Studies on various conditions on Cohen–Macaulay rings and modules, with applications to the representation theory of maximal Cohen–Macaulay modules
(極大Cohen–Macaulay加群の表現論への応用を伴う、Cohen–Macaulay環と加群に対する種々の条件の研究)

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Contents

Preface

Classical representation theory of finite-dimensional algebras has a principle that we can understand algebras by studying modules over them. In this branch of representation theory, it has been investigated what properties modules have, and how restrictions on modules control the structure of a ring.

During the 1960s and 70s, splendid successes in this area were done by many researchers, such as P. Gabriel, M. Auslander, I. Reiten, Y. Drozd, and others, and a potent way was adumbrated to develop the representation theory of commutative rings. Among them, the paper of Gabriel [56] introduced the representations of quivers and his theorem on representation type of quivers is as follows.

Theorem A ([56])**.** *Let Q be a finite connected quiver without loops. The path algebra $k[Q]$ of Q over a field k is of finite representation type if and only if the underlying graph of Q is a Dynkin diagram* A_n*,* D_n*,* E_6*,* E_7*, or* E_8*.*

Here we say that an associative ring A has *finite representation type* if there are only finitely many isomorphism classes of finitely generated indecomposable modules over A.

Dealing with all finitely generated modules, the works of Drozd [46], Ringel [123] and Klingler–Levy [93] tell us that it is hopeless to classify all finitely generated modules except special cases. Therefore it is reasonable to focus on some special class of modules. From this perspective, the works of Drozd–Roĭter [48], Jacobinski [90] and Green–Reiner [67] are remarkable. They introduced the conditions, which is nowadays called "Drozd–Roĭter conditions", to clarify which commutative rings have only finitely many torsion-free modules; see [39, 108, 149] for instance.

A Cohen-Macaulay ring was originally defined to be a commutative ring satisfying the "unmixed theorem", namely, it satisfies certain good property on heights of ideals. By use of homological methods in commutative algebra, the notion of a Cohen-Macaulay ring has been developed with valuable applications in algebraic combinatorics. Over a Cohen–Macaulay ring, maximal Cohen–Macaulay modules are well-behaved. For example, these modules are torsion-free modules, the theory of Auslander–Reiten sequence is worked well for them, and any module can be approximated by them (Auslander–Buchweitz theory [8]); see [108, 149] for details. Gorenstein rings are special class of Cohen-Macaulay rings. Typical examples of Gorenstein rings are complete intersections, including regular local rings and hypersurfaces. The behavior of maximal Cohen–Macaulay modules over a Gorenstein ring is quite interesting. For instance, a celebrated theorem of Buchweitz [20] shows that if R is a Gorenstein ring, then the stable category of Cohen-Macaulay R-modules is triangle equivalent to the singularity category of R.

During the 1970s and 80s, the representation theory of maximal Cohen–Macaulay modules began and grew quickly, inspired by the studies on finite-dimensional algebras. In this theory, maximal Cohen–Macaulay modules over Cohen–Macaulay rings are mainly focused on. A classification of Cohen–Macaulay local rings of finite CM-representation type (i.e. having only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay modules) of Krull dimension two is given by Auslander [6] and Esnault [53] (see also [108, Chapter 6,7]).

In the higher-dimensional case, the full classification of rings of finite CM-representation type is still not known. However, a classification for Gorenstein local ring of finite CM-representation type is provided by the result of Herzog [73], Buchweitz–Greuel–Schreyer [21] and Knörrer [101].

Theorem B ([73], [21], [101])**.** *Let R be a d-dimensional Gorenstein complete local ring with an algebraically closed coefficient field k of characteristic zero. Then R has finite CM-representation type if and only if it is isomorphic to a ring of the form $k[\![x, y, z_2, \dots, z_d]\!]/(f)$, where f is one of the following forms:*

$$\begin{array}{lll}
(\mathrm{A}_n): & x^2 + y^{n+1} + z_2^2 + \cdots + z_d^2, & n \geq 1 \\
(\mathrm{D}_n): & x^2 y + y^{n-1} + z_2^2 + \cdots + z_d^2, & n \geq 4 \\
(\mathrm{E}_6): & x^3 + y^4 + z_2^2 + \cdots + z_d^2 & \\
(\mathrm{E}_7): & x^3 + xy^3 + z_2^2 + \cdots + z_d^2 & \\
(\mathrm{E}_8): & x^3 + y^5 + z_2^2 + \cdots + z_d^2 &
\end{array}$$

Tensor product is an important tool in the study of modules. In general, the tensor product of two maximal Cohen–Macaulay modules may not be maximal Cohen–Macaulay, and this is a difference between rings of Krull-dimension zero and higher-dimensional rings. It is interesting to understand what it means when the tensor product of two finitely generated modules over a local ring is maximal Cohen–Macaulay. Huneke and Wiegand [87] approached this problem by extending the rigidity theorem of Auslander [5] and Lichtenbaum [109]. They solved the probrem for local hypersurfaces, but the general situation is still misterious, even in the case of one-dimensiocal Gorenstein local rings. In order to explore further, they posed a conjecture on the torsion-freeness of modules of the form $M \otimes_R \operatorname{Hom}_R(M, R)$; see Conjecture 1.1.1 for details. It should be remarked that a complete answer for this conjecture is also not yet known even for the ideal case.

In this thesis, we discuss various problems on the representation theory of Cohen–Macaulay rings. We mainly deal with the local case.

In Chapter 1, we give a partial answer to the conjecture of Huneke and Wiegand with consideration on the middle terms of the Auslander–Reiten sequences (Theorem 1.1.2). The key tool of this chapter is some technical lemmas which are generalizations of Roy's results [127].

Another type of an attempt to solve the conjecture was done by Lindo [110]. She verified that the conjecture holds true for any module isomorphic to a trace ideal. After this, she and Pande [111] asked for which ring every ideal is isomorphic to a trace ideal. In Chapter 2, we discuss this question and give several answers. In particular, a complete answer is given in the local case 2.1.4. Note that a relationship between Lindo and Pande's question and stable rings is found by Goto-Isobe-Kumashiro [58]. Our result is proved by one of applications of the technique of finite birational extensions (see Section 2.2). This technique also plays an important role in Chapters 3, 4 and 5.

As it is valuable to focus on torsion-free modules, good restrictions on modules are useful in the representation theory of algebras. We explore suitable restrictions on maximal Cohen–Macaulay modules to develop the representation theory of Cohen–Macaulay rings in Chapter 3, 4, and 5.

In Chapter 3, we turn our attention to maximal Cohen–Macaulay modules that are not locally free on the punctured spectrum. These modules always appear when the ground ring has a non-isolated singularity, for example a local hypersurface of countable CM-representation type (i.e. having only countably but infinitely many isomorphism classes of indecomposable maximal Cohen–Macaulay modules). The starting point of our research in this chapter is the result of Araya, Iima and Takahashi [3]. They observed that local hypersurfaces of countable CM-representation type have only finitely many maximal Cohen–Macaulay modules that are not locally free on the punctured spectrum. Our purpose of this chapter is to clarify whether the converse of this holds true or not. In Theorem 3.5.1, we verify it for Gorenstein local rings of dimension one. As one of the keys to prove this theorem, we use a finite birational extension in order to construct infinitely many indecomposable modules with the desired properties (see the proof of Theorem 3.5.5).

In Chapter 4, we consider two special classes of modules, syzygies of a maximal Cohen–Macaulay module and Ulrich module. When and only when the considering ring is Gorenstein, every maximal Cohen–Macaulay module is a syzygy of some maximal Cohen–Macaulay module. So our motivation is to analyze non-Gorenstein Cohen–Macaulay rings by using such modules. In the one-dimensional case, reflexive modules are syzygies of some maximal Cohen–Macaulay modules, and the converse also holds under some assumption (Lemma 4.1.6). And H. Bass [14] made an observation on reflexive modules below.

Theorem C (cf. [108])**.** *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension one, and $B := \mathrm{End}_R(\mathfrak{m})$ be the endomorphism ring of $\mathfrak{m}$ over R. If M is a reflexive R-module having no free summands, then M has an B-module structure that extends the action of R on M.*

Note that more general result, called the "rejection lemma of Drozd-Kirichenko", is also known; see [47, 77]. We try to extend the observation of Bass and our main result can be said as follows. Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension one with a canonical module and an infinite residue field, and B be the endomorphism ring $\mathrm{End}_R(\mathfrak{m})$ of $\mathfrak{m}$ over R. Denote by $\mathsf{CM}(B)$ and $\Omega\mathsf{CM}^{\times}(R)$ be the categories of maximal Cohen–Macaulay B-modules and of first syzygies of maximal Cohen–Macaulay R-modules without free summands, respectively.

Theorem D (Theorem 4.0.3)**.** *The natural inclusion $R \to B$ induces an equivalence $\mathsf{CM}(B) \cong \Omega\mathsf{CM}^{\times}(R)$ of categories if and only if R is almost Gorenstein.*

Here the class of *almost Gorenstein rings* are introduced by Barucci–Fröberg [15] and Goto–Matsuoka–Phuong [62], as one of the candidates for a class of rings having sufficiently good property next to the Gorenstein rings in dimension one. Later, the theory of almost Gorenstein rings in all dimensions are founded by Goto, Takahashi and Taniguchi [65]. In general, little is known about the category $\Omega\mathsf{CM}^{\times}(R)$, except the case that R is a two-dimensional rational singularity ([40]). Using the above theorem, we can understand $\mathsf{\Omega CM}^{\times}(R)$ by more simpler category $\mathsf{CM}(B)$ when R is an almost Gorenstein ring of dimension one.

In the higher-dimensional case, we compare syzygies of maximal Cohen–Macaulay modules and Ulrich modules. Ulrich modules, which are also called maximally generated maximal Cohen–Macaulay modules, are of some interest in that if the residue field is infinite such modules are liftings of direct sums of copies of the residue field. Let $\mathsf{Ul}(R)$ be the category of Ulrich modules over a local ring R. If R is a local Cohen–Macaulay ring having minimal multiplicity, then we check that the containment $\mathsf{\Omega CM}^{\times}(R) \subseteq \mathsf{Ul}(R)$ holds, as subcategories of the category of R-modules (Proposition 4.1.7). Moreover, we give conditions on rings R to have the equality $\mathsf{\Omega CM}^{\times}(R) = \mathsf{Ul}(R)$ in terms of the typical R-module $\Omega^d k$, the d-th syzygy of the residue field k of R (Theorem 4.3.9 and 4.4.13). This equalty is also considered for cyclic quotient surface singularities by Nakajama and Yoshida [115], while we treat it for rings not have cyclic quotient surface singularity.

In Chapter 5, we return to the case of dimension one, and study the endomorphism rings of the maximal ideals of Gorenstein local rings. The motivation to take case of such rings is the observation (Theorem C) of Bass; over a Gorenstein local ring $(R, \mathfrak{m})$ of dimension one, a maximal Cohen–Macaulay module with no free summands can be regarded as a maximal Cohen–Macaulay module over the endomorphism ring B of $\mathfrak{m}$, and hence studying maximal Cohen–Macaulay modules over B is essential to understand the maximal Cohen–Macaulay modules over R. So our aim in this chapter is to find basic properties of the endomorphism rings of the maximal ideals of Gorenstein local rings of dimension one. We give several characterizations of local rings which appear as the endomorphism rings of the maximal ideals of Gorenstein local rings (Theorem 5.1.4). We also find connections between such rings and almost Gorenstein rings (Corollary 5.1.5).

One of the outcomes in Chapter 6 is a result on classification of subcategories. The classification problem of subcategories has been studied by many researchers; see [55, 79, 117, 140] for instance. One of the motivations of the problem is to understand the category of modules regardless of the representation type. We begin Chapter 6 with introducing the notion of Burch rings. The definition of Burch rings is very simple and it is easy to see that many examples of rings including hypersurfaces and Cohen–Macaulay local rings with minimal multiplingicity are Burch. We see that they have a good property which allows us to classify some resolving subcategories of the category of finitely generated modules over them (Section 6.7). It means that we provide a new class of rings to which the technical machinery of classifying subcategories developed in [103, 140, 141] can be applied.

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Convention

In the rest of this thesis, unless otherwise specified, we adopt the following convention. Rings are commutative and noetherian, and modules are finitely generated. Subcategories are full and strict (i.e., closed under isomorphism). An identity matrix of suitable size is denoted by E. Subscripts and superscripts are often omitted unless there is a risk of confusion.

Definition 0.0.1. Let R be a ring.

(1) An R-module M is *maximal Cohen–Macaulay* if the inequality $\operatorname{depth} M_{\mathfrak{p}} \geq \dim R_{\mathfrak{p}}$ holds for all $\mathfrak{p} \in \operatorname{Spec} R$. Hence, by definition, the zero module is maximal Cohen–Macaulay.

(2) We denote by $\operatorname{\mathsf{mod}} R$ the category of (finitely generated) R-modules, and by $\mathsf{CM}(R)$ the subcategory of $\operatorname{\mathsf{mod}} R$ consisting of maximal Cohen–Macaulay R-modules. For a subcategory $\mathcal{X}$ of $\operatorname{\mathsf{mod}} R$, we denote by $\operatorname{\mathsf{ind}} \mathcal{X}$ the set of isomorphism classes of indecomposable R-modules in $\mathcal{X}$, and by $\operatorname{\mathsf{add}}_R \mathcal{X}$ the *additive closure* of $\mathcal{X}$, that is, the subcategory of $\operatorname{\mathsf{mod}} R$ consisting of direct summands of finite direct sums of objects in $\mathcal{X}$.

(3) A subset S of $\operatorname{Spec} R$ is called *specialization-closed* if $\mathrm{V}(\mathfrak{p}) \subseteq S$ for all $\mathfrak{p} \in S$. This is equivalent to saying that S is a union of closed subsets of $\operatorname{Spec} R$ in the Zariski topology.

(4) Let S be a subset of $\operatorname{Spec} R$. Then it is easy to see that

$$\sup\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in S\} \geq \sup\{n \geq 0 \mid \text{there exists a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } S\},$$

and the equality holds if S is specialization-closed. The *(Krull) dimension* of a specialization-closed subset S of $\operatorname{Spec} R$ is defined as this common number and denoted by $\dim S$.

(5) The *singular locus* of R, denoted by $\operatorname{Sing} R$, is by definition the set of prime ideals $\mathfrak{p}$ of R such that $R_\mathfrak{p}$ is not a regular local ring. It is clear that $\operatorname{Sing} R$ is a specialization-closed subset of $\operatorname{Spec} R$. If R is excellent, then by definition $\operatorname{Sing} R$ is a closed subset of $\operatorname{Spec} R$ in the Zariski topology.

(6) For an $m \times n$ matrix A over R, we denote by $\mathrm{I}_s(A)$ the ideal of R generated by all the s-minors of A. For a linear map ϕ of free R-modules, we define $\mathrm{I}_i(\phi)$ as the ideal $\mathrm{I}_i(A)$, where A is a presentation matrix of ϕ.

Definition 0.0.2. Let $(R, \mathfrak{m}, k)$ be a local ring.

(1) For an R-module M, we denote by $\nu_R(M)$ the minimal number of generators of M, that is, $\nu_R(M) = \dim_k(M \otimes_R k)$.

(2) Let M an R-module and $n \geq 0$ an integer. We denote by $\Omega_R^n M$ (or simply $\Omega^n M$) the n-th *syzygy* of M, i.e., the image of the n-th differential map in the minimal free resolution of M. This is uniquely determined up to isomorphism. In particular, we simply denote by ΩM the first syzygy of M unless otherwise specified.

(3) We denote by $\operatorname{edim} R$ the embedding dimension of R, and by $\operatorname{codepth} R$ the codepth of R, i.e., $\operatorname{codepth} R = \operatorname{edim} R - \operatorname{depth} R$. We say that R is a *hypersurface* if $\operatorname{codepth} R \leq 1$.

(4) The *punctured spectrum* $\operatorname{Spec}^0 R$ of R is the subset $\operatorname{Spec} R \setminus \{\mathfrak{m}\}$ of $\operatorname{Spec} R$. By $\mathsf{CM}_0(R)$ we denote the subcategory of $\mathsf{CM}(R)$ consisting of modules that are locally free on $\operatorname{Spec}^0 R$.

(5) Whenever R is Cohen–Macaulay and admits a canonical module ω, we denote by $(-)^\dagger$ the canonical dual functor $\operatorname{Hom}_R(-, \omega)$.

Chapter 1

The Huneke–Wiegand conjecture and middle terms of almost split sequences

1.1 Introduction

The contents of this chapter is based on [96].

In this chapter, we study the following conjecture of Huneke and Wiegand; see [87, the discussion following the proof of 5.2].

Conjecture 1.1.1 (Huneke and Wiegand [87])**.** Let R be a Gorenstein local domain of dimension one. Let M be a maximal Cohen–Macaulay R-module. If $M \otimes_R \operatorname{Hom}_R(M, R)$ is torsion-free, then M is free.

Huneke and Wiegand [87] showed that this conjecture is true for hypersurfaces. Many other partial answers are known [27, 28, 59, 66, 80, 127], but, the conjecture is still open in general. Let R be a Gorenstein local domain of dimension one. A finitely generated R-module is torsion-free if and only if it is reflexive if and only if it is maximal Cohen–Macaulay. Therefore Conjecture 1.1.1 implies the Auslander–Reiten conjecture for Gorenstein local domains ([30, Proposition 5.10]). Assume that M is a torsion-free R-module. Then it is remarkable that the torsion-freeness of $M \otimes_R \operatorname{Hom}_R(M, R)$ is equivalent to saying that $\operatorname{Ext}^1_R(M, M)$ is zero; see [81, Theorem 5.9].

The main result of this chapter is the following.

Theorem 1.1.2. *Let $(R, \mathfrak{m})$ be a Gorenstein local domain of dimension one. Let M be a nonfree indecomposable torsion-free R-module. Assume that the number of indecomposable summand in the middle term of the almost split sequence ending in M is greater than one. Then one has $\operatorname{Ext}^1_R(M, M) \neq 0$. Hence, Conjecture 1.1.1 holds true for M.*

Remark that Roy [127] showed that for one-dimensional graded complete intersections R satisfying some condition on the a-invariant, the assertion of Theorem 1.1.2 holds. Our result is local (not graded), and we do not assume that the ring is a complete intersection.

In section 2, we give some preliminaries. In section 3, the proof of Theorem 1.1.2 is given.

1.2 Irreducible homomorphisms and almost split sequences

In this section, we prove lemmas needed to prove the main theorem. In the rest of this chapter, let $(R, \mathfrak{m})$ be a commutative Gorenstein henselian local ring.

For R-modules M and N, let $\underline{\mathrm{Hom}}_R(M, N)$ denote the quotient of $\mathrm{Hom}_R(M, N)$ by the set of homomorphisms from M to N factoring through a free R-module. Since R is Gorenstein, the stable category $\underline{\mathsf{CM}}_0(R)$ of $\mathsf{CM}_0(R)$ is a triangulated category. Its morphism set is equal to the stable homset $\underline{\mathrm{Hom}}_R(-,-)$ and its shift functor is the functor taking Ω; see [69, Chapter 1] for instance. Hence we obtain the following lemma.

Lemma 1.2.1. *Let M, N be R-modules in $\mathsf{CM}_0(R)$. Then we have the following isomorphisms.*
(1) $\underline{\mathrm{Hom}}_R(\Omega M, N) \cong \mathrm{Ext}^1_R(M, N)$, (2) $\mathrm{Ext}^1_R(M, N) \cong \mathrm{Ext}^1_R(\Omega M, \Omega N)$,
(3) $\underline{\mathrm{Hom}}_R(M, N) \cong \underline{\mathrm{Hom}}_R(\Omega M, \Omega N)$.

On the set $\underline{\mathrm{Hom}}_R(M, N)$, we also use the following lemma.

Lemma 1.2.2. *Let M, N be R-modules having no free summands and $f\colon M \to N$ be a homomorphism factoring through a free R-module. Then the image $\mathrm{Im}\, f$ of f is contained in $\mathfrak{m}N$.*

Proof. Write $f = hg$ where $g\colon M \to F$ and $h\colon F \to N$ are homomorphisms with a free R-module F. Since M has no free summands, $\mathrm{Im}\, g$ is contained in $\mathfrak{m}F$. Hence $\mathrm{Im}\, f \subseteq h(\mathfrak{m}F) \subseteq \mathfrak{m}N$. ■

Recall that a homomorphism $f\colon X \to Y$ of R-modules is said to be *irreducible* if it is neither a split monomorphism nor a split epimorphism, and for any pair of morphisms g and h such that $f = gh$, either g is a split epimorphism or h a split monomorphism.

Lemma 1.2.3. *Let M, N be R-modules having no free summands and $f, g\colon M \to N$ be homomorphisms. Assume that g factors through a free R-module. Then*

(1) *f is an isomorphism if and only if so is $f + g$.*

(2) *f is a split epimorphism if and only if so is $f + g$.*

(3) *f is a split monomorphism if and only if so is $f + g$.*

(4) *f is irreducible if and only if so is $f + g$.*

Proof. We only need to show one direction; we can view f as $(f + g) - g$.

(1): Assume that f is an isomorphism with an inverse homomorphism $h\colon N \to M$. Then the composite homomorphisms gh factor through some free R-modules. It follows from Lemma 1.2.2 that there are inclusions $\mathrm{Im}\, gh \subseteq \mathfrak{m}M$. By Nakayama's lemma, we see that $(f + g)h$ is a surjective endomorphism of M, and hence are automorphisms. Since h is an isomorphism, it follows that $f + g$ is an isomorphism.

(2): Assume that there exists a homomorphism $s\colon N \to M$ such that $fs = \mathrm{id}_N$. We may apply (1) to the homomorphism $fs + gs$ to see that $(f + g)s$ is also an isomorphism. This means that $f + g$ is a split epimorphism. The item (3) can be checked in the same way.

(4): Assume that f is irreducible. According to the previous part, $f + g$ is neither a split monomorphism nor a split epimorphism. By the assumption, g is a composite ba of homomorphisms $a\colon M \to F$ and $b\colon F \to N$ with a free R-module F. If there is a factorization

$f+g = dc$ for some homomorphisms $c\colon M \to X$ and $d\colon X \to N$, then they induce a decomposition $M \xrightarrow{{}^t[a,c]} F \oplus X \xrightarrow{[-b,d]} N$ of f. By the irreducibility of f, either ${}^t[a,c]$ is a split monomorphism or $[-b,d]$ is a split epimorphism. In the former case, we can take a homomorphism $[p,q]\colon F\oplus X \to N$ such that the composite $pa+qc = [p,q]\circ {}^t[a,c]$ is equal to the identity map of N. Using (1), qc is also an isomorphism. This yields that c is a split monomorphism. In the latter case, we can see that d is a split epimorphism by similar arguments. Thus we conclude that $f+g$ is an irreducible homomorphism. ■

Let M be a nonfree indecomposable module in $\mathsf{CM}_0(R)$. Then there exists an almost split sequence ending in M. Namely, there is a nonsplit short exact sequence

$$0 \to \tau M \xrightarrow{f} E_M \xrightarrow{g} M \to 0$$

in $\mathsf{CM}_0(R)$ such that N is indecomposable and for any maximal Cohen–Macaulay R-module L and a homomorphism $h\colon L \to M$ which is not a split epimorphism, h factors through g; see [149, Chapter 2,3] for details. Note that an almost split sequence ending in M is unique up to isomorphisms of short exact sequences. In particular, for any nonfree indecomposable R-module M in $\mathsf{CM}_0(R)$, the R-modules τM and E_M are unique up to isomorphism.

Lemma 1.2.4. *Let M be a nonfree indecomposable module in $\mathsf{CM}_0(R)$. Consider the almost split sequences*

$$0 \to \tau M \xrightarrow{f} E_M \xrightarrow{g} M \to 0, \quad 0 \to \tau(\Omega M) \to E_{\Omega M} \to \Omega M \to 0$$

ending in M and ΩM. Then $\Omega(E_M)$ is isomorphic to $E_{\Omega M}$ up to free summands.

Proof. By the horseshoe lemma, there exists a short exact sequence $s\colon 0 \to \Omega(\tau M) \xrightarrow{f'} \Omega E_M \oplus P \xrightarrow{g'} \Omega M \to 0$ with some free R-module P. Here, the class $\underline{g'} \in \underline{\mathrm{Hom}}_R(\Omega E_M, \Omega M)$ of g' coincides with the image $\Omega(\underline{g})$ of the class $\underline{g}$ of g under the isomorphism $\Omega\colon \underline{\mathrm{Hom}}_R(E_M, M) \to \underline{\mathrm{Hom}}_R(\Omega E_M, \Omega M)$ in Lemma 1.2.1. We want to show that the sequence s is an almost split sequence ending in ΩM. By Lemma 1.2.3 (2), we see that g' is a split epimorphism if and only if $\underline{g'h} = \mathrm{id}$ for some h in the category $\underline{\mathsf{CM}}_0(R)$. In view of the equivalence $\Omega\colon \underline{\mathsf{CM}}_0(R) \to \underline{\mathsf{CM}}_0(R)$, g' as well as g is not a split surjection. This means that s is not a split exact sequence.

We fix a homomorphism $h'\colon X \to \Omega M$ which is not a split epimorphism. We can use the equivalence $\Omega\colon \underline{\mathsf{CM}}_0(R) \to \underline{\mathsf{CM}}_0(R)$ again to obtain an equality $h' = g'p + rq$ with some homomorphism $p\colon X \to \Omega E_M$, $q\colon X \to F$, $r\colon F \to \Omega M$, where F is a free module. As g' is an epimorphism and F is free, r factors through g. This shows that $h' = g't$ for some $t\colon X \to \Omega E_M$. Consequently, s is an almost split sequence ending in ΩM. ■

Consider the almost split sequence

$$0 \to \tau(M) \to E_M \to M \to 0$$

ended in M. We define a number $\alpha(M)$ to be the number of nonfree indecomposable summand of E_M.

Lemma 1.2.5. *Let M be a nonfree indecomposable module in $\mathsf{CM}_0(R)$. Then $\alpha(M) = \alpha(\Omega^i M)$ for all $i \geq 0$.*

Proof. This is a direct consequence of Lemma 1.2.4. ■

The following two lemmas play key roles in the next section. See [126, Lemma 4.1.8] for details of the lemma below.

Lemma 1.2.6. *Let $f\colon M \to N$ be an irreducible homomorphism such that M and N are indecomposable in $\mathsf{CM}_0(R)$. Assume that $\dim R = 1$. Then f is either injective or surjective.*

Recall that an R-module M has *constant rank n* if one has an isomorphism $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{\oplus n}$ for all associated primes $\mathfrak{p}$ of R.

Lemma 1.2.7. *Let M, N be nonfree indecomposable modules in $\mathsf{CM}_0(R)$ having same constant rank. Let $f\colon M \to N$ be an irreducible monomorphism. Assume that $\dim R = 1$. Then $\operatorname{Cok} f$ is isomorphic to $R/\mathfrak{m}$.*

Proof. By the assumption that f is an irreducible monomorphism, f is not surjective. Hence we can take a maximal proper submodule X of N containing $\operatorname{Im} f$. Remark that the quotient N/X is isomorphic to $R/\mathfrak{m}$ and hence X and N has same constant rank. Since $\dim R = 1$, X is an R-module contained in $\mathsf{CM}_0(R)$. Thus we have a factorization $M \to X \to N$ of f in $\mathsf{CM}_0(R)$. By the irreducibility of f, it follows that either $M \to X$ is a split monomorphism or $X \to N$ is a split epimorphism. As X is proper submodule of N, the later case cannot occur. Therefore, we obtain a split monomorphism $g\colon M \to X$. Then, by the equalities $\operatorname{rank} M = \operatorname{rank} N = \operatorname{rank} X$, g is an isomorphism. This implies the desired isomorphisms $\operatorname{Cok} f \cong N/X \cong R/\mathfrak{m}$. ■

1.3 Proof of the main theorem

In this section, we give a proof of Theorem 1.1.2.

Proof of Theorem 1.1.2. Since R is a Gorenstein local ring of dimension one, $\tau(N)$ is isomorphic to ΩN for any nonfree indecomposable R-module N in $\mathsf{CM}_0(R)$. We assume that M is a nonfree indecomposable R-module in $\mathsf{CM}_0(R)$ satisfying $\operatorname{Ext}_R^1(M, M) = 0$ and want to show that $\alpha(M) = 1$. We see from Lemma 1.2.1 that the isomorphisms

$$\operatorname{Ext}_R^1(\Omega^{i+1}M, \Omega^{i+1}M) \cong \operatorname{Ext}_R^1(M, M) = 0$$

hold for all $i \geq 0$. If $E_{\Omega^i M}$ has a free summand, then $\tau(\Omega^i M) = \Omega^{i+1}M$ has an irreducible homomorphism into R. Hence $\Omega^{i+1}M$ is a direct summand of the maximal ideal $\mathfrak{m}$. Since R is a domain, this means that $\Omega^{i+1}M$ is isomorphic to $\mathfrak{m}$. It follows that $\operatorname{Ext}_R^1(\mathfrak{m}, \mathfrak{m})$ is zero, and so R should be regular. Therefore, we may assume that $E_{\Omega^i M}$ has no free summands for all $i \geq 0$. By lemma 1.2.5, it is enough to show that $\alpha(\Omega^i M) = 1$ for some $i \geq 0$. Thus by replacing M with $\Omega^i M$, we may assume that $\operatorname{rank} M$ is minimal in the set $\{\operatorname{rank} \Omega^i M \mid i \geq 0\}$.

Decompose $E_M = E_1 \oplus \cdots \oplus E_n$ as a direct sum of indecomposable modules and consider the almost split sequence

$$0 \to \Omega M \xrightarrow{f = {}^t(f_1, \dots, f_n)} E_1 \oplus \cdots \oplus E_n \xrightarrow{g = (g_1, \dots, g_n)} M \to 0$$

ended in M, where $f_p\colon \Omega M \to E_p$ and $g_p\colon E_p \to N$ are irreducible homomorphisms and $n = \alpha(M)$. Lemma 1.2.6 guarantees that each of $f_1, \dots, f_n$ and $g_1, \dots, g_n$ is either injective or surjective.

Claim 1. There is a number p such that f_p is injective.

Proof of Claim 1. Suppose that all of the $f_1, \dots, f_n$ are surjective. Then we get equalities $\operatorname{Im} g = \sum_i \operatorname{Im} g_i = \sum_i \operatorname{Im} g_i f_i$. Since $\underline{\operatorname{Hom}}_R(\Omega M, M) = 0$ (Lemma 1.2.1), it follows from Lemma 1.2.2 that $\operatorname{Im} g_i f_i \subseteq \mathfrak{m}M$ for all $i = 1, \dots, n$. This yields that $\operatorname{Im} g \subseteq \mathfrak{m}M$, which contradicts to that g is surjective. □

Claim 2. If there is a number p such that f_p is injective and g_p is surjective, then $\alpha(M) = 1$.

Proof of Claim 2. Suppose that f_p is injective and g_p is surjective. Since $\underline{\operatorname{Hom}}_R(\Omega M, M) = 0$, there is a free R-module F and homomorphisms $a\colon \Omega M \to F$ and $b\colon F \to M$ such that $g_p f_p = ba$. Since F is free and g_p is surjective, we have a factorization $b = g_p c$ with some homomorphism $c\colon F \to E_p$. So we get an equality $g_p(f_p - ca) = 0$. In particular, $f_p - ca$ factors through the kernel $\operatorname{Ker} g_p$ of g_p, i.e. $f_p - ca = ed$ with a homomorphism $d\colon \Omega M \to \operatorname{Ker} g_p$ and the natural inclusion $e\colon \operatorname{Ker} g_p \to E_p$. By Lemma 1.2.3 (2), the homomorphism $f_p - ca\colon M \to E_p$ is also irreducible. Hence either e is a split epimorphism or d is a split monomorphism. In the former case, the equality $\operatorname{Ker} g_p = E_p$ follows. It means that the map g_p is zero. This is a contradiction to the irreducibility of g_p. So it follows that d is a split monomorphism. Then one has $\operatorname{rank} \Omega M \leq \operatorname{rank} \operatorname{Ker} g_p = \operatorname{rank} E_p - \operatorname{rank} M$. This forces that $n = 1$. □

By Claim 1, we already have an integer p such that f_p is a monomorphism. If g_p is surjective, then by Claim 2 it follows that $\alpha(M) = 1$. Therefore, we may suppose that g_p is injective. Then the inequalities $\operatorname{rank} \Omega M \leq \operatorname{rank} E_p \leq \operatorname{rank} M$ hold. By the minimality of $\operatorname{rank} M$, we have $\operatorname{rank} \Omega M = \operatorname{rank} E_p = \operatorname{rank} M$. In this case, we see isomorphisms $\operatorname{Cok} f_p \cong R/\mathfrak{m} \cong \operatorname{Cok} g_p$ by Lemma 1.2.7. Therefore, equalities $\ell(M/\operatorname{Im}(f_p g_p)) = \ell(\operatorname{Cok} f_p) + \ell(\operatorname{Cok} g_p) = 2$ hold (here, $\ell(X)$ denotes the length for an R-module X). By Lemma 1.2.2, $\operatorname{Im}(f_p g_p) \subseteq \mathfrak{m}M$. So it follows that $\ell(M/\mathfrak{m}M) \leq 2$. In other words, M is generated by two elements as an R-module. Since M is nonfree, one has $\operatorname{rank} M = 1$ and $\operatorname{Hom}_R(M, R) \cong \Omega M$. As $\operatorname{rank} \Omega M = \operatorname{rank} M = 1$, we can apply the same argument above for ΩM to see that ΩM is also generated by two elements. Then by [72, Theorem 3.2], one can see that $\operatorname{Ext}^1_R(M, M) \neq 0$, a contradiction. ■

Chapter 2

Rings whose ideals are isomorphic to trace ideals

2.1 Introduction

The contents of this chapter is based on author's parer [100] with R. Takahashi.

This chapter deals with trace ideals of commutative noetherian rings. The notion of trace ideals is classical and fundamental; a lot of studies on this notion have been done in various situations. Many references on trace ideals can be found in [106, 111]. Other than them, for instance, trace ideals play an important role in the proof of a main result of Huneke and Leuschke [84] on the Auslander–Reiten conjecture. Recently, Goto, Isobe and Kumashiro [58] study correspondences of trace ideals with stable ideals and finite birational extensions.

The main purpose of this chapter is to consider a question on trace ideals raised by Lindo and Pande [111]. They prove as their main result that a local ring is an artinian Gorenstein ring if and only if every ideal is a trace ideal, and ask for which ring every ideal is isomorphic to some trace ideal. This question originates the celebrated Huneke–Wiegand conjecture: Lindo [110] shows that for such a Gorenstein domain the conjecture holds. In this chapter, we begin with answering the question for rings with full of zerodivisors, which complements the result of Lindo and Pande.

Theorem 2.1.1. *Let R be a commutative noetherian ring all of whose nonunits are zerodivisors (e.g. a local ring of depth 0). Then the following are equivalent.*

(1) *R is an artinian Gorenstein ring.*

(2) *Every ideal of R is a trace ideal.*

(3) *Every principal ideal of R is a trace ideal.*

(4) *Every ideal of R is isomorphic to a trace ideal.*

(5) *Every principal ideal of R is isomorphic to a trace ideal.*

Next we investigate the question of Lindo and Pande in the case of a local ring of depth one. We prove the following theorem, which states that such a ring as in the question is nothing but a hypersurface singularity of type (A_n), under some mild assumptions. This theorem also removes the assumption of a Gorenstein domain from Lindo's result mentioned above.

Theorem 2.1.2. *Let R be a commutative noetherian local ring of depth 1. Consider the conditions:*

(1) *Every ideal of R is isomorphic to a trace ideal,*

(2) *R is a hypersurface with Krull dimension 1 and multiplicity at most 2,*

(3) *The completion $\widehat{R}$ of R is an (A_n)-singularity of Krull dimension 1 for some $0 \le n \le \infty$.*

Then the implications (1)$\Leftrightarrow$(2) $\Leftarrow$ (3) hold. If the residue field of R is algebraically closed and has characteristic 0, then all the three conditions are equivalent.

Finally, we explore the question of Lindo and Pande in the higher-dimensional case. It turns out that the condition in the question is closely related to factoriality of the ring.

Theorem 2.1.3. *Let R be a commutative noetherian ring. Assume that all maximal ideals of R have height at least 2. Then the following are equivalent.*

(1) *Every ideal of R is isomorphic to a trace ideal.*

(2) *R is a product of factorial rings (i.e., unique factorization domains).*

In particular, when R is local, every ideal of R is isomorphic to a trace ideal if and only if R is factorial.

Combining all the above three theorems, we obtain the following characterization of the local rings whose ideals are isomorphic to trace ideals, which gives a complete answer to the question of Lindo and Pande for local rings.

Corollary 2.1.4. *Let R be a commutative noetherian local ring. Then the following are equivalent.*

(1) *Every ideal of R is isomorphic to a trace ideal.*

(2) *The ring R satisfies one of the following conditions.*

   (a) *R is an artinian Gorenstein ring.*

   (b) *R is a hypersurface of Krull dimension 1 and multiplicity at most 2.*

   (c) *R is a unique factorization domain.*

This chapter is organized as follows. In Section 2, we give a brief survey on finite birational extensions. The tools we mention in this section will be used commonly in the following sections and chapters. In Section 3, we recall the definition of trace ideals and their several basic properties. We also give a couple of observations on the Lindo–Pande condition. In Section 4, we consider characterizing rings that satisfy the Lindo–Pande condition. We state and prove our main results including the theorems introduced above.

2.2 Properties of finite birational extensions

In this section, we collect some basic facts on finite birational extensions, in order to prepare the following sections. Let R be a ring with total quotient ring $Q = Q(R)$. We denote by $(-)^*$ the R-dual functor $\mathrm{Hom}_R(-, R)$. We use $\mathrm{e}(R)$ to denote the multiplicity of R.

We start by remarking an elementary fact, which will be used several times later.

Remark 2.2.1. Let M be an R-submodule of Q. If M is finitely generated, then M is isomorphic to an ideal of R, which can be taken to contain a non-zerodivisor of R if so does M.

Recall that a finitely generated R-module M is called *reflexive* if the natural map $M \to \mathrm{Hom}_A(\mathrm{Hom}_A(M, A), A)$ is an isomorphism. We denote by $\mathsf{Ref}(A)$ the subcategory of reflexive A-modules. In the proofs of our results, it is essential to investigate R-submodules of Q and their colons in Q.

Lemma 2.2.2. *Let M be an R-submodule of Q containing a nonzero divisor c.*

(1) *Let N be an R-submodule of Q. The assignment $(x \mapsto \alpha x) \leftarrow\!\shortmid\, \alpha$ make an isomorphism $\Phi_{M,N} \colon N : M \cong \mathrm{Hom}_R(M, N)$ of R-modules, where the inverse is given by the assignment $f \mapsto \frac{1}{c} f(c)$.*

(2) *Suppose that M is finitely generated. Then M is reflexive if and only if there is an equality $M = R : (R : M)$ in Q.*

Proof. (1) One can show that the equality $f(c)x = cf(x)$ in Q holds for each $x \in M$ by describing x as an element of Q. It is now easy to verify that the two assignments define mutually inverse bijections.

(2) By Remark 2.2.1 we see that $R : M$ contains a non-zerodivisor of R. Applying (1) twice, we have isomorphisms $M^{**} = (M^*)^* \cong (R : M)^* \cong R : (R : M)$. Composition with the canonical homomorphism $M \to M^{**}$ gives a homomorphism $M \to R : (R : M)$, which we observe is nothing but the inclusion map. The assertion immediately follows from this. ■

Remark 2.2.3. In view of Lemma 2.2.2 (1), the endomorphism ring $\mathrm{End}_R(M)$ of a submodule M of Q is identified with the R-subalgebra $M : M$ of Q. Suppose that M is finitely generated. Then the R-module $\mathrm{Hom}_R(M, M)$ is finitely generated, and hence the extension $R \subseteq M : M$ is module-finite. Hence $M : M$ is contained in the integral closure $\overline{R}$ of R in Q. Moreover $M : M$ is semilocal if R is semilocal.

Remark 2.2.4. Let S be a ring extension of R in Q. Let M and N be S-modules such that N is torsion-free as an R-module. Then $\mathrm{Hom}_S(M, N) = \mathrm{Hom}_R(M, N)$.

Proof. Let $f : M \to N$ be an R-homomorphism. Take $a \in S$ and $x \in M$. What we want to show is that $f(ax) = af(x)$. Write $a = \frac{b}{c}$ as an element of Q. We have $c(f(ax) - af(x)) = cf(ax) - caf(x) = f(cax) - caf(x) = f(bx) - bf(x) = 0$. Since N is torsion-free over R, we get $f(ax) - af(x) = 0$. ■

If $(R, \mathfrak{m})$ is local and $\operatorname{depth} R > 0$, then a subring $\mathfrak{m} : \mathfrak{m}$ of Q is semilocal and a module-finite extension of R (Remark 2.2.3). The inclusions $R \subseteq \mathfrak{m} : \mathfrak{m} \subseteq Q$ show that $\mathfrak{m} : \mathfrak{m}$ is a birational extension of R. We give some lemmas on $\mathfrak{m} : \mathfrak{m}$.

Lemma 2.2.5. *Let $(R, \mathfrak{m})$ be a local ring and M be an R-module without free summands. Then the natural inclustion $\mathrm{Hom}_R(M, \mathfrak{m}) \to \mathrm{Hom}_R(M, R)$ is an equality.*

Proof. Since R is local, for any homomorphism $f\colon M \to R$ the image of f is contained in $\mathfrak{m}$ (otherwise it would produce a non-trivial free summand of M). ■

The following lemma is observed by Bass [14]. For the proof, see [108, Lemma 4.9].

Lemma 2.2.6. *Let $(R, \mathfrak{m})$ be a local ring and M be a reflexive R-module without free summands. Then M has an $\mathfrak{m} : \mathfrak{m}$-module structure which is compatible with the action of R on M.*

If R is Cohen–Macaulay of dimension one, we can obtain the following lemma.

Lemma 2.2.7. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension one. Assume R is not a discrete valuation ring. Then*

(a) *$\mathfrak{m}$ has no R-free summands.*

(b) $\mathfrak{m} : \mathfrak{m} = R : \mathfrak{m}$

(c) *$\ell_R(\mathfrak{m} : \mathfrak{m}/R)$ is equal to $r(R)$. In particular, R is properly contained in $\mathfrak{m} : \mathfrak{m}$.*

(d) $\mathfrak{m}$ *is reflexive as an* R*-module.*

(e) $\mathfrak{m} : \mathfrak{m}$ *is reflexive as an* R*-module and has no* R*-free summands.*

Proof. (a) By [9, Corollary 5.7], $\mathfrak{m}$ has no free summand. (b) Combining Lemma 2.2.2, Lemma 2.2.5 and (a), the assertion follows. (c) We look at the long exact sequence

$$0 \to \mathrm{Hom}_R(R/\mathfrak{m}, R) \to \mathrm{Hom}_R(R, R) \xrightarrow{f} \mathrm{Hom}_R(\mathfrak{m}, R) \to \mathrm{Ext}^1_R(R/\mathfrak{m}, R) \to \mathrm{Ext}^1_R(R, R) = 0$$

induced by the short exact sequence $0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0$.

The length of the R-module $\mathrm{Ext}^1_R(R/\mathfrak{m}, R)$ is exactly equal to the type $r(R)$ of R. On the other hand, one can directly check that the natural inclusion $g : R \to \mathfrak{m} : \mathfrak{m} = R : \mathfrak{m}$ satisfies the equality $\Phi_{R,\mathfrak{m}} \circ g = f \circ \Phi_{R,R}$, where $\Phi_{R,\mathfrak{m}} \colon R : \mathfrak{m} \to \mathrm{Hom}_R(\mathfrak{m}, R)$ and $\Phi_{R,R} \colon R : R \to \mathrm{Hom}_R(R, R)$ are isomophirms as in Lemma 2.2.2. It follows that the cokernel $\mathfrak{m} : \mathfrak{m}/R$ of g is isomorphic to $\mathrm{Ext}^1_R(R/\mathfrak{m}, R)$, the cokernel of f. In particular, both two R-module have same length. We thus obtain the equality $\ell_R(\mathfrak{m} : \mathfrak{m}/R) = r(R)$.

(d) By (c), we see that $\mathfrak{m} : \mathfrak{m} = R(\mathfrak{m} : \mathfrak{m}) \neq R$. This means that $R \not\subseteq R : (\mathfrak{m} : \mathfrak{m})$. On the other hand, the inclusion $R \subset \mathfrak{m} : \mathfrak{m}$ induces an inclusion $R : (\mathfrak{m} : \mathfrak{m}) \subseteq R : R = R$. Additionaly, the containment $\mathfrak{m}(\mathfrak{m} : \mathfrak{m}) \subseteq \mathfrak{m} \subseteq R$ shows that $\mathfrak{m} \subseteq R : (\mathfrak{m} : \mathfrak{m})$. Thus we have an equality $\mathfrak{m} = R : (\mathfrak{m} : \mathfrak{m})$. By (b), this is equlity $\mathfrak{m}\ R : (R : \mathfrak{m})$. Thus by Lemma 2.2.2, $\mathfrak{m}$ is reflexive.

(e) Combining (b) and (d), we can see that $R : (R : (\mathfrak{m} : m)) = \mathfrak{m} : \mathfrak{m}$. Thus by Lemma 2.2.2, $\mathfrak{m} : \mathfrak{m}$ is reflexive. If $\mathfrak{m} : \mathfrak{m}$ has a free summand, then its R-dual $\mathfrak{m}$ is also has a free summand. By (a), this is a contradiction. Therefore $\mathfrak{m} : \mathfrak{m}$ has no free summands. ∎

2.3 Trace ideals and the Lindo–Pande condition

We begin with recalling the definition of a trace ideal.

Definition 2.3.1. Let M be an R-module. The *trace* of M is defined as the ideal

$$\mathrm{tr}_R\, M = (f(x) \mid f \in M^*,\, x \in M)$$

of R, that is, each element has the form $\sum_{i=1}^n f_i(x_i)$ with $f_i \in M^*$ and $x_i \in M$. Define the R-linear map

$$\lambda^R_M : M^* \otimes_R M \to R, \qquad f \otimes x \mapsto f(x).$$

Then $\mathrm{tr}_R\, M$ is nothing but the image of λ^R_M. Using this, one can check that if M, N are R-modules with $M \cong N$, then $\mathrm{tr}_R\, M = \mathrm{tr}_R\, N$. An ideal I of R is called an *trace ideal* if $I = \mathrm{tr}\, M$ for some R-module M.

Proof. Let $x_1, \ldots, x_n$ generate M. Write $x_i = \frac{y_i}{z_i}$ as an element of Q, and put $z = z_1 \cdots z_n$. Then zM is an ideal of R. As z is a non-zerodivisor of R, the module M is isomorphic to zM. If M contains a non-zerodivisor r of R, then zM contains the element zr, which is a non-zerodivisor of R. ∎

For R-submodules M, N of Q we denote by MN the R-submodule $\langle xy \mid x \in M,\, y \in N\rangle$ of Q, which consists of the sums of elements of the form xy with $x \in M$ and $y \in N$. Here are several fundamental properties of trace ideals, which will be used throughout the paper.

Proposition 2.3.2. *Let M be an R-submodule of Q containing a non-zerodivisor c of R.*

(1) *There is an equality* $\operatorname{tr} M = (R : M)M$ *in Q.*

(2) *The equality $M = \operatorname{tr} M$ holds in Q if and only if the equality $M : M = R : M$ holds in Q.*

Proof. (1) We can directly check the assertion by using the isomorphism in (1).

(2) By (1) we have only to show that $M = (R : M)M$ if and only if $M : M = R : M$. It is obvious that $M \supseteq (R : M)M$ if and only if $M : M \supseteq R : M$. The implications

$$M \subseteq (R : M)M \Rightarrow M \subseteq R \Rightarrow M : M \subseteq R : M \Rightarrow 1 \in R : M \Rightarrow M \subseteq (R : M)M$$

hold, which shows that $M \subseteq (R : M)M$ if and only if $M : M \subseteq R : M$. ■

Lindo and Pande [111] raise the question asking when each ideal of a given ring is isomorphic to a trace ideal. To consider this question effectively, we give a name to the condition in it.

Definition 2.3.3. We define the *Lindo–Pande condition* (LP) by the following.

(LP) Every ideal of R is isomorphic to some trace ideal of R.

Question 2.3.4 (Lindo–Pande)**.** When does R satisfy (LP)?

Let us give several remarks related to the condition (LP).

Remark 2.3.5. (1) Let M, N be R-modules. If $M \cong \operatorname{tr} N$, then $M \cong \operatorname{tr} M$. Therefore, (LP) is equivalent to saying that each ideal I of R isomorphic to its trace: $I \cong \operatorname{tr} I$.

(2) When R satisfies (LP), any finitely generated R-submodule M of Q admits an isomorphism $M \cong \operatorname{tr} M$.

(3) If R satisfies (LP), then so does R_S for each multiplicatively closed subset S of R. When R is local, if the completion $\widehat{R}$ satisfies (LP), then so does R.

Proof. (1) Taking the traces of both sides of the isomorphism $M \cong \operatorname{tr} N$, we have $\operatorname{tr} M = \operatorname{tr}(\operatorname{tr} N)$. The latter trace coincides with $\operatorname{tr} N$ by [110, Proposition 2.8(iv)]. Hence $\operatorname{tr} M = \operatorname{tr} N \cong M$.

(2) The assertion follows from Remark 2.2.1 and (1).

(3) The assertion on localization is shown by using (1) and [110, Proposition 2.8(viii)]. For the assertion on completion, apply (1) and [50, Exercise 7.5]. ■

Now we recall that an *invertible* R-module is by definition a finitely generated R-module M such that $M_\mathfrak{p} \cong R_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R. The isomorphism classes of invertible R-modules form an abelian group with multiplication $\otimes_R$ and identity $[R]$, which is called the *Picard group* $\operatorname{Pic} R$ of R. The condition (LP) implies the triviality of this group.

Proposition 2.3.6. *If R satisfies* (LP)*, then* $\operatorname{Pic} R = 0$.

Proof. Let M be an invertible R-module. By [50, Theorem 11.6b] the R-module M is isomorphic to an R-submodule of Q, and we get $M \cong \operatorname{tr}_R M$ by Remark 2.3.5(2). By [50, Theorem 11.6a] the map $\lambda_M^R : M^* \otimes_R M \to R$ is an isomorphism, which implies $\operatorname{tr}_R M = R$. Hence we obtain an isomorphism $M \cong R$, and consequently, the Picard group of R is trivial. ■

Recall that a *Dedekind domain* is by definition an integral domain whose nonzero ideals are invertible, or equivalently, a noetherian normal domain of Krull dimension at most one. The above proposition yields a characterization of the Dedekind domains satisfying the Lindo–Pande condition.

Corollary 2.3.7. *A Dedekind domain satisfies* (LP) *if and only if it is a principal ideal domain.*

Proof. Fix a nonzero ideal I of R. If R is a Dedekind domain satisfying (LP), then Proposition 2.3.6 implies $I \cong R$. Conversely, if $I \cong R$, then $\operatorname{tr} I = \operatorname{tr} R = R \cong I$. The assertion now follows. ■

2.4 Characterization of rings satisfying the Lindo–Pande condition

We first consider the Lindo–Pande condition for (not necessarily local) rings whose nonunits are zerodivisors. For this, we need to extend a theorem of Lindo and Pande to non-local rings; the assertion of the following proposition is nothing but [111, Theorem 3.5] in the case where R is local.

Proposition 2.4.1. *The following are equivalent.*

(1) *R is artinian and Gorenstein.*

(2) *Every ideal of R is a trace ideal of R.*

(3) *Every principal ideal of R is a trace ideal of R.*

Proof. Let I be an ideal of R. Then I is a trace ideal if and only if $I = \operatorname{tr}_R I$ by [110, Proposition 2.8(iv)]. In general, I is contained in $\operatorname{tr}_R I$ by [110, Proposition 2.8(iv)] again, which enables us to define the quotient $(\operatorname{tr}_R I)/I$. Using [110, Proposition 2.8(viii)], we see that

$$\begin{aligned} I = \operatorname{tr}_R I &\Leftrightarrow (\operatorname{tr}_R I)/I = 0 \Leftrightarrow ((\operatorname{tr}_R I)/I)_{\mathfrak{p}} = 0 \text{ for all } \mathfrak{p} \in \operatorname{Spec} R \\ &\Leftrightarrow (\operatorname{tr}_{R_{\mathfrak{p}}} I_{\mathfrak{p}})/I_{\mathfrak{p}} = 0 \text{ for all } \mathfrak{p} \in \operatorname{Spec} R \Leftrightarrow I_{\mathfrak{p}} = \operatorname{tr}_{R_{\mathfrak{p}}} I_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in \operatorname{Spec} R. \end{aligned}$$

Thus we can reduce to the local case and apply [111, Theorem 3.5] to deduce the proposition. ■

Using the above proposition, we obtain the following theorem including a criterion for a ring with full of zerodivisors to satisfy the Lindo–Pande condition. Note that in the case where R is local the assumption of the theorem is equivalent to the condition that R has depth zero.

Theorem 2.4.2. *Assume that all non-zerodivisors of R are units. Then the following are equivalent.*

(1) *R is artinian and Gorenstein.*

(2) *Every ideal of R is a trace ideal of R.*

(3) *Every principal ideal of R is a trace ideal of R.*

(4) *Every ideal of R is isomorphic to a trace ideal of R, that is, R satisfies* (LP).

(5) *Every principal ideal of R is isomorphic to a trace ideal of R.*

Proof. The equivalences (1)⇔(2)⇔(3) follow from Proposition 2.4.1, while the implications (2)⇒(4)⇒(5) are obvious. It suffices to show the implication (5)⇒(3).

Assume that (3) does not hold, namely, that there exists a principal ideal (x) of R which is not a trace ideal. Then, in particular, x is nonzero. It follows from (5), Remark 2.3.5(1) and [111, Lemma 2.5] that $(x) \cong \operatorname{tr}(x) = \operatorname{Ann}(\operatorname{Ann}(x))$. Let $\phi : \operatorname{Ann}(\operatorname{Ann}(x)) \to (x)$ be the isomorphism, and $\theta : (x) \to \operatorname{Ann}(\operatorname{Ann}(x))$ the inclusion map. The endomorphism $\phi\theta : (x) \to (x)$ corresponds to an endomorphism $R/\operatorname{Ann}(x) \to R/\operatorname{Ann}(x)$, which is a multiplication map by some element $\overline{a} \in R/\operatorname{Ann}(x)$. Then $\phi\theta$ is the multiplication map by the element $a \in R$. Since $\phi\theta$ is injective, a is a non-zerodivisor on (x). Hence $\operatorname{grade}((a),(x))$ is positive, or in other words, $\operatorname{Hom}_R(R/(a),(x)) = 0$. Taking the R-dual of the isomorphism $(x) \cong R/\operatorname{Ann}(x)$ yields an isomorphism $(x)^* \cong \operatorname{Ann}(\operatorname{Ann}(x))$, and hence $(x) \cong (x)^*$. There are isomorphisms

$$\begin{aligned} 0 = \operatorname{Hom}_R(R/(a),(x)) &\cong \operatorname{Hom}_R(R/(a),(x)^*) \\ &\cong (R/(a) \otimes_R (x))^* = (R/(a) + \operatorname{Ann}(x))^* \cong \operatorname{Ann}((a) + \operatorname{Ann}(x)), \end{aligned}$$

which show that the ideal $(a) + \operatorname{Ann}(x)$ contains a non-zerodivisor of R, which is a unit by the assumption of the theorem. Therefore, $1 = ab + c$ for some $b \in R$ and $c \in \operatorname{Ann}(x)$. We have $\phi(x) = \phi\theta(x) = ax$ and $x = (ab+c)x = abx = b\phi(x)$. Take any element $y \in \operatorname{Ann}(\operatorname{Ann}(x))$. There exists an element $d \in R$ such that $\phi(y) = dx$. Then $\phi(y) = db\phi(x) = \phi(dbx)$, which implies $y = dbx$ as ϕ is injective. Thus y belongs to (x). Consequently, we obtain $(x) = \operatorname{Ann}(\operatorname{Ann}(x)) = \operatorname{tr}(x)$. This contradicts our assumption that (x) is not a trace ideal. We now conclude that (5) implies (3). ■

Next, we study the Lindo–Pande condition for local rings of depth one. We start by showing a lemma on Gorenstein local rings of dimension one. Recall that a local ring R is called a *hypersurface* if R has codepth at most one, i.e., $\operatorname{edim} R - \operatorname{depth} R \le 1$. This is equivalent to saying that the completion of R is isomorphic to the quotient of a regular local ring by a principal ideal. A Cohen–Macaulay local ring is said to have *minimal multiplicity* if the equality $\mathrm{e}(R) = \operatorname{edim} R - \dim R + 1$ holds; see [19, Exercise 4.6.14].

Lemma 2.4.3. *Let R be a 1-dimensional Gorenstein local ring with maximal ideal $\mathfrak{m}$. If $\mathfrak{m} \cong \mathfrak{m}^2$, then R is a hypersurface with $\mathrm{e}(R) \le 2$.*

Proof. Put $s = \operatorname{edim} R$. Note that $\mathfrak{m} \cong \mathfrak{m}^2 \cong \mathfrak{m}^3 \cong \cdots$. Hence $\nu(\mathfrak{m}^i) = s$ for all $i > 0$, and $\ell(R/\mathfrak{m}^{n+1}) = \sum_{i=0}^{n} \ell(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \sum_{i=0}^{n} \nu(\mathfrak{m}^i) = (n+1)s$. Therefore $\mathrm{e}(R) = \lim_{n\to\infty} \frac{1}{n}\ell(R/\mathfrak{m}^{n+1}) = s$. As R has dimension one, it has minimal multiplicity. Since R is Gorenstein, it satisfies $s \le 2$ (see [130, Corollary 3.2]) and so it is a hypersurface. ■

We need one more lemma for our next goal.

Lemma 2.4.4. *Let I be a reflexive ideal of R containing a non-zerodivisor of R, and set $S = I : I$. Assume that the equality $I = \operatorname{tr}_R I$ holds. Then one has an equality $I = \operatorname{tr}_R S$. In particular, if there is an isomorphism $S \cong \operatorname{tr}_R S$ of R-modules, then one has an isomorphism $I \cong S$ of S-modules.*

Proof. First of all, note that I is an S-module. We apply Proposition 2.3.2 several times. We have $S = I : I = R : I$ and $I = R : (R : I) = R : S$. Hence $\operatorname{tr}_R S = (R : S)S = IS = I(I : I) = I$. Therefore, if $S \cong \operatorname{tr}_R S$, then there is an R-isomorphism $I \cong S$, and it is an S-isomorphism by Remark 2.2.4. ■

For each $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, a 1-dimensional *hypersurface singularity of type* (A_n) (or (A_n)-*singularity* for short) is by definition a ring that is isomorphic to the quotient

$$R_n = k[\![x, y]\!]/(x^2 + y^{n+1})$$

of a formal power series ring over a field k, where we set $R_0 = k[\![x]\!]$ and $R_\infty = k[\![x, y]\!]/(x^2)$. It is known that a 1-dimensional (A_n)-singularity has finite (resp. countable) Cohen–Macaulay representation type for $n \in \mathbb{Z}_{\geq 0}$ (resp. $n = \infty$); see [149, Corollary (9.3) and Example (6.5)]. Hence, there exist only at most countably many indecomposable torsion-free modules over such a ring.

Now we can achieve our second purpose of this section, which is to give a characterization of the local rings of depth one that satisfy the Lindo–Pande condition.

Theorem 2.4.5. *Let* $(R, \mathfrak{m}, k)$ *be a local ring with* $\operatorname{depth} R = 1$*. Consider the following conditions.*

(1) *The ring* R *satisfies* (LP)*.*

(2) *The completion* $\widehat{R}$ *satisfies* (LP)*.*

(3) *The ring* R *is a hypersurface with* $\dim R = 1$ *and* $\mathrm{e}(R) \leq 2$*.*

(4) *The completion* $\widehat{R}$ *is a* 1*-dimensional* (A_n)*-singularity for some* $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$*.*

Then the implications (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3) $\Leftarrow$ (4) *hold. If* k *is algebraically closed and has characteristic* 0*, then all the four conditions are equivalent.*

Proof. (4)$\Rightarrow$(3): Since $\widehat{R}$ is a hypersurface, so is R. We see directly from the definition of an (A_n)-singularity that $\mathrm{e}(\widehat{R}) \leq 2$. As the equality $\mathrm{e}(R) = \mathrm{e}(\widehat{R})$ holds in general, we have $\mathrm{e}(R) \leq 2$.

(3)$\Rightarrow$(2): As $\mathrm{e}(\widehat{R}) = \mathrm{e}(R)$, $\dim \widehat{R} = \dim R$ and $\operatorname{depth} \widehat{R} = \operatorname{depth} R$, we may assume that R is complete. Take any ideal I of R. The goal is to prove $I \cong \operatorname{tr} I$.

We begin with the case where I is an $\mathfrak{m}$-primary ideal. Set $S = I : I$. Then S is an intermediate ring of R and Q which is finitely generated as an R-module, and I is also an ideal of S. The proof of Remark 2.2.1 says $zS \subseteq R$ for some non-zerodivisor z of R. By [58, Theorem 3.11], the ring S is Gorenstein. Using Proposition 2.3.2(1) and Remark 2.2.4, we have an S-isomorphism $S = I : I \to \operatorname{Hom}_R(I, I) = \operatorname{Hom}_S(I, I)$ given by $s \mapsto (i \mapsto si)$. Hence I is a *closed* ideal of S in the sense of [18]. It follows from [18, Corollary 3.2] that I is an invertible ideal of S. As $S/\mathfrak{m}S$ is artinian and all maximal ideals of S contain $\mathfrak{m}S$, the ring S is semilocal. We observe $I \cong S$ by [19, Lemma 1.4.4]. Thus it is enough to check that S is isomorphic to its trace as an R-module. Using Proposition 2.3.2, we obtain $\operatorname{tr}_R S = (R : S)S = R : S \cong S^*$. Since R is henselian, S is a product of local rings: we have $S \cong S_1 \times \cdots \times S_r$, where S_i is local for $1 \leq i \leq r$. Each S_i is a localization of S, so it is Gorenstein. Hence $(S_i)^* \cong \omega_{S_i} \cong S_i$ for each i, and therefore $S^* \cong S$. Consequently, we obtain $S \cong \operatorname{tr}_R S$ as desired.

