.1, we recall results by Chen and Lucas which are used to prove the conjecture. In §1.2, we recall the statement of the full conjecture and show their counter-examples. In §1.3, we prove affirmatively the first two formulae while we detect and correct errors in the last two formulae, and prove corrected ones.

1.1. Chen's formula and Lucas's theorem

H. J. Chen proved the following formula for the power sums in [**Chen15**] Theorem 3.1 and Remark 3.2.

Proposition 1.1.1 (Chen's formula). *For $r, s, d \in \mathbb{Z}_{\geq 1}$, the following relation holds.*

$$\Delta_d(r,s) = \sum_{\substack{i+j=r+s \\ q-1|j \\ i,j\geq 1}} \left\{ (-1)^{s-1}\binom{j-1}{s-1} + (-1)^{r-1}\binom{j-1}{r-1} \right\} S_d(i,j).$$

Here we put $\binom{a}{b} = 0$ for $a, b \in \mathbb{Z}_{\geq 0}$ with $a < b$.

We can determine the value of the binomial coefficients modulo p by using Lucas's theorem ([**Lu**, §3]).

Proposition 1.1.2 (Lucas's Theorem). *Let p be a prime number and $m, n \in \mathbb{Z}_{\geq 0}$. Then we have*

$$\binom{m}{n} \equiv \binom{m_0}{n_0} \cdots \binom{m_k}{n_k} \bmod p$$

where $m = m_0 + m_1 p + \cdots + m_k p^k$ and $n = n_0 + n_1 p + \cdots + n_k p^k$ ($m_i, n_i \in \{0, 1, \ldots, p-1\}$ for $i = 0, 1, \ldots, k$) are the p-adic expansions of m and n.

1.2. Lara Rodríguez' full conjecture and counter-examples

Lara Rodríguez conjectured several relations for Thakur's double zeta values in [**La10**]. We recall it in Section 1.2.1. We detect some typos and errors in his formulae in Section 1.2.2.

1.2.1. Statements. The following is one of those conjectures which he called the full conjecture ([**La10**, Conjecture 2.8]). It yields "full" descriptions of the 'sum-shuffle' relation for specific double zeta values (cf. [**La10**, §1]).

CONJECTURE 1.2.1 (Lara Rodríguez' full conjecture). For $n, d \in \mathbb{Z}_{\geq 1}$ and general q, we have

(1.2.1)

$$\Delta_d(q^n+1, q^n) = \mathrm{Int}\left(\frac{2}{q}\right) S_d(2, 2q^n-1) - \sum_{j=1}^{\frac{q^n-1}{q-1}} S_d\Big(3+(j-1)(q-1), 2q^n-2-(j-1)(q-1)\Big).$$

(1.2.2)

$$\Delta_d(q^n-1, q^n+1) = -\sum_{j=1}^{\frac{q^n+q-2}{q-1}} S_d\Big(2+(j-1)(q-1), 2q^n-2-(j-1)(q-1)\Big).$$

(1.2.3)

$$\Delta_d(q^{n-1}, q^n+1) = \mathrm{Int}\left(\frac{2}{q}\right) S_d\Big(2, q^n+q^{n-1}-1\Big) - \sum_{j=1}^{\frac{q^{n-1}-1}{q-1}} S_d\Big(3+(j-1)(q-1), q^n+q^{n-1}-2+(j-1)(q-1)\Big).$$

For $0 \leq i \leq n$, we have

(1.2.4)

$$\begin{aligned}\Delta_d(q^n+1, q^n+1-q^i) = {} & \mathrm{Int}\left(\frac{2}{q}\right) S_d(2, 2q^n-q^i) \\ & - \sum_{j=1}^{\frac{q^n-q^i}{q-1}} S_d\Big(3+(j-1)(q-1), 2q^n-q^i-1-(j-1)(q-1)\Big) \\ & + \sum_{j=\frac{q^n-q^i}{q-1}+1}^{\frac{q^n-q^i}{q-1}} S_d\Big(3+(j-1)(q-1), 2q^n-q^i-1-(j-1)(q-1)\Big).\end{aligned}$$

1.2.2. Remarks and Counter-examples.

REMARK 1.2.2. Actually, in [**La10**, Conjecture 2.8 (2.8.1)], Lara Rodríguez conjectured one more relation

$$\Delta_d(q^n, q^n-1) = -S_d(q^n, q^n-1).$$

However he proved it in his later paper [**La12**, Theorem 6.3].

REMARK 1.2.3. The equation (1.2.1) was stated as [**La10**, (2.8.2)]. In the case when $q = 2$, this coincide with second formula in [**T04**, Section 4.1.3]. The equation (1.2.1) will be affirmatively proven in Theorem 1.3.1.

REMARK 1.2.4. The equation (1.2.2) was stated as [**La10**, (2.8.3)]. In the case when $q = 2$, this coincide with third formula in [**T04**, Section 4.1.3]. Again, the equation (1.2.2) will be affirmatively proven in Theorem 1.3.2.

REMARK 1.2.5. The equation (1.2.3) was stated as (2.8.4) in [**La10**] (in the case when $q = 2$, this coincide with fourth formula in [**T04**, Section 4.1.3]). It looks that (1.2.3) contains a typo, and furthermore it requires an additional term to correct it.

Indeed it is quite curious to expect such an equality among the values with different weights (the sum of the first and the second components of double indices): In the right hand side of the equation (1.2.3), the first term is with weight $q^n+q^{n-1}+1$ while the summand of the second term is with weight $q^n+q^{n-1}+1+2(j-1)(q-1)$. In the case when $q = 2$, $d = 2$ and $n = 3$, the equation (1.2.3) claims

$$\Delta_2(4,9) = S_2(2,11) - S_2(3,10) - S_2(4,11) - S_2(5,12), \tag{1.2.5}$$

while Chen's formula says

$$\begin{aligned} \Delta_2(4,9) &= \sum_{\substack{i+j=13 \\ i,j\geq 1}} \left\{ \binom{j-1}{8} - \binom{j-1}{3} \right\} S_2(i,j) \\ &\equiv S_2(2,11) + S_2(3,10) + S_2(4,9) + S_2(5,8) + S_2(9,4) \bmod 2. \end{aligned} \tag{1.2.6}$$

Therefore we must have

$$\begin{aligned} &S_2(2,11) + S_2(3,10) + S_2(4,11) + S_2(5,12) \\ &- S_2(2,11) - S_2(3,10) - S_2(4,9) - S_2(5,8) - S_2(9,4) \equiv 0 \bmod 2. \end{aligned} \tag{1.2.7}$$

However,

$$\begin{aligned} &S_2(4,11) + S_2(5,12) - S_2(4,9) - S_2(5,8) - S_2(9,4) \\ &\quad \equiv S_2(9,4) + S_2(5,12) + S_2(5,8) + S_2(4,11) + S_2(4,9) \bmod 2 \\ &\quad = S_2(9,4) + S_2(5)\Big(1 + S_1(12)\Big) + S_2(5)\Big(1 + S_1(8)\Big) \\ &\qquad + S_2(4)\Big(1 + S_1(11)\Big) + S_2(4)\Big(1 + S_1(9)\Big) \\ &\quad = S_2(9,4) + S_2(5)\Big(S_1(12) + S_1(8)\Big) + S_2(4)\Big(S_1(11) + S_1(9)\Big) \end{aligned}$$

Each term is calculated to be

$$S_2(9,4) \equiv \frac{\left\{\sum_{i=0}^{33} t^i + t^{32} + t^{31} + t^{30} + t^{26} + t^{25} + t^{22} + t^{17} + t^{16} + t^{15} + t^8 + t^6 + t^5 + t^2\right\}}{t^{22}(t+1)^{19}(t^2+t+1)^9(t^2+1)^5} \cdot (t+1)^6(t^2+1)^5 \bmod 2,$$

$$S_2(5)\Big(S_1(12) + S_1(8)\Big) \equiv \frac{\sum_{i=0}^{29} t^i + t^{26} + t^{25} + t^{23} + t^{21} + t^{20} + t^{19} + t^{16} + t^{13} + t^{12} + t^9 + t^8 + t^5}{t^{22}(t+1)^{19}(t^2+t+1)^9(t^2+1)^5} \cdot (t^2+t+1)^4(t+1)^2 \bmod 2,$$

$$S_2(4)\Big(S_1(11) + S_1(9)\Big) \equiv \frac{\left\{t^{12} + t^5 + t^4 + t^3 + t^2 + t + 1\right\}t^3(t^2+t+1)^8(t^2+1)^5}{t^{22}(t+1)^{19}(t^2+t+1)^9(t^2+1)^5} \bmod 2.$$

The degrees of numerators of $S_2(9,4)$, $S_2(5)\Big(S_1(12) + S_1(8)\Big)$ and $S_2(4)\Big(S_1(11) + S_1(9)\Big)$ are 49, 39 and 41 respectively. Thus we find the degree of each numerator is different while they have the same denominators. Then it follows that $S_2(4,11) + S_2(5,12) - S_2(4,9) - S_2(5,8) - S_2(9,4) \not\equiv 0 \bmod 2$ and this contradicts to (1.2.7). This gives the counter-example of (1.2.3).

Therefore, we may correct (1.2.3) as follows.

(1.2.8)

$$\Delta_d(q^{n-1}, q^n+1) = \mathrm{Int}\left(\frac{2}{q}\right)S_d\Big(2, q^n + q^{n-1} - 1\Big) - \sum_{j=1}^{\frac{q^{n-1}-1}{q-1}} S_d\Big(3 + (j-1)(q-1), q^n + q^{n-1} - 2 - (j-1)(q-1)\Big).$$

However, the above equation is not correct, due to a lack of an additional terms which is explained below: When $q = 3, d = 1$ and $n = 3$, (1.2.8) claims

$$\Delta_1(9,28) = -S_1(3) - S_1(5) - S_1(7) - S_1(9). \tag{1.2.9}$$

But according to Chen's formula, we have

$$\Delta_1(9,28) = \sum_{\substack{i+j=37 \\ 2|j \\ i,j\in\mathbb{Z}_{\geq 1}}} \left\{-\binom{j-1}{27} + \binom{j-1}{8}\right\} S_1(i).$$

By Lucas's theorem, we find that the coefficient of $S_1(i)$'s vanish modulo 3 except $-S_1(3), -S_1(5), -S_1(7), -S_1(9)$ and $S_1(19)$. That is,

$$\Delta_1(9,28) = -S_1(3) - S_1(5) - S_1(7) - S_1(9) + S_1(19). \tag{1.2.10}$$

By the definition of power sum,

$$S_1(19) = \frac{1}{t^{19}} + \frac{1}{(t+1)^{19}} + \frac{1}{(t+2)^{19}} = \frac{t^{19}(t+2)^{19} + (t+1)^{19}(t+2)^{19} + t^{19}(t+1)^{19}}{t^{19}(t+1)^{19}(t+2)^{19}}.$$

The numerator of the right hand side has $2^{19} \equiv -1 \bmod 3$ as a constant term. Therefore $S_1(19)$ does not vanish modulo 3. Thus (1.2.9) contradicts to (1.2.10).

So this suggests that we need additional terms to correct it. In Theorem 1.3.3, we correct the equation (1.2.3) as the equation (1.3.7) and prove it.

REMARK 1.2.6. The equation (1.2.4) was stated as (2.8.5) in [**La10**]. Again, it looks that the equation (1.2.4) contains a typo because the summation of the third term in right hand side runs over the empty sum. We correct the equation (1.2.4) as the equation (1.3.13) and prove it in Theorem 1.3.4.

1.3. Main results

THEOREM 1.3.1. *For n and $d \in \mathbb{Z}_{\geq 1}$, the equation* (1.2.1) *holds.*

PROOF. By Chen's formula, we have

$$\Delta_d(q^n+1, q^n) = \sum_{\substack{i+j=2q^n+1 \\ q-1|j}} a_{j,n} S_d(i,j)$$

where

$$a_{j,n} = (-1)^{q^n-1}\binom{j-1}{q^n-1} + (-1)^{q^n}\binom{j-1}{q^n} = \binom{j-1}{q^n-1} - \binom{j-1}{q^n}.$$

When $0 < j \leq q^n - 1$ with $q-1|j$, it is easily seen that

$$a_{j,n} = 0 - 0 = 0. \tag{1.3.1}$$

When $q^n + q - 2 \leq j \leq 2q^n - 2$ with $q-1|j$, let

$$j - 1 = j_0 + j_1 p + \cdots + j_{ln} p^{ln},$$
$$q^n - 1 = p - 1 + (p-1)p + \cdots + (p-1)p^{ln-1}$$

be the p-adic expansions of $j-1$ and q^n-1. By applying Lucas's theorem, we have

$$\binom{j-1}{q^n-1} = \prod_{k=0}^{ln-1}\binom{j_k}{p-1}\binom{j_{ln}}{0},$$
$$\binom{j-1}{q^n} = \prod_{k=0}^{ln-1}\binom{j_k}{0}\binom{j_{ln}}{1}.$$

Thus it follows that

$$\binom{j-1}{q^n-1} \neq 0 \Leftrightarrow j_k = p-1 \;\; (k \in \{0,1,\cdots,ln-1\}),$$
$$\binom{j-1}{q^n} \neq 0 \Leftrightarrow j_{ln} \neq 0.$$

By the condition $q^n + q - 1 \leq j \leq 2q^n - 2$, we have

$$q^n + q - 1 = p^{ln} + p^l - 1 \leq j \leq p - 2 + (p-1)p + \cdots + (p-1)p^{ln-1} + p^{ln} = 2q^n - 2.$$

So we always have $j_{ln} = 1$. Then $\binom{j_{ln}}{1} = 1$ for j with $q^n + q - 1 \leq j \leq 2q^n - 2$ and $q-1|j$. It follows that

$$\binom{j-1}{q^n} = 1.$$

If $j_k = p-1$ for all $k \in \{0,1,\ldots,ln-1\}$ we have $j-1 = q^n - 1 + q^n = 2q^n - 1$ because we always have $j_{ln} = 1$. This contradicts the condition $q^n + q - 2 \leq j \leq 2q^n - 2$. Thus we have

$$\binom{j-1}{q^n-1} = 0.$$

for j with $q^n + q - 2 \le j \le 2q^n - 2$ and $q - 1|j$. Therefore

$$a_{j,n} = 1 \tag{1.3.2}$$

for j with $q^n + q - 2 \le j \le 2q^n - 2$ and $q - 1|j$.

It remains to analyze $j = 2q^n$, $2q^n - 1$. If $q > 2$, $q - 1$ does not divide j, $a_{j,n}$ is 0. When $q = 2$, $q - 1$ always divides j. Then by using Chen's formula and Lucas's Theorem, we obtain $a_{j,n} = 0$ when $j = 2^{n+1}$ and $a_{j,n} = 1$ when $j = 2^{n+1} - 1$. Therefore $S_d(2, 2q^n - 1)$ appears only when $q = 2$.

This explains the first term dependence on q in (1). The rest contribution to (1) is, by (1.3.1) and (1.3.2),

$$-\sum_{\substack{i+j=2q^n+1\\ q^n+q-2\le j\le 2q^n-2\\ q-1|j}} S_d(i,j) = -\sum_{j=1}^{\frac{q^n-1}{q-1}} S_d(3 + (j-1)(q-1), 2q^n - 2 - (j-1)(q-1)),$$

where the equality follows by the substitutions $3 + (j-1)(q-1)$ for i and $2q^n - 2 - (j-1)(q-1)$ for j. (Note, we have $q^n + q - 2 \le 2q^n - 2 - (q-1)(j-1) \le 2q^n - 2$ and thus $1 \le j \le \frac{q^n-1}{q-1}$.) This finishes the proof of (1).

∎

THEOREM 1.3.2. *For n and $d \in \mathbb{Z}_{\ge 1}$, the equation* (1.2.2) *holds.*

PROOF. By Chen's formula,

$$\Delta_d(q^n - 1, q^n + 1) = -\sum_{\substack{i+j=2q^n\\ q-1|j}} b_{j,n} S_d(i,j)$$

where

$$b_{j,n} = (-1)^{q^n-1}\left\{\binom{j-1}{q^n} + \binom{j-1}{q^n-2}\right\} = \binom{j-1}{q^n} + \binom{j-1}{q^n-2}.$$

By the same arguments to those of the proof for Theorem 1.3.1, we have

$$b_{j,n} = 0 \tag{1.3.3}$$

for $j < q^n - 1$ with $q - 1|j$,

$$b_{j,n} = \binom{q^n-2}{q^n} + \binom{q^n-2}{q^n-2} = 0 + 1 = 1 \tag{1.3.4}$$

for $q^n - 1 \le j \le q^n$ with $q - 1|j$ and

$$b_{j,n} = 1 \tag{1.3.5}$$

for $q^n - 1 \le j \le 2q^n - 2$ with $q - 1|j$.

By (1.3.3), (1.3.4) and (1.3.5), we obtain

$$b_{j,n} = \begin{cases} 0 & \text{if } j < q^n - 1 \text{ with } q-1|j, \\ 1 & \text{if } q^n - 1 \le j \le 2q^n - 2 \text{ with } q-1|j. \end{cases} \tag{1.3.6}$$

Therefore Chen's formula becomes

$$\Delta_d(q^n-1,q^n+1) = -\sum_{\substack{i+j=2q^n\\ q^n-1\le j\le 2q^n-2\\ q-1\mid j}} S_d(i,j).$$

Replacing j with $2q^n-2-(j-1)(q-1)$, we have $q^n-1 \le 2q^n-2-(j-1)(q-1) \le 2q^n-2$ and thus $1 \le j \le \frac{q^n+q-2}{q-1}$. Therefore

$$\Delta_d(q^n-1,q^n+1) = -\sum_{j=1}^{\frac{q^n+q-2}{q-1}} S_d\Big(2+(j-1)(q-1), 2q^n-2-(j-1)(q-1)\Big).$$

Then we obtain the equation (1.2.2). □

THEOREM 1.3.3. *For $d,n \in \mathbb{Z}_{\ge 1}$, we have*

(1.3.7)

$$\begin{aligned}\Delta_d(q^{n-1},q^n+1) = \operatorname{Int}\binom{2}{q} S_d(2,q^n+q^{n-1}-1)\\ -\sum_{j=1}^{\frac{q^{n-1}-1}{q-1}} S_d\Big(3+(j-1)(q-1), q^n+q^{n-1}-2-(j-1)(q-1)\Big)\\ + S_d(2q^{n-1}+1, q^n-q^{n-1}).\end{aligned}$$

PROOF. Since $1 \le j \le \frac{q^{n-1}-1}{q-1}$, we have

$$q^n+q-2 \le q^n+q^{n-1}-2-(j-1)(q-1) \le q^n+q^{n-1}-2.$$

By replacing $q^n+q^{n-1}-2-(j-1)(q-1)$ by j, we see that it is enough to prove

$$\begin{aligned}\Delta_d(q^{n-1},q^n+1) = \operatorname{Int}\binom{2}{q} S_d(2,q^n+q^{n-1}-1) - \sum_{\substack{q^n+q-2\le j\le q^n+q^{n-1}-2\\ q-1\mid j\\ i+j=q^n+q^{n-1}+1}} S_d(i,j)\\ + S_d(2q^{n-1}+1,q^n-q^{n-1}),\end{aligned} \tag{1.3.8}$$

which is a reformulation of (1.3.7).

We note that Chen's formula says

$$\Delta_d(q^{n-1},q^n+1) = \sum_{\substack{i+j=q^n+q^{n-1}+1\\ q-1\mid j\\ i,j\in\mathbb{Z}_{\ge 1}}} c_{j,n} S_d(i,j) \tag{1.3.9}$$

where

$$c_{j,n} = (-1)^{q^n}\binom{j-1}{q^n} + (-1)^{q^{n-1}-1}\binom{j-1}{q^{n-1}-1} = -\binom{j-1}{q^n} + \binom{j-1}{q^{n-1}-1}.$$

When $0 < j-1 \le q^{n-1}-2$ with $q-1 \mid j$, it is easily seen that

$$c_{j,n} = -0+0 = 0. \tag{1.3.10}$$

When $q^{n-1}+q-3 \le j-1 \le q^n-2$ with $q-1\mid j$, it is clear that

$$\binom{j-1}{q^n} = 0.$$

By applying the same method as that of the proof of Theorem 1.3.1, we obtain

$$\binom{j-1}{q^{n-1}-1}=\begin{cases}1 & \text{if } j=q^n-q^{n-1},\\ 0 & \text{if } j\neq q^n-q^{n-1}.\end{cases}$$

Therefore

$$c_{j,n}=\begin{cases}1 & \text{if } j=q^n-q^{n-1},\\ 0 & \text{if } j\neq q^n-q^{n-1}.\end{cases}\tag{1.3.11}$$

When $q^n+q-3\leq j-1\leq q^n+q^{n-1}-3$ with $q-1|j$, by Lucas's theorem, we have

$$c_{j,n}=-1+0=-1.\tag{1.3.12}$$

By (1.3.10), (1.3.11) and (1.3.12), we have

$$c_{j,n}=\begin{cases}0 & \text{if } j\leq q^n-1 \text{ with } q-1|j \text{ and } j\neq q^n-q^{n-1},\\ 1 & \text{if } j=q^n-q^{n-1},\\ -1 & \text{if } q^n+q-2\leq j\leq q^n+q^{n-1}-1 \text{ with } q-1|j.\end{cases}$$

We note that this specializes to the following when $q=2$,

$$c_{j,n}=\begin{cases}1 & \text{if } j=2^{n-1} \text{ or } 2^n\leq j\leq 2^n+2^{n-1}-1,\\ 0 & \text{if } 1\leq j<2^{n-1},\ 2^{n-1}<j<2^n \text{ or } j=2^n+2^{n-1}.\end{cases}$$

Therefore we obtain (1.3.8) by (1.3.9) so the equation (1.3.7) follows.

□

THEOREM 1.3.4. *We set $d,n\in\mathbb{Z}_{\geq 1}$. For $0\leq s\leq n$, the following equation holds.*

$$\begin{aligned}\Delta_d(q^n+1,q^n-q^s+1)=&\operatorname{Int}\left(\frac{2}{q}\right)S_d(2,2q^n-q^s)\\ &-\sum_{j=1}^{\frac{q^n-q^s}{q-1}}S_d\Big(3+(j-1)(q-1),2q^n-q^s-1-(j-1)(q-1)\Big)\\ &+\sum_{j=\frac{q^n-q^s}{q-1}+1}^{\frac{q^n-1}{q-1}}S_d\Big(3+(j-1)(q-1),2q^n-q^s-1-(j-1)(q-1)\Big).\end{aligned}\tag{1.3.13}$$

We remark that when $s=0$ (resp. $s=n$), the third term (resp. the second term) of the right hand side of (1.3.13) means the empty sum. We note that in the case when $s=0$, it recovers (1.2.1).

