Discrete hypergroups derived from graphs
(グラフから得られる離散超群)

Tomohiro Ikkai

Abstract

Wildberger gave a method to construct a finite hermitian hypergroup from a random walk on a certain kind of finite graphs. In this thesis, we formulate his method and reveal that his method is applicable to a random walk on a certain kind of infinite graphs. We also make some observations of finite or infinite graphs on which a random walk produces a hermitian discrete hypergroup. This thesis shows several kinds of examples of graphs on which a random walk produces a discrete hypergroup.

Contents

Acknowledgment. First, I would like to express the most gratitude to Prof. Kohji Matsumoto, the author's supervisor. Thanks to his support I could bring this thesis to completion and have a wonderful student life. I also express my appreciation to Prof. Yusuke Sawada and Mr. Ippei Mimura, who introduced the world of hypergroups to me and studied hypergroups together with me. My deep gratitude goes to Prof. Satoshi Kawakami, Prof. Tatsuya Tsurii and Prof. Satoe Yamanaka, who gave me many advices for my research. Finally, I give sincere thanks to my family for their long time support.

1 Introduction

1.1 Hypergroups

A hypergroup is a locally compact space on which the space of finite regular Borel measures has a convolution structure preserving the probability measures. It is said that the first appearance of a hypergroup was seen in Frobenius's work on group theory around 1900. Axiomatic investigations of hypergroups were begun in 1970s by Dunkl [5], Jewett [7] and Spector [12]. Subsequently many authors developed harmonic analysis and representation theory on hypergroups as well as those on locally compact groups.

Structures of hypergroups appear in spaces of double cosets, spaces of conjugacy classes arising from certain classes of locally compact groups, dual spaces of compact groups etc. In physics, interpreting elementary particles as irreducible representations of groups, irreducible decompositions of their tensor products correspond to convolutions in hypergroups. Thus hypergroups can be used to describe the phenomena in which another particle appears in a certain probability when particles collide. On the other hand, a hypergroup theoretic approach to the 2-dimensional Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u = 0$$

is introduced in Jewett's article [7].

To state the definition of hypergroups in a general form, we prepare a certain topology and some notations. Let X be a locally compact Hausdorff space.

- Let $\mathscr{C}(X)$ denote the space of nonempty compact subsets of X. For subsets A, B of X, we let

  $$\mathscr{C}_A(B) = \{C \in \mathscr{C}(X)\,;\, C \cap A \neq \varnothing, C \subset B\}\,.$$

  The topology on $\mathscr{C}(X)$ generated by the subbasis

  $$\{\mathscr{C}_U(V)\,;\, U \text{ and } V \text{ are open subsets of } X\}$$

  is called *Michael topology.*

- Let $C_c(X)$ be the algebra of the continuous functions $f : X \to \mathbb{C}$ with compact support. A (complex) *Radon measure* on X is a continuous linear functional $\mu : C_c(X) \to \mathbb{C}$. The $\mathbb{C}$-vector space consisting of the Radon measures on X is denoted by $M(X)$.

- We define the norm $\|\cdot\|$ on $M(X)$ by

  $$\|\mu\| = \sup\{|\mu(f)|\,;\, f \in C_c(X),\, \|f\|_\infty \leq 1\}\,,$$

where $\|f\|_\infty = \sup_{x\in X} |f(x)|$. This norm introduces a distance and a topology to $M(X)$. Let

$$M^b(X) = \{\mu \in M(X) \,;\, \|\mu\| < \infty\}\,,$$
$$M^1(X) = \{\mu \in M(X) \,;\, \mu \geq 0,\ \|\mu\| = 1\}\,.$$

We call an element in $M^1(X)$ a *probability measure*.

Definition 1.1. Let K be a nonempty locally compact Hausdorff space. We call K a *hypergroup* if the following seven conditions are fulfilled.

(i) A binary operation

$$\circ : M^b(K) \times M^b(K) \ni (\mu, \nu) \mapsto \mu \circ \nu \in M^b(K)$$

called a *convolution* is defined and $(M^b(K), \circ)$ forms an algebra. (Note that a canonical vector space structure is introduced to $M^b(K)$.)

(ii) For x, $y \in K$, $\varepsilon_x \circ \varepsilon_y \in M^1(K)$ and $\mathrm{supp}\,(\varepsilon_x \circ \varepsilon_y)$ is compact, where ε_x denotes the point measure supported at $\{x\}$.

(iii) The mapping

$$K \times K \ni (x, y) \mapsto \varepsilon_x \circ \varepsilon_y \in M^1(K)$$

is continuous.

(iv) The mapping

$$K \times K \ni (x, y) \mapsto \mathrm{supp}\,(\varepsilon_x \circ \varepsilon_y) \in \mathscr{C}(K)$$

is continuous.

(v) There exists an element $e \in K$ such that

$$\varepsilon_e \circ \varepsilon_x = \varepsilon_x \circ \varepsilon_e = \varepsilon_x$$

for all $x \in K$.

(vi) An *involution* map

$$* : K \ni x \mapsto x^* \in K$$

is defined and satisfies the following conditions (via) and (vib):

(via) The involution map $*$ is a homeomorphism of K onto itself and satisfies $x^{**} = x$ for each $x \in K$.

(vib) For x, $y \in K$,

$$(\varepsilon_x \circ \varepsilon_y)^* = \varepsilon_{y^*} \circ \varepsilon_{x^*},$$

where μ^* denotes the image of μ under the involution.

(vii) For x, $y \in K$, $e \in \mathrm{supp}\,(\varepsilon_x \circ \varepsilon_y)$ if and only if $y = x^*$.

1.2 Discrete hypergroups

Discrete hypergroups, including finite hypergroups, are the main objects in this thesis. We can state the definition of discrete hypergroups without any terms of measure theory and it will be stated in Section 2.1.

Recently, it was shown by Wildberger [15] that finite commutative hypergroups have some connections with number theory, conformal field theory, subfactor theory and other fields.