Next we consider the case where I is not an $\mathfrak{m}$-primary ideal. Then I is contained in some minimal prime $\mathfrak{p}$ of R. When $I = 0$, we have $I = \operatorname{tr} I$ and are done. So we assume $I \neq 0$, which forces R not to be a domain. By Cohen's structure theorem and the assumption that $\mathrm{e}(R) \leq 2$, we can identify R with the ring $S/(f)$, where $(S, \mathfrak{n})$ is a 2-dimensional regular local ring and f is a reducible element in $\mathfrak{n}^2 \setminus \mathfrak{n}^3$. Write $f = gh$ with $g, h \in \mathfrak{n} \setminus \mathfrak{n}^2$. Then g, h are irreducible, and we see that $\operatorname{Min} R = \{gR, hR\}$ (possibly $gR = hR$). Hence $\mathfrak{p}$ is equal to either gR or hR. We also observe $\operatorname{Ann}(gR) = hR \cong R/gR$ and $\operatorname{Ann}(hR) = gR \cong R/hR$. As both R/gR and R/hR are discrete valuation rings, any nonzero submodule of $\mathfrak{p}$ is isomorphic to $\mathfrak{p}$, and therefore we have only to show that $\mathfrak{p} \cong \operatorname{tr} \mathfrak{p}$. Thanks to [111, Corollary 2.9], we obtain $\operatorname{tr} \mathfrak{p} = \operatorname{Ann}(\operatorname{Ann} \mathfrak{p}) = \mathfrak{p}$, which particularly says $\mathfrak{p} \cong \operatorname{tr} \mathfrak{p}$.

(2)⇒(1): This implication immediately follows from Remark 2.3.5(3).

(1)⇒(3): We have $\mathfrak{m} \subseteq \operatorname{tr} \mathfrak{m} \subseteq R$ (see [110, Proposition 2.8(iv)]), and $\mathfrak{m} \cong \operatorname{tr} \mathfrak{m}$ by Remark 2.3.5(1). If $\operatorname{tr} \mathfrak{m} = R$, then $\mathfrak{m} \cong R$, which means that R is a discrete valuation ring, and we are done. Thus we may assume $\mathfrak{m} = \operatorname{tr} \mathfrak{m}$. Put $S = \mathfrak{m} : \mathfrak{m}$. Proposition 2.3.2(3) implies $S = R : \mathfrak{m}$. Applying $(-)^*$ to the exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$ gives rise to an exact sequence $0 \to R \xrightarrow{\phi} \mathfrak{m}^* \to \operatorname{Ext}^1_R(k, R) \to 0$. Note that $\operatorname{Ext}^1_R(k, R) \neq 0$ as $\operatorname{depth} R = 1$. By Proposition 2.3.2(1), the map ϕ can be identified with the inclusion $R \subseteq S$ and so we have $R \neq S$. Choose an element $x \in S \setminus R$ and set $X = R + Rx \subseteq S$. Since $\mathfrak{m}X \subseteq \mathfrak{m}S \subseteq R$, we have $\mathfrak{m} \subseteq R : X$ and

$$\mathfrak{m} = \mathfrak{m}R \subseteq \mathfrak{m}X \subseteq (R : X)X = \operatorname{tr}_R X \subseteq R,$$

where the second equality follows from Proposition 2.3.2(2). Hence $\operatorname{tr}_R X$ coincides with either $\mathfrak{m}$ or R. By Remark 2.3.5(2) we have $X \cong \operatorname{tr}_R X$.

Assume $\operatorname{tr}_R X = R$. Then $X \cong R$, and we find an element $y \in X$ such that $X = Ry$. As $1 \in X$, we have $1 = ay$ for some $a \in R$. Since $y \in S$, we get $\mathfrak{m}y \subseteq \mathfrak{m}$, which shows $a \notin \mathfrak{m}$. Hence a is a unit of R, and we observe $y \in R$. Therefore $X = R$, and x is in R, which contradicts the choice of x.

Thus we have to have $\operatorname{tr}_R X = \mathfrak{m}$, and get an R-isomorphism $X \cong \mathfrak{m}$. This implies that $\mathfrak{m}$ is generated by at most two elements as an R-module. Hence

$$1 = \operatorname{depth} R \leq \dim R \leq \operatorname{edim} R \leq 2.$$

If $\dim R = 2$, then the equality $\dim R = \operatorname{edim} R$ holds, which means that R is a regular local ring. In particular, R is Cohen–Macaulay, and it follows that $1 = \operatorname{depth} R = \dim R = 2$, which is a contradiction. Thus $\dim R = 1$, and we have $\operatorname{edim} R - \dim R \leq 1$, namely, R is a hypersurface.

It remains to prove that R has multiplicity at most 2. According to Lemma 2.4.3, it suffices to show that $\mathfrak{m} \cong \mathfrak{m}^2$. The R-module S is isomorphic to $\operatorname{tr}_R S$ by Remark 2.3.5(2). It follows from Lemma 2.4.4 that $\mathfrak{m} \cong S = R : \mathfrak{m}$. Using Proposition 2.3.2(2), we obtain $\mathfrak{m} = \operatorname{tr}_R \mathfrak{m} = (R : \mathfrak{m})\mathfrak{m} \cong \mathfrak{m}\mathfrak{m} = \mathfrak{m}^2$, as desired. (In general, if a module X is isomorphic to a module Y, then $\mathsf{A}X$ is isomorphic to $\mathsf{A}Y$ for an ideal A.)

(3)⇒(4) (under the assumption that k is algebraically closed and has characteristic 0): Again, we have $\mathrm{e}(\widehat{R}) \leq 2$. Cohen's structure theorem implies that $\widehat{R}$ is isomorphic to a hypersurface of the form $k[\![x, y]\!]/(f)$ with $f \in (x, y) \setminus (x, y)^3$. Changing variables, we can reduce to the case where $f = x$ or $f = x^2$ or $f = x^2 + y^t$ with $t \in \mathbb{Z}_{>0}$; see (i) of [149, Proof of (8.5)] and its preceding part. ■

Remark 2.4.6. Let R be a ring satisfying Theorem 2.4.5(4). Then each ideal of R is a maximal Cohen–Macaulay R-module. The isomorphism classes of indecomposable maximal Cohen–Macaulay R-modules are completely classified; see [149, Proposition (5.11), (9.9) and Example (6.5)]. The implication (4)⇒(1) in Theorem 2.4.5 can also be proved by using this classification (although it is rather complicated).

Combining our Theorems 2.4.2 and 2.4.5, we obtain a remarkable result.

Corollary 2.4.7. *The Lindo–Pande condition* (LP) *implies Serre's condition* (S_2)*.*

Proof. Suppose that R satisfies (LP). Let $\mathfrak{p}$ be a prime ideal of R. The localization $R_\mathfrak{p}$ also satisfies (LP) by Remark 2.3.5(3). We see from Theorems 2.4.2 and 2.4.5 that $R_\mathfrak{p}$ is Cohen–Macaulay when $\operatorname{depth} R_\mathfrak{p} \leq 1$. It is easy to observe from this that R satisfies (S_2). ■

One of the original motivation of the condition (LP) is to seek a new class of rings which Conjecture 1.1.1 holds for. After Theorem 2.4.5, we can check that Conjecture 1.1.1 holds for rings satisfies (LP), by trivial application of [87, Theorem 3.1].

Corollary 2.4.8. *Let R is a local ring of depth one, and suppose that R satisfies* (LP)*. Let M be an R-module having a rank. If $M \otimes_R M^*$ is torsion-free, then M is free. In particular, Conjecture 1.1.1 holds for a ring satisfying* (LP)*.*

Proof. Theorem 2.4.5 implies that R is a 1-dimensional hypersurface. By [87, Theorem 3.1], both M and M^* are torsion-free, and either of them is free. If M^* is free, then so is M by [28, Lemma 2.13]. ■

Our next goal is to study the Lindo–Pande condition for rings having Krull dimension at least two. The following proposition characterize the ideals of normal rings that are isomorphic to trace ideals.

Proposition 2.4.9. *Let M be a finitely generated R-submodule of Q containing a non-zerodivisor of R. Consider the following conditions.*

(1) *M is isomorphic to a trace ideal of R.*

(2) *M^* is isomorphic to R.*

(3) *M is isomorphic to an ideal I of R with* $\operatorname{grade} I \geq 2$ *(i.e.* $\operatorname{Ext}_R^i(R/I, R) = 0$ *for $i < 2$).*

Then the implications (1) $\Longleftarrow$ (2) $\Longleftrightarrow$ (3) *hold. All the three conditions are equivalent if R is normal.*

Proof. In view of Remark 2.2.1, we can replace M with an ideal J of R containing a non-zerodivisor.

(3)⇒(2): Dualizing the natural short exact sequence $0 \to I \to R \to R/I \to 0$ by R induces $I^* \cong R$.

(2)⇒(1): Using Proposition 2.3.2(1)(2), we have $R : J \cong J^* \cong R$, and $\operatorname{tr} J = (R : J)J \cong RJ = J$.

(2)⇒(3): If $J = R$, then we have $\operatorname{grade} J = \infty \geq 2$ and are done. Let $J \neq R$. Then $(R/J)^* = 0$, and dualizing the natural exact sequence $0 \to J \to R \to R/J \to 0$ gives an exact sequence $0 \to R \to J^* \to \operatorname{Ext}^1_R(R/J, R) \to 0$. Combining this with the isomorphism $J^* \cong R$, we find a non-zerodivisor x_1 of R such that $\operatorname{Ext}^1_R(R/J, R) \cong R/(x_1)$. As J annihilates the Ext module, it is contained in the ideal (x_1). Hence we find an ideal J_1 of R such that $J = x_1 J_1$. It is easy to see that J_1 also contains a non-zerodivisor of R. As J_1 is isomorphic to J, we have $J_1^* \cong R$. Thus the argument for J applies to J_1. If $J_1 = R$, then we are done. If $J_1 \neq R$, then we find an ideal J_2 and a non-zerodivisor x_2 with $J_1 = x_2 J_2$. Iterate this procedure, and consider the case where we get ideals J_i and non-zerodivisors x_i such that $J_i = x_{i+1} J_{i+1}$ for all $i \geq 0$. In this case, there is a filtration of ideals of R:

$$J =: J_0 \subseteq J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots .$$

As R is noetherian, this stabilizes: there exists an integer $t \geq 0$ such that $J_t = J_{t+1}$. Hence $J_{t+1} = x_{t+1} J_{t+1}$, and Nakayama's lemma gives rise to an element $r \in R$ such that $1 - r \in (x_{t+1})$ and $r J_{t+1} = 0$. The fact that J_{t+1} contains a non-zerodivisor forces r to be zero, and x_{t+1} is a unit of R. Therefore $\operatorname{Ext}^1_R(R/J_t, R) \cong R/(x_{t+1}) = 0$, and thus $\operatorname{grade} J_t \geq 2$. It remains to note that J is isomorphic to J_t.

Finally, we prove the implication (1)⇒(2) under the additional assumption that R is normal. By Remark 2.3.5(1) the ideal J is isomorphic to its trace $I := \operatorname{tr} J$. As $J \subseteq I$ by [110, Proposition 2.8(iv)], the ideal I contains a non-zerodivisor of R. We have $\operatorname{tr} I = \operatorname{tr}(\operatorname{tr} J) = \operatorname{tr} J = I$ by [110, Proposition 2.8(iv)] again. Using (1) and (3) of Proposition 2.3.2, we get $I^* \cong R : I = I : I$. The ring $I : I$ is a module-finite extension of R in Q, and hence it is integral over R. Since R is normal, we have $I : I = R$. We thus obtain $J^* \cong I^* \cong I : I = R$. ∎

The above proposition yields a characterization of the normal domains that satisfy the Lindo–Pande condition. For a normal domain R we denote by $\operatorname{Cl}(R)$ the *divisor class group* of R.

Corollary 2.4.10. *A ring R is a normal domain satisfying* (LP) *if and only if it is factorial.*

Proof. Let R be a normal domain. Then it follows from [54, Proposition 6.1] that R is factorial if and only if $\operatorname{Cl}(R) = 0$. The zero ideal is a trace ideal as $0 = \operatorname{tr} 0$. Applying Proposition 2.4.9, we observe that R satisfies (LP) if and only if $I^* \cong R$ for all ideals $I \neq 0$. Therefore we have only to show the following two statements (see [132, 2.10]).

(a) Suppose that I^* is isomorphic to R for every nonzero ideal I of R. Let M be a finitely generated reflexive R-module of rank one. Then $[M] = 0$ in $\operatorname{Cl}(R)$.

(b) Let I be a nonzero ideal of R. Then I^* is a reflexive module of rank one.

(a): As M has rank 1 and is torsion-free, it is isomorphic to an ideal $I \neq 0$ of R. Then I^* is isomorphic to R by assumption, and we get isomorphisms $M \cong M^{**} \cong I^{**} \cong R^* \cong R$. Hence $[M] = 0$ in $\operatorname{Cl}(R)$.

(b): The module I has rank 1, and so does I^*. For each R-module X, denote by $\rho(X)$ the canonical homomorphism $X \to X^{**}$. We can directly verify that the composition $(\rho(I))^* \circ \rho(I^*)$ is the identity map of I^*. Hence $I^{***} \cong I^* \oplus E$ for some R-module E. Comparing the ranks, we see that E is torsion. As E is isomorphic to a submodule of the torsion-free module I^{***}, it is zero. Therefore I^* is reflexive. ∎

What we want to do next is to remove from the above corollary the assumption that R is a normal domain. For this, we need to investigate the Lindo–Pande condition for a finite product of rings.

Lemma 2.4.11. *Let $R_1, \dots R_n$ be rings. Then the product ring $R_1 \times \cdots \times R_n$ satisfies* (LP) *if and only if R_i satisfies* (LP) *for all $1 \le i \le n$.*

Proof. The assignment $(M_1, \dots, M_n) \mapsto M_1 \times \cdots \times M_n$ gives an equivalence

$$\textstyle\prod_{i=1}^n(\mathsf{Mod}\, R_i) \cong \mathsf{Mod}(\prod_{i=1}^n R_i)$$

as tensor abelian categories, where for a ring A we denote by $\mathsf{Mod}\, A$ the category of arbitrary A-modules. In particular, we can do the identification

$$\textstyle\prod \operatorname{Hom}_{R_i}(M_i, N_i) = \operatorname{Hom}_{\prod R_i}(\prod M_i, \prod N_i), \qquad \prod(M_i \otimes_{R_i} N_i) = \prod M_i \otimes_{\prod R_i} \prod N_i.$$

Now it is easy to see that for all ideals I_i of R_i with $1 \le i \le n$, one has

$$\textstyle\operatorname{tr}_{\prod R_i}(\prod I_i) = \prod \operatorname{tr}_{R_i} I_i. \tag{2.4.11.1}$$

The "if" part of the lemma directly follows from (2.4.11.1) (see Remark 2.3.5(1)). Applying (2.4.11.1) to the ideal $0 \times \cdots \times 0 \times I_i \times 0 \times \cdots \times 0$ of $\prod R_i$ shows the "only if" part. ■

Now we have reached our third (final) goal of this section, which is to give a criterion for a certain class of rings with Krull dimension at least two to satisfy the Lindo–Pande condition.

Theorem 2.4.12. *Assume that all maximal ideals of R have height at least 2. Then R satisfies* (LP) *if and only if R is a product of factorial rings. In particular, when R is a local ring or an integral domain, it satisfies* (LP) *if and only if it is factorial.*

Proof. The "if" part follows from Corollary 2.4.10 and Lemma 2.4.11. To prove the "only if" part, it suffices to show that R is normal. Indeed, suppose that it is done. Then R is a product $R_1 \times \cdots \times R_n$ of normal domains; see [114, Page 64, Remark]. By Lemma 2.4.11 and Corollary 2.4.10, each R_i is factorial, and the proof is completed.

So let us show that R is normal. As R satisfies (S_2) by Corollary 2.4.7, it is enough to verify that R satisfies (R_1). Fix a prime ideal $\mathfrak{p}$ of R with $\operatorname{ht} \mathfrak{p} \le 1$. What we want to show is that $R_\mathfrak{p}$ is a regular local ring. By assumption, $\mathfrak{p}$ is not a maximal ideal, and we find a prime ideal $\mathfrak{q}$ containing $\mathfrak{p}$ with $\operatorname{ht} \mathfrak{q}/\mathfrak{p} = 1$.

(i) We begin with considering the case where $\operatorname{ht} \mathfrak{p} = 1$. In this case, $\operatorname{ht} \mathfrak{q} \ge 2$. Note that (S_2) localizes, and so does (LP) by Remark 2.3.5(3). Replacing R with $R_\mathfrak{q}$, we may assume that $(R, \mathfrak{m})$ is a local ring with $\dim R = \operatorname{ht} \mathfrak{m} \ge 2$ and $\dim R/\mathfrak{p} = \operatorname{ht} \mathfrak{m}/\mathfrak{p} = 1$. Then $R/\mathfrak{p}$ is a 1-dimensional Cohen–Macaulay local ring. Since R satisfies (S_2), we have $\operatorname{depth} R \ge 2$ and $\mathfrak{p}$ contains a non-zerodivisor of R; see [19, Proposition 1.2.10(a)]. To show that $R_\mathfrak{p}$ is regular, it suffices to prove that $R_\mathfrak{p}$ has embedding dimension at most one.

Let us consider the case $\operatorname{tr} \mathfrak{p} = R$. Then $\mathfrak{p}$ contains a nonzero free summand; see [110, Proposition 2.8(iii)]. We find a non-zerodivisor x of R in $\mathfrak{p}$ and a subideal I of $\mathfrak{p}$ such that $\mathfrak{p} = (x) \oplus I$. Since $(x) \cap I = 0$, we have $xI = 0$, and $I = 0$ as x is a non-zerodivisor. Thus

$\mathfrak{p} = (x)$. In particular, we have $\operatorname{edim} R_{\mathfrak{p}} \le 1$, which is what we want. Consequently, we may assume that $\operatorname{tr} \mathfrak{p}$ is a proper ideal of R.

We claim $\mathfrak{p} = \operatorname{tr} \mathfrak{p}$. Indeed, $\operatorname{tr} \mathfrak{p}$ contains $\mathfrak{p}$ by [110, Proposition 2.8(iv)]. Suppose that the containment is strict. Then $\operatorname{tr} \mathfrak{p}$ is $\mathfrak{m}$-primary as $\operatorname{ht} \mathfrak{m}/\mathfrak{p} = 1$. Apply the depth lemma to the natural exact sequences

$$0 \to \mathfrak{p} \to R \to R/\mathfrak{p} \to 0, \qquad 0 \to \operatorname{tr} \mathfrak{p} \to R \to R/\operatorname{tr} \mathfrak{p} \to 0.$$

We observe $\operatorname{depth} \mathfrak{p} = 2$ and $\operatorname{depth}(\operatorname{tr} \mathfrak{p}) = 1$. Our assumption that R satisfies (LP) and Remark 2.3.5(1) imply that $\mathfrak{p} \cong \operatorname{tr} \mathfrak{p}$, which gives a contradiction. Thus the claim follows.

Next, we claim that $\mathfrak{p}$ is reflexive. In fact, let P be a prime ideal of R. According to [19, Proposition 1.4.1], it is enough to check the following.

(a) If $\operatorname{depth} R_P \le 1$, then $\mathfrak{p}R_P$ is a reflexive R_P-module.

(b) If $\operatorname{depth} R_P \ge 2$, then $\operatorname{depth} \mathfrak{p}R_P \ge 2$.

If P does not contain $\mathfrak{p}$, then $\mathfrak{p}R_P = R_P$. If P contains $\mathfrak{p}$, then P coincides with $\mathfrak{p}$ or $\mathfrak{m}$ as $\operatorname{ht} \mathfrak{m}/\mathfrak{p} = 1$. Recall that $\operatorname{depth} R \ge 2$ and $\operatorname{depth} \mathfrak{p} = 2$. The fact that R satisfies (S_2) especially says $\operatorname{depth} R_{\mathfrak{p}} = 1$. Theorem 2.4.5 and Remark 2.3.5(3) imply that $R_{\mathfrak{p}}$ is a Gorenstein local ring of dimension 1, whence $\mathfrak{p}R_{\mathfrak{p}}$ is a reflexive $R_{\mathfrak{p}}$-module. We now easily see that (a) and (b) hold, and the claim follows.

Set $S = \mathfrak{p} : \mathfrak{p}$. Then $S = R : \mathfrak{p}$ by the above first claim and Proposition 2.3.2(3). It follows from the condition (LP), Remark 2.3.5(2), Lemma 2.4.4 and the above two claims that $S \cong \operatorname{tr}_R S = \mathfrak{p}$. Thus we obtain an S-isomorphism $\mathfrak{p} \cong S$; see Remark 2.2.4. The ideal $\mathfrak{p}$ contains a non-zerodivisor x of S such that $\mathfrak{p} = xS$. Note that x is also a non-zerodivisor of R. If $\mathfrak{p} = xR$, then $\operatorname{edim} R_{\mathfrak{p}} \le 1$ and we are done.

Now, let us suppose that $\mathfrak{p} \ne xR$, and derive a contradiction. Krull's intersection theorem shows $\bigcap_{i>0}(\mathfrak{m}^i + xR) = xR$, which implies $\mathfrak{p} \not\subseteq \mathfrak{m}^t + xR$ for some $t > 0$. Put $I = \mathfrak{p} \cap (\mathfrak{m}^t + xR)$. Notice that I contains the non-zerodivisor x of R and is strictly contained in $\mathfrak{p}$.

We claim that $\mathfrak{p} = \operatorname{tr} I$. Indeed, we have

$$\operatorname{tr} I = (R : I)I = (R : \mathfrak{p})I \subseteq (R : \mathfrak{p})\mathfrak{p} = \operatorname{tr} \mathfrak{p} = \mathfrak{p} = xS = (R : \mathfrak{p})x \subseteq (R : \mathfrak{p})I = \operatorname{tr} I.$$

Here, the first and third equalities follow from Proposition 2.3.2(2). Consider the exact sequence $0 \to I \xrightarrow{f} \mathfrak{p} \to \mathfrak{p}/I \to 0$, where f is the inclusion map. Note that $\mathfrak{p}/I$ has finite length. As $\operatorname{depth} R \ge 2$, the map $f^* : \mathfrak{p}^* \to I^*$ is an isomorphism. It is observed from this and Proposition 2.3.2(1) that the equality $(R : I)I = (R : \mathfrak{p})I$ appearing above holds.

This claim, the condition (LP) and Remark 2.3.5(1) imply $\mathfrak{p} \cong I$. Applying the depth lemma to the exact sequence $0 \to I \to \mathfrak{p} \to \mathfrak{p}/I \to 0$ shows that I has depth 1 as an R-module. However, the R-module $\mathfrak{p}$ has depth 2, and we obtain a desired contradiction. Thus, the proof is completed in the case $\operatorname{ht} \mathfrak{p} = 1$.

(ii) Now we consider the case where $\operatorname{ht} \mathfrak{p} = 0$. We have $\operatorname{ht} \mathfrak{q} \ge 1$. If $\operatorname{ht} \mathfrak{q} = 1$, then $R_{\mathfrak{q}}$ is regular by (i), and so is $R_{\mathfrak{p}} = (R_{\mathfrak{q}})_{\mathfrak{p}R_{\mathfrak{q}}}$, which is what we want. Assume $\operatorname{ht} \mathfrak{q} \ge 2$, and let us derive a contradiction. As in (i), replacing R with $R_{\mathfrak{q}}$, we may assume that $(R, \mathfrak{m})$ is a local ring with $\operatorname{depth} R \ge 2$ and $R/\mathfrak{p}$ is a Cohen–Macaulay local ring of dimension 1. Choosing an element

$y \in \mathfrak{m} \setminus \mathfrak{p}$, we get an exact sequence $0 \to R/\mathfrak{p} \xrightarrow{y} R/\mathfrak{p} \to R/\mathfrak{p}+(y) \to 0$. Since $R/\mathfrak{p}+(y)$ has finite length and R has depth at least two, taking the R-dual yields the isomorphism $(R/\mathfrak{p})^* \xrightarrow{y} (R/\mathfrak{p})^*$. Nakayama's lemma implies $(R/\mathfrak{p})^* = 0$. Hence $\mathfrak{p}$ has positive grade, but this contradicts the fact that $\mathfrak{p}$ is a minimal prime. ■

As an application of the above theorem, we observe that the Lindo–Pande condition does not necessarily ascend along the completion map $R \to \widehat{R}$ for a local ring R.

Corollary 2.4.13. *Let R be a local ring. If $\widehat{R}$ satisfies* (LP)*, then so does R. The converse also holds if* $\operatorname{depth} R \le 1$*, but does not necessarily hold if* $\operatorname{depth} R \ge 2$.

Proof. The descent of (LP) is included in Remark 2.3.5(3), while the ascent for $\operatorname{depth} R \le 1$ is observed from Theorems 2.4.2 and 2.4.5. There exists a factorial local ring R of depth 2 whose completion is not factorial. In fact, Ogoma's famous example [119] of a 2-dimensional factorial local ring without a canonical module is such a ring by [19, Corollaries 3.3.8 and 3.3.19]; see also [118, Example 6.1] and [19, Page 145]. Theorem 2.4.12 implies that this ring R satisfies (LP) but $\widehat{R}$ does not. ■

Chapter 3

Maximal Cohen–Macaulay modules that are not locally free on the punctured spectrum

3.1 Introduction

The contents of this chapter is based on author's work [98] with J. Lyle and R. Takahashi.

Cohen–Macaulay representation theory has been studied widely and deeply for more than four decades. As we stated in Preface, the theorems of Herzog [73] in the 1970s and of Buchweitz, Greuel and Schreyer [21] in the 1980s are recognized as one of the most crucial results in this long history of Cohen–Macaulay representation theory. Both are concerned with Cohen–Macaulay local rings of finite CM-representation type, that is, Cohen–Macaulay local rings possessing finitely many nonisomorphic indecomposable maximal Cohen–Macaulay modules. Buchweitz, Greuel and Schreyer also proved that the local hypersurfaces of countable CM-representation type are precisely the local hypersurfaces of type (A_∞) and (D_∞).

At the beginning of this century, Huneke and Leuschke [82] proved that Cohen–Macaulay local rings of finite CM-representation type have isolated singularities. However, there are ample examples of Cohen–Macaulay local rings not having isolated singularities, including the local hypersurfaces of type (A_∞) and (D_∞) appearing above. Cohen–Macaulay representation theory for non-isolated singularities has been studied by many authors so far; see [4, 22, 77, 88] for instance. It should be remarked that a Cohen–Macaulay local ring with a non-isolated singularity always admits maximal Cohen–Macaulay modules that are *not* locally free on the punctured spectrum. Focusing on these modules, Araya, Iima and Takahashi [3] found out that the local hypersurfaces of type (A_∞) and (D_∞) have *finite* CM_+*-representation type*, that is, there exist only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay modules that are *not* locally free on the punctured spectrum.

In this chapter, we investigate Cohen–Macaulay local rings of finite CM_+-representation type from various viewpoints. Our basic landmark is the following conjecture, which includes the converse of the result of Araya, Iima and Takahashi stated above. We shall give positive results to this conjecture.

Conjecture 3.1.1. Let R be a complete local Gorenstein ring of dimension d not having an isolated singularity. Suppose that R has uncountable algebraically closed coefficient field of characteristic not two. Then the following two conditions are equivalent.

(1) The ring R has finite CM_+-representation type.

(2) There exist a complete regular local ring S and a regular system of parameters $x_0, \dots, x_d$ such that R is isomorphic to

$$S/(x_0^2 + x_2^2 + \cdots + x_d^2) \quad \text{or} \quad S/(x_0^2 x_1 + x_2^2 + \cdots + x_d^2).$$

The implication (2) $\Rightarrow$ (1) holds by [3, Proposition 2.1]. Also, (2) implies that R has countable CM-representation type by [108, Proof of (iii)$\Rightarrow$(i) of Theorem 14.16]. Combining the result of Buchweitz, Greuel and Schreyer, this conjecture says that, under the assumption of the conjecture, finite CM_+-representation type is equivalent to countable CM-representation type.

From now on, we state our main results and the organization of this chapter. Section 3.2 presents some conjectures and questions on finite/countable CM-representation type. Our results are stated in the later sections. In what follows, let R be a Cohen–Macaulay local ring.

In Section 3.3, we consider the (Zariski-)closedness and (Krull) dimension of the singular locus $\operatorname{Sing} R$ of R in connection with the works of Huneke and Leuschke [82, 83]. As we state above, they proved in [82] that if R has finite CM-representation type, then it has an isolated singularity, i.e., $\operatorname{Sing} R$ has dimension at most zero. Also, they showed in [83] that if R is complete or has uncountable residue field, and has countable CM-representation type, then $\operatorname{Sing} R$ has dimension at most one. In relation to these results, we prove the following theorem, whose second assertion extends the result of Huneke and Leuschke [83] from countable CM-representation type to countable CM_+-representation type (i.e., having infinitely but countably many nonisomorphic indecomposable maximal Cohen–Macaulay modules that are not locally free on the punctured spectrum).

Theorem 3.1.2 (Theorem 3.3.2 and Corollary 3.3.3)**.** *Let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring.*

(1) *Suppose that R has finite CM_+-representation type. Then the singular locus* $\operatorname{Sing} R$ *is a finite set. Equivalently, it is a closed subset of* $\operatorname{Spec} R$ *with dimension at most one.*

(2) *Suppose that R has countable CM_+-representation type. Then the set* $\operatorname{Sing} R$ *is at most countable. It has dimension at most one if R is either complete or k is uncountable.*

Furthermore, Huneke and Leuschke [83] proved that if R admits a canonical module and has countable CM-representation type, then the localization $R_\mathfrak{p}$ at each prime ideal $\mathfrak{p}$ of R has at most countable CM-representation type as well. We prove a result on finite CM_+-representation type in the same context.

Theorem 3.1.3 (Theorem 3.3.7). *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring with a canonical module. Suppose that R has finite CM_+-representation type. Then $R_\mathfrak{p}$ has finite CM-representation type for all $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$. In particular, $R_\mathfrak{p}$ has finite CM_+-representation type for all $\mathfrak{p} \in \operatorname{Spec} R$.*

In Section 3.4 we provide various necessary conditions for a given Cohen–Macaulay local ring to have finite CM_+-representation type.

Theorem 3.1.4 (Theorem 3.4.5). *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d > 0$. Let I be an ideal of R such that R/I is maximal Cohen–Macaulay over R. Then R has infinite CM_+-representation type in each of the following cases.*

(1) *The ring R/I has infinite CM_+-representation type.*

(2) *The set $\mathrm{V}(I)$ is contained in $\mathrm{V}(0 : I)$, and either R/I has infinite CM-representation type or $d \geq 2$.*

(3) *The ideal $I + (0 : I)$ is not $\mathfrak{m}$-primary, R/I has infinite CM-representation type, and R/I is either Gorenstein, a domain, or analytically unramified with $d = 1$.*

This theorem may look technical, but it actually gives rise to a lot of restrictions which having finite CM_+-representation type produces, and is used in the later sections. Here we introduce one of the applications of the above theorem.

Theorem 3.1.5 (Theorem 3.4.8). *Let R be a Cohen–Macaulay local ring of dimension $d > 0$. Let I be an ideal of R with $\mathrm{V}(I) \subseteq \mathrm{V}(0 : I)$ such that R/I is maximal Cohen–Macaulay over R. Suppose that R has finite CM_+-representation type. Then one must have $d = 1$. If $I^n = 0$ for some integer $n > 0$, then $\mathsf{CM}(R)$ has dimension at most $n - 1$ in the sense of [43]. If R is Gorenstein, then R is a hypersurface and $\mathsf{D_{sg}}(R)$ has dimension at most $n - 1$ in the sense of [125].*

There are folklore conjectures that a Gorenstein local ring of countable CM-representation type is a hypersurface, and that, for a Cohen–Macaulay local ring R of countable CM-representation type, $\mathsf{CM}(R)$ has dimension at most one. The above theorem gives partial answers to the variants of these folklore conjectures for finite CM_+-representation type.

In Section 3.5, we give a complete answer to Conjecture 3.1.1 in dimension one without the assumption of the conjecture on the coefficient field.

Theorem 3.1.6 (Theorem 3.5.1). *Let R be a homomorphic image of a regular local ring. Suppose that R does not have an isolated singularity but is Gorenstein. If $\dim R = 1$, the following are equivalent.*

(1) *The ring R has finite CM_+-representation type.*

(2) *There exist a regular local ring S and a regular system of parameters x, y such that R is isomorphic to $S/(x^2)$ or $S/(x^2y)$.*

When either of these two conditions holds, the ring R has countable CM-representation type.

In Section 3.6, we explore the higher-dimensional case, that is, we try to understand the Cohen–Macaulay local rings R of finite CM_+-representation type in the case where $\dim R \geq 2$. We prove the following two results in this section.

Theorem 3.1.7 (Corollary 3.6.8). *Let R be a complete local hypersurface of dimension $d \geq 2$ which is not an integral domain. Suppose that R has finite CM_+-representation type. Then one has $d = 2$, and there exist a regular local ring S and elements $x, y \in S$ with $R \cong S/(xy)$ such that $S/(x)$ and $S/(y)$ have finite CM-representation type and $S/(x, y)$ is an integral domain of dimension 1.*

Theorem 3.1.8 (Corollaries 3.6.10 and 3.6.11). *Let R be a 2-dimensional non-normal Cohen–Macaulay complete local domain. Suppose that R has finite CM_+-representation type. Then the integral closure $\overline{R}$ of R has finite CM-representation type. If R is Gorenstein, then R is a hypersurface.*

The former theorem gives a strong restriction of the structure of a hypersurface of finite CM_+-representation type which is not an integral domain. The latter theorem supports the conjecture that a Gorenstein local ring of finite CM_+-representation type is a hypersurface. Note that, under the assumption of the theorem plus the assumption that R is equicharacteristic zero, the integral closure $\overline{R}$ is a quotient surface singularity by the theorem of Auslander [6] and Esnault [53].

3.2 Conjectures and questions

In this section, we present several conjectures and questions which we deal with in later sections. First of all, let us give several definitions of representation types, including that of finite CM_+-representation type, which is the main subject of this chapter.

Definition 3.2.1. Let R be a Cohen–Macaulay ring. Set

$$\mathsf{CM}_+(R) := \mathsf{CM}(R) \setminus \mathsf{CM}_0(R).^1$$

For each $\mathsf{X} \in \{\mathsf{CM}, \mathsf{CM}_0, \mathsf{CM}_+\}$ we say that R has *finite* (resp. *countable*) X-representation type if there exist only finitely (resp. countably) many isomorphism classes of indecomposable modules in $\mathsf{X}(R)$. We say that R has *infinite* (resp. *uncountable*) X-representation type if R does not have finite (resp. countable) X-representation type. Also, R is said to have *bounded* X-representation

[1] The index 0 (resp. +) in $\mathsf{CM}_0(R)$ (resp. $\mathsf{CM}_+(R)$) means that it consists of modules whose nonfree loci have zero (resp. positive) dimension.

type if there exists an upper bound of the multiplicities of indecomposable modules in $\mathsf{X}(R)$, and said to have *unbounded* X-representation type if R does not have bounded X-representation type.

Let R be a complete local hypersurface with uncountable algebraically closed coefficient field of characteristic not two. Buchweitz, Greuel and Schreyer [21, Theorem B] (see also [108, Theorem 14.16]) prove that R has countable CM-representation type if and only if it is either an (A_∞)-singularity or a (D_∞)-singularity. Moreover, when this is the case, they give a complete classification of the indecomposable maximal Cohen–Macaulay R-modules. Using this result, Araya, Iima and Takahashi [3, Theorem 1.1 and Corollary 1.3] prove the following theorem (see [43, Proposition 3.5(3)]), which provides examples of a Cohen–Macaulay local ring of finite CM_+-representation type.

Theorem 3.2.2 (Araya–Iima–Takahashi)**.** *Let R be a complete local hypersurface with uncountable algebraically closed coefficient field of characteristic not two. If R has countable* CM*-representation type, then the following statements hold.*

(1) *The ring R has finite* CM_+*-representation type.*

(2) *There is an inequality* $\dim \mathsf{CM}(R) \le 1$.

By definition, there is a strong connection between finite CM_+-representation type and finite CM-representation type. The first assertion of Theorem 3.2.2 suggests to us that finite CM_+-representation type should also be closely related to countable CM-representation type. Several conjectures have been presented so far concerning finite/countable CM-representation type, and we set the following proposal.

Proposal 3.2.3. One should consider the conjectures on finite/countable CM-representation type for finite CM_+-representation type.

There has been a folklore conjecture on countable CM-representation type probably since the 1980s. Recently, this conjecture has been studied by Stone [134].

Conjecture 3.2.4. A Gorenstein local ring R of countable CM-representation type is a hypersurface.

This conjecture holds true if R has finite CM-representation type; see [149, Theorem (8.15)]. Also, the conjecture holds if R is a complete intersection with algebraically closed uncountable residue field; see [12, Existence Theorem 7.8]. The following example shows that the assumption in the conjecture that R is Gorenstein is necessary.

Example 3.2.5. Let $S = \mathbb{C}[\![x,y,z]\!]/(xy)$. Then S is an (A_∞)-singularity of dimension 2, and has countable CM-representation type by [21, Theorem B]. Let R be the second Veronese subring of S, that is, $R = \mathbb{C}[\![x^2, xy, xz, y^2, yz, z^2]\!] \subseteq S$. Then R is a Cohen–Macaulay non-Gorenstein local ring of dimension 2. We claim that R has countable CM-representation type. Indeed, let $N_1, N_2, \dots$ be the non-isomorphic indecomposable maximal Cohen–Macaulay S-modules. Let M be an indecomposable maximal Cohen–Macaulay R-module. Then $N = \mathrm{Hom}_R(S, M)$ is a maximal Cohen–Macaulay S-module, and one can write $N \cong N_{a_1}^{\oplus b_1} \oplus \cdots \oplus N_{a_t}^{\oplus b_t}$. Since R is a direct summand of S, the module M is a direct summand of N, and hence it is a direct summand of N_{a_i} for some i. The claim follows from this.

Combining Conjecture 3.2.4 with Proposal 3.2.3 gives rise to the following question.

Question 3.2.6. Let R be a Gorenstein local ring which is not an isolated singularity. Suppose that R has finite CM_+-representation type. Then is R a hypersurface?

Here, the assumption that R is not an isolated singularity is necessary. Indeed, if R is an isolated singularity, then $\# \operatorname{ind} \mathsf{CM}_+(R) = 0 < \infty$. Obviously, if Conjecture 3.1.1 holds, then this question is affirmative under the assumption of the conjecture. We shall give answers to Question 3.2.6 in Sections 3.4 and 3.6.

Theorem 3.2.2(2) leads us to the following conjecture.

Conjecture 3.2.7. Let R be a Cohen–Macaulay local ring R of countable CM-representation type. Then there is an inequality $\dim \mathsf{CM}(R) \leq 1$.

This conjecture holds true if R has finite CM-representation type; see [43, Proposition 3.7(1)]. Let R be a Gorenstein local ring. Then the stable category $\underline{\mathsf{CM}}(R)$ of $\mathsf{CM}(R)$ is a triangulated category, and one can consider the *(Rouquier) dimension* $\dim \underline{\mathsf{CM}}(R)$ of $\underline{\mathsf{CM}}(R)$; we refer the reader to [125] for the details. One has $\dim \underline{\mathsf{CM}}(R) \leq \dim \mathsf{CM}(R)$ with equality if R is a hypersurface; see [43, Proposition 3.5]. There seems to be a folklore conjecture asserting that every (noncommutative) selfinjective algebra Λ of tame representation type satisfies the inequality $\dim(\underline{\mathsf{mod}}\, \Lambda) \leq 1$. So Conjecture 3.2.7 is thought of as a Cohen–Macaulay version of this folklore conjecture. Combining Conjecture 3.2.7 with Proposal 3.2.3 leads us to the following question.

Question 3.2.8. Let R be a Cohen–Macaulay local ring of finite CM_+-representation type. Then does one have $\dim \mathsf{CM}(R) \leq 1$?

If Conjecture 3.1.1 holds true, then Question 3.2.8 is affirmative for a Gorenstein local ring satisfying the assumption of the conjecture by [21, Theorem B] and Theorem 3.2.2. We shall give other answers to this question in Section 3.4.

Huneke and Leuschke ([83, Theorem 1.3]) prove the following theorem, which solves a conjecture of Schreyer [133, Conjecture 7.2.3] presented in the 1980s.

Theorem 3.2.9 (Huneke–Leuschke)**.** *Let $(R, \mathfrak{m}, k)$ be an excellent Cohen–Macaulay local ring. Assume that R is complete or k is uncountable. If R has countable CM-representation type, then $\dim \operatorname{Sing} R \leq 1$.*

Indeed, the assumption that R is excellent is unnecessary; see [137, Theorem 2.4]. This result naturally makes us have the following question.

Question 3.2.10. Let R be a Cohen–Macaulay local ring. Suppose that R has finite CM_+-representation type. Then does $\operatorname{Sing} R$ have dimension at most one?

We shall give a complete answer to this question in the next Section 3.3. In fact, we can even prove a stronger statement.

3.3 The closedness and dimension of the singular locus

In this section, we discuss the structure of the singular locus of a Cohen–Macaulay local ring of finite CM_+-representation type. First, we consider what the finiteness of the singular locus means.

Lemma 3.3.1. *Let R be a local ring with maximal ideal $\mathfrak{m}$. The following are equivalent.*

(1) $\operatorname{Sing} R$ *is a finite set.*

(2) $\operatorname{Sing} R$ *is a closed subset of* $\operatorname{Spec} R$ *in the Zariski topology, and has dimension at most one.*

Proof. (2)⇒(1): We find an ideal I of R such that $\operatorname{Sing} R = \mathrm{V}(I)$. As $\operatorname{Sing} R$ has dimension at most one, so does the local ring R/I. Hence $\operatorname{Spec} R/I = \operatorname{Min} R/I \cup \{\mathfrak{m}/I\}$, and this is a finite set.

(1)⇒(2): Write $\operatorname{Sing} R = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. As $\operatorname{Sing} R$ is specialization-closed, it coincides with the finite union $\mathrm{V}(\mathfrak{p}_1) \cup \cdots \cup \mathrm{V}(\mathfrak{p}_n)$ of closed subsets of $\operatorname{Spec} R$. Hence $\operatorname{Sing} R$ is closed.

To show the other assertion, we claim (or recall) that a local ring R of dimension at least two possesses infinitely many prime ideals of height one. Indeed, for any $x \in \mathfrak{m}$ we have $\operatorname{ht}(x) \le 1$ by Krull's principal ideal theorem, that is, (x) is contained in some prime ideal $\mathfrak{p}$ with $\operatorname{ht}\mathfrak{p} \le 1$. This argument shows that $\mathfrak{m} = \bigcup_{\mathfrak{p} \in \operatorname{Spec} R,\, \operatorname{ht}\mathfrak{p} \le 1} \mathfrak{p}$. Now suppose that there exist only finitely many prime ideals of R having height one. Then, since the number of the minimal primes is finite, so is the number of prime ideals of height at most one. Therefore the above union is finite, and by prime avoidance $\mathfrak{m}$ is contained in some $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{ht}\mathfrak{p} \le 1$. This implies $\dim R \le 1$, which is a contradiction. Thus the claim follows.

Now, assume that $\operatorname{Sing} R$ has dimension at least 2. Then $\dim R/\mathfrak{p} \ge 2$ for some $\mathfrak{p} \in \operatorname{Sing} R$. The above claim shows that the ring $R/\mathfrak{p}$ has infinitely many prime ideals of height one, which have the form $\mathfrak{q}/\mathfrak{p}$ with $\mathfrak{q} \in \mathrm{V}(\mathfrak{p})$. Then $\mathfrak{q}$ is also in $\operatorname{Sing} R$, and hence $\operatorname{Sing} R$ contains infinitely many prime ideals. This contradiction shows that the dimension of $\operatorname{Sing} R$ is at most 1. ■

The following theorem clarifies a close relationship between finite/countable CM_+-representation type and finiteness/countablity of the singular locus.

Theorem 3.3.2. *If R is a Cohen–Macaulay local ring of finite (resp. countable) CM_+-representation type, then* $\operatorname{Sing} R$ *is a finite (resp. countable) set.*

Proof. First, let us consider the case where R has finite CM_+-representation type. Write $\operatorname{ind}\mathsf{CM}_+(R) = \{G_1, \dots, G_t\}$, and pick $\mathfrak{p} \in \operatorname{Sing} R \setminus \{\mathfrak{m}\}$. Set $C = \Omega_R^d(R/\mathfrak{p})$. We claim that $\mathfrak{p} = \operatorname{Ann}_R \operatorname{Tor}_1^R(C, C)$. Indeed, $\operatorname{Tor}_1^R(C, C)$ is isomorphic to $T := \operatorname{Tor}_{1+2d}^R(R/\mathfrak{p}, R/\mathfrak{p})$, which is killed by $\mathfrak{p}$. Hence $\mathfrak{p}$ is contained in the annihilator. Also, $T_\mathfrak{p}$ is isomorphic to $\operatorname{Tor}_{1+2d}^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \kappa(\mathfrak{p}))$, which does not vanish as $\mathfrak{p}$ belongs to the singular locus. Hence $\mathfrak{p}$ is in the support of T, and contains the annihilator. Now the claim follows.

Note that $C_\mathfrak{p}$ is stably isomorphic to $\Omega_{R_\mathfrak{p}}^d(\kappa(\mathfrak{p}))$, which is not $R_\mathfrak{p}$-free since $R_\mathfrak{p}$ is singular. This means that C belongs to $\mathsf{CM}_+(R)$, and we get an isomorphism $C \cong G_{l_1}^{\oplus a_1} \oplus \cdots \oplus G_{l_s}^{\oplus a_s} \oplus H$ with $s \ge 1$ and $1 \le l_1 < \cdots < l_s \le t$ and $a_1, \dots, a_s \ge 1$ and $H \in \mathsf{CM}_0(R)$. It is easy to see that

$$\mathfrak{p} = \left(\bigcap_{1 \le i,j \le s} \operatorname{Ann}_R \operatorname{Tor}_1^R(G_{l_i}, G_{l_j})\right) \cap \operatorname{Ann}_R \operatorname{Tor}_1^R(H, M)$$

for some R-module M. Since a prime ideal is irreducible in general, $\mathfrak{p}$ coincides with one of the annihilators in the right-hand side. The module H is locally free on the punctured spectrum, and $\operatorname{Ann}_R \operatorname{Tor}_1^R(H, M)$ contains a power of $\mathfrak{m}$. As $\mathfrak{p}$ is a nonmaximal prime ideal, it cannot coincide with $\operatorname{Ann}_R \operatorname{Tor}_1^R(H, M)$. We thus have $\mathfrak{p} = \operatorname{Ann}_R \operatorname{Tor}_1^R(G_{l_p}, G_{l_q})$ for some p, q. This shows that we have only finitely many such prime ideals $\mathfrak{p}$. Consequently, $\operatorname{Sing} R \setminus \{\mathfrak{m}\}$ is a finite set, and so is $\operatorname{Sing} R$.

We can analogously deal with the case where R has countable CM_+-representation type. In this case, we can write $\operatorname{ind} \mathsf{CM}_+(R) = \{G_1, G_2, G_3, \dots\}$, and for each $\mathfrak{p} \in \operatorname{Sing} R \setminus \{\mathfrak{m}\}$ there exist p, q such that $\mathfrak{p} = \operatorname{Ann}_R \operatorname{Tor}_1^R(G_{l_p}, G_{l_q})$. ■

Theorem 3.3.2 yields the following corollary, which gives a complete answer to Question 3.2.10. We should remark that the second assertion of the corollary highly refines Theorem 3.2.9 due to Huneke and Leuschke.

Corollary 3.3.3. *Let R be a Cohen–Macaulay local ring.*

(1) *If R has finite CM_+-representation type, then $\operatorname{Sing} R$ is closed and has dimension at most one.*

(2) *Suppose that R has countable CM_+-representation type.*

 (a) *If k is uncountable, then $\operatorname{Sing} R$ has dimension at most one.*

 (b) *If R is complete, then $\operatorname{Sing} R$ is closed and has dimension at most one.*

Proof. (1) The assertion follows from Theorem 3.3.2 and Lemma 3.3.1.

(2) Theorem 3.3.2 implies that $\operatorname{Sing} R$ is a countable set. Note that $\operatorname{Sing} R$ is specialization-closed. If R is complete or k is uncountable, then we can apply [137, Lemma 2.2] to deduce that $\dim R/\mathfrak{p} \leq 1$ for all $\mathfrak{p} \in \operatorname{Sing} R$. In case R is complete, $\operatorname{Sing} R$ is closed as well since R is excellent. ■

Next we investigate the relationship of finite CM_+-representation type with localization of the base ring at a prime ideal. For this, we establish two lemmas.

Lemma 3.3.4. *Let R be a local ring. Let M, N, C be R-modules. Suppose that the endomorphism ring $\operatorname{End}_R(C)$ is isomorphic to R, and that C is not a direct summand of M or N. If $M \oplus C^{\oplus a} \cong N \oplus C^{\oplus b}$ for some $a, b \geq 0$, then $M \cong N$ and $a = b$.*

Proof. Without loss of generality, we can assume $a \geq b$. Taking the completions, we get isomorphisms $\widehat{M} \oplus \widehat{C}^{\oplus a} \cong \widehat{N} \oplus \widehat{C}^{\oplus b}$ and $\operatorname{End}_{\widehat{R}}(\widehat{C}) \cong \widehat{R}$. Since $\widehat{R}$ is a local ring, $\widehat{C}$ is an indecomposable $\widehat{R}$-module. Write $\widehat{M} = X \oplus \widehat{C}^{\oplus c}$ and $\widehat{N} = Y \oplus \widehat{C}^{\oplus d}$ with $c, d \geq 0$ integers and X, Y not containing $\widehat{C}$ as a direct summand. Then $X \oplus \widehat{C}^{\oplus(c+a)} \cong Y \oplus \widehat{C}^{\oplus(d+b)}$. Since $\widehat{R}$ is henselian, we can apply the Krull-Schmidt theorem to deduce that $X \cong Y$ and $c + a = d + b$. Hence $d = c + (a - b)$, and we obtain

$$\widehat{N} = Y \oplus \widehat{C}^{\oplus d} \cong X \oplus \widehat{C}^{\oplus(c+(a-b))} = \widehat{M} \oplus \widehat{C}^{\oplus(a-b)} \cong \widehat{L},$$

where $L := M \oplus C^{\oplus(a-b)}$. It follows from [50, Exercise 7.5] that N is isomorphic to L. As C is not a direct summand of N, we must have $a = b$, and therefore $M = L \cong N$. ■

Lemma 3.3.5. *Let R be a local ring. Let M, N be R-modules.*

(1) *If R is a direct summand of $M \oplus N$, then R is a direct summand of either M or N.*

(2) *Assume that R, M, N are maximal Cohen–Macaulay, and that R admits a canonical module ω. If ω is a direct summand of $M \oplus N$, then ω is a direct summand of either M or N.*

Proof. (1) There exists a surjective homomorphism $(f, g) : M \oplus N \to R$. Then we find elements $x \in M$ and $y \in N$ such that $f(x) + g(y) = 1$ in R. Since R is a local ring, either $f(x)$ or $g(y)$ is a unit of R. If $f(x)$ is a unit, then $f : M \to R$ is surjective, which implies that R is a direct summand of M. Similarly, if $g(y)$ is a unit, then R is a direct summand of N.

(2) Write $M \oplus N = \omega \oplus L$ in $\mathsf{mod}\, R$. Applying the canonical dual functor $(-)^\dagger = \mathrm{Hom}_R(-, \omega)$, we get an isomorphism $M^\dagger \oplus N^\dagger \cong R \oplus L^\dagger$. It follows from (1) that R is a direct summand of either $M^\dagger$ or $N^\dagger$. Hence ω is a direct summand of either $M^{\dagger\dagger} = M$ or $N^{\dagger\dagger} = N$. ■

Remark 3.3.6. Recall that an R-module C is *semidualizing* if the natural map $R \to \mathrm{End}_R(C)$ is an isomorphism and $\mathrm{Ext}^i_R(C, C) = 0$ for all $i > 0$. Lemma 3.3.5(2) can be extended to semidualizing modules, that is, the following statement holds: Let R be a Cohen–Macaulay local ring and M, N maximal Cohen–Macaulay R-modules. If a semidualizing R-module C is a direct summand of $M \oplus N$, then C is a direct summand of either M or N. This is shown just by replacing ω with C in the proof of Lemma 3.3.5(2).

Now we can prove the following theorem, which says that finite CM_+-representation type implies finite CM-representation type on the punctured spectrum. This especially shows that finite CM_+-representation type localizes, which should be compared with the result of Huneke and Leuschke [83, Theorem 2.1] asserting that countable CM-representation type localizes under the same assumption as in this theorem. This is also connected with the conjecture that a Cohen–Macaulay local ring with an isolated singularity having countable CM-representation type has finite CM-representation type [83, Page 3006].

Theorem 3.3.7. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring with a canonical module ω. Suppose that R has finite CM_+-representation type. Then $R_\mathfrak{p}$ has finite CM-representation type for all $\mathfrak{p} \in \mathrm{Spec}\, R \setminus \{\mathfrak{m}\}$.*

Proof. Assume that there exists a prime ideal $\mathfrak{p} \neq \mathfrak{m}$ such that $R_\mathfrak{p}$ has infinite CM-representation type. Then the set $\mathsf{ind}\,\mathsf{CM}(R_\mathfrak{p}) \setminus \{\omega_\mathfrak{p}\}$ is infinite, and we can take an infinite subset $\mathcal{N} = \{N_1, N_2, N_3, \dots\}$.

Fix a module $N \in \mathcal{N}$. Then we can choose an R-module L such that $N \cong L_\mathfrak{p}$. Take a *maximal Cohen–Macaulay approximation* of L over R, that is, a short exact sequence

$$\sigma : 0 \to Y \to X \to L \to 0$$

of R-modules such that X is maximal Cohen–Macaulay and Y has finite injective dimension; see [7, Theorem 1.1]. Localization gives an exact sequence $\sigma_\mathfrak{p} : 0 \to Y_\mathfrak{p} \to X_\mathfrak{p} \to N \to 0$. As N is maximal Cohen–Macaulay, $Y_\mathfrak{p}$ is a maximal Cohen–Macaulay $R_\mathfrak{p}$-module of finite injective dimension. It follows from [19, Exercise 3.3.28(a)] that $Y_\mathfrak{p} \cong \omega_\mathfrak{p}^{\oplus n}$ for some $n \geq 0$. The exact sequence $\sigma_\mathfrak{p}$ splits, and we get an isomorphism $X_\mathfrak{p} \cong N \oplus \omega_\mathfrak{p}^{\oplus n}$. Note that $\omega_\mathfrak{p}$ is an indecomposable $R_\mathfrak{p}$-module.

Let $X = X_1 \oplus \cdots \oplus X_m$ be a decomposition of X into indecomposable R-modules. Then there is an isomorphism $(X_1)_\mathfrak{p} \oplus \cdots \oplus (X_m)_\mathfrak{p} \cong N \oplus \omega_\mathfrak{p}^{\oplus n}$. For each i write $(X_i)_\mathfrak{p} = Z_i \oplus \omega_\mathfrak{p}^{\oplus l_i}$ with $l_i \geq 0$ an integer and Z_i not containing $\omega_\mathfrak{p}$ as a direct summand; then Z_i is a maximal Cohen–Macaulay $R_\mathfrak{p}$-module. We get an isomorphism

$$Z_1 \oplus \cdots \oplus Z_m \oplus \omega_\mathfrak{p}^{\oplus(l_1+\cdots+l_m)} \cong N \oplus \omega_\mathfrak{p}^{\oplus n}.$$

The module $Z_1 \oplus \cdots \oplus Z_m$ does not contain $\omega_\mathfrak{p}$ as a direct summand by Lemma 3.3.5(2), while N is an indecomposable $R_\mathfrak{p}$-module with $N \ncong \omega_\mathfrak{p}$. Using Lemma 3.3.4, we see that $Z_1 \oplus \cdots \oplus Z_m \cong N$ and $l_1 + \cdots + l_m = n$. We may assume that $Z_1 \cong N$ and $Z_2 = \cdots = Z_m = 0$. It holds that $(X_1)_\mathfrak{p} \cong N \oplus \omega_\mathfrak{p}^{\oplus l_1}$.

Suppose that $(X_1)_\mathfrak{p}$ is $R_\mathfrak{p}$-free. Then so are N and $\omega_\mathfrak{p}$, and we have $N \cong R_\mathfrak{p} \cong \omega_\mathfrak{p}$, which contradicts the choice of N. Hence $(X_1)_\mathfrak{p}$ is not $R_\mathfrak{p}$-free, which implies that $X_1 \in \mathsf{CM}_+(R)$.

Thus we have shown that for each integer $i \geq 1$ there exist an integer $n_i \geq 0$ and a module $C_i \in \operatorname{ind} \mathsf{CM}_+(R)$ such that $(C_i)_\mathfrak{p} \cong N_i \oplus \omega_\mathfrak{p}^{\oplus n_i}$. Assume that $C_i \cong C_j$ for some $i \neq j$. Then $N_i \oplus \omega_\mathfrak{p}^{\oplus n_i} \cong N_j \oplus \omega_\mathfrak{p}^{\oplus n_j}$, and by Lemma 3.3.4 we see that $N_i \cong N_j$ (and $n_i = n_j$), contrary to the choice of $\mathcal{N}$. Hence $C_i \ncong C_j$ for all $i \neq j$, and we conclude that R has infinite CM_+-representation type. This contradiction completes the proof of the theorem. ∎

Remark 3.3.8. In Corollary 3.3.3(1) we proved that the singular locus of a Cohen–Macaulay local ring of finite CM_+-representation type has dimension at most one. As an application of Theorem 3.3.7, we can get another proof of this statement under the assumption that R admits a canonical module.

Let R be a d-dimensional Cohen–Macaulay local ring with a canonical module, and suppose that R has finite CM_+-representation type. Then $R_\mathfrak{p}$ has finite CM-representation type for all nonmaximal prime ideals $\mathfrak{p}$ of R by Theorem 3.3.7. In particular, $R_\mathfrak{p}$ has an isolated singularity for all such $\mathfrak{p}$ by [82, Corollary 2]. This implies that $R_\mathfrak{q}$ is a regular local ring in codimension $d-2$, and therefore $\dim \operatorname{Sing} R \leq 1$.

3.4 Necessary conditions for finite CM_+-representation type

In this section, we explore necessary conditions for a Cohen–Macaulay local ring to have finite CM_+-representation type. For this purpose we begin with stating and showing a couple of lemmas.

Lemma 3.4.1. *Let R be a local ring.*

(1) *The subcategory of* $\operatorname{mod} R$ *consisting of periodic modules is closed under finite direct sums: if the R-modules $M_1, \dots, M_n$ are periodic, then so is $M_1 \oplus \cdots \oplus M_n$.*

(2) *Let $0 \to M_1 \to \cdots \to M_n \to 0$ be an exact sequence in* $\operatorname{mod} R$. *Let $r \geq 0$ and $1 \leq t \leq n$ be integers. If $\operatorname{cx}_R(M_i) \leq r$ for all $1 \leq i \leq n$ with $i \neq t$, then $\operatorname{cx}_R(M_t) \leq r$.*

Proof. (1) We have only to show the assertion for $n = 2$. Let M_1, N_2 be periodic R-modules, so $\Omega^a M_1 \cong M_1$ and $\Omega^b M_2 \cong M_2$ for some $a, b > 0$. Let l be the least common multiple of a, b. Then $\Omega^l M_1 \cong M_1$ and $\Omega^l M_2 \cong M_2$. Hence $\Omega^l(M_1 \oplus M_2) \cong \Omega^l M_1 \oplus \Omega^l M_2 \cong M_1 \oplus M_2$.

(2) It suffices to show the statement when $n = 3$. Suppose that M_2, M_3 have complexity at most r. Then we find $p, q \in \mathbb{R}_{>0}$ such that $\beta_j^R(M_2) \le pj^{r-1}$ and $\beta_j^R(M_3) \le qj^{r-1}$ for $j \gg 0$. The induced exact sequence $\mathrm{Tor}_{j+1}^R(M_3, k) \to \mathrm{Tor}_j^R(M_1, k) \to \mathrm{Tor}_j^R(M_2, k)$ shows that $\beta_j^R(M_1) \le \beta_j^R(M_2) + \beta_{j+1}^R(M_3) \le (p + qr)j^{r-1}$ for $j \gg 0$. Therefore we obtain $\mathrm{cx}_R(M_3) \le r$. The other cases are handled similarly. ∎

Let R be a local ring. A subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is called *resolving* if $\mathcal{X}$ contains R and closed under

- direct summands: if $X \in \mathcal{X}$ and $M \in \operatorname{mod} R$ is a direct summand of X, then $M \in \mathcal{X}$;
- extensions: for an exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{mod} R$, if $L, N \in \mathcal{X}$, then $M \in \mathcal{X}$;
- syzygies: if $X \in \mathcal{X}$, then $\Omega_R X \in \mathcal{X}$.

Typical examples of resolving subcategories of $\operatorname{mod} R$ are $\mathsf{CM}(R)$ and $\mathsf{CM}_0(R)$.

The subcategory $\mathsf{CM}_+(R)$ of $\operatorname{mod} R$ is stable under syzygies.

Lemma 3.4.2. *Let R be a local ring. Let $0 \to N \to F \to M \to 0$ be an exact sequence in $\operatorname{mod} R$ such that F is free and M is maximal Cohen–Macaulay. Then M belongs to $\mathsf{CM}_+(R)$ if and only if so does N.*

Proof. Note that all the modules N, F, M are maximal Cohen–Macaulay. Hence the assertion is equivalent to saying that M belongs to $\mathsf{CM}_0(R)$ if and only if so does N. The "if" part follows from the fact that $\mathsf{CM}_0(R)$ is resolving. To show the "only if" part, assume that N is in $\mathsf{CM}_0(R)$. Let $\mathfrak{p}$ be a nonmaximal prime ideal of R. Then $N_\mathfrak{p}$ is $R_\mathfrak{p}$-free, and we see that the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ has projective dimension at most 1. Note that $M_\mathfrak{p}$ is maximal Cohen–Macaulay over $R_\mathfrak{p}$. The Auslander–Buchsbaum formula implies that $M_\mathfrak{p}$ is free. Hence M is in $\mathsf{CM}_0(R)$. ∎

We state some containments among indecomposable maximal Cohen–Macaulay modules over Cohen–Macaulay local rings, one of which is a homomorphic image of the other.

Proposition 3.4.3. *Let R be a Cohen–Macaulay local ring of dimension d. Let I be an ideal of R such that R/I is a maximal Cohen–Macaulay R-module. Then the following statements hold.*

(1) $\operatorname{ind} \mathsf{CM}(R/I)$ *is contained in* $\operatorname{ind} \mathsf{CM}(R)$.