PROOF. We apply the same method as that of the proof of Theorem 1.3.1.

We have $q^n+q-q^s-1\leq 2q^n-q^s-1-(j-1)(q-1)\leq 2q^n-q^s-1$ when $1\leq j\leq\frac{q^n-1}{q-1}$. Replacing $2q^n-q^s-1-(j-1)(q-1)$ with j, we see it is enough to prove

$$\begin{aligned}\Delta_d(q^n+1,q^n+1-q^s)=&\operatorname{Int}\left(\frac{2}{q}\right)S_d(2,2q^n-q^s)\\ &-\sum_{\substack{q^n+q-2\leq j\leq 2q^n-q^s-1\\ i+j=2q^n-q^s+2\\ q-1|j}}S_d(i,j)+\sum_{\substack{q^n-q^s+q-1\leq j\leq q^n-1\\ i+j=2q^n-q^s+2\\ q-1|j}}S_d(i,j).\end{aligned}\tag{1.3.14}$$

Chen's formula says

$$\Delta_d(q^n+1, q^n-q^s+1) = \sum_{\substack{i+j=2q^n-q^s+2\\ q-1|j}} d_{j,n,s} S_d(i,j).$$

where

$$d_{j,n,s} = (-1)^{q^n-q^s}\binom{j-1}{q^n-q^s} + (-1)^{q^n}\binom{j-1}{q^n} = \binom{j-1}{q^n-q^s} - \binom{j-1}{q^n}.$$

When $s=0$, we have (1.3.13) because it is equivalent to (1.2.1).

When $s=n$, Chen's formula becomes

$$\Delta_d(q^n+1, 1) = \sum_{\substack{i+j=q^n+2\\ q-1|j}} d_{j,n,n} S_d(i,j).$$

It is easily seen that

$$d_{j,n,n} = 1 - 0 = 1$$

for all j with $q-1 \le j \le q^n-1$ and $q-1|j$. Thus we have

$$\Delta_d(q^n+1, 1) = \sum_{\substack{q-1\le j\le q^n-1\\ i+j=q^n+2\\ q-1|j}} S_d(i,j).$$

Hence we get (1.3.14) and therefore the equation (1.3.13) holds in this case.

So we may assume that $1 \le s \le n-1$.

When $1 \le j \le q^n - q^s$ or $j = 2q^n - q^s + 1$ with $q-1|j$ (this only holds when $q=2$), it is easily seen that

$$d_{j,n,s} = 0. \tag{1.3.15}$$

When $q^n - q^s + 1 \le j \le q^n - 1$ with $q-1|j$, it is clear that $\binom{j-1}{q^n} = 0$. In this case, by applying Lucas's theorem, we have

$$d_{j,n,s} = 1 \tag{1.3.16}$$

for all j with $q^n - q^s + 1 \le j \le q^n - 1$ and $q-1|j$ (this only holds for $q=2$ when $q>2$, it should be $q^n - q^s + q - 1 \le j \le q^n - 1$ and $q-1|j$).

When $q^n \le j \le 2q^n - q^s$ with $q-1|j$ (this only holds for $q=2$, when $q>2$, it should be $q^n + q - 2 \le j \le 2q^n - q^s - 1$ with $q-1|j$), again by Lucas's theorem, it follows that

$$d_{j,n,s} = -1. \tag{1.3.17}$$

Therefore by (1.3.15), (1.3.16) and (1.3.17), we obtain

$$d_{j,n,s} = \begin{cases} 0 & \text{if } 1 \le j \le q^n - q^s \text{ or } j = 2q^n - q^s + 1 \text{ with } q-1|j, \\ 1 & \text{if } q^n - q^s + 1 \le j \le q^n - 1 \text{ with } q-1|j, \\ -1 & \text{if } q^n \le j \le 2q^n - q^s \text{ with } q-1|j. \end{cases}$$

We note that this specializes to the following when $q=2$,

$$d_{j,n,s} = \begin{cases} 0 & \text{if } 1 \le j \le 2^n - 2^s \text{ or } j = 2^{n+1} - 2^s + 1, \\ 1 & \text{if } 2^n - 2^s + 1 \le j \le 2^{n+1} - 2^s. \end{cases}$$

So (1.3.14) holds in this case by Chen's formula. Therefore (1.3.13) follows. $\square$

Summing all of the equation (1.2.1), (1.2.2), (1.3.7) and (1.3.13) over d respectively and noting that all hold trivially for $d = 0$, we obtain the following corollary.

COROLLARY 1.3.5. *The following 'sum-shuffle product' formula holds:*

(1.3.18)

$$\begin{aligned}\zeta(q^n+1)\zeta(q^n) =&\zeta(q^n+1, q^n) + \zeta(q^n, q^n+1) + \zeta(2q^n+1) + \mathrm{Int}\left(\frac{2}{q}\right)\zeta(2, 2q^n-1)\\ &- \sum_{j=1}^{\frac{q^n-1}{q-1}} \zeta\Big(3+(j-1)(q-1), 2q^n-2-(j-1)(q-1)\Big),\end{aligned}$$

(1.3.19)
$$\begin{aligned}\zeta(q^n-1)\zeta(q^n+1) =&\zeta(q^n-1, q^n+1) + \zeta(q^n+1, q^n-1) + \zeta(2q^n)\\ &- \sum_{j=1}^{\frac{q^n+q-2}{q-1}} \zeta\Big(2+(j-1)(q-1), 2q^n-2-(j-1)(q-1)\Big),\end{aligned}$$

(1.3.20)

$$\begin{aligned}\zeta(q^{n-1})\zeta(q^n+1) =&\zeta(q^{n-1}, q^n+1) + \zeta(q^n+1, q^{n-1}) + \zeta(q^n+q^{n-1}+1)\\ &+ \mathrm{Int}\left(\frac{2}{q}\right)\zeta(2, q^n+q^{n-1}-1)\\ &- \sum_{j=1}^{\frac{q^{n-1}-1}{q-1}} \zeta\Big(3+(j-1)(q-1), q^n+q^{n-1}-2-(j-1)(q-1)\Big)\\ &+ \zeta(2q^{n-1}+1, q^n-q^{n-1}),\end{aligned}$$

and for $0 \le s \le n$,

(1.3.21)

$$\begin{aligned}\zeta(q^n+1)\zeta(q^n+1-q^s) =&\zeta(q^n+1, q^n+1-q^s) + \zeta(q^n+1-q^s, q^n+1) + \zeta(2q^n+2-q^s)\\ &+ \mathrm{Int}\left(\frac{2}{q}\right)\zeta(2, 2q^n-q^s)\\ &- \sum_{j=1}^{\frac{q^n-q^s}{q-1}} \zeta\Big(3+(j-1)(q-1), 2q^n-q^s-1-(j-1)(q-1)\Big)\\ &+ \sum_{j=\frac{q^n-q^s}{q-1}+1}^{\frac{q^n-1}{q-1}} \zeta\Big(3+(j-1)(q-1), 2q^n-q^s-1-(j-1)(q-1)\Big).\end{aligned}$$

CHAPTER 2

On multi-poly-Bernoulli-Carlitz numbers

In this chapter, we introduce multi-poly-Bernoulli numbers in characteristic p. In characteristic 0 case, Imatomi-Kaneko-Takeda **[IKT14]** introduced multi-poly-Bernoulli numbers which are generalization of Bernoulli numbers and showed several formulae. In particullar, they showed an explicit formula of multi-poly-Bernoulli numbers by using factorials and Stirling numbers of the second kind. Further they showed the relations which connect multi-poly-Bernoulli numbers with Kaneko-Zagier's finite multiple zeta values. We describe the characteristic p analogues of their results by using characteristic p counterparts of finite multiple zeta values, factorials and Stirling numbers of the second kind. In §2.1.10, we recall the definition of Chang-Mishiba's finite multiple zeta values, that is, characteristic p analogue of Kaneko-Zagier's finite multiple zeta values. In §2.1.2, we recall the definitions of (finite) multiple polylogarithms in characteristic p. In §2.2 we recall characteristic p analogue of Bernoulli numbers and Stirling numbers of the second kind, that is, Bernoulli-Carlitz numbers and Stirling-Carlitz numbers of the second kind. Then we introduce multi-poly-Bernoulli numbers in characteristic p. In §2.3, we prove main results which consist of an explicit formula to express characteristic p analogue of multi-poly-Bernoulli numbers in terms of Carlitz factorials and Stirling-Carlitz numbers of the second kind, a relation which connects multi-poly-Bernoulli numbers in characteristic p with Chang-Mishiba's finite multiple zeta values and several relations between them.

2.1. Definitions

2.1.1. Definition of finite multiple zeta values in characteristic p. In this subsection, we recall the characteristic p analogue of finite multiple zeta values which were introduced in **[CM17]**.

In 1935, L. Carlitz **[Ca35]** considered an analogue of the *Riemann zeta values* in characteristic p which we call the *Carlitz zeta values.* For $s \in \mathbb{N}$, they are defined by

$$\zeta_A(s) := \sum_{a \in A_+} \frac{1}{a^s} \in k_\infty.$$

D. S. Thakur **[T10]** generalized this definition to that of *multiple zeta values in A*, which are defined for $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$,

$$\zeta_A(\mathfrak{s}) := \sum_{\substack{\deg a_1 > \cdots > \deg a_r \geq 0 \\ a_1, \dots, a_r \in A_+}} \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in k_\infty.$$

Also, Chang-Mishiba and D. S. Thakur concerned v-adic variant (**[CM19, CM, T10]**) and finite variant (**[CM17, T17]**). In this paper, we consider Chang and Mishiba's finite variant (**[CM17]**).

DEFINITION 2.1.1 ([**CM17**, (2.1)]). We set a k-algebra as follows:

$$\mathscr{A}_k := \prod_{\wp} A/\wp A \Big/ \bigoplus_{\wp} A/\wp A$$

where $\wp$ runs over all monic irreducible polynomials in A. For $\mathfrak{s} := (s_1, \ldots, s_r) \in \mathbb{N}^r$ and a monic irreducible polynomial $\wp \in A$, *finite multiple zeta values* are defined as follows:

$$\zeta_{\mathscr{A}_k}(\mathfrak{s}) := (\zeta_{\mathscr{A}_k}(\mathfrak{s})_\wp) \in \mathscr{A}_k$$

where

$$\zeta_{\mathscr{A}_k}(\mathfrak{s})_\wp := \sum_{\substack{\deg \wp > \deg a_1 > \cdots > \deg a_r \geq 0 \\ a_1, \ldots, a_r \in A_+}} \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in A/\wp A.$$

2.1.2. Definition of finite Carlitz multiple polylogarithms. In 2014, C.-Y. Chang [**C14**] introduced the Carlitz multiple polylogarithms as characteristic p analogues of the multiple polylogarithms.

DEFINITION 2.1.2 ([**C14**, Definition 5.1.1]). For $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, the *Carlitz multiple polylogarithms* are defined as the following series of r-variables $z_1, \ldots, z_r$:

$$Li_{\mathfrak{s}}(z_1, \ldots, z_r) := \sum_{i_1 > \cdots > i_r \geq 0} \frac{z_1^{q^{i_1}} \cdots z_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \in k[[z_1, \ldots, z_r]].$$

REMARK 2.1.3. We recover the *Carlitz logarithms* in the case of $r = 1$ and $s_1 = 1$

$$\log_C(z) := \sum_{i \geq 0} \frac{z^{q^i}}{L_i} \in k[[z]].$$

In [**CM17**], C.-Y. Chang and Y. Mishiba introduced finite Carlitz multiple polylogarithms, a finite variant of the Carlitz multiple polylogarithms.

DEFINITION 2.1.4 ([**CM17**, (3.1)]). For $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ and r-tuple of variables $\mathfrak{z} = (z_1, \ldots, z_r)$, *finite Carlitz multiple polylogarithms* are defined as follows:

$$Li_{\mathscr{A}_k, \mathfrak{s}}(\mathfrak{z}) := (Li_{\mathscr{A}_k, \mathfrak{s}}(z_1, \ldots, z_r)_\wp) \in \mathscr{A}_{k, \mathfrak{z}}$$

where

$$Li_{\mathscr{A}_k, \mathfrak{s}}(z_1, \ldots, z_r)_\wp := \sum_{\deg \wp > i_1 > \cdots > i_r \geq 0} \frac{z_1^{q^{i_1}} \cdots z_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \bmod \wp \in A[z_1, \ldots, z_r]/\wp A.$$

Here $\mathscr{A}_{k, \mathfrak{z}}$ is the following quotient ring

$$\mathscr{A}_{k, \mathfrak{z}} := \prod_{\wp} A[\mathfrak{z}]/\wp A[\mathfrak{z}] \Big/ \bigoplus_{\wp} A[\mathfrak{z}]/\wp A[\mathfrak{z}]$$

(we put $A[\mathfrak{z}] := A[z_1, \ldots, z_r]$).

REMARK 2.1.5. In the above definition, we remark that $\wp$ does not divide L_i for $i < \deg \wp$.

In [**CM17**], they established an explicit formula expressing $\zeta_{\mathscr{A}_k}(\mathfrak{s})$ as a k-linear combination of $Li_{\mathscr{A}_k, \mathfrak{s}}(z_1, \ldots, z_r)_\wp$ evaluated at some integral points. Before we recall it, let us prepare the Anderson-Thakur polynomial.

DEFINITION 2.1.6 ([**AT90**, (3.7.1)]). Let θ, t, x be independent variables. For $n \in \mathbb{Z}_{\geq 0}$, *Anderson-Thakur polynomial* $H_n \in A[t]$ is defined by

$$\left\{1 - \sum_{i=0}^{\infty} \frac{\prod_{j=1}^{i}\left(t^{q^i} - \theta^{q^j}\right)}{D_i|_{\theta=t}} x^{q^i}\right\}^{-1} = \sum_{n=0}^{\infty} \frac{H_n}{\Gamma_{n+1}|_{\theta=t}} x^n.$$

REMARK 2.1.7. We note that $H_n = 1$ for $0 \leq n \leq q-1$.

NOTATION 2.1.8. For r-tuple $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, let $H_{s_i - 1} = \sum_{j=0}^{m_i} u_{ij} t^j \quad (u_{ij} \in A, 1 \leq i \leq r)$ and then, we set following symbols which are introduced in [**CM17**]:

$$J_{\mathfrak{s}} := \{0, 1, \dots, m_1\} \times \cdots \times \{0, 1, \dots, m_r\}.$$

For each $\mathbf{j} = (j_1, \dots, j_r) \in J_{\mathfrak{s}}$, we set

$$\mathbf{u_j} := (u_{1j_1}, \dots, u_{rj_r}) \in A^r,$$

and

$$a_{\mathbf{j}} := a_{\mathbf{j}}(t) := t^{j_1 + \cdots + j_r}.$$

EXAMPLES 2.1.9. We note that when $\mathfrak{s} = (s_1, \dots, s_r) = (1, \dots, 1)$, by Remark 3.2.2, we have $J_{\mathfrak{s}} = \{0\} \times \cdots \times \{0\}$ and $\mathbf{u_j} = (1, \dots, 1)$ for $\mathbf{j} \in J_{\mathfrak{s}}$.

The following equation was obtained by C.-Y. Chang and Y. Mishiba in [**CM17**].

PROPOSITION 2.1.10 ([**CM17**, p.1056]). *For $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, let $\wp \in A$ be a monic irreducible polynomial which satisfy $\wp \nmid \Gamma_{s_1} \cdots \Gamma_{s_r}$. Then we have*

$$\zeta_{\mathscr{A}_k}(\mathfrak{s})_{\wp} = \frac{1}{\Gamma_{s_1} \cdots \Gamma_{s_r}} \sum_{\mathbf{j} \in J_{\mathfrak{s}}} a_{\mathbf{j}}(\theta) Li_{\mathscr{A}_k, \mathfrak{s}}(\mathbf{u_j})_{\wp}.$$

2.2. Multi-poly-Bernoulli-Carlitz numbers

In this section, we define multi-poly-Bernoulli-Carlitz numbers which are characteristic p analogues of multi-poly-Bernnoulli numbers.

In 1935, L. Carlitz [**Ca35**] introduced the Bernoulli-Carlitz numbers, characteristic p analogues of the Bernoulli numbers by using the Carlitz factorials $\Pi(n)$ and the *Carlitz exponentials*

$$e_C(z) := \sum_{i \geq 0} \frac{z^{q^i}}{D_i}$$

as follows.

DEFINITION 2.2.1 ([**Ca35**]). For $n \in \mathbb{Z}_{\geq 0}$, the *Bernoulli-Carlitz numbers* BC_n are the elements of k defined by

$$\sum_{n=0}^{\infty} BC_n \frac{z^n}{\Pi(n)} := \frac{z}{e_C(z)}.$$

In [**Ca37**], L. Carlitz obtained the following:

$$BC_n = 0 \quad \text{for } (q-1) \nmid n.$$

In 2016, H. Kaneko and T. Komatsu [**KK16**] introduced the Stirling-Carlitz numbers of the first and second kind as an analogue of the Stirling numbers which were introduced in (0.1.7). We recall below those of the second kind.

Definition 2.2.2 ([**KK16**, (15)]). For $m \in \mathbb{Z}_{\geq 0}$, the *Stirling-Carlitz numbers of the second kind* $\left\{ {n \atop m} \right\}_C \in k$ are defined by

$$\sum_{n=0}^{\infty} \left\{ {n \atop m} \right\}_C \frac{z^n}{\Pi(n)} := \frac{(e_C(z))^m}{\Pi(m)}.$$

In addition, they [**KK16**] showed that

$$\left\{ {n \atop 0} \right\}_C = 0 \ (n \geq 1), \quad \left\{ {n \atop m} \right\}_C = 0 \ (n < m), \quad \left\{ {n \atop n} \right\}_C = 1 \ (n \geq 0) \tag{2.2.1}$$

and the following property.

Proposition 2.2.3 ([**KK16**, Proposition 8]). *For $n, m \in \mathbb{Z}_{>0}$ with $\lambda(n) > \lambda(m)$,*

$$\left\{ {n \atop m} \right\}_C = 0$$

here we noted $\lambda(n) := \sum_i n_i$ where n_i are the digits of q-adic expansion $n = \sum_i n_i q^i$.

By using the Stirling-Carlitz numbers of the second kind, they obtained the following proposition as a characteristic p analogue of (0.1.6).

Proposition 2.2.4 ([**KK16**, Theorem 2]). *For $n \in \mathbb{Z}_{\geq 0}$, we have*

$$BC_n = \sum_{j=0}^{\infty} \frac{(-1)^j D_j}{L_j^2} \left\{ {n \atop q^j - 1} \right\}_C.$$

Remark 2.2.5. In [**KK16**], they put L_j by $\prod_{i=1}^{j} (\theta^{q^i} - \theta)$. But the above equation is same to their equation (cf. [**KK16**, (20)]) due to the appearance of L_j^2.

Remark 2.2.6. By the definition of D_m, we have

$$D_m^q = \prod_{i=0}^{m-1} (\theta^{q^m} - \theta^{q^i})^q = \prod_{i=0}^{m-1} (\theta^{q^{m+1}} - \theta^{q^{i+1}}) = \prod_{i'=1}^{m} (\theta^{q^{m+1}} - \theta^{q^{i'}}) = -\frac{D_{m+1}}{(\theta - \theta^{q^{m+1}})}.$$

Thus we obtain

$$D_m^{q-1} = -\frac{D_{m+1}}{D_m(\theta - \theta^{q^{m+1}})}.$$

By the definition of Carlitz factorial, L_j and the above equation, we have the following:

$$\Pi(q^j - 1) = \prod_{m=0}^{j-1} D_m^{q-1} = \prod_{m=0}^{j-1} -\frac{D_{m+1}}{D_m(\theta - \theta^{q^{m+1}})} = (-1)^j \frac{D_j}{L_j} \quad (j \in \mathbb{Z}_{\geq 0}). \tag{2.2.2}$$

Thus we may write the formula in Theorem 2.2.4 as follows:

$$BC_n = \sum_{j=0}^{\infty} \frac{\Pi(q^j - 1)}{L_j} \left\{ {n \atop q^j - 1} \right\}_C.$$

Next we introduce multi-poly-Bernoulli-Carlitz numbers (MPBCNs) as characteristic p analogues of MPBNs (Definition 0.1.4). It is defined by the following generating function.

Definition 2.2.7. For $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, $\mathbf{j} = (j_1, \ldots, j_r) \in J_{\mathfrak{s}}$ (for $J_{\mathfrak{s}}$, see Notation 2.1.8), we define *multi-poly-Bernoulli-Carlitz numbers* (MPBCNs for short) $BC_n^{\mathfrak{s},\mathbf{j}}$ to be elements of k as follows:

$$\sum_{n \geq 0} BC_n^{\mathfrak{s},\mathbf{j}} \frac{z^n}{\Pi(n)} := \frac{Li_{\mathfrak{s}}(e_C(z) u_{1j_1}, u_{2j_2}, \ldots, u_{rj_r})}{e_C(z)}. \tag{2.2.3}$$

REMARK 2.2.8. In the case when $r = 1$ and $s_1 = 1$ in the above definition, we have $Li_{\mathfrak{s}}(z_1, \ldots, z_r) = \log_C(z)$ and $J_{\mathfrak{s}} = \{0\}, u_{1j_1} = u_{10} = 1$ since $H_{s_1-1} = H_0 = 1$. Hence we recover the Definition 2.2.1 by

$$\sum_{n\geq 0} BC_n^{(1),(0)} \frac{z^n}{\Pi(n)} = \frac{\log_C(e_C(z))}{e_C(z)} = \frac{z}{e_C(z)}.$$

This is the one we have seen in Definition 2.2.1 so we have

$$BC_n^{(1),(0)} = BC_n. \tag{2.2.4}$$

PROPOSITION 2.2.9. *For $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, $\mathbf{j} = (j_1, \ldots, j_r) \in J_{\mathfrak{s}}$ and $n \in \mathbb{Z}_{\geq 0}$ with $(q-1) \nmid n$, we have*

$$BC_n^{\mathfrak{s},\mathbf{j}} = 0.$$

PROOF. Let g be a generator of $\mathbb{F}_q^\times$ then we have

$$g^n = 1 \Leftrightarrow (q-1) | n. \tag{2.2.5}$$

By the definition, it follows that $e_C(gz) = ge_C(z)$. Then by (2.2.3) and Definition 2.1.2, we have

$$\sum_{n\geq 0} BC_n^{\mathfrak{s},\mathbf{j}} \frac{(gz)^n}{\Pi(n)} = \frac{Li_{\mathfrak{s}}(e_C(gz)u_{1j_1}, u_{2j_2}, \ldots, u_{rj_r})}{e_C(gz)} = \sum_{i_1 > \cdots > i_r \geq 0} e_C(gz)^{q^{i_1}-1} \frac{u_{1j_1}^{q^{i_1}} \cdots u_{rj_r}^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}}$$

by using $e_C(gz) = ge_C(z)$ and (2.2.5),

$$\begin{aligned} &= \sum_{i_1 > \cdots > i_r \geq 0} e_C(z)^{q^{i_1}-1} \frac{u_{1j_1}^{q^{i_1}} \cdots u_{rj_r}^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} = \frac{Li_{\mathfrak{s}}(e_C(z)u_{1j_1}, u_{2j_2}, \ldots, u_{rj_r})}{e_C(z)} \\ &= \sum_{n\geq 0} BC_n^{\mathfrak{s},\mathbf{j}} \frac{z^n}{\Pi(n)}. \end{aligned}$$

By comparing the coefficients of z^n, we have $g^n BC_n^{\mathfrak{s},\mathbf{j}} = BC_n^{\mathfrak{s},\mathbf{j}}$. Therefore we obtain the following by (2.2.5):

$$BC_n^{\mathfrak{s},\mathbf{j}} = 0 \quad \text{for } (q-1) \nmid n.$$

□

The MPBNs are defined for $s_i \in \mathbb{Z}$, on the other hand our MPBCNs are defined for $s_i \in \mathbb{N}$. It is because in Definition 2.2.7, we use u_{ij_i}, the coefficients of H_{s_i-1} which are defined for $s_i \in \mathbb{N}$. We remark that we do not have two kinds of MPBCNs as we do in Definition 0.1.4.