One of the problems on discrete hypergroups is determining the structures of them. Structures of finite hypergroups of order two and three are completely determined. Indeed, those of order two are classically known, and those of order three were investigated by Wildberger [16]. However, few examples of finite hypergroups of order four or greater were known; It is known that there exist non-commutative hypergroups of order four, and a recent result by Matsuzawa, Ohno, Suzuki, Tsurii and Yamanaka [11] gives some examples of non-commutative hypergroups of order five.

A significant problem to study the structures of hypergroups is the extension problem. The extension problem is to find hypergroups K which make the sequence of hypergroup homomorphisms

$$1 \to H \to K \to L \to 1$$

exact when H and L are given. A significant role is played by the "signed action" to solve the extension problem.

1.3 Motivation and main results

Wildberger [14], [15] gave a way to construct a finite commutative hypergroup from a random walk on a certain kind of finite graphs. By his method, we have some examples of hypergroups of large order, but they are all contained in a category of "polynomial hypergroups."

It is known that not all finite graphs produce hypergroups. Wildberger mentioned that a random walk on any strong regular graph and on any distance transitive graph produces a hypergroup and suggested that a random walk on any distance-regular graphs produces a finite hypergroup. (We do not discuss on distance transitive graphs in this thesis, but it is known that distance transitive graphs are always distance-regular.) It is a natural problem that on what kind of graphs a random walk produces a (finite) hypergroup. First the authors verify that a random walk on a distance-regular graph certainly produces a finite hypergroup.

Distance-regular graphs can be defined by using terms of association schemes. Wildberger [15] gave a way to construct a finite hypergroup from the Bose-Mesner algebra of an association scheme without using any graph theoretical terms, so that we have two ways to construct hypergroups from a distance-regular graph. The authors also verify that these two ways produce the same hypergroup, which is proved in Section 3.1.

The authors discovered that Wildberger's method also works for some infinite graphs, and a random walk on such a graph produces a countably infinite but discrete hypergroup. This thesis is the first attempt to construct infinite hypergroups from random walks on graphs. In Section 3.2 and Section 4.2, we give several examples of infinite regular graphs on which a random walk produces a discrete hypergroup. The following theorem is one of the main results and gives examples of infinite regular graphs on which a random walk produces a discrete hypergroup.

Theorem 1.2. (a) For any $k \in \mathbb{N}$ with $k \geq 2$, a random walk on the infinite k-regular tree produces a discrete hypergroup.

(b) A random walk on the Cayley graph $\mathrm{Cay}\left(\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}), \{(\pm 1, \overline{0}), (0, \overline{1})\}\right)$ produces a discrete hypergroup. (The symbols $\overline{0}$ and $\overline{1}$ denote the residue classes of 0 and 1 modulo 2, respectively.)

The definitions of the infinite k-regular tree and Cayley graphs will be given in Section 3.2.

In addition, another infinite graph named "linked-triangle graph" is discovered to be an example on which a random walk produces a discrete hypergroup.

It is a graph theoretical problem to decide whether a random walk on a given graph produces a discrete hypergroup. The purpose of this thesis is to formulate Wildberger's construction as a graph theoretical problem in the above sense and to establish a fundamental theory on that problem.

1.4 Contents of the thesis

This thesis is based on the author's article [6].

Contents of this thesis are as follows: Section 2 of this thesis is devoted to definitions, notations and basic facts on discrete hypergroups, graphs and association schemes. Definition 2.1 and Example 2.3 are added to the contents of [6]. Contents of Section 2 are all classically known but some of them are slightly modified to make them suitable for this thesis.

In Section 3, we recall Wildberger's construction of a discrete hypergroup from a random walk on a graph in an extended form including the case when the graph is infinite. Section 3.1 gathers some fundamental propositions to formulate our problems and provides a proof that a random walk on a distance-regular graph always produces a discrete hypergroup. Section 3.2 furnishes two kinds of examples of finite distance-regular graphs (complete graphs and platonic solids) and two kinds of examples of infinite ones (infinite regular trees and the "linked-triangle graph"). The distance-regularity of finite graphs is a classical concept, and it is already known that random walks on complete graphs and platonic solids produce discrete hypergroups. However, the distance-regularity of infinite graphs has not been discussed even in graph theory. The authors of [6] proved that the two kinds of infinite graphs are certainly distance-regular and the claim (a) of Theorem 1.2, and proofs of them will be given in that section. In particular, the "linked-triangle graph," constructed in that section, was an unexpected example on which a random walk produce a discrete hypergroup. The author found an error in the proof of Lemma 3.11 and [6] and revised it into Lemma 3.11 in this thesis.

We will consider non-distance-regular graphs in Section 4. In Section 4.1, further observations than those in Section 3.1 will be made. One can meet some examples of non-distance-regular graphs (prisms, complete bipartite graphs, the infinite ladder graph and some intriguing finite graphs) in Section 4.2. These are new examples on which random walks produce discrete hypergroups. We will mention the essence of a proof of Theorem 1.2 (b) in the same section.

Contents of Section 3 and 4 are quoted from the author's article [6] and revised.

2 Preliminaries

Let $\mathbb{N}$ be the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ be the ring of rational integers, $\mathbb{R}$ be the field of real numbers, $\mathbb{R}^\times$ be the multiplicative group of non-zero real numbers, $\mathbb{R}_+$ be the set of non-negative real numbers and $\mathbb{C}$ be the field of complex numbers, respectively.

As preliminaries to arguments in the following sections, we will give some definitions, notations and fundamental propositions from hypergroup theory and graph theory in this section.

First, we provide some notations used in the following arguments.

- For a set X, we let $|X|$ denote the number of elements in X. For an infinite set X, we just say $|X| = \infty$ and do not consider its cardinality.
- We are to use the convention that $\infty > n$ for any $n \in \mathbb{N}_0$.
- Let X be an arbitrary nonempty set and $\mathcal{A}$ be a commutative ring. A map $\varphi : X \times X \to \mathcal{A}$ is identified with an $\mathcal{A}$-valued $X \times X$ matrix $A_\varphi = (\varphi(x, y))_{x,y \in X}$. The set of $\mathcal{A}$-valued $X \times X$ matrices is denoted by $\mathrm{Mat}_X(\mathcal{A})$.