(2) $\operatorname{ind} \mathsf{CM}_+(R/I)$ *is contained in* $\operatorname{ind} \mathsf{CM}_+(R)$.

(3) $\operatorname{ind} \mathsf{CM}(R/I)$ *is contained in* $\operatorname{ind} \mathsf{CM}_+(R)$, *if* $\mathrm{V}(I) \subseteq \mathrm{V}(0 : I)$.

Proof. Let M be an indecomposable maximal Cohen–Macaulay R/I-module. The definition of indecomposability says $M \ne 0$. The equalities $\operatorname{depth} M = \dim R/I = \dim R$ imply M is a maximal Cohen–Macaulay R-module. It is directly checked that M is indecomposable as an R-module. Now (1) follows.

Let $\mathfrak{p}$ be a prime ideal of R such that $M_\mathfrak{p} \cong (R_\mathfrak{p})^{\oplus n}$ for some $n \ge 0$. If $n = 0$, then $M_\mathfrak{p} = 0$. If $n > 0$, then $IR_\mathfrak{p} = 0$ since $IM = 0$, and hence $M_\mathfrak{p} \cong R_\mathfrak{p}^{\oplus n} = (R/I)_\mathfrak{p}^{\oplus n}$.

Let us consider the case where M is in $\mathsf{CM}_+(R/I)$. Then there is a prime ideal $\mathfrak{q}$ of R with $I \subseteq \mathfrak{q} \neq \mathfrak{m}$ such that $M_\mathfrak{q}$ is not $(R/I)_\mathfrak{q}$-free. Letting $\mathfrak{p} := \mathfrak{q}$ in the above argument, we observe that $M_\mathfrak{q}$ is not $R_\mathfrak{q}$-free (note that the zero module is free). Thus M is in $\mathsf{CM}_+(R)$, and (2) follows.

Next we consider the case where M is in $\mathsf{CM}_0(R)$. As $\dim M = \dim R/I = d > 0$, there is a nonmaximal prime ideal $\mathfrak{r}$ of R such that $M_\mathfrak{r} \neq 0$. Letting $\mathfrak{p} := \mathfrak{r}$ in the above argument, we have $IR_\mathfrak{r} = 0$. Hence $\mathfrak{r}$ is not in the support of the R-module I, which is equivalent to saying that $\mathfrak{r}$ does not contain $(0 : I)$. On the other hand, $\mathfrak{r}$ is in the support of the R-module M, which implies that $\mathfrak{r}$ contains I. Thus $\mathrm{V}(I)$ is not contained in $\mathrm{V}(0 : I)$. We now observe that (3) holds. ■

The lemma below says finite CM-representation type is equivalent to finite CM_0-representation type.

Lemma 3.4.4. *Let R be a Cohen–Macaulay local ring. If R has infinite CM-representation type, then R has infinite CM_0-representation type.*

Proof. Suppose that R has finite CM_0-representation type. Then by [43, Corollary 1.2] it is an isolated singularity. Hence $\mathsf{CM}(R) = \mathsf{CM}_0(R)$, and we have $\mathsf{ind}\,\mathsf{CM}(R/I) = \mathsf{ind}\,\mathsf{CM}_0(R/I)$, which is a finite set. This contradicts the assumption that R has infinite CM-representation type. ■

Now we can prove the first main result of this section, which gives various necessary conditions for a Cohen–Macaulay local ring to have finite CM_+-representation type.

Theorem 3.4.5. *Let R be a Cohen–Macaulay local ring of dimension $d > 0$. Let I be an ideal of R, and assume that R/I is a maximal Cohen–Macaulay R-module. Then R has infinite CM_+-representation type in each of the following cases.*

(1) *R/I has infinite CM_+-representation type.*

(2) $\mathrm{V}(I) \subseteq \mathrm{V}(0 : I)$ *and*

  (a) *R/I has infinite CM-representation type, or*

  (b) $d \geq 2$.

(3) $\mathrm{ht}(I + (0 : I)) < d$, *$R/I$ has infinite CM-representation type, and*

  (a) *R/I is a Gorenstein ring, or*

  (b) *R/I is a domain, or*

  (c) *$d = 1$ and R/I is analytically unramified, or*

  (d) *$d = 1$, k is infinite, and R/I is equicharacteristic and reduced.*

Proof. (1)&(2a) These assertions immediately follow from (2) and (3) of Proposition 3.4.3, respectively.

(2b) In view of (2a), we may assume that R/I has finite CM-representation type. It follows from [82, Corollary 2] that R/I is an isolated singularity. As $d \geq 2$, the ring R/I is a (normal) domain. Hence $\mathfrak{p} := I$ is a prime ideal of R. As $\dim R/\mathfrak{p} = d$, the prime ideal $\mathfrak{p}$ is minimal.

The assumption $\mathrm{V}(\mathfrak{p}) \subseteq \mathrm{V}(0 : \mathfrak{p})$ implies $(0 :_R \mathfrak{p}) \subseteq \mathfrak{p}$. Localizing this inclusion at $\mathfrak{p}$, we get an inclusion $(0 :_{R_\mathfrak{p}} \mathfrak{p}R_\mathfrak{p}) \subseteq \mathfrak{p}R_\mathfrak{p}$, which particularly says that $R_\mathfrak{p}$ is not a field. Therefore $\mathfrak{p}$ belongs to $\operatorname{Sing} R$.

Suppose that R has finite CM_+-representation type. Then Corollary 3.3.3(1) implies that $\operatorname{Sing} R$ has dimension at most one. In particular, we obtain $d = \dim R/\mathfrak{p} \leq 1$, which is a contradiction. Consequently, R has infinite CM_+-representation type.

(3) We find a nonmaximal prime ideal $\mathfrak{p}$ of R that contains the ideal $I + (0 : I)$. Then, as $\mathfrak{p}$ contains I, the prime ideal $\mathfrak{p}/I$ of R/I is defined, which is not maximal. Also, since $\mathfrak{p}$ contains $(0 : I)$ as well, we see that $IR_\mathfrak{p}$ is a nonzero proper ideal of $R_\mathfrak{p}$.

We establish several claims.

Claim 3. Let $M \in \operatorname{ind} \mathsf{CM}_0(R/I)$ with $M_\mathfrak{p} \neq 0$. Then $M \in \operatorname{ind} \mathsf{CM}_+(R)$.

Proof of Claim. Proposition 3.4.3(1) implies $M \in \operatorname{ind} \mathsf{CM}(R)$. There exists an integer $n \geq 0$ such that

$$M_\mathfrak{p} = M_{\mathfrak{p}/I} \cong (R/I)^{\oplus n}_{\mathfrak{p}/I} = (R/I)^{\oplus n}_\mathfrak{p} = (R_\mathfrak{p}/IR_\mathfrak{p})^{\oplus n}.$$

Since $M_\mathfrak{p}$ is nonzero, we have to have $n > 0$. Since $IR_\mathfrak{p}$ is a nonzero proper ideal of $R_\mathfrak{p}$, we have that $M_\mathfrak{p}$ is not a free $R_\mathfrak{p}$-module. We now conclude that M belongs to $\operatorname{ind} \mathsf{CM}_+(R)$. □

Claim 4. When R/I be Gorenstein, for each $M \in \operatorname{ind} \mathsf{CM}_0(R/I)$, either M or $\Omega_{R/I} M$ is in $\operatorname{ind} \mathsf{CM}_+(R)$.

Proof of Claim. If $M_\mathfrak{p} \neq 0$, then $M \in \operatorname{ind} \mathsf{CM}_+(R)$ by Claim 3. Let $M_\mathfrak{p} = 0$. There is an exact sequence $0 \to N \to (R/I)^{\oplus n} \to M \to 0$, where we set $N := \Omega_{R/I} M$ and $n := \nu_{R/I}(M) > 0$. Localization at $\mathfrak{p}$ gives an isomorphism $N_\mathfrak{p} \cong (R_\mathfrak{p}/IR_\mathfrak{p})^{\oplus n}$. As $n > 0$ and $IR_\mathfrak{p}$ is a proper ideal, the module $N_\mathfrak{p}$ is nonzero. Since R/I is Gorenstein and $\mathsf{CM}_0(R/I)$ is a resolving subcategory of $\operatorname{mod} R/I$, the module N also belongs to $\operatorname{ind} \mathsf{CM}_0(R/I)$; see [149, Lemma (8.17)]. Using Claim 3 again, we obtain $N \in \operatorname{ind} \mathsf{CM}_+(R)$. □

Claim 5. There is an inclusion

$$\{M \in \operatorname{ind} \mathsf{CM}_0(R/I) \mid M \text{ has a rank as an } R/I\text{-module}\} \subseteq \operatorname{ind} \mathsf{CM}_+(R).$$

Proof of Claim. Take M from the left-hand side. Since the R/I-module M is maximal Cohen–Macaulay, its annihilator has grade 0. Hence M has positive rank, and we see that $\operatorname{Supp}_{R/I} M = \operatorname{Spec} R/I$. Therefore $M_\mathfrak{p} = M_{\mathfrak{p}/I}$ is nonzero. It follows from Claim 3 that M belongs to $\operatorname{ind} \mathsf{CM}_+(R)$. □

(3a) Suppose that R has finite CM_+-representation type, namely, $\operatorname{ind} \mathsf{CM}_+(R)$ is a finite set. Lemma 3.4.4 guarantees that the set $\operatorname{ind} \mathsf{CM}_0(R/I)$ is infinite, and hence the set difference

$$\mathcal{S} := \operatorname{ind} \mathsf{CM}_0(R/I) \setminus \operatorname{ind} \mathsf{CM}_+(R)$$

is infinite as well. Thus we can choose a (countably) infinite subset $\{M_1, M_2, M_3, \dots\}$ of $\mathcal{S}$. By Claim 4 we see that $\Omega_{R/I} M_i$ belongs to $\operatorname{ind} \mathsf{CM}_+(R)$ for all i. Note that $\Omega_{R/I} M_i \not\cong \Omega_{R/I} M_j$ for all distinct i, j since R/I is Gorenstein and M_i, M_j are maximal Cohen–Macaulay over R/I.

It follows that $\operatorname{ind} \mathsf{CM}_+(R)$ is an infinite set, which is a contradiction. Thus R has infinite CM_+-representation type.

(3b) Since R/I is a domain, every R/I-module has a rank. Claim 5 implies that $\operatorname{ind} \mathsf{CM}_0(R/I)$ is contained in $\operatorname{ind} \mathsf{CM}_+(R)$, while $\operatorname{ind} \mathsf{CM}_0(R/I)$ is an infinite set by Lemma 3.4.4. It follows that R has infinite CM_+-representation type.

(3c) Note that $\mathsf{CM}(R/I) = \mathsf{CM}_0(R/I)$. Since R/I is analytically unramified, it follows from [108, Theorem 4.10] that the left-hand side of the inclusion in Claim 5 is infinite, and so is the right-hand side $\operatorname{ind} \mathsf{CM}_+(R)$, that is, R has infinite CM_+-represenation type.

(3d) Since k is infinite and R/I is equicharacteristic, we can apply [108, Theorem 17.10] to deduce that if R/I has unbounded CM-representation type, then the left-hand side of the inclusion in Claim 5 is infinite (as R/I is reduced), and we are done. Hence we may assume that R/I has bounded CM-representation type. By [108, Theorems 10.1 and 17.10] the completion $\widehat{R/I}$ has infinite and bounded CM-representation type. According to [108, Theorem 17.9], the ring $\widehat{R/I}$ is isomorphic to one of the following three rings.

$$k[\![x,y]\!]/(x^2), \quad k[\![x,y]\!]/(x^2y), \quad k[\![x,y,z]\!]/(yz, x^2-xz, xz-z^2).$$

The indecomposable maximal Cohen–Macaulay modules over these rings are classified; one can find complete lists of those modules in [21, Propositions 4.1 and 4.2] and [108, Example 14.23]. We can check by hand that each of these rings has an infinite family of nonisomorphic indecomposable maximal Cohen–Macaulay modules of rank 1. This family of modules is extended from a family of R/I-modules by [107, Corollary 2.2], and these are nonisomorphic indecomposable maximal Cohen–Macaulay R/I-modules of rank 1. Again, the left-hand side of the inclusion in Claim 5 is infinite, and the proof is completed. ■

Two irreducible elements p, q of an integral domain R are said to be *distinct* if $pR \neq qR$. Applying our Theorem 3.4.5, we can obtain the following corollary, which is a basis in the next Section 3.5 to obtain a stronger result (Theorem 3.5.1).

Corollary 3.4.6. *Let $(S, \mathfrak{n})$ be a regular local ring of dimension two. Take an element $0 \neq f \in \mathfrak{n}$ and set $R = S/(f)$. Suppose that R is not an isolated singularity but has finite CM_+-representation type. Then f has one of the following forms:*

$$f = \begin{cases} p^2qr & \text{where } p,q,r \text{ are distinct irreducibles} \\ & \text{with } S/(pqr) \text{ having finite } \mathsf{CM}\text{-representation type}, \\ p^2q & \text{where } p \neq q \text{ are irreducibles with } S/(pq) \text{ having finite } \mathsf{CM}\text{-representation type}, \\ p^2 & \text{where } p \text{ is an irreducible with } S/(p) \text{ having finite } \mathsf{CM}\text{-representation type}. \end{cases}$$

Proof. As S is factorial, we can write $f = p_1^{a_1} \cdots p_n^{a_n}$, where $p_1, \dots, p_n$ are distinct irreducible elements and $n, a_1, \dots, a_n$ are positive integers. If $a_1 = \cdots = a_n = 1$, then R is reduced, and hence it is an isolated singularity, which is a contradiction. Thus we may assume $a_1 \geq 2$.

Put $x := p_1 \cdots p_n \in R$. We have

$$(x) + (0:x) = (p_1 \cdots p_n, p_1^{a_1-1}p_2^{a_2-1} \cdots p_n^{a_n-1}) \subseteq (p_1),$$

and hence $\operatorname{ht}((x) + (0:x)) = 0 < 1$. Taking advantage of Theorem 3.4.5(3a), we observe that $R/(x)$ has finite CM-representation type. Also, $R/(x) = S/(p_1 \cdots p_n)$ has multiplicity at least n. By [108, Theorem 4.2 and Proposition 4.3] we see that $n \leq 3$.

Assume either $a_1 \geq 3$ or $a_l \geq 2$ for some $l \geq 2$, say $l = 2$. Then put $x := p_1^2 p_2 \cdots p_n \in R$. We have

$$(x) + (0 : x) = (p_1^2 p_2 \cdots p_n, p_1^{a_1-2} p_2^{a_2-1} \cdots p_n^{a_n-1}) \subseteq \begin{cases} (p_1) & (\text{if } a_1 \geq 3), \\ (p_2) & (\text{if } a_2 \geq 2) \end{cases}$$

and hence $\mathrm{ht}((x) + (0 : x)) = 0 < 1$. The ring $R/(x) = S/(p_1^2 p_2 \cdots p_n)$ is not reduced, so it is not an isolated singularity. By [82, Corollary 2], it has infinite CM-representation type. Theorem 3.4.5(3a) implies that R has infinite CM_+-representation type, which is a contradiction. Thus $a_1 = 2$ and $a_2 = \cdots = a_n = 1$.

Getting toghther all the above arguments completes the proof of the corollary. ■

To give applications of Theorem 3.4.5, we establish a lemma.

Lemma 3.4.7. *Let R be a Gorenstein local ring of finite CM_+-representation type. Then for all $M \in \mathsf{ind}\,\mathsf{CM}_+(R)$ one has $\mathrm{cx}_R M = 1$.*

Proof. As R is Gorenstein, $\Omega^i M \in \mathsf{ind}\,\mathsf{CM}_+(R)$ for all $i \geq 0$ by Lemma 3.4.2 and [149, Lemma 8.17]. Since $\mathsf{ind}\,\mathsf{CM}_+(R)$ is a finite set, $\Omega^t M$ is periodic for some $t \geq 0$. Hence M has complexity at most one. As M is in $\mathsf{CM}_+(R)$, it has to have infinite projective dimension. Thus the complexity of M is equal to one. ■

Let R be a ring. We denote by $\mathsf{D_{sg}}(R)$ the *singularity category* of R, that is, the Verdier quotient of the bounded derived category of finitely generated R-modules by perfect complexes. For an R-module M, we denote by $\mathrm{NF}_R(M)$ the *nonfree locus* of M, that is, the set of prime ideals $\mathfrak{p}$ of R such that $M_\mathfrak{p}$ is nonfree as an $R_\mathfrak{p}$-module. Now we prove the following result by using Theorem 3.4.5.

Theorem 3.4.8. *Let R be a Cohen–Macaulay local ring of dimension $d > 0$. Let I be an ideal of R with $\mathrm{V}(I) \subseteq \mathrm{V}(0 : I)$, and assume that R/I is a maximal Cohen–Macaulay R-module. Suppose that R has finite CM_+-representation type. Then:*

(1) *One has $d = 1$.*

(2) *If $I^n = 0$, then $\dim \mathsf{CM}(R) \leq n - 1$.*

(3) *If R is Gorenstein, then R is a hypersurface and $\dim \mathsf{D_{sg}}(R) \leq n - 1$.*

Proof. (1) This is a direct consequence of Theorem 3.4.5(2b).

(2) It follows from Theorem 3.4.5(2a) that R/I has finite CM-representation type. Hence there exists a maximal Cohen–Macaulay R/I-module G such that $\mathsf{CM}(R/I) = \mathsf{add}_{R/I}\, G$. Take any maximal Cohen–Macaulay R-module M and put $M_0 := M$. For each integer $0 \leq i \leq n - 1$ we have an exact sequence $0 \to (0 :_{M_i} I) \xrightarrow{f_i} M_i \to M_{i+1} \to 0$, where f_i is the inclusion map.

Let us show that for all $0 \leq i \leq n - 1$ the R-module M_i is maximal Cohen–Macaulay and annihilated by I^{n-i}. We use induction on i. It clearly holds in the case $i = 0$, so let $i \geq 1$. Applying the functor $\mathrm{Hom}_R(-, M_{i-1})$ to the natural exact sequence $0 \to I \to R \to R/I \to 0$ induces an exact sequence $0 \to (0 :_{M_{i-1}} I) \xrightarrow{f_{i-1}} M_{i-1} \to \mathrm{Hom}_R(I, M_{i-1})$, and hence M_i is identified with a submodule of $\mathrm{Hom}_R(I, M_{i-1})$. The induction hypothesis implies that M_{i-1}

is maximal Cohen–Macaulay and $I^{n-i-1}M_{i-1} = 0$. Then $\mathrm{Hom}_R(I, M_{i-1})$ has positive depth (see [19, Exercise 1.4.19]), and so does M_i. Since $d = 1$ by (1), the R-module M_i is maximal Cohen–Macaulay. Also, $I^{n-i}M_{i-1}$ is contained in $(0 :_{M_{i-1}} I)$, which implies that I^{n-i} annihilates $M_{i-1}/(0 :_{M_{i-1}} I) = M_i$.

Thus, for each $0 \le i \le n-1$ the submodule $(0 :_{M_i} I)$ of M_i is also maximal Cohen–Macaulay (as $d = 1$ again). Since it is killed by I, it is a maximal Cohen–Macaulay R/I-module. Therefore $(0 :_{M_i} I)$ belongs to $\mathsf{add}_R G = [G]_1$ for all $0 \le i \le n-1$. Using that fact that $M_0 = M$ and $M_n = 0$, we easily observe that M belongs to $[G]_n$. It is concluded that $\mathsf{CM}(R) = [G]_n$, which means that $\dim \mathsf{CM}(R) \le n-1$.

(3) We claim that the R-module R/I has complexity at most one. Indeed, we have

$$\mathrm{NF}_R(R/I) = \mathrm{V}(I + (0 : I)) = \mathrm{V}(I) \cap \mathrm{V}(0 : I) = \mathrm{V}(I),$$

where the first equality follows from [140, Proposition 1.15(4)]. As I is not $\mathfrak{m}$-primary, $\mathrm{NF}_R(R/I)$ contains a nonmaximal prime ideal of R. Hence R/I is in $\mathsf{CM}_+(R)$. Since R/I is a local ring, it is an indecomposable R-module, and therefore $R/I \in \mathsf{ind}\,\mathsf{CM}_+(R)$. It is seen from Lemma 3.4.7 that R/I has complexity at most one as an R-module. Now the claim follows.

Let X be an indecomposable R/I-module which is a direct summand of $C := \Omega^d_{R/I} k$. Proposition 3.4.3(3) implies that X belongs to $\mathsf{ind}\,\mathsf{CM}_+(R)$. As in the proof of the first claim, $\Omega^i_R X$ belongs to $\mathsf{ind}\,\mathsf{CM}_+(R)$ for all $i \ge 0$, and $\Omega^n_R X$ is periodic for some $n \ge 0$. Therefore, we find an integer $m \ge 0$ such that $\Omega^m_R C$ is periodic; see Lemma 3.4.1. This implies that C has complexity at most one. There is an exact sequence

$$0 \to C \to (R/I)^{\oplus r_{m-1}} \to \cdots \to (R/I)^{\oplus r_2} \to (R/I)^{\oplus r_1} \to R/I \to k \to 0.$$

As $\mathrm{cx}_R C \le 1$ and $\mathrm{cx}_R(R/I) \le 1$, we get $\mathrm{cx}_R k \le 1$. By [10, Theorem 8.1.2] the ring R is a hypersurface. The last assertion follows from [20, Theorem 4.4.1] and [43, Proposition 3.5(3)]. ■

The above theorem gives rise to the two corollaries below. Note that the theorem and the two corollaries all give answers to Questions 3.2.6 and 3.2.8.

Corollary 3.4.9. *Let R be a Cohen–Macaulay local ring of dimension $d > 0$ possessing an element $x \in R$ with $(0 : x) = (x)$. Suppose that R has finite CM_+-representation type. Then $d = 1$ and $\dim \mathsf{CM}(R) \le 1$. If R is Gorenstein, then R is a hypersurface and $\dim \mathsf{D_{sg}}(R) \le 1$.*

Proof. We have $x^2 = 0$. The sequence $\cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \cdots$ is exact, which implies that $R/(x)$ is a maximal Cohen–Macaulay R-module. The assertions follow from Theorem 3.4.8. ■

Corollary 3.4.10. *Let R be a Gorenstein non-reduced local ring of dimension one. If R has finite CM_+-representation type, then R is a hypersurface.*

Proof. Since R does not have an isolated singularity, $\mathrm{Sing}\, R$ contains a nonmaximal prime ideal $\mathfrak{p}$. It is easy to see that $(R/\mathfrak{p})_\mathfrak{p} = \kappa(\mathfrak{p})$ is not $R_\mathfrak{p}$-free, and we also have $\mathrm{V}(\mathfrak{p}) = \{\mathfrak{p}, \mathfrak{m}\} \subseteq \mathrm{Supp}_R(\mathfrak{p}) = \mathrm{V}(0 : \mathfrak{p})$ as $\mathfrak{p}R_\mathfrak{p} \ne 0$. Lemma 3.4.7 implies that the R-module $R/\mathfrak{p}$ has complexity at most 1, and the local ring R is a hypersurface by virtue of Theorem 3.4.8(3). ■

3.5 The one-dimensional hypersurfaces of finite CM_+-representation type

The purpose of this section is to prove the following theorem.

Theorem 3.5.1. *Let R be a homomorphic image of a regular local ring. Suppose that R does not have an isolated singularity but is Gorenstein. If $\dim R = 1$, then the following are equivalent.*

(1) *The ring R has finite CM_+-representation type.*

(2) *There exist a regular local ring S and a regular system of parameters x, y such that R is isomorphic to $S/(x^2)$ or $S/(x^2y)$.*

When either of these two conditions holds, the ring R has countable CM-representation type.

In fact, the last assertion and the implication (2) $\Rightarrow$ (1) follow from [21, Propositions 4.1 and 4.2] and [3, Proposition 2.1], respectively. The implication (1) $\Rightarrow$ (2) is an immediate consequence of the combination of Corollaries 3.4.6, 3.4.10 with Theorems 3.5.5, 3.5.11, 3.5.12 shown in this section. Note by Theorem 3.2.2 that the above theorem guarantees that under the assumption that R is a complete Gorenstein local ring of dimension one, Question 3.2.8 has an affirmative answer.

We establish three subsections, whose purposes are to prove Theorems 3.5.5, 3.5.11 and 3.5.12, respectively.

3.5.1 The hypersurface $S/(p^2)$

For a ring A we denote by $\mathrm{NZD}(A)$ the set of non-zerodivisors of A, and by $\mathrm{Q}(A)$ the total quotient ring $A_{\mathrm{NZD}(A)}$ of A. A ring extension $A \subseteq B$ is called *birational* if $B \subseteq \mathrm{Q}(A)$.

Lemma 3.5.2. *Let $A \subseteq B$ be a birational extension. Let M be a B-module which is torsion-free as an A-module. If M is indecomposable as a B-module, then M is indecomposable as an A-module as well.*

Proof. The assertion follows by Remark 2.2.4. ■

Let A be a ring and M an A-module. We denote by $\underline{\mathrm{End}}_A(M)$ the quotient of $\mathrm{End}_A(M)$ by the endomorphisms factoring through projective A-modules. For a flat A-algebra B one has $\underline{\mathrm{End}}_A(M) \otimes_A B \cong \underline{\mathrm{End}}_B(M \otimes_A B)$; this can be shown by using [149, Lemma 3.9].

Lemma 3.5.3. *Let $A \subseteq B$ be a finite birational extension of 1-dimensional Cohen–Macaulay local rings. Then $\mathsf{ind}\,\mathsf{CM}_+(B)$ is contained in $\mathsf{ind}\,\mathsf{CM}_+(A)$.*

Proof. Let $M \in \mathsf{ind}\,\mathsf{CM}_+(B)$. Then $\operatorname{depth}_A M = \operatorname{depth}_B M > 0$, which shows that M is maximal Cohen–Macaulay as an A-module. Lemma 3.5.2 implies $M \in \mathsf{ind}\,\mathsf{CM}(A)$. Set $Q = \mathrm{Q}(A) = \mathrm{Q}(B)$. Applying the functor $Q \otimes_A -$ to the inclusions $A \subseteq B \subseteq Q$ yields $B \otimes_A Q = Q$. Hence we have

$$M \otimes_B Q = M \otimes_B (B \otimes_A Q) = M \otimes_A Q, \quad \underline{\mathrm{End}}_A(M) \otimes_A Q \cong \underline{\mathrm{End}}_Q(M \otimes_A Q) \cong \underline{\mathrm{End}}_Q(M \otimes_B Q).$$

Since M is in $\mathsf{CM}_+(B)$, there is a minimal prime P of B such that M_P is not B_P-free. Note that $M_P = (M \otimes_B Q) \otimes_Q Q_P$ and $Q_P = B_P$. Hence $M \otimes_B Q$ is not Q-projective, and we obtain $\underline{\mathrm{End}}_Q(M \otimes_B Q) \neq 0$. Therefore $\underline{\mathrm{End}}_A(M) \otimes_A Q$ is nonzero, which means that the A-module $\underline{\mathrm{End}}_A(M)$ is not torsion. Thus $\mathrm{Supp}_A(\underline{\mathrm{End}}_A(M))$ contains a minimal prime of A, which implies that M belongs to $\mathsf{CM}_+(A)$. Consequently, we obtain $M \in \mathsf{ind}\,\mathsf{CM}_+(A)$, and the lemma follows. ■

The following lemma is a consequence of [149, Corollary 7.6], which is used not only now but also later.

Lemma 3.5.4. *Let $(S, \mathfrak{n})$ be a regular local ring and $x \in \mathfrak{n}$, and set $R = S/(x)$. Then*

$$\{M \in \mathsf{CM}(R) \mid M \text{ is cyclic}\}/_{\cong} = \{R/yR \mid y \in S \text{ with } x \in yS\}/_{\cong}.$$

In particular, there exist only finitely many nonisomorphic indecomposable cyclic maximal Cohen–Macaulay R-modules.

Now we can achieve the purpose of this subsection.

Theorem 3.5.5. *Let $(S, \mathfrak{n})$ be a regular local ring of dimension two, and let $p \in \mathfrak{n}^2$ be an irreducible element. Then $R = S/(p^2)$ has infinite CM_+-representation type.*

Proof. Take any element $t \in \mathfrak{n}$ that is regular on R. We consider the S-algebra $T = S[z]/(tz - p, z^2)$, where z is an indeterminate over S. We establish two claims.

Claim 1. The ring T is a local complete intersection of dimension 1 and codimension 2 with t being a system of parameters.

Proof of Claim. It is clear that $T = S[\![z]\!]/(tz - p, z^2)$, which shows that T is a local ring, and $\dim T = \dim S[\![z]\!] - \mathrm{ht}(tz - p, z^2) \geq 3 - 2 = 1$ by Krull's Hauptidealsatz. We have $T/tT = S[\![z]\!]/(t, p, z^2) = (S/(t, p))[\![z]\!]/(z^2)$. As $S/(t, p)$ is artinian, so is T/tT. Hence $\dim T = 1$ and t is a system of parameters of T, and thus T is a complete intersection (the equalities $\dim S[\![z]\!] = 3$ and $\dim T = 1$ imply $\mathrm{ht}(tz - p, z^2) = 2$, whence $tz - p, z^2$ is a regular sequence). As $(tz - p, z^2) \subseteq \mathfrak{n}^2$, the local ring T has codimension 2. □

Claim 2. The ring R is naturally embedded in T, and this embedding is a finite birational extension.

Proof of Claim. Let $\phi : S \to T$ be the natural map and put $I = \mathrm{Ker}\,\phi$. As $p^2 = t^2 z^2 = 0$ in T, we have $(p^2) \subseteq I$. Hence the map ϕ factors as $S \twoheadrightarrow R \twoheadrightarrow S/I \hookrightarrow T$. It is seen that T is an R-module generated by $1, z$ and S/I is an R-submodule of T. Since T has positive depth by Claim 1, so does S/I. Thus S/I is a maximal Cohen–Macaulay cyclic module over the hypersurface R, and Lemma 3.5.4 implies that I coincides with either (p) or (p^2). If $I = (p)$, then $T = T/pT = S[z]/(tz, p, z^2)$, which contradicts the fact following from Claim 1 that t is T-regular. We get $I = (p^2)$, which means the map $R \to T$ is injective.

Let C be the cokernel of the injection $R \hookrightarrow T$. Then C is generated by z as an R-module. Note that $tz = p = 0$ in C. Hence C is a torsion R-module, which means $C \otimes_R \mathrm{Q}(R) = 0$. Thus $(\mathrm{Q}(R) \to T \otimes_R \mathrm{Q}(R)) = (R \hookrightarrow T) \otimes_R \mathrm{Q}(R)$ is an isomorphism, while the natural map $T \to T \otimes_R \mathrm{Q}(R)$ is injective as T is maximal Cohen–Macaulay over R by Claim 1. Thus the embedding $R \hookrightarrow T$ is birational. □

By Claim 1, the ring T is a complete intersection, which implies that the element z^2 is regular on the ring $S[z]/(tz-p)$ and so is z. It is easy to check that $(0 :_T z) = zT$. Claim 1 also guarantees that T is not a hypersurface. It follows from Corollary 3.4.9 that T has infinite CM_+-representation type. Combining Claim 2 with Lemma 3.5.3, we obtain the inclusion $\operatorname{\mathsf{ind}} \mathsf{CM}_+(T) \subseteq \operatorname{\mathsf{ind}} \mathsf{CM}_+(R)$. We now conclude that R has infinite CM_+-representation type, and the proof of the theorem is completed. ∎

3.5.2 The hypersurface $S/(p^2qr)$

Setup 3.5.6. Throughout this subsection, let $(S, \mathfrak{n})$ be a 2-dimensional regular local ring and p, q, r pairwise distinct irreducible elements of S. Let $R = S/(p^2qr)$ be a local hypersurface of dimension 1. Setting $\mathfrak{p} = pR$, $\mathfrak{q} = qR$, $\mathrm{R} = rR$ and $\mathfrak{m} = \mathfrak{n}R$, one has $\operatorname{Spec} R = \{\mathfrak{p}, \mathfrak{q}, \mathrm{R}, \mathfrak{m}\}$. For each $i \geq 1$ we define matrices

$$A_i = \begin{pmatrix} p & 0 & r^i \\ 0 & pq & p \\ 0 & 0 & pr \end{pmatrix}, \qquad B_i = \begin{pmatrix} pqr & 0 & -qr^i \\ 0 & pr & -p \\ 0 & 0 & pq \end{pmatrix}$$

over S. Put $M_i = \operatorname{Cok}_S A_i$ and $N_i = \operatorname{Cok}_S B_i$.

Lemma 3.5.7. (1) *For every $i \geq 1$ it holds that $M_i, N_i \in \mathsf{CM}_+(R)$, $\Omega_R M_i = N_i$ and $\Omega_R N_i = M_i$.*

(2) *For all positive integers $i \neq j$, one has $M_i \not\cong M_j$ and $N_i \not\cong N_j$ as R-modules.*

Proof. (1) It is clear that $A_iB_i = B_iA_i = p^2qrE$. Hence A_i, B_i give a matrix factorization of p^2qr over S, and we have $M_i, N_i \in \mathsf{CM}(R)$, $\Omega_R M_i = N_i$ and $\Omega_R N_i = M_i$; see [149, Chapter 7]. Note that q, r are units and $p^2 = 0$ in $R_\mathfrak{p} = S_{(p)}/p^2S_{(p)}$. There are isomorphisms

$$(M_i)_\mathfrak{p} \cong \operatorname{Cok} \begin{pmatrix} p & 0 & r^i \\ 0 & p & p \\ 0 & 0 & p \end{pmatrix} \cong \operatorname{Cok} \begin{pmatrix} p & 0 & 1 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \cong \operatorname{Cok} \begin{pmatrix} 0 & 0 & 1 \\ 0 & p & 0 \\ -p^2 & 0 & p \end{pmatrix} \cong \operatorname{Cok} \begin{pmatrix} 0 & 0 & 1 \\ 0 & p & 0 \\ 0 & 0 & 0 \end{pmatrix} \cong \operatorname{Cok} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \cong R_\mathfrak{p} \oplus \kappa(\mathfrak{p}),$$

where all the cokernels are over $R_\mathfrak{p}$. Therefore $M_i \in \mathsf{CM}_+(R)$, and we get $N_i \in \mathsf{CM}_+(R)$ by Lemma 3.4.2.

(2) Suppose that there is an R-isomorphism $M_i \cong M_j$. It then holds that $\operatorname{Fitt}_2(M_i) = \operatorname{Fitt}_2(M_j)$, which means $(p, r^i)R = (p, r^j)R$. This implies that $(\overline{r}^i) = (\overline{r}^j)$ in the integral domain $R/\mathfrak{p} = S/(p)$. Since $\overline{r} \neq \overline{0}$ in this ring, we get $i = j$. If $N_i \cong N_j$, then $M_i \cong \Omega_R N_i \cong \Omega_R N_j \cong M_j$ by (1), and we get $i = j$. ∎

Lemma 3.5.8. *There is an equality*

$$\{M \in \mathsf{CM}_+(R) \mid M \text{ is cyclic}\}/_{\cong} = \{R/(p),\ R/(pq),\ R/(pr),\ R/(pqr)\}/_{\cong}.$$

Proof. Let M be a cyclic R-module with $M \in \mathsf{CM}_+(R)$. It follows from Lemma 3.5.4 that M is isomorphic to R/fR for some element $f \in S$ which divides p^2qr in S. The localizations $R_\mathfrak{q}, R_\mathfrak{r}$ are fields, and hence $M_\mathfrak{p}$ is not $R_\mathfrak{p}$-free. As $p^2 = 0$ in $R_\mathfrak{p} = S_{(p)}/p^2S_{(p)}$, it is observed that $f \in pS \setminus p^2S$. Thus $f \in \{p, pq, pr, pqr\}$. Conversely, for any $g \in \{p, pq, pr, pqr\}$ we have $(R/gR)_\mathfrak{p} \cong \kappa(\mathfrak{p})$ and get $R/gR \in \mathsf{CM}_+(R)$. ∎

Lemma 3.5.9. *Let $i \geq 1$ be an integer. Then neither* $\operatorname{Cok}_{S/(pq)}\begin{pmatrix} p & r^i \\ 0 & p \end{pmatrix}$ *nor* $\operatorname{Cok}_{S/(pr)}\begin{pmatrix} p \\ qr^i \end{pmatrix}$ *contains $S/(p)$ as a direct summand.*

Proof. (1) Set $T = S/(pq)$ and $C = \operatorname{Cok}_T\begin{pmatrix} p & r^i \\ 0 & p \end{pmatrix}$. Consider the sequence

$$T^{\oplus 2} \xleftarrow{\begin{pmatrix} p & r^i \\ 0 & p \end{pmatrix}} T^{\oplus 2} \xleftarrow{\left(\begin{smallmatrix} q \\ 0 \end{smallmatrix}\right)} T$$

of homomorphisms of free T-modules. Clearly, this is a complex. Let $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in T^{\oplus 2}$ be such that $\begin{pmatrix} p & r^i \\ 0 & p \end{pmatrix}\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. In S we have $pa + r^i b = pqc$ and $pb = pqd$ for some $c, d \in S$, and get $b = qd$ and $pa + r^i qd = pqc$. Hence $pa \in qS \in \operatorname{Spec} S$ and $a \in qS$; we find $e \in S$ with $a = qe$. Then $pqe + r^i qd = pqc$, and $pe + r^i d = pc$. Therefore $r^i d \in pS \in \operatorname{Spec} S$, and $d \in pS$; we find $f \in S$ with $d = pf$ and get $b = qpf$. In $T^{\oplus 2}$ we have $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \left(\begin{smallmatrix} qe \\ pqf \end{smallmatrix}\right) = \left(\begin{smallmatrix} qe \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} q \\ 0 \end{smallmatrix}\right)(e)$. It follows that the above sequence is exact, and the sequence

$$\cdots \xrightarrow{p} T \xrightarrow{q} T \xrightarrow{p} T \xrightarrow{\left(\begin{smallmatrix} q \\ 0 \end{smallmatrix}\right)} T^{\oplus 2} \xrightarrow{\begin{pmatrix} p & r^i \\ 0 & p \end{pmatrix}} T^{\oplus 2} \to C \to 0$$

gives a minimal free resolution of the T-module C.

Now, assume that $S/(p) = T/pT$ is a direct summand of C. Then $C \cong T/pT \oplus T/I$ for some ideal I of T. There are equalities of Betti numbers

$$2 = \beta_1^T(C) = \beta_1^T(T/pT \oplus T/I) = \beta_1^T(T/pT) + \beta_1^T(T/I) = 1 + \beta_1^T(T/I),$$

and we get $\beta_1^T(T/I) = 1$. This means I is a nonzero proper principal ideal of T; we write $I = gT$ where g is a nonzero nonunit of T. The uniqueness of a minimal free resolution yields a commutative diagram

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{p} & T & \xrightarrow{q} & T & \xrightarrow{p} & T & \xrightarrow{\left(\begin{smallmatrix} q \\ 0 \end{smallmatrix}\right)} & T^{\oplus 2} & \xrightarrow{\left(\begin{smallmatrix} p & 0 \\ 0 & g \end{smallmatrix}\right)} & T^{\oplus 2} & \longrightarrow & C & \longrightarrow 0 \\
 & & \cong\big\downarrow u_3 & & \cong\big\downarrow u_2 & & \cong\big\downarrow u_1 & & \cong\big\downarrow \left(\begin{smallmatrix} t_1 & t_2 \\ t_3 & t_4 \end{smallmatrix}\right) =: t & & \cong\big\downarrow \left(\begin{smallmatrix} s_1 & s_2 \\ s_3 & s_4 \end{smallmatrix}\right) =: s & & \Vert & \\
\cdots & \xrightarrow[p]{} & T & \xrightarrow[q]{} & T & \xrightarrow[p]{} & T & \xrightarrow[\left(\begin{smallmatrix} q \\ 0 \end{smallmatrix}\right)]{} & T^{\oplus 2} & \xrightarrow[\left(\begin{smallmatrix} p & r^i \\ 0 & p \end{smallmatrix}\right)]{} & T^{\oplus 2} & \longrightarrow & C & \longrightarrow 0
\end{array}$$

whose vertical maps are isomorphisms. As s, t are isomorphisms, their determinants $s_1 s_4 - s_2 s_3$ and $t_1 t_4 - t_2 t_3$ are units of T. The commutativity of the diagram shows $s_3 p = p t_3$ and $t_3 q = 0$ in T, which imply $s_3 - t_3 \in (0 :_T p) = qT$ and $t_3 \in (0 :_T q) = pT$. Hence s_3 is a nonunit of T, and therefore s_1, s_4 are units of T. Again from the commutativity of the diagram we get $s_4 g = p t_4$ and $s_2 g = p t_2 + r^i t_4$ in T, which give $p(s_2 s_4^{-1} t_4 - t_2) = r^i t_4$. Hence $r^i t_4 \in pT \in \operatorname{Spec} T$ and $t_4 \in pT$. We now get $t_1 t_4 - t_2 t_3$ is in pT, which contradicts the fact that it is a unit of T. Consequently, $S/(p)$ is not a direct summand of C.

(2) Put $T = S/(pr)$ and $C = \operatorname{Cok}_T\begin{pmatrix} p \\ qr^i \end{pmatrix}$. We have $\operatorname{Spec} T = \{pT, rT, \mathfrak{n}T\}$. Since $(p, qr^i)T$ is not contained in pT or rT, it is $\mathfrak{n}T$-primary and has positive grade. Hence the sequence

$$0 \to T \xrightarrow{\left(\begin{smallmatrix} p \\ qr^i \end{smallmatrix}\right)} T^{\oplus 2} \to C \to 0$$

is exact, which gives a minimal free resolution of the T-module C. This implies $\operatorname{pd}_R C = 1$.

Suppose that $S/(p) = T/pT$ is a direct summand of C. Then T/pT has projective dimension at most one, which contradicts the fact that its minimal free resolution is $\cdots \xrightarrow{p} T \xrightarrow{q} T \xrightarrow{p} T \to T/pT \to 0$. It follows that $S/(p)$ is not a direct summand of C. ■

Lemma 3.5.10. (1) *The ring $S/(p,q)$ is artinian, and hence the number $\ell\ell(S/(p,q))$ is finite.*

(2) *Let $n \geq \ell\ell(S/(p,q))$ be a positive integer.*

(i) *If $X \in \mathsf{CM}_+(R)$ is a cyclic direct summand of M_n, then X is isomorphic to $R/(pqr)$.*

(ii) *If $Y \in \mathsf{CM}_+(R)$ is a cyclic direct summand of N_n, then Y is isomorphic to $R/(pqr)$.*

Proof. (1) The factoriality of S shows that pS is a prime ideal of S. As $pS \neq qS$, we have $\operatorname{ht}(p,q)S > \operatorname{ht} pS = 1$. Since S has dimension two, the ideal $(p,q)S$ is $\mathfrak{n}$-primary. Thus $S/(p,q)S$ is an artinian ring.

(2i) There is an R-module Z such that $M_n \cong X \oplus Z$. According to Lemma 3.5.8, it holds that $X \cong R/(f)$ for some $f \in \{p, pq, pr, pqr\}$. There are isomorphisms

$$R/(f,r) \oplus Z/rZ \cong M_n/rM_n \cong \operatorname{Cok}_{R/(r)} \begin{pmatrix} p & 0 & 0 \\ 0 & pq & p \\ 0 & 0 & 0 \end{pmatrix} \cong \operatorname{Cok}_{R/(r)} \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} \cong (R/(p,r))^{\oplus 2} \oplus R/(r).$$

Taking the completions and using the Krull–Schmidt property and [50, Exercise 7.5], we observe that the ideal $(f,r)R$ coincides with either $(p,r)R$ or rR. Hence $f \neq pq$. Similarly, there are isomorphisms

$$\begin{aligned} R/(f,q) \oplus Z/qZ \cong M_n/qM_n \cong \operatorname{Cok}_{R/(q)} \begin{pmatrix} p & 0 & r^n \\ 0 & 0 & p \\ 0 & 0 & pr \end{pmatrix} &\cong \operatorname{Cok}_{R/(q)} \begin{pmatrix} p & 0 & r^n \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} \\ &\cong R/(q) \oplus \operatorname{Cok}_{R/(q)} \begin{pmatrix} p & r^n \\ 0 & p \end{pmatrix}. \end{aligned}$$

The assumption $n \geq \ell\ell(S/(p,q))$ implies $r^n \in \mathfrak{n}^n \subseteq (p,q)S$. We observe from this that $\operatorname{Cok}_{R/(q)} \begin{pmatrix} p & r^n \\ 0 & p \end{pmatrix} \cong \operatorname{Cok}_{R/(q)} \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$, and obtain an isomorphism $R/(f,q) \oplus Z/qZ \cong R/(q) \oplus (R/(p,q))^{\oplus 2}$. It follows that $(f,q)R$ coincides with either qR or $(p,q)R$, which implies $f \neq pr$. Finally, consider the isomorphisms

$$\begin{aligned} R/(f,pq) \oplus Z/pqZ \cong M_n/pqM_n \cong \operatorname{Cok}_{R/(pq)} \begin{pmatrix} p & 0 & r^n \\ 0 & 0 & p \\ 0 & 0 & pr \end{pmatrix} &\cong \operatorname{Cok}_{R/(pq)} \begin{pmatrix} p & 0 & r^n \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} \\ &\cong R/(pq) \oplus \operatorname{Cok}_{R/(pq)} \begin{pmatrix} p & r^n \\ 0 & p \end{pmatrix}. \end{aligned}$$

If $f = p$, then $R/(f,pq) = R/(p)$ and we see that this is a direct summand of $\operatorname{Cok}_{R/(pq)} \begin{pmatrix} p & r^n \\ 0 & p \end{pmatrix}$, which contradicts Lemma 3.5.9. Thus $f \neq p$, and we conclude that $f = pqr$.

(2ii) We go along the same lines as the proof of (2i). We have $N_n \cong Y \oplus Z$ for some $Z \in \mathsf{mod}\, R$, and get $Y \cong R/(f)$ for some $f \in \{p, pq, pr, pqr\}$ by Lemma 3.5.8. The isomorphisms

$$\begin{aligned} R/(f,r) \oplus Z/rZ \cong N_n/rN_n \cong \operatorname{Cok}_{R/(r)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & pq \end{pmatrix} \cong \operatorname{Cok}_{R/(r)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} \cong R/(p,r) \oplus (R/(r))^{\oplus 2}, \\ R/(f,q) \oplus Z/qZ \cong N_n/qN_n \cong \operatorname{Cok}_{R/(q)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & pr & p \\ 0 & 0 & 0 \end{pmatrix} \cong \operatorname{Cok}_{R/(q)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} \cong R/(p,q) \oplus (R/(q))^{\oplus 2} \end{aligned}$$

show that (f, q) (resp. (f, r)) coincides with either (p, q) or (q) (resp. either (p, r) or (r)), which implies $f \neq pq, pr$. We also have isomorphisms

$$R/(f, pr) \oplus Z/prZ \cong N_n/prN_n \cong \operatorname{Cok}_{R/(pr)} \left(\begin{smallmatrix} 0 & 0 & qr^n \\ 0 & 0 & p \\ 0 & 0 & pq \end{smallmatrix}\right) \cong \operatorname{Cok}_{R/(pr)} \left(\begin{smallmatrix} 0 & 0 & qr^n \\ 0 & 0 & p \\ 0 & 0 & 0 \end{smallmatrix}\right)$$
$$\cong \operatorname{Cok}_{R/(pr)} \left(\begin{smallmatrix} p \\ qr^n \end{smallmatrix}\right) \oplus R/(pr).$$

Using Lemma 3.5.9, we see that $f \neq p$, and obtain $f = pqr$. ∎

The purpose of this subsection is now fulfilled.

Theorem 3.5.11. *Let S be a regular local ring of dimension two. Let p, q, r be distinct irreducible elements of S. Then $R = S/(p^2qr)$ has infinite CM_+-representation type.*

Proof. We assume that R has finite CM_+-representation type, and derive a contradiction. It follows from Lemma 3.5.7(1) that there exists an integer $a \geq 1$ such that both M_i and N_i are decomposable for all $i \geq a$; we write $M_i \cong X_i \oplus Y_i$ for some R-modules X_i, Y_i with $\nu(X_i) = 1$ and $\nu(Y_i) = 2$. In view of Lemmas 3.5.4 and 3.5.7(2), we see that there exists an integer $b \geq a$ such that Y_h is indecomposable for all $h \geq b$ and that $Y_i \not\cong Y_j$ for all $i, j \geq b$ with $i \neq j$. Then, we have to have $Y_i \in \mathsf{CM}_0(R)$ for all $i \geq b$, and hence $X_i \in \mathsf{CM}_+(R)$ for all $i \geq b$ (by Lemma 3.5.7(1)). Putting $c = \max\{b, \ell\ell(S/(p, q))\}$ and applying Lemma 3.5.10(2i), we obtain that X_i is isomorphic to $R/(pqr)$ for all $i \geq c$. There are isomorphisms

$$N_i \cong \Omega_R M_i \cong \Omega_R X_i \oplus \Omega_R Y_i \cong \Omega_R(R/(pqr)) \oplus \Omega_R Y_i \cong R/(p) \oplus \Omega_R Y_i,$$

where the first isomorphism follows from Lemma 3.5.7(1). Since $R/(p)$ is in $\mathsf{CM}_+(R)$, it follows from Lemma 3.5.10(2ii) that $R/(p) \cong R/(pqr)$, which is absurd. ∎

3.5.3 The hypersurface $S/(p^2q)$

The goal of this subsection is to prove the following theorem.

Theorem 3.5.12. *Let $(S, \mathfrak{n})$ be a 2-dimensional regular local ring. Let p, q be distinct irreducible elements of S. Suppose that $R = S/(p^2q)$ has finite CM_+-representation type. Then $p, q \notin \mathfrak{n}^2$.*

Note that the rings R and R/p^2R are local hypersurfaces of dimension one. If $p \in \mathfrak{n}^2$, then $R/p^2R = S/(p^2)$ has infinite CM_+-representation type by Theorem 3.5.5, and so does R by Theorem 3.4.5(1), which contradicts the assumption of the theorem. Hence $p \notin \mathfrak{n}^2$, and p is a member of a regular system of parameters of S. Thus we establish the following setting.

Setup 3.5.13. Throughout the remainder of this subsection, let $(S, \mathfrak{n})$ be a regular local ring of dimension two. Let x, y be a regular system of parameters of S, namely, $\mathfrak{n} = (x, y)$. Let $h \in \mathfrak{n}^2$ be an irreducible element, and write $h = x^2s + xyt + y^2u$ with $s, t, u \in S$. Let $R = S/(x^2h)$ be a local hypersurface of dimension one. One has $\operatorname{Spec} R = \{\mathfrak{p}, \mathfrak{q}, \mathfrak{m}\}$, where we set $\mathfrak{p} = xR$, $\mathfrak{q} = hR$ and $\mathfrak{m} = \mathfrak{n}R$. For each integer $i \geq 1$ we define matrices

$$A_i = \begin{pmatrix} x & 0 & y^i \\ 0 & xy & x \\ 0 & xh & 0 \end{pmatrix}, \qquad B_i = \begin{pmatrix} xh & -y^ih & y^{i+1} \\ 0 & 0 & x \\ 0 & xh & -xy \end{pmatrix}$$

over S. We put $M_i = \operatorname{Cok}_S A_i$ and $N_i = \operatorname{Cok}_S B_i$.

In what follows, we argue along similar lines as in the previous subsection.

Lemma 3.5.14 (cf. Lemma 3.5.7)**.** (1) *Let $i \geq 1$ be an integer. The modules M_i and N_i belong to $\mathsf{CM}_+(R)$, and it holds that $\Omega_R M_i = N_i$ and $\Omega_R N_i = M_i$.*

(2) *Let $i, j \geq 1$ be integers with $i \neq j$. One then have $M_i \ncong M_j$ and $N_i \ncong N_j$ as R-modules.*

Proof. (1) We have $A_iB_i = B_iA_i = x^2hE$. The matrices A_i, B_i give a matrix factorization of x^2h over S. We have that M_i, N_i are maximal Cohen–Macaulay R-modules with $\Omega_R M_i = N_i$ and $\Omega_R N_i = M_i$. Note that y, h are units and $x^2 = 0$ in $R_\mathfrak{p} = S_{(x)}/x^2S_{(x)}$. We have

$$\begin{aligned}(M_i)_\mathfrak{p} &\cong \mathrm{Cok}_{R_\mathfrak{p}} \begin{pmatrix} x & 0 & y^i \\ 0 & xy & x \\ 0 & x & 0 \end{pmatrix} \cong \mathrm{Cok}_{R_\mathfrak{p}} \begin{pmatrix} x & 0 & y^i \\ 0 & 0 & x \\ 0 & x & 0 \end{pmatrix} \cong \mathrm{Cok}_{R_\mathfrak{p}} \begin{pmatrix} x & 0 & 1 \\ 0 & 0 & x \\ 0 & x & 0 \end{pmatrix} \cong \mathrm{Cok}_{R_\mathfrak{p}} \begin{pmatrix} 0 & 0 & 1 \\ -x^2 & 0 & x \\ 0 & x & 0 \end{pmatrix} \\ &= \mathrm{Cok}_{R_\mathfrak{p}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & x \\ 0 & x & 0 \end{pmatrix} \cong \mathrm{Cok}_{R_\mathfrak{p}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & x & 0 \end{pmatrix} \cong \mathrm{Cok}_{R_\mathfrak{p}} \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \cong R_\mathfrak{p} \oplus \kappa(\mathfrak{p}),\end{aligned}$$

which shows that $M_i \in \mathsf{CM}_+(R)$, and Lemma 3.4.2 implies $N_i \in \mathsf{CM}_+(R)$ as well.

(2) If $M_i \cong M_j$, then $(x, y^i)R = \mathrm{Fitt}_2(M_i) = \mathrm{Fitt}_2(M_j) = (x, y^j)R$, and $(\overline{y}^i) = (\overline{y}^j)$ in the discrete valuation ring $R/xR = S/(x)$ with $\overline{y}$ a uniformizer, which implies $i = j$. As N_i, N_j are the first syzygies of M_i, M_j by (1), we see that if $N_i \cong N_j$, then $i = j$. ■

Lemma 3.5.15 (cf. Lemma 3.5.8)**.** *It holds that*

$$\{M \in \mathsf{CM}_+(R) \mid M \text{ is cyclic}\}/_{\cong} = \{R/(x),\, R/(xh)\}/_{\cong}.$$

Proof. It is easy to see that neither $(R/(x))_\mathfrak{p}$ nor $(R/(xh))_\mathfrak{p}$ is $R_\mathfrak{p}$-free. Let $M \in \mathsf{CM}_+(R)$ be cyclic. As $R_\mathfrak{q}$ is a field, $M_\mathfrak{p}$ is not $R_\mathfrak{p}$-free. Using Lemma 3.5.4, we get $M \cong R/fR$ for some $f \in S$ with $f \mid x^2h$, $x \mid f$ and $x^2 \nmid f$. Hence, either $f = x$ or $f = xh$ holds. ■

Lemma 3.5.16 (cf. Lemma 3.5.10)**.** *Let $i \geq 1$ be an integer. Let C be a cyclic R-module with $C \in \mathsf{CM}_+(R)$. If C is a direct summand of either M_i or N_i, then C is isomorphic to $R/(xh)$.*

Proof. (1) First, consider the case where C is a direct summand of M_i. Assume that C is not isomorphic to $R/(xh)$. Then $C \cong R/(x)$ by Lemma 3.5.15. Application of the functor $- \otimes_R R/(xy)$ shows that $C/xyC = C \cong R/(x)$ is a direct summand of

$$M_i/xyM_i = \mathrm{Cok}_{R/(xy)} \begin{pmatrix} x & 0 & y^i \\ 0 & 0 & x \\ 0 & xh & 0 \end{pmatrix} \cong \mathrm{Cok}_{R/(xy)} \begin{pmatrix} x & y^i & 0 \\ 0 & x & 0 \\ 0 & 0 & xh \end{pmatrix} \cong \mathrm{Cok}_{R/(xy)} \begin{pmatrix} x & y^i \\ 0 & x \end{pmatrix} \oplus R/(xy, xh).$$

As $(x) \neq (xy, xh)$, we have $R/(x) \ncong R/(xy, xh)$ and hence $R/(x)$ is a direct summand of $\mathrm{Cok}_{R/(xy)} \begin{pmatrix} x & y^i \\ 0 & x \end{pmatrix}$. Note that $R/(xy) = S/(x^2h, xy) = S/(x^2(x^2s + xyt + y^2u), xy) = S/(x^4s, xy)$. Put $T := R/(xy, x^4) = S/(x^4, xy)$. Applying the functor $- \otimes_R R/(x^4)$, we see that $T/(x) = R/(x)$ is a direct summand of $L := \mathrm{Cok}_T \begin{pmatrix} x & y^i \\ 0 & x \end{pmatrix}$. Write $L = T/(x) \oplus D$ with $D \in \mathsf{mod}\, T$. It is easy to verify that the sequence

$$0 \leftarrow L \leftarrow T^{\oplus 2} \xleftarrow{\begin{pmatrix} x & y^i \\ 0 & x \end{pmatrix}} T^{\oplus 2} \xleftarrow{\begin{pmatrix} y & x^3 & 0 \\ 0 & 0 & x^3 \end{pmatrix}} T^{\oplus 3}$$

is exact, and we observe $D \cong T/(v)$ for some $v \in T$. Uniqueness of a minimal free resolution gives rise to a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longleftarrow & L & \longleftarrow & T^{\oplus 2} & \xleftarrow{\left(\begin{smallmatrix} x & y^i \\ 0 & x \end{smallmatrix}\right)} & T^{\oplus 2} & \xleftarrow{\left(\begin{smallmatrix} y & x^3 & 0 \\ 0 & 0 & x^3 \end{smallmatrix}\right)} & T^{\oplus 3} \\
 & & \downarrow \cong & & \cong \downarrow \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right) & & \cong \downarrow \left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right) & & \\
0 & \longleftarrow & T/(x) \oplus T/(v) & \longleftarrow & T^{\oplus 2} & \xleftarrow[\left(\begin{smallmatrix} x & 0 \\ 0 & v \end{smallmatrix}\right)]{} & T^{\oplus 2} & &
\end{array}$$

with vertical maps being isomorphisms. The elements $a_1a_4 - a_2a_3$ and $b_1b_4 - b_2b_3$ are units of T. We have $a_1y^i + a_2x = xb_2$ and $a_1x = xb_1$ in T. Hence $a_1y^i \in (x) \in \operatorname{Spec} T$, which implies $a_1 \in (x)$. Also, $a_1 - b_1 \in (0 : x) = (x^3, y)$, which implies $b_1 \in (x, y)$. It follows that a_2, a_3, b_2, b_3 are units of T. The equality $a_3x = vb_3$ implies that $(x) = (v)$ in T. We obtain isomorphisms

$$T/(x^3, y) \oplus T/(x^3) \cong \operatorname{Cok}_T \left(\begin{smallmatrix} y & x^3 & 0 \\ 0 & 0 & x^3 \end{smallmatrix}\right) \cong \Omega_T L \cong (x) \oplus (v) \cong (x)^{\oplus 2} \cong (T/(x^3, y))^{\oplus 2}.$$

It follows that $(x^3) = (x^3, y)$ in T, which is a contradiction. Consequently, C is isomorphic to $R/(xh)$.

(2) Next we consider the case where C is a direct summand of N_i. The proof is analogous to that of (1). Again, assume $C \not\cong R/(xh)$. Then $C \cong R/(x)$ by Lemma 3.5.15. Set $T := R/(xh) = S/(xh)$. Applying $- \otimes_R T$, we see that $R/(x) = T/(x)$ is a direct summand of

$$N_i/xhN_i = \operatorname{Cok}_T \begin{pmatrix} 0 & -y^ih & y^{i+1} \\ 0 & 0 & x \\ 0 & 0 & -xy \end{pmatrix} \cong \operatorname{Cok}_T \begin{pmatrix} 0 & -y^ih & y^{i+1} \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} \cong T \oplus \operatorname{Cok}_T \begin{pmatrix} y^ih & y^{i+1} \\ 0 & x \end{pmatrix},$$

which implies that $T/(x)$ is a direct summand of $L := \operatorname{Cok}_T \begin{pmatrix} y^ih & y^{i+1} \\ 0 & x \end{pmatrix}$. There are an isomorphism $L \cong T/(x) \oplus T/(v)$ with $v \in T$ and a commutative diagram:

$$\begin{array}{ccccccc}
0 & \longleftarrow & L & \longleftarrow & T^{\oplus 2} & \xleftarrow{\left(\begin{smallmatrix} y^ih & y^{i+1} \\ 0 & x \end{smallmatrix}\right)} & T^{\oplus 2} \\
 & & \downarrow \cong & & \cong \downarrow \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right) & & \cong \downarrow \left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right) \\
0 & \longleftarrow & T/(x) \oplus T/(v) & \longleftarrow & T^{\oplus 2} & \xleftarrow[\left(\begin{smallmatrix} x & 0 \\ 0 & v \end{smallmatrix}\right)]{} & T^{\oplus 2}
\end{array}$$

Note that $\operatorname{Spec} T = \{(x), (h), \mathfrak{n}T\}$. We have $(h) \ni a_1y^ih = xb_1 \in (x)$, which implies $a_1 \in (x)$ and $b_1 \in (h)$. As $a_1a_4 - a_2a_3$ and $b_1b_4 - b_2b_3$ are units, so are a_2, a_3, b_2, b_3. The equalities $a_3y^ih = vb_3$ and $a_3y^{i+1} + a_4x = vb_4$ imply $a_3y^i(b_3^{-1}hb_4 - y) = a_4x \in (x)$, which gives $b_3^{-1}hb_4 - y \in (x)$. Hence $y \in (x, h) = (x, x^2s + xyt + y^2u) = (x, y^2u)$ in T, which is a contradiction. Thus $C \cong R/(xh)$. ■

Lemma 3.5.17 (cf. Theorem 3.5.11)**.** *The ring R has infinite CM_+-representation type.*

Proof. Assume contrarily that R has finite CM_+-representation type. Then, by (1) and (2) of Lemma 3.5.14, there exists an integer $a \geq 1$ such that M_i is decomposable for all integers $i \geq a$.

Suppose that for some $i \geq 1$ the module M_i has a cyclic direct summand $C \in \mathsf{CM}_+(R)$. Then C is isomorphic to $R/(xh)$ by Lemma 3.5.16, and $\Omega_R C = R/(x)$ is a direct summand of $\Omega_R M_i = N_i$ by Lemma 3.5.14(1). Applying Lemma 3.5.16 again, we have to have $R/(x) \cong R/(xh)$, which is a contradiction.