2.3. Several properties of multi-poly-Bernoulli-Carlitz numbers

In this section, we obtain characteristic p analogues of some results in [**IKT14**]. We prove characteristic p analogues of Proposition 0.1.6-0.1.9. The following theorem is a characteristic p analogue of Proposition 0.1.6.

THEOREM 2.3.1. *For $r \in \mathbb{N}$, $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, $\mathbf{j} = (j_1, \ldots, j_r) \in J_{\mathfrak{s}}$ and $n \in \mathbb{Z}_{\geq 0}$,*

$$BC_n^{\mathfrak{s},\mathbf{j}} = \sum_{\log_q(n+1) \geq i_1 > \cdots > i_r \geq 0} \Pi(q^{i_1}-1) \left\{ \begin{matrix} n \\ q^{i_1}-1 \end{matrix} \right\}_C \frac{u_{1j_1}^{q^{i_1}} \cdots u_{rj_r}^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}}. \tag{2.3.1}$$

PROOF. By Definition 2.1.2, the right hand side of (2.2.3) is translated as follows.

$$\frac{Li_{\mathfrak{s}}(e_C(z)u_{1j_1}, u_{2j_2}, \dots, u_{rj_r})}{e_C(z)} = \sum_{i_1>\cdots>i_r\geq 0} e_C(z)^{q^{i_1}-1} \frac{u_{1j_1}^{q^{i_1}} \cdots u_{rj_r}^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}}$$

by Definition 2.2.2 for $m = q^{i_1} - 1$,

$$\begin{aligned}
&= \sum_{i_1>\cdots>i_r\geq 0} \sum_{n\geq 0} \Pi(q^{i_1}-1) \begin{Bmatrix} n \\ q^{i_1}-1 \end{Bmatrix}_C \frac{z^n}{\Pi(n)} \frac{u_{1j_1}^{q^{i_1}} \cdots u_{rj_r}^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \\
&= \sum_{n\geq 0} \sum_{i_1>\cdots>i_r\geq 0} \Pi(q^{i_1}-1) \begin{Bmatrix} n \\ q^{i_1}-1 \end{Bmatrix}_C \frac{u_{1j_1}^{q^{i_1}} \cdots u_{rj_r}^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \frac{z^n}{\Pi(n)} \\
&= \sum_{n\geq 0} \sum_{\log_q(n+1)\geq i_1>\cdots>i_r\geq 0} \Pi(q^{i_1}-1) \begin{Bmatrix} n \\ q^{i_1}-1 \end{Bmatrix}_C \frac{u_{1j_1}^{q^{i_1}} \cdots u_{rj_r}^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \frac{z^n}{\Pi(n)}.
\end{aligned}$$

Then by Definition 2.2.7, we have

$$\sum_{n\geq 0} BC_n^{\mathfrak{s},\mathbf{j}} \frac{z^n}{\Pi(n)} = \sum_{n\geq 0} \sum_{\log_q(n+1)\geq i_1>\cdots>i_r\geq 0} \Pi(q^{i_1}-1) \begin{Bmatrix} n \\ q^{i_1}-1 \end{Bmatrix}_C \frac{u_{1j_1}^{q^{i_1}} \cdots u_{rj_r}^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \frac{z^n}{\Pi(n)}.$$

By comparing the coefficients of z^n, we obtain the formula (2.3.1). □

From (2.2.1) and (2.3.1), we easily deduce that $BC_n^{\mathfrak{s},\mathbf{j}} = 0$ if $n < q^{r-1} - 1$. We note that for $x, y \in \mathbb{N}$, all digits of the q-adic expansion of $q^x - 1$ and $q^y - 1$ are $q - 1$. Therefore we have

$$\begin{Bmatrix} q^x - 1 \\ q^y - 1 \end{Bmatrix}_C = \begin{cases} 0 & \text{if } x \neq y, \\ 1 & \text{if } x = y, \end{cases} \tag{2.3.2}$$

by Proposition 2.2.3 and (2.2.1). Thus we have the following from Theorem 2.3.1.

COROLLARY 2.3.2. *For $m \in \mathbb{N}$, we have*

$$BC_{q^m-1}^{\mathfrak{s},\mathbf{j}} = \sum_{m>i_2>\cdots>i_r\geq 0} \Pi(q^m - 1) \frac{u_{1j_1}^{q^m} u_{2j_2}^{q^{i_2}} \cdots u_{rj_r}^{q^{i_r}}}{L_m^{s_1} L_{i_2}^{s_2} \cdots L_{i_r}^{s_r}}. \tag{2.3.3}$$

REMARK 2.3.3. When $r = 1$ and $s_1 = 1$, we have $H_{s_1-1} = H_0 = 1$. Then $J_{\mathfrak{s}} = \{0\}, u_{1j_1} = u_{10} = 1$ hence we have

$$BC_n^{(1),(0)} = \sum_{\log_q(n+1)\geq i_1\geq 0} \Pi(q^{i_1}-1) \begin{Bmatrix} n \\ q^{i_1}-1 \end{Bmatrix}_C \frac{1}{L_{i_1}}$$

by using (2.2.2),

$$= \sum_{\log_q(n+1)\geq i_1\geq 0} (-1)^{i_1} \frac{D_{i_1}}{L_{i_1}^2} \begin{Bmatrix} n \\ q^{i_1}-1 \end{Bmatrix}_C.$$

Therefore by Remark 2.2.8 our Theorem 2.3.1 includes H. Kaneko and T. Komatsu's result (Proposition 2.2.4) in the case of $r = 1$ and $s_1 = 1$.

We obtain the following relation between the MPBCNs and the Bernoulli-Carlitz numbers for the tuple $(1,\dots,1)$ as a characteristic p analogue of Proposition 0.1.7.

Theorem 2.3.4. *For $r \geq 1$ and $n \geq q^{r-1}-1$, we have*

(2.3.4)
$$BC_n^{\overbrace{(1,\dots,1)}^{r},\overbrace{(0,\dots,0)}^{r}} = \sum_{\log_q(n+1)\geq i_1>\dots>i_r\geq 0} \begin{Bmatrix} n \\ q^{i_1}-1 \end{Bmatrix}_C BC_{q^{i_1}-1} \frac{BC_{q^{i_2}-1}}{\Pi(q^{i_2}-1)} \cdots \frac{BC_{q^{i_r}-1}}{\Pi(q^{i_r}-1)}.$$

Proof. Let us first prove an equation

$$\frac{BC_{q^i-1}}{\Pi(q^i-1)} = \frac{1}{L_i} \tag{2.3.5}$$

for each $i \geq 0$. It follows from Proposition 2.2.4 that we have

$$BC_{q^i-1} = \sum_{j=0}^{\infty} \frac{(-1)^j D_j}{L_j^2} \begin{Bmatrix} q^i-1 \\ q^j-1 \end{Bmatrix}_C.$$

The right hand side is translated as follows:

$$\sum_{j=0}^{\infty} \frac{(-1)^j D_j}{L_j^2} \begin{Bmatrix} q^i-1 \\ q^j-1 \end{Bmatrix}_C = \frac{(-1)^i D_i}{L_i^2} = \frac{\Pi(q^i-1)}{L_i}.$$

The first equality follows from Proposition 2.2.3, the second one follows from (2.2.2). Then we have the equation (2.3.5).

It follows from Theorem 2.3.1 that we have

$$BC_n^{\overbrace{(1,\dots,1)}^{r},\overbrace{(0,\dots,0)}^{r}} = \sum_{\log_q(n+1)\geq i_1>\dots>i_r\geq 0} \begin{Bmatrix} n \\ q^{i_1}-1 \end{Bmatrix}_C \frac{\Pi(q^{i_1}-1)}{L_{i_1}\cdots L_{i_r}}.$$

By using the equation (2.3.5) to the right hand side,

$$\begin{aligned} &BC_n^{\overbrace{(1,\dots,1)}^{r},\overbrace{(0,\dots,0)}^{r}} \\ &= \sum_{\log_q(n+1)\geq i_1>\dots>i_r\geq 0} \begin{Bmatrix} n \\ q^{i_1}-1 \end{Bmatrix}_C \Pi(q^{i_1}-1) \frac{BC_{q^{i_1}-1}}{\Pi(q^{i_1}-1)} \frac{BC_{q^{i_2}-1}}{\Pi(q^{i_2}-1)} \cdots \frac{BC_{q^{i_r}-1}}{\Pi(q^{i_r}-1)}. \end{aligned}$$

Therefore we obtain the desired equation (2.3.4). ☐

Next, before we see a characteristic p analogue of Proposition 0.1.9, we prepare the following lemma.

Lemma 2.3.5. *When $r \geq 2$, we have the following equation for $\mathfrak{s} = (s_1,\dots,s_r) \in \mathbb{N}^r$, $\mathbf{j} \in J_{\mathfrak{s}}$ and $m \geq r-1$.*

$$BC_{q^m-1}^{\mathfrak{s},\mathbf{j}} = BC_{q^m-1}^{(s_1),(j_1)} \sum_{i=r-2}^{m-1} \frac{1}{\Pi(q^i-1)} BC_{q^i-1}^{(s_2,\dots,s_r),(j_2,\dots,j_r)}. \tag{2.3.6}$$

Proof. By using Theorem 2.3.1, we have

$$BC_{q^m-1}^{\mathfrak{s},\mathbf{j}} = \sum_{m\geq i_1>\dots>i_r\geq 0} \begin{Bmatrix} q^m-1 \\ q^{i_1}-1 \end{Bmatrix}_C \Pi(q^{i_1}-1) \frac{u_{1j_1}^{q^{i_1}} \cdots u_{rj_r}^{q^{i_r}}}{L_{i_1}^{s_1}\cdots L_{i_r}^{s_r}}.$$

Then by using (2.3.2), we have

$$BC^{\mathfrak{s},\mathbf{j}}_{q^m-1} = \sum_{m>i_2>\cdots>i_r\geq 0} \Pi(q^m-1)\frac{u_{1j_1}^{q^m}u_{2j_2}^{q^{i_2}}\cdots u_{rj_r}^{q^{i_r}}}{L_m^{s_1}L_{i_2}^{s_2}\cdots L_{i_r}^{s_r}} = \Pi(q^m-1)\frac{u_{1j_1}^{q^m}}{L_m^{s_1}}\sum_{m>i_2>\cdots>i_r\geq 0}\frac{u_{2j_2}^{q^{i_2}}\cdots u_{rj_r}^{q^{i_r}}}{L_{i_2}^{s_2}\cdots L_{i_r}^{s_r}}.$$

By using Theorem 2.3.1, we have

$$\begin{aligned} BC^{\mathfrak{s},\mathbf{j}}_{q^m-1} &= BC^{(s_1),(j_1)}_{q^m-1}\sum_{m>i_2>\cdots>i_r\geq 0}\frac{u_{2j_2}^{q^{i_2}}\cdots u_{rj_r}^{q^{i_r}}}{L_{i_2}^{s_2}\cdots L_{i_r}^{s_r}} \\ &= BC^{(s_1),(j_1)}_{q^m-1}\sum_{i=r-2}^{m-1}\frac{1}{\Pi(q^i-1)}\sum_{i>i_3>\cdots>i_r\geq 0}\Pi(q^i-1)\frac{u_{2j_2}^{q^{i}}u_{3j_3}^{q^{i_3}}\cdots u_{rj_r}^{q^{i_r}}}{L_{i}^{s_2}L_{i_3}^{s_3}\cdots L_{i_r}^{s_r}} \end{aligned} \tag{2.3.7}$$

again by using Theorem 2.3.1,

$$= BC^{(s_1),(j_1)}_{q^m-1}\sum_{i=r-2}^{m-1}\frac{1}{\Pi(q^i-1)}BC^{(s_2,\ldots,s_r),(j_2,\ldots,j_r)}_{q^i-1}.$$

Then we obtain the desired equation (2.3.6). □

The following result is an analogue of Proposition 0.1.9 which provides the connection between MPBCNs and finite multiple zeta values in characteristic p case.

THEOREM 2.3.6. *For* $\mathfrak{s} = (s_1,\ldots,s_r) \in \mathbb{N}^r$ *and a monic irreducible polynomial* $\wp \in A$ *so that* $\wp \nmid \Gamma_{s_1}\cdots\Gamma_{s_r}$*, we have the following:*

$$\zeta_{\mathscr{A}_k}(\mathfrak{s})_\wp = \frac{1}{\Gamma_{s_1}\cdots\Gamma_{s_r}}\sum_{\mathbf{j}\in J_\mathfrak{s}} a_\mathbf{j}(\theta)\sum_{i=r-1}^{\deg\wp-1}\frac{1}{L_i}\frac{BC^{\mathfrak{s},\mathbf{j}}_{q^i-1}}{BC_{q^i-1}} \mod \wp. \tag{2.3.8}$$

For $\mathfrak{s} = (\underbrace{1,\ldots,1}_{d},s_1,\ldots,s_r) \in \mathbb{N}^{r+d}$ $(d\geq 0)$ *and a monic irreducible polynomial* $\wp \in A$ *so that* $\wp \nmid \Gamma_{s_1}\cdots\Gamma_{s_r}$*, we have the following:*

$$\zeta_{\mathscr{A}_k}(\mathfrak{s})_\wp = \frac{1}{\Gamma_{s_1}\cdots\Gamma_{s_r}}\sum_{\mathbf{j}'\in J_{\mathfrak{s}'}} a_{\mathbf{j}'}(\theta)\sum_{\deg\wp>i_0>\cdots>i_d\geq r-1}\frac{1}{L_{i_0}\cdots L_{i_d}}\frac{BC^{\mathfrak{s}',\mathbf{j}'}_{q^{i_d}-1}}{BC_{q^{i_d}-1}} \mod \wp. \tag{2.3.9}$$

Here we put $\mathfrak{s}' = (s_1,\ldots,s_r)$.

We remark that both sides of the equation (2.3.8) become 0 for $\deg\wp < r$ (resp. (2.3.9), $\deg\wp < d+r$).

Proof. We first prove that the equation (2.3.8). From Definition 2.1.2, the equation (2.3.3) and (2.3.5) we have

$$\begin{aligned}
Li_{\mathscr{A}_k,\mathfrak{s}}(\mathbf{u_j})_\wp &= \sum_{\deg\wp > i_1 > \cdots > i_r \geq 0} \frac{u_{1j_1}^{q^{i_1}} \cdots u_{rj_r}^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \\
&= \sum_{i=r-1}^{\deg\wp - 1} \frac{1}{\Pi(q^i - 1)} \sum_{i > i_2 > \cdots > i_r \geq 0} \Pi(q^i - 1) \frac{u_{1j_1}^{q^{i_1}} u_{2j_2}^{q^{i_2}} \cdots u_{rj_r}^{q^{i_r}}}{L_{i_1}^{s_1} L_{i_2}^{s_2} \cdots L_{i_r}^{s_r}} \\
&= \sum_{i=r-1}^{\deg\wp - 1} \frac{1}{\Pi(q^i - 1)} BC_{q^i-1}^{\mathfrak{s}', \mathbf{j}'} \\
&= \sum_{i=r-1}^{\deg\wp - 1} \frac{1}{L_i} \frac{BC_{q^i-1}^{\mathfrak{s}', \mathbf{j}'}}{BC_{q^i-1}}.
\end{aligned}$$

By our assumption $\wp \nmid \Gamma_{s_1} \cdots \Gamma_{s_r}$ we may apply Proposition 2.1.10 and obtain the desired formula (2.3.8).

Next we prove the equation (2.3.9). By using (2.3.8) for $\mathfrak{s} = (\underbrace{1, \ldots, 1}_{d}, s_1, \ldots, s_r)$, the $\wp$-part of $\zeta(\mathfrak{s})_\wp$ is computed as

$$\begin{aligned}
\zeta(\mathfrak{s})_\wp &= \frac{1}{\Gamma_1^d \Gamma_{s_1} \cdots \Gamma_{s_r}} \sum_{\mathbf{j} \in J_\mathfrak{s}} a_\mathbf{j}(\theta) \sum_{i=d+r-1}^{\deg\wp - 1} \frac{1}{L_i} \frac{BC_{q^i-1}^{\mathfrak{s}, \mathbf{j}}}{BC_{q^i-1}} \\
&= \frac{1}{\Gamma_1^d \Gamma_{s_1} \cdots \Gamma_{s_r}} \sum_{\mathbf{j} \in J_\mathfrak{s}} a_\mathbf{j}(\theta) \sum_{i_0=d+r-1}^{\deg\wp - 1} \frac{BC_{q^{i_0}-1}^{\mathfrak{s}, \mathbf{j}}}{\Pi(q^{i_0} - 1)}.
\end{aligned}$$

Then by using Lemma 2.3.5, $\frac{BC_{q^{i_0}-1}^{\mathfrak{s},\mathbf{j}}}{\Pi(q^{i_0}-1)}$ is computed as

$$\begin{aligned}
\frac{BC_{q^{i_0}-1}^{\mathfrak{s},\mathbf{j}}}{\Pi(q^{i_0} - 1)} &= \frac{BC_{q^{i_0}-1}^{(1),(0)}}{\Pi(q^{i_0} - 1)} \sum_{i_1=d+r-2}^{i_0-1} \frac{BC_{q^{i_1}-1}^{(1,\ldots,1,s_1,\ldots,s_r),(0,\ldots,0,j_1,\ldots,j_r)}}{\Pi(q^{i_1} - 1)} \\
&= \frac{BC_{q^{i_0}-1}^{(1),(0)}}{\Pi(q^{i_0} - 1)} \sum_{i_1=d+r-2}^{i_0-1} \frac{BC_{q^{i_1}-1}^{(1),(0)}}{\Pi(q^{i_1} - 1)} \sum_{i_2=d+r-3}^{i_1-1} \frac{BC_{q^{i_2}-1}^{(1,\ldots,1,s_1,\ldots,s_r),(0,\ldots,0,j_1,\ldots,j_r)}}{\Pi(q^{i_2} - 1)} \\
&= \frac{BC_{q^{i_0}-1}^{(1),(0)}}{\Pi(q^{i_0} - 1)} \sum_{i_1=d+r-2}^{i_0-1} \frac{BC_{q^{i_1}-1}^{(1),(0)}}{\Pi(q^{i_1} - 1)} \sum_{i_2=d+r-3}^{i_1-1} \frac{BC_{q^{i_2}-1}^{(1),(0)}}{\Pi(q^{i_2} - 1)} \sum_{i_3=d+r-4}^{i_2-1} \\
&\qquad \cdots \sum_{i_d=r-1}^{i_{d-1}-1} \frac{BC_{q^{i_d}-1}^{(s_1,\ldots,s_r),(j_1,\ldots,j_r)}}{\Pi(q^{i_d} - 1)} \\
&= \frac{1}{L_{i_0}} \sum_{i_1=d+r-2}^{i_0-1} \frac{1}{L_{i_1}} \sum_{i_2=d+r-3}^{i_1-1} \frac{1}{L_{i_2}} \sum_{i_3=d+r-4}^{i_2-1} \cdots \sum_{i_d=r-1}^{i_{d-1}-1} \frac{1}{L_{i_d}} \frac{BC_{q^{i_d}-1}^{(s_1,\ldots,s_r),(j_1,\ldots,j_r)}}{BC_{q^{i_d}-1}}
\end{aligned}$$

$$= \sum_{\substack{i_0>i_1>\cdots>i_d \\ i_l\geq d+r-l-1 \text{ for each } l}} \frac{1}{L_{i_0}L_{i_1}\cdots L_{i_d}} \frac{BC^{(s_1,\dots,s_r),(j_1,\dots,j_r)}_{q^{i_d}-1}}{BC_{q^{i_d}-1}}$$

$$= \sum_{i_0>i_1>\cdots>i_d\geq r-1} \frac{1}{L_{i_0}L_{i_1}\cdots L_{i_d}} \frac{BC^{(s_1,\dots,s_r),(j_1,\dots,j_r)}_{q^{i_d}-1}}{BC_{q^{i_d}-1}}.$$

Thus by $\Gamma_1 = 1$, we have

$$\frac{1}{\Gamma_{s_1}\cdots\Gamma_{s_r}} \sum_{\mathbf{j}\in J_{\mathfrak{s}}} a_{\mathbf{j}}(\theta) \sum_{\deg\wp>i_0>\cdots>i_d\geq r-1} \frac{1}{L_{i_0}\cdots L_{i_d}} \frac{BC^{(s_1,\dots,s_r),(j_1,\dots,j_r)}_{q^{i_d}-1}}{BC_{q^{i_d}-1}} = \zeta_{\mathscr{A}_k}(\mathfrak{s})_\wp \bmod \wp.$$

For $\mathfrak{s} = (1,\dots,1,s_1,\dots,s_r)$, we have $J_{\mathfrak{s}} = \{0\}\times\cdots\{0\}\times\{0,1,\dots,m_1\}\times\cdots\times\{0,1,\dots,m_r\}$ so $a_{\mathbf{j}}(\theta) = \theta^{j_1+\cdots+j_r}$ for $\mathbf{j} = (0,\dots,0,j_1,\dots,j_r) \in J_{\mathfrak{s}}$ and thus $a_{\mathbf{j}}(\theta)$ depends only on $\mathbf{j}' = (j_1,\dots j_r) \in J_{\mathfrak{s}'}$. Therefore the above equation is rewritten as follows:

$$\frac{1}{\Gamma_{s_1}\cdots\Gamma_{s_r}} \sum_{\mathbf{j}'\in J_{\mathfrak{s}'}} a_{\mathbf{j}'}(\theta) \sum_{\deg\wp>i_0>\cdots>i_d\geq r-1} \frac{1}{L_{i_0}\cdots L_{i_d}} \frac{BC^{\mathfrak{s}',\mathbf{j}'}_{q^{i_d}-1}}{BC_{q^{i_d}-1}} = \zeta_{\mathscr{A}_k}(\mathfrak{s})_\wp \bmod \wp.$$

Thus we obtain the equation (2.3.9). □

We remark that the relation (2.3.9) is a generalization of (2.3.8).