  The sum and the product of two matrices $A = (A_{x,y})_{x,y\in X}$, $B = (B_{x,y})_{x,y\in X} \in \mathrm{Mat}_X(\mathcal{A})$ is defined in a usual way like

  $$A + B = (A_{x,y} + B_{x,y})_{x,y\in X} .$$

  On the other hand, note that the product of two matrices $A = (A_{x,y})_{x,y\in X}$, $B = (B_{x,y})_{x,y\in X} \in \mathrm{Mat}_X(\mathcal{A})$ can be defined in a usual way like

  $$AB = \left(\sum_{z\in X} A_{x,z} B_{z,y} \right)_{x,y\in X} .$$

  only if all of the entries of AB take finite values. In particular, when A has finitely many non-zero entries in each rows and each columns, we can define the n-th power A^n of A with $n \in \mathbb{N}_0$. (As usual, A^0 is defined as the identity matrix $E = (\delta_{x,y})_{x,y\in X}$, where $\delta_{x,y}$ is the Kronecker delta.)

2.1 Discrete Hypergroups

We treat only "discrete hypergroups" in this thesis. In general, hypergroups are equipped with some topologies and require several complicated conditions. Such conditions for hypergroups with the discrete topologies can be simplified. For a general definition, see Definition 1.1.

First, we give the definition of an involution map on a $\mathbb{C}$-algebra.

Definition 2.1. Let $\mathcal{A}$ be a $\mathbb{C}$-algebra with the unit $1_{\mathcal{A}}$ and $* : \mathcal{A} \ni x \mapsto x^* \in \mathcal{A}$ be a map. We say $*$ to be an *involution map* if the following four conditions are all satisfied.

(i) For any x, $y \in \mathcal{A}$, $(x+y)^* = x^* + y^*$.

(ii) For any x, $y \in \mathcal{A}$, $(xy)^* = y^* x^*$.

(iii) For any $x \in \mathcal{A}$ and $\alpha \in \mathbb{C}$, $(\alpha x)^* = \overline{\alpha} x^*$.

(iv) For any $x \in \mathcal{A}$, $x^{**} = x$.

It is easily seen from uniqueness of the unit that $1_{\mathcal{A}}^* = 1_{\mathcal{A}}$.

Let K be an arbitrary set and consider the free vector space

$$\mathbb{C}K = \bigoplus_{x \in K} \mathbb{C}x = \left\{ \sum_{i=1}^{n} a_i x_i \,;\, n \in \mathbb{N}, a_1, \cdots, a_n \in \mathbb{C}, x_1, \cdots, x_n \in K \right\},$$

generated by K. We note that the ℓ^1-norm $\|\cdot\|_1 : \mathbb{C}K \to \mathbb{R}_+$ such that

$$\|\mu\|_1 = \sum_{x \in K} |\mu(x)| \ \ (\mu \in \mathbb{C}K),$$

where $\mu(x)$ denotes the coefficient of x as μ is expressed by a linear combination of elements of K, induces a normed space structure into $\mathbb{C}K$. For $\mu \in \mathbb{C}K$, we define the *support* of μ, denoted by $\mathrm{supp}\,(\mu)$, as

$$\mathrm{supp}\,(\mu) = \{x \in K \,;\, \mu(x) \neq 0\}.$$

Here is the definition of discrete hypergroups.

Definition 2.2. We call a nonempty set K a *discrete hypergroup* if the following four conditions are fulfilled.

(i) A binary operation

$$\circ : \mathbb{C}K \times \mathbb{C}K \ni (\mu, \nu) \mapsto \mu \circ \nu \in \mathbb{C}K,$$

called a *convolution*, and a map

$$* : \mathbb{C}K \ni \mu \mapsto \mu^* \in \mathbb{C}K$$

are defined, and $(\mathbb{C}K, \circ, *)$ forms an associative $*$-algebra (i.e. $\mathbb{C}$-algebra equipped with an involution map) with the neutral element $e \in K$.

(ii) The convolution $x \circ y$ of two elements x, $y \in K$ satisfies the following conditions (iia) and (iib):

(iia) For all x, y, $z \in K$, $(x \circ y)(z) \in \mathbb{R}_+$.

(iib) For all x, $y \in K$, $\|x \circ y\|_1 = 1$.

(iii) The involution $*$ maps K onto K itself.

(iv) For x, $y \in K$, the neutral element e belongs to $\operatorname{supp}(x \circ y)$ if and only if $y = x^*$.

A discrete hypergroup K is said to be *finite* if K is a finite set, to be *commutative* if $(\mathbb{C}K, \circ)$ is a commutative algebra and to be *hermitian* if the restriction of the involution $*|_K$ is the identity map on K.

It can be easily verified that a hermitian discrete hypergroup must be commutative.

The structure of a discrete hypergroup K can be determined by results of the operations $\circ$ and $*$ for elements of K. In particular, under the assumption that K is a hermitian discrete hypergroup, we can compute a convolution of arbitrary two elements in $\mathbb{C}K$ if we know the results of computations of $x \circ y$ for all x, $y \in K$ since the convolution $\circ$ is bilinear. In other words, if the convolution of two elements x, $y \in K$ is expressed as

$$x \circ y = \sum_{z \in K} P^z_{x,y} z \tag{2.1}$$

with some $P^z_{x,y} \in \mathbb{R}_+$, we can express the convolution of any two elements of $\mathbb{C}K$ as a linear combination of elements of K by using the coefficients $P^z_{x,y}$. In this thesis, the identities such as (2.1), defining convolutions of two elements of K, are called the *structure identities* of K.

Example 2.3. (i) A discrete group G can be regarded as a discrete hypergroup. More precisely, defining a convolution on $\mathbb{C}G$ as the bilinear extension of the multiplication $G \times G \ni (g, h) \mapsto gh \in G$ and the involution on $\mathbb{C}G$ as the conjugate-linear extension of the inversion $G \ni g \mapsto g^{-1} \in G$, one can check that G satisfies the conditions (i)-(iv) in Definition 2.2. These operations are the same as those of the group algebra $\mathbb{C}[G]$. This is the reason why the concept of discrete hypergroups is a generalization of the concept of discrete groups.

(ii) This example is quoted from Wildberger's article [15].