Thus M_i has no cyclic direct summand belonging to $\mathsf{CM}_+(R)$ for all $i \geq 1$. This means that for every $i \geq a$ the R-module M_i has an indecomposable direct summand $Y_i \in \mathsf{CM}_+(R)$ with $\nu(Y_i) = 2$. This, in turn, contradicts the assumption that R has finite CM_+-representation type. ■

Now the purpose of this subsection is readily accomplished:

Proof of Theorem 3.5.12. The theorem is an immediate consequence of Lemma 3.5.17 and what we state just after the theorem. ■

3.6 On the higher-dimensional case

In this section, we explore the higher-dimensional case: we consider Cohen–Macaulay local rings R with $\dim R \geq 2$ and having finite CM_+-representation type. In particular, we give various results supporting Conjecture 3.1.1. We begin with presenting an example by using a result obtained in Section 4.

Example 3.6.1. Let S be a regular local ring with a regular system of parameters x, y, z. Then $R = S/(xyz)$ has infinite CM_+-representation type.

Proof. Let $I = (xy)$ be an ideal of R. Then $(0 : I) = (z)$ in R, and $\operatorname{ht}(I + (0 : I)) = \operatorname{ht}(xy, z) = 1 < 2 = \dim R$. The ring $R/I = S/(xy)$ is a 2-dimensional hypersurface which does not have an isolated singularity. We see by [82, Corollary 2] that R/I has infinite CM-representation type. It follows from Theorem 3.4.5(3a) that R has infinite CM_+-representation type. ■

We consider constructing from a given hypersurface of infinite CM_+-representation type another hypersurface of infinite CM_+-representation type. For this we establish the following lemma, which provides a version of Knörrer's periodicity theorem for $\mathsf{CM}_+(R)$.

Lemma 3.6.2. *Let $(S, \mathfrak{n})$ be a regular local ring, and let $f, g \in S$. Let $R = S/(f)$ and $R^\sharp = S[\![x]\!]/(f + x^2 g)$ be hypersurfaces with x an indeterminate over S. Then the following statements hold.*

(1) *There is an additive functor*

$$\Phi : \mathsf{CM}_+(R) \to \mathsf{CM}_+(R^\sharp), \qquad \operatorname{Cok}(A, B) \mapsto \operatorname{Cok}\left(\left(\begin{smallmatrix} A & -xE \\ xgE & B \end{smallmatrix}\right), \left(\begin{smallmatrix} B & xE \\ -xgE & A \end{smallmatrix}\right)\right).$$

(2) *Let $M \in \mathsf{ind}\,\mathsf{CM}_+(R)$ and put $N = \Phi(M)$. Then one has either $N \in \mathsf{ind}\,\mathsf{CM}_+(R^\sharp)$ or $N \cong X \oplus Y$ for some $X, Y \in \mathsf{ind}\,\mathsf{CM}_+(R^\sharp)$.*

Proof. (1) It holds that $\left(\begin{smallmatrix} A & -xE \\ xgE & B \end{smallmatrix}\right)\left(\begin{smallmatrix} B & xE \\ -xgE & A \end{smallmatrix}\right) = \left(\begin{smallmatrix} B & xE \\ -xgE & A \end{smallmatrix}\right)\left(\begin{smallmatrix} A & -xE \\ xgE & B \end{smallmatrix}\right) = (f + x^2 g)E$. If $(V, W) : (A, B) \to (A', B')$ is a morphism of matrix factorizations of f over S, then $\left(\left(\begin{smallmatrix} V & 0 \\ 0 & W \end{smallmatrix}\right), \left(\begin{smallmatrix} W & 0 \\ 0 & V \end{smallmatrix}\right)\right) : \left(\left(\begin{smallmatrix} A & -xE \\ xgE & B \end{smallmatrix}\right), \left(\begin{smallmatrix} B & xE \\ -xgE & A \end{smallmatrix}\right)\right) \to \left(\left(\begin{smallmatrix} A' & -xE \\ xgE & B' \end{smallmatrix}\right), \left(\begin{smallmatrix} B' & xE \\ -xgE & A' \end{smallmatrix}\right)\right)$ is a morphism of matrix factorizations of $f + x^2 g$ over $S[\![x]\!]$. We observe that Φ defines an additive functor from $\mathsf{CM}(R)$ to $\mathsf{CM}(R^\sharp)$.

Fix $M \in \mathsf{CM}_+(R)$. Let (A, B) be the corresponding matrix factorization. Set N to be the corresponding module $\operatorname{Cok}_{S[\![x]\!]}\left(\begin{smallmatrix} A & -xE \\ xgE & B \end{smallmatrix}\right)$ via Φ. There is a nonmaximal prime ideal $\mathfrak{p}$ of S such that $M_\mathfrak{p}$ is not $R_\mathfrak{p}$-free. Put $\mathfrak{q} = \mathfrak{p}S[\![x]\!] + xS[\![x]\!]$. We see that $\mathfrak{q}$ is a nonmaximal prime ideal of $S[\![x]\!]$. Suppose that $N_\mathfrak{q} \cong (R^\sharp)_\mathfrak{q}^{\oplus n}$ for some n. Then

$$R_\mathfrak{p}^{\oplus n} \cong ((R^\sharp/xR^\sharp)^{\oplus n})_\mathfrak{q} \cong N_\mathfrak{q}/xN_\mathfrak{q} \cong \operatorname{Cok}_{S[\![x]\!]_\mathfrak{q}} \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \cong \operatorname{Cok}_{S_\mathfrak{p}} A \oplus \operatorname{Cok}_{S_\mathfrak{p}} B \cong M_\mathfrak{p} \oplus (\Omega_R M)_\mathfrak{p},$$

which implies that $M_\mathfrak{p}$ is $R_\mathfrak{p}$-free, a contradiction. Therefore $N_\mathfrak{q}$ is not $(R^\sharp)_\mathfrak{q}$-free, and we obtain $N \in \mathsf{CM}_+(R^\sharp)$. Thus Φ induces an additive functor from $\mathsf{CM}_+(R)$ to $\mathsf{CM}_+(R^\sharp)$.

(2) Let (A, B) be the matrix factorization which gives M. Then $N = \operatorname{Cok}_{S[\![x]\!]} \left(\begin{smallmatrix} A & -xE \\ xgE & B \end{smallmatrix}\right)$. Suppose that N is decomposable. Then $N \cong X \oplus Y$ for some nonzero modules $X, Y \in \mathsf{CM}(R^\sharp)$. It holds that

$$X/xX \oplus Y/xY \cong N/xN \cong \operatorname{Cok}_S \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \cong \operatorname{Cok}_S A \oplus \operatorname{Cok}_S B \cong M \oplus \Omega_R M.$$

Since R is Gorenstein, not only M but also $\Omega_R M$ is indecomposable; see [149, Lemma 8.17]. Nakayama's lemma guarantees that X/xX and Y/xY are nonzero, and both X and Y have to be indecomposable. We may assume that $M \cong X/xX$ and $\Omega_R M \cong Y/xY$. Take a nonmaximal prime ideal $\mathfrak{p}$ of S such that $M_\mathfrak{p}$ is not $R_\mathfrak{p}$-free. Then $\mathfrak{q} := \mathfrak{p}S[\![x]\!] + xS[\![x]\!]$ is a nonmaximal prime ideal of $S[\![x]\!]$ as in the proof of (1). We easily see that the $R_\mathfrak{p}$-module $(\Omega_R M)_\mathfrak{p}$ is not free. Now it follows that neither $X_\mathfrak{q}$ nor $Y_\mathfrak{q}$ is free over $(R^\sharp)_\mathfrak{q}$, which shows that $X, Y \in \mathsf{CM}_+(R^\sharp)$. ■

Infinite CM_+-representation type ascends from R to $R^\sharp$.

Proposition 3.6.3. *Let $(S, \mathfrak{n})$ be a regular local ring and $f, g \in S$. Let $R = S/(f)$ and $R^\sharp = S[\![x]\!]/(f + x^2 g)$ be hypersurfaces with x an indeterminate. If R has infinite CM_+-representation type, then so does $R^\sharp$.*

Proof. Pick any $M_1 \in \operatorname{ind} \mathsf{CM}_+(R)$. The set $\operatorname{ind} \mathsf{CM}_+(R) \setminus \{M_1, \Omega M_1\}$ is infinite, and we pick any M_2 in this set. The set $\operatorname{ind} \mathsf{CM}_+(R) \setminus \{M_1, \Omega M_1, M_2, \Omega M_2\}$ is infinite, and we pick any M_3 in it. Iterating this procedure, we obtain modules $M_1, M_2, M_3, \dots$ in $\operatorname{ind} \mathsf{CM}_+(R)$ such that $M_i \ncong M_j$ and $M_i \cong \Omega M_j$ for all $i \neq j$. We put $N_i = \Phi M_i$ for each i, where Φ is the functor defined in Lemma 3.6.2. Then by the lemma N_i is either in $\operatorname{ind} \mathsf{CM}_+(R^\sharp)$ or isomorphic to $X_i \oplus Y_i$ for some $X_i, Y_i \in \operatorname{ind} \mathsf{CM}_+(R^\sharp)$.

Assume $N_i \cong N_j$ for some $i \neq j$. Then, as we saw in the proof of the lemma, there are isomorphisms $M_i \oplus \Omega M_i \cong N_i/xN_i \cong N_j/xN_j \cong M_j \oplus \Omega M_j$ and the modules $M_i, \Omega M_i, M_j, \Omega M_j$ are indecomposable. This contradicts the choice of these modules. Hence we have $N_i \ncong N_j$ for all $i \neq j$.

Suppose that there are only a finite number, say n, of indecomposable modules in $\mathsf{CM}_+(R)$. Then it is seen that the set $\{N_1, N_2, N_3, \dots\}/_{\cong}$ has cardinality at most $n + \binom{n+1}{2}$, which is a contradiction. We now conclude that $R^\sharp$ has infinite CM_+-representation type, and the proof of the proposition is completed. ■

Here is an application of Proposition 3.6.3.

Corollary 3.6.4. *Let R be a 2-dimensional complete local hypersurface with algebraically closed residue field k of characteristic 0 and not having an isolated singularity. Suppose that R has multiplicity at most 2. If R has finite CM_+-representation type, then $R \cong k[\![x, y, z]\!]/(f)$ with $f = x^2 + y^2$ or $f = x^2 + y^2 z$, and hence R has countable CM-representation type.*

Proof. If $\mathrm{e}(R) = 1$, then R is regular, which contradicts the assumption that R does not have an isolated singularity. Hence $\mathrm{e}(R) = 2$, and the combination of Cohen's structure theorem and the Weierstrass preparation theorem shows $R \cong k[\![x, y, z]\!]/(x^2 + g)$ for some $g \in k[\![y, z]\!]$; see [149, Proof of Theorem 8.8]. It follows from Proposition 3.6.3 that the 1-dimensional hypersurface $S := k[\![y, z]\!]/(g)$ has finite CM_+-representation type. By virtue of Theorem 3.5.1, we obtain $g = y^2$ or $g = y^2 z$ after changing variables (i.e., after applying a k-algebra automorphism of $k[\![y, z]\!]$). We observe that R is isomorphic to either $k[\![x, y, z]\!]/(x^2 + y^2)$ or $k[\![x, y, z]\!]/(x^2 + y^2 z)$. It follows from [108, Propositions 14.17 and 14.19] that R has countable CM-representation type. ■

Proposition 3.6.3 can provide a lot of examples of hypersurfaces of infinite CM_+-representation type of higher dimension. The following example is not covered by this proposition or any other general result given in this chapter.

Example 3.6.5. Let S be a regular local ring with a regular system of parameters x, y, z. Let

$$f = x^n + x^2 ya + y^2 b$$

be an irreducible element of S with $n \geq 4$ and $a, b \in S$. Then the hypersurface $R = S/(f)$ has infinite CM_+-representation type.

Proof. Putting $g = x^2 a + yb$, we have $f = x^n + yg$. For each integer $i \geq 0$ we define a pair of matrices $A_i = \begin{pmatrix} x^2 & xz^i \\ 0 & -x^2 \end{pmatrix}$ and $B_i = \begin{pmatrix} x^{n-2} & x^{n-3}z^i \\ 0 & -x^{n-2} \end{pmatrix}$, which gives a matrix factorization of x^n over S and $S/(y)$. Define another pair of matrices $A_i^\sharp = \begin{pmatrix} A_i & -yE \\ gE & B_i \end{pmatrix}$ and $B_i^\sharp = \begin{pmatrix} B_i & yE \\ -gE & A_i \end{pmatrix}$. These form a matrix factorization of f over S, and hence $M_i := \mathrm{Cok}_S(A_i^\sharp)$ is a maximal Cohen–Macaulay R-module. There are equalities

$$\mathrm{Fitt}_3^S(M_i) = \mathrm{I}_1(A_i^\sharp) = (x^2, xz^i, x^{n-2}, x^{n-3}z^i, y, g)S = (x^2, xz^i, y)S$$

of ideals of S, where we use $n \geq 4$.

Suppose that $M_i \cong M_j$ for some $i < j$. Then $(x^2, xz^i, y)S = (x^2, xz^j, y)S$ and $(x^2, xz^i)\overline{S} = (x^2, xz^j)\overline{S}$, where $\overline{S} := S/(y)$ is a regular local ring having the regular system of parameters x, z. Hence $z^i \in (x, z^j)\overline{S}$ and $z^i \in z^j\widetilde{S}$ where $\widetilde{S} := \overline{S}/x\overline{S}$ is a discrete valuation ring with z a uniformizer. This gives a contradiction, and we see that $M_i \not\cong M_j$ for all $i \neq j$.

Let $\mathfrak{p} = (x, y)S \in \operatorname{Spec} S$, and fix an integer $i \geq 0$. Note that all the entries of A_i, B_i are in $\mathfrak{p}$ since $n \geq 4$. It follows from [149, Remark 7.5] that the $R_\mathfrak{p}$-module $(M_i)_\mathfrak{p}$ does not have a nonzero free summand. Since f is assumed to be irreducible, R is an integral domain. Hence each nonzero direct summand X of the maximal Cohen–Macaulay R-module M_i has positive rank, and hence has full support. Therefore $X_\mathfrak{p} \neq 0$, and thus all the indecomposable direct summands of M_i belong to $\operatorname{ind}\mathsf{CM}_+(R)$. Since all the M_i are generated by four elements, it is observed that $\operatorname{ind}\mathsf{CM}_+(R)$ is an infinite set. ■

To prove our next result, we prepare a lemma on unique factorization domains.

Lemma 3.6.6. *Let R be a Cohen–Macaulay factorial local ring with* $\dim R \geq 3$. *Let I be an ideal of R generated by two elements. Then* $\operatorname{depth} R/I > 0$.

Proof. We write $I = (x, y)R$ and put $g = \gcd\{x, y\}$. Then $x = gx'$ and $y = gy'$ for some $x', y' \in R$, and we set $I' = (x', y')R$. There is an exact sequence $0 \to R/I' \xrightarrow{g} R/I \to R/gR \to 0$ of R-modules. As R is Cohen–Macaulay, we have $\operatorname{depth} R \geq 3$ and $\operatorname{ht} I' = \operatorname{grade} I'$. Since R is a domain and $g \neq 0$, we have $\operatorname{depth} R/gR = \operatorname{depth} R - 1 \geq 2$. If $\operatorname{ht} I' = 1$, then I' is contained in a principal prime ideal, which contradicts the fact that x', y' are coprime. Hence $\operatorname{ht} I' = 2$, and the sequence x', y' is R-regular. It follows that $\operatorname{depth} R/I' = \operatorname{depth} R - 2 \geq 1$, and the depth lemma implies $\operatorname{depth} R/I \geq 1$. ■

Now we can prove the following theorem, which provides the shape of a hypersurface of infinite CM_+-representation type.

Theorem 3.6.7. *Let $(S, \mathfrak{n})$ be a regular local ring and $x, y \in \mathfrak{n}$. Suppose that the ideal (x, y) of S is neither prime nor $\mathfrak{n}$-primary. Then $R = S/(xy)$ has infinite CM_+-representation type.*

Proof. Lemma 3.6.6 guarantees that there exists an $S/(x, y)$-regular element $a \in \mathfrak{n}$. Take a minimal prime $\mathfrak{p}$ of (x, y). Since (x, y) is not prime, we can choose an element $b \in \mathfrak{p} \setminus (x, y)$. Set $z_n = a^n b$ for each n. The matrices $A_n = \left(\begin{smallmatrix} x & z_n \\ 0 & -y \end{smallmatrix}\right)$ and $B_n = \left(\begin{smallmatrix} y & z_n \\ 0 & -x \end{smallmatrix}\right)$ with $n \geq 1$ form a matrix factorization of xy over S, and $M_n = \operatorname{Cok}_S A_n$ is a maximal Cohen–Macaulay R-module. Put $I_n := \mathrm{I}_1(A_n) = (x, y, z^n) \subseteq S$. Since the I_n are pairwise distinct, the M_n are pairwise nonisomorphic. If M_n is decomposable, it decomposes into two cyclic R-modules, while Lemma 3.5.4 says that there are only finitely many such cyclic modules up to isomorphism. Thus we find infinitely many n such that M_n is indecomposable. Since (x, y, z_n) is contained in $\mathfrak{p}$, each $(M_n)_\mathfrak{p}$ has no nonzero free summand by [149, (7.5.1)]. In particular, we have $M_n \in \mathsf{CM}_+(R)$. Now it is seen that R has infinite CM_+-representation type. ■

Applying the above theorem, we can obtain a couple of restrictions for a hypersurface of dimension at least 2 which is not an integral domain but has finite CM_+-representation type.

Corollary 3.6.8. *Let R be a complete local hypersurface of dimension $d \geq 2$ which is not a domain. Suppose that R has finite CM_+-representation type. Then one has $d = 2$, and there exist a complete regular local ring S of dimension 3 and elements $x, y \in S$ satisfying the following conditions.*

(1) *R is isomorphic to $S/(xy)$.*

(2) *$S/(x)$ and $S/(y)$ have finite CM-representation type.*

(3) *$S/(x, y)$ is a domain of dimension 1.*

Proof. Corollary 3.3.3(1) says that R satisfies Serre's condition (R_{d-2}). Suppose $d \geq 3$. Then R satisfies (R_1), and hence it is normal. In particular, R is a domain, contrary to our assumption. Therefore, we have to have $d = 2$. Cohen's structure theorem yields $R \cong S/fS$ for some 3-dimensional complete regular local ring $(S, \mathfrak{n})$ and $f \in \mathfrak{n} \setminus \mathfrak{n}^2$. As R is not a domain, there are elements $x, y \in S$ with $f = xy$. Since $\dim S = 3$, the ideal $(x, y)S$ is not $\mathfrak{n}$-primary. Hence $\dim S/(x, y)S = 1$, and $S/(x, y)S$ is a domain by Theorem 3.6.7. We have $\dim R = \dim R/xR = 2$, $(0 :_R x) = yR$ and $\operatorname{ht}(xR + (0 :_R x)) < 2$. It follows from Theorem 3.4.5(3a) that S/xS has finite CM-representation type, and similarly so does S/yS. ■

Proposition 3.6.3 gives an ascent property of infinite CM_+-representation type. Now we presents a descent property of infinite CM_+-representation type.

Theorem 3.6.9. *Let $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a finite local homomorphism of Cohen–Macaulay local rings of dimension d such that S is a domain. Set $\mathfrak{p} = \operatorname{Ker} \phi$ and assume the following.*

(a) *The induced embedding $R/\mathfrak{p} \hookrightarrow S$ is birational.*

(b) *There exists $\mathfrak{q} \in \mathrm{V}(\mathfrak{p}) \setminus \{\mathfrak{m}\}$ such that $R_\mathfrak{q}$ is not a direct summand of $S_\mathfrak{q}$.*

If S has infinite CM-representation type, then R has infinite CM_+-representation type.

Proof. We prove the theorem by establishing several claims.

Claim 1. Let $X \neq 0$ be an R-submodule of a maximal Cohen–Macaulay S-module M. Then $X_\mathfrak{q} \neq 0$.

Proof of Claim. Assume $X_\mathfrak{q} = 0$. Then there exists an element $s \in \operatorname{Ann}_R X$ such that $s \notin \mathfrak{q}$. As $\mathfrak{p} \subseteq \mathfrak{q}$, we have $s \notin \mathfrak{p}$, which means $\phi(s) \neq 0$. Choose a nonzero element $x \in X$. Since s annihilates X, we have $0 = s \cdot x = \phi(s)x$ in M. This contradicts the fact that M is torsion-free over the domain S. □

Claim 2. Let $M \in \mathsf{CM}_0(S)$. Let X be an indecomposable R-module which is a direct summand of M. Then $X \in \operatorname{ind} \mathsf{CM}_+(R)$.

Proof of Claim. As $\operatorname{depth}_R M = \operatorname{depth}_S M \geq d$, we have $M \in \mathsf{CM}(R)$ and hence $X \in \operatorname{ind} \mathsf{CM}(R)$. To show the claim, it suffices to verify that $X_\mathfrak{q}$ is not $R_\mathfrak{q}$-free.

Take an exact sequence $\sigma : 0 \to \Omega_S M \to S^{\oplus n} \to M_\mathfrak{q} \to 0$. Since M belongs to $\mathsf{CM}_0(S)$, the S-module $E := \operatorname{Ext}^1_S(M, \Omega_S M)$ has finite length. The induced field extension $k \hookrightarrow l$ is finite because so is the homomorphism ϕ, and hence E also has finite length as an R-module. As $\mathfrak{q}$ is a nonmaximal prime ideal of R, we have $0 = E_\mathfrak{q} = \operatorname{Ext}^1_{S_\mathfrak{q}}(M_\mathfrak{q}, (\Omega_S M)_\mathfrak{q})$, and the exact sequence $\sigma_\mathfrak{q} : 0 \to (\Omega_S M)_\mathfrak{q} \to S_\mathfrak{q}^{\oplus n} \to M_\mathfrak{q} \to 0$ corresponds to an element in this Ext module. Hence $\sigma_\mathfrak{q}$ has to split, and $M_\mathfrak{q}$ is a direct summand of $S_\mathfrak{q}^{\oplus n}$ as an $S_\mathfrak{q}$-module. (Note that $S_\mathfrak{q}$ is not necessarily a local ring.) The $S_\mathfrak{q}$-module $X_\mathfrak{q}$ is a direct summand of $M_\mathfrak{q}$, which is nonzero by Claim 2.

Suppose that $X_\mathfrak{q}$ is $R_\mathfrak{q}$-free. Then $R_\mathfrak{q}$ is a direct summand of $S_\mathfrak{q}^{\oplus n}$ in $\operatorname{mod} R_\mathfrak{q}$. As $R_\mathfrak{q}$ is a local ring, we can apply Lemma 3.3.5(1) to deduce that $R_\mathfrak{q}$ is a direct summand of $S_\mathfrak{q}$. This contradicts the assumption of the theorem, and thus $X_\mathfrak{q}$ is not $R_\mathfrak{q}$-free. □

Claim 3. One has the inclusion $\operatorname{ind} \mathsf{CM}_0(S) \subseteq \operatorname{ind} \mathsf{CM}_+(R)$.

Proof of Claim. Take $M \in \operatorname{ind} \mathsf{CM}_0(S)$. Lemma 3.5.2 implies that M is indecomposable as an $R/\mathfrak{p}$-module, and it is indecomposable as an R-module. Taking $X := M$ in Claim 2, we have $M \in \operatorname{ind} \mathsf{CM}_+(R)$. □

It follows from Lemma 3.4.4 that S has infinite CM_0-representation type. Claim 3 implies that R has infinite CM_+-representation type, and the proof of the theorem is completed. ■

We obtain an application of the above theorem, which gives an answer to Question 3.2.6. For a ring R we denote by $\overline{R}$ the integral closure of R. Recall that a typical example of a henselian Nagata ring is a complete local ring.

Corollary 3.6.10. *Let R be a 2-dimensional henselian Nagata Cohen–Macaulay non-normal local ring. Suppose that R has finite CM_+-representation type. Then the following statements hold.*

(1) *There exists a minimal prime $\mathfrak{p}$ of R such that the integral closure $\overline{R/\mathfrak{p}}$ has finite CM-representation type. In particular, if R is a domain, then $\overline{R}$ has finite CM-representation type.*

(2) *If R is Gorenstein, then R is a hypersurface.*

Proof. By Corollary 3.3.3(1) the singular locus of R has dimension at most one, so that R satisfies Serre's condition (R_0). As R is Cohen–Macaulay, it is reduced. Let $S = \overline{R}$ be the integral closure of R. We have a decomposition $S = \overline{R/\mathfrak{p}_1} \oplus \cdots \oplus \overline{R/\mathfrak{p}_n}$ as R-modules, where $\operatorname{Min} R = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ (see [85, Corollary 2.1.13]). Since R is Nagata, the extension $R \subseteq S$ is finite. The ring S is normal and has dimension two, so it is Cohen–Macaulay.

We claim that if $\mathfrak{p}$ is a nonmaximal prime ideal of R such that $S_\mathfrak{p}$ is $R_\mathfrak{p}$-free, then $R_\mathfrak{p}$ is a regular local ring. In fact, if $\operatorname{ht}\mathfrak{p} = 0$, then $R_\mathfrak{p}$ is a field. Let $\operatorname{ht}\mathfrak{p} = 1$. The induced map $\operatorname{Spec} S \to \operatorname{Spec} R$ is surjective, and we find a prime ideal P of S such that $P \cap R = \mathfrak{p}$. We easily see $\operatorname{ht} P = 1$. As S is normal, S_P is regular. The induced map $R_\mathfrak{p} \to S_P$ factors as $R_\mathfrak{p} \xrightarrow{a} S_\mathfrak{p} \xrightarrow{b} S_P$, where a is a finite free extension, and b is flat since $S_P = (S_\mathfrak{p})_{PS_\mathfrak{p}}$. Hence $R_\mathfrak{p} \to S_P$ is a flat local homomorphism. As S_P is regular, so is $R_\mathfrak{p}$.

Since R does not have an isolated singularity, there exists a nonmaximal prime ideal $\mathfrak{p}$ of R such that $R_\mathfrak{p}$ is not regular. The claim implies that $S_\mathfrak{p}$ is not $R_\mathfrak{p}$-free, whence $S \in \mathsf{CM}_+(R)$. There exists an integer $1 \le l \le n$ such that $T := \overline{R/\mathfrak{p}_l}$ belongs to $\mathsf{CM}_+(R)$.

Put $\mathfrak{p} := \mathfrak{p}_l \in \operatorname{Min} R$. The ring $R/\mathfrak{p}$ is also Nagata, and the extension $R/\mathfrak{p} \subseteq T$ is finite and birational. The ring T is a 2-dimensional henselian normal local domain, whence it is a Cohen–Macaulay. Choose a nonmaximal prime ideal $\mathfrak{q}$ of R such that $T_\mathfrak{q}$ is not $R_\mathfrak{q}$-free. If $\mathfrak{p}$ is not contained in $\mathfrak{q}$, then $(R/\mathfrak{p})_\mathfrak{q} = \kappa(\mathfrak{p})_\mathfrak{q} = 0$ and $T_\mathfrak{q} = 0$, which particularly says that $T_\mathfrak{q}$ is $R_\mathfrak{q}$-free, a contradiction. Hence $\mathfrak{p} \subseteq \mathfrak{q}$.

Suppose that $R_\mathfrak{q}$ is a direct summand of $T_\mathfrak{q}$. Then there is an isomorphism $T_\mathfrak{q} \cong R_\mathfrak{q} \oplus X$ of $R_\mathfrak{q}$-modules. Since $T_\mathfrak{q}$ is annihilated by $\mathfrak{p}$, so is $R_\mathfrak{q}$. We have ring extensions $R_\mathfrak{q} = (R/\mathfrak{p})_\mathfrak{q} \subseteq T_\mathfrak{q} \subseteq \kappa(\mathfrak{p})$, which especially says that $R_\mathfrak{q}$ is a domain and that $T_\mathfrak{q}$ has rank one as an $R_\mathfrak{q}$-module. Hence the $R_\mathfrak{q}$-module X has rank zero, and it is easy to see that $X = 0$. We get $T_\mathfrak{q} \cong R_\mathfrak{q}$, which contradicts the choice of $\mathfrak{q}$. Consequently, $T_\mathfrak{q}$ does not have a direct summand isomorphic to $R_\mathfrak{q}$.

Now, application of Theorem 3.6.9 proves the assertion (1). To show (2), we consider the T-module $U = \Omega_T^2(T/\mathfrak{m}T)$. Fix any nonzero direct summand X of U or T in $\operatorname{\mathsf{mod}} R$. Note that $T = \overline{R/\mathfrak{p}}$ is a torsion-free module over $R/\mathfrak{p}$. Since U is a submodule of a nonzero free T-module, U is also torsion-free over $R/\mathfrak{p}$, and so is X. We easily see from this that $X_\mathfrak{q} \neq 0$. The module $X_\mathfrak{q}$ is a direct summand of $U_\mathfrak{q} \cong T_\mathfrak{q}^{\oplus \operatorname{edim} R - 1}$. As $R_\mathfrak{q}$ is not a direct summand of $T_\mathfrak{q}$, it is not a direct summand of $X_\mathfrak{q}$. In particular, X belongs to $\mathsf{CM}_+(R)$. Thus, all the indecomposable direct summands of U and of T in $\operatorname{\mathsf{mod}} R$ belong to $\operatorname{\mathsf{ind}} \mathsf{CM}_+(R)$, and it follows from Lemma

3.4.7 that they have complexity at most one. Hence U and T have complexity at most one over R, and so does $T/\mathfrak{m}T$. We obtain $\operatorname{cx}_R k \le 1$, and R is a hypersurface by [10, Theorem 8.1.2]. ■

The above result yields a strong restriction for finite CM_+-representation type in dimension two.

Corollary 3.6.11. *Let R be a 2-dimensional non-normal Gorenstein complete local ring. If R has finite CM_+-representation type, then the integral closure $\overline{R}$ of R has finite CM-representation type.*

Proof. If R is a domain, then the assertion follows from Corollary 3.6.10(1). Hence let us assume that R is not a domain. By Corollary 3.6.10(2) the ring R is a hypersurface. We can apply Corollary 3.6.8 to see that there exists a 3-dimensional regular local ring S and elements $x, y \in S$ such that R is isomorphic to $S/(xy)$ and $S/(x), S/(y)$ have finite CM-representation type. Note by [82, Corollary 2] that $S/(x), S/(y)$ are normal. As in the beginning of the proof of Corollary 3.6.10, the ring R is reduced. Hence $(x) \neq (y)$, and we have an isomorphism $\overline{R} \cong \overline{S/(x)} \times \overline{S/(y)} = S/(x) \times S/(y)$; see [85, Corollary 2.1.13]. There is a natural category equivalence $\operatorname{\mathsf{mod}} \overline{R} \cong \operatorname{\mathsf{mod}} S/(x) \times \operatorname{\mathsf{mod}} S/(y)$, which induces a category equivalence $\mathsf{CM}(\overline{R}) \cong \mathsf{CM}(S/(x)) \times \mathsf{CM}(S/(y))$. It is observed from this that $\overline{R}$ has finite CM-representation type. ■

The converse of Corollary 3.6.11 does not necessarily hold, as the following example says.

Example 3.6.12. Let $R = k[\![x, y, z]\!]/(x^4 - y^3 z)$ be a quotient of the formal power series ring $k[\![x, y, z]\!]$ over a field k. Then R is a 2-dimensional complete non-normal local hypersurface. The assignment $x \mapsto s^3 t$, $y \mapsto s^4$, $z \mapsto t^4$ gives an isomorphism from R to the subring $S = k[\![s^4, s^3 t, t^4]\!]$ of the formal power series ring $T = k[\![s, t]\!]$. The integral closure of S is the fourth Veronese subring $k[\![s^4, s^3 t, s^2 t^2, s t^3, t^4]\!]$ of T, which has finite CM-representation type by [108, Theorem 6.3]. Hence $\overline{R}$ has finite CM-representation type. However, as $x^4 - y^3 z = x^4 + x^2 y \cdot 0 + y^2(-yz)$, the ring R does not have finite CM_+-representation type by Example 3.6.5.

Remark 3.6.13. The integral closure has to actually be regular (under the assumptions of Corollary 3.6.11) provided that our conjecture that countable CM-representation type is equivalent to finite CM_+-representation type holds true in this setting.

Chapter 4

Ulrich modules over Cohen–Macaulay local rings

with minimal multiplicity

Introduction

The contents of this chapter is based on author's work [95] and author's paper [99] with R. Takahashi.

The notion of an *Ulrich module*, which is also called a *maximally generated (maximal) Cohen–Macaulay module*, has first been studied by Ulrich [144], and widely investigated in both commutative algebra and algebraic geometry; see [17, 26, 37, 63, 64, 76, 92, 115] for example. In [144] the natural question is posed to ask whether Ulrich modules exist over any Cohen–Macaulay local ring R. A lot of partial affirmative answers to this question have been obtained so far. One of them states that the conjecture holds whenever R has minimal multiplicity ([17]). Thus, in this paper, mainly assuming that R has minimal multiplicity, we are interested in what we can say about the structure of Ulrich R-modules.

We begin with exploring the number and generation of Ulrich modules. The following theorem is a special case of our main results in this direction (Ω denotes the first syzygy).

Theorem 4.0.1. *Let $(R, \mathfrak{m}, k)$ be a d-dimensional complete Cohen–Macaulay local ring.*

(1) *Assume that R is normal with $d = 2$ and $k = \mathcal{C}$ and has minimal multiplicity. If R does not have a rational singularity, then there exist infinitely many indecomposable Ulrich R-modules.*

(2) *Suppose that R has an isolated singularity. Let M, N be maximal Cohen–Macaulay R-modules with $\mathrm{Ext}^i_R(M, N) = 0$ for all $1 \le i \le d-1$. If either M or N is Ulrich, then so is $\mathrm{Hom}_R(M, N)$.*

(3) *Let $\boldsymbol{x} = x_1, \ldots, x_d$ be a system of parameters of R such that $\mathfrak{m}^2 = \boldsymbol{x}\mathfrak{m}$. If M is an Ulrich R-module, then so is $\Omega(M/x_iM)$ for all $1 \le i \le d$. If one chooses M to be indecomposable and not to be a direct summand of $\Omega^d k$, then one finds an indecomposable Ulrich R-module not isomorphic to M among the direct summands of the modules $\Omega(M/x_iM)$.*

Next, we relate the Ulrich modules with the syzygies of maximal Cohen–Macaulay modules. To state our result, we fix some notation. Let R be a Cohen–Macaulay local ring with canonical module ω. We denote by $\mathsf{mod}\,R$ the category of finitely generated R-modules, and by $\mathsf{Ul}(R)$ and $\Omega\mathsf{CM}^{\times}(R)$ the full subcategories of Ulrich modules and first syzygies of maximal Cohen–Macaulay modules without free summands, respectively. Denote by $(-)^{\dagger}$ the canonical dual $\mathrm{Hom}_R(-, \omega)$. Then $\mathsf{Ul}(R)$ is closed under $(-)^{\dagger}$, and contains $\Omega\mathsf{CM}^{\times}(R)$ if R has minimal multiplicity. The module $\Omega^d k$ belongs to $\Omega\mathsf{CM}^{\times}(R)$, and hence $\Omega^d k, (\Omega^d k)^{\dagger}$ belong to $\mathsf{Ul}(R)$. Thus it is natural to ask when the conditions in the theorem below hold, and we actually answer this.

Theorem 4.0.2. *Let R be a d-dimensional singular Cohen–Macaulay local ring with residue field k and canonical module ω, and assume that R has minimal multiplicity. Consider the following conditions.*

(1) *The equality* $\mathsf{Ul}(R) = \Omega\mathsf{CM}^\times(R)$ *holds.*

(2) *The category* $\Omega\mathsf{CM}^\times(R)$ *is closed under* $(-)^\dagger$.

(3) *The module* $(\Omega^d k)^\dagger$ *belongs to* $\Omega\mathsf{CM}^\times(R)$.

(4) *There is an isomorphism* $\Omega^d k \cong (\Omega^d k)^\dagger$.

(5) *The local ring* R *is almost Gorenstein (see Definition 4.3.8).*

Then (1)–(3) are equivalent and (4) implies (1). If $d > 0$ *and* k *is infinite, then (1) implies (5). If* $d = 1$ *and* k *is infinite, then (1)–(5) are equivalent. If* R *is complete normal with* $d = 2$ *and* $k = \mathbb{C}$*, then (1)–(4) are equivalent unless* R *has a cyclic quotient singularity.*

As the first step to prove this theorem, we give the following theorem. This is a one-dimensional version of the theorem above, but we don't assume that the ring has minimal multiplicity.

Theorem 4.0.3. *Let* B *be the endomorphism ring* $\mathrm{End}_R(\mathfrak{m})$ *of* $\mathfrak{m}$ *over* R*. Assume that* $d = 1$*,* R *has a canonical module, and* k *is infinite. Then the followings are equivalent.*

(1) *the natural inclusion* $R \to B$ *induces an equivalence* $\mathsf{CM}(B) \cong \Omega\mathsf{CM}^\times(R)$ *of categories.*

(2) R *is almost Gorenstein.*

(3) $B \cong \mathfrak{m}^\dagger$ *as* R*-modules.*

Finally, we study the structure of the category $\mathsf{Ul}(R)$ of Ulrich R-modules as an exact category in the sense of Quillen [122]. We prove that if R has minimal multiplicity, then $\mathsf{Ul}(R)$ admits an exact structure with enough projective/injective objects.

Theorem 4.0.4. *Let* R *be a* d*-dimensional Cohen–Macaulay local ring with residue field* k *and canonical module, and assume that* R *has minimal multiplicity. Let* $\mathcal{S}$ *be the class of short exact sequences* $0 \to L \to M \to N \to 0$ *of* R*-modules with* L, M, N *Ulrich. Then* $(\mathsf{Ul}(R), \mathcal{S})$ *is an exact category having enough projective objects and enough injective objects with* $\mathrm{proj}\,\mathsf{Ul}(R) = \mathsf{add}\,\Omega^d k$ *and* $\mathrm{inj}\,\mathsf{Ul}(R) = \mathsf{add}(\Omega^d k)^\dagger$.

The organization of this paper is as follows. In Section 4.1, we deal with a question of Cuong on the number of indecomposable Ulrich modules. We prove the first assertion of Theorem 4.0.1 to answer this question in the negative. In Section 4.2, we consider how to generate Ulrich modules from given ones, and prove the second and third assertions of Theorem 4.0.1. In Section 4.3, we compare Ulrich modules with syzygies of maximal Cohen–Macaulay modules, and prove Theorem 4.0.2; in fact, we obtain more equivalent and related conditions. The final Section 4.4 is devoted to giving applications of the results obtained in Section 4.3. In this section we study the cases of dimension one and two, and exact structures of Ulrich modules, and prove the rest assertions of Theorem 4.0.2 and Theorem 4.0.4.

4.1 A question of Cuong

In the rest of this chapter, let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring of Krull dimension d. In this section, we consider a question raised by Cuong [27] on the number of Ulrich modules over Cohen–Macaulay local rings with minimal multiplicity. First of all, let us recall the definitions of an Ulrich module and minimal multiplicity.

Definition 4.1.1. (1) An R-module M is called *Ulrich* if M is Cohen–Macaulay with $e(M) = \mu(M)$.

(2) The ring R is said to have *minimal multiplicity* if $e(R) = \operatorname{edim} R - \dim R + 1$.

An Ulrich module is also called a *maximally generated (maximal) Cohen–Macaulay* module. There is always an inequality $e(R) \geq \operatorname{edim} R - \dim R + 1$, from which the name of minimal multiplicity comes. If k is infinite, then R has minimal multiplicity if and only if $\mathfrak{m}^2 = Q\mathfrak{m}$ for some parameter ideal Q of R. See [19, Exercise 4.6.14] for details of minimal multiplicity.

The following question has been raised by Cuong [27].

Question 4.1.2 (Cuong)**.** If R is non-Gorenstein and has minimal multiplicity, then are there only finitely many indecomposable Ulrich R-modules?

To explore this question, we start by introducing notation, which is used throughout the paper.

Notation 4.1.3. We denote by $\mathsf{mod}\, R$ the category of finitely generated R-modules. We use the following subcategories of $\mathsf{mod}\, R$:

$$\begin{aligned}
\mathsf{Ul}(R) &= \{M \in \mathsf{CM}(R) \mid M \text{ is Ulrich}\},\\
\Omega\mathsf{CM}(R) &= \left\{ M \in \mathsf{CM}(R) \,\middle|\, \begin{array}{c} M \text{ is the kernel of an epimorphism from a} \\ \text{free module to a maximal Cohen–Macaulay module} \end{array} \right\},\\
\Omega\mathsf{CM}^{\times}(R) &= \{M \in \Omega\mathsf{CM}(R) \mid M \text{ does not have a (nonzero) free summand}\}.
\end{aligned}$$

Remark 4.1.4. (1) The subcategories $\mathsf{CM}(R), \mathsf{Ul}(R), \Omega\mathsf{CM}(R), \Omega\mathsf{CM}^{\times}(R)$ of $\mathsf{mod}\, R$ are closed under finite direct sums and direct summands.

(2) One has $\Omega\mathsf{CM}(R) \cup \mathsf{Ul}(R) \subseteq \mathsf{CM}(R) \subseteq \mathsf{mod}\, R$.

Here we make a remark to reduce to the case where the residue field is infinite.

Remark 4.1.5. Consider the faithfully flat extension $S := R[t]_{\mathfrak{m}R[t]}$ of R. Then we observe that:

(1) If X is a module in $\Omega\mathsf{CM}^{\times}(R)$, then $X \otimes_R S$ is in $\Omega\mathsf{CM}^{\times}(S)$.

(2) A module Y is in $\mathsf{Ul}(R)$ if and only if $Y \otimes_R S$ is in $\mathsf{Ul}(S)$ (see [85, Lemma 6.4.2]).

The converse of (1) also holds true; we prove this in Corollary 4.3.4.

For any finitely generated R-module M, we denote by $\operatorname{Tr} M$ the Auslander transpose of M. For an integer $n \geq 1$, we define $\mathcal{F}_n(R) = \{M \mid \operatorname{Ext}^i_R(\operatorname{Tr} M, R) = 0 \text{ for } i = 1, \ldots, n\}$ as a full subcategory of $\mathsf{mod}(R)$. A module M in $\mathcal{F}_n$ is called *n-torsionfree*.

We have the following characterization of modules in $\mathsf{CM}(R)$, $\Omega\mathsf{CM}(R)$, or $\mathsf{Ref}(R)$.

Lemma 4.1.6. (1) *One has* $\mathsf{Ref}(R) = \mathcal{F}_2(R)$.

(2) *Assume that* $R_\mathfrak{p}$ *is Gorenstein for every prime ideal* $\mathfrak{p}$ *of* R *with* $\operatorname{ht}\mathfrak{p}$ *being at most* $d-1$. *Then* $\mathcal{F}_d(R) = \mathsf{CM}(R)$ *and* $\mathcal{F}_{d+1}(R) = \Omega\mathsf{CM}(R)$.

(3) *Assume* $d = 1$ *and* R *is generically Gorenstein (i.e.* $R_\mathfrak{p}$ *is Gorenstein for any minimal prime* $\mathfrak{p}$ *of* R*). Then* $\mathsf{Ref}(R) = \Omega\mathsf{CM}(R)$.

Proof. (1) See [108, Proposition 12.5] for instance. (2) See [51, Theorems 3.6 and 3.8] to prove the equality $\mathcal{F}_d(A) = \mathsf{CM}(A)$ holds. The proof of [88, Proposition 2.4] shows that the equality $\mathcal{F}_{d+1}(A) = \Omega\mathsf{CM}(A)$ holds. (3) This is a combination of (1) and (2). ■

If R has minimal multiplicity, then all syzygies of maximal Cohen–Macaulay modules are Ulrich:

Proposition 4.1.7. *Suppose that* R *has minimal multiplicity. Then* $\Omega\mathsf{CM}^\times(R)$ *is contained in* $\mathsf{Ul}(R)$.

Proof. By Remark 4.1.5 we may assume that k is infinite. Since R has minimal multiplicity, we have $\mathfrak{m}^2 = Q\mathfrak{m}$ for some parameter ideal Q of R. Let M be a Cohen–Macaulay R-module. There is a short exact sequence $0 \to \Omega M \to R^{\oplus n} \to M \to 0$, where n is the minimal number of generators of M. Since M is Cohen–Macaulay, taking the functor $R/Q \otimes_R -$ preserves the exactness; we get a short exact sequence

$$0 \to \Omega M/Q\Omega M \xrightarrow{f} (R/Q)^{\oplus n} \to M/QM \to 0.$$

The map f factors through the inclusion map $X := \mathfrak{m}(R/Q)^{\oplus n} \to (R/Q)^{\oplus n}$, and hence there is an injection $\Omega M/Q\Omega M \to X$. As X is annihilated by $\mathfrak{m}$, so is $\Omega M/Q\Omega M$. Therefore $\mathfrak{m}\Omega M = Q\Omega M$, which implies that ΩM is Ulrich. ■

As a direct consequence of [40, Corollary 3.3], we obtain the following proposition.

Proposition 4.1.8. *Let* R *be a* 2*-dimensional normal excellent henselian local ring with algebraically closed residue field of characteristic* 0*. Then there exist only finitely many indecomposable modules in* $\Omega\mathsf{CM}(R)$ *if and only if* R *has a rational singularity.*

Combining the above propositions yields the following result.

Corollary 4.1.9. *Let* R *be a* 2*-dimensional normal excellent henselian local ring with algebraically closed residue field of characteristic* 0*. Suppose that* R *has minimal multiplicity and does not have a rational singularity. Then there exist infinitely many indecomposable Ulrich* R*-modules. In particular, Quenstion 4.1.2 has a negative answer.*

Proof. Proposition 4.1.8 implies that $\Omega\mathsf{CM}(R)$ contains infinitely many indecomposable modules, and so does $\mathsf{Ul}(R)$ by Proposition 4.1.7. ■

Here is an example of a non-Gorenstein ring satisfying the assumption of Corollary 4.1.9, which concludes that the question of Cuong is negative.

Example 4.1.10. Let $B = \mathcal{C}[x, y, z, t]$ be a polynomial ring with $\deg x = \deg t = 3$, $\deg y = 5$ and $\deg z = 7$. Consider the 2×3-matrix $M = \begin{pmatrix} x & y & z \\ y & z & x^3 - t^3 \end{pmatrix}$ over B, and let I be the ideal of B generated by 2×2-minors of M. Set $A = B/I$. Then A is a nonnegatively graded $\mathcal{C}$-algebra as I is homogeneous. By virtue of the Hilbert–Burch theorem ([19, Theorem 1.4.17]), A is a 2-dimensional Cohen–Macaulay ring, and x, t is a homogeneous system of parameters of A. Directly calculating the Jacobian ideal J of A, we can verify that A/J is Artinian. The Jacobian criterion implies that A is a normal domain. The quotient ring A/tA is isomorphic to the numerical semigroup ring $\mathcal{C}[H]$ with $H = \langle 3, 5, 7 \rangle$. Since this ring is not Gorenstein (as H is not symmetric), neither is A. Let $a(A)$ and $F(H)$ stand for the a-invariant of A and the Frobenius number of H, respectively. Then

$$a(A) + 3 = a(A) + \deg(t) = a(A/tA) = F(H) = 4,$$

where the third equality follows from [129, Theorem 3.1]. Therefore we get $a(A) = 1 \not< 0$, and A does not have a rational singularity by the Flenner–Watanabe criterion (see [108, Page 98]).

Let A' be the localization of A at A_+, and let R be the completion of the local ring A'. Then R is a 2-dimensional complete (hence excellent and henselian) normal non-Gorenstein local domain with residue field $\mathcal{C}$. The maximal ideal $\mathfrak{m}$ of R satisfies $\mathfrak{m}^2 = (x, t)\mathfrak{m}$, and thus R has minimal multiplicity. Having a rational singularity is preserved by localization since A has an isolated singularity, while it is also preserved by completion. Therefore R does not have a rational singularity.

We have seen that Question 4.1.2 is not true in general. However, in view of Corollary 4.1.9, we wonder if having a rational singularity is essential. Thus, we pose a modified question.

Question 4.1.11. Let R be a 2-dimensional normal local ring with a rational singularity. Then does R have only finitely many indecomposable Ulrich modules?

Proposition 4.1.8 leads us to an even stronger question:

Question 4.1.12. If $\Omega\mathsf{CM}(R)$ contains only finitely many indecomposable modules, then does $\mathsf{Ul}(R)$ so?

4.2 Generating Ulrich modules

In this section, we study how to generate Ulrich modules from given ones. First of all, we consider using the Hom functor to do it.

Proposition 4.2.1. *Let M, N be Cohen–Macaulay R-modules such that $\operatorname{Ext}^i_R(M, N) = 0$ for all $1 \le i \le d - 1$. If either M or N is Ulrich, then so is $\operatorname{Hom}_R(M, N)$.*

Proof. Take a free resolution

$$\cdots \to F_{d+1} \to F_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

of the R-module M. Dualizing this by N and using the assumption on Ext, we get an exact sequence

$$0 \to \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(F_0, N) \to \operatorname{Hom}_R(F_1, N) \to \cdots \to \operatorname{Hom}_R(F_{d-1}, N) \to \operatorname{Hom}_R(F_d, N).$$

The depth lemma implies that $\operatorname{Hom}_R(M,N)$ is a Cohen–Macaulay R-module.

We may assume that k is infinite by Remark 4.1.5(2), so that we can find a reduction Q of $\mathfrak{m}$ which is a parameter ideal of R. Write $Q=(x_1,\dots,x_d)$.

We show the assertion by induction on d. Let us consider the case $d=1$. There are exact sequences

$$0 \to N \xrightarrow{x_1} N \to N/QN \to 0, \qquad 0 \to M \xrightarrow{x_1} M \to M/QM \to 0,$$

which induce injections

$$\begin{aligned}
&\operatorname{Hom}_R(M,N)/Q\operatorname{Hom}_R(M,N) \hookrightarrow \operatorname{Hom}_R(M,N/QN),\\
&\operatorname{Hom}_R(M,N)/Q\operatorname{Hom}_R(M,N) \hookrightarrow \operatorname{Ext}_R^1(M/QM,N).
\end{aligned}$$

If N (resp. M) is Ulrich, then N/QN (resp. M/QM) is annihilated by $\mathfrak{m}$, and so is $\operatorname{Hom}_R(M,N/QN)$ (resp. $\operatorname{Ext}_R^1(M/QM,N)$). In either case, the quotient module $\operatorname{Hom}_R(M,N)/Q\operatorname{Hom}_R(M,N)$ is annihilated by $\mathfrak{m}$, which shows that $\operatorname{Hom}_R(M,N)$ is an Ulrich R-module.

Next we consider the case $d \geq 2$. Clearly, M/x_1M and N/x_1N are Cohen–Macaulay $R/(x_1)$-modules. There are isomorphisms

$$\begin{aligned}
\operatorname{Ext}^i_{R/(x_1)}(M/x_1M,N/x_1N) &\cong \operatorname{Ext}_R^{i+1}(M/x_1M,N)\\
&\cong \begin{cases} 0 & (1 \leq i \leq d-2),\\ \operatorname{Hom}_R(M,N)/x_1\operatorname{Hom}_R(M,N) & (i=0),\end{cases}
\end{aligned}$$

where the first isomorphism follows from [19, Lemma 3.1.16], and the second isomorphism is shown by using the exact sequence $0 \to M \xrightarrow{x_1} M \to M/x_1M \to 0$ and our assumption on Ext. Furthermore, it is easily observed that if M (resp. N) is an Ulrich R-module, then M/x_1M (resp. N/x_1N) is an Ulrich $R/(x_1)$-module. The induction hypothesis implies that $\operatorname{Hom}_{R/(x_1)}(M/x_1M,N/x_1N)$ is an Ulrich $R/(x_1)$-module, which is isomorphic to $\operatorname{Hom}_R(M,N)/x_1\operatorname{Hom}_R(M,N)$. Now it is deduced that $\operatorname{Hom}_R(M,N)$ is an Ulrich R-module. ■

As an immediate consequence of Proposition 4.2.1, we obtain the following corollary, which is a special case of [63, Theorem 5.1].

Corollary 4.2.2. *Suppose that R admits a canonical module. If $M \in \mathsf{Ul}(R)$, then $M^\dagger \in \mathsf{Ul}(R)$.*

Next, we consider taking extensions of given Ulrich modules to obtain a new one.

Proposition 4.2.3. *Let Q be a parameter ideal of R which is a reduction of $\mathfrak{m}$. Let M,N be Ulrich R-modules, and take any element $a \in Q$. Let $\sigma : 0 \to M \to E \to N \to 0$ be an exact sequence, and consider the multiplication $a\sigma : 0 \to M \to X \to N \to 0$ as an element of the R-module $\operatorname{Ext}_R^1(N,M)$. Then X is an Ulrich R-module.*

Proof. It follows from [132, Theorem 1.1] that the exact sequence

$$a\sigma \otimes_R R/aR : \ 0 \to M/aM \to X/aX \to N/aN \to 0$$

splits; we have an isomorphism $X/aX \cong M/aM \oplus N/aN$. Applying the functor $- \otimes_{R/aR} R/Q$, we get an isomorphism $X/QX \cong M/QM \oplus N/QN$. Since M,N are Ulrich, the modules $M/QM, N/QN$ are k-vector spaces, and so is X/QX. Hence X is also Ulrich. ■

As an application of the above proposition, we give a way to make an Ulrich module over a Cohen–Macaulay local ring with minimal multiplicity.

Corollary 4.2.4. *Let Q be a parameter ideal of R such that $\mathfrak{m}^2 = Q\mathfrak{m}$. Let M be an Ulrich R-module. Then for each R-regular element $a \in Q$, the syzygy $\Omega(M/aM)$ is also an Ulrich R-module.*

Proof. There is an exact sequence $\sigma : 0 \to \Omega M \to R^{\oplus n} \to M \to 0$, where n is a minimal number of generators of M. We have a commutative diagram

$$\begin{array}{rcccccccccc}
 & & & & & 0 & & 0 & & & \\
 & & & & & \uparrow & & \uparrow & & & \\
 & & & & & M/aM & = & M/aM & & & \\
 & & & & & \uparrow & & \uparrow & & & \\
\sigma : & & 0 & \longrightarrow & \Omega M & \longrightarrow R^{\oplus n} & \longrightarrow & M & \longrightarrow & 0 & \\
 & & & & \| & \uparrow & & \uparrow{\scriptstyle a} & & & \\
a\sigma : & & 0 & \longrightarrow & \Omega M & \longrightarrow X & \longrightarrow & M & \longrightarrow & 0 & \\
 & & & & & \uparrow & & \uparrow & & & \\
 & & & & & 0 & & 0 & & &
\end{array}$$

with exact rows and columns. Since the minimal number of generators of M/aM is equal to n, the middle column shows $X \cong \Omega(M/aM)$. Propositions 4.1.2 and 4.2.3 show that X is Ulrich, and we are done. ■

Remark 4.2.5. In Corollary 4.2.4, if the parameter ideal Q annihilates the R-module $\operatorname{Ext}^1_R(M, \Omega M)$, then we have $a\sigma = 0$, and $\Omega(M/aM) \cong M \oplus \Omega M$. Hence, in this case, the operation $M \mapsto \Omega(M/aM)$ does not produce an essentially new Ulrich module.

Next, we investigate the annihilators of Tor and Ext modules.

Proposition 4.2.6. *For an R-module M one has*

$$\begin{aligned}
\operatorname{Ann}_R \operatorname{Ext}^1_R(M, \Omega M) &= \textstyle\bigcap_{i>0,\, N \in \mathsf{mod}\, R} \operatorname{Ann}_R \operatorname{Ext}^i_R(M, N) \\
= \operatorname{Ann}_R \operatorname{Tor}^R_1(M, \operatorname{Tr} M) &= \textstyle\bigcap_{i>0,\, N \in \mathsf{mod}\, R} \operatorname{Ann}_R \operatorname{Tor}^R_i(M, N).
\end{aligned}$$

Proof. It is clear that

$$\begin{aligned}
I &:= \textstyle\bigcap_{i>0,\, N \in \mathsf{mod}\, R} \operatorname{Ann}_R \operatorname{Ext}^i_R(M, N) \subseteq \operatorname{Ann}_R \operatorname{Ext}^1_R(M, \Omega M) \\
J &:= \textstyle\bigcap_{i>0,\, N \in \mathsf{mod}\, R} \operatorname{Ann}_R \operatorname{Tor}^R_i(M, N) \subseteq \operatorname{Ann}_R \operatorname{Tor}^R_1(M, \operatorname{Tr} M).
\end{aligned}$$

It is enough to show that $\operatorname{Ann} \operatorname{Ext}^1(M, \Omega M) \cup \operatorname{Ann} \operatorname{Tor}_1(M, \operatorname{Tr} M)$ is contained in $I \cap J$.

(1) Take any element $a \in \operatorname{Ann}_R \operatorname{Ext}^1_R(M, \Omega M)$. The proof of [89, Lemma 2.14] shows that the multiplication map $(M \xrightarrow{a} M)$ factors through a free module, that is, $(M \xrightarrow{a} M) = (M \xrightarrow{f} F \xrightarrow{\pi} M)$ with F free. Hence, for all $i > 0$ and $N \in \mathsf{mod}\, R$ we have commutative diagrams:

$$\begin{array}{ccccc}
\operatorname{Tor}_i(M,N) & & \xrightarrow{\quad a \quad} & & \operatorname{Tor}_i(M,N) \\
 & {\scriptstyle \operatorname{Tor}_i(f,N)} \searrow & & \nearrow {\scriptstyle \operatorname{Tor}_i(\pi,N)} & \\
 & & \operatorname{Tor}_i(F,N) & &
\end{array}
\qquad
\begin{array}{ccccc}
\operatorname{Ext}^i(M,N) & & \xrightarrow{\quad a \quad} & & \operatorname{Ext}^i(M,N) \\
 & {\scriptstyle \operatorname{Ext}^i(\pi,N)} \searrow & & \nearrow {\scriptstyle \operatorname{Ext}^i(f,N)} & \\
 & & \operatorname{Ext}^i(F,N) & &
\end{array}$$

As $\operatorname{Tor}_i(F, N) = \operatorname{Ext}^i(F, N) = 0$, the element a is in $I \cap J$.

(2) Let $b \in \operatorname{Ann}_R \operatorname{Tor}_1^R(M, \operatorname{Tr} M)$. By [149, Lemma (3.9)], the element b annihilates $\underline{\operatorname{Hom}}_R(M, M)$. Hence the map $b \cdot \mathrm{id}_M$, which is nothing but the multiplication map $(M \xrightarrow{b} M)$, factors through a free R-module. Similarly to (1), we get b is in $I \cap J$. ■

Definition 4.2.7. We denote by $\mathsf{A}M$ the ideal in the above proposition.

Note that $\mathsf{A}M = R$ if and only if M is a free R-module.

For an R-module M we denote by $\operatorname{\mathsf{add}} M$ the subcategory of $\operatorname{\mathsf{mod}} R$ consisting of direct summands of finite direct sums of copies of M.

With the notation of Remark 4.2.5, we are interested in when the operation $M \mapsto \Omega(M/aM)$ actually gives rise to an essentially new Ulrich module. The following result presents a possible way: if we choose an indecomposable Ulrich module M that is not a direct summand of $\Omega^d k$, then we find an indecomposable Ulrich module not isomorphic to M among the direct summands of the modules $\Omega(M/x_iM)$.

Proposition 4.2.8. *Suppose that R is henselian. Let $Q = (x_1, \dots, x_d)$ be a parameter ideal of R which is a reduction of $\mathfrak{m}$. Let M be an indecomposable Ulrich R-module. If M is a direct summand of $\Omega(M/x_iM)$ for all $1 \le i \le d$, then M is a direct summand of $\Omega^d k$.*

Proof. For all integer $1 \le i \le d$ the module $\operatorname{Ext}_R^1(M, \Omega M)$ is a direct summand of $\operatorname{Ext}_R^1(\Omega(M/x_iM), \Omega M)$. The latter module is annihilated by x_i since it is isomorphic to $\operatorname{Ext}_R^2(M/x_iM, \Omega M)$. Hence Q is contained in $\operatorname{Ann}_R \operatorname{Ext}_R^1(M, \Omega M) = \mathsf{A}M$, and therefore $Q \operatorname{Ext}_R^{>0}(M, N) = 0$ for all $N \in \operatorname{\mathsf{mod}} R$. It follows from [142, Corollary 3.2(1)] that M is a direct summand of $\Omega^d(M/QM)$. As M is Ulrich, the module M/QM is a k-vector space, and $\Omega^d(M/QM)$ belongs to $\mathsf{add}(\Omega^d k)$, whence so does M. Since R is henselian and M is indecomposable, the Krull–Schmidt theorem implies that M is a direct summand of $\Omega^d k$. ■

4.3 Comparison of $\mathsf{Ul}(R)$ with $\Omega\mathsf{CM}^\times(R)$

In this section, we study the relationship of the Ulrich R-modules with the syzygies of Cohen–Macaulay R-modules. We begin with giving equivalent conditions for a given maximal Cohen–Macaulay module to be a syzygy of a maximal Cohen–Macaulay module, after stating an elementary lemma.

Lemma 4.3.1. *Let M, N be R-modules. The evaluation map* $\mathrm{ev} : M \otimes_R \operatorname{Hom}_R(M, N) \to N$ *is surjective if and only if there exists an epimorphism $(f_1, \dots, f_n) : M^{\oplus n} \to N$.*

Proof. The "only if" part follows by taking an epimorphism $R^{\oplus n} \to \operatorname{Hom}_R(M, N)$ and tensoring M. To show the "if" part, pick any element $y \in N$. Then we have $y = f_1(x_1) + \cdots + f_n(x_n)$ for some $x_1, \dots, x_n \in M$. Therefore $y = \mathrm{ev}(\sum_{i=1}^n x_i \otimes f_i))$, and we are done. ■

Proposition 4.3.2. *Let R be a Cohen–Macaulay local ring with canonical module ω. Then the following are equivalent for a Cohen–Macaulay R-module M.*

(1) $M \in \Omega\mathsf{CM}(R)$.

(2) $\underline{\operatorname{Hom}}_R(M, \omega) = 0$.

(3) *There exists a surjective homomorphism* $\omega^{\oplus n} \to \mathrm{Hom}_R(M, \omega)$.

(4) *The natural homomorphism* $\Phi : \omega \otimes_R \mathrm{Hom}_R(\omega, \mathrm{Hom}_R(M, \omega)) \to \mathrm{Hom}_R(M, \omega)$ *is surjective.*

(5) M *is torsionless and* $\mathrm{Tr}\,\Omega\,\mathrm{Tr}\,M$ *is Cohen–Macaulay.*

(6) $\mathrm{Ext}^1_R(\mathrm{Tr}\,M, R) = \mathrm{Ext}^1_R(\mathrm{Tr}\,\Omega\,\mathrm{Tr}\,M, \omega) = 0$.