CHAPTER 3

Alternating multiple zeta values in positive characteristic

In this chapter, we introduce alternating multiple zeta values in positive characteristic and show their various properties. In §3.1, we introduce a characteristic p analogue of alternating multiple zeta values by using alternating power sums. We also express those values by using Anderson-Thakur polynomials. In §3.2, we obtain several fundamental properties of alternating multiple zeta values in positive characteristic. In §3.2.1, we prove that characteristic p analogue of alternating multiple zeta values is non-vanishing by using an inequality shown by Thakur [**T09a**]. In §3.2.2, we prove sum-shuffle relations among them by showing alternating analogue of Chen's formula and by using induction on the depth of alternating multiple zeta values in characteristic p. In §3.3, we construct certain pre-t-motives which express characteristic p analogue of alternating multiple zeta values. In §3.4, we show a linear independence property of alternating multiple zeta values in characteristic p by using the Anderson-Brownawell-Papanikolas criterion and an alternating version of the MZ-property.

3.1. Definitions

In the characteristic p case, A is an analogue of the integer ring $\mathbb{Z}$. Thus we may consider $A^\times = \mathbb{F}_q^\times$ as an analogue of $\mathbb{Z}^\times = \{\pm 1\}$.

We define alternating power sums and alternating multiple zeta values in positive characteristic along the construction of multiple zeta values in characteristic p (MZVs in short) by Thakur. For $s \in \mathbb{N}$, $\epsilon \in \mathbb{F}_q^\times$ and $d \in \mathbb{Z}_{\geq 0}$, we define the *alternating power sums* by

$$S_d(s;\epsilon) := \epsilon^d S_d(s) = \sum_{a \in A_{d+}} \frac{\epsilon^d}{a^s} \in k.$$

The above alternating power sums are extended inductively as follows.

For $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$ and $d \in \mathbb{Z}_{\geq 0}$, we define

$$S_{<d}(\mathfrak{s};\boldsymbol{\epsilon}) := \sum_{d > d_1 > \cdots > d_r \geq 0} S_{d_1}(s_1;\epsilon_1) \cdots S_{d_r}(s_r;\epsilon_r) \in k$$

and

$$\begin{aligned} S_d(\mathfrak{s};\boldsymbol{\epsilon}) &:= S_d(s_1;\epsilon_1) S_{<d}(s_2,\dots,s_r;\epsilon_2,\dots,\epsilon_r) \\ &:= S_d(s_1;\epsilon_1) \sum_{d > d_2 > \cdots > d_r \geq 0} S_{d_2}(s_2;\epsilon_2) \cdots S_{d_r}(s_r;\epsilon_r) \in k. \end{aligned} \tag{3.1.1}$$

When $r - 1 > d$, $S_d(\mathfrak{s};\boldsymbol{\epsilon}) = 0$ since it is empty sum. By using these alternating power sums, characteristic p analogue of alternating multiple zeta values (AMZVs in short) (cf. (0.2.12)) are interpreted as follows.

Definition 3.1.1. For $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$ we define the *alternating multiple zeta values* by the following series:

$$\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon}) := \sum_{d \geq 0} S_d(\mathfrak{s}; \boldsymbol{\epsilon}) \in k_\infty. \tag{3.1.2}$$

Remark 3.1.2. We remark that $\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon})$ specializes to $\zeta_A(\mathfrak{s})$ when $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_r) = (1, \dots, 1)$.

Remark 3.1.3. In writing this paper, the author got to know that there exist the colored variant of MZVs by communicating with Thakur. They were defined by his students Qibin Shen and Shuhui Shi as follows. For $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, and $n \in \mathbb{N}$, let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_r) \in (\overline{\mathbb{F}_q}^\times)^r$ so that each ξ_i is n-th root of unity. Then level n colored MZVs are the following series:

$$\zeta_A(\mathfrak{s}; \boldsymbol{\xi}) := \sum_{\substack{a_1, a_2, \dots, a_r \in A_+ \\ \deg a_1 > \deg a_2 > \cdots > \deg a_r \geq 0}} \frac{\xi_1^{\deg a_1} \xi_2^{\deg a_2} \cdots \xi_r^{\deg a_r}}{a_1^{s_1} a_2^{s_2} \cdots a_r^{s_r}}. \tag{3.1.3}$$

This $\zeta_A(\mathfrak{s}; \boldsymbol{\xi})$ includes AMZVs as level $q-1$ case and we remark that the $q-1$ case was defined by the author and Shen-Shi independently.

In this paper, we write n-fold Frobenius twisting as follows.

$$\begin{aligned} \mathbb{C}_\infty((t)) &\to \mathbb{C}_\infty((t)) \\ f := \sum_i a_i t^i &\mapsto \sum_i a_i^{q^n} t^i =: f^{(n)}. \end{aligned}$$

Moreover, we fix a fundamental period $\tilde{\pi}$ of the Carlitz module (see [**G96, T04**]). We define the following power series.

$$\Omega = \Omega(t) := (-\theta)^{-q/(q-1)} \prod_{i=1}^{\infty} (1 - t/\theta^{q^i}) \in \mathbb{C}_\infty[[t]]$$

where $(-\theta)^{1/(q-1)}$ is a fixed $(q-1)$st root of $-\theta$ so that $\frac{1}{\Omega(\theta)} = \tilde{\pi}$ ([**ABP04, AT09**]). Here, we also introduce another expression of $\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon})$ by using the following theorem which was shown by Anderson and Thakur.

Theorem 3.1.4 ([**AT90**]). *For each $s \in \mathbb{N}$, there exists an unique polynomial $H_s = H_s(t) \in A[t]$ such that*

$$(H_{s-1}\Omega^s)^{(d)}|_{t=\theta} = \Gamma_s S_d(s)/\tilde{\pi}^s \tag{3.1.4}$$

for all $d \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{N}$. Moreover, when we regard H_s as a polynomial of θ over $\mathbb{F}_q[t]$ by $A[t] = \mathbb{F}_q[t][\theta]$ then

$$\deg_\theta H_s \leq \frac{sq}{q-1}. \tag{3.1.5}$$

This polynomial H_s is the Anderson-Thakur polynomial which we recalled in Definition 28. From (3.1.2) and (3.1.4), we obtain the following expression of $\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon})$;

$$\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon}) = \frac{\tilde{\pi}^{s_1 + \cdots s_r}}{\Gamma_{s_1} \cdots \Gamma_{s_r}} \sum_{d_1 > \cdots > d_r \geq 0} \epsilon_1^{d_1} (H_{s_1 - 1}\Omega^{s_1})^{(d_1)}|_{t=\theta} \cdots \epsilon_r^{d_r} (H_{s_r - 1}\Omega^{s_r})^{(d_r)}|_{t=\theta}. \tag{3.1.6}$$

3.2. Fundamental properties

In this section, we prove the non-vanishing and sum-shuffle relation of AMZVs.

3.2.1. Non-vanishing property of AMZVs. We show the non-vanishing property as the following theorem by using valuation of power sums which evaluated by Thakur [**T09a**].

THEOREM 3.2.1. *For any* $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ *and* $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$, $\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon})$ *are non-vanishing.*

PROOF. From (3.1.2), we can write $\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon})$ as follows.

$$\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon}) = \sum_{d_1 > d_2 > \cdots > d_r \geq 0} \epsilon_1^{d_1} \epsilon_2^{d_2} \cdots \epsilon_r^{d_r} S_{d_1}(s_1) S_{d_2}(s_2) \cdots S_{d_r}(s_r)$$

On the other hand, in [**T09a**], Thakur showed that

$$\deg_\theta S_d(k) > \deg_\theta S_{d+1}(k).$$

Therefore we have

$$\begin{aligned} |\zeta_A(s_1, \dots, s_r; \epsilon_1, \dots, \epsilon_r)|_\infty &= |\sum_{d_1 > d_2 > \cdots > d_r \geq 0} \epsilon_1^{d_1} \epsilon_2^{d_2} \cdots \epsilon_r^{d_r} S_{d_1}(s_1) S_{d_2}(s_2) \cdots S_{d_r}(s_r)|_\infty \\ &= |S_{r-1}(s_1) S_{r-2}(s_2) \cdots S_0(s_r)|_\infty \\ &\neq 0 \end{aligned}$$

by using $\deg_\theta S_0(k) = 0$ and $\deg_\theta S_d(k) < 0$ $(k > 0, d > 0)$ in [**T09a**, §2.2.3.]. Thus $\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon})$ are non-vanishing. □

3.2.2. Sum-shuffle relation for AMZVs. In this section, we give sum-shuffle relations for our $\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon})$. This kind of relations show that products of two AMZVs are expressed by $\mathbb{F}_p$-linear combination of AMZVs with preserving their weights. From the relation, $\zeta_A(\mathfrak{s}; \boldsymbol{\epsilon})$ form an $\mathbb{F}_p$-algebra.

For the products of power sums $S_d(s)$, the following formula was shown by Chen [**Chen15**].

PROPOSITION 3.2.2 ([**Chen15**, Theorem 3.1]). *For* $s_1, s_2 \in \mathbb{N}$, *we have*

$$S_d(s_1) S_d(s_2) - S_d(s_1 + s_2) = \sum_{\substack{0 < j < s_1 + s_2 \\ q-1 | j}} \Delta^j_{s_1, s_2} S_d(s_1 + s_2 - j, j)$$

where

$$\Delta^j_{s_1, s_2} = (-1)^{s_1 - 1} \binom{j-1}{s_1 - 1} + (-1)^{s_2 - 1} \binom{j-1}{s_2 - 1}. \tag{3.2.1}$$

The key idea to prove Proposition 3.2.2 is the following partial fraction decomposition

$$\frac{1}{a^{s_1} b^{s_2}} = \sum_{0 < j < s_1 + s_2} \left\{ \frac{(-1)^{s_1 - 1} \binom{j-1}{s_1 - 1}}{a^{s_1 + s_2 - j} (a - b)^j} + \frac{(-1)^{s_2 - 1} \binom{j-1}{s_2 - 1}}{b^{s_1 + s_2 - j} (a - b)^j} \right\}$$

for $a, b \in A \backslash \{0\}$. By using the above decomposition, she obtained the following formula in the proof of [**Chen15**, Theorem3.1]:

(3.2.2)

$$\sum_{\substack{a \neq b \in A_+ \\ \deg a = \deg b}} \frac{1}{a^{s_1} b^{s_2}} = \sum_{\substack{0 < j < s_1 + s_2 \\ (q-1) | j}} \left\{ \sum_{\substack{a, b \in A_+ \\ \deg a > \deg b}} \frac{(-1)^{s_1 - 1} \binom{j-1}{s_1 - 1}}{a^{s_1 + s_2 - j} b^j} + \sum_{\substack{a, b \in A_+ \\ \deg b > \deg a}} \frac{(-1)^{s_2 - 1} \binom{j-1}{s_2 - 1}}{b^{s_1 + s_2 - j} a^j} \right\}.$$

Chen's method is applied to prove the following lemma.

LEMMA 3.2.3. *For $s_1, s_2 \in \mathbb{N}$, $\epsilon_1, \epsilon_2 \in \mathbb{F}_q^\times$ and $d \in \mathbb{Z}_{\geq 0}$, we have*

(3.2.3)
$$S_d(s_1;\epsilon_1)S_d(s_2;\epsilon_2) - S_d(s_1+s_2;\epsilon_1\epsilon_2) = \sum_{\substack{0<j<s_1+s_2\\ q-1\mid j}} \Delta^j_{s_1,s_2} S_d(s_1+s_2-j, j; \epsilon_1\epsilon_2, 1)$$

PROOF. We give a proof based on the idea in [**Chen15**].

$$\begin{aligned}
&S_d(s_1;\epsilon_1)S_d(s_2;\epsilon_2) - S_d(s_1+s_2;\epsilon_1\epsilon_2)\\
&\quad= \sum_{a\in A_{d+}} \frac{\epsilon_1^d}{a^{s_1}} \sum_{b\in A_{d+}} \frac{\epsilon_2^d}{b^{s_2}} - S_d(s_1+s_2;\epsilon_1\epsilon_2)\\
&\quad= \sum_{\substack{a,b\in A_+\\ d=\deg a>\deg b\geq 0}} \frac{(\epsilon_1\epsilon_2)^d}{a^{s_1}b^{s_2}} + \sum_{\substack{a,b\in A_+\\ d=\deg b>\deg a\geq 0}} \frac{(\epsilon_1\epsilon_2)^d}{b^{s_2}a^{s_1}} + \sum_{\substack{a=b\in A_{d+}\\ d=\deg a=\deg b}} \frac{(\epsilon_1\epsilon_2)^d}{a^{s_1}b^{s_2}}\\
&\qquad+ \sum_{\substack{a\neq b\in A_+\\ d=\deg a=\deg b}} \frac{(\epsilon_1\epsilon_2)^d}{a^{s_1}b^{s_2}} - S_d(s_1+s_2;\epsilon_1\epsilon_2).
\end{aligned}$$

By (3.1.1), we have $\sum_{\substack{a=b\in A_+\\ d=\deg a=\deg b}} \frac{(\epsilon_1\epsilon_2)^d}{a^{s_1}b^{s_2}} = S_d(s_1+s_2;\epsilon_1\epsilon_2)$ and thus

$$\begin{aligned}
&\sum_{\substack{a,b\in A_+\\ d=\deg a>\deg b\geq 0}} \frac{(\epsilon_1\epsilon_2)^d}{a^{s_1}b^{s_2}} + \sum_{\substack{a,b\in A_+\\ d=\deg b>\deg a\geq 0}} \frac{(\epsilon_1\epsilon_2)^d}{b^{s_2}a^{s_1}} + \sum_{\substack{a=b\in A_{d+}\\ d=\deg a=\deg b}} \frac{(\epsilon_1\epsilon_2)^d}{a^{s_1}b^{s_2}}\\
&\qquad+ \sum_{\substack{a\neq b\in A_+\\ d=\deg a=\deg b}} \frac{(\epsilon_1\epsilon_2)^d}{a^{s_1}b^{s_2}} - S_d(s_1+s_2;\epsilon_1\epsilon_2)\\
&\quad= S_d(s_1,s_2;\epsilon_1\epsilon_2,1) + S_d(s_2,s_1;\epsilon_1\epsilon_2,1) + \sum_{\substack{a\neq b\in A_+\\ d=\deg a=\deg b\geq 0}} \frac{(\epsilon_1\epsilon_2)^d}{a^{s_1}b^{s_2}}.
\end{aligned}$$

By (3.2.2), we have

$$\begin{aligned}
\sum_{\substack{a\neq b\in A_+\\ d=\deg a=\deg b\geq 0}} \frac{(\epsilon_1\epsilon_2)^d}{a^{s_1}b^{s_2}} &= \sum_{\substack{0<j<s_1+s_2\\ q-1\mid j}} \left\{ \sum_{\substack{x,y\in A_+\\ d=\deg x>\deg y}} \frac{(-1)^{s_1-1}\binom{j-1}{s_1-1}(\epsilon_1\epsilon_2)^d}{x^{s_1+s_2-j}y^j} \right.\\
&\qquad\left. + \sum_{\substack{x,y\in A_+\\ d=\deg y>\deg x}} \frac{(-1)^{s_2-1}\binom{j-1}{s_2-1}(\epsilon_1\epsilon_2)^d}{y^{s_1+s_2-j}x^j} \right\}\\
&= \sum_{\substack{0<j<s_1+s_2\\ q-1\mid j}} \Delta^j_{s_1,s_2} S_d(s_1+s_2-j, j;\epsilon_1\epsilon_2, 1).
\end{aligned}$$

Therefore the relation (3.2.3) is proved. □

REMARK 3.2.4. In Lemma 3.2.3, coefficients $\Delta^j_{s_1,s_2}$ are independent of d.

Preceding to the next argument, we introduce an expression which is used in the rest of this section. For any index $\mathfrak{s} = (s_1, s_2, \dots, s_r) \in \mathbb{N}^r$, we can write $\mathfrak{s} = (s_1, \mathfrak{s}')$ where $\mathfrak{s}' = (s_2, \dots, s_r) \in \mathbb{N}^r$ (resp. $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$) and when $r = 1$, we set $\mathfrak{s}' = \phi$ (resp. $\boldsymbol{\epsilon}' = \phi$) and further $S_d(\mathfrak{s}';\boldsymbol{\epsilon}') := 1$.

Next we prepare the following lemma to show sum-shuffle relations for alternating power sums in general depth.

LEMMA 3.2.5. *For* $\mathfrak{a} := (a_1, a_2, \dots, a_r) \in \mathbb{N}^r$*,* $\mathfrak{b} := (b_1, b_2, \dots, b_s) \in \mathbb{N}^s$*,* $\boldsymbol{\epsilon} := (\epsilon_1, \epsilon_2, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$ *and* $\boldsymbol{\lambda} := (\lambda_1, \lambda_2, \dots, \lambda_s) \in (\mathbb{F}_q^\times)^s$*, we may express the product* $S_{<d}(\mathfrak{a}; \boldsymbol{\epsilon}) S_{<d}(\mathfrak{b}; \boldsymbol{\lambda})$ *as follows:*

$$S_{<d}(\mathfrak{a}; \boldsymbol{\epsilon}) S_{<d}(\mathfrak{b}; \boldsymbol{\lambda}) = \sum_i f_i S_{<d}(c_{i1}, \dots, c_{il_i}; \mu_{i1}, \dots, \mu_{il_i}) \tag{3.2.4}$$

for some $c_{ij} \in \mathbb{N}$*,* $\mu_{ij} \in \mathbb{F}_q^\times$ *so that* $\sum_{m=1}^r a_m + \sum_{n=1}^s b_n = \sum_{h=1}^{l_i} c_h$*,* $\prod_{m=1}^r \epsilon_m \prod_{n=1}^s \lambda_n = \prod_{h=1}^{l_i} \mu_h$*,* $l_i \leq r + s$ *and* $f_i \in \mathbb{F}_p$ *which are independent of* d *for each* i*.*

PROOF. We proceed to prove the result by induction on $\mathrm{dep}(\mathfrak{a}) + \mathrm{dep}(\mathfrak{b}) (= r + s) > 2$:

$$\begin{aligned}
&S_{<d}(\mathfrak{a}; \boldsymbol{\epsilon}) S_{<d}(\mathfrak{b}; \boldsymbol{\lambda}) \\
&\quad = \left(\sum_{0 \leq m_1 < d} S_{m_1}(\mathfrak{a}; \boldsymbol{\epsilon}) \right) \left(\sum_{0 \leq n_1 < d} S_{n_1}(\mathfrak{b}; \boldsymbol{\lambda}) \right) = \sum_{0 \leq m_1, n_1 < d} S_{m_1}(\mathfrak{a}; \boldsymbol{\epsilon}) S_{n_1}(\mathfrak{b}; \boldsymbol{\lambda}) \\
&\quad = \sum_{0 \leq n_1 < m_1 < d} S_{m_1}(\mathfrak{a}; \boldsymbol{\epsilon}) S_{n_1}(\mathfrak{b}; \boldsymbol{\lambda}) + \sum_{0 \leq m_1 < n_1 < d} S_{m_1}(\mathfrak{a}; \boldsymbol{\epsilon}) S_{n_1}(\mathfrak{b}; \boldsymbol{\lambda}) \\
&\qquad + \sum_{0 \leq n_1 = m_1 < d} S_{m_1}(\mathfrak{a}; \boldsymbol{\epsilon}) S_{n_1}(\mathfrak{b}; \boldsymbol{\lambda}) \\
&\quad = \sum_{0 \leq m_1 < d} \left(S_{m_1}(a_1; \epsilon_1) S_{<m_1}(\mathfrak{a}'; \boldsymbol{\epsilon}') \sum_{0 \leq n_1 < m_1} S_{n_1}(\mathfrak{b}; \boldsymbol{\lambda}) \right) \\
&\qquad + \sum_{0 \leq n_1 < d} \left(S_{n_1}(b_1; \lambda_1) S_{<n_1}(\mathfrak{b}'; \boldsymbol{\lambda}') \sum_{0 \leq m_1 < n_1} S_{m_1}(\mathfrak{a}; \boldsymbol{\epsilon}) \right) \\
&\qquad + \sum_{0 \leq m_1 = n_1 < d} S_{m_1}(a_1; \epsilon_1) S_{<m_1}(\mathfrak{a}'; \boldsymbol{\epsilon}') S_{n_1}(b_1; \lambda_1) S_{<n_1}(\mathfrak{b}'; \boldsymbol{\lambda}') \\
&\quad = \sum_{0 \leq m_1 < d} S_{m_1}(a_1; \epsilon_1) S_{<m_1}(\mathfrak{a}'; \boldsymbol{\epsilon}') S_{<m_1}(\mathfrak{b}; \boldsymbol{\lambda}) \\
&\qquad + \sum_{0 \leq n_1 < d} S_{n_1}(b_1; \lambda_1) S_{<n_1}(\mathfrak{b}'; \boldsymbol{\lambda}') S_{<n_1}(\mathfrak{a}; \boldsymbol{\epsilon}) \\
&\qquad + \sum_{0 \leq m_1 < d} S_{m_1}(a_1; \epsilon_1) S_{<m_1}(\mathfrak{a}'; \boldsymbol{\epsilon}') S_{m_1}(b_1; \lambda_1) S_{<m_1}(\mathfrak{b}'; \boldsymbol{\lambda}')
\end{aligned}$$

by applying induction hypothesis to the first, second series of power sums and applying Lemma 3.2.3 to the third series in the right hand side of last equality, we have

$$\begin{aligned}
&= \sum_{0\le m_1<d} S_{m_1}(a_1;\epsilon_1)\sum_i f_i S_{<m_1}(c_{i1},\dots,c_{il_i};\mu_{i1},\dots,\mu_{il_i})\\
&\quad+\sum_{0\le n_1<d} S_{n_1}(b_1;\lambda_1)\sum_i f'_i S_{<n_1}(c'_{i1},\dots,c'_{il_i};\mu'_{i1},\dots,\mu'_{il_i})\\
&\quad+\sum_{0\le m_1<d}\Bigg(\sum_{\substack{0<j<a_1+b_1\\ q-1|j}} \Delta^j_{a_1,b_1} S_{m_1}(a_1+b_1-j,j;\epsilon_1\lambda_1,1)+S_{m_1}(a_1+b_1;\epsilon_1\lambda_1)\Bigg)\\
&\qquad\qquad\cdot S_{<m_1}(\mathfrak{a}';\boldsymbol{\epsilon}')S_{<m_1}(\mathfrak{b}';\boldsymbol{\lambda}')\\
&= \sum_{0\le m_1<d} S_{m_1}(a_1;\epsilon_1)\sum_i f_i S_{<m_1}(c_{i1},\dots,c_{il_i};\mu_{i1},\dots,\mu_{il_i})\\
&\quad+\sum_{0\le n_1<d} S_{n_1}(b_1;\lambda_1)\sum_i f'_i S_{<n_1}(c'_{i1},\dots,c'_{il_i};\mu'_{i1},\dots,\mu'_{il_i})\\
&\quad+\sum_{0\le m_1<d}\Bigg(\sum_{\substack{0<j<a_1+b_1\\ q-1|j}} \Delta^j_{a_1,b_1} S_{m_1}(a_1+b_1-j;\epsilon_1\lambda_1)S_{<m_1}(j;1)+S_{m_1}(a_1+b_1;\epsilon_1\lambda_1)\Bigg)\\
&\qquad\qquad\cdot S_{<m_1}(\mathfrak{a}';\boldsymbol{\epsilon}')S_{<m_1}(\mathfrak{b}';\boldsymbol{\lambda}').
\end{aligned}$$