Let G be a finite group with conjugacy classes $C_0 = \{1_G\}$, C_1, $\cdots$, C_n and consider formal sums $X_i = \sum_{g \in C_i} g \in \mathbb{C}G$ for $i = 0, 1, \cdots, n$. There exist non-negative integers $N^k_{i,j}$ for any i, j, $k \in \{0, 1, \cdots, n\}$ such that

$$X_i X_j = \sum_{k=0}^{n} N^k_{i,j} X_k.$$

We let $x_i = |C_i|^{-1} X_i$ and $x_i^* = |C_i|^{-1} \sum_{g \in C_i} g^{-1}$ for $i = 0, 1, \cdots, n$. Then $\{x_0, x_1, \cdots, x_n\}$ becomes a discrete hypergroup with the structure identities

$$x_i \circ x_j := x_i x_j = \sum_{k=0}^{n} \frac{N_{i,j}^k |C_k|}{|C_i| |C_j|} x_k.$$

This is called the *class hypergroup* of G.

(iii) This example is quoted from Lasser's article [9].

Given sequences $\{a_n\}_{n \in \mathbb{N}_0} \subset \mathbb{R}^\times$, $\{b_n\}_{n \in \mathbb{N}_0} \subset \mathbb{R}$ and $\{c_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^\times$ of real numbers satisfying

$$\begin{aligned} a_0 + b_0 &= 1, \\ a_n + b_n + c_n &= 1 \quad (n \geq 1), \end{aligned}$$

consider a sequence $\{R_n\}_{n \in \mathbb{N}_0}$ of one variable polynomials given by

$$\begin{aligned} R_0(x) &= 1, \\ R_1(x) &= \frac{1}{a_0}(x - b_0), \\ R_1(x) R_n(x) &= a_n R_{n+1}(x) + b_n R_n(x) + c_n R_{n-1}(x) \quad (n \geq 1). \end{aligned}$$

Then we have

$$R_m(x) R_n(x) = \sum_{k=|m-n|}^{m+n} g(m, n; k) R_k(x)$$

for all $m, n \in \mathbb{N}_0$ where $g(m, n; k)$'s are suitable real numbers with $\sum_{k=|m-n|}^{m+n} g(m, n; k) = 1$. If all of $g(m, n; k)$'s are non-negative, $\{R_n\}_{n \in \mathbb{N}_0}$ becomes a hermitian discrete hypergroup under the convolution $R_m \circ R_n := R_m R_n$. This hypergroup is called a *polynomial hypergroup* on $\mathbb{N}_0$.

2.2 Distance-regular Graphs

In our context, all graphs are supposed to be undirected and to have neither any loops nor any multiple edges (i.e. to be simple graphs). We consider only *connected graphs*, in which, for any two vertices v, w, there exists a path from v to w, but we will handle both finite graphs and infinite graphs.

Let $X = (V, E)$ be a graph with the vertex set V and the edge set E. For two vertices v, $w \in V$, the *distance* $d(v, w)$ between v and w is defined as the length of the shortest paths from v to w. Note that the distance function $d : V \times V \to \mathbb{N}_0$ allows V to be a metric space, that is, the function $d(\cdot,\cdot)$ satisfies the following three properties:

$$d(v,w) = 0 \Longleftrightarrow v = w \ (v, w \in V), \tag{2.2}$$

$$d(v,w) = d(w,v) \ (v, w \in V), \tag{2.3}$$

$$d(v,w) \leq d(v,u) + d(u,w) \ (u, v, w \in V). \tag{2.4}$$

It is clear that two vertices v and w are adjacent if and only if $d(v,w) = 1$. A path in X is called a *geodesic* if its length equals to the distance between the initial vertex and the terminal vertex. The *eccentricity* $e(v) \in \mathbb{N}_0 \cup \{\infty\}$ of $v \in V$ is defined as

$$e(v) = \sup\{d(v,w)\ ;\ w \in V\}. \tag{2.5}$$

If the set $\{d(v,w)\ ;\ w \in V\}$ is unbounded above, then we define $e(v) = \infty$. This happens only if X is infinite. The *radius* $\mathrm{rad}(X) \in \mathbb{N}_0 \cup \{\infty\}$ and the *diameter* $\mathrm{diam}(X) \in \mathbb{N}_0 \cup \{\infty\}$ of X are defined as

$$\mathrm{rad}(X) = \inf\{e(v)\,;\ v \in V\},$$
$$\mathrm{diam}(X) = \sup\{e(v)\,;\ v \in V\}.$$

Clearly, that $\mathrm{rad}(X) = \infty$ implies that $\mathrm{diam}(X) = \infty$. A *complete graph*, in which distinct vertices are mutually adjacent, is of diameter one, and a non-complete graph is of diameter two or greater. We say X to be *self-centered* if X is finite and every $v \in V$ satisfies that $e(v) = \mathrm{rad}(X)$. For each $i \in \mathbb{N}_0$ and $v \in V$, let $\Gamma_i(v)$ denote the set

$$\Gamma_i(v) = \{w \in V\,;\ d(v,w) = i\}.$$

It immediately follows that $\Gamma_i(v) = \varnothing$ if $i > \mathrm{diam}(X)$. The number of elements in $\Gamma_1(v)$ is called the *degree* of v. The graph X is said to be k-*regular* for some $k \in \mathbb{N}$ if every vertex $v \in V$ is of degree k. We are going to deal with graphs of which every vertex has a finite degree in this thesis.

An *automorphism* of a graph $X = (V, E)$ is a bijection $\varphi : V \to V$ such that, for any two vertices v, $w \in V$, $\varphi(v)$ and $\varphi(w)$ are adjacent in X if and only if v and w are adjacent in X. A graph X is said to be *vertex-transitive* if, for any two vertices v, $w \in V$, there exists an automorphism φ of X such that $\varphi(v) = w$.

The "distance-regular graphs" play a key role in the following part. Here we recall the definition of such graphs, and some examples will appear in Section 3.2.

Definition 2.4. A (finite or infinite) connected graph $X = (V, E)$ is said to be *distance-regular* if the following three conditions are fulfilled.

(i) There exists $b_0 \in \mathbb{N}$ such that $|\Gamma_1(v)| = b_0$ for any $v \in V$.