(7) $\mathrm{Tor}^R_1(\mathrm{Tr}\,M, \omega) = 0$.

Proof. (1) ⇒ (2): By the assumption, there is an exact sequence $0 \to M \to F \to N \to 0$ such that N is Cohen–Macaulay and F is free. Take $f \in \mathrm{Hom}_R(M, \omega)$. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow & & \| & & \\ 0 & \longrightarrow & \omega & \longrightarrow & W & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

with exact rows. Since N is Cohen–Macaulay, we have $\mathrm{Ext}^1_R(N, \omega) = 0$. Hence the second row splits, and f factors through F. This shows $\underline{\mathrm{Hom}}_R(M, \omega) = 0$.

(2) ⇒ (1): There is an exact sequence $0 \to M \xrightarrow{f} \omega^{\oplus m} \to N \to 0$ such that N is Cohen–Macaulay. Since $\underline{\mathrm{Hom}}_R(M, \omega^{\oplus m}) = \underline{\mathrm{Hom}}_R(M, \omega)^{\oplus m} = 0$, there are a free R-module F, homomorphisms $g : M \to F$ and $h : F \to \omega^{\oplus m}$ such that $f = hg$. We get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{g} & F & \longrightarrow & L & \longrightarrow & 0 \\ & & \| & & \downarrow h & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{f} & \omega^{\oplus m} & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

with exact rows. The secound square is a pullback-pushout diagram, which gives an exact sequence $0 \to F \to L \oplus \omega^{\oplus m} \to N \to 0$. This shows that L is Cohen–Macaulay, and hence $M \in \Omega\mathsf{CM}(R)$.

(2) ⇔ (7): This equivalence follows from [149, Lemma (3.9)].

(1) ⇒ (3): Let $0 \to M \to R^{\oplus n} \to N \to 0$ be an exact sequence with F free. Applying $(-)^\dagger$, we have an exact sequence $0 \to N^\dagger \to \omega^{\oplus n} \to M^\dagger \to 0$.

(3) ⇒ (1): There is an exact sequence $0 \to K \to \omega^{\oplus n} \to M^\dagger \to 0$. It is seen that K is Cohen–Macaulay. Taking $(-)^\dagger$ gives an exact sequence $0 \to M \to R^{\oplus n} \to K^\dagger \to 0$, which shows $M \in \Omega\mathsf{CM}(R)$.

(3) ⇔ (4): This follows from Lemma 4.3.1.

(5) ⇔ (6): The module $\mathrm{Tr}\,\Omega\,\mathrm{Tr}\,M$ is Cohen–Macaulay if and only if $\mathrm{Ext}^i_R(\mathrm{Tr}\,\Omega\,\mathrm{Tr}\,M, \omega) = 0$ for all $i > 0$. One has $\mathrm{Ext}^1_R(\mathrm{Tr}\,M, R) = 0$ if and only if M is torsionless, if and only if $M \cong \Omega\,\mathrm{Tr}\,\Omega\,\mathrm{Tr}\,M$ up to free summands; see [8, Theorem (2.17)]. Hence $\mathrm{Ext}^i_R(\mathrm{Tr}\,\Omega\,\mathrm{Tr}\,M, \omega) = \mathrm{Ext}^{i-1}_R(M, \omega) = 0$ for all $i > 1$.

(1) ⇔ (5): This equivalence follows from a similar argument in the proof of [88, Proposition 2.4]. ■

Remark 4.3.3. The equivalence (1) ⇔ (5) in Proposition 4.3.2 holds without the assumption that R admits a canonical module. Indeed, its proof does not use the existence of a canonical module.

The property of being a syzygy of a maximal Cohen–Macaulay module (without free summand) is preserved under faithfully flat extension.

Corollary 4.3.4. *Let $R \to S$ be a faithfully flat homomorphism of Cohen–Macaulay local rings. Let M be a Cohen–Macaulay R-module. Then $M \in \Omega\mathsf{CM}^{\times}(R)$ if and only if $M \otimes_R S \in \Omega\mathsf{CM}^{\times}(S)$.*

Proof. Using Remark 4.3.3, we see that $M \in \Omega\mathsf{CM}(R)$ if and only if $\operatorname{Ext}^1_R(\operatorname{Tr}_R M, R) = 0$ and $\operatorname{Tr}_R \Omega_R \operatorname{Tr}_R M$ is Cohen–Macaulay. Also, M has a nonzero R-free summand if and only if the evaluation map $M \otimes_R \operatorname{Hom}_R(M, R) \to R$ is surjective by Lemma 4.3.1. Since the latter conditions are both preserved under faithfully flat extension, they are equivalent to saying that $M \otimes_R S \in \Omega\mathsf{CM}(S)$ and that $M \otimes_R S$ has a nonzero S-free summand, respectively. Now the assertion follows. ■

Next we state and prove a couple of lemmas. The first one concerns Ulrich modules and syzygies of maximal Cohen–Macaulay modules with respect to short exact sequences.

Lemma 4.3.5. *Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules.*

(1) *If L, M, N are in $\mathsf{Ul}(R)$, then the equality $\mu(M) = \mu(L) + \mu(N)$ holds.*

(2) *Suppose that L, M, N are in $\mathsf{CM}(R)$. Then:*
(a) *If M is in $\mathsf{Ul}(R)$, then so are L and N.* (b) *If M is in $\Omega\mathsf{CM}^{\times}(R)$, then so is L.*

Proof. (1) We have $\mu(M) = e(M) = e(L) + e(N) = \mu(L) + \mu(N)$.

(2) Assertion (a) follows by [17, Proposition (1.4)]. Let us show (b). As M is in $\Omega\mathsf{CM}^{\times}(R)$, there is an exact sequence $0 \to M \xrightarrow{\beta} R^{\oplus a} \xrightarrow{\gamma} C \to 0$ with C Cohen–Macaulay. As M has no free summand, γ is a minimal homomorphism. In particular, $\mu(C) = a$. The pushout of β and γ gives a commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \beta} & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & R^{\oplus a} & \xrightarrow{\delta} & D & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \gamma} & & \downarrow & & \\
 & & & & C & = & C & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

with exact rows and columns. We see that $a = \mu(C) \le \mu(D) \le a$, which implies that δ is a minimal homomorphism. Hence $L = \Omega D \in \Omega\mathsf{CM}^{\times}(R)$. ■

The following lemma is used to reduce to the case of a lower dimensional ring.

Lemma 4.3.6. *Let $Q = (x_1, \dots, x_d)$ be a parameter ideal of R that is a reduction of $\mathfrak{m}$. Let M be a Cohen–Macaulay R-module. Then M is an Ulrich R-module if and only if M/x_iM is an Ulrich R/x_iR-module.*

Proof. Note that Q/x_iR is a reduction of $\mathfrak{m}/x_iR$. We see that $(\mathfrak{m}/x_iR)(M/x_iM) = (Q/x_iR)(M/x_iM)$ if and only if $\mathfrak{m}M = QM$. Thus the assertion holds. ■

Now we explore syzygies of the residue field of a Cohen–Macaulay local ring with minimal multiplicity.

Lemma 4.3.7. *Assume that R is singular and has minimal multiplicity.*

(1) *One has $\Omega^d_R k \in \Omega\mathsf{CM}^\times(R)$. In particular, $\Omega^d_R k$ is an Ulrich R-module.*

(2) *There is an isomorphism $\Omega^{d+1}_R k \cong (\Omega^d_R k)^{\oplus n}$ for some $n \geq 0$.*

(3) *Let $Q = (x_1, \dots, x_d)$ be a parameter ideal of R with $\mathfrak{m}^2 = Q\mathfrak{m}$, and suppose that $d \geq 1$. Then $\Omega^1_R(\Omega^i_{R/(x_1)} k) \cong \Omega^{i+1}_R k$ for all $i \geq 0$. In particular, $\Omega^1_R(\Omega^{d-1}_{R/(x_1)} k) \cong \Omega^d_R k$.*

(4) *For each $M \in \mathsf{Ul}(R)$ there exists a surjective homomorphism $(\Omega^d_R k)^{\oplus n} \to M$ for some $n \geq 0$.*

Proof. (1)(2) We may assume that k is infinite; see Remark 4.1.5. So we find a parameter ideal $Q = (x_1, \dots, x_d)$ of R with $\mathfrak{m}^2 = Q\mathfrak{m}$. The module $\mathfrak{m}/Q$ is a k-vector space, and there is an exact sequence $0 \to k^{\oplus n} \to R/Q \to k \to 0$. Taking the dth syzygies gives an exact sequence

$$0 \to (\Omega^d k)^{\oplus n} \to R^{\oplus t} \to \Omega^d k \to 0.$$

Since $\Omega^d k$ has no free summand by [136, Theorem 1.1], we obtain $\Omega^d k \in \Omega\mathsf{CM}^\times(R)$ and $(\Omega^d k)^{\oplus n} \cong \Omega^{d+1} k$. The last assertion of (1) follows from this and Proposition 4.1.7.

(3) Set $x = x_1$. We show that $\Omega(\Omega^i_{R/xR} k) \cong \Omega^{i+1} k$ for all $i \geq 0$. We may assume $i \geq 1$; note then that x is $\Omega^i k$-regular. By [136, Corollary 5.3] we have an isomorphism $\Omega^i k / x\Omega^i k \cong \Omega^i_{R/xR} k \oplus \Omega^{i-1}_{R/xR} k$. Hence

$$\Omega^i k \oplus \Omega^{i+1} k \cong \Omega(\Omega^i k / x\Omega^i k) \cong \Omega(\Omega^i_{R/xR} k) \oplus \Omega(\Omega^{i-1}_{R/xR} k), \tag{4.3.7.1}$$

where the first isomorphism follows from the proof of Corollary 4.2.4. There is an exact sequence $0 \to \Omega^i_{R/xR} k \to (R/xR)^{\oplus a_{i-1}} \to \cdots \to (R/xR)^{\oplus a_0} \to k \to 0$ of R/xR-modules, which gives an exact sequence

$$0 \to \Omega(\Omega^i_{R/xR} k) \to R^{\oplus b_{i-1}} \to \cdots \to R^{\oplus b_0} \to \Omega k \to 0$$

of R-modules. This shows $\Omega(\Omega^i_{R/xR} k) \cong \Omega^{i+1} k \oplus R^{\oplus u}$ for some $u \geq 0$, and similarly we have an isomorphism $\Omega(\Omega^{i-1}_{R/xR} k) \cong \Omega^i k \oplus R^{\oplus v}$ for some $v \geq 0$. Substituting these in (4.3.7.1), we see $u = v = 0$ and obtain an isomorphism $\Omega(\Omega^i_{R/xR} k) \cong \Omega^{i+1} k$.

(4) According to Lemma 4.3.1 and Remark 4.1.5, we may assume that k is infinite. Take a parameter ideal $Q = (x_1, \dots, x_d)$ of R with $\mathfrak{m}^2 = Q\mathfrak{m}$. We prove this by induction on d. If $d = 0$, then M is a k-vector space, and there is nothing to show. Assume $d \geq 1$ and set $x = x_1$. Clearly, R/xR has minimal multiplicity. By Lemma 4.3.6, M/xM is an Ulrich R/xR-module. The induction hypothesis gives an exact sequence $0 \to L \to (\Omega^{d-1}_{R/xR} k)^{\oplus n} \to M/xM \to 0$ of R/xR-modules. Lemma 4.3.5(2) shows that L is also an Ulrich R/xR-module, while Lemma 4.3.5(1) implies

$$\mu_{R/xR}(L) + \mu_{R/xR}(M/xM) = \mu_{R/xR}((\Omega^{d-1}_{R/xR} k)^{\oplus n}).$$

Note that $\mu_R(X) = \mu_{R/xR}(X)$ for an R/xR-module X. Thus, taking the first syzygies over R, we get an exact sequence of R-modules:

$$0 \to \Omega L \to \Omega(\Omega^{d-1}_{R/xR} k)^{\oplus n} \to \Omega(M/xM) \to 0.$$

From the proof of Corollary 4.2.4 we see that there is an exact sequence $0 \to \Omega M \to \Omega(M/xM) \to M \to 0$, while $\Omega(\Omega^{d-1}_{R/xR}k)$ is isomorphic to $\Omega^d k$ by (3). Consequently, we obtain a surjection $(\Omega^d k)^{\oplus n} \to M$. ■

Here we recall the definition of an almost Gorenstein local ring, which is introduced in [65].

Definition 4.3.8. Let R be a Cohen–Macaulay local ring of dimension d with canonical module ω. Then R is called *almost Gorenstein* if there exists an exact sequence

$$0 \to R \to \omega \to C \to 0$$

of finitely generated R-modules such that $e(C) = \mu(C)$.

We have reached the stage to state and prove the main result of this section.

Theorem 4.3.9. *Let R be a d-dimensional Cohen–Macaulay local ring with residue field k and canonical module ω. Suppose that R has minimal multiplicity. Then the following are equivalent.*

(1) *The equality $\Omega\mathsf{CM}^\times(R) = \mathsf{UI}(R)$ holds.*

(2) *For an exact sequence $M \to N \to 0$ in $\mathsf{CM}(R)$, if $M \in \Omega\mathsf{CM}^\times(R)$, then $N \in \Omega\mathsf{CM}^\times(R)$.*

(3) *The category $\Omega\mathsf{CM}^\times(R)$ is closed under $(-)^\dagger$.*

(4) *The module $(\Omega^d k)^\dagger$ belongs to $\Omega\mathsf{CM}^\times(R)$.* (4') *The module $(\Omega^d k)^\dagger$ belongs to $\Omega\mathsf{CM}(R)$.*

(5) *One has $\underline{\mathrm{Hom}}_R((\Omega^d k)^\dagger, \omega) = 0$.*

(6) *One has $\mathrm{Tor}_1^R(\mathrm{Tr}((\Omega^d k)^\dagger), \omega) = 0$.*

(7) *One has $\mathrm{Ext}_R^{d+1}(\mathrm{Tr}((\Omega^d k)^\dagger), R) = 0$ and R is locally Gorenstein on the punctured spectrum.*

(8) *The natural homomorphism $\omega \otimes_R \mathrm{Hom}_R(\omega, \Omega^d k) \to \Omega^d k$ is surjective.*

(9) *There exists a surjective homomorphism $\omega^{\oplus n} \to \Omega^d k$.*

If d is positive, k is infinite and one of the above nine conditions holds, then R is almost Gorenstein.

Proof of the equivalence of (1)–(9). (1) ⇒ (2): This follows from Lemma 4.3.5(2).

(2) ⇒ (3): Let M be an R-module in $\Omega\mathsf{CM}^\times(R)$. Then $M \in \mathsf{UI}(R)$ by Proposition 4.1.7, and hence $M^\dagger \in \mathsf{UI}(R)$ by Corollary 4.2.2. It follows from Lemma 4.3.7(4) that there is a surjection $(\Omega^d k)^{\oplus n} \to M^\dagger$. Since $(\Omega^d k)^{\oplus n}$ is in $\Omega\mathsf{CM}^\times(R)$ by Lemma 4.3.7(1), the module $M^\dagger$ is also in $\Omega\mathsf{CM}^\times(R)$.

(3) ⇒ (4): Lemma 4.3.7(1) says that $\Omega^d k$ is in $\Omega\mathsf{CM}^\times(R)$, and so is $(\Omega^d k)^\dagger$ by assumption.

(4) ⇒ (1): The inclusion $\Omega\mathsf{CM}^\times(R) \subseteq \mathsf{UI}(R)$ follows from Proposition 4.1.7. Take any module M in $\mathsf{UI}(R)$. Then $M^\dagger$ is also in $\mathsf{UI}(R)$ by Corollary 4.2.2. Using Lemma 4.3.7(4), we get an exact sequence $0 \to X \to (\Omega^d k)^{\oplus n} \to M^\dagger \to 0$ of maximal Cohen–Macaulay modules, which induces an exact sequence $0 \to M \to (\Omega^d k)^{\dagger\oplus n} \to X^\dagger \to 0$. The assumption and Lemma 4.3.5(2) imply that M is in $\Omega\mathsf{CM}^\times(R)$.

(4) $\Leftrightarrow$ (4'): As R is singular, by [136, Corollary 4.4] the module $(\Omega^d k)^\dagger$ does not have a free summand.

(4') $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9): These equivalences follow from Proposition 4.3.2.

(4') $\Leftrightarrow$ (7): We claim that, under the assumption that R is locally Gorenstein on the punctured spectrum, $(\Omega^d k)^\dagger \in \Omega\mathsf{CM}(R)$ if and only if $\operatorname{Ext}_R^{d+1}(\operatorname{Tr}((\Omega^d k)^\dagger), R) = 0$. In fact, since $(\Omega^d k)^\dagger$ is Cohen–Macaulay, it satisfies Serre's condition (S_d). Therefore it is d-torsionfree, that is, $\operatorname{Ext}_R^i(\operatorname{Tr}((\Omega^d k)^\dagger), R) = 0$ for all $1 \le i \le d$; see [113, Theorem 2.3]. Hence, $\operatorname{Ext}_R^{d+1}(\operatorname{Tr}((\Omega^d k)^\dagger), R) = 0$ if and only if $(\Omega^d k)^\dagger$ is $(d+1)$-torsionfree, if and only if it belongs to $\Omega\mathsf{CM}(R)$ by [113, Theorem 2.3] again. Thus the claim follows.

According to this claim, it suffices to prove that if (4') holds, then R is locally Gorenstein on the punctured spectrum. For this, pick any nonmaximal prime ideal $\mathfrak{p}$ of R. There are exact sequences

$$0 \to \Omega^d k \to R^{\oplus a_{d-1}} \to \cdots \to R^{\oplus a_0} \to k \to 0, \qquad 0 \to (\Omega^d k)_\mathfrak{p} \to R_\mathfrak{p}^{\oplus a_{d-1}} \to \cdots \to R_\mathfrak{p}^{\oplus a_0} \to 0.$$

We observe that $(\Omega^d k)_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module with $\operatorname{rank}_{R_\mathfrak{p}}((\Omega^d k)_\mathfrak{p}) = \sum_{i=0}^{d-1}(-1)^i a_{d-1-i} = \operatorname{rank}_R(\Omega^d k)$. The module $\Omega^d k$ has positive rank as it is torsionfree, and we see that $(\Omega^d k)_\mathfrak{p}$ is a nonzero free $R_\mathfrak{p}$-module. Since we have already shown that (4') implies (9), there is a surjection $\omega^{\oplus n} \to \Omega^d k$. Localizing this at $\mathfrak{p}$, we see that $\omega_\mathfrak{p}^{\oplus n}$ has an $R_\mathfrak{p}$-free summand, which implies that the $R_\mathfrak{p}$-module $R_\mathfrak{p}$ has finite injective dimension. Thus $R_\mathfrak{p}$ is Gorenstein.

So far we have proved the equivalence of the conditions (1)–(9). It remains to prove that R is almost Gorenstein under the assumption that d is positive, k is infinite and (1)–(9) all hold. We prove this assertion after the proof of Theorem 4.0.3; see section 4.4.1. ■

Remark 4.3.10. When $d \ge 2$, it holds that

$$\operatorname{Ext}_R^{d+1}(\operatorname{Tr}((\Omega^d k)^\dagger), R) \cong \operatorname{Ext}_R^{d-1}(\operatorname{Hom}_R(\omega, \Omega^d k), R).$$

Thus Theorem 4.3.9(7) can be replaced with the condition that $\operatorname{Ext}_R^{d-1}(\operatorname{Hom}_R(\omega, \Omega^d k), R) = 0$.

Indeed, using the Hom-$\otimes$ adjointness twice, we get isomorphisms

$$\operatorname{Hom}_R(\omega, \Omega^d k) \cong \operatorname{Hom}_R(\omega, (\Omega^d k)^{\dagger\dagger}) \cong \operatorname{Hom}_R((\Omega^d k)^\dagger \otimes_R \omega, \omega) \cong \operatorname{Hom}_R((\Omega^d k)^\dagger, \omega^\dagger) \cong (\Omega^d k)^{\dagger *},$$

and $(\Omega^d k)^{\dagger *}$ is isomorphic to $\Omega^2 \operatorname{Tr}((\Omega^d k)^\dagger)$ up to free summand.

We have several more conditions related to the equality $\Omega\mathsf{CM}^\times(R) = \mathsf{UI}(R)$.

Corollary 4.3.11. *Let R be as in Theorem 4.3.9. Consider the following conditions:*

(1) $(\Omega^d k)^\dagger \cong \Omega^d k$, (2) $(\Omega^d k)^\dagger \in \mathsf{add}(\Omega^d k)$, (3) $\mathsf{A}(\Omega^d k)^\dagger = \mathfrak{m}$, (4) $\Omega\mathsf{CM}^\times(R) = \mathsf{UI}(R)$.
It then holds that (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longrightarrow$ (4).

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. The proof of Proposition 4.2.8 shows that if an Ulrich R-module M satisfies $\mathsf{A}M = \mathfrak{m}$, then M is in $\mathsf{add}(\Omega^d k)$. This shows (3) $\Rightarrow$ (2). Proposition 4.3.7(1) says that $\Omega^d k$ is in $\Omega\mathsf{CM}^\times(R)$, and so is $(\Omega^d k)^\dagger$ by assumption. Theorem 4.3.9 shows (2) $\Rightarrow$ (4). ■

We close this section by constructing an example by applying the above corollary.

Example 4.3.12. Let $S = \mathcal{C}[[x, y, z]]$ be a formal power series ring. Let G be the cyclic group $\frac{1}{2}(1, 1, 1)$, and let $R = S^G$ be the invariant (i.e. the second Veronese) subring of S. Then $\Omega\mathsf{CM}^{\times}(R) = \mathsf{UI}(R)$. In fact, by [149, Proposition (16.10)], the modules R, ω, $\Omega\omega$ are the nonisomorphic indecomposable Cohen–Macaulay R-modules and $(\Omega\omega)^{\dagger} \cong \Omega\omega$. By [136, Theorem 4.3] the module $\Omega^2\mathcal{C}$ does not have a nonzero free or canonical summand. Hence $\Omega^2\mathcal{C}$ is a direct sum of copies of $\Omega\omega$, and thus $(\Omega^2\mathcal{C})^{\dagger} \cong \Omega^2\mathcal{C}$. The equality $\Omega\mathsf{CM}^{\times}(R) = \mathsf{UI}(R)$ follows from Corollary 4.3.11.

4.4 Applications

This section is devoted to stating applications of our main theorems obtained in the previous section.

4.4.1 The case of dimension one

We begin with studying the case where R has dimension 1.

Let B be the endomorphism ring $\operatorname{End}_R(\mathfrak{m})$ of $\mathfrak{m}$. By Remark 2.2.3, B is identified with $\mathfrak{m} : \mathfrak{m}$, a subring of $Q(R)$. Furthermore, B is Cohen–Macaulay of Krull dimension one, semilocal, and module-finite over R. Note that $\mathsf{CM}(B)$ can be considered as a subcategory of $\mathsf{CM}(R)$ via the inclusion $R \to B$.

If ω exists, then we can give an equivalent condition to the equality $\mathsf{CM}(B) = \Omega\mathsf{CM}^{\times}(R)$ by using the canonical dual $(-)^{\dagger}$.

We prepare the following three lemmas about $\Omega\mathsf{CM}(R)$. The first lemma follows from [148, Lemma 2.1].

Lemma 4.4.1. *Let M be a Cohen–Macaulay R-module. Then ΩM has no free summand.*

Lemma 4.4.2. *Let M be an R-module in $\Omega\mathsf{CM}^{\times}(R)$. Then there is an exact sequence*

$$0 \to M \to \mathfrak{m}^{\oplus n} \to N \to 0$$

of modules in $\mathsf{CM}(B)$.

Proof. As M is in $\Omega\mathsf{CM}(R)$, we have an exact sequence $0 \to M \xrightarrow{\alpha} R^{\oplus n} \to N' \to 0$ with a maximal Cohen–Macaulay R-module N. Since M has no free summand, there is a homomorphism $\beta : M \to \mathfrak{m}$ such that $\alpha = i \circ \beta$, where i is the natural inclusion $\mathfrak{m}^{\oplus n} \to R^{\oplus n}$. Let N be the cokernel of β. We have the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\alpha} & R^{\oplus n} & \longrightarrow & N' & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle i} & & \uparrow & & \\
0 & \longrightarrow & M & \xrightarrow{\beta} & \mathfrak{m}^{\oplus n} & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $\beta \in \operatorname{Hom}_R(M, \mathfrak{m}) = \operatorname{Hom}_B(M, \mathfrak{m})$, N is an B-module. The exactness of $0 \to N \to N'$ implies that N is maximal Cohen–Macaulay over R. Thus N is in $\mathsf{CM}(B)$. ■

Lemma 4.4.3. *Assume that R has a canonical module ω. Then the equality $\mathsf{CM}(B) = \Omega\mathsf{CM}^{\times}(R)$ holds if and only if $B^{\dagger} \in \Omega\mathsf{CM}^{\times}(R)$.*

Proof. Since $B^{\dagger} = \mathrm{Hom}_R(B, \omega)$ is an B-module contained in $\mathsf{CM}(B)$, The "only if" part is clear. Now we assume $B^{\dagger} \in \Omega\mathsf{CM}_{\mathcal{P}}(R)$. Let M be in $\mathsf{CM}(B)$. Taking a free cover of $M^{\dagger}$ over B, we get an exact sequence $0 \to N \to B^{\oplus n} \to M^{\dagger} \to 0$ with some B-module N. Since $M^{\dagger}, B \in \mathsf{CM}(R)$, N is also in $\mathsf{CM}(R)$ by the depth lemma. Applying $(-)^{\dagger}$ to this sequence, we have an exact sequence $0 \to M \to (B^{\dagger})^{\oplus n} \to N^{\dagger} \to 0$. Using Lemma 4.3.5, M is in $\Omega\mathsf{CM}^{\times}(R)$. This shows that $\mathsf{CM}(B) = \Omega\mathsf{CM}^{\times}(R)$. ■

If the completion $\widehat{R}$ of R is generically Gorenstein, then R has a canonical module by [62, Proposition 2.7]. In this situation, we see in the next lemma that the condition $\mathsf{CM}(B) = \Omega\mathsf{CM}^{\times}(R)$ is stable under flat local extension.

Corollary 4.4.4. *Let $\varphi : (R, \mathfrak{m}) \to (R', \mathfrak{m}')$ be a flat local homomorphism such that $\mathfrak{m}R' = \mathfrak{m}'$. Assume that the completion $\widehat{R}$ of R is generically Gorenstein. Then $\mathsf{CM}(B) = \Omega\mathsf{CM}^{\times}(R)$ if and only if $\mathsf{CM}(\mathrm{End}_{R'}(\mathfrak{m}')) = \Omega\mathsf{CM}^{\times}(R')$.*

Proof. Let $B' = \mathrm{End}_{R'}(\mathfrak{m}')$. Note that $\widehat{R'}$ is also generically Gorenstein by [62, Proposition 2.12]. In addition, $\omega \otimes_R R'$ is a canonical module of R'. Therefore, by Lemma 4.4.3, $\mathsf{CM}(B) = \Omega\mathsf{CM}^{\times}(R)$ if and only if $B^{\dagger} \in \Omega\mathsf{CM}^{\times}(R)$, and $\mathsf{CM}(B') = \Omega\mathsf{CM}^{\times}(R')$ if and only if $\mathrm{Hom}_{R'}(B', \omega \otimes_R R') \in \Omega\mathsf{CM}^{\times}(R)$. Here $B' = B \otimes_R R'$ and hence $\mathrm{Hom}_{R'}(B', \omega \otimes_R R') = (B^{\dagger}) \otimes_R R'$. Thus the asertion follows by Corollary 4.3.4. ■

Using the above lemma, we can replace R with the completion $\widehat{R}$. We have one more equivalent condition to being $\mathsf{CM}(B) = \Omega\mathsf{CM}^{\times}(R)$.

Lemma 4.4.5. *Assume that the completion $\widehat{R}$ of R is generically Gorenstein. Then $B^{\dagger} \in \Omega\mathsf{CM}_s(R)$ if and only if $B^{\dagger} \cong \mathfrak{m}$.*

Proof. Thanks to Corollary 4.4.4, we can assume that R is complete. If $B^{\dagger} \cong \mathfrak{m}$, then we have $B^{\dagger} \in \Omega\mathsf{CM}_{\mathcal{P}}(R)$. Conversely, we assume $B^{\dagger} \in \Omega\mathsf{CM}^{\times}(R)$. Using Lemma 4.4.2, we get an exact sequence

$$0 \to B^{\dagger} \xrightarrow{\alpha} \mathfrak{m}^{\oplus m} \to N \to 0 \tag{4.4.5.1}$$

of modules in $\mathsf{CM}(B)$. By the Krull–Schmidt theorem for R, we have a unique decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_n$, where $\mathfrak{m}_i$ are indecomposable R-modules. Then we obtain $B = \mathrm{End}_R(\mathfrak{m}_1) \times \cdots \times \mathrm{End}_R(\mathfrak{m}_n)$ as an R-algebra. The components $B_i = \mathrm{End}_R(\mathfrak{m}_i)$ of B are local rings because of the indecomposability of $\mathfrak{m}_i$. Set $\mathfrak{n}_i$ the maximal ideal of B corresponding to the maximal ideal of B_i. Note that the localization $(B^{\dagger})_{\mathfrak{n}_i} = (\mathrm{Hom}_R(B, \omega))_{\mathfrak{n}_i}$ is the canonical module of B_i and the localization $\mathfrak{m}_{\mathfrak{n}_i}$ is equal to $(\mathfrak{m}_i)_{\mathfrak{n}_i}$. Thus, after localizing at $\mathfrak{n}_i$, the sequence (4.4.5.1) becomes split exact and $(B^{\dagger})_{\mathfrak{n}_i}$ is a direct summand of $(\mathfrak{m}_i)_{\mathfrak{n}_i}^{\oplus m}$. The modules $(B^{\dagger})_{\mathfrak{n}_i}$ and $(\mathfrak{m}_i)_{\mathfrak{n}_i}$ are both indecomposable. Hence we obtain an isomorphism $(B^{\dagger})_{\mathfrak{n}_i} \cong (\mathfrak{m}_i)_{\mathfrak{n}_i}$ by the Krull-Schmidt theorem. The homomorphism α is a split injection, since it becomes a split injection after localizing at $\mathfrak{n}_i$ for all $i = 1, \ldots, n$. Therefore $B^{\dagger}$ is isomorphic a direct summand of $\mathfrak{m}^{\oplus m}$. Set $B^{\dagger} \cong \mathfrak{m}_1^{\oplus a_1} \oplus \cdots \oplus \mathfrak{m}_n^{\oplus a_n}$. Then the localization at $\mathfrak{n}_i$ shows that $a_i = 1$. Consequently, $B^{\dagger} \cong \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_n = \mathfrak{m}$. ■

The following lemma will be used to prove Theorem 4.0.3.

Lemma 4.4.6. *Let A be a ring with total quotient ring T, $\overline{A}$ be the integral closure of A in T, and X be an A-submodule of $\overline{A}$ containing A. If there is an isomorphism $\phi : A \to X$ of A-modules, then $X = A$.*

Proof. Let $i\colon A \to X$ be the inclusion homomorphism. Then $\phi^{-1} \circ i\colon A \to A$ is an endomorphism of A. Hence it is a multiplication map by r for some $r \in A$. Since $r = \phi^{-1} \circ i : A \to A$ is injective, $1/r$ is in T. We have $1 = i(1) = \phi(r) = r\phi(1)$ in $\overline{A}$ and hence $1/r = \phi(1) \in \overline{A}$. It means that $1/r$ is integral over A. Therefore we have an equation of integral dependence

$$(1/r)^n + a_1(1/r)^{n-1} + \cdots + a_n = 0,$$

where $a_i \in A$ for all $i = 1, \ldots, n$. Multiplying r^n, we get $1 + r(a_1 + \cdots + a_n r^{n-1}) = 0$. This equation yields that r is a unit of A. Thus the endomorphism $r = \phi^{-1} \circ i : A \to A$ is an automorphism, i is an isomorphism, and $A = X$. ■

Assume that R is complete and has a infinite residue field. Then there is an R-submodule K of $Q(R)$ such that $R \subset K \subset \overline{R}$, and as an R-module, K is a canonical module of R; see [62, Corollary 2.9]. Using this module K, we can give a proof of Theorem 4.0.3.

Proof of Theorem 4.0.3. (1) $\Leftrightarrow$ (3): This can be shown by Lemma 4.4.3 and Lemma 4.4.5.

(3) $\Rightarrow$ (2): We may assume that R is not Gorenstein. Then $R : \mathfrak{m} \subseteq K : \mathfrak{m} \subseteq R[K] \subseteq \overline{R}$ by [62, Corollary 3.8]. On the other hand, $\mathfrak{m}^\dagger = K : \mathfrak{m}$ is isomorphic to $B = R : \mathfrak{m}$ by the assumption. Applying Lemma 4.4.6 to $X = K : \mathfrak{m}$ and $A = B$, we obtain $K : \mathfrak{m} = B$. Thus we see that $\mathfrak{m} = K : (K : \mathfrak{m}) = K : B$. This yields that $\mathfrak{m} : K = (K : B) : K = (K : K) : B = R : B = \mathfrak{m}$. In particular, we have inclusions $\mathfrak{m}K \subset \mathfrak{m} \subset R$. This implies that R is almost Gorenstein by [62, Theorem 3.11].

(2) $\Rightarrow$ (3): We may assume that R is not Gorenstein again. [62, Theorem 3.16] says that $(B =)\mathfrak{m} : \mathfrak{m} = K : \mathfrak{m}$. Then we can check that $\mathfrak{m} = K : (K : \mathfrak{m}) = K : B$. In particular, $\mathfrak{m} \cong B^\dagger$. Taking the canonical duals, we get an isomorphism $\mathfrak{m}^\dagger \cong B$. ■

The remaining part of the proof of Theorem 4.3.9. It remains to prove that R is almost Gorenstein under the assumption that d is positive, k is infinite and (1)–(9) all hold. We use induction on d. Let $d = 1$. Let Q be the total quotient ring of R, and set $B = \mathrm{End}_R(\mathfrak{m})$. Let K be an R-module with $K \cong \omega$ and $R \subseteq K \subseteq \overline{R}$ in Q, where $\overline{R}$ is the integral closure of R. Using [121, Proposition 2.5], we have:

$$\mathfrak{m} \cong \mathrm{Hom}_R(\mathfrak{m}, R) = B \quad \text{and} \quad \mathfrak{m}^\dagger \cong \mathrm{Hom}_R(\mathfrak{m}, K) \cong (K :_Q \mathfrak{m}). \tag{4.4.6.1}$$

By (4) the module $\mathfrak{m}^\dagger$ belongs to $\Omega\mathsf{CM}^\times(R)$. It follows from Theorem 4.0.3 that R is almost Gorenstein; note that the completion of R also has Gorenstein punctured spectrum by (4').

Let $d > 1$. Since $(\Omega^d k)^\dagger \in \Omega\mathsf{CM}(R)$, there is an exact sequence $0 \to (\Omega^d k)^\dagger \to R^{\oplus m} \to N \to 0$ for some $m \geq 0$ and $N \in \mathsf{CM}(R)$. Choose a parameter ideal $Q = (x_1, \ldots, x_d)$ of R satisfying the equality $\mathfrak{m}^2 = Q\mathfrak{m}$, and set $\overline{(-)} = (-) \otimes_R R/(x_1)$. An exact sequence

$$0 \to \overline{(\Omega^d k)^\dagger} \to \overline{R}^{\oplus m} \to \overline{N} \to 0$$

is induced, which shows that $\overline{(\Omega^d k)^\dagger}$ is in $\Omega\mathsf{CM}(\overline{R})$. Applying $(-)^\dagger$ to the exact sequence $0 \to \Omega^d k \xrightarrow{x} \Omega^d k \to \overline{\Omega^d k} \to 0$ and using [19, Lemma 3.1.16], we obtain isomorphisms

$$\overline{(\Omega^d k)^\dagger} \cong \mathrm{Ext}^1_R(\overline{\Omega^d k}, \omega) \cong \mathrm{Hom}_{\overline{R}}(\overline{\Omega^d k}, \overline{\omega}).$$

The module $\Omega^{d-1}_{\overline{R}} k$ is a direct summand of $\overline{\Omega^d k}$ by [136, Corollary 5.3], and hence $\mathrm{Hom}_{\overline{R}}(\Omega^{d-1}_{\overline{R}} k, \overline{\omega})$ is a direct summand of $\mathrm{Hom}_{\overline{R}}(\overline{\Omega^d k}, \overline{\omega})$. Summarizing these, we observe that $\mathrm{Hom}_{\overline{R}}(\Omega^{d-1}_{\overline{R}} k, \overline{\omega})$ belongs to $\Omega\mathsf{CM}(\overline{R})$. Since $\overline{R}$ has minimal multiplicity, we can apply the induction hypothesis to $\overline{R}$ to conclude that $\overline{R}$ is almost Gorenstein, and so is R by [65, Theorem 3.7]. ■

Corollary 4.4.7. *Let $(R, \mathfrak{m}, k)$ be a 1-dimensional Cohen–Macaulay local ring with k infinite and canonical module ω. Suppose that R has minimal multiplicity, and set $(-)^\dagger = \mathrm{Hom}_R(-, \omega)$. Then*

$$\Omega\mathsf{CM}^\times(R) = \mathsf{UI}(R) \iff \mathfrak{m}^\dagger \in \Omega\mathsf{CM}^\times(R) \iff \mathfrak{m}^\dagger \cong \mathfrak{m} \iff R \textit{ is almost Gorenstein.}$$

Proof. Call the four conditions (i)–(iv) from left to right. The implications (i) ⇔ (ii) ⇒ (iv) are shown by Theorem 4.3.9, while (iii) ⇔ (iv) by Theorem 4.0.3 and (4.4.6.1). Lemma 4.3.7(1) shows (iii) ⇒ (ii). ■

Now we pose a question related to Question 4.1.2.

Question 4.4.8. Can we classify 1-dimensional Cohen–Macaulay local rings R with minimal multiplicity (and infinite residue field) satisfying the condition $\#\,\mathsf{ind}\,\mathsf{UI}(R) < \infty$?

If R has finite Cohen–Macaulay representation type (that is, if $\#\,\mathsf{ind}\,\mathsf{CM}(R) < \infty$), then of course this question is affirmative. However, we do not have any partial answer other than this. The reader may wonder if the condition $\#\,\mathsf{ind}\,\mathsf{UI}(R) < \infty$ implies the equality $\Omega\mathsf{CM}^\times(R) = \mathsf{UI}(R)$. Using the above theorem, we observe that this does not necessarily hold:

Example 4.4.9. Let $R = k[[t^3, t^7, t^8]]$ be (the completion of) a numerical semigroup ring, where k is an algebraically closed field of characteristic zero. Then R is a Cohen–Macaulay local ring of dimension 1 with minimal multiplicity. It follows from [76, Theorem A.3] that $\#\,\mathsf{ind}\,\mathsf{UI}(R) < \infty$. On the other hand, R is not almost-Gorenstein by [62, Example 4.3], so $\Omega\mathsf{CM}^\times(R) \neq \mathsf{UI}(R)$ by Corollary 4.4.7.

4.4.2 The case of dimension two

From now on, we consider the case where R has dimension 2. We recall the definition of a Cohen–Macaulay approximation. Let R be a Cohen–Macaulay local ring with canonical module. A homomorphism $f : X \to M$ of R-modules is called a *Cohen–Macaulay approximation* (of M) if X is Cohen–Macaulay and any homomorphism $f' : X' \to M$ with X' being Cohen–Macaulay factors through f. It is known that f is a (resp. minimal) Cohen–Macaulay approximation if and only if there exists an exact sequence

$$0 \to Y \xrightarrow{g} X \xrightarrow{f} M \to 0$$

of R-modules such that X is Cohen–Macaulay and Y has finite injective dimension (resp. and that X, Y have no common direct summand along g). For details of Cohen–Macaulay approximations, we refer the reader to [108, Chapter 11].

The module E appearing in the following remark is called the *fundamental module* of R.

Remark 4.4.10. Let $(R, \mathfrak{m}, k)$ be a 2-dimensional Cohen–Macaulay local ring with canonical module ω.

(1) There exists a nonsplit exact sequence

$$0 \to \omega \to E \to \mathfrak{m} \to 0 \tag{4.4.10.1}$$

which is unique up to isomorphism. This is because $\mathrm{Ext}_R^1(\mathfrak{m}, \omega) \cong \mathrm{Ext}_R^2(k, \omega) \cong k$.

(2) The module E is Cohen–Macaulay and uniquely determined up to isomorphism.

(3) The sequence (4.4.10.1) gives a minimal Cohen–Macaulay approximation of $\mathfrak{m}$.

(4) There is an isomorphism $E \cong E^\dagger$. In fact, applying $(-)^\dagger$ to (4.4.10.1) induces an exact sequence

$$0 \to \mathfrak{m}^\dagger \to E^\dagger \to R \to \mathrm{Ext}_R^1(\mathfrak{m}, \omega) \to \mathrm{Ext}_R^1(E, \omega) = 0.$$

Applying $(-)^\dagger$ to the natural exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$ yields $\mathfrak{m}^\dagger \cong \omega$, while $\mathrm{Ext}_R^1(\mathfrak{m}, \omega) \cong k$. We get an exact sequence $0 \to \omega \to E^\dagger \to \mathfrak{m} \to 0$, and the uniqueness of (4.4.10.1) shows $E^\dagger \cong E$.

To prove the main result of this section, we prepare two lemmas. The first one relates the fundamental module of a 2-dimensional Cohen–Macaulay local ring R with $\mathsf{Ul}(R)$ and $\Omega\mathsf{CM}^\times(R)$.

Lemma 4.4.11. *Let $(R, \mathfrak{m}, k)$ be a 2-dimensional Cohen–Macaulay local ring with canonical module ω and fundamental module E.*

(1) *Assume that R has minimal multiplicity. Then E is an Ulrich R-module.*

(2) *For each module $M \in \Omega\mathsf{CM}^\times(R)$ there exists an exact sequence $0 \to M \to E^{\oplus n} \to N \to 0$ of R-modules such that N is Cohen–Macaulay.*

Proof. (1) We may assume that k is infinite by Remark 4.1.5(2). Let $Q = (x, y)$ be a parameter ideal of R with $\mathfrak{m}^2 = Q\mathfrak{m}$. We have $\mathfrak{m}/x\mathfrak{m} \cong \mathfrak{m}/(x) \oplus k$; see [136, Corollary 5.3]. Note that $(\mathfrak{m}/(x))^2 = y(\mathfrak{m}/(x))$. By [148, Corollary 2.5] the minimal Cohen–Macaulay approximation of $\mathfrak{m}/x\mathfrak{m}$ as an $R/(x)$-module is E/xE. In view of the proof of [108, Proposition 11.15], the minimal Cohen–Macaulay approximations of $\mathfrak{m}/(x)$ and k as $R/(x)$-modules are $\mathfrak{m}/(x)$ and $\mathrm{Hom}_{R/(x)}(\mathfrak{m}/(x), \omega/x\omega)$, respectively. Thus we get an isomorphism

$$E/xE \cong \mathfrak{m}/(x) \oplus \mathrm{Hom}_{R/(x)}(\mathfrak{m}/(x), \omega/x\omega).$$

In particular, E/xE is an Ulrich $R/(x)$-module by Lemma 4.3.7(1) and Corollary 4.2.2. It follows from Lemma 4.3.6 that E is an Ulrich R-module.

(2) Take an exact sequence $0 \to M \xrightarrow{f} R^{\oplus n} \xrightarrow{e} L \to 0$ such that L is Cohen–Macaulay. As M has no free summand, the homomorphism e is minimal. This means that f factors through the

natural inclusion $i : \mathfrak{m}^{\oplus n} \to R^{\oplus n}$, that is, $f = ig$ for some $g \in \mathrm{Hom}_R(M, \mathfrak{m}^{\oplus n})$. The direct sum $p : E^{\oplus n} \to \mathfrak{m}^{\oplus n}$ of copies of the surjection $E \to \mathfrak{m}$ (given by (4.4.10.1)) is a Cohen–Macaulay approximation. Hence there is a homomorphism $h : M \to E^{\oplus n}$ such that $g = ph$. We get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{f} & R^{\oplus n} & \longrightarrow & L & \longrightarrow & 0 \\ & & \| & & {\scriptstyle ip}\uparrow & & \uparrow & & \\ 0 & \longrightarrow & M & \xrightarrow{h} & E^{\oplus n} & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

with exact rows. This induces an exact sequence $0 \to E^{\oplus n} \to R^{\oplus n} \oplus N \to L \to 0$, and therefore N is a Cohen–Macaulay R-module. ■

A short exact sequence of Ulrich modules is preserved by certain functors:

Lemma 4.4.12. *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of modules in $\mathsf{Ul}(R)$. Then it induces exact sequences of R-modules*

(a) $0 \to X \otimes_R k \to Y \otimes_R k \to Z \otimes_R k \to 0$,

(b) $0 \to \mathrm{Hom}_R(Z, k) \to \mathrm{Hom}_R(Y, k) \to \mathrm{Hom}_R(X, k) \to 0$, *and*

(c) $0 \to \mathrm{Hom}_R(Z, (\Omega^d k)^\dagger) \to \mathrm{Hom}_R(Y, (\Omega^d k)^\dagger) \to \mathrm{Hom}_R(X, (\Omega^d k)^\dagger) \to 0$.

Proof. The sequence $X \otimes_R k \to Y \otimes_R k \to Z \otimes_R k \to 0$ is exact and the first map is injective by Lemma 4.3.5(1). Hence (a) is exact, and so is (b) by a dual argument. In what follows, we show that (c) is exact. We first note that $(\Omega^d k)^\dagger$ is a minimal Cohen–Macaulay approximation of k; see the proof of [108, Proposition 11.15]. Thus there is an exact sequence $0 \to I \to (\Omega^d k)^\dagger \to k \to 0$ such that I has finite injective dimension. As $\mathsf{Ul}(R) \subseteq \mathsf{CM}(R)$, we have $\mathrm{Ext}^1_R(M, I) = 0$ for all $M \in \{X, Y, Z\}$. We obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(Y, I) & \longrightarrow & \mathrm{Hom}_R(Y, (\Omega^d k)^\dagger) & \longrightarrow & \mathrm{Hom}_R(Y, k) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(X, I) & \longrightarrow & \mathrm{Hom}_R(X, (\Omega^d k)^\dagger) & \longrightarrow & \mathrm{Hom}_R(X, k) & \longrightarrow & 0 \end{array}$$

with exact rows, where α is surjective. The exactness of (b) implies that γ is surjective. By the snake lemma β is also surjective, and therefore (c) is exact. ■

Now we can state and show our main result in this section.

Theorem 4.4.13. *Let R be a 2-dimensional complete singular normal local ring with residue field $\mathbb{C}$ and having minimal multiplicity. Suppose that R does not have a cyclic quotient singularity. Then:*

$$(\Omega^d k)^\dagger \cong \Omega^d k \iff (\Omega^d k)^\dagger \in \mathsf{add}(\Omega^d k) \iff \mathsf{A}(\Omega^d k)^\dagger = \mathfrak{m} \iff \Omega\mathsf{CM}^\times(R) = \mathsf{Ul}(R).$$

Proof. In view of Corollary 4.3.11, it suffices to show that if R does not have a cyclic quotient singularity, then the fourth condition implies the first one. By virtue of [149, Theorem 11.12] the fundamental module E is indecomposable. Applying Lemma 4.4.11(2) to $(\Omega^d k)^\dagger$, we have an exact sequence $0 \to (\Omega^d k)^\dagger \xrightarrow{\alpha} E^{\oplus n} \to N \to 0$ such that N is Cohen–Macaulay. Since E is

Ulrich by Lemma 4.4.11(1), so are all the three modules in this sequence by Lemma 4.3.5(2). Thus we can apply Lemma 4.4.12 to see that the induced map

$$\mathrm{Hom}_R(\alpha, (\Omega^d k)^\dagger) : \mathrm{Hom}_R(E^{\oplus n}, (\Omega^d k)^\dagger) \to \mathrm{Hom}_R((\Omega^d k)^\dagger, (\Omega^d k)^\dagger)$$

is surjective. This implies that α is a split monomorphism, and $(\Omega^d k)^\dagger$ is isomorphic to a direct summand of $E^{\oplus n}$. Since E is indecomposable, it folllows that $(\Omega^d k)^\dagger$ is isomorphic to $E^{\oplus m}$ for some m. We obtain

$$(\Omega^d k)^\dagger \cong E^{\oplus m} \cong (E^\dagger)^{\oplus m} \cong (\Omega^d k)^{\dagger\dagger} \cong \Omega^d k,$$

where the second isomorphism follows by Remark 4.4.10(4). ∎

Remark 4.4.14. Let R be a cyclic quotient surface singularity over $\mathcal{C}$. Nakajima and Yoshida [115, Theorem 4.5] give a necessary and sufficient condition for R to be such that the number of nonisomorphic indecomposable Ulrich R-modules is equal to the number of nonisomorphic nonfree indecomposable special Cohen–Macaulay R-modules. By [88, Corollary 2.9], the latter is equal to the number of isomorphism classes of indecomposable modules in $\Omega\mathsf{CM}^\times(R)$. Therefore, they actually gives a necessary and sufficient condition for R to satisfy $\Omega\mathsf{CM}^\times(R) = \mathsf{Ul}(R)$.

Using our Theorem 4.4.13, we give some examples of a quotient surface singularity over $\mathcal{C}$ to consider Ulrich modules over them.

Example 4.4.15. (1) Let $S = \mathcal{C}[[x, y]]$ be a formal power series ring. Let G be the cyclic group $\frac{1}{3}(1, 1)$, and let $R = S^G$ be the invariant (i.e. the third Veronese) subring of S. Then $\Omega\mathsf{CM}^\times(R) = \mathsf{Ul}(R)$. This follows from [115, Theorem 4.5] and Remark 4.4.14, but we can also show it by direct caluculation: we have

$$\mathrm{Cl}(R) = \{[R], [\omega], [\mathfrak{p}]\} \cong \mathbb{Z}/3\mathbb{Z},$$

where $\omega = (x^3, x^2y)R$ is a canonical ideal of R, and $\mathfrak{p} = (x^3, x^2y, xy^2)R$ is a prime ideal of height 1 with $[\omega] = 2[\mathfrak{p}]$. Since the second Betti number of $\mathcal{C}$ over R is 9, we see $\Omega^2\mathcal{C} \cong \mathfrak{p}^{\oplus 3}$. As $[\mathfrak{p}^\dagger] = [\omega] - [\mathfrak{p}] = [\mathfrak{p}]$, we have $\mathfrak{p}^\dagger \cong \mathfrak{p}$ and $(\Omega^2\mathcal{C})^\dagger \cong \Omega^2\mathcal{C}$. Theorem 4.4.13 shows $\Omega\mathsf{CM}^\times(R) = \mathsf{Ul}(R)$.

(2) Let $S = \mathcal{C}[[x, y]]$ be a formal power series ring. Let G be the cyclic group $\frac{1}{8}(1, 5)$, and let $R = S^G$ be the invariant subring of S. With the notation of [115], the Hirzebruch-Jung continued fraction of this group is $[2, 3, 2]$. It follows from [115, Theorem 4.5] and Remark 4.4.14 that $\Omega\mathsf{CM}^\times(R) \neq \mathsf{Ul}(R)$.

4.4.3 An exact structure of the category of Ulrich modules

Finally, we consider realization of the additive category $\mathsf{Ul}(R)$ as an exact category in the sense of Quillen [122]. We begin with recalling the definition of an exact category given in [91, Appendix A].

Definition 4.4.16. Let $\mathcal{A}$ be an additive category. A pair (i, d) of composable morphisms

$$X \xrightarrow{i} Y \xrightarrow{d} Z$$

is *exact* if i is the kernel of d and d is the cokernel of i. Let $\mathcal{E}$ be a class of exact pairs closed under isomorphism. The pair $(\mathcal{A}, \mathcal{E})$ is called an *exact category* if the following axioms hold. Here, for each $(i, d) \in \mathcal{E}$ the morphisms i and d are called an *inflation* and a *deflation*, respectively.

(Ex0) $1 : 0 \to 0$ is a deflation.

(Ex1) The composition of deflations is a deflation.

(Ex2) For each morphism $f : Z' \to Z$ and each deflation $d : Y \to Z$, there is a pullback diagram as in the left below, where d' is a deflation.

(Ex2$^{\mathrm{op}}$) For each morphism $f : X \to X'$ and each inflation $i : X \to Y$, there is a pushout diagram as in the right below, where i' is an inflation.

$$\begin{array}{ccc} Y' & \xrightarrow{d'} & Z' \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{d} & Z \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle f}\downarrow & & \downarrow \\ X' & \xrightarrow{i'} & Y' \end{array}$$

We can equip a structure of an exact category with our $\mathsf{Ul}(R)$ as follows.

Theorem 4.4.17. *Let R be a d-dimensional Cohen–Macaulay local ring with residue field k and canonical module, and assume that R has minimal multiplicity. Let $\mathcal{S}$ be the class of exact sequences $0 \to L \to M \to N \to 0$ of R-modules with L, M, N Ulrich. Then $\mathsf{Ul}(R) = (\mathsf{Ul}(R), \mathcal{S})$ is an exact category having enough projective objects and enough injective objects with* $\operatorname{proj} \mathsf{Ul}(R) = \mathsf{add}(\Omega^d k)$ *and* $\operatorname{inj} \mathsf{Ul}(R) = \mathsf{add}((\Omega^d k)^\dagger)$.

Proof. We verify the axioms in Definition 4.4.16.

(Ex0): This is clear.

(Ex1): Let $d : Y \to Z$ and $d' : Z \to W$ be deflations. Then there is an exact sequence $0 \to X \to Y \xrightarrow{d'd} W \to 0$ of R-modules. Since Y is in $\mathsf{Ul}(R)$ and $X, W \in \mathsf{CM}(R)$, it follows from that $X \in \mathsf{Ul}(R)$. Thus this sequence belongs to $\mathcal{S}$, and $d'd$ is a deflation.

(Ex2): Let $f : Z' \to Z$ be a homomorphism in $\mathsf{Ul}(R)$ and $d : Y \to Z$ a deflation in $\mathcal{S}$. Then we get an exact sequence $0 \to Y' \to Y \oplus Z' \xrightarrow{(d,f)} Z \to 0$. Since $Y \oplus Z' \in \mathsf{Ul}(R)$ and $Y', Z \in \mathsf{CM}(R)$, Lemma 4.3.5(2) implies $Y' \in \mathsf{Ul}(R)$. Make an exact sequence $0 \to X' \to Y' \xrightarrow{d'} Z' \to 0$. As $Y' \in \mathsf{Ul}(R)$ and $X', Z' \in \mathsf{CM}(R)$, the module Z' is in $\mathsf{Ul}(R)$ by Lemma 4.3.5(2) again. Thus d' is a deflation.

(Ex2$^{\mathrm{op}}$): We can check this axiom by the opposite argument to (Ex2).

Now we conclude that $(\mathsf{Ul}(R), \mathcal{S})$ is an exact category. Let us prove the remaining assertions. Lemma 4.4.12(c) yields the injectivity of $(\Omega^d k)^\dagger$. Since $(-)^\dagger$ gives an exact duality of $(\mathsf{Ul}(R), \mathcal{S})$, the module $\Omega^d k$ is a projective object. We also observe from Lemma 4.3.7 and Corollary 4.2.2 that $(\mathsf{Ul}(R), \mathcal{S})$ has enough projective objects with $\operatorname{proj} \mathsf{Ul}(R) = \mathsf{add}(\Omega^d k)$, and has enough injective objects with $\operatorname{inj} \mathsf{Ul}(R) = \mathsf{add}((\Omega^d k)^\dagger)$ by the duality $(-)^\dagger$. ■

Remark 4.4.18. Let $(R, \mathfrak{m})$ be 1-dimensional Cohen–Macaulay local ring with infinite residue field. Let (t) be a minimal reduction of $\mathfrak{m}$. Then $\mathsf{Ul}(R) = \mathsf{CM}\left(R\left[\frac{\mathfrak{m}}{t}\right]\right)$ by [76, Proposition A.1]. This equality actually gives an equivalence $\mathsf{Ul}(R) \cong \mathsf{CM}(R[\frac{\mathfrak{m}}{t}])$ of categories, since Hom-sets do not change; see [108, Proposition 4.14]. Thus the usual exact structure on $\mathsf{CM}(R[\frac{\mathfrak{m}}{t}])$ coincides with the exact structure on $\mathsf{Ul}(R)$ given above via this equivalence.

Chapter 5

Local rings with self-dual maximal ideal

5.1 Introduction

The contents of this chapter is based on author's work [97].

Let R be a Cohen-Macaulay local ring with a canonical module ω. For an R-module M, we denote by $M^\dagger$ the R-module $\mathrm{Hom}_R(M, \omega)$. The R-module M is called *self-dual* if there exists an isomorphism $M \xrightarrow{\cong} M^\dagger$ of R-modules. Note that the self-duality of R-modules is independent of the choice of ω.

Let R and S be artinian local rings such that S maps onto R. Denote by $c_S(R)$ the colength $\ell_S(S) - \ell_S(R)$. In the case that S is Gorenstein, the integer $c_S(R)$ is used to estimate homological properties of R, for example, see [105, Theorem 7.5]. Ananthnarayan [1] introduced the *Gorenstein colength* $g(R)$ of an artinian local ring $(R, \mathfrak{m}, k)$ to be the following integer

$$g(R) := \min\{c_S(R) \mid S \text{ is a Gorenstein artinian local ring mapping onto } R\}.$$

The number $g(R)$ measures how close R is to a Gorenstein ring. Clearly, $g(R)$ is zero if and only if R is Gorenstein. One can see that $g(R) = 1$ if and only if R is non-Gorenstein and $R \cong S/\mathrm{soc}(S)$ for an artinian Gorenstein ring S. These rings are called *Teter rings*. On Teter rings, the following characterization is known, which is an improvement of Teter's result [143]. This was proved by Huneke-Vraciu [86] under the assumption that $1/2 \in R$ and $\mathrm{soc}(R) \subseteq \mathfrak{m}^2$, and later Ananthnarayan-Avramov-Moore [2] removed the assumption $\mathrm{soc}(R) \subseteq \mathfrak{m}^2$. See also the result of Elias-Takatsuji [52].

Theorem 5.1.1 (Huneke-Vraciu, Ananthnarayan-Avramov-Moore, Elias-Takatuji)**.** *Let* $(R, \mathfrak{m}, k)$ *be an artinian local ring such that either* R *contains* $1/2$ *or* R *is equicharacteristic with* $\mathrm{soc}(R) \subseteq \mathfrak{m}^2$*. Then the following are equivalent.*

(1) $g(R) \le 1$.

(2) Either R *is Gorenstein or* $\mathfrak{m} \cong \mathfrak{m}^\dagger$.

(3) Either R *is Gorenstein or there exists a surjective homomorphism* $\omega \to \mathfrak{m}$.

Moreover, Ananthnarayan [1] extended this theorem to the case $g(R) \le 2$ as follows.

Theorem 5.1.2 (Ananthnarayan)**.** *Let $(R, \mathfrak{m})$ be an artinian local ring. Write $R \cong T/I$ where $(T, \mathfrak{m}_T)$ is a regular local ring and I is an ideal of T. Suppose $I \subseteq \mathfrak{m}_T^6$ and $1/2 \in R$. Then the following are equivalent.*

(1) $g(R) \leq 2$.

(2) There exists a self-dual ideal $\mathfrak{a} \subseteq R$ such that $l(R/\mathfrak{a}) \leq 2$.

In this paper, we try to extend the notion of Gorenstein colengths and the above results to the case that R is a one-dimensional Cohen-Macaulay local ring.

For a local ring $(R, \mathfrak{m})$, we denote by $Q(R)$ the total quotient ring of R. An extension $S \subseteq R$ of local rings is called *birational* if $R \subseteq Q(S)$. In this case, R and S have same total quotient ring.

Let $(S, \mathfrak{n}) \subseteq (R, \mathfrak{m})$ be an extension of local rings. Suppose $\mathfrak{n} = \mathfrak{m} \cap S$. Then $S \subseteq R$ is called *residually rational* if there is an isomorphism $S/\mathfrak{n} \cong R/\mathfrak{m}$ induced by the natural inclusion $S \to R$. For example, if $S \subseteq R$ is module-finite and $S/\mathfrak{n}$ is algebraically closed, then it automatically follows that $S \subseteq R$ is residually rational. We introduce an invariant $bg(R)$ for local rings R as follows, which is the infimum of Gorenstein colengths in birational maps.

Definition 5.1.3. For a local ring R, we define

$$bg(R) := \inf \left\{ \ell_S(R/S) \,\middle|\, \begin{array}{l} S \text{ is Gorenstein and } S \subseteq R \text{ is a module-finite} \\ \text{residually rational birational map of local rings} \end{array} \right\}.$$

We will state the main results of this paper by using this invariant. The first one is the following theorem, which gives a one-dimensional analogue of Theorem 5.1.1.

Theorem 5.1.4. *Let $(R, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring having a canonical module ω. Consider the following conditions.*

(1) $bg(R) \leq 1$.

(2) Either R is Gorenstein or there exists a Gorenstein local ring $(S, \mathfrak{n})$ of dimension one such that $R \cong \mathrm{End}_S(\mathfrak{n})$.

(3) Either R is Gorenstein or $\mathfrak{m} \cong \mathfrak{m}^\dagger$.

(4) Either R is Gorenstein or there is a short exact sequence $0 \to \omega \to \mathfrak{m} \to k \to 0$.

(5) There is an ideal I of R such that $I \cong \omega$ (i.e. I is a canonical ideal of R) and $l(R/I) \leq 2$.

Then the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) *hold. The direction* (5) $\Rightarrow$ (1) *also holds if R contains an infinite field k as a subalgebra which is isomorphic to $R/\mathfrak{m}$ via the projection $R \to R/\mathfrak{m}$, i.e. R has an infinite coefficient field $k \subseteq R$.*

The existence of a canonical ideal I of R with $\ell_R(R/I) = 2$ is considered by Dibaei-Rahimi [48]. Using their notion, the condition (5) above is equivalent to the condition that $\min(S_{\mathfrak{C}_R}) \leq 2$.

We also remark that Bass's idea [14] tells us the importance of the endomorpshism ring $\mathrm{End}_S(\mathfrak{n})$ of the maximal ideal $\mathfrak{n}$ of a Gorenstein local ring S of dimension one. He shew that any torsion-free S-module without non-zero free summand can be regarded as a module over

$\mathrm{End}_S(\mathfrak{n})$. So we can analyze Cohen-Macaulay representations of R via the ring $\mathrm{End}_S(\mathfrak{n})$ (see also [108, Chapter 4]).