Again by applying induction hypothesis for $S_{<m_1}(\mathfrak{a}';\boldsymbol{\epsilon}')S_{<m_1}(\mathfrak{b}';\boldsymbol{\lambda}')$, the above quantity equals

$$\begin{aligned}
&= \sum_{0\le m_1<d}\sum_i f_i S_{m_1}(a_1,c_{i1},\dots,c_{il_i};\epsilon_1,\mu_{i1},\dots,\mu_{il_i})\\
&\quad+\sum_{0\le n_1<d}\sum_i f'_i S_{n_1}(b_1,c'_{i1},\dots,c'_{il_i};\lambda_1,\mu'_{i1},\dots,\mu'_{il_i})\\
&\quad+\sum_{0\le m_1<d}\sum_{\substack{0<j<a_1+b_1\\ q-1|j}} \Delta^j_{a_1,b_1}\sum_i g_i S_{m_1}(a_1+b_1-j,e_{i1},\dots,e_{il_i};\epsilon_1\lambda_1,\eta_{i1},\dots,\eta_{il_i})\\
&+\sum_{0\le m_1<d}\sum_i g'_i S_{m_1}(a_1+b_1,e'_{i1},\dots,e'_{il_i};\epsilon_1\lambda_1,\eta'_{i1},\dots,\eta'_{il_i})
\end{aligned}$$

for some $g_i,\ g'_i\in\mathbb{F}_p$. Therefore the product $S_{<d}(\mathfrak{a};\boldsymbol{\epsilon})S_{<d}(\mathfrak{b};\boldsymbol{\lambda})$ is expressed by an $\mathbb{F}_p$-linear combination of $S_{<d}(-;-)$ with the desired conditions. □

To prove the sum-shuffle result in Theorem 3.2.8 the following is the key ingredient.

THEOREM 3.2.6. *For* $\mathfrak{a}:=(a_1,a_2,\dots,a_r)\in\mathbb{N}^r$, $\mathfrak{b}:=(b_1,b_2,\dots,b_s)\in\mathbb{N}^s$, $\boldsymbol{\epsilon}:=(\epsilon_1,\epsilon_2,\dots,\epsilon_r)\in(\mathbb{F}_q^\times)^r$ *and* $\boldsymbol{\lambda}:=(\lambda_1,\lambda_2,\dots,\lambda_s)\in(\mathbb{F}_q^\times)^s$, *we may express the product* $S_d(\mathfrak{a};\boldsymbol{\epsilon})S_d(\mathfrak{b};\boldsymbol{\lambda})$ *as follows:*

$$S_d(\mathfrak{a};\boldsymbol{\epsilon})S_d(\mathfrak{b};\boldsymbol{\lambda})=\sum_i f'_i S_d(c_{i1},\dots,c_{il_i};\mu_{i1},\dots,\mu_{il_i}) \tag{3.2.5}$$

for some $c_{ij}\in\mathbb{N}$, $\mu_{ij}\in\mathbb{F}_q^\times$ *so that* $\sum_{m=1}^r a_m+\sum_{n=1}^s b_n=\sum_{h=1}^{l_i}c_{ih}$, $\prod_{m=1}^r\epsilon_m\prod_{n=1}^s\lambda_n=\prod_{h=1}^{l_i}\mu_{ih}$, $l_i\le r+s$ *and* $f'_i\in\mathbb{F}_p$ *for each* i.

PROOF. From the decomposition which is described in (3.1.1), we have

$$S_d(\mathfrak{a};\boldsymbol{\epsilon})S_d(\mathfrak{b};\boldsymbol{\lambda})=S_d(a_1;\epsilon_1)S_{<d}(\mathfrak{a}';\boldsymbol{\epsilon}')S_d(b_1;\lambda_1)S_{<d}(\mathfrak{b}';\boldsymbol{\lambda}').$$

By using the equation (3.2.4) to the product $S_{<d}(\mathfrak{a}';\boldsymbol{\epsilon}')S_{<d}(\mathfrak{b}';\boldsymbol{\lambda}')$, we have

$$S_d(\mathfrak{a};\boldsymbol{\epsilon})S_d(\mathfrak{b};\boldsymbol{\lambda}) = S_d(a_1;\epsilon_1)S_d(b_1;\lambda_1)\left(\sum_i f_i S_{<d}(c_{i1},\dots,c_{il_i};\mu_{i1},\dots,\mu_{il_i})\right)$$

Using the equation (3.2.3), the quantity above is equal to

$$\begin{aligned}
&= \sum_{\substack{0<j<a_1+b_1\\ q-1|j}} \left(\Delta^j_{a_1,b_1} S_d(a_1+b_1-j,j;\epsilon_1\lambda_1,1) + S_d(a_1+b_1;\epsilon_1\lambda_1)\right)\\
&\qquad\cdot\left(\sum_i f_i S_{<d}(c_{i1},\dots,c_{il_i};\mu_{i1},\dots,\mu_{il_i})\right)\\
&= \sum_{\substack{0<j<a_1+b_1\\ q-1|j}} \left(\Delta^j_{a_1,b_1} S_d(a_1+b_1-j;\epsilon_1\lambda_1)S_{<d}(j;1) + S_d(a_1+b_1;\epsilon_1\lambda_1)\right)\\
&\qquad\cdot\left(\sum_i f_i S_{<d}(c_{i1},\dots,c_{il_i};\mu_{i1},\dots,\mu_{il_i})\right)\\
&= \sum_{\substack{0<j<a_1+b_1\\ q-1|j}} \sum_i f_i\Delta^j_{a_1,b_1} S_d(a_1+b_1-j;\epsilon_1\lambda_1)\\
&\qquad\cdot S_{<d}(j;1)S_{<d}(c_{i1},\dots,c_{il_i};\mu_{i1},\dots,\mu_{il_i})\\
&\quad+\sum_{\substack{0<j<a_1+b_1\\ q-1|j}} \sum_i f_i S_d(a_1+b_1;\epsilon_1\lambda_1)S_{<d}(c_{i1},\dots,c_{il_i};\mu_{i1},\dots,\mu_{il_i}).
\end{aligned}$$

Applying (3.2.4) to the product $S_{<d}(j;1)S_{<d}(c_{i1},\dots,c_{il_i};\mu_{i1},\dots,\mu_{il_i})$, we have

$$\begin{aligned}
S_d(\mathfrak{a};\boldsymbol{\epsilon})S_d(\mathfrak{b};\boldsymbol{\lambda}) &= \sum_{\substack{0<j<a_1+b_1\\ q-1|j}} \sum_i f_i\Delta^j_{a_1,b_1} S_d(a_1+b_1-j;\epsilon_1\lambda_1)\\
&\qquad\cdot\left(\sum_m g_m S_{<d}(e_{m1},\dots,e_{mn_m};\nu_{m1},\dots,\nu_{mn_m})\right)\\
&\quad+\sum_{\substack{0<j<a_1+b_1\\ q-1|j}} \sum_i f_i S_d(a_1+b_1;\epsilon_1\lambda_1)S_{<d}(c_{i1},\dots,c_{il_i};\mu_{i1},\dots,\mu_{il_i})\\
&= \sum_{\substack{0<j<a_1+b_1\\ q-1|j}} \sum_{i,m} f_i g_m\Delta^j_{a_1,b_1} S_d(a_1+b_1-j,e_{m1},\dots,e_{mn_m};\epsilon_1\lambda_1,\nu_{m1},\dots,\nu_{mn_m})\\
&\quad+\sum_{\substack{0<j<a_1+b_1\\ q-1|j}} \sum_i f_i S_d(a_1+b_1,c_{i1},\dots,c_{il_i};\epsilon_1\lambda_1,\mu_{i1},\dots,\mu_{il_i}).
\end{aligned}$$

Thus we show that $S_d(\mathfrak{a};\boldsymbol{\epsilon})S_d(\mathfrak{b};\boldsymbol{\lambda})$ is expressed by with the desired formulation. □

REMARK 3.2.7. In Theorem 3.2.6, coefficients $f_i \in \mathbb{F}_p$ are independent of d by Lamma 3.2.5.

The main result on sum-shuffle relations for AMZVs are the following.

THEOREM 3.2.8. *For* $\mathfrak{a} := (a_1, a_2, \dots, a_r) \in \mathbb{N}^r$, $\mathfrak{b} := (b_1, b_2, \dots, b_s) \in \mathbb{N}^s$, $\boldsymbol{\epsilon} := (\epsilon_1, \epsilon_2, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$ *and* $\boldsymbol{\lambda} := (\lambda_1, \lambda_2, \dots, \lambda_s) \in (\mathbb{F}_q^\times)^s$, *we may express the product* $\zeta_A(\mathfrak{a}; \boldsymbol{\epsilon})\zeta_A(\mathfrak{b}; \boldsymbol{\lambda})$ *as follows:*

$$\zeta_A(\mathfrak{a}; \boldsymbol{\epsilon})\zeta_A(\mathfrak{b}; \boldsymbol{\lambda}) = \sum_i f_i'' \zeta_A(c_{i1}, \dots, c_{il_i}; \mu_{i1}, \dots, \mu_{il_i}) \tag{3.2.6}$$

for some $c_{ij} \in \mathbb{N}$ *and* $\mu_{ij} \in \mathbb{F}_q^\times$ *so that* $\sum_{m=1}^r a_m + \sum_{n=1}^s b_n = \sum_{h=1}^{l_i} c_{ih}$, $\prod_{m=1}^r \epsilon_m \prod_{n=1}^s \lambda_n = \prod_{h=1}^{l_i} \mu_{ih}$, $l_i \leq r + s$ *and* $f_i'' \in \mathbb{F}_p$ *for each* i.

PROOF. By the set theoretical inclusion-exclusion principle, we can express $\sum_{d \geq 0} S_d(\mathfrak{a}; \boldsymbol{\epsilon}) S_d(\mathfrak{b}; \boldsymbol{\lambda})$ as follows:

$$\begin{aligned}
&\sum_{d \geq 0} S_d(\mathfrak{a}; \boldsymbol{\epsilon}) S_d(\mathfrak{b}; \boldsymbol{\lambda}) \\
&= \left(\sum_{d_1 \geq 0} S_{d_1}(\mathfrak{a}; \boldsymbol{\epsilon}) \right) \left(\sum_{e_1 \geq 0} S_{e_1}(\mathfrak{b}; \boldsymbol{\lambda}) \right) - \sum_{d_1 > e_1 \geq 0} S_{d_1}(\mathfrak{a}; \boldsymbol{\epsilon}) S_{e_1}(\mathfrak{b}; \boldsymbol{\lambda}) - \sum_{e_1 > d_1 \geq 0} S_{e_1}(\mathfrak{b}; \boldsymbol{\lambda}) S_{d_1}(\mathfrak{a}; \boldsymbol{\epsilon}) \\
&= \left(\sum_{d_1 \geq 0} S_{d_1}(\mathfrak{a}; \boldsymbol{\epsilon}) \right) \left(\sum_{e_1 \geq 0} S_{e_1}(\mathfrak{b}; \boldsymbol{\lambda}) \right) - \sum_{d_1 > 0} S_{d_1}(\mathfrak{a}; \boldsymbol{\epsilon}) S_{<d_1}(\mathfrak{b}; \boldsymbol{\lambda}) - \sum_{e_1 > 0} S_{e_1}(\mathfrak{b}; \boldsymbol{\lambda}) S_{<e_1}(\mathfrak{a}; \boldsymbol{\epsilon}) \\
&= \left(\sum_{d_1 \geq 0} S_{d_1}(\mathfrak{a}; \boldsymbol{\epsilon}) \right) \left(\sum_{e_1 \geq 0} S_{e_1}(\mathfrak{b}; \boldsymbol{\lambda}) \right) \\
&\quad - \sum_{d_1 > 0} S_{d_1}(a_1; \epsilon_1) S_{<d_1}(\mathfrak{a}'; \boldsymbol{\epsilon}') S_{<d_1}(\mathfrak{b}; \boldsymbol{\lambda}) - \sum_{e_1 > 0} S_{e_1}(b_1; \lambda_1) S_{<e_1}(\mathfrak{b}'; \boldsymbol{\lambda}') S_{<e_1}(\mathfrak{a}; \boldsymbol{\epsilon}).
\end{aligned}$$

By (3.2.4), the above is equal to

$$\begin{aligned}
&= \left(\sum_{d_1 \geq 0} S_{d_1}(\mathfrak{a}; \boldsymbol{\epsilon}) \right) \left(\sum_{e_1 \geq 0} S_{e_1}(\mathfrak{b}; \boldsymbol{\lambda}) \right) - \sum_{d_1 > 0} \sum_i g_i S_{d_1}(a_1, m_{i1}, \dots, m_{il_i}; \epsilon_1, \mu_{i1}, \dots, \mu_{il_i}) \\
&\qquad - \sum_{e_1 > 0} \sum_i h_i S_{e_1}(b_1, n_{i1}, \dots, n_{il_i}; \lambda_1, \eta_{i1}, \dots, \eta_{il_i}).
\end{aligned}$$

for some g_i, $h_i \in \mathbb{F}_p$. Combining the expression of $S_d(\mathfrak{a}; \boldsymbol{\epsilon}) S_d(\mathfrak{b}; \boldsymbol{\lambda})$ in (3.2.5) with the above, we have the following identity

$$\begin{aligned}
\sum_{d \geq 0} \sum_i f_i' S_d(c_{i1}, \dots, c_{il_i}; \mu_{i1}, \dots, \mu_{il_i}) &= \sum_{d_1 \geq 0} S_{d_1}(\mathfrak{a}; \boldsymbol{\epsilon}) \sum_{e_1 \geq 0} S_{e_1}(\mathfrak{b}; \boldsymbol{\lambda}) \\
&\quad - \sum_{d_1 > 0} \sum_i g_i S_{d_1}(a_1, m_{i1}, \dots, m_{il_i}; \epsilon_1, \mu_{i1}, \dots, \mu_{il_i}) \\
&\quad - \sum_{e_1 > 0} \sum_i h_i S_{e_1}(b_1, n_{i1}, \dots, n_{il_i}; \lambda_1, \eta_{i1}, \dots, \eta_{il_i}).
\end{aligned}$$

The coefficients f_i', g_i, h_i are independent of d by Theorem 3.2.6. Thus by (3.1.2), we obtain

$$\begin{aligned}
\sum_i f_i' \zeta_A(c_{i1}, \dots, c_{il_i}; \mu_{i1}, \dots, \mu_{il_i}) &= \zeta_A(\mathfrak{a}; \boldsymbol{\epsilon}) \zeta_A(\mathfrak{b}; \boldsymbol{\lambda}) \\
&\quad - \sum_i g_i \zeta_A(a_1, m_{i1}, \dots, m_{il_i}; \epsilon_1, \mu_{i1}, \dots, \mu_{il_i}) \\
&\quad - \sum_i h_i \zeta_A(b_1, n_{i1}, \dots, n_{il_i}; \lambda_1, \eta_{i1}, \dots, \eta_{il_i}).
\end{aligned}$$

Thus we have

$$\begin{aligned}\zeta_A(\mathfrak{a};\boldsymbol{\epsilon})\zeta_A(\mathfrak{b};\boldsymbol{\lambda}) = &\sum_i f_i'\zeta_A(c_{i1},\dots,c_{il_i};\mu_{i1},\dots,\mu_{il_i})\\ &+\sum_i g_i\zeta_A(a_1,m_{i1},\dots,m_{il_i};\epsilon_1,\mu_{i1},\dots,\mu_{il_i})\\ &+\sum_i h_i\zeta_A(b_1,n_{i1},\dots,n_{il_i};\lambda_1,\eta_{i1},\dots,\eta_{il_i}).\end{aligned}$$

By Lemma 3.2.5 and Theorem 3.2.6, the indices and coefficients in the above equation satisfy desired conditions.

Therefore we obtain the sum-shuffle relation for $\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})$. □

By Theorem 3.2.8, the $\mathbb{F}_p$-linear span of our AMZVs form an $\mathbb{F}_p$-algebra.

EXAMPLES 3.2.9. For some q, $(\mathfrak{a};\boldsymbol{\epsilon})$ and $(\mathfrak{b};\boldsymbol{\lambda})$, the product $\zeta_A(\mathfrak{a};\boldsymbol{\epsilon})\zeta_A(\mathfrak{b};\boldsymbol{\lambda})$ is explicitly computed as follows.

When $q=3$, $(\mathfrak{a};\boldsymbol{\epsilon})=(2;1)$ and $(\mathfrak{b};\boldsymbol{\lambda})=(1,2;2,2)$,

$$\begin{aligned}\zeta_A(2;1)\zeta_A(1,2;2,2) = &\zeta_A(3,2;2,2)-\zeta_A(1,2,2;2,2,1)+\zeta_A(1,2,2;2,1,2)\\ &+\zeta_A(1,4;2,2)+\zeta_A(2,1,2;1,2,2)\end{aligned}$$

When $q=5$, $(\mathfrak{a};\boldsymbol{\epsilon})=(2;3)$ and $(\mathfrak{b};\boldsymbol{\lambda})=(3;1)$,

$$\zeta_A(2;3)\zeta_A(3;1)=\zeta_A(5;3)+\zeta_A(2,3;3,1)+\zeta_A(3,2;1,3).$$

Later we give an explicit sum-shuffle relation for a product of any depth 2 AMZV and depth 1 AMZV in Appendix.

3.3. Period interpretation of AMZVs

In this section, we show that each $\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})$ appears as a period of certain pre-t-motive basing on the idea in [**AT09**].

We denote the ring $\overline{k}(t)[\sigma,\sigma^{-1}]$ the non-commutative Laurent polynomial ring in σ with coefficients in $\overline{k}(t)$ subject to the following relations,

$$\sigma f = f^{(-1)}\sigma,\quad f\in\overline{k}(t).$$

We denote $\mathbb{E}$ to be the ring consisting of formal power series

$$\sum_{n=0}^{\infty}a_nt^n\in\overline{k}[[t]]$$

such that

$$\lim_{n\to\infty}\sqrt[n]{|a_n|_\infty}=0,\quad [k_\infty(a_0,a_1,\dots):k_\infty]<\infty.$$

We note that the former condition guarantees that such a series is an entire function, that is, it has an infinite radius of convergence with respect to the absolute value $|\cdot|_\infty$ thus $\mathbb{E}\subset\mathbb{T}$. The latter condition guarantees that for any $t_0\in\overline{k_\infty}$ the value of such a series at $t=t_0$ belongs again to $\overline{k_\infty}$. We note that

$$\Omega\in\mathbb{E}. \tag{3.3.1}$$

Therefore $\Omega\in\mathbb{T}$.

3.3.1. Review of pre-t-motive. First we recall the notions of pre-t-motives. For more detail, see [**P08**].

DEFINITION 3.3.1 ([**P08**, §3.2.1]). A *pre-t-motive* M is a left $\overline{k}(t)[\sigma,\sigma^{-1}]$-module that is finite dimensional over $\overline{k}(t)$.

Let M be a pre-t-motive dimension r over $\overline{k}(t)$ and Φ be representing matrix of multiplication by σ on M with respect to a given basis of $\mathbf{m}$ of M, then M is rigid analytically trivial if and only if there is a matrix $\Psi \in \mathrm{GL}_r(\mathbb{L})$ ($\mathbb{L}$ is a quotient field of $\mathbb{T}$) satisfying

$$\Psi^{(-1)} = \Phi\Psi.$$

Here we define $\Psi^{(-1)}$ by $(\Psi^{(-1)})_{ij} := (\Psi_{ij})^{(-1)}$. Such matrix Ψ is called *rigid analytic trivialization* of Φ and if all the entries of Ψ are convergent at $t=\theta$, and entries of $\Psi|_{t=\theta}$ are called *periods* of M.