(ii) For each integer i with $1 \le i < \operatorname{diam}(X)$, there exist $b_i \in \mathbb{N}$ and $c_i \in \mathbb{N}$ such that $|\Gamma_{i+1}(v) \cap \Gamma_1(w)| = b_i$ and $|\Gamma_{i-1}(v) \cap \Gamma_1(w)| = c_i$ for any v, $w \in V$ with $d\,(v, w) = i$.

(iii) If $s = \operatorname{diam}(X) < \infty$, there exists $c_s \in \mathbb{N}$ such that $|\Gamma_{s-1}(v) \cap \Gamma_1(w)| = c_s$ for any v, $w \in V$ with $d\,(v, w) = s$.

For a distance-regular graph X, the sequence $(b_0, b_1, b_2, \cdots ; c_1, c_2, c_3, \cdots)$ of integers is called the *intersection array* of X.

Note that a distance-regular graph is b_0-regular.

2.3 Association Schemes

Distance-regular graphs are important in combinatorics: Its vertex set has a structure of an "association scheme." The concept of association schemes is one of the most important concepts in algebraic combinatorics. They are useful for the unification of graph theory, coding theory and the theory of combinatorial designs.

Definition 2.5. Let Y be a set and $\mathcal{R} = \{R_i\}_{i \in I}$ be a partition of $Y \times Y$ (i.e. R_i's are mutually disjoint subsets of $Y \times Y$ and $Y \times Y = \bigcup_{i \in I} R_i$), which consists of countably many non-empty sets. The pair $(Y, \mathcal{R})$ is called a (*symmetric*) *association scheme* if the following conditions are all satisfied.

(i) The diagonal set $\{(y, y)\,;\, y \in Y\}$ of $Y \times Y$ is an entry of $\mathcal{R}$.

(ii) If $(x, y) \in R_i$, then $(y, x) \in R_i$.

(iii) For any i, j, $k \in I$, there exists $p^k_{i,j} \in \mathbb{N}_0$ such that

$$|\{z \in Y\,;\, (x, z) \in R_i, (z, y) \in R_j\}| = p^k_{i,j}$$

for each pair $(x, y) \in R_k$.

The numbers $p^k_{i,j}$ in the condition (iii) are called the *intersection numbers* of $(Y, \mathcal{R})$.

Given a connected graph $X = (V, E)$, we can find a canonical partition of $V \times V$; Let

$$I = \begin{cases} \{0, 1, \cdots, \operatorname{diam}(X)\} & (\operatorname{diam}(X) \in \mathbb{N}), \\ \mathbb{N}_0 & (\operatorname{diam}(X) = \infty), \end{cases} \tag{2.6}$$

$$R_i = \{(v, w) \in V \times V\,;\, d\,(v, w) = i\} \quad (i \in I) \tag{2.7}$$

and $\mathcal{R}(X) = \{R_i\}_{i \in I}$. If every vertex of X has a finite degree, then R_i's are all non-empty sets and $\mathcal{R}(X)$ forms a partition of $V \times V$ (see Proposition 3.1). Furthermore, we find that, by (2.2),

$$R_0 = \{(v, v)\,;\, v \in V\}$$

and that, by (2.3),

$$(v, w) \in R_i \Longrightarrow (w, v) \in R_i$$

for any $i \in I$.

We prove that distance-regular graphs can be characterized by the condition (iii) in Definition 2.5. This is a classical result in the finite graph case, so that a proof of the following proposition for the case when X is a finite graph is introduced in [1]. Even if X is infinite, the same method as that on [1] works.

Proposition 2.6. Let $X = (V, E)$ be a connected graph and $s = \operatorname{diam}(X) \in \mathbb{N}_0 \cup \{\infty\}$. Then the following statements are equivalent.

(a) The graph X is distance-regular.

(b) The pair $(V, \mathcal{R}(X))$ forms an association scheme.

Proof. First, suppose that X satisfies the condition (b). Then, for two vertices v, $w \in V$, that $d\,(v, w) = k$ is equivalent to that $(v, w) \in R_k$, and we have

$$\Gamma_j(v) \cap \Gamma_1(w) = \{u \in V\,;\, (v, u) \in R_j, (u, w) \in R_1\}\,.$$

Hence we can find that X is a distance-regular graph with the intersection array $(p^0_{1,1}, p^1_{2,1}, \cdots, p^{i-1}_{i,1}, \cdots; p^1_{0,1}, p^2_{1,1}, \cdots, p^i_{i-1,1}, \cdots)$ (the i-th entry of the former array is $p^{i-1}_{i,1}$, and that of the latter array is $p^i_{i-1,1}$).

Conversely, suppose that X is a distance-regular graph with the intersection array $(b_0, b_1, b_2, \cdots; c_1, c_2, c_3, \cdots)$. Since the conditions (i) and (ii) in Definition 2.5 hold, we now check that the condition (iii) holds.

Consider the family of matrices $\{A^{(i)}\}_{i\in I} \subset \mathrm{Mat}_V(\mathbb{Z})$ whose entries $A^{(i)}_{v,w}$ are defined as

$$A^{(i)}_{v,w} = \begin{cases} 1 & ((v, w) \in R_i), \\ 0 & ((v, w) \notin R_i). \end{cases} \tag{2.8}$$

Then, we find that

$$A^{(i)}A^{(1)} = b_{i-1}A^{(i-1)} + (b_0 - b_i - c_i)A^{(i)} + c_{i+1}A^{(i+1)} \tag{2.9}$$

for any $i \in I$ since

$$\begin{aligned} \left(A^{(i)}A^{(1)}\right)_{v,w} &= |\{u \in V\,;\, (v, u) \in R_i, (u, w) \in R_1\}| \\ &= |\Gamma_i(v) \cap \Gamma_1(w)| \\ &= \begin{cases} b_{i-1} & ((v, w) \in R_{i-1}), \\ b_0 - b_i - c_i & ((v, w) \in R_i), \\ c_{i+1} & ((v, w) \in R_{i+1}), \\ 0 & (\text{otherwise}). \end{cases} \end{aligned}$$

Here, we use the convention that $b_{-1} = b_s = 0$ and $c_0 = c_{s+1} = 0$. The equation (2.9) implies that $(A^{(1)})^n$ can be written as a linear combination of $A^{(k)}$'s for any $n \in \mathbb{N}$.