As a corollary, we can characterize Cohen-Macaulay local rings R of dimension one having minimal multiplicity and satisfying $bg(R) \leq 1$. To give the statement of our corollary, we recall some definitions. For a local ring R, we denote by $e(R)$ the multiplicity of R, $r(R)$ the Cohen–Macaulay type, and by $\operatorname{edim} R$ the embedding dimension of R. According to Goto-Matsuoka-Phuong [62], a local ring R is called *almost Gorenstein*, if R posseses a canonical ideal I of R such that $e_1(I) \leq r(R)$, where $e_1(I)$ is the first Hilbert coefficient of I. A Gorenstein ring of dimension one satisfying $e(S) = \operatorname{edim} S + 1$ are called *a ring of almost minimal multiplicity* or a *Gorenstein ring of minimal multiplicity*, and studied by J. D. Sally [130]. The invariant $\rho(R)$ is the *canonical index* of R, introduced by Ghezzi-Goto-Hong-Vasconcelos [57].

Corollary 5.1.5. *Let $(R, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring. Consider the following conditions.*

(1) $bg(R) \leq 1$ and R has minimal multiplicity.

(2) Either $e(R) \leq 2$ or R is almost Gorenstein with $bg(R) = 1$.

(3) $\mathfrak{m} \cong \mathfrak{m}^\dagger$ and R is almost Gorenstein.

(4) $\mathfrak{m} \cong \mathfrak{m}^\dagger$ and R has minimal multiplicity.

(5) R is almost Gorenstein and has minimal multiplicity.

(6) There exists a Gorenstein local ring $(S, \mathfrak{n})$ of dimension one such that $e(S) \leq \operatorname{edim} S + 1$ and $R \cong \mathrm{End}_S(\mathfrak{n})$.

(7) $\mathfrak{m} \cong \mathfrak{m}^\dagger$ and $\rho(R) \leq 2$.

Then $(1) \Leftrightarrow (2) \Rightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5)$ holds. If $R/\mathfrak{m}$ is infinite, then $(5) \Leftrightarrow (7)$ and $(6) \Rightarrow (4)$ hold. If R has an infinite coefficient field $k \subseteq R$, then all the conditions are equivalent.

The second main theorem of this paper is the following, which is a one-dimensional analogue of Theorem 5.1.2.

Theorem 5.1.6. *Let $(R, \mathfrak{m})$ be a complete one-dimensional Cohen-Macaulay local ring. Consider the following conditions.*

(1) $bg(R) \leq 2$.

(2) There exists a self-dual ideal $\mathfrak{a} \subseteq R$ such that $\ell_R(R/\mathfrak{a}) \leq 2$.

Then (1) implies (2). The implication $(2) \Rightarrow (1)$ also holds if R has an infinite coefficient field $k \subseteq R$.

In the view of Theorem 5.1.4, local rings with self-dual maximal ideal are naturally constructed from Gorenstein local rings, and so their ubiquity is certified. It is interesting to consider what good properties they have compared to Gorenstein rings. In section 3, we have an observation that a Cohen-Macaulay local ring $(R, \mathfrak{m})$ is nearly Gorenstein (see Definition 5.3.4

for the definition) if $\mathfrak{m}$ is self-dual. The converse of this is not true in general, however, we have the following result. Here $\mathfrak{m} : \mathfrak{m}$ is a subring of $Q(R)$ consisting of the elements a satisfying $a\mathfrak{m} \subseteq \mathfrak{m}$.

Theorem 5.1.7. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring of dimension one. Put $B = \mathfrak{m} : \mathfrak{m}$. Assume k is infinite.*

(1) If B is local with Cohen-Macaulay type two and R is nearly Gorenstein, then R is almost Gorenstein and does not satisfy $\mathfrak{m} \cong \mathfrak{m}^\dagger$.

(2) If B is local with Cohen-Macaulay type three and R is nearly Gorenstein, then either R is almost Gorenstein or $\mathfrak{m} \cong \mathfrak{m}^\dagger$.

We will provide a proof of Theorem 5.1.7 in section 3. One should compare this theorem with the following result of Goto-Matsuoka-Phuong [62, Theorem 5.1].

Theorem 5.1.8 (Goto-Matsuoka-Phuong)**.** *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring of dimension one. Put $B = \mathfrak{m} : \mathfrak{m}$. Then B is Gorenstein if and only if R is almost Gorenstein and has minimal multiplicity.*

In section 4, we deal with numerical semigroup rings having self-dual maximal ideal. The definition of UESY-semigroups was given by [124]. These numerical semigroups are exactly the semigroups obtained by adding one element to a symmetric numerical semigroup. We will show that a numerical semigroup ring has self-dual maximal ideal if and only if the corresponding numerical semigroup is UESY. After that, we also prove that the rings of UESY-numerical semigroup have quasi-decomposable maximal ideal. According to [116], an ideal I of R is called *quasi-decomposable* if there exists a regular sequence $\underline{x} = x_1, \dots, x_t$ such that $I/(\underline{x})$ is decomposable as an R-module. Local rings with quasi-decomposable maximal ideal have some interesting properties; we can classify thich subcategories of the singularity category with some assumption on the punctured spectrum ([116, Theorem 4.5]), and we have results on the vanishings of Ext and Tor ([116, Section 6]).

In section 5, we characterize the endomorphism ring of a local hypersurface of dimension one, using Theorem 5.1.4.

5.2 Proof of Theorem 5.1.4 and 5.1.6

In this section, we prove Theorem 5.1.4 and 5.1.6. Let $(R, \mathfrak{m})$ be a Noetherian local ring with total quotient ring $Q(R)$. Denote by $\widetilde{R}$ the integral closure of R in $Q(R)$. By Remark 2.2.3, $\mathrm{End}_R(\mathfrak{m})$ is identified with $\mathfrak{m} : \mathfrak{m}$, which is a subring of $\widetilde{R}$. Furthermore, $\mathfrak{m} : \mathfrak{m}$ is semilocal and module-finite over R.

We give the following lemma in order to use in the proof of Theorem 5.1.4.

Lemma 5.2.1. *Let $(S, \mathfrak{n}) \subsetneq (R, \mathfrak{m})$ be a module-finite birational extension of one-dimensional local rings. Assume R is reflexive as an S-module. Then we have birational extensions $S \subsetneq \mathfrak{n} : \mathfrak{n} \subseteq R$.*

Proof. Note that S is not a discrete valuation ring, and so S is properly contained in $\mathfrak{n} : \mathfrak{n}$ (Lemma 2.2.7). By the assumption, $R = S : (S : R)$.

We use the following claim.

Claim 4. The S-modules R and $S : R$ have no nonzero S-free summands.

Proof of Claim 4. First consider the case when R has a S-free summand, that is, $R \cong S \oplus X$ for some S-module X. Then $R \otimes_S Q(S) \cong Q(S) \oplus X \otimes_S Q(S)$. Since $S \subseteq R$ is finite birational, $R \otimes_S Q(S) \cong Q(R) = Q(S)$. Therefore, we obtain that $Q(S)$ is isomorphic to $Q(S) \oplus X \otimes_S Q(S)$. Since $Q(S)$ is artinian, we may use the Krull–Schmidt theorem for $Q(S)$ to show that $X \otimes_S Q(S)$. In particular, X is a torsion S-module. Howeber, X is a submodule of torsionfree S-module R, and hence X itself is torsionfree. Thus X should be a zero module. This shows that $R \cong S$. As R is a finite module over S, the ring-extension $R \subseteq S$ is integral. Thus R is contained in the integral closure of S in $Q(S)$. By Lemma 4.4.6, it follows that $R = S$. This is a contradiction.

Now suppose that $S : R$ has an S-free summand. Since $R = S : (S : R) \cong \mathrm{Hom}_S(S : R, S)$, it follows that R has an S-free summand, too. This is a contradiction. ■

By Claim 4 and Lemma 2.2.6, R has an $\mathfrak{n} : \mathfrak{n}$-module structure compatible with the action of S. Therefore, we get equalities $R = (\mathfrak{n} : \mathfrak{n})R$ and $(\mathfrak{n} : \mathfrak{n}) \subseteq (\mathfrak{n} : \mathfrak{n})R \subseteq R$. ■

Now we can explain the proof of the direction 5.1.4 (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) of Theorem 5.1.4.

Proof of Theorem 5.1.4 (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5). (1) $\Rightarrow$ (2): Assume $bg(R) \leq 1$. If $bg(R) = 0$, then R is Gorenstein, and there is nothing to prove. We may assume $bg(R) = 1$. Then there is a Gorenstein local ring $(S, \mathfrak{n})$ and module-finite residually rational birational extension $S \subsetneq R$ with $\ell_S(R/S) = 1$. Since S is Gorenstein and R is maximal Cohen–Macaulay over S, R is reflexive as an S-module (see [19, Theorem 3.3.10] for instance). By the previous lemma, we have $S \subsetneq \mathfrak{n} : \mathfrak{n} \subseteq R$. Therefore, it should follows that $\ell_S(R/\mathfrak{n} : \mathfrak{n}) = 0$, in other words, $R = \mathfrak{n} : \mathfrak{n} = \mathfrak{n} : S$.

(2) $\Rightarrow$ (3): We may assume S is not a discrete valuation ring, (otherwise $R \cong S$ and hence R is Gorenstein). Identify R with $\mathfrak{n} : \mathfrak{n}$. By Lemma 2.2.7, one has $\ell_S(R/S) = 1$. Hence we have that the colength $\ell_S(\mathfrak{m}/\mathfrak{n})$ of the inclusion $\mathfrak{n} \subseteq \mathfrak{m}$ is less than or equal to 1. it is easy to check that $\mathfrak{m}/\mathfrak{n}$ is an R-module. And we have a calculation $\ell_R(\mathfrak{m}/\mathfrak{n}) \times \ell_S(R/\mathfrak{m}) = \ell_S(\mathfrak{m}/\mathfrak{n}) = 1$. Thus it follows that $\mathfrak{m}/\mathfrak{n}$ has dimension one as a vector space over $R/\mathfrak{m}$. Fix a preimage $t \in R$ of a basis $\bar{t}$ of $\mathfrak{m}/\mathfrak{n}$. Then $\mathfrak{m} = \mathfrak{n} + Rt$ and $\mathfrak{m}^2 = \mathfrak{n}^2 + \mathfrak{m}t \subseteq \mathfrak{n}$. This means $\mathfrak{m} \subseteq S : \mathfrak{m}$. We have another inclusion $S : \mathfrak{m} \subseteq S : \mathfrak{n}$. Using Lemma 2.2.5, we see that $R = \mathfrak{n} : \mathfrak{n} = S : \mathfrak{n}$. It also holds that $Rt \not\subseteq S$ (otherwise $\mathfrak{m} = \mathfrak{n} + Rt \subseteq S$). These observation yield that $S : \mathfrak{m} = \mathfrak{m}$. The fractional ideal $S : \mathfrak{m}$ is isomorphic to $\mathrm{Hom}_S(\mathfrak{m}, S) \cong \mathrm{Hom}_R(\mathfrak{m}, \mathrm{Hom}_S(R, S))$. Now as S is Gorenstein and $S \subseteq R$ is a local homomorphism which makes R a finite S-module, ω is isomorphic to $\mathrm{Hom}_R(R, S)$ [19, Theorem 3.3.7 (b)]. Thus $S : \mathfrak{m}$ is isomorphic to $\mathrm{Hom}_R(\mathfrak{m}, \omega) = \mathfrak{m}^\dagger$. We conclude that $\mathfrak{m} \cong \mathfrak{m}^\dagger$.

(3) $\Rightarrow$ (4): Applying the functor $(-)^\dagger$ to the short exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$, we see that the resulting exact sequence is $0 \to \omega \to \mathfrak{m}^\dagger \to \mathrm{Ext}^1_R(k, \omega) \cong k \to 0$. Replacing $\mathfrak{m}^\dagger$ by $\mathfrak{m}$, using the assumption $\mathfrak{m} \cong \mathfrak{m}^\dagger$, we get the desired exact sequence.

(4) $\Rightarrow$ (3): Applying the functor $(-)^\dagger$ to the short exact sequence $0 \to \omega \to \mathfrak{m} \to k \to 0$, we get an exact sequence $0 \to \mathfrak{m}^\dagger \to R \to \operatorname{Ext}^1_R(k, \omega) \cong k \to 0$. Then, the image of $\mathfrak{m}^\dagger$ in R must be equal to $\mathfrak{m}$ and hence one has an isomorphism $\mathfrak{m}^\dagger \cong \mathfrak{m}$.

(4) $\Rightarrow$ (5): The exact sequence $0 \to \omega \to \mathfrak{m} \to k \to 0$ yields that there is an ideal $I \cong \omega$ such that the colength $\ell_R(\mathfrak{m}/I)$ is one. The equality $\ell_R(R/I) = 2$ immediately follows from the above.

(5) $\Rightarrow$ (4): Take an ideal $I \cong \omega$ such that $l(R/I) \leq 2$. If $I = R$, then R is Gorenstein and there is nothing to prove. So we may suppose that $I \subseteq \mathfrak{m}$. If $I = \mathfrak{m}$, then $\mathfrak{m} \cong \omega$. Take a regular element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ of R. Then $\omega/x\omega \cong \mathfrak{m}/x\mathfrak{m}$, and $\mathfrak{m}/x\mathfrak{m}$ is isomorphic to $k \oplus \mathfrak{m}/(x)$ (see [145, Lemma 2] for instance). On the other hand, $\omega/x\omega$ is a canonical module of $R/(x)$. Thus $\omega/x\omega$ is indecomposable. This implies that $\mathfrak{m}/(x) = 0$, i.e. $\mathfrak{m} = (x)$. In particular, R is a discrete valuation ring. Now we deal with an assumption that $I \subsetneq \mathfrak{m}$. The inequality $l(R/I) \leq 2$ implies that the equality $l(\mathfrak{m}/I) = 1$. Thus the exact sequence $0 \to I \to \mathfrak{m} \to k \to 0$ is induced. ■

All that remains is to show the direction (5) $\Rightarrow$ (1). Let $(R, \mathfrak{m})$ be a Noetherian local ring containing a coefficient field $k \cong R/\mathfrak{m}$. Let $I \subset R$ be a fractional ideal such that $\ell_R(R/I) < \infty$. Put $k+I := \{a+b \mid a \in k, b \in I\} \subseteq R$, which is a k-subalgebra of R. Then, since $\dim_k R/(k+I) \leq \ell_R(R/I) < \infty$, R is finitely generated as a $k+I$-module and hence $k+I$ is Noetherian by Hilbert basis theorem. By the lying over property of $k+I \subseteq R$ (see [50, Proposition 4.15] for instance), any maximal ideal of $k+I$ is contained in $\mathfrak{m}$. Therefore $(k+I) \cap \mathfrak{m} = I$ is the unique maximal ideal of $k+I$. It also holds that $k+I$ contains a regular element in its maximal ideal I. Since we have inequalities

$$\begin{aligned} \ell_{k+I}(R/(k+I)) &= \ell_{k+I}(R/I) - \ell_{k+I}((k+I)/I) \\ &= \ell_{k+I}(R/I) - 1 \\ = \ell_R(R/I)\ell_{k+I}(R/\mathfrak{m}) < \infty, \end{aligned}$$

$R/(k+I)$ is torsion $k+I$-module. Thus $R/(k+I) \otimes_{k+I} Q(k+I) = 0$. This implies that $R \otimes_{k+I} Q(k+I) = Q(k+I)$, equivalently $Q(R) = Q(k+I)$. Consequently the ring extension $k+I \subseteq R$ is module-finite residually rational and birational.

Lemma 5.2.2. *Let $(R, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring. Assume R has a canonical ideal $I \cong \omega$ such that $l(R/I) = 2$. Put $S = k + I$. Then S is Gorenstein, and the colength $\ell_S(R/S)$ is equal to* 1.

Proof. S is local with a maximal ideal $\mathfrak{n} = I$. The extension $S \subseteq R$ is module-finite, residually rational and birational. Since I is a canonical ideal, we have $I : I = R$. Equivalently, $\mathfrak{n} : \mathfrak{n} = R$. In particular, the colength $\ell_S(R/S)$ is equal to the Cohen-Macaulay type of S (Lemma 2.2.7). Since R and S have same residue field k, we can see the equalities $\ell_S(R/S) = l_S(\mathfrak{m}/\mathfrak{n}) = \ell_R(\mathfrak{m}/I)$. On the other hand, we have

$$\ell_R(\mathfrak{m}/I) = \ell_R(R/I) - \ell_R(R/\mathfrak{m}) = 2 - 1 = 1.$$

It follows that S has Cohen-Macaulay type 1, that is, S is Gorenstein. Moreover, the colength $\ell_S(R/S)$ is equal to $\ell_R(\mathfrak{m}/I) = 1$. ■

Proof of Theorem 5.1.4 (5) ⇒ (1). Assume there is a canonical ideal I such that $\ell_R(R/I) \leq 2$. If $\ell_R(R/I) \leq 1$, then $I = R$ or $\mathfrak{m}$. In both of these cases, R should be Gorenstein (in the case of $I = \mathfrak{m}$, see the proof of Theorem 1.4 (5)⇒ (4)).

Thus we only need to consider the case $\ell_R(R/I) = 2$. By previous lemma, the ring $S := k+I$ is Gorenstein and the colength $\ell_S(R/S)$ is 1. This shows $bg(R) \leq 1$. ■

We put the following lemma here, which will be used in the proof of Corollary 5.1.5.

Lemma 5.2.3. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay generically Gorenstien local ring of dimension one having a canonical module. Assume R is not a discrete valuation ring. Then*

(1) *R has minimal multiplicity if and only if $\mathfrak{m} \cong \mathfrak{m} : \mathfrak{m}$.*

(2) *R is almost Gorenstein in the sense of [62] if and only if $\mathfrak{m}^\dagger \cong \mathfrak{m} : \mathfrak{m}$.*

Proof. See [121, Proposition 2.5] and Theorem 4.0.3 respectively. ■

We give a proof of Corollary 5.1.5 as follows.

Proof of Corollary 5.1.5. The implications (3) ⇔ (4) ⇔ (5) follow immediately from Lemma 5.2.3.

(1) ⇔ (2): In the case $bg(R) = 0$, R is Gorenstein and has minimal multiplicity, and thus $e(R) \leq 2$ (see [130, 3.2. Corollary]). The converse also holds. Now suppose $bg(R) = 1$. Then by Theorem 5.1.4, $\mathfrak{m}$ is isomorphic to $\mathfrak{m}^\dagger$. Therefore, R has minimal multiplicity if and only if R is almost Gorenstein.

(1) ⇒ (3): Clear.

Now assume the residue field $R/\mathfrak{m}$ is infinite.

(6) ⇒ (4): Obviously, $S/\mathfrak{n}$ is also infinite. If $e(S) \leq \operatorname{edim} S$, then $e(S) \leq 2$. Using [108, Theorem A.29 (iii)], we see that

$$\begin{aligned} e(R) &= \max\{\ell_R(X/\mathfrak{m}) \mid R \subseteq \text{ is a finite birational extension}\} \\ &\leq \max\{\ell_S(X/\mathfrak{n}) \mid S \subseteq \text{ is a finite birational extension}\} = e(S), \end{aligned}$$

and so we have an inequality $e(R) \leq 2$. This says that R is Gorenstein and has minimal multiplicity. So we may assume $e(S) = \operatorname{edim} S + 1$.

Take a minimal reduction (t) of $\mathfrak{n}$ and a preimage $\delta \in \mathfrak{n}^2$ of a generator of the socle of $S/(t)$.Then $\mathfrak{n}^3 = t\mathfrak{n}^2$, $\ell_S(\mathfrak{n}^2/t\mathfrak{n}) = 1$ (see [130, Proof of (3.4)]) and $(t) :_S \mathfrak{n} = (t) + S\delta$. The equality $\ell_S(\mathfrak{n}^2/t\mathfrak{n}) = 1$ implies $\mathfrak{n}^2 = t\mathfrak{n} + S\delta$.

Now we claim the following.

Claim 5. $\operatorname{End}_S(\mathfrak{n}) \cong (t) :_S \mathfrak{n}/t$.

Proof of the claim. Recall that $\operatorname{End}_S(\mathfrak{n})$ is isomorphic to $\mathfrak{n} : \mathfrak{n}$. We want to show the equality $\mathfrak{n} : \mathfrak{n} = (t) :_S \mathfrak{n}/t$ as subsets of $Q(S)$. The containment $((t) :_S \mathfrak{n})\mathfrak{n} \subseteq (t)$ shows that $((t) :_S \mathfrak{n})/t$ is contained in $S : \mathfrak{n}$. By Lemma 2.2.7, $S : \mathfrak{n}$ coincides with $\mathfrak{n} : \mathfrak{n}$. In particular, the inclusion "⊇" holds.

Since t is in $\mathfrak{n}$, we have $t(\mathfrak{n} : \mathfrak{n}) \subseteq \mathfrak{n} \subset S$. Thus the inequality $t(\mathfrak{n} : \mathfrak{n})\mathfrak{n} \subseteq t\mathfrak{n} \subseteq (t)$ shows that the inclusion "⊆". ■

Therefore

$$R \cong \mathrm{End}_S(\mathfrak{n}) \cong (t) :_S \mathfrak{n}/t = S + S(\delta/t).$$

Identify R with $S + S(\delta/t)$. Since R is local and $\delta^2 \in \mathfrak{n}^4 = t^2\mathfrak{n}^2$, (δ/t) cannot be a unit of R. This shows $\mathfrak{m} = \mathfrak{n} + S(\delta/t)$. By this equality, we also have an isomorphism $R/\mathfrak{m} \cong S/\mathfrak{n}$ induced by $S \subseteq R$. Observe the following equalities

$$t\mathfrak{m} = t\mathfrak{n} + S\delta = \mathfrak{n}^2$$

and

$$\mathfrak{m}^2 = (\mathfrak{n} + S(\delta/t))^2 = \mathfrak{n}^2 + \mathfrak{n}(\delta/t) + S(\delta/t)^2.$$

Then $\delta^2 \in \mathfrak{n}^4 = t^2\mathfrak{n}^2$ implies $S(\delta/t)^2 \subseteq \mathfrak{n}^2$, and $\mathfrak{n}\delta \subseteq \mathfrak{n}^3 = t\mathfrak{n}^2$ implies $\mathfrak{n}(\delta/t) \subseteq \mathfrak{n}^2$. So $\mathfrak{m}^2 = \mathfrak{n}^2 = t\mathfrak{m}$. This means that R has minimal multiplicity.

It remains to show that $\mathfrak{m} \cong \mathfrak{m}^\dagger$. By Theorem 5.1.4, it holds that either R is Gorenstein or $\mathfrak{m} \cong \mathfrak{m}^\dagger$. In the case that R is Gorenstein, it holds that $e(R) \leq 2$ and so $\mathfrak{m}$ is self-dual by [121, Theorem 2.6].

(5) $\Rightarrow$ (7): Assume R is almost Gorenstein and has minimal multiplicity. Then we already saw that m is self-dual (Lemma 5.2.3). It follows from [62, Theorem 3.16] that $\rho(R) \leq 2$.

(7) $\Rightarrow$ (5): Recall that $\rho(R)$ is the reduction number of a canonical ideal of R ([57, Definition 4.2]). So if $\rho(R) \leq 1$, then R is Gorenstein ([62, Theorem 3.7]). It means that $R \cong \omega$. We may assume that R is not a discrete valuation ring. Therefore $\mathfrak{m} : \mathfrak{m} = R : \mathfrak{m}$ by Lemma 2.2.7, and so we have $\mathfrak{m} : \mathfrak{m} \cong \mathrm{Hom}_R(\mathfrak{m}, R) \cong \mathrm{Hom}_R(\mathfrak{m}, \omega) = \mathfrak{m}^\dagger$. Since $\mathfrak{m}$ is self-dual, this yields that $\mathfrak{m} : \mathfrak{m} \cong \mathfrak{m}$. Using Lemma 5.2.3, we deduce that R has minimal multiplicity.

Assume $\rho(R) = 2$. Combining [45, Theorem 3.5 (b), Proposition 3.8] and Theorem 5.1.4, we obtain that R is almost Gorenstein and has minimal multiplicity.

Finally, we deal with the assumption that R contains a infinite field k isomorphic to $R/\mathfrak{m}$ via $R \to R/\mathfrak{m}$.

(4) $\Rightarrow$ (1): This follows directly from Theorem 5.1.4.

(1) $\Rightarrow$ (6): First we consider the case that R is Gorenstein (i.e. $bg(R) = 0$). In this case, $e(R) \leq 2$ and $\mathrm{edim}\, R \leq 2$ by the assumption. Take a minimal reduction Rt of $\mathfrak{m}$. Then $\mathfrak{m}^2 = t\mathfrak{m}$. In particular, $\ell_R(\mathfrak{m}/I) = \ell_R(\mathfrak{m}/I + \mathfrak{m}^2) \leq 1$. Put $I = Rt$ and $S = k + I$. Then the ring-extension $S \subseteq R$ is module-finite, residually rational and birational. Since $I : I = R$ and $\ell_R(\mathfrak{m}/I) \leq 1$, we can see that S is Gorenstein and $\mathrm{End}_S(I) \cong R$ by the similar argument in the proof of 5.1.4 (3) $\Rightarrow$ (1). Furthermore, one has an equality $tI = I^2$, which particularly show that S has minimal multiplicity, that is, $e(S) = \mathrm{edim}\, S$.

Now consider the case that $bg(R) = 1$. Repeating the proof of Theorem 5.1.4 (3) $\Rightarrow$ (1), there is a canonical ideal I such that if we let $S = k + I$, then S is Gorenstein local and $R \cong \mathrm{End}_S(\mathfrak{n})$, where $\mathfrak{n}$ is the maximal ideal of S. Since R is Almost Gorenstein, it was shown in [62, Theorem 3.16] that there is a minimal reduction $Q = (t) \subseteq I$ of I in R such that $\ell_R(I^2/QI) \leq 1$ and $QI^2 = I^3$. Then it follows that $\ell_S(I^2/QI) \leq 1$. Using [130, Proposition 3.3], the equality $e(S) = \mathrm{edim}\, S + 1$ holds. ∎

We give here an example of a ring R with $bg(R) = 1$.

Example 5.2.4. Let $R = k[[t^3, t^4, t^5]]$ and $S = k[[t^3, t^4]]$ be numerical semigroup rings, where k is a field. Then the natural inclusion $S \subseteq R$ is a module-finite birational extension of local rings with the same coefficient field. The colength $\ell_S(R/S)$ is equal to 1. Since R is non-Gorenstein and S is Gorenstein, we have $bg(R) = 1$.

We now turn to estimate the invariant $bg(R)$ in general. Suppose there exists a self-dual fractional ideal of R. Then we have an upper bound of $bg(R)$ as follows.

Lemma 5.2.5. *Let $(R, \mathfrak{m})$ be a complete one-dimensional Cohen-Macaulay local ring. Assume R contains an infinite coefficient field $k \cong R/\mathfrak{m}$. Let $I \subseteq R$ be a fractional ideal of R. If I is self-dual, then we have $bg(R) \leq l(R/I)$. In other words, the following inequality holds*

$$bg(R) \leq \inf\{\ell_R(R/I) \mid I \cong I^\dagger\}.$$

Proof. In the case $I = R$, the self-duality of I implies R is Gorenstein. So we may assume $I \subseteq \mathfrak{m}$. Take a non-zero divisor $t \in I$, and Put $B = k + I$. Then $B \subseteq R$ is a module-finite extension and I is the maximal ideal of local ring B. Remark that B is also complete, and so a canonical module ω_B of B exists. Since $B \subseteq R$ is birational, the R-isomorphism $I \to I^\dagger$ is also a B-isomorphism. We also have an isomorphism $\mathrm{Hom}_B(R, \omega_B) \cong \omega_R$, which yields isomorphisms

$$\mathrm{Hom}_B(I, \omega_B) \cong \mathrm{Hom}_B(I \otimes_R R, \omega_B) \cong \mathrm{Hom}_R(I, \mathrm{Hom}_B(R, \omega_B)) \cong \mathrm{Hom}_R(I, \omega_R) \cong I^\dagger.$$

This says that $I^\dagger$ is isomorphic to the canonical dual of I over B. By Theorem 5.1.4, $bg(B) \leq 1$, that is, there is a Gorenstein ring S and module-finite birational extension $S \subseteq B$. Then $S \subseteq R$ is also a module-finite birational extension. The calculation

$$\ell_S(R/S) = \ell_S(R/B) + \ell_S(B/S) = \ell_R(\mathfrak{m}/I) + 1 = \ell_R(R/I)$$

shows that $bg(R) \leq \ell_R(R/I)$. ■

As a corollary of this, we can see the finiteness of $bg(R)$ in the analytically unramified case.

Corollary 5.2.6. *Let $(R, \mathfrak{m})$ be a complete one-dimensional local ring. Assume R contains an infinite coefficient field. If there exists a module-finite birational extension $R \subseteq T$ with a Gorenstein ring T, Then $bg(R) \leq l(R/aT)$ for any non-zero divisor $a \in T : R$ of T. Moreover, if R is analytically unramified, then $bg(R) \leq l(R/R : \widetilde{R}) < \infty$, where $\widetilde{R}$ is the integral closure of R in $Q(R)$.*

Proof. Since T is Gorenstein, the R-module $aT \cong T$ is self-dual. So we can apply Lemma 5.2.5 for $I = aT$. If R is analytically unramified, $\widetilde{R}$ of R in $Q(R)$ is Gorenstein, and $R \subseteq \widetilde{R}$ is finite birational. The conductor $R : \widetilde{R}$ is a nonzero and satisfies $R : \widetilde{R} \otimes_R Q(R) = Q(R)$. Thus $R : \widetilde{R}$ has constant rank one and contains a non-zero divisor of R. In particular, $R : \widehat{R}$ is torsion-free over $\widetilde{R}$. As $\widetilde{R}$ is reduced and integrally closed in its total ring of quotients, its localization at any maximal ideal $\mathfrak{p}$ is a discrete valuation ring. Therefore $(R : \widetilde{R})_{\mathfrak{p}}$ is a free module of rank one for any $\mathfrak{p}$. Since $\widetilde{R}$ is semilocal, it follows that $R : \widetilde{R}$ is free of rank one over $\widetilde{R}$. This means that $R : \widetilde{R} \cong \widetilde{R}$. Applying Lemma 5.2.5 for $I = R : \widetilde{R}$, we have an inequality $bg(R) \leq l(R/R : \widetilde{R}) < \infty$. ■

Remark 5.2.7. Ananthnarayan [1] shows the following inequalities hold for an artinian local ring R.

$$\ell_R(R/\omega^*(\omega)) \leq \min\{\ell_R(R/I) \mid I \cong I^\dagger\} \leq g(R). \tag{5.2.7.1}$$

Here $\omega^*(\omega)$ is the trace ideal of ω; see Definition 5.3.4.

As analogies of these inequalities, the followings are natural questions.

Question 5.2.8. Let $(R, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring. Does an inequality

$$bg(R) \geq \inf\{\ell_R(R/I) \mid I \cong I^\dagger\}$$

hold true?

Question 5.2.9. Let $(R, \mathfrak{m})$ be a one-dimensional generically Gorenstein local ring. Does an inequality $\ell_R(R/\omega^*(\omega)) \leq bg(R)$ hold true?

By our main theorems 5.1.4 and 5.1.6, Question 5.2.8 is affirmative for R with $bg(R) \leq 2$. Question 5.2.9 has positive answer given in Proposition 5.3.6 if $bg(R) \leq 1$.

We now return to prove the Theorem 5.1.6.

Proof of Theorem 5.1.6. (2) $\Rightarrow$ (1): This is a consequence of Lemma 5.2.5 by letting $I = \mathfrak{a}$.

(1) $\Rightarrow$ (2): In the case $bg(R) \leq 1$, assertion follows by Theorem 5.1.4. So we may assume $bg(R) = 2$. Take a Gorenstein local ring $(S, \mathfrak{n})$ and module-finite residually rational birational extension $S \subset R$ satisfying $\ell_S(R/S) = 2$. Note that, since S is Gorenstein and R is a maximal Cohen–Macaulay S-module, R is reflexive.

Let B be the ring $\mathfrak{n} : \mathfrak{n}$. By Lemma 5.2.1 and Lemma 2.2.7, we have $\ell_S(B/S) = 1$ and $S \subsetneq B \subseteq R$. Therefore $\ell_B(R/B) = 1$. As in the paragraph before Lemma 5.2.2, the lying over property of the exntension $B \subseteq R$ shows that B is local. Let $\mathfrak{m}_B$ be the maximal ideal of B and fix a preimage $t \in R$ of a basis $\overline{t}$ of the one-dimensional vector space R/B over $B/\mathfrak{m}_B$. By the relation $\mathfrak{m}_B t \subseteq B$ yields that $t \in B : \mathfrak{m}_B = \mathfrak{m}_B : \mathfrak{m}_B$. Therefore $R = B + Bt \subseteq \mathfrak{m}_B : \mathfrak{m}_B$. In particular, $R\mathfrak{m}_B \subseteq \mathfrak{m}_B$. This says that $\mathfrak{m}_B$ is an ideal of R. Since $bg(B) = 1$, $\mathfrak{m}_B$ is a self-dual ideal of B by Theorem 5.1.4. Fix a canonical module ω_B of B. Then $\operatorname{Hom}_B(B, \omega_B)$ is a canonical module of R.

$$\operatorname{Hom}_B(\mathfrak{m}_B, \omega_B) \cong \operatorname{Hom}_B(\mathfrak{m}_B \otimes_R R, \omega_B) \cong \operatorname{Hom}_R(\mathfrak{m}_B, \operatorname{Hom}_B(R, \omega_B)) \cong \operatorname{Hom}_R(\mathfrak{m}_B, \omega_R).$$

Thus, it is also self-dual as R-module. One can also have equalities

$$\ell_R(R/\mathfrak{m}_B) = \ell_B(R/B) + \ell_B(B/\mathfrak{m}_B) = 2.$$

■

Remark 5.2.10. Let $(R, \mathfrak{m})$ be a one-dimensional local ring. Assume R is complete, equicharacteristic and $bg(R) = n < \infty$. If there exists a Cohen-Macaulay local ring $(B, \mathfrak{m}_B)$ with $bg(B) = 1$ and module-finite residually rational birational extensions $B \subseteq R \subseteq \mathfrak{m}_B : \mathfrak{m}_B$ such that $\ell_B(R/B) + 1 \leq n$. Then, by the same argument of proof of Theorem 5.1.6, it follows that $\mathfrak{m}_B$ is a self-dual ideal of R satisfying $\ell_R(R/\mathfrak{m}_B) \leq n$. In this case, Question 5.2.8 is affirmative for R.

5.3 The self-duality of the maximal ideal

In this section, we collect some properties of local rings $(R, \mathfrak{m})$ with $\mathfrak{m} \cong \mathfrak{m}^\dagger$.

Lemma 5.3.1. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with a canonical module. Assume $\mathfrak{m} \cong \mathfrak{m}^\dagger$. Then*

(1) $\dim R \leq 1$.

(2) *Let $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ be a non-zero divisor of R. Then $R/(x)$ also has self-dual maximal ideal.*

(3) $\mathrm{edim}(R) = r(R) + \dim R$.

Proof. (1) Suppose $\dim R \geq 2$ and ω is a canonical module of R. Applying $(-)^\dagger$ to the exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$, we get an exact sequence

$$0 \to \mathrm{Hom}_R(k, \omega) \to \omega \to \mathfrak{m}^\dagger \to \mathrm{Ext}^1_R(k, \omega).$$

By the assumption $\dim R \geq 2$ yields that $\mathrm{Hom}_R(k, \omega) = 0 = \mathrm{Ext}^1_R(k, \omega)$ and hence $\mathfrak{m}^\dagger \cong \omega$.

From the isomorphism $\mathfrak{m} \cong \mathfrak{m}^\dagger$, it follows that $\mathfrak{m} \cong R$, i. e. $\mathfrak{m}$ is a principal ideal. This shows that $\dim R \leq 1$, which is a contradiction. Thus, it must be $\dim R \leq 1$.

(2) Applying the functor $\mathrm{Hom}_R(-, \omega)$ to the exact sequence $0 \to \mathfrak{m} \xrightarrow{x} \mathfrak{m} \to \mathfrak{m}/x\mathfrak{m} \to 0$, we get an exact sequence

$$0 \to \mathrm{Hom}_R(\mathfrak{m}/x\mathfrak{m}, \omega) \to \mathfrak{m}^d ag \xrightarrow{x} \mathfrak{m}^\dagger \to \mathrm{Ext}^1_R(\mathfrak{m}/x\mathfrak{m}, \omega) \to \mathrm{Ext}^1_R(\mathfrak{m}, \omega).$$

Since $\dim R$ is less than or equal to 1 by (1), and $x \in \mathfrak{m}$ is a non-zero divisor, it follows that $\dim R = 1$. Thus $\mathfrak{m}$ is a maximal Cohen–Macaulay R-module, which yields the equality $\mathrm{Ext}^1_R(\mathfrak{m}, \omega) = 0$. The equalities $\mathrm{Hom}_R(\mathfrak{m}/x\mathfrak{m}, \omega) = 0$ and $\mathrm{Ext}^1_R(\mathfrak{m}/x\mathfrak{m}, \omega) \cong \mathrm{Hom}_{R/(x)}(\mathfrak{m}/x\mathfrak{m}, \omega/x\omega)$ also hold (see [19, Lemma 1.2.4]). Thus we get an isomorphism $\mathfrak{m}^\dagger/x\mathfrak{m}^\dagger \cong \mathrm{Hom}_{R/(x)}(\mathfrak{m}/x\mathfrak{m}, \omega/x\omega)$. From this isomorphism and $\mathfrak{m} \cong \mathfrak{m}^\dagger$, isomorphisms $\mathfrak{m}/x\mathfrak{m} \cong \mathfrak{m}^\dagger/x\mathfrak{m}^\dagger \cong \mathrm{Hom}_R/(x)(\mathfrak{m}/x\mathfrak{m}, \omega/x\omega)$ are induced. By [145, Lemma 2], we have an isomorphism $\mathfrak{m}/x\mathfrak{m} \cong R/m \oplus \mathfrak{m}/(x)$. Therefore we obtain isomorphisms

$$R/m \oplus \mathfrak{m}/(x) \cong \mathrm{Hom}_{R/(x)}(R/m \oplus \mathfrak{m}/(x), \omega/x\omega) \cong \mathrm{Hom}_{R/(x)}(R/m, \omega/x\omega) \oplus \mathrm{Hom}_{R/(x)}(\mathfrak{m}/(x), \omega/x\omega).$$

Remark that $\omega/x\omega$ is a canonical module of $R/(x)$ (see [19, Theorem 3.3.5]). Thus we have $\mathrm{Hom}_R/(x)(R/m, \omega/x\omega) \cong R/\mathfrak{m}$. Then it follows that

$$R/m \oplus \mathfrak{m}/(x) \cong R/\mathfrak{m} \oplus \mathrm{Hom}_R/(x)(\mathfrak{m}/(x), \omega/x\omega).$$

By the Krull–Schmidt theorem for $R/(x)$, this yields that $\mathfrak{m}/(x) \cong \mathrm{Hom}_R/(x)(\mathfrak{m}/(x), \omega/x\omega)$, which means the self-duality of the maximal ideal $\mathfrak{m}/(x)$ of $R/(x)$.

(3) Suppose that $\dim R = 0$. Then by $\mathfrak{m} \cong \mathfrak{m}^\dagger$ and [19, Proposition 3.3.11], $\mu(\mathfrak{m}) = \mu(\mathfrak{m}^\dagger) = r(\mathfrak{m})$. Here for an R-module X, $\mu_R(X)$ denotes the minimal number of generators of X. Since $\dim \mathfrak{m} = \dim R = 0$, $r(\mathfrak{m})$ (resp. $r(R)$) is equal to $\dim_{R/\mathfrak{m}}(\mathrm{soc}\mathfrak{m})$ (resp. $\dim_{R/\mathfrak{m}}(\mathrm{soc}R)$), and so the equality $\mathrm{soc}\mathfrak{m} = \mathrm{soc}R$ implies that $r(\mathfrak{m}) = r(R)$. Thus we have $\mathrm{edim}\, R = \mu(\mathfrak{m}) = r(\mathfrak{m}) = r(R) + \dim R$.

Now suppose that $\dim R > 0$. Then (1) shows that $\dim R = 1$. Since R is Cohen–Macaulay, we can take a non-zero divisor $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ of R. Then, by (2), $R/(x)$ has self-dual maximal ideal. Since $\dim R/(x) = 0$, it follows that $\operatorname{edim} R/(x) = r(R/(x))$. Note that $r(R/(x)) = r(R)$ and $\operatorname{edim} R/(x) = \mu_{R/(x)}\mathfrak{m}/(x) = \mu_R(\mathfrak{m}) - 1$. Thus we have equalities

$$\operatorname{edim} R = \mu(\mathfrak{m}) = \operatorname{edim} R/(x) + 1 = r(R) + 1$$

. ■

When $\dim R \geq 2$, the maximal ideal $\mathfrak{m}$ cannot be self-dual. However, we suggest the following generalization of the self-duality of the maximal ideal in higher dimensional case.

Proposition 5.3.2. $(R, \mathfrak{m})$ *be a non-Gorenstein Cohen-Macaulay local ring of dimension* $d > 0$ *having an infinite residue field. Assume* R *has a canonical ideal* I *satisfying* $e(R/I) = 2$. *Then there is a regular sequence* $\underline{x} = x_1, \ldots, x_{d-1}$ *such that* $R/(\underline{x})$ *has self-dual maximal ideal.*

Proof. Since R/I is Cohen-Macaulay of dimension $d-1$ ([108, Proposition 11.6]), we can take a minimal reduction $\underline{y} = y_1, \ldots, y_{d-1}$ of the maximal ideal $\mathfrak{m}/I$ in R/I. Then the length $\ell((R/I)/(\underline{y}))$ is equal to $e(R/I)(\leq 2)$. Let $\underline{x} = x_1, \ldots, x_{d-1}$ be a preimage of $\underline{y}$ in R. As I is unmixed, we can take $\underline{x}$ as a regular sequence in R. The tensor product $I' = I \otimes R/(\underline{x})$ is naturally isomorphic a canonical ideal of $R' = R/(\underline{x})$. The quotient R'/I' has length $l(R/(I + \underline{x})) = l((R/I)/(\underline{y})) \leq 2$. Therefore R' has self-dual maximal ideal by Theorem 5.1.4. ■

Example 5.3.3. Let $R = k[[x^3, x^2y, xy^2, y^3]]$ be the third Veronese subring of $k[[x, y]]$. Then $I = (x^3, x^2y)R$ is a canonical ideal of R. The quotient ring R/I is isomorphic to $k[[s, t]]/(s^2)$, and hence $e(R/I) = 2$.

Go back to the subject on self-duality of the maximal ideal. Recall the notion of trace ideal of an R-module and nearly Gorensteiness of local rings (see [74]).

Definition 5.3.4. Let R be a commutative ring. For an R-module M, the *trace ideal* $M^*(M)$ of M in R is defined to be the ideal $\sum_{f \in \operatorname{Hom}_R(M,R)} \operatorname{Im} f \subseteq R$.

A Cohen-Macaulay local ring $(R, \mathfrak{m})$ with a canonical module ω is called *nearly Gorenstein* if $\omega^*(\omega) \supseteq \mathfrak{m}$.

Lemma 5.3.5. *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring with a canonical module. The following are equivalent.*

(1) R *is nearly Gorenstein.*

(2) there is a surjective homomorphism $\omega^{\oplus n} \to \mathfrak{m}$ *for some* n.

Moreover, if $\dim R \leq 1$, *then we can add the following conditions.*

(3) there is a short exact sequence $0 \to \mathfrak{m}^\dagger \to R^{\oplus n} \to M \to 0$ *for some* n *and maximal Cohen-Macaulay module* M.

(4) there is a short exact sequence $0 \to \mathfrak{m}^\dagger \to \mathfrak{m}^{\oplus n} \to M \to 0$ *for some* n *and maximal Cohen-Macaulay module* M.

Proof. (1) ⇔ (2): By the definition of trace ideals, there is a surjection $\omega^{\oplus n} \to \omega^*(\omega)$ for some n. So the equivalence immediately follows.

Now assume $\dim R \leq 1$. Then the maximal ideal $\mathfrak{m}$ is maximal Cohen-Macaulay as an R-module. So the condition (2) is equivalent to that there is a short exact sequence $0 \to M \to \omega^{\oplus n} \to \mathfrak{m} \to 0$ for some n and maximal Cohen-Macaulay module M. Taking the canonical duals, the equivalence of (2) and (3) follows.

We turn the equivalence of (3) and (4). We may assume R is not a discrete valuation ring, and hence both $\mathfrak{m}$ and $\mathfrak{m}^\dagger$ are not free R-modules. Assume that (3) holds. The condition (3) means that $\mathfrak{m}^\dagger$ is a syzygy of a maximal Cohen–Macaulay module. Thus by Lemma 4.4.2, there is a short exact sequence $0 \to \mathfrak{m}^d ag \to \mathfrak{m}^{\oplus n} \to M' \to 0$ with some maximal Cohen–Macaulay R-module M'. This shows the implication (3) ⇒ (4).

Conversely, suppose that (4) holds. Then we may use Lemma 4.3.5 to show that (3) holds, since $\mathfrak{m}$ is a syzygy of a maximal Cohen–Macaulay module by Lemma 2.2.7. ■

Proposition 5.3.6. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with a canonical module. Assume $\mathfrak{m} \cong \mathfrak{m}^\dagger$. Then*

(1) R is nearly Gorenstein.

(2) If R is non-Gorenstein and 2 is invertible in R, then R is G-regular.

Proof. We already saw that $\dim R \leq 1$ from Lemma 5.3.5.

(1) In the case of $\dim R = 0$, we have a short exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$ and hence we can apply Lemma 5.3.5 (3) ⇒ (1).

In the case of $\dim R = 1$, we may assume R is not a , . Applying Lemma 5.3.5 to the short exact sequence in Theorem 5.1.4 (4), it follows that R is nearly Gorenstein.

(2) In the case that $\dim R = 0$, the statement is proved in [135, Corollary 3.4]. So we may assume $\dim R = 1$. Take $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ a non-zero divisor. Thanks to Lemma 3.1, the maximal ideal of $R/(x)$ is self-dual. Then $R/(x)$ is G-regular by [135, Corollary 3.4]. It follows from [138, Proposition 4.2] that R is also G-regular.

■

Example 5.3.7. Let $R = k[[t^4, t^5, t^7]]$. Then R is almost Gorenstein local ring of dimension one. Therefore, R is G-regular and nearly Gorenstein. On the other hand, R does not have minimal multiplicity, and hence $\mathfrak{m}$ is not self-dual. This shows that the converse of Proposition 5.3.6 doesn't hold true in general.

Example 5.3.8. The associated graded ring $\mathsf{gr}_{\mathfrak{m}}(R)$ of a local ring $(R, \mathfrak{m})$ with self-dual maximal ideal need not be Cohen-Macaulay, for example, $R = k[[t^4, t^5, t^{11}]]$.

We use the notion of minimal faithful modules. The definition of them is given in below.

Definition 5.3.9. Let R be a commutative ring. An R-module M is called *minimal faithful* if it is faithful and no proper submodule or quotient module is faithful.

Example 5.3.10. For an artinian local ring R, the R-module R and a canonical module ω of R (i.e. injective hull of the residue field) are minimal faithful.

The following is proved by Bergman [16, Corollary 2].

Lemma 5.3.11 (Bergman)**.** *Let A, B and C be finite-dimensional vector spaces over a field k. and $f\colon A \times B \to C$ be a bilinear map. Assume the following conditions.*

(1) *any nonzero element a of A induces a nonzero map $f(-, a)\colon B \to C$*

(2) *For any proper submodule $i\colon B' \to B$, there is a nonzero element $a \in A$ such that $f(i(-), a)\colon B' \to C$ is a zero map.*

(3) *For any proper quotient module $p\colon C \to C'$ there is a nonzero element $a \in A$ such that the map $p \circ f(-, a)\colon B \to C'$ is a zero map.*

Then $\dim_k A \geq \dim_k B + \dim_k C - 1$.

To have an application of Lemma for modules, we need the following lemmas.

Lemma 5.3.12. *Let R be a commutative ring, n be a positive integer, M, N be R-modules and $f = [f_1, \ldots, f_n]^t\colon N \to M^{\oplus n}$ be an R-homomorphism. Assume that N is Artinian. Then f is injective if and only if for any nonzero element $a \in \operatorname{soc}(N)$, there exists i such that $f_i(a) \neq 0$.*

Proof. We can see that $\operatorname{Ker} f = \operatorname{Ker} f_1 \cap \cdots \cap \operatorname{Ker} f_n$. Thus f is injective if and only if $\operatorname{Ker} f_1 \cap \cdots \cap \operatorname{Ker} f_n = 0$. Since N is Artinian, the latter condition is equivalent to that $\operatorname{soc}(N) \cap \operatorname{Ker} f_1 \cap \cdots \cap \operatorname{Ker} f_n = 0$. ■

Lemma 5.3.13. *Let $(R, \mathfrak{m}, k)$ be an artinian local ring and M, N be finitely generated R-modules. Assume M is minimal faithful and N is Artinian. If n is the smallest positive integer such that exists an injective homomorphism $f = [f_1, \ldots, f_n]\colon N \to M^{\oplus n}$, then the k-subspace B of $\operatorname{Hom}_R(N, M) \otimes_R k$ generated by the image of $f_1, \ldots, f_n$ has a dimension exactly equal to n over k.*

Proof. We only need to show $\dim_k B \geq n$. Assume there is a equation $f_1 = a_1 f_2 + \cdots + a_n f_n + g$ for some $a_2, \ldots, a_n \in R$ and $g \in \mathfrak{m} \operatorname{Hom}_R(N, M)$. Then for any element $a \in \operatorname{soc}(N)$, $g(a) = 0$. So $n \geq 2$ and $f(a) \neq 0$ implies there exists $i \geq 2$ such that $f_i(a) \neq 0$. This particular says that the homomorphism $[f_2, \ldots, f_n]\colon N \to M^{\oplus n-1}$ also an injection by Lemma 5.3.12, which is a contradiction to our assumption on n. ■

The following lemma is a generalization of the result of Gulliksen [68, Lemma 2].

Lemma 5.3.14. *Let $(R, \mathfrak{m}, k)$ be an artinian local ring and M, N be finitely generated faithful R-modules. Assume M is minimal faithful. If there exists an injective homomorphism $N \to M^{\oplus n}$ for some n, then $\dim_k \operatorname{soc}(M) \leq \dim_k \operatorname{soc}(N)$ and equality holds if and only if $N \cong M$.*

Proof. Let n be the minimal integer such that there is an injective map $N \to M^{\oplus n}$. Take a injective map $N \xrightarrow{[f_1, \ldots, f_n]^t} M^{\oplus n}$ and set B the k-subspace of $\operatorname{Hom}_R(N, M) \otimes_R k$ generated by the image of $f_1, \ldots, f_n$. Then $\dim_k B = n$ by Lemma 5.3.13. By letting $A = \operatorname{soc}(N)$ and $C = \operatorname{soc}(M)$, we have a bilinear map $A \times B \to C$ over k satisfying the assumption (1) and (2) of Lemma 5.3 in view of Lemma 5.3.12 and 5.3.13. We also verify the condition (3) of Lemma as follows. Assume (3) is not satisfied. Then there is a subspace C' of C such that any nonzero element a of A induces a nonzero map $p \circ f(-,a)\colon B \to C/C'$, where $p\colon C \to C/C'$ is the natural surjection. Since $C/C' \subseteq M/C'$ as an R-module, we obtain an injective map $g\colon N \xrightarrow{q\circ f_1,\dots,q\circ f_n} (M/C')^{\oplus n}$, where $q\colon M \to M/C'$ is also the natural surjection. Since N is faithful, there is an injective map h from R to some copies $N^{\oplus m}$ of N. Taking a composition of h and $g^{\oplus m}$, one has an injective map from R to $(M/C')^{\oplus mn}$. In particular, M/C' is a faithful R-module, which contradicts the assumption that M is minimal faithful.

Therefore, we can apply Lemma 5.3 and get an equality $\dim A \geq \dim B + \dim C - 1$. It follows that $\dim \operatorname{soc}(N) - \dim \operatorname{soc}(M) \geq n - 1 \geq 0$. If the equalities hold, then $n = 1$ and N is isomorphic to a submodule of M. By the minimality of M, one has $N \cong M$. ∎

We also give some basic properties of minimal faithful modules.

Lemma 5.3.15. *Let $(R, \mathfrak{m}, k)$ be an artinian local ring. Then*

(1) *Any minimal faithful R-module is indecomposable.*

(2) *Assume R has Cohen-Macaulay type at most three. Then $\ell_R(R) \leq \ell_R(M)$ for all faithful R-module M. In particular, a faithful R-module M is minimal faithful if $\ell_R(M) = \ell_R(R)$.*

Proof. (1): Let M be a minimal faithful R-module, and assume that M decomposes as direct sum $M = M_1 \oplus M_2$ of R-modules. the faithfulness of M yields that $\operatorname{Ann}(M_1) \cap \operatorname{Ann}(M_2) = 0$. Take minimal generators $x_1, \dots, x_n$ of M_1 and $y_1, \dots, y_m$ of M_2. Without loss of generality, we may assume $n \leq m$. Then the submodule N of $M = M_1 \oplus M_2$ generated by the elements $x_1 + y_1, \dots, x_n + y_n, 0 + y_{n+1}, \dots, 0 + y_m$ is proper and faithful. This contradicts that M is minimal faithful. (2): This follows by [68, Theorem 1]. ∎

Definition 5.3.16. Let $(R, \mathfrak{m}, k)$ be a commutative ring. A fractional ideal I of R is called *closed* [18] if the natural homomorphism $R \to \operatorname{Hom}_R(I, I)$ is an isomorphism.

Example 5.3.17. Let $(R, \mathfrak{m}, k)$ be a one-dimensional Cohen-Macaulay local ring. Set $B = \mathfrak{m} : \mathfrak{m}$. Then $\mathfrak{m}$ is closed as a fractional ideal of B.

Lemma 5.3.18. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional Cohen-Macaulay local ring having a canonical module and I be a fractional ideal of R. Then the following are equivalent.*

(1) I is closed.

(2) $I^\dagger$ is closed.

(3) There is a surjective homomorphism $I^{\oplus n} \to \omega$ for some n.

(4) There is a short exact sequence $0 \to R \to I^{\oplus n} \to M \to 0$ for some n and maximal Cohen-Macaulay R-module M.

Proof. See [18, Proposition 2.1]. Note that (4) follows by the canonical dual of (3), since I is maximal Cohen-Macaulay as an R-module. ∎

Proof of Theorem 1.7. Take a minimal reduction (t) of $\mathfrak{m}_B$. The assumption that R is nearly Gorenstein implies that any localization of ω_R at a non-maximal prime ideal is free of rank one ([74, Proposition 2.3]). This yields that $\omega_R \otimes_R Q(R) = Q(R)$. On the other hand, $\mathfrak{m} \otimes_B Q(B) = \mathfrak{m} \otimes_R Q(R) = Q(R) = Q(B)$ since $\mathfrak{m}$ contains a regular element. Thus one has

$$\begin{aligned}\mathfrak{m}^\dagger \otimes_B Q(B) = \operatorname{Hom}_R(\mathfrak{m}, \omega_R) \otimes_B Q(B) &= \operatorname{Hom}_R(\mathfrak{m}, \omega_R) \otimes_R Q(R) \\ &= \operatorname{Hom}_{Q(R)}(\mathfrak{m} \otimes_R Q(R), \omega_R \otimes_R Q(R)) \\ &= \operatorname{Hom}_{Q(R)}(Q(R), Q(R)) = Q(R) = Q(B).\end{aligned}$$

This shows that $\mathfrak{m}$ and $\mathfrak{m}^\dagger$ has constant rank one as a B-module. In particular, we have equalities $\ell_B(B/tB) = e(B) = \ell_B(\mathfrak{m}/t\mathfrak{m}) = \ell_B(\mathfrak{m}^\dagger/t\mathfrak{m}^\dagger)$. Note that $\mathfrak{m}/t\mathfrak{m}$ and $\mathfrak{m}^\dagger/t\mathfrak{m}^\dagger$ are faithful over B/tB (see [18, Proposition 3.3]). As B/tB has Cohen-Macaulay type less than or equal to three in both case (1) and (2), Lemma 5.3.15 ensures that $\mathfrak{m}/t\mathfrak{m}$ and $\mathfrak{m}^\dagger/t\mathfrak{m}^\dagger$ are minimal faithful over B/tB. Consider the exact sequence

$$0 \to \mathfrak{m}^\dagger \xrightarrow{\phi} \mathfrak{m}^{\oplus n} \to M \to 0$$

as in Lemma 5.3.5. Then $\phi \in \operatorname{Hom}_R(\mathfrak{m}^\dagger, \mathfrak{m}^{\oplus n}) = \operatorname{Hom}_B(\mathfrak{m}^\dagger, \mathfrak{m}^{\oplus n})$. Therefore $M = \operatorname{Cok}\phi$ is also a B-module and it is torsion-free over B as well as over R. Moreover, the above sequence is an exact sequence of B-modules and B-homomorphisms. Tensoring B/tB to this sequence, we have a short exact sequence

$$0 \to \mathfrak{m}^\dagger/t\mathfrak{m}^\dagger \xrightarrow{\phi \otimes B/tB} (\mathfrak{m}/t\mathfrak{m})^{\oplus n} \to M/tM \to 0 \tag{5.3.18.1}$$

of B/tB-modules.

(1): Applying Lemma 5.3.14 to the sequence (5.3.18.1) and using [68, Lemma 2], we obtain the inequalities

$$1 \le \dim_k \operatorname{soc}(\mathfrak{m}/t\mathfrak{m}) \le \dim_k \operatorname{soc}(\mathfrak{m}^\dagger/t\mathfrak{m}^\dagger) \le r_B(B) = 2.$$

So one has either $\mathsf{soc}(\mathfrak{m}/t\mathfrak{m}) = 1$ or $\mathsf{soc}(\mathfrak{m}^\dagger/t\mathfrak{m}^\dagger) = 2$. In the former case, $\mathfrak{m}^\dagger$ must be a cyclic B-module and hence $\mathfrak{m}^\dagger \cong B$. R is almost Gorenstein. So one has either $\dim_k \operatorname{soc}(\mathfrak{m}/t\mathfrak{m}) = 1$ or $\dim_k \dim_k \operatorname{soc}(\mathfrak{m}/t\mathfrak{m}) = \dim_k \operatorname{soc}(\mathfrak{m}^\dagger/t\mathfrak{m}^\dagger) = 2$. In the former case, we have $1 = r_B(\mathfrak{m}) = \mu_B(\mathfrak{m}^\dagger)$, which shows that $\mathfrak{m}^\dagger$ must be a cyclic B-module. Thus $\mathfrak{m}^\dagger \cong B$. This yields that R is almost Gorenstein by Lemma 5.2.3. Suppose that $\dim_k \operatorname{soc}(\mathfrak{m}/t\mathfrak{m}) = \dim_k \operatorname{soc}(\mathfrak{m}^\dagger/t\mathfrak{m}^\dagger) = 2$.

Take a system of minimal generator $a_1, \dots, a_m$ of $\mathfrak{m}/t\mathfrak{m}$ over B/tB, and for each $i = 1, \dots m$ we set a homomorphism $f_i \colon B/tB \to \mathfrak{m}/t\mathfrak{m}$, which sends 1 to a_i. Since $\mathfrak{m}/t\mathfrak{m}$ is faithful over B/tB, The homomorphism $B/tB \xrightarrow{[f_1,\dots,f_m]^t} (\mathfrak{m}/t\mathfrak{m})^{\oplus m}$ is injective. Then by Lemma 5.3.14, the equality $\dim_k \operatorname{soc}(\mathfrak{m}/t\mathfrak{m}) = 2 = \dim_k \operatorname{soc}(B/tB)$ induces an isomorphism $\mathfrak{m}/t\mathfrak{m} \cong B/tB$. This shows that $\mathfrak{m}$ is a cyclic B-module. Thus we have $\mathfrak{m} \cong B$. Similar argument shows that $\dim_k \operatorname{soc}(\mathfrak{m}^\dagger/t\mathfrak{m}^\dagger) = 2 = \dim_k \operatorname{soc}(B/tB)$ implies that $\mathfrak{m}^\dagger \cong B$. Then R is almost Gorenstein and has minimal multiplicity by Lemma 5.2.3, and so B has type one by Theorem 5.1.8, a contradiction.