Anderson and Thakur obtained period interpretation of MZVs in [**AT09**], that is, they showed that MZVs appear as entries of $\Psi|_{t=\theta}$ for Ψ which is a rigid analytic trivialization of the rigid analytically trivial pre-t-motive with dimension $r+1$ over $\overline{k}(t)$ multiplication by σ is represented by the following matrix Φ:

$$\Phi := \begin{pmatrix} (t-\theta)^{s_1+\cdots+s_r} & 0 & 0 & \cdots & 0 \\ (t-\theta)^{s_1+\cdots+s_r}H_{s_1-1}^{(-1)} & (t-\theta)^{s_2+\cdots+s_r} & 0 & \cdots & 0 \\ 0 & (t-\theta)^{s_2+\cdots+s_r}H_{s_2-1}^{(-1)} & \ddots & & \vdots \\ \vdots & & \ddots & (t-\theta)^{s_r} & 0 \\ 0 & \cdots & 0 & (t-\theta)^{s_r}H_{s_r-1}^{(-1)} & 1 \end{pmatrix}.$$

They obtained the rigid analytic trivialization of Φ as the following matrix:

$$\Psi := \begin{pmatrix} \Omega^{s_1+\cdots+s_r} & & & & & \\ L(s_1)\Omega^{s_2+\cdots+s_r} & \Omega^{s_2+\cdots+s_r} & & & & \\ \vdots & L(s_2)\Omega^{s_3+\cdots+s_r} & \ddots & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ L(s_1,\dots,s_{r-1})\Omega^{s_r} & L(s_2,\dots .s_{r-1})\Omega^{s_r} & & \ddots & \Omega^{s_r} & \\ L(s_1,\dots,s_r) & L(s_2,\dots,s_r) & \cdots & \cdots & L(s_r) & 1 \end{pmatrix}.$$

Here we define

$$L(s_1,\dots,s_r) := \sum_{d_1>\cdots>d_r\geq 0} (H_{s_1-1}\Omega^{s_1})^{(d_1)}\cdots(H_{s_r-1}\Omega^{s_r})^{(d_r)}.$$

By using $\Omega|_{t=\theta} = \tilde{\pi}^{-1}$ and (3.1.4), we obtain the following for $1\leq i\leq j\leq r$:

$$L(s_i,\dots,s_j)|_{t=\theta} = \frac{\Gamma_{s_i}\cdots\Gamma_{s_j}}{\tilde{\pi}^{s_i+\cdots+s_j}}\sum_{d_i>\cdots>d_j\geq 0} S_{d_i}(s_i)\cdots S_{d_j}(s_j) = \frac{\Gamma_{s_i}\cdots\Gamma_{s_j}}{\tilde{\pi}^{s_i+\cdots+s_j}}\zeta_A(s_i,\dots,s_j).$$

Therefore it is known that MZVs are periods of the rigid analytically trivial pre-t-motive M defined by Φ.

3.3.2. AMZVs as periods. Next we give a period interpretation of AMZVs
.

DEFINITION 3.3.2. Given $\gamma_i \in \overline{\mathbb{F}_q}^{\times}$ $(i=1,\dots,r)$, a fixed $(q-1)$st root of $\epsilon_i \in \mathbb{F}_q^{\times}$, we let M be the pre-t-motive such that $\dim_{\overline{k}(t)} M = r+1$ and for the fixed $\overline{k}(t)$-basis $\mathbf{m}$ of M which satisfy

$$\sigma\mathbf{m} = \Phi\mathbf{m}$$

where Φ is the following matirix:

$$\begin{pmatrix} (t-\theta)^{s_1+\cdots+s_r} & 0 & 0 & \cdots & 0 \\ \gamma_1^{(-1)}(t-\theta)^{s_1+\cdots+s_r}H_{s_1-1}^{(-1)} & (t-\theta)^{s_2+\cdots+s_r} & 0 & \cdots & 0 \\ 0 & \gamma_2^{(-1)}(t-\theta)^{s_2+\cdots+s_r}H_{s_2-1}^{(-1)} & \ddots & & \vdots \\ \vdots & & \ddots & (t-\theta)^{s_r} & 0 \\ 0 & \cdots & 0 & \gamma_r^{(-1)}(t-\theta)^{s_r}H_{s_r-1}^{(-1)} & 1 \end{pmatrix}.$$

Here $H_{s_i-1} \in A[t]$ is the Anderson-Thakur polynomial.

For the above pre-t-motive, we can give a matrix Ψ which has a relation with Φ as the following proposition.

Proposition 3.3.3. *For the pre-t-motive M defined in Definition 3.3.2, there exists $\Psi \in \mathrm{GL}_{r+1}(\overline{k}[t])$ so that it satisfies $\Psi^{(-1)} = \Phi\Psi$.*

Proof. For the matrix Φ in Definition 3.3.2, we show that the matrix $\Psi \in \mathrm{GL}_{r+1}(\overline{k}[t])$ satisfies $\Psi^{(-1)} = \Phi\Psi$ and which is given as follows:

(3.3.2)
$$\Psi := \begin{pmatrix} \Omega^{s_1+\cdots+s_r} & & & & \\ \gamma_1 L(s_1)\Omega^{s_2+\cdots+s_r} & \Omega^{s_2+\cdots+s_r} & & & \\ \vdots & \gamma_2 L(s_2)\Omega^{s_3+\cdots+s_r} & \ddots & & \\ \vdots & \vdots & \ddots \quad \ddots & & \\ \gamma_1\cdots\gamma_{r-1}L(s_1,\dots,s_{r-1})\Omega^{s_r} & \gamma_2\cdots\gamma_{r-1}L(s_2,\dots s_{r-1})\Omega^{s_r} & \ddots & \Omega^{s_r} & \\ \gamma_1\cdots\gamma_r L(s_1,\dots,s_r) & \gamma_2\cdots\gamma_r L(s_2,\dots,s_r) & \cdots \quad \cdots & \gamma_r L(s_r) & 1 \end{pmatrix}.$$

Here we define the following:

$$\begin{aligned} L(s_1,\dots,s_r) &:= L(s_1,\dots,s_r;\epsilon_1,\dots,\epsilon_r) \\ &:= \sum_{d_1>\cdots>d_r\geq 0} \epsilon_1^{d_1}(H_{s_1-1}\Omega^{s_1})^{(d_1)}\cdots\epsilon_r^{d_r}(H_{s_r-1}\Omega^{s_r})^{(d_r)}. \end{aligned}$$

We also note that the following equations hold by their definitions.

$$\gamma_i^{(-1)} = \epsilon_i^{-1}\gamma_i, \tag{3.3.3}$$

$$\Omega^{(-1)} = (t-\theta)\Omega, \tag{3.3.4}$$

$$\begin{aligned} L(s_1,\dots,s_r)^{(-1)} &= \left(\prod_{n=0}^{r}\epsilon_n\right)L(s_1,\dots,s_r) \\ &\quad + \left(\prod_{n=0}^{r-1}\epsilon_n\right)(\Omega^{s_r}H_{s_r-1})^{(-1)}L(s_1,\dots,s_{r-1}). \end{aligned} \tag{3.3.5}$$

Now we show the validity of the equation

$$\Psi^{(-1)} = \Phi\Psi$$

by comparing their entries.

For $1 \leq i \leq r+1$, and $i \leq j \leq r+1$, it is obvious that (i,j)-th entry of the left hand side $\Psi_{ij}^{(-1)}$ is equal to the (i,j)-th entry of the right hand side $(\Phi\Psi)_{ij}$ by (3.3.4).

In the rest of this proof, we set $\prod_{n=i}^{j}\gamma_n = 1$ for $0 \leq j < i$ since it is empty product. For $2 \leq i \leq r$ and $1 \leq j < i$, $\Psi_{ij}^{(-1)}$ is transformed as follows.

$$\Psi_{ij}^{(-1)} = \left(\prod_{n=j}^{i-1}(\gamma_n)^{(-1)}\right)(t-\theta)^{s_i+\cdots+s_r}\Omega^{s_i+\cdots+s_r}L(s_j,\ldots,s_{i-1})^{(-1)}$$

by using the equation (3.3.5),

$$\begin{aligned} =&\left(\prod_{n=j}^{i-1}\gamma_n^{(-1)}\right)\left\{(t-\theta)\Omega\right\}^{s_i+\cdots+s_r}\left(\prod_{n=j}^{i-1}\epsilon_n\right)L(s_j,\ldots,s_{i-1}) \\ &+\left(\prod_{n=j}^{i-1}\gamma_n^{(-1)}\right)\left\{(t-\theta)\Omega\right\}^{s_i+\cdots+s_r}\left(\prod_{n=j}^{i-2}\epsilon_n\right)(\Omega^{s_{i-1}}H_{s_{i-1}-1})^{(-1)}L(s_j,\ldots,s_{i-2})\end{aligned}$$

by using the equation (3.3.3),

$$\begin{aligned} =&\left(\prod_{n=j}^{i-1}\gamma_n\right)\left\{(t-\theta)\Omega\right\}^{s_i+\cdots+s_r}L(s_j,\ldots,s_{i-1}) \\ &+\gamma_{i-1}^{(-1)}\left(\prod_{n=j}^{i-2}\gamma_n\right)\left\{(t-\theta)\Omega\right\}^{s_i+\cdots+s_r}(\Omega^{s_{i-1}}H_{s_{i-1}-1})^{(-1)}L(s_j,\ldots,s_{i-2})\end{aligned}$$

by using the equation (3.3.4),

$$\begin{aligned} =&\left(\prod_{n=j}^{i-1}\gamma_n\right)\left\{(t-\theta)\Omega\right\}^{s_i+\cdots+s_r}L(s_j,\ldots,s_{i-1}) \\ &+\gamma_{i-1}^{(-1)}\left(\prod_{n=j}^{i-2}\gamma_n\right)\left\{(t-\theta)\Omega\right\}^{s_{i-1}+\cdots+s_r}(H_{s_{i-1}-1})^{(-1)}L(s_j,\ldots,s_{i-2}).\end{aligned}$$

On the other hand, $(\Phi\Psi)_{ij}$ is the product of i-th row of Φ

$$\begin{pmatrix} \overbrace{0 & \cdots & 0}^{i-2} & \gamma_{i-1}^{(-1)}(t-\theta)^{s_{i-1}+\cdots+s_r}H_{s_{i-1}-1}^{(-1)} & (t-\theta)^{s_i+\cdots+s_r} & 0 & \cdots & 0 \end{pmatrix}$$

and j-th column of Ψ

$$(3.3.6)\qquad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \left(\prod_{n=j}^{j-1}\gamma_n\right)\Omega^{s_j+\cdots+s_r} \\ \left(\prod_{n=j}^{j}\gamma_n\right)\Omega^{s_{j+1}+\cdots+s_r}L(s_j) \\ \vdots \\ \left(\prod_{n=j}^{r-1}\gamma_n\right)\Omega^{s_r}L(s_j,\ldots,s_{r-1}) \\ \left(\prod_{n=j}^{r}\gamma_n\right)L(s_j,\ldots,s_r) \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} j-1 \\ \\ \\ \\ \\ \\ \end{matrix}.$$

That is,

$$(\Phi\Psi)_{ij} = \gamma_{i-1}^{(-1)}(t-\theta)^{s_{i-1}+\cdots+s_r} H_{s_{i-1}-1}^{(-1)} \left(\prod_{n=j}^{i-2} \gamma_n\right) \Omega^{s_{i-1}+\cdots+s_r} L(s_j,\dots,s_{i-2})$$
$$+ (t-\theta)^{s_i+\cdots+s_r} \left(\prod_{n=j}^{i-1} \gamma_n\right) \Omega^{s_i+\cdots+s_r} L(s_j,\dots,s_{i-1}).$$

Thus we have $(\Phi\Psi)_{ij} = \Psi_{ij}^{(-1)}$. □

As we will see in Lemma 3.4.3 later, the matrix Ψ in (3.3.2) belongs to $\mathrm{Mat}_{r+1}(\mathbb{T})$ then we obtain that $\Psi \in \mathrm{GL}_{r+1}(\mathbb{L})$ from (3.3.1) and $\det\Psi = \Omega^{\sum_{i=1}^r d_i} \neq 0$ where $d_i = s_i + \cdots + s_r$. Therefore Ψ is a rigid analytic trivialization of Φ and we call each entry of $\Psi|_{t=\theta}$ a period of M. Thus we have the following result by the above proposition.

Theorem 3.3.4. *For $\mathfrak{s} = (s_1,\dots,s_r) \in \mathbb{N}^r$ and $\boldsymbol{\epsilon} = (\epsilon_1,\dots,\epsilon_r) \in (\mathbb{F}_q^\times)^r$, $\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})$ are periods of the pre-t-motive M in Definition 3.3.2.*

Proof. By using $\Omega|_{t=\theta} = \tilde{\pi}^{-1}$ and (3.1.4), we obtain

$$L(\mathfrak{s})|_{t=\theta} = \frac{\Gamma_{s_1}\cdots\Gamma_{s_r}}{\tilde{\pi}^{s_1+\cdots+s_r}} \sum_{d_1>\cdots>d_r\geq 0} \epsilon_1^{d_1}\cdots\epsilon_r^{d_r} S_{d_1}(s_1)\cdots S_{d_r}(s_r) \tag{3.3.7}$$
$$= \frac{\Gamma_{s_1}\cdots\Gamma_{s_r}}{\tilde{\pi}^{s_1+\cdots+s_r}} \zeta_A(\mathfrak{s};\boldsymbol{\epsilon}).$$

Therefore by the matrix Ψ in (3.3.2), $\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})$ are periods of the pre-t-motive M in Definition 3.3.2. □

3.4. Linear independence of monomials of AMZVs

In this section, we show that AMZVs form an weight-graded algebra as an application of their period interpretation in the former section. The proof is shown along the method which was invented by Chang [**C14**].

3.4.1. ABP-criterion. In our proof, we need to use ABP criterion (ABP stands for Anderson-Brownawell-Papanikolas), which was introduced in [**ABP04**].

The ABP criterion is stated as the following theorem.

Theorem 3.4.1 ([**ABP04**, Theorem 3.1.1]). *Fix $\Phi \in \mathrm{Mat}_d(\overline{k}[t])$ so that $\det\Phi = c(t-\theta)^s$ for some $c \in \overline{k}^\times$ and some $s \in \mathbb{Z}_{\geq 0}$. Suppose that there exists a vector $\psi \in \mathrm{Mat}_{d\times 1}(\mathbb{E})$ satisfies*

$$\psi^{(-1)} = \Phi\psi.$$

For every $\rho \in \mathrm{Mat}_{1\times d}(\overline{k})$ such that $\rho\psi(\theta) = 0$, there is a $P \in \mathrm{Mat}_{1\times d}(\overline{k}[t])$ so that $P(\theta) = \rho$ and $P\psi = 0$.

From Proposition 3.3.3, it is a simple matter to verify that the matrices Φ in Definition 3.3.2 satisfy the conditions. We thus only need to show that $\Psi \in \mathrm{Mat}_{r+1}(\mathbb{E})$. This is necessary in applying Theorem 3.4.1 to our AMZVs. We use the following proposition which was given in [**ABP04**].

Proposition 3.4.2 ([**ABP04**, Proposition 3.1.3]). *Suppose*

$$\Phi \in \mathrm{Mat}_l(\overline{k}[t]), \quad \psi \in \mathrm{Mat}_{l\times 1}(\mathbb{T})$$

such that

$$\det\Phi|_{t=0} \neq 0, \quad \psi^{(-1)} = \Phi\psi.$$

Then

$$\psi \in \mathrm{Mat}_{l\times 1}(\mathbb{E}).$$

By using this proposition, we prove the following lemma whose proof is based on [**C14**, Lemma 5.3.1].

LEMMA 3.4.3. *Let $\Psi \in \mathrm{GL}_{r+1}(\overline{k}[t])$ be as in Proposition 3.3.3. Then the following holds:*

$$\Psi \in \mathrm{Mat}_{r+1}(\mathbb{E}).$$

PROOF. To apply Proposition 3.4.2, we first prove that $\Psi \in \mathrm{Mat}_{r+1}(\mathbb{T})$. By (3.3.1), $\Omega \in \mathbb{T}$ thus it is sufficient to show that each $\left(\prod_{n=i}^{j} \gamma_n\right) L(\mathfrak{s};\boldsymbol{\epsilon})$ belongs to $\mathbb{T}$ where $\mathfrak{s} = (s_i, \dots, s_j)$ and $\boldsymbol{\epsilon} = (\epsilon_i, \dots, \epsilon_j)$ for $1 \le i \le j \le r$. When $|t|_\infty \le 1$, we have the following for $\sum_{n=0}^{m} a_n t^n \in \overline{k}[t]$:

$$\|\sum_{n=0}^{m} a_n t^n\|_\infty = \max_{m\ge n\ge 0}\{|a_n|_\infty\} \ge \max_{m\ge n\ge 0}\{|a_n t^n|_\infty\} \ge |\sum_{n=0}^{m} a_n t^n|_\infty \ge 0.$$

Thus if $\|\sum_{n=0}^{m} a_n t^n\|_\infty \to 0$ as $m \to \infty$, $\sum_{n=0}^{\infty} a_n t^n$ converges. Then for the following series

$$\begin{aligned}\left(\prod_{n=i}^{j}\gamma_n\right)L(\mathfrak{s};\boldsymbol{\epsilon}) &= \sum_{d_i>\cdots>d_j\ge 0} \gamma_i \epsilon_i^{d_i}(H_{s_i-1}\Omega^{s_i})^{(d_i)}\cdots\gamma_j\epsilon_j^{d_j}(H_{s_j-1}\Omega^{s_j})^{(d_j)} \\ &= \Omega^{s_i+\cdots+s_j}\sum_{d_i>\cdots>d_j\ge 0}\frac{(\gamma_i\epsilon_i^{d_i})^{\frac{1}{q^{d_i}}}H_{s_i-1}^{(d_i)}\cdots(\gamma_j\epsilon_j^{d_j})^{\frac{1}{q^{d_j}}}H_{s_j-1}^{(d_j)}}{\left((t-\theta^q)\cdots(t-\theta^{q^{d_i}})\right)^{s_i}\cdots\left((t-\theta^q)\cdots(t-\theta^{q^{d_j}})\right)^{s_j}},\end{aligned}$$

we need to show that when $|t|_\infty \le 1$,

$$\frac{\|(\gamma_i\epsilon_i^{d_i})^{\frac{1}{q^{d_i}}}H_{s_i-1}^{(d_i)}\cdots(\gamma_j\epsilon_j^{d_j})^{\frac{1}{q^{d_j}}}H_{s_j-1}^{(d_j)}\|_\infty}{\|\left((t-\theta^q)\cdots(t-\theta^{q^{d_i}})\right)^{s_i}\cdots\left((t-\theta^q)\cdots(t-\theta^{q^{d_j}})\right)^{s_j}\|_\infty} \to 0$$

as $0 \le d_j < \cdots < d_i \to \infty$ (d_i goes to infinity preserving $0 \le d_j < \cdots < d_i$). By $\epsilon_n \in \mathbb{F}_q^\times$ and $\gamma_n \in \overline{\mathbb{F}_q}^\times$, we have $|(\gamma_n\epsilon_n^{d_n})^{\frac{1}{q^{d_n}}}|_\infty = 1$ and thus

$$\|(\gamma_n\epsilon_n^{d_n})^{\frac{1}{q^{d_n}}}H_{s_n-1}^{(d_n)}\|_\infty = \|H_{s_n-1}^{(d_n)}\|_\infty. \tag{3.4.1}$$

Moreover, by $|t|_\infty < |\theta|_\infty$ we have $|t-\theta^{q^n}|_\infty = |\theta^{q^n}|_\infty$ and then

$$\|\left((t-\theta^q)\cdots(t-\theta^{q^{d_i}})\right)^{s_i}\cdots\left((t-\theta^q)\cdots(t-\theta^{q^{d_j}})\right)^{s_j}\|_\infty = |(\theta^q\cdots\theta^{q^{d_i}})^{s_i}\cdots(\theta^q\cdots\theta^{q^{d_j}})^{s_j}|_\infty.$$

For each $i \le n \le j$, we have

(3.4.2)
$$\frac{1}{|\theta^{q+\cdots+q^{d_n}}|_\infty^{s_n}} = \frac{1}{q^{s_n(q^{d_n}+q^{d_n-1}+\cdots+q)}} = \frac{1}{q^{qs_n(q^{d_n-1}+q^{d_n-2}+\cdots+1)}} = \frac{q^{qs_n/(q-1)}}{(q^{qs_n/(q-1)})^{q^{d_n}}}.$$

By using (3.1.5), (3.4.1) and (3.4.2) we have

$$\begin{aligned}
&\frac{\|(\gamma_i\epsilon_i^{d_i})^{\frac{1}{q^{d_i}}}H_{s_i-1}^{(d_i)}\cdots(\gamma_j\epsilon_j^{d_j})^{\frac{1}{q^{d_j}}}H_{s_j-1}^{(d_j)}\|_\infty}{\|\big((t-\theta^q)\cdots(t-\theta^{q^{d_i}})\big)^{s_i}\cdots\big((t-\theta^q)\cdots(t-\theta^{q^{d_j}})\big)^{s_j}\|_\infty}\\
&=\Big(\|H_{s_i-1}\|_\infty^{q^{d_i}}/|\theta^{q+\cdots+q^{d_i}}|_\infty^{s_i}\Big)\cdots\Big(\|H_{s_j-1}\|_\infty^{q^{d_j}}/|\theta^{q+\cdots+q^{d_j}}|_\infty^{s_j}\Big)\\
&\le\left(\Big(q^{\frac{s_i-1}{q-1}q}\Big)^{q^{d_i}}q^{\frac{qs_i}{q-1}}/\Big(q^{\frac{qs_i}{q-1}}\Big)^{q^{d_i}}\right)\cdots\left(\Big(q^{\frac{s_j-1}{q-1}q}\Big)^{q^{d_j}}q^{\frac{qs_j}{q-1}}/\Big(q^{\frac{qs_j}{q-1}}\Big)^{q^{d_j}}\right)\\
&=q^{\frac{q}{q-1}(s_i+\cdots+s_j)}\big(q^{-s_i-\frac{1}{q-1}}\big)^{q^{d_i}}\cdots\big(q^{-s_j-\frac{1}{q-1}}\big)^{q^{d_j}}.
\end{aligned}$$

It is clear that $(q^{-s_i-\frac{1}{q-1}})^{q^{d_i}}\cdots(q^{-s_j-\frac{1}{q-1}})^{q^{d_j}}\to 0$ as $0\le d_j<\cdots<d_i\to\infty$.

Thus we obtain the desired conclusion and therefore

$$\Psi\in\mathrm{Mat}_{r+1}(\mathbb{T}). \tag{3.4.3}$$

Now we complete the proof. Let $\psi_i\in\mathrm{Mat}_{r+1\times 1}(\mathbb{T})$ $(1\le i\le r+1)$ be the column of the matrix Ψ. Then by using Proposition 3.3.3, we can show that $\Phi\in\mathrm{Mat}_{r+1}(\overline{k}[t])$ in Definition 3.3.2 and ψ_i satisfy $\psi_i^{(-1)}=\Phi\psi_i$ for each i. Furthermore, Φ satisfies $\det\Phi|_{t=0}=(-\theta)^{\sum_{i=1}^r d_i}\neq 0$ where $d_i=s_i+\cdots+s_r$. Thus we may apply Proposition 3.4.2 to Φ and ψ_i and therefore we obtain

$$\Psi\in\mathrm{Mat}_{r+1}(\mathbb{E}).$$

□

3.4.2. MZ property for AMZVs. First we verify that AMZVs satisfy the following lemma which is alternating analogue of MZ property in **[C14]**.