On the other hand, from (2.9), we can verify by induction on i that each $A^{(i)}$ can be expressed as a polynomial of $A^{(1)}$ with rational number coefficients. Therefore, it turns out that the product $A^{(i)}A^{(j)}$ of two matrices can be written as a linear combination of $A^{(k)}$'s. Moreover, the coefficients in the linear combination expressing $A^{(i)}A^{(j)}$ must be all non-negative integers.

Let $p^k_{i,j} \in \mathbb{N}_0$ denote the coefficient of $A^{(k)}$ in the linear combination expressing $A^{(i)}A^{(j)}$. (Note that $A^{(k)}$'s are linearly independent over $\mathbb{C}$.) Then, for any $(v, w) \in R_k$, we obtain that

$$\begin{aligned} &|\{u \in V\,;\,(v,u) \in R_i, (u,w) \in R_j\}| \\ =&\left(A^{(i)}A^{(j)}\right)_{v,w} \\ =&\sum_{l\in I} p^l_{i,j} A^{(l)}_{v,w} \\ =&p^k_{i,j} \end{aligned} \tag{2.10}$$

since

$$A^{(l)}_{v,w} = \begin{cases} 1 & (l = k), \\ 0 & (l \neq k). \end{cases}$$

The calculations (2.10) warrant that $(V, \mathcal{R}(X))$ satisfies the condition (iii) in Definition 2.5 and that $(V, \mathcal{R}(X))$ forms an association scheme. □

In general, it is difficult to express $p^k_{i,j}$'s in terms of entries b_i's and c_i's of the intersection array.

Here are several useful formulas on intersection numbers $p^k_{i,j}$ of an association scheme. Those are known as classical results, and one can find the same formulas on some textbooks, [4] etc. Nevertheless, the author could not find any proofs of the following identities except for (e), which is written in [4]. The proofs of the following identities consist of only elementary calculus in combinatorics.

Proposition 2.7. Let $(Y, \mathcal{R} = \{R_i\}_{i\in I})$ be an association scheme and $R_0 \in \mathcal{R}$ be the diagonal set of $Y \times Y$. Then, for any $i, j, k, m \in I$, the following identities hold.

(a) $p^k_{0,j} = \delta_{j,k}$.

(b) $p^0_{i,j} = \delta_{i,j} p^0_{j,j}$.

(c) $p^k_{i,j} = p^k_{j,i}$.

(d) $\sum_{j\in I} p^k_{i,j} = p^0_{i,i}$.

(e) $\sum_{l\in I} p^l_{i,j} p^m_{l,k} = \sum_{l\in I} p^l_{j,k} p^m_{i,l}$.

(f) $p^k_{i,j} p^0_{k,k} = p^j_{i,k} p^0_{j,j}$.

Proof. For the identity (a), take $(x, y) \in R_k$. Then, we have

$$p^k_{0,j} = |\{z \in Y\,;\, (x, z) \in R_0, (z, y) \in R_j\}|\,.$$

The set of which we are counting elements is empty when $j \neq k$ and equals to $\{x\}$ when $j = k$, so that we obtain (a).

For the identity (b), take $(x, x) \in R_0$. (The set R_0 is the diagonal set of $Y \times Y$.) Then, we have

$$p^0_{i,j} = |\{z \in Y\,;\, (x, z) \in R_i, (z, x) \in R_j\}|\,.$$

The desired identity is derived from the condition (ii) in Definition 2.5.

The identity (c) is also derived from the condition (ii) in Definition 2.5.

For the identity (d), take $(x, y) \in R_k$. Since $\mathcal{R}$ is a partition of $Y \times Y$, we have

$$\begin{aligned}
&\sum_{j\in I} p^k_{i,j}\\
=&\sum_{j\in I} |\{z \in Y\,;\, (x, z) \in R_i, (z, y) \in R_j\}|\\
=&\,|\{z \in Y\,;\, (x, z) \in R_i\}|\\
=&p^0_{i,i}.
\end{aligned}$$

To verify the identity (e), take $(x, y) \in R_m$ and count elements in the set

$$S = \{(z, u) \in Y \times Y\,;\, (x, z) \in R_i, (z, u) \in R_j, (u, y) \in R_k\}\,.$$

One can see that

$$p^l_{i,j} p^m_{l,k} = |\{(z, u) \in Y \times Y\,;\, (x, u) \in R_l, (u, y) \in R_k, (x, z) \in R_i, (z, u) \in R_j\}|\,,$$

and it follows that $\sum_{l\in I} p^l_{i,j} p^m_{l,k} = |S|$. On the other hand, one finds that

$$p^l_{j,k} p^m_{i,l} = |\{(z, u) \in Y \times Y\,;\, (x, z) \in R_i, (z, y) \in R_l, (z, u) \in R_j, (u, y) \in R_k\}|$$

holds, and it follows that $\sum_{l\in I} p^l_{j,k} p^m_{i,l} = |S|$. Combining these results, we obtain the desired identity.

We can substitute $m = 0$ into the identity (e) and use (b) and (c) to get the identity (f). □

Finally, we recall the definition of "strongly regular graphs," a special class of distance-regular graphs.

Definition 2.8. A *strongly regular graph* is a distance-regular graph with diameter two.

Strongly regular graphs are represented by four parameters (n, k, λ, μ), where $n = |V|$, $k = b_0 = p^0_{1,1}$, $\lambda = p^1_{1,1}$ and $\mu = c_2 = p^2_{1,1}$. Hence the strongly regular graph with parameters (n, k, λ, μ) is called "(n, k, λ, μ)-strongly regular graph." (In this thesis, we regard complete graphs as not strongly regular since the parameter μ is not well-defined for complete graphs.)

3 Hermitian discrete hypergroups derived from distance-regular graphs

3.1 Wildberger's construction

Wildberger [15] gave a way to construct finite hermitian discrete hypergroups from a certain class of finite graphs. Coefficients appearing in the structure identities of such a discrete hypergroup are given by certain probabilities coming from a random walk on a corresponding graph. Actually, his method is applicable to some infinite graphs, so that we formulate his method in a general form. Here, we recall Wildberger's construction of a hermitian discrete hypergroup from a random walk on a graph.