(2): Applying Lemma 5.3.14 to the sequence (5.3.18.1), we obtain the inequalities

$$1 \le \dim_k \operatorname{soc}(\mathfrak{m}/t\mathfrak{m}) \le \dim_k \operatorname{soc}(\mathfrak{m}^\dagger/t\mathfrak{m}^\dagger) \le r_B(B) = 3.$$

In the case that $\dim_k \operatorname{soc}(\mathfrak{m}/t\mathfrak{m}) = 1$ or $\dim_k \operatorname{soc}(\mathfrak{m}^\dagger/t\mathfrak{m}^\dagger) = 3 = \dim_k \operatorname{soc}(B/tB)$, it follows by same argument as in (1) that $\mathfrak{m}^\dagger \cong B$ and R is almost Gorenstein. So we only need to consider the case $\dim_k \operatorname{soc}(\mathfrak{m}/t\mathfrak{m}) = \dim_k \operatorname{soc}(\mathfrak{m}^\dagger/t\mathfrak{m}^\dagger) = 2$. In this case, $\mathfrak{m}/t\mathfrak{m}$ should be isomorphic to $\mathfrak{m}^\dagger/t\mathfrak{m}^\dagger$ by lemma 5.3.14. Put $\phi = [\phi_1, \dots, \phi_n]^t \colon \mathfrak{m}^\dagger \to \mathfrak{m}^{\oplus n}$ and so $\phi \otimes B/tB = [\phi_1 \otimes B/tB, \dots, \phi_n \otimes B/tB]^t$. Consider the canonical dual $(\phi \otimes B/tB)^\dagger \colon (\mathfrak{m}/t\mathfrak{m})^{\dagger \oplus n} \to (\mathfrak{m}/t\mathfrak{m})$, which is surjective. Since $\mathfrak{m}/t\mathfrak{m}$ is indecomposable (Lemma 5.3.15), the Nakayama's lemma indicates that $\mathsf{jac}(\operatorname{End}(\mathfrak{m}/t\mathfrak{m})) \cdot (\mathfrak{m}/t\mathfrak{m}) \neq \mathfrak{m}/t\mathfrak{m}$. Therefore, one of the endomorphism $(\phi_1 \otimes B/tB)^\dagger, \dots, (\phi_n \otimes B/tB)^\dagger$ of $\mathfrak{m}/t\mathfrak{m}$ must be not contained in $\mathsf{jac}(\operatorname{End}(\mathfrak{m}/t\mathfrak{m}))$, otherwise $(\phi \otimes B/tB)$ cannot be surjective. This means that one of the $\phi_1 \otimes B/tB, \dots, \phi_n \otimes B/tB$ is an isomorphism. Say $\phi_i \otimes B/tB$ is an isomorphism. Then the B-homomorphism $\phi_i \colon \mathfrak{m}^\dagger \to \mathfrak{m}$ is also surjective. Both $\mathfrak{m}$ and $\mathfrak{m}^\dagger$ have constant rank, ϕ_i must be an isomorphism. This shows that $\mathfrak{m} \cong \mathfrak{m}^\dagger$. ■

Corollary 5.3.19. *Let $(R, \mathfrak{m}, k)$ be a complete Cohen-Macaulay local ring of dimension one with a canonical module. Assume $B \coloneqq \mathfrak{m} : \mathfrak{m}$ is local and k is infinite. If R is nearly Gorenstein with multiplicity $e(R) \leq 4$, then either R is almost Gorenstein or $\mathfrak{m} \cong \mathfrak{m}^\dagger$.*

Proof. Take a minimal reduction (t) of R. We have that $B \otimes_R Q(R) = Q(R)$, this means that B has a constant rank as an R-module. Then the multiplicity $e(\mathfrak{m}, B) = \ell_R(B/tB)$ of B as an R-module is equal to 4. If B is a discrete valuation ring, then the statement follows by Theorem 5.1.8. So we may assume that B/tB is not a field. Deduce that

$$4 = \ell_R(B/tB) \geq \ell_B(B/tB) > \dim_k \operatorname{soc}(B/tB) = r(B).$$

Now we can apply Theorem 5.1.7 and attain the desired statement. ■

Example 5.3.20. Let $R = k[[t^5, t^6, t^7]]$. Then R is nealy Gorenstein and has multiplicity 5, however, neither R is almost Gorenstein nor $\mathfrak{m} \cong \mathfrak{m}^\dagger$.

5.4 Numerical semigroup rings

In this section, we deal with the numerical semigroup rings $(R, \mathfrak{m})$ having an isomorphism $\mathfrak{m} \cong \mathfrak{m}^\dagger$. We begin the section with recalling preliminaries on numerical semigroup. Let $H \subsetneq \mathbb{N}$ be a numerical semigroup. The set of *pseudo-Frobenius numbers* $\mathsf{PF}(H)$ of H is consisting of integers $a \in \mathbb{N} \setminus H$ such that $a + b \in H$ for any $b \in H \setminus \{0\}$. Then the maximal element $\mathsf{F}(H)$ of $\mathsf{PF}(H)$ is the Frobenius number of H. Set $H' = H \cup \{\mathsf{F}(H)\}$. Then H' is also a numerical semigroup. A numerical semigroup of the form $H' = H \cup \{\mathsf{F}(H)\}$ for some symmtric numerical semigroup H (see [104] for the definition of symmtric numerical semigroups) is called *UESY-semigroup (unitary extension of a symmetric semigroup)*, which is introduced in [124]. Note that $\mathsf{F}(H)$ is a minimal generator of $H' = H \cup \{\mathsf{F}(H)\}$. For a numerical semigoup H and a field k, the *numerical semigoup ring* of H over k is the subring $k[[\{t^a \mid a \in H\}]]$ of $k[[t]]$, where t is an indeterminate.

Lemma 5.4.1. *Let H be a numerical semigroup, k is an infinite field and $(R, \mathfrak{m})$ is the numerical semigroup ring $k[[H]]$. Then the following are equivalent.*

(1) $\mathfrak{m}$ is self-dual.

(2) H is a UESY-semigroup.

Proof. (1) ⇒ (2): In the case that H is symmetric, or equivalently R is Gorenstein (see [104]), it follows by Corollary 1.5 (3)⇒ (2) that $e(R) \leq 2$. Then there is an odd integer a such that $H = \langle 2, a\rangle$. It can be checked that the subsemigroup $H' = \langle 2, a+2\rangle$ of H is symmetric, and $H \setminus H' = \{a\}$. Thus H is UESY.

We may assume that H is not symmetric. By Theorem 5.1.4, there is a Gorenstein local subring $(S, \mathfrak{n})$ of R such that $R = \mathfrak{n} : \mathfrak{n}$. Take a value semigroup $v(S)$ of S, where v is the normalized valuation of $k[[t]]$, that is, v takes t to $1 \in \mathbb{Z}$. Then $H = v(R)$, and $v(S)$ is symmetric by the result of Kunz [104]. Since $R\mathfrak{n} \subset \mathfrak{n}$, $v(R) \setminus v(S)$ is contained in $\mathsf{PF}(v(S))$. Since $v(S)$ is symmetric, $\mathsf{PF}(v(S)) = \{F(v(S))\}$. Thus one has $v(S) \subseteq H \subseteq v(S) \cup \{F(v(S))\}$. Therefore, H should be equal to $v(S) \cup \{F(v(S))\}$. In particular, H is UESY.

(2) ⇒ (1): Describe H as $H = H' \cup \{\mathsf{F}(H')\}$ with a symmtric numerical semigoup H'. Set $S = k[[H']]$. Then $\mathrm{End}_S(\mathfrak{m}_S)$ is isomorphic to

$$\begin{aligned}\mathfrak{m}\colon \mathfrak{m} &= (t^a \mid a \in \mathbb{Z} \text{ such that for any } b \in H' \setminus \{0\},\ a+b \in H')S \\ &= (t^a \mid a \in H' \cup PF(H'))S = (t^a \mid a \in H)S = R.\end{aligned}$$

Thus by our theorem (Theorem 5.1.4), the maximal ideal $\mathfrak{m}$ of R is self-dual. ■

Proposition 5.4.2. *Let $H = \langle a_1, \ldots, a_n\rangle$ be a symmetric numerical semigroup minimally generated by $\{a_i\}$ with $2 < a_1 < a_2 < \cdots < a_n$ and $H' \coloneqq H \cup \{\mathsf{F}(H)\}$. Put $S = k[[H]]$ over an infinite field k and $R = k[[H']]$. Then the maximal ideal of R is quasi-decomposable.*

Proof. Denote by $\mathfrak{m}_R$ the maximal ideal of R. We will prove that the maximal ideal $\mathfrak{m}_R/(t^{a_1})$ of $R/(t^{a_1})$ has a direct summand I generated by the image of $t^{\mathsf{F}(H)}$, and $I \cong k$ as an R-module. Since $t^{\mathsf{F}(H)}$ is a minimal generator of $\mathfrak{m}_R$, it is enough to show that $\mathfrak{m}_R t^{\mathsf{F}(H)} \subseteq t^{a_1}R$. So what we need to show is that $\mathsf{F}(H) + a_i - a_1 \in H$ for all $i \neq 1$ and $2\mathsf{F}(H) - a_1 \in H$. These follow by the fact that $\mathsf{F}(H)$ is the largest number in $\mathbb{N} \setminus H$ and the inequalities $a_i - a_i > 0$ and $\mathsf{F}(H) - a_1 > 0$. ■

5.5 Further characterizations

The goal of this section is to give characterizations of local rings R such that there exists a one-dimensional local hypersurface $(S, \mathfrak{n})$ such that $R \cong \mathrm{End}_S(\mathfrak{n})$.

Proposition 5.5.1. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension one. Assume that R has a canonical module and infinite coefficient field k. Then the followings are equivalent.*

(1) *There is a local hypersurface $(S, \mathfrak{n})$ such that $R \cong \mathrm{End}_S(\mathfrak{n})$.*

(2) *$e(R) \leq 2$, or R has type 2 and a canonical ideal I such that $I^2 = \mathfrak{m}I$ and $\ell_R(R/I) = 2$.*

(3) *$e(R) \leq 2$, or R has embedding dimension 3, and a canonical ideal I such that $I^2 = \mathfrak{m}^2$.*

Proof. (1)⇒(2): Assume $e(R) > 2$ and R satisfies (1). Then R is not Gorenstein, and $I \coloneqq \mathfrak{n}$ is a canonical ideal of R. Since S is a hypersurface and not a , , $\ell_R(I/I^2) = \ell_S(\mathfrak{n}/\mathfrak{n}^2) = 2$. It forces the equality $I^2 = \mathfrak{m}I$, since I is not a principal ideal.

(2)⇒(1): Consider the case that $e(R) \leq 2$. Then by the proof of Corollary 5.1.5 (1) ⇒ (6), there is a Gorenstein local ring $(S, \mathfrak{n})$ such that $R \cong \operatorname{End}_S(\mathfrak{n})$ and $e(S) = \operatorname{edim} S$. In particular, $e(S) \leq 2$ and S is a hypersurface. Now consider the case that R has type 2 and a canonical ideal I such that $I^2 = \mathfrak{m}I$ and $\ell_R(R/I) = 2$. One has equalities $\ell_R(I/I^2) = \ell_R(I/\mathfrak{m}I) = 2$. Put $S \coloneqq k + I$. Then S is Gorenstein local with a maximal ideal $\mathfrak{n} \coloneqq I$, and $R \cong \operatorname{End}_S(\mathfrak{n})$ (Lemma 5.2.2). We can compute the embedding dimension $\operatorname{edim} S$ as follows:

$$\operatorname{edim} S = \ell_S(\mathfrak{n}/\mathfrak{n}^2) = \ell_R(I/I^2) = 2.$$

Therefore, S should be a hypersurface. (2)⇒ (3): We may assume R has type 2. By the implication (2)⇒(1), we can calculate the embedding dimension of R as $\operatorname{edim} R \leq \operatorname{edim} S + 1 = 3$, where $(S, \mathfrak{n})$ is a hypersurface with $R \cong \operatorname{End}_S(\mathfrak{n})$. Since R is not Gorenstein, $\operatorname{edim} R$ should be equal to 3. This means $\ell_R(\mathfrak{m}/\mathfrak{m}^2) = 3$. On the other hand, one has

$$\ell_R(\mathfrak{m}/I^2) = \ell_R(\mathfrak{m}/I) + \ell_R(I/I^2) = 1 + \ell_R(I/\mathfrak{m}I) = 1 + 2 = 3.$$

So the inclusion $I^2 \subseteq \mathfrak{m}^2$ yields that $I^2 = \mathfrak{m}^2$. The direction (3)⇒(2) also follows by similar calculations. ■

Question 5.5.2. For a Cohen-Macaulay local ring $(R, \mathfrak{m})$ of dimension one, when is there a local complete intersection $(S, \mathfrak{n})$ with an isomorphism $R \cong \operatorname{End}_S(\mathfrak{n})$?

Chapter 6

Burch ideals and Burch rings

6.1 Introduction

The contents of this chapter is based on author's work [41] with H. Dao and R. Takahashi.

This chapter introduces and studies a class of ideals and their affiliated rings which we call Burch ideals and Burch rings. While their definitions are quite simple, our investigation shows that they enjoy remarkable ideal-theoretic and homological properties. These properties allow us to link them to many classes of ideals and rings in the literature, and consequently strengthen numerous old results as well as establish new ones.

Let us make a brief remark on our motivation and historical context. The project originated from our effort to understand a beautiful result by Burch on homological properties of ideals below ([23, Theorem 5(ii) and Corollary 1(ii)]).

Theorem 6.1.1 (Burch)**.** *Let $(R, \mathfrak{m})$ be a local ring. Let I be an ideal of R with $\mathfrak{m}I \neq \mathfrak{m}(I :_R \mathfrak{m})$.*

(1) *Let M be a finitely generated R-module. If* $\operatorname{Tor}_n^R(R/I, M) = \operatorname{Tor}_{n+1}^R(R/I, M) = 0$ *for some positive integer n, then M has projective dimension at most n.*

(2) *If I has finite projective dimension, then R is regular.*

Lindsay Burch[1] was a PhD student of David Rees, and she wrote several (short) papers that have had a sizable impact on two active corners of commutative algebra: homological theory and integral closure of ideals. Perhaps most researchers in the field know of her work via the frequently used Hilbert–Burch Theorem ([23]), her construction of ideals with only three-generators while possessing arbitrarily complicated homological behavior ([24]), and the Burch inequality on analytic spreads ([25]). The ideas of Burch's particular result above, while less well-known, have resurfaced in the work of several authors which also motivated our work, see [34, 36, 102, 105, 135]. However, it has appeared to us that what was known previously is just the tip of an iceberg, and led us to formally make the following definitions.

Let $(R, \mathfrak{m})$ be a local ring. We define an ideal I of R to be a *Burch ideal* if $\mathfrak{m}I \neq \mathfrak{m}(I :_R \mathfrak{m})$. We also define *Burch rings of depth zero* to be those local rings whose completions are quotients of regular local rings by Burch ideals. Then we further define *Burch rings* of positive depth as local rings which "deform" to Burch rings of depth zero; see Section 6.2 for the precise definitions.

It is not hard to see that the class of Burch ideals contains other well-studied classes: integrally closed ideals of codepth zero (under mild conditions), $\mathfrak{m}$-full ideals, weakly $\mathfrak{m}$-full ideals, etc.

One of our main results characterizes Burch ideals and Burch rings of depth zero:

Theorem 6.1.2 (Theorem 6.4.1)**.** *Let $(R, \mathfrak{m}, k)$ be a local ring and $I \neq \mathfrak{m}$ an ideal of R. Then I is Burch if and only if the second syzygy $\Omega_{R/I}^2 k$ of k over R/I contains k as a direct summand.*

From this, we can quickly deduce a characterization of Gorenstein Burch ideals, which extends results on integrally closed or $\mathfrak{m}$-full ideals in [60, 61]. In fact, our proofs allow us to completely characterized modules over Burch rings of depth zero whose some higher syzygies contain the residue field as a direct summand, as follows:

Theorem 6.1.3 (Theorem 6.4.5)**.** *Let $(R, \mathfrak{m}, k)$ be a Burch ring of depth zero. Let M be a finitely generated R-module. The following are equivalent:*

(1) *The ideal $\mathrm{I}(M)$ generated by all entries of the matrices ∂_i, $i > 0$ in a minimal free resolution (F, ∂) of M is equal to $\mathfrak{m}$.*

(2) *The R-module k is a direct summand of $\Omega_R^r M$ for some $r \geq 2$.*

[1] We are grateful to Rodney Sharp and Edmund Robertson for providing us with the following brief biography of Burch: Lindsay Burch was born in 1939. She did her first degree at Girton College, Cambridge from 1958 to 1961. She then went to Exeter University to study for a Ph.D. advised by David Rees. She was appointed to Queen's College, Dundee in 1964 before the award of her Ph.D. which wasn't until 1967 for her thesis "Homological algebra in local rings". At the time she was appointed to Queen's College it was a college of the University of St Andrews but later, in 1967, it became a separate university, the University of Dundee. Burch continued to work in the Mathematics Department of the University of Dundee until at least 1978. She then took up computing and moved to a computing position at Keele University near Stafford in the north of England. She remained there until she retired and she still lives near Keele University.

Our work reveals some interesting connections between Burch ideals/rings and concepts studied by other authors in quite different contexts. For instance, we show that in codimension two, artinian almost Gorenstein rings as introduced by Huneke–Vraciu [86] (also studied in [135]) are Burch; see Proposition 6.6.10. Over a regular local ring, the "Burchness" of an ideal I imposes a strong condition on the matrix at the end of a minimal free resolution of I, a condition that also appeared in the work of Corso–Goto–Huneke–Polini–Ulrich [34] on iterated socles. That connection led us to obtain a refinement of their result in Theorem 6.6.2.

We also study Burch rings of higher depth, especially their homological and categorical aspects. We completely classify Burch rings which are fibre products in Proposition 6.6.15. The Cohen–Macaulay rings of minimal multiplicity are Burch. Non-Gorenstein Burch rings turn out to be *G-regular* in Theorem 6.7.7, in the sense that all the totally reflexive modules are free. Moreover, we show an explicit result on vanishing behavior of Tor for any pair of modules.

Theorem 6.1.4 (Corollary 6.7.13)**.** *Let R be a Burch ring of depth t. Let M, N be finitely generated R-modules. Assume that there exists an integer $l \geq \max\{3, t+1\}$ such that $\operatorname{Tor}_i^R(M, N) = 0$ for all $l + t \leq i \leq l + 2t + 1$. Then either M or N has finite projective dimension.*

To state our last main result in this introduction, recall that the *singularity category* $\mathsf{D_{sg}}(R)$ is by definition the triangulated category given as the Verdier quotient of the bounded derived category of finitely generated R-modules by perfect complexes. Under some assumptions, one can classify all the thick subcategories of $\mathsf{D_{sg}}(R)$ for a Burch ring R.

Theorem 6.1.5 (Theorem 6.7.10)**.** *Let R be a singular Cohen–Macaulay Burch ring. Suppose that on the punctured spectrum R is either locally a hypersurface or locally has minimal multiplicity. Then there is a one-to-one correspondence between the thick subcategories of $\mathsf{D_{sg}}(R)$ and the specialization-closed subsets of* $\operatorname{Sing} R$*.*

Next we describe the structure of the chaper as well as other notable results. In Section 6.2 we state our convention, basic definitions and preliminary results. Section 6.3 is devoted to giving a sufficient condition for a module to have a second syzygy having a cyclic direct summand (Proposition 6.3.4). This is a generalization of [105, Lemma 4.1], and has an application to provide an exact pair of zero divisors (Corollary 6.3.6). These materials are used in Section 6.4 and are perhaps of independent interest.

In Section 6.5, we focus on the study of Burch rings of positive depth. We verify that the class of Gorenstein Burch rings coincides with that of hypersurfaces (Proposition 6.5.1). Cohen–Macaulay local rings of minimal multiplicity with infinite residue field are Burch (Proposition 6.5.2). Quotients of polynomial rings by perfect ideals with linear resolution are Burch (Proposition 6.5.6). We also consider the subtle question of whether the Burch property is preserved by cutting down by *any* regular sequence consisting of minimal generators of $\mathfrak{m}$. Remarkably, this holds for Cohen–Macaulay local rings of dimension one with minimal multiplicity (Proposition 6.5.5). However, the answer turns out to be negative in general (Example 6.5.8).

In Section 6.6 we focus more deeply on Burch ideals in a regular local ring. We give a complete characterization in dimension two and link Burch rings and Burch ideals to various other concepts. Moreover, we give a characterization of the Burch local rings $(R, \mathfrak{m}, k)$ with $\mathfrak{m}^3 = 0$ in terms of a Betti number of k, the embedding dimension and type of R (Theorem

6.6.12). We also characterize the Burch monomial ideals of regular local rings (Proposition 6.6.4).

In Section 6.7, we explore the homological and categorical aspects of Burch rings. We find out the significant property of Burch rings that every module of infinite projective dimension contains a high syzygy of the residue field in its resolving closure (Proposition 6.7.6). We apply this and make an analogous argument as in [116] to classify various subcategories.

6.2 Definitions and basic properties of Burch ideals and rings

For a local ring $(R, \mathfrak{m}, k)$, we denote by $r(R)$ the (Cohen–Macaulay) type of R, and by K^R the Koszul complex of R, i.e., the Koszul complex of a minimal system of generators of $\mathfrak{m}$. We set $\mathrm{K}^R = 0$ when R is a field.

The remaining of this section deals with the formal notion of Burch ideals and Burch rings and their basic properties.

Definition 6.2.1. Let $(R, \mathfrak{m})$ be a local ring. We define a *Burch ideal* as an ideal I with $\mathfrak{m}I \neq \mathfrak{m}(I :_R \mathfrak{m})$. Note by definition that any Burch ideal I of R satisfies $\operatorname{depth} R/I = 0$.

Here are some quick examples of Burch ideals. Many more examples will follow from our results later.

Example 6.2.2. (1) Let (R, xR) be a discrete valuation ring. Then (x^n) is a Burch ideal of R for all $n \geq 1$, since $x(x^n) = (x^{n+1}) \neq (x^n) = x(x^{n-1}) = x((x^n) :_R (x))$.

(2) Let I be an ideal of a local ring $(R, \mathfrak{m})$. Put $J = \mathfrak{m}I$ and suppose $J \neq 0$. Then $\mathfrak{m}(J :_R \mathfrak{m}) = J \neq \mathfrak{m}J$, so J is a Burch ideal of R.

(3) By the previous item, if $(R, \mathfrak{m})$ has positive depth then $I = \mathfrak{m}^t$ is Burch for any $t \geq 1$. More generally, if $\mathfrak{m}^{t+1} \subseteq I \subseteq \mathfrak{m}^t$, then I is Burch if and only if $I : \mathfrak{m} \neq \mathfrak{m}^t$ and $I\mathfrak{m} \neq \mathfrak{m}^{t+1}$. Using this one can show that the set of Burch ideals is Zariski-open in $\mathrm{Grass}_k(r, \mathfrak{m}^t/\mathfrak{m}^{t+1})$, for each $r = \dim_k I/\mathfrak{m}^{t+1}$.

(4) Let $(R, \mathfrak{m})$ be a local ring of positive depth. Let I be an integrally closed ideal of R. Then $\mathfrak{m}I :_R \mathfrak{m} = I$ by the determinantal trick, so it is Burch. See Proposition 6.2.3 below.

The following proposition gives some basic characterizations of Burch ideals.

Proposition 6.2.3. *Let $(R, \mathfrak{m})$ be a local ring and I an ideal of R. The following are equivalent.*
(1) *I is a Burch ideal.* (2) $(I :_R \mathfrak{m}) \neq (\mathfrak{m}I :_R \mathfrak{m})$. (3) $\operatorname{Soc}(R/I) \cdot \mathfrak{m}/I\mathfrak{m} \neq 0$.
(4) $\operatorname{depth} R/I = 0$ *and* $r(R/\mathfrak{m}I) \neq r(R/I) + \mu(I)$.
(5) *$I\widehat{R}$ is a Burch ideal of $\widehat{R}$, where $\widehat{R}$ is the completion of R.*

Proof. (1) ⇔ (2): If $(I :_R \mathfrak{m}) = (\mathfrak{m}I :_R \mathfrak{m})$, then $\mathfrak{m}(I :_R \mathfrak{m}) = \mathfrak{m}(\mathfrak{m}I :_R \mathfrak{m}) = \mathfrak{m}I$. Conversely, if $\mathfrak{m}I = \mathfrak{m}(I :_R \mathfrak{m})$, then $(\mathfrak{m}I :_R \mathfrak{m}) = (\mathfrak{m}(I :_R \mathfrak{m}) :_R \mathfrak{m}) = (I :_R \mathfrak{m})$.

(1) ⇔ (3): As $\operatorname{Soc} R/I = (I :_R \mathfrak{m})/I$, we have $\operatorname{Soc} R/I \cdot \mathfrak{m}/I\mathfrak{m} = 0$ if and only if $\mathfrak{m}(I :_R \mathfrak{m}) = \mathfrak{m}I$.

(2) ⇔ (4): There are inclusions $\mathfrak{m}I \subseteq I \subseteq (\mathfrak{m}I :_R \mathfrak{m}) \subseteq (I :_R \mathfrak{m})$, which especially says that $(\mathfrak{m}I :_R \mathfrak{m}) \neq (I :_R \mathfrak{m})$ implies $\operatorname{depth} R/I = 0$. We have $\ell((I :_R \mathfrak{m})/\mathfrak{m}I) = \ell((I :_R \mathfrak{m})/I) +$

$\ell(I/\mathfrak{m}I) = r(R/I) + \mu(I)$ if $\operatorname{depth} R/I = 0$, and $\ell((\mathfrak{m}I :_R \mathfrak{m})/\mathfrak{m}I) = r(R/\mathfrak{m}I)$. Thus, under the assumption $\operatorname{depth} R/I = 0$, the equalities $(I :_R \mathfrak{m}) = (\mathfrak{m}I :_R \mathfrak{m})$ and $r(R/\mathfrak{m}I) = r(R/I) + \mu(I)$ are equivalent.

(1) $\Leftrightarrow$ (5): It is clear that $\mathfrak{m}I = \mathfrak{m}(I :_R \mathfrak{m})$ if and only if $\widehat{\mathfrak{m}}I = \widehat{\mathfrak{m}}(I :_{\widehat{R}} \widehat{\mathfrak{m}})$. ■

Recall that an ideal I of a local ring $(R, \mathfrak{m})$ is $\mathfrak{m}$-*full* (resp. *weakly* $\mathfrak{m}$-*full*) if $(\mathfrak{m}I :_R x) = I$ for some $x \in \mathfrak{m}$ (resp. $(\mathfrak{m}I :_R \mathfrak{m}) = I$). Clearly, every $\mathfrak{m}$-full ideal is weakly $\mathfrak{m}$-full. The notion of $\mathfrak{m}$-full ideals has been studied by many authors so far; see [33, 60, 61, 146, 147] for instance. Notably, it is fundamental to figure out the connections between $\mathfrak{m}$-full ideals and another class of ideals. For example, $\mathfrak{m}$-primary integrally closed ideals are $\mathfrak{m}$-full or equal to the nilradical of R under the assumption that the residue field k is infinite; see [60, Theorem (2.4)]. There are many related classes of ideals, such as ideals satisfying the Rees property, contracted ideals and basically full ideals. See [78, 128] for the hierarchy of these classes. The notion of weakly $\mathfrak{m}$-full ideals is introduced in [29, Definition 3.7]. The class of weakly $\mathfrak{m}$-full ideals coincide with that of basically full ideals if they are $\mathfrak{m}$-primary; see [70, Theorem 2.12]. The following corollary is immediate from the implication (2) $\Rightarrow$ (1) in the above proposition.

Corollary 6.2.4. *Let $(R, \mathfrak{m})$ be a local ring. Let I be an ideal of R such that* $\operatorname{depth} R/I = 0$*. If I is weakly $\mathfrak{m}$-full, then it is Burch.*

Let $f : (S, \mathfrak{n}, k) \to (R, \mathfrak{m}, k)$ be a surjective homomorphism of local rings, and set $I = \operatorname{Ker} f$. Choi [31] defines the invariant

$$c_R(S, f) = \dim_k(\mathfrak{n}(I :_S \mathfrak{n})/\mathfrak{n}I).$$

Clearly, an ideal I of a local ring $(S, \mathfrak{n})$ is Burch if and only if Choi's invariant $c_{S/I}(S, \pi)$ is positive, where π is the canonical surjection $S \to S/I$. We give a description of Choi's invariant for a regular local ring.

Proposition 6.2.5. *Let $(R, \mathfrak{m}, k)$ be a local ring, $(S, \mathfrak{n}, k)$ a regular local ring, and $f : S \to R$ a surjective homomorphism with kernel I. Then*

$$c_R(S, f) = \begin{cases} \dim_k \operatorname{Soc} R + \dim_k \mathrm{H}_1(\mathrm{K}^R) - \operatorname{edim} R - \dim_k \mathrm{H}_1(\mathrm{K}^{R'}) + \operatorname{edim} R' & (\textit{if } I \neq \mathfrak{n}), \\ \dim_k \mathfrak{n}/\mathfrak{n}^2 & (\textit{if } I = \mathfrak{n}), \end{cases}$$

where $R' = R/\operatorname{Soc} R$.

Proof. Put $J = (I :_S \mathfrak{n})$. We may assume $I \neq \mathfrak{n}$, and hence $J \neq S$. Then there are equalities

$$\begin{aligned} c_R(S, f) = \dim_k \mathfrak{n}J/\mathfrak{n}I &= \ell(J/I) + (\ell(I/\mathfrak{n}I) - \ell(\mathfrak{n}/\mathfrak{n}^2)) - (\ell(J/\mathfrak{n}J) - \ell(\mathfrak{n}/\mathfrak{n}^2)) \\ &= \dim_k \operatorname{Soc} R + (\dim_k \mathrm{H}_1(\mathrm{K}^R) - \operatorname{edim} R) - (\dim_k \mathrm{H}_1(\mathrm{K}^{R'}) - \operatorname{edim} R'). \end{aligned}$$

Now the proof of the proposition is completed. ■

The above result especially says that in the case where $I \neq \mathfrak{n}$ the number $c_R(S, f)$ is determined by the target R of the surjection f. Thus the following result is immediately obtained.

Corollary 6.2.6 (cf. [31, Theorem 2.4])**.** *Let R be a local ring that is not a field. Let $(S_1, \mathfrak{n}_1)$ and $(S_2, \mathfrak{n}_2)$ be regular local rings, and $f_i : S_i \to R$ surjective homomorphisms for $i = 1, 2$. Then the equality $c_R(S_1, f_1) = c_R(S_2, f_2)$ holds. In particular,* $\operatorname{Ker} f_1$ *is Burch if and only if so is* $\operatorname{Ker} f_2$.

We are now ready to define Burch rings.

Definition 6.2.7. Let $(R, \mathfrak{m})$ be a local ring of depth t. Denote by $\widehat{R}$ the $\mathfrak{m}$-adic completion of R. We say that R is *Burch* if there exist a maximal $\widehat{R}$-regular sequence $\boldsymbol{x} = x_1, \dots, x_t$ in $\widehat{R}$, a regular local ring S and a Burch ideal I of S such that $\widehat{R}/(\boldsymbol{x}) \cong S/I$.

Remark 6.2.8. If I is a Burch ideal of a local ring $(R, \mathfrak{m})$, then R/I is a Burch ring of depth zero. Indeed, $I\widehat{R}$ is a Burch ideal of $\widehat{R}$ by Proposition 6.2.3. Take a Cohen presentation $\widehat{R} \cong S/J$, where $(S, \mathfrak{n})$ is a regular local ring. Let I' be the ideal of S such that $I' \supseteq J$ and $I'/J = I\widehat{R}$. Then one can easily verify that $\mathfrak{n}I' \neq \mathfrak{n}(I' :_S \mathfrak{n})$, that is, I' is a Burch ideal of S. Note that the completion of the local ring R/I is isomorphic to S/I'. Hence R/I is a Burch ring of depth zero.

Let R be a local ring. The *codimension* and *codepth* of R are defined by

$$\operatorname{codim} R = \operatorname{edim} R - \dim R, \qquad \operatorname{codepth} R = \operatorname{edim} R - \operatorname{depth} R.$$

Then R is said to be a *hypersurface* if $\operatorname{codepth} R \leq 1$. This is equivalent to saying that the completion $\widehat{R}$ of R is isomorphic to $S/(f)$ for some regular local ring S and some element $f \in S$.

Example 6.2.9. If R is a hypersurface, then it is a Burch ring. Indeed, take a regular sequence $\boldsymbol{x}$ in $\widehat{R}$ such that $\widehat{R}/(\boldsymbol{x})$ is an artinian local ring with $\operatorname{edim} \widehat{R}/(\boldsymbol{x}) \leq 1$. Then $\widehat{R}/(\boldsymbol{x})$ is isomorphic to the quotient ring of a discrete valuation ring S by a nonzero ideal I. By Example 6.2.2(1), the ideal I of S is Burch.

We define the invariant c_R of a local ring $(R, \mathfrak{m}, k)$ by

$$c_R = \dim_k \operatorname{Soc} R + \dim_k \mathrm{H}_1(\mathrm{K}^R) - \operatorname{edim} R - \dim_k \mathrm{H}_1(\mathrm{K}^{R'}) + \operatorname{edim} R'.$$

Here, we set $R' = R/\operatorname{Soc} R$, and adopt the convention that $\dim_k \mathrm{H}_1(\mathrm{K}^{R'}) = 0 = \operatorname{edim} R'$ in the case where $R' = 0$ (i.e. R is a field). Then we can characterize the Burch rings of depth zero:

Lemma 6.2.10. *Let $(R, \mathfrak{m}, k)$ be a local ring. Then $c_R = c_{\widehat{R}}$, and the following are equivalent.*

(1) *R is a Burch ring and* $\operatorname{depth} R = 0$. (3) $c_R \neq 0$.

(2) *$\widehat{R}$ is a Burch ring and* $\operatorname{depth} R = 0$. (4) $c_R > 0$.

Moreover, if R is not a field but a Burch ring of depth zero and isomorphic to S/I for some regular local ring $(S, \mathfrak{n})$ and some ideal I of S, then I is a Burch ideal of S.

Proof. The numbers $\dim_k \operatorname{Soc} R$, $\dim_k \mathrm{H}_1(\mathrm{K}^R)$, $\operatorname{edim} R$, $\dim_k \mathrm{H}_1(\mathrm{K}^{R'}), \operatorname{edim} R'$ are preserved by the completion of R. In particular, one has $c_R = c_{\widehat{R}}$. Furthermore, take a Cohen presentation $\widehat{R} \cong S/I$ with a complete regular local ring S. Letting $\pi : S \to S/I$ be the natural surjection, we have $c_{\widehat{R}} = c_R(S, \pi)$. This especially shows that c_R is nonnegative. Now we show the equivalence of (1)–(4). It is obvious that (1) and (3) are equivalent to (2) and (4), respectively. The equivalence of (2) and (3) follows from Proposition 6.2.5. Finally, we show the last assertion.

Suppose that R is Burch of depth zero and that $R \cong S/I$, where S is a regular local ring and I is an ideal of S. Then $\widehat{R} \cong T/J$ for some regular local ring T and a Burch ideal J of T. There are surjections from the regular local rings $\widehat{S}$ (the completion of S) and T to the local ring $\widehat{S}/I\widehat{S} \cong \widehat{R} \cong T/J$, and the kernel of the latter is the Burch ideal J. Corollary 6.2.6 implies that $I\widehat{S}$ is a Burch ideal of $\widehat{S}$, and I is a Burch ideal of S by Proposition 6.2.3. ■

We end this section by proving an useful characterization of Burch ideals when depth $R > 1$. The only if direction is known for $\mathfrak{m}$-full ideals; see [147, Corollary 7].

Lemma 6.2.11. *Let $(R, \mathfrak{m})$ be a local ring of depth > 1. An ideal I of R is Burch if and only if there exists a non-zerodivisor $a \in \mathfrak{m}$ such that $R/\mathfrak{m}$ is a direct summand of the R-module I/aI.*

Proof. Assume that I is Burch. Then there exist $a \in \mathfrak{m}$ and $b \in (I :_R \mathfrak{m})$ such that $ab \in I \setminus \mathfrak{m}I$. We have $a \notin \mathfrak{m}^2$, since otherwise $ab \in \mathfrak{m}^2(I :_R \mathfrak{m}) = \mathfrak{m}I$. As $b\mathfrak{m} \subseteq I$, it holds that $ab\mathfrak{m} \subseteq aI$. We can define an R-homomorphism $f : R/\mathfrak{m} \to I/aI$ by $f(\overline{1}) = \overline{ab}$. As $ab \notin \mathfrak{m}I$, the element $\overline{ab}$ is a part of a minimal system of generators of I/aI, and hence f is a split monomorphism.

Conversely, assume that there is a split monomorphism $f : R/\mathfrak{m} \to I/aI$, where $a \in R$ is a non-zerodivisor. Let $c \in I$ be the preimage of $f(\overline{1}) \in I/aI$. Then $c\mathfrak{m} \subseteq aI \subseteq (a)$. The assumption depth $R > 1$ implies depth $R/(a) > 0$. Hence c has to be in (a), that is, there exists $b \in R$ with $c = ab$. Observe $ab\mathfrak{m} = c\mathfrak{m} \subseteq aI$. Then a being non-zerodivisor yields $b\mathfrak{m} \in I$. In other words, $b \in (I :_R \mathfrak{m})$. The image of $ab = c$ is a part of a minimal system of generators of I/aI, and we have $ab \notin \mathfrak{m}I$. Thus $\mathfrak{m}(I :_R \mathfrak{m}) \neq \mathfrak{m}I$, which means that I is a Burch ideal. ■

Remark 6.2.12. It is worth noting that Lemma 6.2.11 can be used to give a quick proof of Theorem 6.1.1 when depth $R > 1$ and $n > 1$. Namely, if $\operatorname{Tor}_n^R(R/I, M) = \operatorname{Tor}_{n+1}^R(R/I, M) = 0$ then it follows that $\operatorname{Tor}_n^R(I/aI, M) = 0$, which implies that $\operatorname{Tor}_n^R(k, M) = 0$.

6.3 Cyclic direct summands of second syzygies

The main purpose of this section is to study sufficient conditions for an R-module to have a cyclic direct summand in its second syzygy. They will be used in the proofs of Section 6.4 and are perhaps of independent interest. In fact, some of our proofs were motivated by the work of Kustin-Vraciu ([105]) and Striuli-Vraciu ([135]) which focused on different but related problems.

We start by some simple criteria for a homomorphism $f : R \to M$ to be a split monomorphism.

Lemma 6.3.1. *Let $(R, \mathfrak{m})$ be a local ring of depth zero. Let $f : R \to M$ be a homomorphism of R-modules. Assume one of the following conditions holds.*

(a) *R is Gorenstein.* (b) *M is free.* (c) *M is a syzygy (i.e., a submodule of a free module). Then the followings are equivalent.*

(1) *f is a split monomorphism.* (2) *f is a monomorphism.* (3) *$f(\operatorname{Soc} R) \neq 0$.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are clear. To show (3) ⇒ (1), put $C = \operatorname{Cok} f$.

(a) As R is Gorenstein, we have $\operatorname{Soc} R \cong R/\mathfrak{m}$. The equality $f(\operatorname{Soc} R) \neq 0$ implies $\operatorname{Ker} f \cap \operatorname{Soc} R = 0$. Hence $\operatorname{Ker} f = 0$, and f is injective. As $\operatorname{Ext}_R^1(C, R) = 0$, the map f is split injective.

(b) If f is not split injective, then $\operatorname{Im} f$ is contained in $\mathfrak{m}M$ by the assumption that M is free. This yields that the inclusions $\operatorname{Ker} f \supseteq \operatorname{Ann}(\mathfrak{m}M) \supseteq \operatorname{Soc} R$ hold.

(c) Let $g : M \to F$ be a monomorphism with F free. The composition $gf : R \to F$ satisfies $gf(\operatorname{Soc} R) \neq 0$. By the previous argument, gf is split injective. There is a retraction $r : F \to R$ with $rgf = \mathrm{id}_R$. We see that $rg : M \to R$ is a retraction of f. Therefore f is split injective. ■

Next we consider R-homomorphisms from a cyclic R-module to an R-module.

Lemma 6.3.2. *Let R be a ring, I an ideal of R and M an R-module. Consider an R-homomorphism $f : R/I \to M$. Then f is split injective if and only if the composition map $pf : R/I \to M/IM$ is split injective, where $p : M \to M/IM$ is the natural surjection.*

Proof. Suppose f is split injective. Then there is an R-homomorphism $g : M \to R/I$ such that $gf = \mathrm{id}_{R/I}$. On the other hand, g factor through $p : M \to M/IM$, that is $g = g'p$ for some $g' : M/IM \to R/I$. So we see that g' is a retraction of pf. Next, suppose pf is split injective. Then there is an R-homomorphism $h : R/I \to M/IM$ such that $hpf = \mathrm{id}_{R/I}$. Thus $hp : M \to R/I$ is a retraction of f. ■

The following lemma is well-known; we state it for the convenience of the reader.

Lemma 6.3.3. *Let $R^n \xrightarrow{d} R^m \to M \to 0$ be exact. If $\mathrm{I}_1(d) \subseteq I$, then M/IM is R/I-free.*

Proof. The tensored sequence $(R/I)^n \xrightarrow{d\otimes R/I} (R/I)^m \to M/IM \to 0$ is exact. Since $\mathrm{I}_1(d)$ is contained in I, we see that $d \otimes R/I = 0$, and hence $M \cong (R/I)^m$. ■

We generalize [105, Lemma 4.1] as follows.

Proposition 6.3.4. *Let $(S, \mathfrak{n}, k)$ be a local ring and $I \subseteq J$ ideals of S. Set $R = S/I$. Let $\cdots \to R^q \xrightarrow{\overline{C}} R^p \xrightarrow{\overline{B}} R^n \xrightarrow{\overline{A}} R^m \to M \to 0$ be a minimal R-free resolution of an R-module M, where $A, B, C, \ldots$ are matrices over S. Assume that J satisfies either of the following conditions.*

(a) *$J \supseteq \mathrm{I}_1(A) + \mathrm{I}_1(C)$.* (b) *$J \supseteq \mathrm{I}_1(A)$ and S/J is Gorenstein.*

If $(I :_S J) \not\subseteq (IJ :_S (J :_S \mathfrak{n})\,\mathrm{I}_1(A))$, then S/J is a direct summand of $\Omega^2_R M$.

Proof. For each integer i, let J_i be the ideal of S generated by the entries of the ith column of A. Then $\mathrm{I}_1(A) = J_1 + \cdots + J_n$, and $(I :_S J) \not\subseteq (IJ :_S (J :_S \mathfrak{n})\,\mathrm{I}_1(A)) = (IJ :_S (J :_S \mathfrak{n})J_1) \cap \cdots \cap (IJ :_S (J :_S \mathfrak{n})J_n)$. Hence $(I :_S J) \not\subseteq (IJ :_S (J :_S \mathfrak{n})J_s)$ for some s. Choose an element $u \in (I :_S J) \setminus (IJ :_S (J :_S \mathfrak{n})J_s)$ and let $v \in R^n$ be the image of $u \cdot e_s$, where e_s is the sth unit vector of S^n. Since $Ju \subseteq I$ and $\mathrm{I}_1(A) \subseteq J$, v is in $\operatorname{Ker} \overline{A} = \Omega^2_R M =: X$. We can define an R-homomorphism $f : S/J \to X$ by $f(\overline{1}) = v$.

Now we want to show f is split injective. By Lemma 6.3.2, it is enough to verify so is the induced map $f' = pf : S/J \to X/\overline{J}X$. By Lemmas 6.3.1 and 6.3.3, it suffices to check $f'(\operatorname{Soc} S/J) \neq 0$.

Since $u \notin ((IJ) :_S (J :_S \mathfrak{n})J_s)$, we can choose an element $a \in (J :_S \mathfrak{n})$ such that $auJ_s \not\subseteq IJ$. Remark that $a \notin J$, otherwise one has $au \in I$, which forces auJ_S to be contained in IJ. Let $\overline{a}$ be the image of a in S/J. We have that $0 \neq \overline{a} \in \operatorname{Soc} S/J$. If $f'(\overline{a}) \neq 0$, then $av \in \overline{J}X$. Then there exist elements $x \in JR^p$ and $y \in IR^n$ such that $aue_s = Bx + y$. Observe that $auAe_s = ABx + Ay \in IJR^m$. So we obtain the inclusion $auJ_s \subseteq IJ$, which is contradiction. Thus $f'(\overline{a}) = 0$ and we conclude that f is split injective. ■

As a corollary, we have the following restatement of [105, Lemma 4.1].

Corollary 6.3.5. *Let $(S, \mathfrak{n}, k)$ be a local ring and I an ideal of S. Set $R = S/I$ and consider a minimal R-free presentation $R^n \xrightarrow{\overline{A}} R^m \to M \to 0$ of an R-module M, where A is an $m \times n$ matrix over S and $\overline{A}$ is the corresponding matrix over R. If $(I :_S \mathfrak{n}) \nsubseteq (\mathfrak{n}I :_S \mathrm{I}_1(A))$, then k is a direct summand of $\Omega_R^2 M$.*

Recall that a module M over a ring R is called *totally reflexive* if the natural map $M \to M^{**}$ is an isomorphism and $\mathrm{Ext}_R^i(M, R) = \mathrm{Ext}_R^i(M^*, R) = 0$ for all $i > 0$, where $(-)^* = \mathrm{Hom}_R(-, R)$. Over a Cohen–Macaulay local ring, a totally reflexive module is a maximal Cohen–Macaulay module, and the converse holds as well over a Gorenstein local ring.

Also, recall that a pair (x, y) of elements of a ring R is called an *exact pair of zerodivisors* if the equalities $(0 :_R x) = yR$ and $(0 :_R y) = xR$ hold. This is equivalent to saying that the sequence $\cdots \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} \cdots$ is exact. It is easy to see that for each exact pair of zerodivisors (x, y) the R-modules R/xR and R/yR are totally reflexive.

The following result is another application of Proposition 6.3.4.

Corollary 6.3.6. *Let $(S, \mathfrak{n}, k)$ be a local ring and $I \subseteq J$ be $\mathfrak{n}$-primary ideals of S. Assume that $S/I, S/J$ are Gorenstein and that $(I :_S J) \nsubseteq (IJ) :_S ((J :_S \mathfrak{n})J)$. Then there exist elements $a, b \in S$ such that $J = I + (a)$, $(I :_S J) = I + (b)$, and $(\overline{a}, \overline{b})$ is an exact pair of zerodivisors of S/I.*

Proof. Put $R = S/I$. Consider a minimal R-free resolution $\cdots \to R^n \xrightarrow{\overline{A}} R \to S/J \to 0$ of the R-module S/J. Clearly, the equality $\mathrm{I}_1(A) + I = J$ holds. We can derive from Proposition 6.3.4 that the R-module $\Omega_R^2(S/J)$ has a direct summand isomorphic to S/J. Since R is Gorenstein and the R-module S/J is indecomposable, $\Omega_R^2(S/J)$ is also indecomposable. This implies that $\Omega_R^2(S/J) \cong S/J$, that is, the sequence $0 \to S/J \to R^n \to R \to S/J \to 0$ is exact. We have $\ell(R^n) + \ell(S/J) = \ell(R) + \ell(S/J)$, which yields $n = 1$. Thus the ideal J/I of R is principal, and we find $a \in R$ with $J/I = aR$. As $(0 :_R a) = \Omega_R^1(J/I) \cong S/J$, the ideal $(0 :_R a)$ of R is also principal. Taking a generator b of $(0 :_R a)$, we get an exact pair of zerodivisors (a, b) of R. ■

6.4 Proof of Theorem 6.4.1 and some applications

This section concerns with a surprising characterization of Burch rings of depth zero below, and some applications.

Theorem 6.4.1. *Let $(R, \mathfrak{m}, k)$ be a local ring that is not a field. Then R is a Burch ring of depth zero if and only if k is isomorphic to a direct summand of its second syzygy $\Omega_R^2 k$.*

We shall delay the proof until the end of this section. First, note that we can interpret Corollary 6.3.5 in terms of Burch rings as follows. Here we use the notation $\mathrm{I}_1(M)$ for an R-module M to be the ideal $\mathrm{I}_1(A)$ where A is a matrix in a minimal free presentation $F \xrightarrow{A} G \to M \to 0$ of M. Remark that $\mathrm{I}_1(M)$ is independent of the choice of A (see [19, page 21] for instance).

Proposition 6.4.2. *Let $(R, \mathfrak{m}, k)$ be a Burch ring of depth zero that is not a field. Let M be an R-module with $\mathrm{I}_1(M) = \mathfrak{m}$. Then k is a direct summand of $\Omega_R^2 M$. In particular, k is a direct summand of $\Omega_R^2 k$.*

Proof. By [108, Corollary 1.15], the module $\Omega_R^2 M$ contains k as a direct summand if and only if so does $\Omega_R^2 M \otimes_R \widehat{R} \cong \Omega_{\widehat{R}}^2 (M \otimes_R \widehat{R})$. Hence we may assume that R is complete, and then there is a regular local ring $(S, \mathfrak{n})$ and a Burch ideal $I \subset \mathfrak{n}^2$ such that $R \cong S/I$. Consider a minimal R-free presentation $R^n \xrightarrow{\overline{A}} R^m \to M \to 0$ of an R-module M, where A is a matrix over S and $\overline{A}$ is A modulo I. Then we see that $\mathrm{I}_1(\overline{A}) = \mathrm{I}_1(M) = \mathfrak{m}$, which implies that $\mathrm{I}_1(A) = \mathfrak{n}$. Hence $(I :_S \mathfrak{n}) \nsubseteq (\mathfrak{n}I :_S \mathrm{I}_1(A))$, and thus k is a direct summand of $\Omega_R^2 M$ by Corollary 6.3.5. ■

Here is an immediate consequence of the above proposition.

Corollary 6.4.3. *Let $(R, \mathfrak{m}, k)$ be an artinian Burch ring. Then there exists an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that k is a direct summand of the ideal $(0 :_R x)$ of R.*

Proof. Let $x_1, \dots, x_n$ be a minimal system of generators of $\mathfrak{m}$. There is an exact sequence

$$0 \to \bigoplus_{i=1}^{n} (0 :_R x_i) \to R^n \xrightarrow{\partial} R^n \to \bigoplus_{i=1}^{n} R/(x_i) \to 0 \quad \text{with} \quad \partial = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}.$$

This shows $\mathrm{I}_1(\partial) = \mathfrak{m}$ and $\Omega^2(\bigoplus_{i=1}^n R/(x_i)) = \bigoplus_{i=1}^n (0 :_R x_i)$. Proposition 6.4.2 implies that k is a direct summand of $\bigoplus_{i=1}^n (0 :_R x_i)$. Since R is artinian, it is henselian. The Krull–Schmidt theorem shows that k is a direct summand of $(0 :_R x_i)$ for some i. ■

The following theorem classifies $\mathfrak{m}$-primary Gorenstein Burch ideals.

Theorem 6.4.4. *Let $(R, \mathfrak{m})$ be a local ring and I an $\mathfrak{m}$-primary ideal. The following are equivalent.*
(1) *I is a Burch ideal of R and R/I is Gorenstein.*
(2) *I is weakly $\mathfrak{m}$-full and R/I is Gorenstein.* (3) *I is $\mathfrak{m}$-full and R/I is Gorenstein.*
(4) *$I = (x_1^r, x_2, \dots, x_n)$ with $x_1, \dots, x_n$ a minimal system of generators of $\mathfrak{m}$ and $n, r > 0$.*

Proof. It follows from [61, Proposition (2.4)] that (3) is equivalent to (4), while it is obvious that (3) implies (2) and (2) implies (1). Assume (1) to deduce (4). Remark 6.2.8 shows that R/I is a Burch ring. Proposition 6.4.2 implies that k is a direct summand of $\Omega_{R/I}^2 k$. As $\Omega_{R/I}^2 k$ is indecomposable (see [149, Lemma (8.17)] for instance),, we get $k \cong \Omega_{R/I}^2 k$, whence R/I is a hypersurface. Thus $\mathfrak{m}/I$ is cyclic. Choose an element $x_1 \in \mathfrak{m}$ such that $\overline{x_1}$ is a minimal generator of $\mathfrak{m}/I$. Then x_1 is a minimal generator of $\mathfrak{m}$, and $\mathfrak{m} = I + (x_1)$. There is a unique integer $r > 0$ with $x_1^r \in I$ and $x_1^{r-1} \notin I$. Choose $x_2, \dots, x_n \in I$ so that $\overline{x_2}, \dots, \overline{x_n}$ is a minimal system of generators of $I(R/(x_1)) = \mathfrak{m}/(x_1)$. We see that $x_1, x_2, \dots, x_n$ is a minimal system of generators of $\mathfrak{m}$. Clearly, I contains $J := (x_2, \dots, x_n)$. Note that every $\mathfrak{m}/J$-primary ideal is a power of $\mathfrak{m}/J = ((x_1) + J)/J$. As $x_1^r \in I$ and $x_1^{r-1} \notin I$, we get $I/J = ((x_1^r) + J)/J$. This shows $I = (x_1^r, x_2, \dots, x_n)$. ■

We now characterize the modules over a Burch ring having the residue field as a direct summand of some high syzygy.

Theorem 6.4.5. *Let $(R, \mathfrak{m}, k)$ be a Burch local ring of depth zero which is not a field. Let M be an R-module. Take a minimal free resolution (F, ∂) of M. The following are equivalent.*

(1) *One has $\sum_{i>0} \mathrm{I}_1(\partial_i) = \mathfrak{m}$.* (2) *$k$ is a direct summand of $\Omega_R^r M$ for some $r \geq 2$.*

In particular, if $\sum_{i>0} \mathrm{I}_1(\partial_i) = \mathfrak{m}$, then there exists an integer $i \geq 3$ such that $\mathrm{I}_1(\partial_i) = \mathfrak{m}$.

Proof. (2) $\Rightarrow$ (1): The minimal presentation matrix A of $\Omega_R^r M$ is equivalent to $\left(\begin{smallmatrix} B & 0 \\ 0 & C \end{smallmatrix}\right)$, where B and C are the minimal presentation matrices of k and N, respectively. Hence $\mathrm{I}_1(\partial_{r+1}) = \mathrm{I}_1(A) = \mathrm{I}_1(B) + \mathrm{I}_1(C) = \mathfrak{m} + \mathrm{I}_1(C) = \mathfrak{m}$, which shows $\sum_{i>0} \mathrm{I}_1(\partial_i) = \mathfrak{m}$.

(1) $\Rightarrow$ (2): We may assume that R is complete, and hence there is a regular local ring $(S, \mathfrak{n})$ and a Burch ideal $I \subseteq S$ with $R \cong S/I$. For each $i > 0$ we identify ∂_i with a matrix over R, and let d_i be a matrix over S lifting ∂_i. Then $\mathfrak{n} = \sum_{i>0} \mathrm{I}_1(d_i) + I$. The noetherian property shows $\mathfrak{n} = \mathrm{I}_1(d_1) + \cdots + \mathrm{I}_1(d_n) + I$ for some $n > 0$. Hence $(\mathfrak{n}I :_R \mathfrak{n}) = (\mathfrak{n}I :_R \mathrm{I}_1(d_1) + \cdots + \mathrm{I}_1(d_n) + I) = (\mathfrak{n}I :_R \mathrm{I}_1(d_1)) \cap \cdots \cap (\mathfrak{n}I :_R \mathrm{I}_1(d_n)) \cap (\mathfrak{n}I : I)$. Since I is Burch, we have $(I :_R \mathfrak{n}) \nsubseteq (\mathfrak{n}I :_R \mathfrak{n})$ by Proposition 6.2.3. In particular I is nonzero, and we see that $(I :_R \mathfrak{n}) \subseteq \mathfrak{n} = (\mathfrak{n}I :_R I)$. We obtain $(I :_R \mathfrak{n}) \nsubseteq (\mathfrak{n}I :_R \mathrm{I}_1(d_t))$ for some $1 \leq t \leq n$. It follows from Corollary 6.3.5 that k is a direct summand of the cokernel of ∂_t, which is $\Omega_R^{t+1} M$. ■

Let k be a field. A local ring R is said to be a *fibre product* (over k) provided that it is of the form

$$R \cong S \times_k T = \{(s, t) \in S \times T \mid \pi_S(s) = \pi_T(t)\},$$

where $(S, \mathfrak{m}_S)$ and $(T, \mathfrak{m}_T)$ are local rings with common residue field k, and $\pi_S \colon S \to k$ and $\pi_T \colon T \to k$ are the natural surjections. The set $S \times_k T$ is a local ring with maximal ideal $\mathfrak{m}_{S \times_k T} = \mathfrak{m}_S \oplus \mathfrak{m}_T$ and residue field k. Conversely, a local ring R with decomposable maximal ideal $\mathfrak{m}_R = I \oplus J$ is a fibre product since $R \cong (R/I) \times_k (R/J)$. These observations are due to Ogoma [120, Lemma 3.1].

We can now complete the proof of Theorem 6.4.1.

Proof of Theorem 6.4.1. The "only if" part is a direct consequence of Proposition 6.4.2.

We consider the "if" part. Again we may assume that R is complete. Take a Cohen presentation $R \cong S/I$, where $(S, \mathfrak{n})$ is a regular local ring and I is an ideal of S contained in $\mathfrak{n}^2$. If $(I :_S \mathfrak{n}) \nsubseteq \mathfrak{n}^2$, then there is an element $x \in (\mathfrak{m} \cap \operatorname{Soc} R) \setminus \mathfrak{m}^2$. One has a decomposition $\mathfrak{m} = J \oplus (x)$, which means that R is of the form $S \times_k T$ with $\operatorname{edim} T = 1$. Then R is Burch by Example 6.2.9 and Proposition 6.6.14. Thus we may assume that $(I :_S \mathfrak{n}) \subseteq \mathfrak{n}^2$. Suppose that I is not Burch, so that $\mathfrak{n}(I :_S \mathfrak{n}) = \mathfrak{n}I$. We aim to show that $\operatorname{Soc} \Omega_R^2 k \subseteq \mathfrak{m}\Omega_R^2 k$. Take minimal generators $x_1, \dots, x_e$ of $\mathfrak{n}$. There is a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega_S^2 k & \longrightarrow & \Omega_R^2 k & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I^e & \longrightarrow & S^e & \longrightarrow & R^e & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \longrightarrow & \mathfrak{n} & \longrightarrow & \mathfrak{m} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & I/\mathfrak{n}I & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

of S-modules with exact rows and columns. Applying the snake lemma, we get an exact sequence

$$\Omega_S^2 k \to \Omega_R^2 k \xrightarrow{\delta} I/\mathfrak{n}I \to 0, \qquad (6.4.5.1)$$

where δ sends each element $a \in \Omega_R^2 k$ whose preimage in S^e is ${}^{\mathrm{t}}(a_1, \dots, a_e)$ to the image of $\sum_i x_i a_i$ in $I/\mathfrak{n}I$. Now consider element $a \in \operatorname{Soc} \Omega_R^2 k$. This means that the preimage ${}^{\mathrm{t}}(a_1, \dots, a_e) \in S^e$ of a satisfies $a_i \in (I :_S \mathfrak{n})$ for all i. Therefore, the element $\sum_i x_i a_i \in S$ is contained in $\mathfrak{n}(I :_S \mathfrak{n}) = \mathfrak{n}I$. This yields that $\delta(a) = 0$. By the exact sequence (6.4.5.1), we can take the preimage $(a_1, \dots, a_e) \in S^e$ of a to be contained in $\Omega_S^2 k$. We already have ${}^{\mathrm{t}}(a_1, \dots, a_e) \in (I :_S \mathfrak{n})S^e \subseteq \mathfrak{n}^2 S^e$. It follows that ${}^{\mathrm{t}}(a_1, \dots, a_e) \in \Omega_S^2 k \cap \mathfrak{n}^2 S^e \subseteq \mathfrak{n}\Omega_S^2 k$, see [75, Theorems 3.7 and 4.1] for the second containment. Consequently, the element a is contained in $\mathfrak{m}\Omega_R^2 k$. This allows us to conclude that if $\operatorname{Soc} \Omega_R^2 k \not\subseteq \mathfrak{m}\Omega_R^2 k$ then I is a Burch ideal, and hence R is a Burch ring. ■

In view of Theorem 6.4.1, one may wonder if an artinian local ring R is Burch if the residue field k is a direct summand of $\Omega^n k$ for some $n \geq 3$. This is not true in general:

Example 6.4.6. Let k be a field, and consider the ring $R = k[\![x, y]\!]/I$, where $I = (x^4, x^2y^2, y^4)$. The minimal free resolution of k is

$$0 \leftarrow k \leftarrow R \xleftarrow{(x\ y)} R^2 \xleftarrow{\begin{pmatrix} -y & xy^2 & x^3 & 0 \\ x & 0 & 0 & y^3 \end{pmatrix}} R^4 \xleftarrow{\begin{pmatrix} xy^2 & 0 & x^3 & 0 & 0 & y^3 & 0 & 0 \\ y & x & 0 & 0 & 0 & 0 & y^2 & 0 \\ 0 & 0 & y & x & 0 & 0 & 0 & y^2 \\ 0 & 0 & 0 & 0 & y & -x & 0 & 0 \end{pmatrix}} R^8 \leftarrow \cdots .$$

We have $\operatorname{Soc} \Omega^3 k = \operatorname{Soc} R^4 = (x^3y, xy^3)R^4$. The column vector $z := {}^{\mathrm{t}}(x^3y, 0, 0, 0) = y \cdot {}^{\mathrm{t}}(x^3, 0, y, 0) - {}^{\mathrm{t}}(0, 0, y^2, 0)$ is in $\operatorname{Soc} \Omega^3 k \setminus \mathfrak{m}\Omega^3 k$. The assignment $1 \mapsto z$ makes a split monomorphism $k \to \Omega^3 k$, and k is a direct summand of $\Omega^3 k$. However, R is not Burch as one can easily check the equality $\mathfrak{m}(I :_R \mathfrak{m}) = \mathfrak{m}I$.

6.5 Burch rings of positive depth

In this section, we study Burch rings of positive depth. First of all, let us investigate what Gorenstein Burch rings are.

Proposition 6.5.1. *A local ring is Burch and Gorenstein if and only if it is a hypersurface.*

Proof. Let R be a local ring of dimension d. If R is hypersurface, then R is clearly Gorenstein, and it is also Burch by Example 6.2.9. Conversely, suppose that R is Burch and Gorenstein. Then there exists a system of parameters $\boldsymbol{x} = x_1, \dots, x_d$ such that $\widehat{R}/(\boldsymbol{x})$ is an artinian Gorenstein Burch local ring. By definition, there exist a regular local ring $(S, \mathfrak{n})$ and a Burch ideal I of S such that $\widehat{R}/(\boldsymbol{x}) \cong S/I$. By Theorem 6.4.4, there are a minimal system of generators $y_1, \dots, y_n$ of $\mathfrak{n}$ with $n > 0$ and an integer $r > 0$ such that $I = (y_1^r, y_2, \dots, y_n)$. In particular, $S/I \cong \widehat{R}/(\boldsymbol{x})$ is a hypersurface, and so is R. ■

A Cohen–Macaulay local ring R is said to have *minimal multiplicity* if $e(R) = \operatorname{codim} R + 1$.

Proposition 6.5.2. *Let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring with minimal multiplicity, assume that k is infinite. Then R is Burch.*

Proof. We can find a general system of parameters $\underline{x}$ such that $A = R/(\underline{x})$ is artinian and still has minimal multiplicity. This simply means that $\mathfrak{m}_A^2 = 0$, so the first syzygy of k is a k-vector space. Thus A is Burch by 6.4.1 and so is R. ■

Remark 6.5.3. A Cohen–Macaulay local ring with minimal multplicity is a typical example of a Golod local ring. In view of Proposition 6.5.2, the reader may wonder if a Golod local ring is Burch. This is not true in general; the ring R given in Example 6.4.6 is not Burch but Golod by [11, 1.4.3 and 2.1]. Also the converse doesn't holds in general. Indeed, let $R = k[x, y, z, w]/\mathfrak{m}J$, where $\mathfrak{m} = (x, y, z, w)$ and $J = (x^2, y^2, z^2, w^2)$ in $k[x, y, z, w]$. This is the example of non-Golod ring R given in [44, Example 2.1]. However, It is Burch by Exmaple 6.2.2 (2).

We establish a lemma to prove our next result on Burch rings.