Lemma 3.4.4. *For a given AMZV $\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})$ with $\mathrm{wt}(\mathfrak{s})=w$ and $\mathrm{dep}(\mathfrak{s})=r$, there exists $\Phi\in\mathrm{Mat}_{r+1}(\overline{k}[t])$ and $\psi\in\mathrm{Mat}_{(r+1)\times 1}(\mathbb{E})$ with $r\ge 1$ such that:*

(i) *$\psi^{(-1)}=\Phi\psi$ and Φ satisfies the condition of Theorem 3.4.1;*
(ii) *the last column of Φ is of the form $(0,\dots,0,1)^{\mathrm{tr}}$;*
(iii) *for some $a\in\overline{\mathbb{F}}_q^\times$ and $b\in k^\times$, $\psi(\theta)$ is of the form with specific first and last entries*

$$\psi(\theta)=\left(\frac{1}{\tilde{\pi}^w},\dots,a\frac{b\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})}{\tilde{\pi}^w}\right)^{\mathrm{tr}};$$

(iv) *for any $N\in\mathbb{N}$ and some $c\in\mathbb{F}_q^\times$, $\psi(\theta^{q^N})$ is of the form*

$$\psi(\theta^{q^N})=\left(0,\dots,0,ac^N\Big(\frac{b\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})}{\tilde{\pi}^w}\Big)^{q^N}\right)^{\mathrm{tr}}.$$

Proof. We may take matrices $\Phi\in\mathrm{Mat}_{r+1}(\overline{k}[t])$ and $\Psi\in\mathrm{Mat}_{r+1}(\mathbb{E})$ $(r\ge 1)$ for each $\zeta_A(\mathfrak{s};\boldsymbol{\epsilon})$ with $\mathrm{wt}(\mathfrak{s})=w$ and $\mathrm{dep}(\mathfrak{s})=r$ as in Definition 3.3.2 and (3.3.2) respectively. Let us denote $\psi_i\in\mathrm{Mat}_{(r+1)\times 1}(\mathbb{E})$ $(1\le i\le r+1)$ the i-th column of Ψ. From Definition 3.3.2, Proposition 3.3.3 and Lemma 3.4.3, it is evident that Φ and each ψ_i satisfy (i)-(iii). Thus it is enough to check the condition (iv) for each ψ_i.

For each i $(1 \le i \le r)$, we put

$$L(\mathfrak{s})_i := \sum_{d_i > \cdots > d_r \ge 0} \epsilon_i^{d_i} \cdots \epsilon_r^{d_r} (\Omega^{s_i} H_{s_i - 1})^{(d_i)} \cdots (\Omega^{s_r} H_{s_r - 1})^{(d_r)},$$
$$L(\mathfrak{s})_{i, \ge N} := \sum_{d_i > \cdots > d_r \ge N} \epsilon_i^{d_i} \cdots \epsilon_r^{d_r} (\Omega^{s_i} H_{s_i - 1})^{(d_i)} \cdots (\Omega^{s_r} H_{s_r - 1})^{(d_r)},$$
$$L(\mathfrak{s})_{i, < N} := \sum_{\substack{d_i > \cdots > d_r \\ d_r < N}} \epsilon_i^{d_i} \cdots \epsilon_r^{d_r} (\Omega^{s_i} H_{s_i - 1})^{(d_i)} \cdots (\Omega^{s_r} H_{s_r - 1})^{(d_r)}.$$

It is obvious that $L(\mathfrak{s})_i = L(\mathfrak{s})_{i, \ge N} + L(\mathfrak{s})_{i, < N}$. By the definition, Ω has simple zero at $t = \theta^{q^N}$ and its $(s_i + \cdots + s_r)$-th power has $s_i + \cdots + s_r$ order of zero at $t = \theta^{q^N}$ while the denominator of each term in $L(\mathfrak{s})_{i, < N}$ has at most $s_i + \cdots + s_{r-1}$ order of zero at $t = \theta^{q^N}$. Then we have $L(\mathfrak{s})_{i, < N}|_{t = \theta^{q^N}} = 0$ and thus $L(\mathfrak{s})_i|_{t = \theta^{q^N}} = L(\mathfrak{s})_{i, \ge N}|_{t = \theta^{q^N}}$. Here we note that $L(\mathfrak{s})_{i, \ge N}|_{t = \theta^{q^N}}$ converges because of both $\Omega^{s_i + \cdots + s_r}$ and denominator of each term has $s_i + \cdots + s_r$ order of zero at $t = \theta^{q^N}$. We have the following equalities:

$$L(\mathfrak{s})_i|_{t = \theta^{q^N}} = L(\mathfrak{s})_{i, \ge N}|_{t = \theta^{q^N}} = \sum_{d_i > \cdots > d_r \ge N} \epsilon_i^{d_i} \cdots \epsilon_r^{d_r} \left\{ (\Omega^{s_i} H_{s_i - 1})^{(d_i)} \cdots (\Omega^{s_r} H_{s_r - 1})^{(d_r)} \right\} |_{t = \theta^{q^N}}$$
$$= \sum_{d_i > \cdots > d_r \ge 0} \epsilon_i^{d_i + N} \cdots \epsilon_r^{d_r + N} \left\{ (\Omega^{s_i} H_{s_i - 1})^{(d_i + N)} \cdots (\Omega^{s_r} H_{s_r - 1})^{(d_r + N)} \right\} |_{t = \theta^{q^N}}$$
$$= \epsilon_i^N \cdots \epsilon_r^N \sum_{d_i > \cdots > d_r \ge 0} \epsilon_i^{d_i} \cdots \epsilon_r^{d_r} \left\{ (\Omega^{s_i} H_{s_i - 1})^{(d_i + N)} \cdots (\Omega^{s_r} H_{s_r - 1})^{(d_r + N)} \right\} |_{t = \theta^{q^N}}$$
$$= \epsilon_i^N \cdots \epsilon_r^N \Big(\sum_{d_i > \cdots > d_r \ge 0} \epsilon_i^{d_i} \cdots \epsilon_r^{d_r} \left\{ (\Omega^{s_i} H_{s_i - 1})^{(d_i)} \cdots (\Omega^{s_r} H_{s_r - 1})^{(d_r)} \right\} |_{t = \theta} \Big)^{q^N}.$$

Let $(\Omega^{s_i} H_{s_i - 1})^{(d_i)} \cdots (\Omega^{s_r} H_{s_r - 1})^{(d_r)} = \sum_{n \ge 0} a_n t^n \in \mathbb{C}_\infty[[t]]$, then the last equality in the above is shown by the following:

$$\Big\{ \Big(\sum_{n \ge 0} a_n t^n \Big)^{(N)} \Big\} |_{t = \theta^{q^N}} = \sum_{n \ge 0} a_n^{q^N} \theta^{n q^N} = \Big\{ \Big(\sum_{n \ge 0} a_n t^n \Big) |_{t = \theta} \Big\}^{q^N}.$$

Thus we have

$$L(\mathfrak{s})_i|_{t = \theta^{q^N}} = \epsilon_i^N \cdots \epsilon_r^N L(\mathfrak{s})_i^{q^N}.$$

Therefore by using $\Omega|_{t = \theta^{q^N}} = 0$, (3.3.6) and (3.3.7), indeed we obtain

$$\psi_i(\theta^{q^N}) = \left(0, \ldots, 0, a c^N \Big(\frac{b \zeta_A(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)}{\tilde{\pi}^{w_i}} \Big)^{q^N} \right)^{\mathrm{tr}}$$

where $\mathfrak{s}_i = (s_i, \ldots, s_r)$, $\boldsymbol{\epsilon}_i = (\epsilon_i, \ldots, \epsilon_r)$, $w_i = \mathrm{wt}(\mathfrak{s}_i)$, $a = \prod_{j=i}^r \gamma_j$, $b = \Gamma_{s_i} \cdots \Gamma_{s_r}$ and $c = \prod_{j=i}^r \epsilon^j$ for each i. □

Next we show that monomials of AMZVs also satisfy Lemma 3.4.4 by using the same method in proof of [**C14**, Proposition 3.4.4].

Proposition 3.4.5. *We let $\zeta_A(\mathfrak{s}_1; \boldsymbol{\epsilon}_1), \ldots, \zeta_A(\mathfrak{s}_n; \boldsymbol{\epsilon}_n)$ AMZVs with weights w_1, $\ldots, w_n$ respectively and let $m_1, \ldots, m_n \in \mathbb{Z}_{\ge 0}$. Then there exist matrices $\Phi \in \mathrm{Mat}_d(\overline{k}[t])$ and $\psi \in \mathrm{Mat}_{d \times 1}(\mathbb{E})$ with $d \ge 2$ so that $(\Phi, \psi, \zeta_A(\mathfrak{s}_1; \boldsymbol{\epsilon}_1)^{m_1} \cdots \zeta_A(\mathfrak{s}_n; \boldsymbol{\epsilon}_n)^{m_n})$ satisfies (i)-(iv) in Lemma 3.4.4.*

PROOF. We take triple $(\Phi_i, \psi_i, \zeta_A(\mathfrak{s}_i; \boldsymbol{\epsilon}_i))$ which satisfies Lemma 3.4.4 for each i. Then we consider the Kronecker product $\otimes$ (See [**S05**, Chapter 8]) of Φ_i and ψ_i respectively as following:

$$\Phi := \Phi_1^{\otimes m_1} \otimes \cdots \otimes \Phi_n^{\otimes m_n}, \quad \psi := \psi_1^{\otimes m_1} \otimes \cdots \otimes \psi_n^{m_n}.$$

By our assumption, $(\Phi_i, \psi_i, \zeta_A(\mathfrak{s}_i; \boldsymbol{\epsilon}_i))$ satisfy Lemma 3.4.4 and thus by using the property of Kronecker product which involves matrix multiplication (cf. [**S05**, Theorem 7.7]), the triple $(\Phi, \psi, \zeta_A(\mathfrak{s}_1; \boldsymbol{\epsilon}_1)^{m_1} \cdots \zeta_A(\mathfrak{s}_n; \boldsymbol{\epsilon}_n)^{m_n})$ does so. $\square$

DEFINITION 3.4.6. Let $\zeta_A(\mathfrak{s}_1; \boldsymbol{\epsilon}_1), \ldots, \zeta_A(\mathfrak{s}_n; \boldsymbol{\epsilon}_n)$ be AMZVs of $\mathrm{wt}(\mathfrak{s}_i) = w_i$ $(i = 1, \ldots, n)$. For $m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}$ not all zero, we define the total weight of the monomial $\zeta_A(\mathfrak{s}_1; \boldsymbol{\epsilon}_1)^{m_1} \cdots \zeta_A(\mathfrak{s}_n; \boldsymbol{\epsilon}_n)^{m_n}$ as

$$\sum_{i=1}^{n} m_i w_i.$$

For $w \in \mathbb{N}$, we denote AZ_w the set of monomials of AMZVs with total weight w.

We note that AZ_w is finite set.

Now we prove the linear independence of monomials of AMZVs.

THEOREM 3.4.7. *Let $w_1, \ldots, w_l \in \mathbb{N}$ be distinct. We suppose that V_i is a k-linearly independent subset of AZ_{w_i} for $i = 1, \ldots, l$. Then the following union*

$$\{1\} \bigcup_{i=1}^{l} V_i$$

is a linearly independent set over $\overline{k}$, that is, there are no nontrivial $\overline{k}$-linear relation among elements of $\{1\} \bigcup_{i=1}^{l} V_i$.

PROOF. We may assume that $w_l > \cdots > w_1$ without loss of generality. For each $i = 1, \ldots, l$, AZ_{w_i} is a finite set by definition and thus its subset V_i is also finite. Let V_i consist of $\{Z_{i1}, \ldots, Z_{im_i}\}$ where $Z_{ij} \in AZ_{w_i} (j = 1, \ldots, m_i)$ are the same total weight w_i. The proof is by induction on weight w_l.

We require on the contrary that $\{1\} \bigcup_{i=1}^{l} V_i$ is $\overline{k}$-linearly dependent set. Then we may also assume that there are nontrivial $\overline{k}$-linear relations

$$a_0 \cdot 1 + a_{11} Z_{11} + \cdots + a_{1m_1} Z_{1m_1} + \cdots + a_{l1} Z_{l1} + \cdots + a_{lm_l} Z_{lm_l} = 0,$$

we may take $a_0, a_{11}, \ldots, a_{lm_l} \in \overline{k}$ with $a_{li} \neq 0$ for some $i = 1, \ldots, m_l$.

We proceed our proof by assuming the existence of nontrivial $\overline{k}$-linear relations between V_l and $\{1\} \bigcup_{i=1}^{l-1} V_i$.

For $1 \leq i \leq l$ and $1 \leq j \leq m_l$, by combining Proposition 3.3.3 with Proposition 3.4.5, there exist the matrices

$$\Phi_{ij} \in \mathrm{Mat}_{d_{ij}}(\overline{k}[t]) \quad \text{and} \quad \psi_{ij} \in \mathrm{Mat}_{d_{ij} \times 1}(\mathbb{E}) \tag{3.4.4}$$

so that $d_{ij} \geq 2$ and each $(\Phi_{ij}, \psi_{ij}, Z_{ij})$ satisfy Lemma 3.4.4.

For the matrix Φ_{ij} and the column vector ψ_{ij}, we define the following block diagonal matrix and the column vector

$$\tilde{\Phi} := \bigoplus_{i=1}^{l} \left(\bigoplus_{j=1}^{m_i} (t - \theta)^{w_l - w_i} \Phi_{ij} \right) \quad \text{and} \quad \tilde{\psi} := \bigoplus_{i=1}^{l} \left(\bigoplus_{j=1}^{m_i} \Omega^{w_l - w_i} \psi_{ij} \right).$$

In this proof, we define the direct sum of any column vectors $\mathbf{v}_1, \ldots, \mathbf{v_m}$ whose entries belong to $\mathbb{C}_\infty((t))$ by $\bigoplus_{i=1}^{m} \mathbf{v}_i := (\mathbf{v}_1^{\mathrm{tr}}, \ldots, \mathbf{v}_m^{\mathrm{tr}})^{\mathrm{tr}}$.

By the requirement, $\{1\}\bigcup_{i=1}^{l} V_i$ is a linearly dependent over $\overline{k}$. Thus there exists a nonzero vector

$$\rho = (\mathbf{v}_{11}, \dots, \mathbf{v}_{1m_1}, \dots, \mathbf{v}_{l1}, \dots, \mathbf{v}_{lm_l})$$

such that

$$\begin{aligned}\rho \cdot (\tilde{\psi}|_{t=\theta}) &= \rho \cdot \bigoplus_{i=1}^{l}\bigoplus_{j=1}^{m_i}\left(\frac{1}{\tilde{\pi}^{w_l}}, \dots, a\frac{bZ_{ij}}{\tilde{\pi}^{w_l}}\right)^{\mathrm{tr}} \\ &= \frac{1}{\tilde{\pi}^{w_l}}(\mathbf{v}_{11}, \dots, \mathbf{v}_{1m_1}, \dots, \mathbf{v}_{l1}, \dots, \mathbf{v}_{lm_l})\bigoplus_{i=1}^{l}\bigoplus_{j=1}^{m_i}\left(1, \dots, abZ_{ij}\right)^{\mathrm{tr}} = 0,\end{aligned}$$

where $\mathbf{v}_{ij} \in \mathrm{Mat}_{1\times d_{ij}}(\overline{k})$ for $1 \le i \le l$ and $1 \le j \le m_i$. Then we have nontrivial $\overline{k}$-linear relation

$$(\mathbf{v}_{11}, \dots, \mathbf{v}_{1m_1}, \dots, \mathbf{v}_{l1}, \dots, \mathbf{v}_{lm_l})\bigoplus_{i=1}^{l}\bigoplus_{j=1}^{m_i}\left(1, \dots, abZ_{ij}\right)^{\mathrm{tr}} = 0.$$

In the beginning of this proof, we assumed that there exists nontrivial $\overline{k}$-linear relations between V_l and $\{1\}\bigcup_{i=1}^{l-1} V_i$ and then for some $1 \le s \le m_l$, the last entry of $\mathbf{v}_{ls}$ is nonzero. Since the last entry in $\mathbf{v}_{li}$ is coefficient of abZ_{li} for $1 \le i \le m_l$ in the above relation. By using Theorem 3.4.1, we have $\mathbf{F} := (\mathbf{f}_{11}, \dots, \mathbf{f}_{1m_1}, \dots, \mathbf{f}_{l1}, \dots, \mathbf{f}_{lm_l})$ where $\mathbf{f}_{ij} = (f_{i1}, \dots, f_{id_{ij}}) \in \mathrm{Mat}_{1\times d_{ij}}(\overline{k}[t])$ for $1 \le i \le l$, $1 \le j \le m_i$ and it satisfies

$$\mathbf{F}\tilde{\psi} = 0 \quad \text{and} \quad \mathbf{F}|_{t=\theta} = \rho.$$

The last entry of $\mathbf{f}_{ls}$ is a nontrivial polynomial because the last entry of $\mathbf{v}_{ls}$ is not zero. We choose a sufficiently large $N \in \mathbb{Z}$ so that $\mathbf{f}_{ls}|_{t=\theta^{q^N}} \neq 0$. We rewrite the equation $(\mathbf{F}\tilde{\psi})|_{t=\theta^{q^N}} = 0$ by using $\Omega|_{t=\theta^{q^N}} = 0$, Lemma 3.4.4 (iv) and the definition of $\tilde{\psi}$ as follows:

$$\begin{aligned}&(\mathbf{F}\tilde{\psi})|_{t=\theta^{q^N}} \\ &= (\mathbf{f}_{11}, \dots, \mathbf{f}_{1m_1}, \dots, \mathbf{f}_{l1}, \dots, \mathbf{f}_{lm_l})|_{t=\theta^{q^N}}\bigoplus_{i=1}^{l}\bigoplus_{j=1}^{m_i}\Omega^{w_l-w_i}|_{t=\theta^{q^N}}\left(0, \dots, 0, a_jc_j^N\left(\frac{b_jZ_{ij}}{\tilde{\pi}^{w_i}}\right)^{q^N}\right)^{\mathrm{tr}} \\ &= (\mathbf{f}_{11}, \dots, \mathbf{f}_{1m_1}, \dots, \mathbf{f}_{l1}, \dots, \mathbf{f}_{lm_l})|_{t=\theta^{q^N}}\bigoplus_{j=1}^{m_l}\left(0, \dots, 0, a_jc_j^N\left(\frac{b_jZ_{lj}}{\tilde{\pi}^{w_l}}\right)^{q^N}\right)^{\mathrm{tr}} \\ &= (f_{11}, \dots, f_{1d_{11}}, \dots, f_{l1}, \dots, f_{ld_{lm_l}})|_{t=\theta^{q^N}}\bigoplus_{j=1}^{m_l}\left(0, \dots, 0, a_jc_j^N\left(\frac{b_jZ_{lj}}{\tilde{\pi}^{w_l}}\right)^{q^N}\right)^{\mathrm{tr}} \\ &= \sum_{j=1}^{m_l}(f_{ld_{lj}}|_{t=\theta^{q^N}})a_jc_j^N\left(\frac{b_jZ_{lj}}{\tilde{\pi}^{w_l}}\right)^{q^N} = 0.\end{aligned}$$

Thus we obtain the following nontrivial $\overline{k}$-linear relation with some $f_{ld_{ls}} \neq 0$:

$$\sum_{j=1}^{m_l}(f_{ld_{lj}}|_{t=\theta^{q^N}})a_jc_j^N\left(b_jZ_{lj}\right)^{q^N} = 0.$$

Therefore by taking q^Nth root of the above $\overline{k}$-linear relation, we get a nontrivial relation for $\{Z_{l1},\dots,Z_{lm_l}\}$ as follows.

$$\sum_{j=1}^{m_l}\Big\{(f_{ld_{lj}}|_{t=\theta^{q^N}})a_jc_j^N\Big\}^{\frac{1}{q^N}}b_jZ_{lj}=0.$$

This shows that V_l is a $\overline{k}$-linearly dependent set. Then by using Lemma 3.4.8, we can show that V_l is a k-linearly dependent subset. However, it contradicts the condition saying that V_l is the k-linearly independent set. Therefore we obtain the claim. $\square$

LEMMA 3.4.8. *Let V be a finite $\overline{k}$-linearly dependent subset of AZ_w. Then V is a finite k-linearly dependent subset of AZ_w.*

PROOF. We put $V=\{Z_1,\dots,Z_m\}$. Without loss of generality, we may assume that $m\geq 2$ by Theorem 3.2.1 and

$$\dim_{\overline{k}}\mathrm{Span}_{\overline{k}}\{V\}=m-1. \tag{3.4.5}$$

Again we may assume that $Z_1\in\mathrm{Span}_{\overline{k}}\{Z_2,\dots,Z_m\}$, then by the assumption (3.4.5), $\{Z_2,\dots,Z_m\}$ is a linearly independent set over $\overline{k}$. As in the proof of our previous theorem, we take the matrix Φ_j, and the column vector ψ_j $(1\leq j\leq m)$ so that the triple (Φ_j,ψ_j,Z_j) satisfying Lemma 3.4.4 (i)-(iv), we define block diagonal matrix Φ and column vector ψ as follows.

$$\Phi:=\bigoplus_{j=1}^m\Phi_j\quad\text{and}\quad\psi:=\bigoplus_{j=1}^m\psi_j.$$

In the above, we again define the direct sum of column vectors $\mathbf{v}_1,\dots,\mathbf{v}_\mathbf{m}$ whose entries belong to $\mathbb{C}_\infty((t))$ by $\bigoplus_{i=1}^m\mathbf{v}_i:=(\mathbf{v}_1^{\mathrm{tr}},\dots,\mathbf{v}_m^{\mathrm{tr}})^{\mathrm{tr}}$. By definition, each (Φ_j,ψ_j,Z_j) satisfy Lemma 3.4.4 (i)-(iv) and then we have

$$\psi|_{t=\theta}=\bigoplus_{j=1}^m\left(\frac{1}{\tilde{\pi}^w},\dots,a_j\frac{b_jZ_j}{\tilde{\pi}^w}\right)^{\mathrm{tr}} \tag{3.4.6}$$

for some $a_j\in\overline{\mathbb{F}}_q^\times$, $b_j\in k^\times$ and

$$\psi|_{t=\theta^{q^N}}=\bigoplus_{j=1}^m\left(0,\dots,0,a_jc_j^N\Big(\frac{b_jZ_j}{\tilde{\pi}^w}\Big)^{q^N}\right)^{\mathrm{tr}}. \tag{3.4.7}$$

for some $c_j\in\mathbb{F}_q^\times$, $N\in\mathbb{N}$. By using Theorem 3.4.1, there exist row vectors $\mathbf{f}_j=(f_{j,1},\dots,f_{j,d_j})\in\mathrm{Mat}_{1\times d_j}(\overline{k}[t])$ $(j=1,\dots,m)$ so that if we put $\mathbf{F}=(\mathbf{f}_1,\dots,\mathbf{f}_m)$, then we have

$$\mathbf{F}\psi=0,\ f_{1,d_1}|_{t=\theta}=1\text{ and }f_{j,i}|_{t=\theta}=0\text{ for }1\leq i<d_j$$

since $f_{1,d_1}|_{t=\theta}$ is coefficient of $a_1b_1Z_1/(\tilde{\pi}^w)$ in $(\mathbf{F}\psi)|_{t=\theta}=0$ and by the assumption (3.4.5), Z_1 is expressed by nontrivial $\overline{k}$-linear combinations of $Z_2,\dots,Z_m$. We write $\mathbf{F}':=(1/f_{1,d_1})\mathbf{F}$ and $d:=\sum_{j=1}^m d_j$. Note that the vector $\mathbf{F}'$ is of the form

$$\mathbf{F}'=(f'_{1,1},\dots,f'_{1,d_1},\dots,f'_{m,1},\dots,f'_{m,d_m})\in\mathrm{Mat}_{1\times d}(\overline{k}(t))$$

where $f'_{1,d_1}=1$. We have the following from $\mathbf{F}\psi=0$ and $f_{j,i}|_{t=\theta}=0$ for $1\leq i<d_j$:

$$\mathbf{F}'\psi=0\text{ and }f'_{j,i}|_{t=\theta}=0\ \text{ for all }1\leq i<d_j. \tag{3.4.8}$$