Given a finite or infinite connected graph $X = (V, E)$, all of whose vertices have finite degree, and given a vertex $v_0 \in V$ as a "base point," we consider the canonical partition $\mathcal{R}(X) = \{R_i\}_{i \in I}$ of $V \times V$ defined as (2.6) - (2.7).

Now, we define a convolution $\circ = \circ_{v_0} : \mathbb{C}\mathcal{R}(X) \times \mathbb{C}\mathcal{R}(X) \to \mathbb{C}\mathcal{R}(X)$ for the base point v_0. We let $P_{i,j}^k$ for i, j, $k \in I$ denote the following probability: Consider a 'jump' to a random vertex $w \in \Gamma_j(v)$ after a 'jump' to a random vertex $v \in \Gamma_i(v_0)$. Let $P_{i,j}^k$ denote the probability that w belongs to $\Gamma_k(v_0)$. These probabilities $P_{i,j}^k$ are explicitly given by

$$P_{i,j}^k = \frac{1}{|\Gamma_i(v_0)|} \sum_{v \in \Gamma_i(v_0)} \frac{|\Gamma_j(v) \cap \Gamma_k(v_0)|}{|\Gamma_j(v)|}. \tag{3.1}$$

In general, the probabilities $P_{i,j}^k$ depend on the choice of the base point v_0.

We note that the denominators $|\Gamma_i(v_0)|$ and $|\Gamma_j(v)|$ can be zero so that $P_{i,j}^k$'s are not necessarily well-defined, but we can determine when $P_{i,j}^k$ are well-defined.

Proposition 3.1. Let $X = (V, E)$ be a connected graph all of whose vertices have finite degrees and the base point $v_0 \in V$ be arbitrarily given.

(a) When X is infinite, $P_{i,j}^k$ are all well-defined.

(b) Suppose that X is finite. Then, $P_{i,j}^k$ are all well-defined if and only if X is a self-centered graph.

Proof. First, we suppose that X is infinite. It suffices to show that $\Gamma_j(v) \neq \varnothing$ for any $v \in V$ and any $j \in I$.

If $\Gamma_j(v) = \varnothing$ for some $v \in V$ and some $j \in I$, we would find that $\Gamma_{j'}(v) = \varnothing$ for any $j' \in I$ such that $j' \geq j$. Indeed, a geodesic $v = v_0 \to v_1 \to \cdots \to v_{j'-1} \to v_{j'} = w$ from v to $w \in \Gamma_{j'}(v)$ should pass through a vertex $v_j \in \Gamma_j(v)$ when $j' \geq j$. Then, we would have $V = \bigcup_{l=0}^{j-1} \Gamma_l(v)$.

On the other hand, by our assumption that all vertices of X have finite degrees, we could find that $\Gamma_l(v)$ must be a finite set for every $l \in \{0, 1, \cdots, j-1\}$. We had that the vertex set V, a union of finitely many finite sets, should be a finite set, but this is a contradiction to the assumption that V is an infinite set, so that the claim (a) of the proposition has been proved.

Next, we suppose that X is finite and $P_{i,j}^k$ are all well-defined. We will show that $e(v) = \mathrm{diam}(X)$ for every $v \in V$.

If there would exist a vertex $v \in V$ such that $e(v) < \mathrm{diam}(X)$, we had $\Gamma_{\mathrm{diam}(X)}(v) = \varnothing$. Then, setting $d(v_0, v) = i$ and $\mathrm{diam}(X) = s$, we would find that

$$P_{i,s}^k = \frac{1}{|\Gamma_i(v_0)|} \sum_{w \in \Gamma_i(v_0)} \frac{|\Gamma_s(w) \cap \Gamma_k(v_0)|}{|\Gamma_s(w)|}$$

cannot be well-defined since $v \in \Gamma_i(v_0)$ and $|\Gamma_s(v)| = 0$. Accordingly, we find that $e(v) = \mathrm{diam}(X)$ for every $v \in V$. This means that all of $e(v)$ coincide one another and we obtain that $e(v) = \mathrm{rad}(X)$ for every $v \in V$.

Conversely, suppose that X is finite and self-centered. A similar argument to the above one yields that $e(v) = \mathrm{diam}(X)$ for every $v \in V$. Then, we have that $\Gamma_j(v) \neq \varnothing$ for every $v \in V$ and $j \in I$. Indeed, a geodesic starting from $v \in V$ of the length equal to $\mathrm{diam}(X)$ passes through a vertex belonging to $\Gamma_j(v)$ for each $j \in I$. This implies that the denominators of fractions in (3.1) never vanish. $\square$

When $P_{i,j}^k$ are all well-defined, we define a bilinear convolution $\circ = \circ_{v_0}$ on $\mathbb{C}\mathcal{R}(X)$ by the following identities:

$$R_i \circ_{v_0} R_j = \sum_{k \in I} P_{i,j}^k R_k \ (i, j \in I).$$

Equipped with a convolution, $\mathbb{C}\mathcal{R}(X)$ becomes a unital $\mathbb{C}$-algebra which is not necessarily associative or commutative. (The existence of the neutral element will be shown below.) In addition, we define a conjugate-linear map $* : \mathbb{C}\mathcal{R}(X) \to \mathbb{C}\mathcal{R}(X)$ by $*|_{\mathcal{R}(X)} = \mathrm{id}_{\mathcal{R}(X)}$, which is expected to be an involution on $\mathbb{C}\mathcal{R}(X)$. This map $*$ becomes an involution on $\mathbb{C}\mathcal{R}(X)$ if and only if $\mathbb{C}\mathcal{R}(X)$ is associative and commutative.

It can be easily verified that

$$P_{i,0}^k = \delta_{i,k} = \begin{cases} 1 & (k = i), \\ 0 & (k \neq i), \end{cases} \quad P_{0,j}^k = \delta_{j,k} = \begin{cases} 1 & (k = j), \\ 0 & (k \neq j), \end{cases}$$

so that $R_0 \in \mathcal{R}(X)$ should be the neutral element with respect to the convolution $\circ_{v_0}$ on $\mathbb{C}\mathcal{R}(X)$. One can also verify the following statements.