Lemma 6.5.4. *Let $(R, \mathfrak{m}, k)$ be a 1-dimensional Cohen–Macaulay local ring with minimal multiplicity. Then there exists an isomorphism $\mathfrak{m}^* \cong \mathfrak{m}$, where $(-)^* = \operatorname{Hom}_R(-, R)$.*

Proof. If R is a discrete valuation ring, then $\mathfrak{m} \cong R$, and hence $\mathfrak{m}^* \cong \mathfrak{m}$. So we assume that R is not a discrete valuation ring. Since R has minimal multiplicity, by [112, Lemma 1.11], there is an R-regular element $x \in \mathfrak{m}$ such that $\mathfrak{m}^2 = x\mathfrak{m}$. Let Q be the total quotient ring of R. We have

$$\mathfrak{m}^* = \operatorname{Hom}_R(\mathfrak{m}, R) \cong \operatorname{Hom}_R(\mathfrak{m}, xR) \cong (xR :_Q \mathfrak{m}) \supseteq \mathfrak{m},$$

where the second isomorphism follows from [100, Proposition 2.4(1)] for instance. For each element $\frac{a}{s} \in (xR :_Q \mathfrak{m})$, we have $ax \in a\mathfrak{m} \subseteq sxR$, which implies $a \in sR$ as x is R-regular, and hence $\frac{a}{s} \in R$. Therefore $(xR :_Q \mathfrak{m})$ is an ideal of R containing $\mathfrak{m}$. Since R is not a discrete valuation ring, it is a proper ideal. We get $(xR :_Q \mathfrak{m}) = \mathfrak{m}$, and consequently $\mathfrak{m}^* \cong \mathfrak{m}$. ■

Cohen–Macaulay rings of dimension 1 with minimal multiplicity have a remarkable property.

Proposition 6.5.5. *Let $(R, \mathfrak{m}, k)$ be a 1-dimensional Cohen–Macaulay local ring with minimal multiplicity. Then the quotient artinian ring $R/(x)$ is a Burch ring for any parameter $x \in \mathfrak{m} \setminus \mathfrak{m}^2$.*

Proof. If R is regular, then it is a discrete valuation ring, and x is a uniformizer. Hence $R/(x)$ is a field, and it is Burch. Thus we assume that R is singular. Applying $(-)^* = \operatorname{Hom}_R(-, R)$ to the natural exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$, we get an exact sequence $0 \to R \to \mathfrak{m}^* \to k^{\oplus r} \to 0$, where r is the type of R. Making the pullback diagram of the map $\mathfrak{m}^* \to k^{\oplus r}$ and the natural surjection $R^{\oplus r} \to k^{\oplus r}$, we obtain an exact sequence $0 \to \mathfrak{m}^{\oplus r} \to R^{\oplus (r+1)} \to \mathfrak{m}^* \to 0$. As R is singular, $\mathfrak{m}^{\oplus r}$ does not have a nonzero free summand by [49, Corollary 1.3]. We get an isomorphism $\mathfrak{m}^{\oplus r} \cong \Omega(\mathfrak{m}^*)$. Combining this with Lemma 6.5.4 yields $\mathfrak{m}^{\oplus r} \cong \Omega\mathfrak{m} \cong \Omega^2 k$. Since x is an R-regular element in $\mathfrak{m} \setminus \mathfrak{m}^2$, there is a split exact sequence $0 \to k \to \mathfrak{m}/x\mathfrak{m} \to \mathfrak{m}/(x) \to 0$, which induces $\mathfrak{m}/x\mathfrak{m} \cong k \oplus \mathfrak{m}/(x)$. We obtain isomorphisms of $R/(x)$-modules

$$\begin{aligned} k^{\oplus r} \oplus (\mathfrak{m}/(x))^{\oplus r} &\cong (\mathfrak{m}/x\mathfrak{m})^{\oplus r} \cong \Omega^2 k / x\Omega^2 k \\ &\cong \Omega_{R/(x)}(\mathfrak{m}/x\mathfrak{m}) \cong \Omega_{R/(x)} k \oplus \Omega_{R/(x)}(\mathfrak{m}/(x)) \cong \Omega_{R/(x)} k \oplus \Omega^2_{R/(x)} k, \end{aligned}$$

where the third isomorphism holds since there is an exact sequence $0 \to \Omega^2 k \to R^{\oplus n} \to \mathfrak{m} \to 0$ with $n = \operatorname{edim} R$, which induces an exact sequence $0 \to \Omega^2 k / x\Omega^2 k \to (R/(x))^{\oplus n} \to \mathfrak{m}/x\mathfrak{m} \to 0$. As $R/(x)$ is an artinian local ring, it is henselian. The Krull–Schmidt theorem implies that k is a direct summand of either $\Omega_{R/(x)} k$ or $\Omega^2_{R/(x)} k$. In the former case, applying $\Omega_{R/(x)}(-)$ shows that k is a direct summand of $\Omega^2_{R/(x)} k$. Theorem 6.4.1 concludes that $R/(x)$ is a Burch ring. ∎

Proposition 6.5.6. *Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over an infinite field and $I \subset S$ is a homogenous ideal such that S/I is Cohen-Macaulay and I has a linear resolution. Then $R = (S/I)_{\mathfrak{m}}$ is Burch where $\mathfrak{m} = (x_1, \ldots, x_n)$.*

Proof. Let $A = S/I$ and $(l_1, \ldots, l_d)$ be a general linear system of parameters on A. We write $A/(l_1, \ldots, l_d)A$ as T/J where T is a polynomial ring in $n-d$ variables over k and J is a zero-dimensional ideal. Then J still has linear resolution. Assume I (and J) are generated in degree t, then the regularity of J is t, but since J is zero-dimensional, the socle degree of J is $t-1$. Thus $J = \mathfrak{n}^t$ where $\mathfrak{n}$ is the irrelevant ideal of T, and so R is Burch by definition and Example 6.2.2. ∎

Example 6.5.7. There are many examples satisfying the conditions of Proposition 6.5.6. For example, let $m \geq n$ and let $I = I_n \subset k[x_{ij}] = S$ be the ideal generated by maximal minors in a m by n matrix of indeterminates. Then it is well-known that S/I is Cohen-Macaulay with $\dim S/I = (m+1)(n-1)$ and the a-invariant of S/I is $-m(n-1)$ (see [19]). It follows that the regularity of I is n, so it has linear resolution.

Another source of examples are Stanley-Reisner rings of "facet constructible" or "stacked" simplicial complexes, see [42, Theorem 4.1 and 4.4].

We will show in Corollary 6.7.9 that if x is a regular element of a local ring $(R, \mathfrak{m})$ such that $R/(x)$ is Burch, then $x \notin \mathfrak{m}^2$. It is natural to ask whether the quotient ring R/Q of a Burch ring R is again Burch for any ideal Q generated by regular sequence consisting of elements in $\mathfrak{m} \setminus \mathfrak{m}^2$. This is true if R is either a hypersurface or a Cohen–Macaulay local ring of dimension one with minimal multiplicity, as we saw in Propositions 6.5.1 and 6.5.5. The example below says that the question is not always affirmative.

Example 6.5.8. Let k be a field, and let $R = k[\![x,y,z]\!]/\,\mathrm{I}_2\left(\begin{smallmatrix} x^2 & y & z^2 \\ y & z^2 & x^2 \end{smallmatrix}\right)$. The Hilbert–Burch theorem implies that R is a Cohen–Macaulay local ring of dimension 1. The ring R is a Burch ring since so is the artinian quotient ring $R/(x) = k[\![y,z]\!]/(y^2, yz^2, z^4)$. However, the artinian ring $R/(y) = k[\![x,z]\!]/(x^4, x^2z^2, z^4)$ is not Burch. By Theorem 6.4.1, the R-module k is a direct summand of $\Omega^2_{R/(x)}k$, but not a direct summand of $\Omega^2_{R/(y)}k$. Incidentally, the module k is a direct summand of $\Omega^3_{R/(y)}k$ by Example 6.4.6.

To show our next result on Burch rings, we prepare a lemma on cancellation of free summands.

Lemma 6.5.9. *Let R be a local ring. Let M, N be R-modules having no nonzero free summand. If $M \oplus R^{\oplus a} \cong N \oplus R^{\oplus b}$ for some $a, b \geq 0$, then $M \cong N$ and $a = b$.*

Proof. We may assume $a \geq b$. Taking the completions, we get isomorphisms $\widehat{M} \oplus \widehat{R}^{\oplus a} \cong \widehat{N} \oplus \widehat{R}^{\oplus b}$. Write $\widehat{M} = X \oplus \widehat{R}^{\oplus c}$ and $\widehat{N} = Y \oplus \widehat{R}^{\oplus d}$ with $c, d \geq 0$ integers and X, Y having no nonzero free summand. Then $X \oplus \widehat{R}^{\oplus(c+a)} \cong Y \oplus \widehat{R}^{\oplus(d+b)}$. As $\widehat{R}$ is henselian, we can apply the Krull-Schmidt theorem to deduce $X \cong Y$ and $c + a = d + b$. Hence $d = c + (a - b)$, and we get $\widehat{N} = Y \oplus \widehat{R}^{\oplus d} \cong X \oplus \widehat{R}^{\oplus(c+(a-b))} = \widehat{M} \oplus \widehat{R}^{\oplus(a-b)} \cong \widehat{L}$, where $L := M \oplus R^{\oplus(a-b)}$. It follows from [108, Corollary 1.15] that N is isomorphic to L. Since N has no nonzero free summand, we must have $a = b$, and therefore $M = L \cong N$. ■

The following result is a higher-dimensional version of the "only if" part of Theorem 6.4.1.

Proposition 6.5.10. *Let $(R, \mathfrak{m}, k)$ be a singular Burch ring of depth t, Then $\Omega^t k$ is a direct summand of $\Omega^{t+2}k$.*

Proof. We prove the proposition by induction on t. The case $t = 0$ follows from Lemma 6.2.10, so let $t \geq 1$. There is an R-sequence $\boldsymbol{x} = x_1, \dots, x_t$ such that $R/(\boldsymbol{x})$ is a Burch ring of depth zero. Hence $R/(x_1)$ is a Burch ring of dimension $d - 1$. The induction hypothesis implies that $\Omega^{t-1}_{R/(x_1)}k$ is a direct summand of $\Omega^{t+1}_{R/(x_1)}k$. Taking the syzygy over R, we see that $\Omega_R\Omega^{t-1}_{R/(x_1)}k$ is a direct summand of $\Omega_R\Omega^{t+1}_{R/(x_1)}k$. For each $n \geq 0$ there is an exact sequence $0 \to \Omega^n_{R/(x_1)}k \to P_{n-1} \to \cdots \to P_1 \to P_0 \to k \to 0$ with each P_i being a direct sum of copies of $R/(x_1)$, which gives rise to an exact sequence

$$0 \to \Omega_R\Omega^n_{R/(x_1)}k \to \Omega_R P_{n-1} \oplus R^{\oplus e_{n-1}} \to \cdots \to \Omega_R P_1 \oplus R^{\oplus e_1} \to \Omega_R P_0 \oplus R^{\oplus e_0} \to \Omega_R k \to 0$$

with $e_i \geq 0$ for $0 \leq i \leq n-1$. Note that each $\Omega_R P_i$ is a free R-module. The above sequence shows that $\Omega^{n+1}_R k = \Omega^n_R(\Omega_R k)$ is isomorphic to $\Omega_R\Omega^n_{R/(x_1)}k$ up to free R-summands. We obtain an R-isomorphism $\Omega^{n+1}_R k \oplus R^{\oplus e} \cong \Omega_R\Omega^n_{R/(x_1)}k$ with $e \geq 0$. Thus, for some $a, b \geq 0$ we have that $\Omega^t_R k \oplus R^{\oplus a}$ is a direct summand of $\Omega^{t+2}_R k \oplus R^{\oplus b}$. Since R is singular, it follows from [49, Corollary 1.3] that $\Omega^i_R k$ has no nonzero free summand for all $i \geq 0$. Applying Lemma 6.5.9, we observe that $\Omega^t_R k$ is a direct summand of $\Omega^{t+2}_R k$. ■

We pose a question asking whether or not the converse of Proposition 6.5.10 holds true.

Question 6.5.11. Does there exist a non-Burch local ring $(R, \mathfrak{m}, k)$ of depth t such that $\Omega^t k$ is a direct summand of $\Omega^{t+2}k$?

6.6 Some classes of Burch ideals and rings

In this section, we study in a regular local ring and give a complete characterization in dimension two. We also give a simple characterization of monomial Burch ideals. We compare Burch rings to other classes of rings: radical cube zero, almost Gorenstein, nearly Gorenstein, and fibre products.

Over a two-dimensional regular local ring $(R, \mathfrak{m})$, the Burch ideals I are characterized in terms of the minimal numbers of generators of I and $\mathfrak{m}I$.

Lemma 6.6.1. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension two, and let I be an $\mathfrak{m}$-primary ideal of R. Then I is a Burch ideal of R if and only if $\mu(\mathfrak{m}I) < 2\mu(I)$.*

Proof. It follows from the Hilbert–Burch theorem that $\mu(I) = r(R/I)+1$ and $\mu(\mathfrak{m}I) = r(R/\mathfrak{m}I)+1$. The assertion follows from the equivalence (1) $\Leftrightarrow$ (2) in Proposition 6.2.3. ■

Now we can show the following theorem, which particularly gives a characterization of the Burch ideals of two-dimensional regular local rings in terms of minimal free resolutions. Compare this theorem with the result of Corso, Huneke and Vasconcelos [35, Lemma 3.6].

Theorem 6.6.2. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension d. Let I be an $\mathfrak{m}$-primary ideal of R. Take a minimal free resolution $0 \to F_d \xrightarrow{\varphi_d} F_{d-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to R/I \to 0$ of the R-module R/I. Consider the following conditions.*

(1) *The ideal I is Burch.*

(2) *There exist a regular system of parameters $x_1, \dots, x_d$ and an integer $r > 0$ such that $\mathrm{I}_1(\varphi_d) = (x_1^r, x_2, \dots, x_d)$.*

(3) *One has $(I :_R \mathfrak{m})^2 \neq I(I :_R \mathfrak{m})$.*

Then the implication (1) $\Rightarrow$ (2) holds. If R contains a field, then the implication (3) $\Rightarrow$ (2) holds. If $d = 2$, then the implication (2) $\Rightarrow$ (1) holds as well.

Proof. We first show that (1) implies (2). We may assume $d \geq 2$, so that R has depth greater than 1. By Lemma 6.2.11 and its proof, there is a non-zerodivisor $x_1 \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that I/x_1I contains the residue field $R/\mathfrak{m}$ as a direct summand. Tensoring $R/(x)$ with the complex $F = (0 \to F_d \to \cdots \to F_0 \to 0)$, we get a minimal free resolution

$$(0 \to F_d/x_1F_d \xrightarrow{\varphi_d \otimes S/(x_1)} F_{d-1}/F_{d-1} \to \cdots \to F_2/x_1F_2 \to F_1/x_1F_1 \to 0)$$

of I/x_1I over $R/(x_1)$. As $R/\mathfrak{m}$ is a direct summand of I/x_1I, a minimal $R/(x_1)$-free resolution G of $R/\mathfrak{m}$ is a direct summand of the above complex. Since G is isomorphic to the Koszul complex $\mathrm{K}^{R/(x_1)}$ of $R/(x_1)$, the ideal $\mathrm{I}_1(\varphi_d \otimes R/(x_1))$ of $R/(x_1)$ contains the maximal ideal $\mathfrak{m}/(x_1)$. Therefore $\mathrm{I}_1(\varphi_d)$ contains elements $x_2, \dots, x_d$ such that $x_1, x_2, \dots, x_d$ form a regular system of parameters of R. Since the radical of $\mathrm{I}_1(\varphi_d)$ contains I, it is an $\mathfrak{m}$-primary ideal. It follows that there is an integer $r > 0$ such that $x_1^r \in \mathrm{I}_1(\varphi_d)$ but $x_1^{r-1} \notin \mathrm{I}_1(\varphi_d)$. We obtain $\mathrm{I}_1(\varphi_d) = (x_1^r, x_2, \dots, x_d)$, and (2) follows.

Next, under the assumption that R contains a field, we prove that (3) implies (2). We use an analogue of the proof of [34, Theorem 2.4]. After completion, we may assume that R is a formal power series ring over a field k. Suppose that (2) does not hold. Then $d \geq 2$ and we can take an ideal L containing $\mathrm{I}_1(\varphi_d)$ such that there is a regular system of parameters $x_1, \ldots, x_d$ with $L = (x_1^2, x_1x_2, x_2^2, x_3, \ldots, x_d)$. By [34, Proposition 2.1], an isomorphism $(I :_R L)/I \cong \omega_{R/L} \otimes_R F_d$ and its retraction $(I :_R \mathfrak{m})/I \cong \omega_{R/\mathfrak{m}} \otimes_R F_d$ are given. Note that the canonical module $\omega_{R/L}$ of R/L is isomorphic to $(0 :_{\mathrm{E}_R(k)} L)$. The module $\mathrm{E}_R(k)$ is identified with $k[x_1, x_1^{-1}, \ldots, x_d^{-1}]/N$, where N is the subspace spaned by the monomials not in $k[x_1^{-1}, \ldots, x_d^{-1}]$. Under this identification, $\omega_{R/L} = (0 :_R L)$ is generated by the monomials x_1^{-1} and x_2^{-1}. Set $M = \{x_1^{-1}, x_2^{-1}\}$. Then $x_1M = \{1\} = x_2M$ generates $\omega_{R/\mathfrak{m}}$. Also, either $x_1w = 0$ or $x_2w = 0$ holds for all $w \in M$. We may apply [34, Proposition 2.3] as in the proof of [34, Theorem 2.4] to get $(I :_R \mathfrak{m})^2 = I(I :_R \mathfrak{m})$, contrary to (3). We have shown that (3) implies (2).

Finally, assuming $d = 2$, we prove (2) implies (1). As the entries of φ_2 are contained in $\mathfrak{m}$, we have an exact sequence $0 \to F_2 \xrightarrow{\varphi_2} \mathfrak{m}F_1 \to \mathfrak{m}I \to 0$. This induces an exact sequence $F_2/\mathfrak{m}F_2 \xrightarrow{\varphi_2 \otimes_R R/\mathfrak{m}} \mathfrak{m}F_1/\mathfrak{m}^2F_1 \to \mathfrak{m}I/\mathfrak{m}^2I \to 0$. Suppose that (2) holds. Then $\varphi_2 \otimes_R R/\mathfrak{m} \neq 0$, and $\dim_{R/\mathfrak{m}}(\mathfrak{m}I/\mathfrak{m}^2I) < \dim_{R/\mathfrak{m}}(\mathfrak{m}F_1/\mathfrak{m}^2F_1)$. Note that $\dim_{R/\mathfrak{m}}(\mathfrak{m}I/\mathfrak{m}^2I) = \mu(I)$ and $\dim_{R/\mathfrak{m}}(\mathfrak{m}F_1/\mathfrak{m}^2F_1) = 2\mu(I)$. Lemma 6.6.1 shows that I is a Burch ideal, that is, (1) holds. ■

Example 6.6.3. (1) Let $I = (x^4, y^4, z^4, x^2y, y^2z, z^2x)$ be an ideal of $(R, \mathfrak{m}) = k[\![x, y, z]\!]$. Then one can check that $(I :_R \mathfrak{m}) = (x^4, x^3z, x^2y, xy^3, xyz, xz^2, y^4, y^2z, yz^3, z^4)$, and so $(I :_R \mathfrak{m})^2 \neq I(I :_R \mathfrak{m})$. However, I is not Burch. This gives a counterexample of the implication (3) $\Rightarrow$ (1) in Theorem 6.6.2.

(2) Let $I = (x^4, y^4, x^3y, xy^3)$ be an ideal of $(R, \mathfrak{m}) = k[\![x, y]\!]$. Then $(I :_R \mathfrak{m}) = (x^3, x^2y^2, y^3)$. We see that $(I :_R \mathfrak{m})^2 = I(I :_R \mathfrak{m})$ and I is Burch. This shows that the implication (1) $\Rightarrow$ (3) in Theorem 6.6.2 is not affirmative, even when R has dimension two.

We provide some characterizations of Burchness for monomial ideals of regular local rings.

Proposition 6.6.4. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension d. Let $x_1, \ldots, x_d$ be a regular system of parameters of R, and let I be a monomial ideal (in the x_is) of R. Then I is Burch if and only if there exist a monomial $m \in I \setminus \mathfrak{m}I$ and an integer $1 \leq i \leq d$ such that $x_i \mid m$ and $m(x_j/x_i) \in I$ for all $1 \leq j \leq d$.*

Proof. Since I is a Burch ideal, we have $\mathfrak{m}I \neq \mathfrak{m}(I :_R \mathfrak{m})$. Therefore, there is a monomial $m' \in (I :_R \mathfrak{m})$ and an integer i such that $x_im' \notin \mathfrak{m}I$. It also holds that $x_jm' \in I$ for all $j = 1, \ldots, d$. So the element $m := x_im'$ satisfies $m(x_j/x_i) \in I$ for all $j = 1, \ldots, d$. ■

Corollary 6.6.5. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension 2 with a regular system of parameters x, y. Let $I = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, \ldots, x^{a_n}y^{b_n})$ be a monomial ideal with $a_1 > a_2 > \cdots > a_n$ and $b_1 < b_2 < \cdots < b_n$. Then I is a Burch ideal of R if and only if $a_i = a_{i+1} + 1$ or $b_i = b_{i+1} - 1$ for some $i = 1, \ldots, n$.*

Proof. By Proposition 6.6.4, the ideal I is Burch if and only if $x^{a_i}y^{b_i}(y/x) \in I$ or $x^{a_i}y^{b_i}(x/y) \in I$ for some $i = 1, \ldots, n$. Equivalently, either $x^{a_i-1}y^{b_i+1} \in I$ or $x^{a_i+1}y^{b_i-1} \in I$ holds for some $i = 1, \ldots, n$. Since $a_{i+1} \leq a_i - 1 < a_i < a_i + 1 \leq a_{i-1}$ and $b_{i-1} \leq b_i - 1 < b_i < b_i + 1 \leq b_{i+1}$, the condition is equivalent to saying that $b_i + 1 = b_{i+1}$ or $a_i + 1 = a_{i-1}$ for some $i = 1, \ldots, n$. ■

Next, we discuss the relationship between Burch rings and several classes of rings studied previously in the literature.

The following notions are introduced in [74, 135].

Definition 6.6.6 (Herzog–Hibi–Stamate)**.** Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring with canonical module ω. Then R is called *nearly Gorenstein* if $\operatorname{tr}\omega$ contains $\mathfrak{m}$.

Definition 6.6.7 (Striuli–Vraciu)**.** Let $(R,\mathfrak{m})$ be an artinian local ring. Then R is called *almost Gorenstein*[2] if $(0:_R (0:_R I)) \subseteq (I:_R \mathfrak{m})$ for all ideals I of R.

It follows from [86, Proposition 1.1] that artinian nearly Gorenstein local rings are almost Gorenstein.

We want to consider the relationship of Burchness with near Gorensteinness and almost Gorensteinness. For this, we establish two lemmas.

Lemma 6.6.8. *Let $(R,\mathfrak{m},k)$ be a non-Gorenstein artinian almost Gorenstein local ring. Let $R^n \xrightarrow{A} R^m \to E \to 0$ be a minimal R-free presentation of the R-module $E = \mathrm{E}_R(k)$. One then has $\mathrm{I}_1(A) = \mathfrak{m}$.*

Proof. Choose an artinian Gorenstein local ring $(S,\mathfrak{n})$ and an ideal I of S such that $R \cong S/I$. We identify E with $(0:_S I)$ via the isomorphisms $E \cong \operatorname{Hom}_S(R,S) \cong (0:_S I)$. Let $x_1,\dots,x_m$ be a minimal system of generators of E. By [135, Lemma 1.2] we have $\mathfrak{n} = ((x_1):_S (x_2,\dots,x_m)) + ((x_2,\dots,x_m):_S x_1)$. We find a matrix B over S with m rows such that $\mathrm{I}_1(B) = \mathfrak{n}$ and $\begin{pmatrix} x_1 & \cdots & x_m \end{pmatrix} B = 0$. We find a matrix C over R such that the matrix $\overline{B}$ over R corresponding to B is equal to AC. We have $\mathfrak{m} = \mathrm{I}_1(\overline{B}) = \mathrm{I}_1(A\cdot C) \subseteq \mathrm{I}_1(A) \subseteq \mathfrak{m}$, which implies $\mathrm{I}_1(A) = \mathfrak{m}$. ■

Lemma 6.6.9. *Let $(R,\mathfrak{m})$ be a regular local ring of dimension d, and let $I \subseteq \mathfrak{m}^2$ be an ideal of R. Take a minimal free resolution $0 \to F_d \xrightarrow{\varphi_d} F_{d-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to R/I \to 0$ of the R-module R/I. If R/I is artinian, non-Gorenstein and almost Gorenstein, then $\mathrm{I}_1(\varphi_d) = \mathfrak{m}$.*

Proof. Set $A = R/I$ and $E = \mathrm{E}_A(k)$. Then the sequence $(F_{d-1}/IF_{d-1})^* \xrightarrow{(\varphi_d \otimes A)^*} (F_d/IF_d)^* \to E \to 0$ gives a minimal A-free presentation of E, where $(-)^* = \operatorname{Hom}_A(-,A)$. Note that $\operatorname{rank}_A(F_d/IF_d)^* = r(A) = \mu(E)$. Lemma 6.6.8 implies $\mathrm{I}_1((\varphi_d \otimes A)^*) = \mathfrak{m}$, which shows $\mathrm{I}_1(\varphi_d) + I = \mathfrak{m}$. The desired result follows from Nakayama's lemma. ■

We can show an artinian almost Gorenstein local ring of embedding dimension two is Burch.

Proposition 6.6.10. *Let $(R,\mathfrak{m})$ be a regular local ring of dimension 2 and I an ideal of R. Assume that R/I is a non-Gorenstein artinian almost Gorenstein ring. Then I is a Burch ideal of R.*

Proof. Take a minimal free resolution $0 \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \to R/I \to 0$ of the R-module R/I. It follows from Lemma 6.6.9 that $I_1(\varphi_2) = \mathfrak{m}$. Since R has dimension two, we can use the implication (2)⇒(1) in Theorem 6.6.2 to have that I is Burch. ■

[2]There is another notion of an almost Gorenstein ring; see [65].

Remark 6.6.11. One may hope a non-Gorenstein nearly Gorenstein local ring is Burch, but this is not necessarily true. Indeed, let $(R, \mathfrak{m})$ be a 1-dimensional nearly Gorenstein local ring (e.g. $R = k[\![t^3, t^4, t^5]\!] \subseteq k[\![t]\!]$ with k a field). Take a regular element $x \in \mathfrak{m}^2$, and set $A = R/(x)$. Then A is nearly Gorenstein by [74, Proposition 2.3(b)], but A is not a Burch ring by Corollary 6.7.9.

Next, we deal with local rings the cube of whose maximal ideal is zero. The following gives a characterization of Burchness for such rings.

Theorem 6.6.12. *Let $(R, \mathfrak{m}, k)$ be a local ring with $\mathfrak{m}^3 = 0$. Then R is a Burch ring if and only if there is an inequality $\beta_2(k) > (\operatorname{edim} R)^2 - r(R)$.*

Proof. Put $e = \operatorname{edim} R$ and $r = r(R)$. By Theorem 6.4.1, the ring R is Burch if and only if k is a direct summand of $\Omega^2 k$, if and only if $\operatorname{Soc} \Omega^2 k \not\subseteq \mathfrak{m}\Omega^2 k$. There is a short exact sequence $0 \to \Omega^2 k \to R^e \to \mathfrak{m} \to 0$, which gives an inclusion $\Omega^2 k \subseteq \mathfrak{m}R^e$ and an equality $\operatorname{Soc} \Omega^2 k = \operatorname{Soc} R^e$. Since $\mathfrak{m}^3 = 0$, we have an inclusion $\mathfrak{m}\Omega^2 k \subseteq \operatorname{Soc} \Omega^2 k$. Thus R is Burch if and only if $\ell(\operatorname{Soc} \Omega^2 k) > \ell(\mathfrak{m}\Omega^2 k)$. There are equalities

$$\begin{aligned}\beta_2(k) &= \ell(\Omega^2 k) - \ell(\mathfrak{m}\Omega^2 k) = \ell(R^e) - \ell(\mathfrak{m}) - \ell(\mathfrak{m}\Omega^2 k) = (e-1)\ell(\mathfrak{m}) + e - \ell(\mathfrak{m}\Omega^2 k) \\ &= (e-1)(e + \ell(\mathfrak{m}^2)) + e - \ell(\mathfrak{m}\Omega^2 k) = e^2 + (e-1)\ell(\mathfrak{m}^2) - \ell(\mathfrak{m}\Omega^2 k).\end{aligned}$$

On the other hand, there is an inclusion $\Omega^2 k \subseteq \mathfrak{m}^e$, which induces an inclusion $\mathfrak{m}\Omega^2 k \subseteq (\mathfrak{m}^2)^e$. Thus one has $\ell(\mathfrak{m}\Omega^2 k) \leq e\ell(\mathfrak{m}^2) \leq er = \ell(\operatorname{Soc} \Omega^2 k)$. If $\ell(\mathfrak{m}^2) < \ell(\operatorname{Soc} R) = r$, then we see that $\ell(\operatorname{Soc} \Omega^2 k) > \ell(\mathfrak{m}\Omega^2 k)$. The above equalities show that $\beta_2(k) \geq e^2 - \ell(\mathfrak{m}^2) > e^2 - r$. Therefore, we may assume $\ell(\mathfrak{m}^2) = r$. We obtain $\beta_2(k) = e^2 - r + er - \ell(\mathfrak{m}\Omega^2 k)$. It follows that $\beta_2 > e^2 - r$ if and only if $er - \ell(\mathfrak{m}\Omega^2 k) > 0$. The latter condition is equivalent to $\ell(\operatorname{Soc} \Omega^2 k) > \ell(\mathfrak{m}\Omega^2 k)$. ■

Let R be a local ring with maximal ideal $\mathfrak{m}$. An element $x \in \mathfrak{m}$ is called a *Conca generator* of $\mathfrak{m}$ if $x^2 = 0$ and $\mathfrak{m}^2 = x\mathfrak{m}$. This notion has been introduced in [13]. Note that the condition $\mathfrak{m}^3 = 0$ is necessary for R to possess a Conca generator.

Corollary 6.6.13. *Let $(R, \mathfrak{m}, k)$ be a local ring with $\mathfrak{m}^3 = 0$ and $\operatorname{Soc} R \subseteq \mathfrak{m}^2$. If R is a Burch ring, then R has no Conca generator.*

Proof. If R has a Conca generator, then the Poincaré series $P_k(t) = \sum \beta_i t^i$ is of the form $\frac{1}{1-et+rt^2}$ by [13, Theorem 1.1]. In particular, $\beta_2(k) = e^2 - r$. Thus R is not Burch by Theorem 6.6.12. ■

Next, we consider the Burchness of a fibre product.

Let S, T be local rings having common residue field k. We say that the fibre product $S \times_k T$ is *nontrivial* if $S \neq k \neq T$. It holds that $\operatorname{depth} S \times_k T = \min\{\operatorname{depth} S, \operatorname{depth} T, 1\}$; see [106, Remarque 3.3].

We consider the Burchness of the fibre product $S \times_k T$. We compute some invariants.

Lemma 6.6.14. *Let $R = S \times_k T$ be a nontrivial fibre product, where $(S, \mathfrak{m}_S, k)$ and $(T, \mathfrak{m}_T, k)$ are local rings. Then the following equalities hold.*

(1) $\operatorname{edim} R = \operatorname{edim} S + \operatorname{edim} T$.

(2) $\dim_k \operatorname{Soc} R = \dim_k \operatorname{Soc} S + \dim_k \operatorname{Soc} T$.

(3) $\dim_k \mathrm{H}_1(\mathrm{K}^R) = \dim_k \mathrm{H}_1(\mathrm{K}^S) + \dim_k \mathrm{H}_1(\mathrm{K}^T) + \operatorname{edim} S \cdot \operatorname{edim} T$.

(4) $c_R = c_S + c_T + \operatorname{edim} S \cdot \operatorname{edim} T - \operatorname{edim}(S/\operatorname{Soc} S) \cdot \operatorname{edim}(T/\operatorname{Soc} T)$.

Proof. (1)(2) These equalities can be checked directly.

(3) One has $\beta_2^R(k) = \beta_2^S(k) + \beta_2^T(k) + 2\operatorname{edim} S \cdot \operatorname{edim} T$ and $\dim_k \mathrm{H}_1(\mathrm{K}^R) = \beta_2^R(k) - \binom{\operatorname{edim} R}{2}$; see [102] and [19, Theorem 2.3.2] for example. Thus there are equalities

$$\begin{aligned}\dim_k \mathrm{H}_1(\mathrm{K}^R) &= \beta_2^R(k) - \tbinom{\operatorname{edim} R}{2} = \beta_2^S(k) + \beta_2^T(k) + 2\operatorname{edim} S \cdot \operatorname{edim} T - \tbinom{\operatorname{edim} R}{2} \\ &= \dim_k \mathrm{H}_1(\mathrm{K}^S) - \tbinom{\operatorname{edim} S}{2} + \dim_k \mathrm{H}_1(\mathrm{K}^T) - \tbinom{\operatorname{edim} T}{2} + 2\operatorname{edim} S \cdot \operatorname{edim} T - \tbinom{\operatorname{edim} R}{2} \\ &= \dim_k \mathrm{H}_1(\mathrm{K}^{R_1}) + \dim_k \mathrm{H}_1(\mathrm{K}^{R_2}) + \operatorname{edim} S \cdot \operatorname{edim} T.\end{aligned}$$

(4) Put $R' = R/\operatorname{Soc} R$, $S' = S/\operatorname{Soc} S$ and $T' = T/\operatorname{Soc} T$. Then $R' \cong S' \times T'$ unless $S = k$ or $T = k$. Using (1), (2) and (3), we can calculate c_R as follows:

$$\begin{aligned}c_R &= \dim_k \operatorname{Soc} R + \dim \mathrm{H}_1(\mathrm{K}^R) - \operatorname{edim} R - \dim \mathrm{H}_1(\mathrm{K}^{R'}) + \operatorname{edim} R' \\ &= \dim_k \operatorname{Soc} S + \dim_k \operatorname{Soc} T + \dim_k \mathrm{H}_1(\mathrm{K}^S) + \dim_k \mathrm{H}_1(\mathrm{K}^{R_2}) + \operatorname{edim} S \cdot \operatorname{edim} T \\ &\quad - \operatorname{edim} S - \operatorname{edim} T - \dim_k \mathrm{H}_1(\mathrm{K}^{S'}) - \dim_k \mathrm{H}_1(\mathrm{K}^{T'}) - \operatorname{edim} S' \cdot \operatorname{edim} T' + \operatorname{edim} S' + \operatorname{edim} T' \\ &= c_S + c_T + \operatorname{edim} S \cdot \operatorname{edim} T - \operatorname{edim} S' \cdot \operatorname{edim} T'.\end{aligned}$$ ■

Using the above lemma, we can characterize the Burch fibre products.

Proposition 6.6.15. *Let $R = S \times_k T$ be a nontrivial fibre product, where $(S, \mathfrak{m}_S, k)$ and $(T, \mathfrak{m}_T, k)$ are local rings. Then R is a Burch ring if and only if*

(a) $\operatorname{depth} R > 0$, *or* (b) $\operatorname{depth} R = 0$ *and either S or T is a Burch ring of depth zero.*

Proof. First we deal with the case where $\operatorname{depth} R = 0$. Lemma 6.2.10 shows that R is Burch if and only if $c_R > 0$. Note that the integers c_S, c_T and $N := \operatorname{edim} S \cdot \operatorname{edim} T - \operatorname{edim}(S/\operatorname{Soc} S) \cdot \operatorname{edim}(T/\operatorname{Soc} T)$ are always nonnegative. By Lemmas 6.6.14(4), the positivity of c_S or c_T implies that R is Burch. Conversely, assume that R is Burch. Then by Lemma 6.6.14(4) again, one of the three integers c_S, c_T, N is positive. If c_S or c_T is positive, then S or T is Burch. When $N > 0$, either $\operatorname{edim} S > \operatorname{edim} S/\operatorname{Soc} S$ or $\operatorname{edim} T > \operatorname{edim} T/\operatorname{Soc} T$ holds. Without loss of generality, we may assume that $\operatorname{edim} S > \operatorname{edim} S/\operatorname{Soc} S$. This inequality means that there is an element $x \in (\mathfrak{m}_S \cap \operatorname{Soc} S) \setminus \mathfrak{m}_S^2$. Then $\mathfrak{m}_S = I \oplus (x)$ for some ideal I. We see that $S \cong S/(x) \times_k S/I$ and $\operatorname{edim} S/I \leq 1$. Example 6.2.9 implies that S/I is Burch, and so is S.

Next, we consider the case where $\operatorname{depth} R > 0$. In this case, we have $\operatorname{depth} S > 0$, $\operatorname{depth} T > 0$ and $\operatorname{depth} R = 1$. Take regular elements $x \in \mathfrak{m}_S \setminus \mathfrak{m}_S^2$ and $y \in \mathfrak{m}_T \setminus \mathfrak{m}_T^2$. The element $x - y \in \mathfrak{m}_R = \mathfrak{m}_S \oplus \mathfrak{m}_T$ is also a regular element of R. The equalities $x\mathfrak{m}_R = x\mathfrak{m}_S = (x-y)\mathfrak{m}_S$ show that the image $\overline{x} \in R/(x-y)$ of x is in $\operatorname{Soc} R/(x-y)$. We have $\mathfrak{m}_R/(x-y) = (\overline{x}) \oplus I$ for some ideal I of $R/(x-y)$. Hence $R/(x-y)$ is isomorphic to the fibre product $U \times_k V$ of local rings over their common residue field k such that $\operatorname{edim} V \leq 1$. As V is Burch by Example 6.2.9, it follows that so is $R/(x-y)$, and hence so is R. ■

Example 6.6.16. Let $R = k[x,y]/(x^a, xy, y^b)$ with k a field and $a, b \geq 1$. Then R is a Burch ring. In fact, R is isomorphic to the fibre product of $k[x]/(x^a)$ and $k[y]/(y^b)$ over k. By Example 6.2.9, the rings $k[x]/(x^a)$ and $k[y]/(y^b)$ are Burch, and so is R by Proposition 6.6.15.

6.7 Homological and categorical properties of Burch rings

In this section, we explore some homological and categorical aspects of Burch rings. They come in several flavors. We prove a classification theorem of subcategories over Burch rings. We also prove that non-Gorenstein Burch rings are G-regular in the sense of [138], and that nontrivial consecutive vanishings of Tor over Burch rings cannot happen. We begin with recalling the definition of resolving subcategories.

Definition 6.7.1. Let R be a ring. A subcategory $\mathcal{X}$ of $\mathsf{mod}\,R$ is *resolving* if the following hold.

(1) The projective R-modules belong to $\mathcal{X}$.

(2) Let M be an R-module and N a direct summand of M. If M is in $\mathcal{X}$, then so is N.

(3) For an exact sequence $0 \to L \to M \to N \to 0$, if L and N are in $\mathcal{X}$, then so is M.

(4) For an exact sequence $0 \to L \to M \to N \to 0$, if M and N are in $\mathcal{X}$, then so is L.

Note that (1) can be replaced by the condition that $\mathcal{X}$ contains R. Also, (4) can be replaced by the condition that if M is an R-module in $\mathcal{X}$, then so is ΩM. For an R-module C, we denote by $\operatorname{res}_R C$ the *resolving closure* of C, the smallest resolving subcategory of $\mathsf{mod}\,R$ containing C.

We establish a couple of lemmas to prove Proposition 6.7.6. The first lemma is used as a base result of this section, which is essentially shown in [139, Proposition 4.2]. For an R-module M we denote by $\mathrm{NF}(M)$ the *nonfree locus* of M, that is, the set of prime ideals $\mathfrak{p}$ of R such that $M_\mathfrak{p}$ is nonfree as an $R_\mathfrak{p}$-module.

Lemma 6.7.2. *Let $(R, \mathfrak{m})$ be a local ring, M a nonfree R-module, and x an element in $\mathfrak{m}$.*

(1) *There exists a short exact sequence $0 \to \Omega M \to M(x) \to M \to 0$ such that $x \in \mathrm{I}_1(M(x)) \subseteq \mathfrak{m}$ and $\operatorname{pd}_R M(x) \geq \operatorname{pd}_R M$. In particular, $M(x)$ belongs to $\operatorname{res}_R M$.*

(2) *For each $\mathfrak{p} \in \mathrm{V}(x) \cap \mathrm{NF}(M)$ one has $\mathrm{V}(\mathfrak{p}) \subseteq \mathrm{NF}(M(x)) \subseteq \mathrm{NF}(M)$ and $\mathrm{D}(x) \cap \mathrm{NF}(M(x)) = \emptyset$.*

Proof. (1) Let $\cdots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\pi} M \to 0$ be a minimal free resolution of M. Taking the mapping cone of the multiplication map of the complex F by x, we get an exact sequence

$$\cdots \to F_3 \oplus F_2 \xrightarrow{\left(\begin{smallmatrix} d_3 & x \\ 0 & -d_2 \end{smallmatrix}\right)} F_2 \oplus F_1 \xrightarrow{\left(\begin{smallmatrix} d_2 & x \\ 0 & -d_1 \end{smallmatrix}\right)} F_1 \oplus F_0 \xrightarrow{\left(\begin{smallmatrix} d_1 & x \\ 0 & -\pi \end{smallmatrix}\right)} F_0 \oplus M \xrightarrow{(\pi\ x)} M \to 0.$$

Set $M(x) = \operatorname{Im}\left(\begin{smallmatrix} d_1 & x \\ 0 & -\pi \end{smallmatrix}\right) = \operatorname{Cok}\left(\begin{smallmatrix} d_2 & x \\ 0 & -d_1 \end{smallmatrix}\right)$. The free resolution of $M(x)$ given by truncating the above sequence is minimal. We see that $x \in \mathrm{I}_1(M(x)) \subseteq \mathfrak{m}$ as M is nonfree, and that $\operatorname{pd}_R M(x) \geq \operatorname{pd}_R M$. The following pullback diagram gives an exact sequence as in the assertion.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega M & \xrightarrow{f} & F_0 & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle x} & & \\ 0 & \longrightarrow & \Omega M & \longrightarrow & M(x) & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

(2) The module $M(x)$ fits into the pushout diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega M & \xrightarrow{f} & F_0 & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle x} & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega M & \longrightarrow & M(x) & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

Using the same argument as in the proof of [139, Proposition 4.2], we observe that $\mathrm{V}(\mathfrak{p}) \subseteq \mathrm{NF}(M(x)) \subseteq \mathrm{NF}(M)$ and $\mathrm{D}(x) \cap \mathrm{NF}(M(x)) = \emptyset$ hold. ■

Lemma 6.7.3. *Let $(R, \mathfrak{m})$ be a local ring and M an R-module. Let $W \subseteq \mathrm{NF}(M)$ be a closed subset of $\operatorname{Spec} R$. Then there exists an R-module X such that $\operatorname{pd}_R X \geq \operatorname{pd}_R M$ and $\mathrm{NF}(X) = W$.*

Proof. The assertion follows from the proof of [139, Theorem 4.3] by replacing [139, Lemma 4.2] used there with our Lemma 6.7.2. ■

Lemma 6.7.4. *Let $(R, \mathfrak{m})$ be a local ring and M a nonfree R-module. Then there is an exact sequence $0 \to (\Omega M)^n \to N \to M^n \to 0$ with $n \geq 1$, $\mathrm{I}_1(N) = \mathfrak{m}$ and $\operatorname{pd}_R N \geq \operatorname{pd}_R M$. In particular, $N \in \operatorname{res}_R M$.*

Proof. Let $x_1, \dots, x_n$ be a minimal system of generators of $\mathfrak{m}$. According to Lemma 6.7.2, for each i there exists an exact sequence $0 \to \Omega M \to M(x_i) \to M \to 0$ such that $x_i \in \mathrm{I}_1(M(x_i)) \subseteq \mathfrak{m}$ and $\operatorname{pd}_R M(x_i) \geq \operatorname{pd}_R M$. Putting $N = \bigoplus_{i=1}^n M(x_i)$, we obtain an exact sequence $0 \to (\Omega M)^n \to N \to M^n \to 0$ with $\mathrm{I}_1(N) = \sum_{i=1}^n \mathrm{I}_1(M(x_i)) = \mathfrak{m}$ and $\operatorname{pd}_R N \geq \operatorname{pd}_R M$. ■

Lemma 6.7.5. *Let R be a local ring. Let M be an R-module that is locally free on the punctured spectrum of R.*

(1) *For every $X \in \operatorname{res}_{\widehat{R}} \widehat{M}$ there exists $Y \in \operatorname{res}_R M$ such that X is a direct summand of $\widehat{Y}$.*

(2) *Let N be an R-module. If $\widehat{N} \in \operatorname{res}_{\widehat{R}} \widehat{M}$, then $N \in \operatorname{res}_R M$.*

Proof. (1) Let $\mathcal{C}$ be the subcategory of $\operatorname{mod} \widehat{R}$ consisting of direct summands of the completions of modules in $\operatorname{res}_R M$. We claim that $\mathcal{C}$ is a resolving subcategory of $\operatorname{mod} \widehat{R}$ containing $\widehat{M}$. Indeed, since R, M are in $\operatorname{res}_R M$, the completions $\widehat{R}, \widehat{M}$ are in $\mathcal{C}$. For each $E \in \mathcal{C}$, there exists $D \in \operatorname{res}_R M$ such that E is a direct summand of $\widehat{D}$. The module $\Omega_{\widehat{R}} E$ is a direct summand of $\Omega_{\widehat{R}} \widehat{D} = \widehat{\Omega_R D}$. As $\Omega_R D \in \operatorname{res}_R M$, we have $\Omega_R E \in \mathcal{C}$. Let $0 \to A \to B \to C \to 0$ be an exact sequence of $\widehat{R}$-modules with $A, C \in \mathcal{C}$. Then A, C are direct summands of $\widehat{V}, \widehat{W}$ for some $V, W \in \operatorname{res}_R M$, respectively. Writing $A \oplus A' = \widehat{V}$ and $C \oplus C' = \widehat{W}$, we get an exact sequence $\sigma : 0 \to \widehat{V} \to B' \to \widehat{W} \to 0$, where $B' = A' \oplus B \oplus C'$. The exact sequence σ corresponds to an element of $\operatorname{Ext}^1_{\widehat{R}}(\widehat{W}, \widehat{V}) = \widehat{\operatorname{Ext}^1_R(W, V)}$. Since M is locally free on the punctured spectrum of R, so are V and W. Hence $\operatorname{Ext}^1_R(W, V)$ has finite length as an R-module, and is complete. This implies that there exists an exact sequence $\tau : 0 \to V \to U \to W \to 0$ of R-modules such that $\widehat{\tau} \cong \sigma$. Thereofore U is in $\operatorname{res}_R M$ and B' is isomorphic to $\widehat{U}$. Thus B belongs to $\mathcal{C}$, and the claim follows. The claim shows that $\mathcal{C}$ contains $\operatorname{res}_{\widehat{R}} \widehat{M}$. Hence X is in $\mathcal{C}$, which shows the assertion.

(2) By (1) there is an R-module $Y \in \operatorname{res}_R M$ such that $\widehat{N}$ is a direct summand of $\widehat{Y}$. Thanks to [108, Corollary 1.15(i)], the module N is a direct summand of Y. Hence N belongs to $\operatorname{res}_R M$. ■

Now we can show the proposition below, which yields a significant property of Burch rings. This is also used in the proofs of Theorem 6.7.7 and 6.7.10.

Proposition 6.7.6. *Let R be a Burch local ring of depth t with residue field k. Let M be an R-module of infinite projective dimension. Then $\Omega^t k$ belongs to $\operatorname{res}_R M$.*

Proof. We begin with proving the proposition when R is complete and $t = 0$. As M has infinite projective dimension, Lemma 6.7.4 gives rise to an R-module $N \in \operatorname{res}_R M$ with $\mathrm{I}_1(N) = \mathfrak{m}$. Proposition 6.4.2 implies that k is a direct summand of $\Omega^2_R N$. As $\Omega^2_R N$ is in $\operatorname{res}_R M$, so is k.

Now, let us consider the case where R is complete and $t > 0$. By definition, there is a maximal regular sequence $\boldsymbol{x}$ of R such that $R/(\boldsymbol{x})$ is a Burch ring of depth 0. Note that $\Omega^t M \in \operatorname{res}_R M$. For all $i > 0$ we have $\operatorname{Tor}_i^R(\Omega^t M, R/(\boldsymbol{x})) = \operatorname{Tor}_{i+t}^R(M, R/(\boldsymbol{x})) = 0$, which says that $\boldsymbol{x}$ is a regular sequence on $\Omega^t M$. The $R/(\boldsymbol{x})$-module $\Omega^t M/\boldsymbol{x}\Omega^t M$ has infinite projective dimension by [19, Lemma 1.3.5]. The case $t = 0$ implies that k belongs to $\operatorname{res}_{R/(\boldsymbol{x})} \Omega^t M/\boldsymbol{x}\Omega^t M$. It follows from [140, Lemma 5.8] that $\Omega^t_R k \in \operatorname{res}_R \Omega^t M \subseteq \operatorname{res}_R M$.

Finally, we consider the case where R is not complete. Lemma 6.7.3 gives an R-module $X \in \operatorname{res}_R M$ with $\operatorname{pd}_R X = \infty$ and $\operatorname{NF}(X) = \{\mathfrak{m}\}$. As $\widehat{R}$ is Burch and $\operatorname{pd}_{\widehat{R}} \widehat{X} = \operatorname{pd}_R X = \infty$, the above argument yields $\Omega^t_{\widehat{R}} k \in \operatorname{res}_{\widehat{R}} \widehat{X}$. Using Lemma 6.7.5, we see $\Omega^t k \in \operatorname{res}_R X$, and $\Omega^t k \in \operatorname{res}_R M$. ■

Non-Gorenstein Burch rings admit only trivial totally reflexive modules. Recall that a local ring R is called *G-regular* if every totally reflexive R-module is free.

Theorem 6.7.7. *Let R be a non-Gorenstein Burch local ring. Then R is G-regular.*

Proof. By taking the completion and using [138, Corollary 4.7], we may assume that R is complete. Let $\mathcal{G}$ be the category of totally reflexive R-modules. Then $\mathcal{G}$ is a resolving subcategory of $\operatorname{mod} R$ by [32, (1.1.10) and (1.1.11)]. If R is not G-regular, that is, there is a nonfree R-module M in $\mathcal{G}$, then Proposition 6.7.6 shows that $\mathcal{G}$ contains the R-module $\Omega^d k$, where $d = \dim R$. In other words, $\Omega^d k$ is totally reflexive. This especially says that the R-module k has finite G-dimension, and R is Gorenstein; see [32, (1.4.9)]. This contradiction shows that R is G-regular. ■

Remark 6.7.8. The converse of Theorem 6.7.7 does not necessarily hold. In fact, the non-trivial fibre product $R = S \times_k T$ of non-Burch local rings S, T is non-Burch. However, thanks to [116, Lemma 4.4], the same argument of the proof of Theorem 6.7.7 works, and hence R is G-regular.

As a corollary of Theorem 6.7.7, "embedded deformations" of Burch rings are never Burch.

Corollary 6.7.9. *Let $(R, \mathfrak{m})$ be a singular local ring. Let $x \in \mathfrak{m}^2$ be an R-regular element. Then the local ring $R/(x)$ is not Burch.*

Proof. The proof of [138, Proposition 4.6] gives rise to an endomorphism $\delta : R^n \to R^n$ such that $\delta^2 = x \cdot \mathrm{id}_{R^n}$ and $\operatorname{Im}\delta \subseteq \mathfrak{m}R^n$. It is easy to see that δ is injective, and we have an exact sequence $0 \to R^n \xrightarrow{\delta} R^n \to C \to 0$ with $xC = 0$. This induces an exact sequence $\cdots \xrightarrow{\overline{\delta}} (R/(x))^n \xrightarrow{\overline{\delta}} (R/(x))^n \xrightarrow{\overline{\delta}} (R/(x))^n \xrightarrow{\overline{\delta}} \cdots$ of $R/(x)$-modules whose $R/(x)$-dual is exact as well. Since $\operatorname{Im}\overline{\delta} = C$, the $R/(x)$-module C is totally reflexive. As $\operatorname{Im}\delta \subseteq \mathfrak{m}R^n$, we see that C is not $R/(x)$-free. Hence $R/(x)$ is not G-regular.

Suppose that $R/(x)$ is Burch. Then Theorem 6.7.7 implies that $R/(x)$ is Gorenstein. By Proposition 6.5.1, the ring $R/(x)$ is a hypersurface. We have

$$1 \geq \operatorname{codepth} R/(x) = \operatorname{edim} R/(x) - \operatorname{depth} R/(x) = \operatorname{edim} R - (\dim R - 1) = \operatorname{codim} R + 1,$$

where the second equality follows from the assumption that x is not in $\mathfrak{m}^2$. We get $\operatorname{codim} R = 0$, which means that R is regular, contrary to our assumption. ■

Let $(R, \mathfrak{m})$ be a local ring. We denote by $\operatorname{Spec}^0 R$ the punctured spectrum of R. For a property $\mathbb{P}$, we say that $\operatorname{Spec}^0 R$ *satisfies* $\mathbb{P}$ if $R_\mathfrak{p}$ satisfies $\mathbb{P}$ for all $\mathfrak{p} \in \operatorname{Spec}^0 R$. We denote by $\mathsf{CM}(R)$ the subcategory of $\mathsf{mod}\, R$ consisting of maximal Cohen–Macaulay modules. Also, $\mathsf{D^b}(R)$ stands for the bounded derived category of $\mathsf{mod}\, R$, and $\mathsf{D_{sg}}(R)$ the *singularity category* of R, that is, the Verdier quotient of $\mathsf{D^b}(R)$ by perfect complexes. Note that $\mathsf{D^b}(R)$ and $\mathsf{D_{sg}}(R)$ have the structure of a triangulated category. A *thick* subcategory of a triangulated category is by definition a triangulated subcategory closed under direct summands. The following theorem gives rise to classifications of several kinds of subcategories over Burch rings. For the unexplained notations and terminologies appearing in the theorem, we refer to [116, §2].

Theorem 6.7.10. *Let $(R, \mathfrak{m})$ be a singular Cohen–Macaulay Burch local ring.*

(1) *Suppose that $\operatorname{Spec}^0 R$ is either a hypersurface or has minimal multiplicity. Then there is a commutative diagram of mutually inverse bijections:*

$$\begin{array}{ccc}
\left\{\begin{array}{c}\textit{Resolving subcategories of}\\ \mathsf{mod}\,R \textit{ contained in } \mathsf{CM}(R)\end{array}\right\} & \underset{\mathrm{NF}_{\mathsf{CM}}^{-1}}{\overset{\mathrm{NF}}{\rightleftarrows}} & \left\{\begin{array}{c}\textit{Specialization-closed}\\ \textit{subsets of } \operatorname{Sing} R\end{array}\right\}\\
\Vert & & \mathrm{IPD}\ \uparrow\downarrow\ \mathrm{IPD}^{-1}\\
\left\{\begin{array}{c}\textit{Thick subcategories of}\\ \mathsf{CM}(R) \textit{ containing } R\end{array}\right\} & \underset{\mathrm{rest}_{\mathsf{CM}(R)}}{\overset{\mathrm{thick}_{\mathsf{mod}\,R}}{\rightleftarrows}} & \left\{\begin{array}{c}\textit{Thick subcategories of}\\ \mathsf{mod}\,R \textit{ containing } R\end{array}\right\}\\
\mathrm{rest}_{\mathsf{CM}(R)}\ \uparrow\downarrow\ \mathrm{thick}_{\mathsf{D_{sg}}(R)} & & \mathrm{rest}_{\mathsf{mod}\,R}\ \uparrow\downarrow\ \mathrm{thick}_{\mathsf{D^b}(R)}\\
\left\{\begin{array}{c}\textit{Thick subcategories of}\\ \mathsf{D_{sg}}(R)\end{array}\right\} & \underset{\pi}{\overset{\pi^{-1}}{\rightleftarrows}} & \left\{\begin{array}{c}\textit{Thick subcategories of}\\ \mathsf{D^b}(R) \textit{ containing } R\end{array}\right\}
\end{array}$$

(2) *Assume that R is excellent and admits a canonical module ω. Suppose that $\operatorname{Spec}^0 R$ has finite CM-representation type. Then there is a commutative diagram of mutually inverse bijections:*

$$
\begin{array}{ccc}
\left\{\begin{array}{c}\textit{Resolving subcategories}\\ \textit{of } \mathsf{mod}\, R \textit{ contained in}\\ \mathsf{CM}(R) \textit{ and containing } \omega\end{array}\right\} & \underset{\mathrm{NF}_{\mathsf{CM}}^{-1}}{\overset{\mathrm{NF}}{\rightleftarrows}} & \left\{\begin{array}{c}\textit{Specialization-closed}\\ \textit{subsets of } \operatorname{Sing} R\\ \textit{containing } \operatorname{NG} R\end{array}\right\}\\
\Big\| & & \mathrm{IPD} \big\uparrow\big\downarrow \mathrm{IPD}^{-1}\\
\left\{\begin{array}{c}\textit{Thick subcategories of}\\ \mathsf{CM}(R) \textit{ containing}\\ R \textit{ and } \omega\end{array}\right\} & \underset{\mathrm{rest}_{\mathsf{CM}(R)}}{\overset{\mathrm{thick}_{\mathsf{mod}\, R}}{\rightleftarrows}} & \left\{\begin{array}{c}\textit{Thick subcategories of}\\ \mathsf{mod}\, R \textit{ containing}\\ R \textit{ and } \omega\end{array}\right\}\\
\mathrm{rest}_{\mathsf{CM}(R)} \big\uparrow\big\downarrow \mathrm{thick}_{\mathsf{D_{sg}}(R)} & & \mathrm{rest}_{\mathsf{mod}\, R} \big\uparrow\big\downarrow \mathrm{thick}_{\mathsf{D^b}(R)}\\
\left\{\begin{array}{c}\textit{Thick subcategories of}\\ \mathsf{D_{sg}}(R) \textit{ containing } \omega\end{array}\right\} & \underset{\pi}{\overset{\pi^{-1}}{\rightleftarrows}} & \left\{\begin{array}{c}\textit{Thick subcategories of}\\ \mathsf{D^b}(R) \textit{ containing}\\ R \textit{ and } \omega\end{array}\right\}
\end{array}
$$

Proof. The proof of [116, Theorem 4.5] uses [116, Lemma 4.4]. Replace this lemma with our Proposition 6.7.6. Then the same argument works, and the theorem follows. ■

Example 6.7.11. We have the following list of examples of non-Gorenstein Cohen–Macaulay local rings not having isolated singularities, where ◦ and × mean "Yes" and "No" respectively.

[141, Example]	R	$\dim R$	Burch	$\operatorname{Spec}^0 R$		
				hypersurface	min. mult.	finite CM rep. type
7.1	$\frac{k[\![x,y,z]\!]}{(x^2,xz,yz)}$	1	◦	◦	◦	◦
7.2	$\frac{k[\![x,y,z]\!]}{(x^2,xy,y^2)}$	1	◦	×	◦	×
7.3	$\frac{k[\![x,y,z]\!]}{(xy,z^2,zw,w^3)}$	1	×	×	◦	×
7.4	$\frac{k[\![x,y,z]\!]}{(x^2-yz,xy,y^2)}$	1	◦	◦	×	◦
7.5	$\frac{k[\![x,y,z,w]\!]}{(xy,xz,yz)}$	2	◦	×	◦	◦

The assertions are shown in [141, Examples 7.1–7.5], except those on the Burch property. As to the first, second, fourth and fifth rings R are Burch since the quotient of a system of parameters is isomorphic to $k[x,y]/(x^2,xy,y^2)$, which is an artinian Burch ring by Example 6.6.16. As for the third ring R, note that (x,y) is an exact pair of zerodivisors. Hence it is not G-regular, and not Burch by Theorem 6.7.7.

Now we discuss the vanishing of Tor modules over Burch rings. The following result is a simple consequence of Lemmas 6.2.10 and 6.7.4.

Proposition 6.7.12. *Let $(R,\mathfrak{m},k)$ be a Burch ring of depth zero, and let M,N be R-modules. If $\operatorname{Tor}_l^R(M,N)=\operatorname{Tor}_{l+1}^R(M,N)=0$ for some $l\geq 3$, then either M or N is a free R-module.*

Proof. We may assume that R is complete. Assume that M is nonfree. Since $\operatorname{depth} R=0$, the R-module M has infinite projective dimension. By Lemma 6.7.4, there is a short exact sequence $0\to(\Omega M)^n\to X\to M^n\to 0$, where X satisfies $\mathrm{I}_1(X)=\mathfrak{m}$. It induces an exact sequence

$0 \to (\Omega^3 M)^n \to \Omega^2 X \oplus F \to (\Omega^2 M)^n \to 0$ with F free. We also have $\mathrm{Tor}_{l-2}(\Omega^2 M, N) = \mathrm{Tor}_{l-2}(\Omega^3 M, N) = 0$, which implies that $\mathrm{Tor}_{l-2}(\Omega^2 X, N) = 0$. Lemma 6.2.10 implies that k is a direct summand of $\Omega^2 X$, as R is Burch. We see that $\mathrm{Tor}_{l-2}(k, N)$ vanishes. This shows that N has finite projective dimension, and so it is R-free. ■

We can prove the following by applying a similar argument as in the proof of [116, Corollary 6.5], where we use Proposition 6.7.12 instead of [116, Corollary 6.2].

Corollary 6.7.13. *Let $(R, \mathfrak{m}, k)$ be a Burch ring of depth t. Let M, N be R-modules. Assume that there exists an integer $l \geq \max\{3, t+1\}$ such that $\mathrm{Tor}_i^R(M, N) = 0$ for all $l+t \leq i \leq l+2t+1$. Then either M or N has finite projective dimension.*

Remark 6.7.14. Using an analogous argument as in the proof of [116, Corollary 6.6], one can also prove a variant of Corollary 6.7.13 regarding Ext modules.

We state a remark on the ascent of Burchness along a flat local homomorphism.

Remark 6.7.15. Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism of local rings. Even if the rings R and $S/\mathfrak{m}S$ are Burch, S is not necessarily Burch. In fact, consider the natural injection

$$\phi : R = k[x, y]/(x^2, xy, y^2) \hookrightarrow k[x, y, t]/(x^2, xy, y^2, t^2) = S.$$

Then ϕ is a flat local homomorphism. The artinian local rings R and $S/\mathfrak{m}S = k[t]/(t^2)$ are Burch by Examples 6.6.16 and 6.2.2(1). The ring S is not G-regular since (t, t) is an exact pair of zerodivisors of S. Theorem 6.7.7 implies that S is not Burch.

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