By using Lemma 3.4.4, we obtain $\mathbf{F}'^{(-1)}\Phi\psi = (\mathbf{F}'\psi)^{(-1)} = 0$ and thus

$$\mathbf{F}'\psi - \mathbf{F}'^{(-1)}\Phi\psi = (\mathbf{F}' - \mathbf{F}'^{(-1)}\Phi)\psi = 0. \tag{3.4.9}$$

The last column of the matrix Φ_j is $(0,\dots,0,1)^{\mathrm{tr}}$ for each j and consequently, the d_1-th entry of row vector $\mathbf{F}' - \mathbf{F}'^{(-1)}\Phi$ is zero since the d_1-th entry of the row vectors $\mathbf{F}'$ and $\mathbf{F}'^{(-1)}\Phi$ are 1. The $\sum_{i=1}^{j} d_i$-th column of Φ is

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \sum_{i=1}^{j} d_i \\ \\ \\ \\ . \end{array}$$

Then the $\sum_{i=1}^{j} d_i$-th entry of the row vector $\mathbf{F}'^{(-1)}\Phi$ is $f'^{(-1)}_{j,d_j}$. Thus the $\sum_{i=1}^{j} d_i$-th entry of $\mathbf{F}' - \mathbf{F}'^{(-1)}\Phi$ is written as follows.

$$f'_{j,d_j} - f'^{(-1)}_{j,d_j} \quad \text{for } j = 1,\dots,m.$$

We have the equation $f'_{j,d_j} - f'^{(-1)}_{j,d_j} = 0$ for $j = 2,\dots,m$. Indeed, if there exist some $2 \le j \le m$ so that $f'_{j,d_j} - f'^{(-1)}_{j,d_j} \ne 0$, we can derive the contradiction in the following way:

Let us take sufficiently large $N \in \mathbb{N}$ so that $(f'_{j,d_j} - f'^{(-1)}_{j,d_j})|_{t=\theta^{q^N}} \ne 0$ and all entries of $(\mathbf{F}' - \mathbf{F}'^{(-1)}\Phi)$ are regular at $t = \theta^{q^N}$. By using (3.4.7) and substituting $t = \theta^{q^N}$ in (3.4.9), we obtain

$$\begin{aligned}
\Big\{(\mathbf{F}' - \mathbf{F}'^{(-1)}\Phi)\psi\Big\}\big|_{t=\theta^{q^N}} &= \Big\{(\mathbf{F}' - \mathbf{F}'^{(-1)}\Phi)\Big\}\big|_{t=\theta^{q^N}} \bigoplus_{j=1}^{m} \left(0,\dots,0,a_j c_j^N \Big(\frac{b_j Z_j}{\tilde{\pi}^w}\Big)^{q^N}\right)^{\mathrm{tr}} \\
&= \sum_{j=1}^{m} (f'_{j,d_j} - f'^{(-1)}_{j,d_j})|_{t=\theta^{q^N}}\, a_j c_j^N \Big(\frac{b_j Z_j}{\tilde{\pi}^w}\Big)^{q^N} \\
&= \frac{1}{\tilde{\pi}^w} \sum_{j=1}^{m} (f'_{j,d_j} - f'^{(-1)}_{j,d_j})|_{t=\theta^{q^N}}\, a_j c_j^N \Big(b_j Z_j\Big)^{q^N} = 0.
\end{aligned}$$

Thus combining with the $f'_{1,d_1} - f'^{(-1)}_{1,d_1} = 1 - 1 = 0$, we obtain a nontrivial $\overline{k}$-linear relations among $Z_2^{q^N},\dots,Z_m^{q^N}$ as follows:

$$\sum_{j=2}^{m} (f'_{j,d_j} - f'^{(-1)}_{j,d_j})|_{t=\theta^{q^N}}\, a_j c_j^N \Big(b_j Z_j\Big)^{q^N} = 0.$$

Then by taking q^Nth root of the relation, we obtain the following nontrivial $\overline{k}$-linear relation among $Z_2,\dots,Z_m$:

$$\sum_{j=2}^{m} \Big\{(f'_{j,d_j} - f'^{(-1)}_{j,d_j})|_{t=\theta^{q^N}}\, a_j c_j^N\Big\}^{\frac{1}{q^N}} b_j Z_j = 0.$$

This contradicts our assumption saying that $\{Z_2,\dots,Z_m\}$ is a $\overline{k}$-linearly independent set.

Therefore we get ${f'}_{j,d_j} - {f'}^{(-1)}_{j,d_j} = 0$ for $j = 2, \dots, m$ and this equation shows the following:

$$
{f'}_{j,d_j} \in k \quad (j = 2, \dots, m). \tag{3.4.10}
$$

By substituting $t = \theta$ in the equation $\mathbf{F}'\psi = 0$ and using (3.4.6), (3.4.8), we obtain the following equalities:

$$
\begin{aligned}
(\mathbf{F}'\psi)|_{t=\theta} &= (0, \dots, 0, f'_{1,d_1}, \dots, 0, \dots, 0, f'_{m,d_m})|_{t=\theta} \bigoplus_{j=1}^{m} \left(\frac{1}{\tilde{\pi}^w}, \dots, a_j \frac{b_j Z_j}{\tilde{\pi}^w} \right)^{\mathrm{tr}} \\
&= \sum_{j=1}^{m} ({f'}_{j,d_j}|_{t=\theta}) a_j \frac{b_j Z_j}{\tilde{\pi}^w} = \frac{1}{\tilde{\pi}^w} \sum_{j=1}^{m} ({f'}_{j,d_j}|_{t=\theta}) a_j b_j Z_j = 0.
\end{aligned}
$$

By $f'_{1,d_1} = 1$ and (3.4.10), we have the following nontrivial k-linear relation among $Z_1, \dots, Z_m$:

$$
\sum_{j=1}^{m} ({f'}_{j,d_j}|_{t=\theta}) a_j b_j Z_j = 0.
$$

Therefore we obtain the claim. □

The above theorem provides an alternating analogue of Theorem 2.2.1 in [**C14**]. Finally, we define the following notations and state the result.

NOTATION 3.4.9. We denote $\overline{\mathcal{AZ}}_w$ (resp. $\mathcal{AZ}_w$) be the $\bar{k}$-vector space (resp. k-vector space) spanned by weight w AMZVs. By Theorem 3.2.8, we derive $\mathcal{AZ}_w \cdot \mathcal{AZ}_{w'} \subseteq \mathcal{AZ}_{w+w'}$. We also note the $\bar{k}$-algebra $\overline{\mathcal{AZ}}$ (resp. k-algebra $\mathcal{AZ}$) generated by AMZVs.

THEOREM 3.4.10. *We have the following:*

(i) *$\overline{\mathcal{AZ}}$ forms an weight-graded algebra, that is, $\overline{\mathcal{AZ}} = \bar{k} \bigoplus_{w \in \mathbb{N}} \overline{\mathcal{AZ}}_w$,*
(ii) *$\overline{\mathcal{AZ}}$ is defined over k, that is, we have the canonical map $\bar{k} \otimes_k \mathcal{AZ} \to \overline{\mathcal{AZ}}$ which is bijective.*

This is characteristic p version of alternating version of Conjecture 0.1.3. A direct consequence of Theorem 3.4.10 is the following transcendence result.

COROLLARY 3.4.11. *Each AMZV $\zeta_A(\mathfrak{a}; \boldsymbol{\epsilon})$ is transcendental over k.*

Appendix. Explicit sum-shuffle relations in a lower depth case

We provide several steps to calculate an explicit sum-shuffle relation for $\zeta_A(\mathfrak{a}; \boldsymbol{\epsilon})\zeta_A(\mathfrak{b}; \boldsymbol{\lambda})$ with $\mathfrak{a} := (a_1, a_2) \in \mathbb{N}^2$, $\mathfrak{b} := (b_1) \in \mathbb{N}$, $\boldsymbol{\epsilon} := (\epsilon_1, \epsilon_2) \in (\mathbb{F}_q^\times)^2$ and $\boldsymbol{\lambda} := (\lambda_1) \in (\mathbb{F}_q^\times)$.

(I) Express $\zeta_A(\mathfrak{a}; \boldsymbol{\epsilon})\zeta_A(\mathfrak{b}; \boldsymbol{\lambda})$ in alternating power sums by using (3.1.1) and (3.1.2):

$$
\begin{aligned}
\zeta_A(\mathfrak{a}; \boldsymbol{\epsilon})\zeta_A(\mathfrak{b}; \boldsymbol{\lambda}) &= \sum_{d \geq 0} S_d(\mathfrak{a}; \boldsymbol{\epsilon}) S_d(\mathfrak{b}; \boldsymbol{\lambda}) + \sum_{d > e \geq 0} S_d(\mathfrak{a}; \boldsymbol{\epsilon}) S_e(\mathfrak{b}; \boldsymbol{\lambda}) + \sum_{e > d \geq 0} S_e(\mathfrak{b}; \boldsymbol{\lambda}) S_d(\mathfrak{a}; \boldsymbol{\epsilon}) \\
&= \sum_{d \geq 0} S_d(\mathfrak{a}; \boldsymbol{\epsilon}) S_d(\mathfrak{b}; \boldsymbol{\lambda}) + \sum_{d > e \geq 0} S_d(\mathfrak{a}; \boldsymbol{\epsilon}) S_e(\mathfrak{b}; \boldsymbol{\lambda}) + \sum_{e \geq 0} S_e(\mathfrak{b}, \mathfrak{a}; \boldsymbol{\lambda}, \boldsymbol{\epsilon}).
\end{aligned}
$$

(II) Express $S_d(\mathfrak{a}; \boldsymbol{\epsilon}) S_d(\mathfrak{b}; \boldsymbol{\lambda})$ and $\sum_{d > e \geq 0} S_d(\mathfrak{a}; \boldsymbol{\epsilon}) S_e(\mathfrak{b}; \boldsymbol{\lambda})$ as an $\mathbb{F}_p$-linear combination of $S_{d_1}(-;-)$ by using (3.1.1), (3.2.3) and Remark 3.2.3 (for $\Delta^{j_1}_{a_1,b_1}$, $\Delta^{j_2}_{j_1,a_2}$ and $\Delta^{j_3}_{a_2,b_1}$, see (3.2.1)).

For $\sum_{d\ge 0} S_d(\mathfrak{a};\boldsymbol{\epsilon})S_d(\mathfrak{b};\boldsymbol{\lambda})$, we have the following:

$$
\begin{aligned}
&\sum_{d\ge 0} S_d(\mathfrak{a};\boldsymbol{\epsilon})S_d(\mathfrak{b};\boldsymbol{\lambda}) = \sum_{d_1\ge 0} S_{d_1}(\mathfrak{a};\boldsymbol{\epsilon})S_{d_1}(\mathfrak{b};\boldsymbol{\lambda}) = \sum_{d_1\ge 0} S_{d_1}(a_1;\epsilon_1)S_{<d_1}(a_2;\epsilon_2)S_{d_1}(b_1;\lambda_1)\\
&= \sum_{d_1\ge 0} S_{d_1}(a_1;\epsilon_1)S_{d_1}(b_1;\lambda_1) \sum_{d_1>d_2\ge 0} S_{d_2}(a_2;\epsilon_2)\\
&= \sum_{d_1\ge 0}\Big(\sum_{\substack{a_1+b_1>j_1>0\\ q-1\mid j_1}} \Delta^{j_1}_{a_1,b_1} S_{d_1}(a_1+b_1-j_1, j_1;\epsilon_1\lambda_1,1)\\
&\qquad\qquad + S_{d_1}(a_1+b_1;\epsilon_1\lambda_1)\Big) \sum_{d_1>d_2\ge 0} S_{d_2}(a_2;\epsilon_2)\\
&= \sum_{d_1\ge 0}\left(\sum_{\substack{a_1+b_1>j_1>0\\ q-1\mid j_1}} \Delta^{j_1}_{a_1,b_1} S_{d_1}(a_1+b_1-j_1;\epsilon_1\lambda_1) \sum_{d_1>d_2'\ge 0} S_{d_2'}(j_1;1)\right.\\
&\qquad\qquad \left. + S_{d_1}(a_1+b_1;\epsilon_1\lambda_1)\right) \sum_{d_1>d_2\ge 0} S_{d_2}(a_2;\epsilon_2)\\
&= \sum_{d_1\ge 0}\left(\sum_{\substack{a_1+b_1>j_1>0\\ q-1\mid j_1}} \Delta^{j_1}_{a_1,b_1} S_{d_1}(a_1+b_1-j_1;\epsilon_1\lambda_1) \left\{ \sum_{d_1>d_2'=d_2\ge 0} S_{d_2'}(j_1;1)S_{d_2}(a_2;\epsilon_2)\right.\right.\\
&\qquad \left. + \sum_{d_1>d_2'>d_2\ge 0} S_{d_2'}(j_1;1)S_{d_2}(a_2;\epsilon_2) + \sum_{d_1>d_2>d_2'\ge 0} S_{d_2}(a_2;\epsilon_2)S_{d_2'}(j_1;1)\right\}\\
&\qquad \left. + \sum_{d_1\ge 0} S_{d_1}(a_1+b_1, a_2;\epsilon_1\lambda_1,\epsilon_2)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{d_1\ge 0}\left(\sum_{\substack{a_1+b_1>j_1>0\\ q-1\mid j_1}} \Delta^{j_1}_{a_1,b_1} S_{d_1}(a_1+b_1-j_1;\epsilon_1\lambda_1)\right.\\
&\qquad \left\{ \sum_{d_1>d_2'\ge 0}\Big(\sum_{\substack{j_1+a_2>j_2>0\\ q-1\mid j_2}} \Delta^{j_2}_{j_1,a_2} S_{d_2'}(j_1+a_2-j_2, j_2;\epsilon_2,1) + S_{d_2'}(a_2+j_1;\epsilon_2)\Big)\right.\\
&\qquad \left. + \sum_{d_1>d_2'\ge 0} S_{d_2'}(j_1,a_2;1,\epsilon_2) + \sum_{d_1>d_2\ge 0} S_{d_2}(a_2,j_1;\epsilon_2,1)\right\}\\
&\qquad \left. + \sum_{d_1\ge 0} S_{d_1}(a_1+b_1,a_2;\epsilon_1\lambda_1,\epsilon_2)\right)\\
&= \sum_{\substack{a_1+b_1>j_1>0\\ q-1\mid j_1}} \Delta^{j_1}_{a_1,b_1}\left(\sum_{\substack{j_1+a_2>j_2>0\\ q-1\mid j_2}} \Delta^{j_2}_{j_1,a_2} \sum_{d_1\ge 0} S_{d_1}(a_1+b_1-j_1, j_1+a_2-j_2, j_2;\epsilon_1\lambda_1,\epsilon_2,1)\right.\\
&\qquad + \sum_{d_1\ge 0} S_{d_1}(a_1+b_1-j_1, j_1+a_2;\epsilon_1\lambda_1,\epsilon_2) + \sum_{d_1\ge 0} S_{d_1}(a_1+b_1-j_1, j_1, a_2;\epsilon_1\lambda_1,1,\epsilon_2)\\
&\qquad \left. + \sum_{d_1\ge 0} S_{d_1}(a_1+b_1-j_1, a_2, j_1;\epsilon_1\lambda_1,\epsilon_2,1)\right) + \sum_{d_1\ge 0} S_{d_1}(a_1+b_1,a_2;\epsilon_1\lambda_1,\epsilon_2).
\end{aligned}
$$

For $\sum_{d>e\geq 0} S_d(\mathfrak{a};\boldsymbol{\epsilon})S_e(\mathfrak{b};\boldsymbol{\lambda})$, we have the following:

$$\begin{aligned}
&\sum_{d>e\geq 0} S_d(\mathfrak{a};\boldsymbol{\epsilon})S_e(\mathfrak{b};\boldsymbol{\lambda}) = \sum_{d_1>e_1\geq 0} S_{d_1}(\mathfrak{a};\boldsymbol{\epsilon})S_{e_1}(\mathfrak{b};\boldsymbol{\lambda}) = \sum_{d_1>e_1\geq 0} S_{d_1}(a_1;\epsilon_1)S_{<d_1}(a_2;\epsilon_2)S_{e_1}(b_1;\lambda_1)\\
&= \sum_{d_1\geq 0} S_{d_1}(a_1;\epsilon_1)\sum_{d_1>d_2\geq 0} S_{d_2}(a_2;\epsilon_2)\sum_{d_1>e_1\geq 0} S_{e_1}(b_1;\lambda_1)\\
&= \sum_{d_1\geq 0} S_{d_1}(a_1;\epsilon_1)\Bigg(\sum_{d_1>d_2=e_1\geq 0} S_{d_2}(a_2;\epsilon_2)S_{e_1}(b_1;\lambda_1) + \sum_{d_1>d_2>e_1\geq 0} S_{d_2}(a_2;\epsilon_2)S_{e_1}(b_1;\lambda_1)\\
&\qquad\qquad + \sum_{d_1>e_1>d_2\geq 0} S_{e_1}(b_1;\lambda_1)S_{d_2}(a_2;\epsilon_2)\Bigg)\\
&= \sum_{d_1\geq 0} S_{d_1}(a_1;\epsilon_1)\Bigg(\sum_{d_1>d_2=e_1\geq 0}\Bigg\{\sum_{\substack{a_2+b_1>j_3>0\\ q-1|j_3}} \Delta^{j_3}_{a_2,b_1} S_{d_2}(a_2+b_1-j_3,j_3;\epsilon_2\lambda_1,1)\\
&\qquad +S_{d_2}(a_2+b_1;\epsilon_2\lambda_1)\Bigg\} + \sum_{d_1>d_2>\geq 0} S_{d_2}(a_2,b_1;\epsilon_2,\lambda_1) + \sum_{d_1>e_1>\geq 0} S_{e_1}(b_1,a_2;\lambda_1,\epsilon_2)\Bigg)\\
&= \sum_{\substack{a_2+b_1>j_3>0\\ q-1|j_3}} \Delta^{j_3}_{a_2,b_1}\sum_{d_1\geq 0} S_{d_1}(a_1,a_2+b_1-j_3,j_3;\epsilon_1,\epsilon_2\lambda_1,1) + \sum_{d_1\geq 0} S_{d_1}(a_1,a_2+b_1;\epsilon_1,\epsilon_2\lambda_1)\\
&\qquad + \sum_{d_1\geq 0} S_{d_1}(a_1,a_2,b_1;\epsilon_1,\epsilon_2,\lambda_1) + \sum_{d_1\geq 0} S_{d_1}(a_1,b_1,a_2;\epsilon_1,\lambda_1,\epsilon_2).
\end{aligned}$$

(III) Use step (II) and (3.1.2). Then we obtain a sum-shuffle relation for $\zeta_A(\mathfrak{a};\boldsymbol{\epsilon})\zeta_A(\mathfrak{b};\boldsymbol{\lambda})$ with explicit coefficients.

$$\begin{aligned}
&\zeta_A(\mathfrak{a};\boldsymbol{\epsilon})\zeta_A(\mathfrak{b};\boldsymbol{\lambda}) = \sum_{d\geq 0} S_d(\mathfrak{a};\boldsymbol{\epsilon})S_d(\mathfrak{b};\boldsymbol{\lambda}) + \sum_{d>e\geq 0} S_d(\mathfrak{a};\boldsymbol{\epsilon})S_e(\mathfrak{b};\boldsymbol{\lambda}) + \sum_{e\geq 0} S_e(\mathfrak{b},\mathfrak{a};\boldsymbol{\lambda},\boldsymbol{\epsilon})\\
&= \sum_{\substack{a_1+b_1>j_1>0\\ q-1|j_1}} \Delta^{j_1}_{a_1,b_1}\Bigg(\sum_{\substack{j_1+a_2>j_2>0\\ q-1|j_2}} \Delta^{j_2}_{j_1,a_2}\zeta_A(a_1+b_1-j_1,j_1+a_2-j_2,j_2;\epsilon_1\lambda_1,\epsilon_2,1)\\
&\qquad + \zeta_A(a_1+b_1-j_1,j_1+a_2;\epsilon_1\lambda_1,\epsilon_2) + \zeta_A(a_1+b_1-j_1,j_1,a_2;\epsilon_1\lambda_1,1,\epsilon_2)\\
&\qquad + \zeta_A(a_1+b_1-j_1,a_2,j_1;\epsilon_1\lambda_1,\epsilon_2,1)\Bigg) + \zeta_A(a_1+b_1,a_2;\epsilon_1\lambda_1,\epsilon_2)\\
&+ \sum_{\substack{a_2+b_1>j_3>0\\ q-1|j_3}} \Delta^{j_3}_{a_2,b_1}\zeta_A(a_1,a_2+b_1-j_3,j_3;\epsilon_1,\epsilon_2\lambda_1,1) + \zeta_A(a_1,a_2+b_1;\epsilon_1,\epsilon_2\lambda_1)\\
&\qquad +\zeta_A(a_1,a_2,b_1;\epsilon_1,\epsilon_2,\lambda_1) + \zeta_A(a_1,b_1,a_2;\epsilon_1,\lambda_1,\epsilon_2)\\
&+ \zeta_A(\mathfrak{b},\mathfrak{a};\boldsymbol{\lambda},\boldsymbol{\epsilon}).
\end{aligned}$$

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