Proposition 3.2. Let $X = (V, E)$ be a connected graph all of whose vertices have finite degrees and the base point $v_0 \in V$ be arbitrarily given. Suppose that $P_{i,j}^k$ are well-defined for all i, j, $k \in I$.

(a) For any $i, j \in I$, we have that $R_i \circ_{v_0} R_j \in \mathbb{C}\mathcal{R}(X)$ satisfies the following statements:

$$(R_i \circ_{v_0} R_j)(R_k) \in \mathbb{R}_+ \text{ for all } k \in I, \tag{3.2}$$

$$\|R_i \circ_{v_0} R_j\|_1 = 1, \tag{3.3}$$

$$\operatorname{supp}(R_i \circ_{v_0} R_j) \subset \left\{R_{|i-j|}, R_{|i-j|+1}, \cdots, R_{i+j}\right\}. \tag{3.4}$$

(b) For $i, j \in I$, the neutral element R_0 belongs to $\operatorname{supp}(R_i \circ_{v_0} R_j)$ if and only if $i = j$.

Proof. The first assertion (3.2) immediately follows from (3.1), the definition of $P_{i,j}^k$.

Next, we show the third assertion. When $P_{i,j}^k > 0$, we can find a vertex $v \in \Gamma_i(v_0)$ such that $\Gamma_j(v) \cap \Gamma_k(v_0) \neq \varnothing$. Then, any $w \in \Gamma_j(v) \cap \Gamma_k(v_0)$ should satisfy that

$$\begin{aligned} |i-j| &= |d(v_0, v) - d(w, v)| \\ &\leq k = d(v_0, w) \\ &\leq d(v_0, v) + d(v, w) = i + j. \end{aligned}$$

This means that $\operatorname{supp}(R_i \circ_{v_0} R_j) \subset \{R_{|i-j|}, R_{|i-j|+1}, \cdots, R_{i+j}\}$.

Since $\{\Gamma_j(v) \cap \Gamma_k(v_0)\,;\, k \in I\}$ forms a partition of $\Gamma_j(v)$, we can compute $\|R_i \circ_{v_0} R_j\|_1$ to be

$$\begin{aligned} \|R_i \circ_{v_0} R_j\|_1 &= \sum_{k=|i-j|}^{i+j} P_{i,j}^k \\ &= \sum_{k=|i-j|}^{i+j} \frac{1}{|\Gamma_i(v_0)|} \sum_{v \in \Gamma_i(v_0)} \frac{|\Gamma_j(v) \cap \Gamma_k(v_0)|}{|\Gamma_j(v)|} \\ &= \sum_{v \in \Gamma_i(v_0)} \frac{1}{|\Gamma_i(v_0)| \cdot |\Gamma_j(v)|} \sum_{k=|i-j|}^{i+j} |\Gamma_j(v) \cap \Gamma_k(v_0)| \\ &= 1. \end{aligned}$$

We have shown the claim (a) in the above arguments.

Now, we prove the claim (b). If $i = j$, the base point v_0 belongs to $\Gamma_i(v) \cap \Gamma_0(v_0)$ for every $v \in \Gamma_i(v_0)$, so we have that $P_{i,i}^0 > 0$. Next, we suppose that $R_0 \in \operatorname{supp}(R_i \circ_{v_0} R_j)$. Then, $P_{i,j}^0 > 0$ holds, and this yields that there exists a vertex $v \in \Gamma_i(v_0)$ such that $\Gamma_j(v) \cap \Gamma_0(v_0) \neq \varnothing$. This causes v_0 to lie in $\Gamma_j(v)$. Therefore, we have that $i = d(v, v_0) = j$. $\square$

Appealing to Proposition 3.2, one finds that $\mathcal{R}(X)$ becomes a hermitian discrete hypergroup if and only if the convolution $\circ_{v_0}$ is associative and commutative. The convolution $\circ_{v_0}$ is not always associative or commutative, and it is a problem when $\circ_{v_0}$ is both associative and commutative.

An answer to this problem was given by Wildberger [15]; If X is a strongly regular graph, then $\mathcal{R}(X)$ becomes a hermitian discrete hypergroup. Let $X = (V, E)$ be an (n, k, λ, μ)-strongly regular graph. Then the structure identities of $\mathcal{R}(X) = \{R_0, R_1, R_2\}$ are given by

$$R_1 \circ R_1 = \frac{1}{k} R_0 + \frac{\lambda}{k} R_1 + \frac{k - \lambda - 1}{k} R_2, \tag{3.5}$$

$$R_1 \circ R_2 = R_2 \circ R_1 = \frac{\mu}{k} R_1 + \frac{k - \mu}{k} R_2, \tag{3.6}$$

$$R_2 \circ R_2 = \frac{1}{n - k - 1} R_0 + \frac{k - \mu}{n - k - 1} R_1 + \frac{n + \mu - 2k - 2}{n - k - 1} R_2, \tag{3.7}$$

which are independent of a choice of the base point v_0. (Since R_0 is the neutral element of $\mathbb{C}\mathcal{R}(X)$, we omit the identities for R_0.) The commutativity immediately follows from (3.5) – (3.7), and the associativity can be revealed by direct calculations. (Using Proposition 4.4 curtails necessary calculations.)

Similarly, if X is distance-regular, we find that $\mathcal{R}(X)$ becomes a hermitian discrete hypergroup, whose structure is independent of v_0. Since

$$\begin{aligned} |\Gamma_j(v) \cap \Gamma_k(v_0)| &= |\{w \in V\,;\, d(v, w) = j, d(v, v_0) = k\}| \\ &= |\{w \in V\,;\, (v, w) \in R_j, (w, v_0) \in R_k\}| \\ &= p^i_{j,k} \end{aligned}$$

when $d(v, v_0) = i$, we can compute $P^k_{i,j}$ to be

$$P^k_{i,j} = \frac{1}{|\Gamma_i(v_0)|} \sum_{v \in \Gamma_i(v_0)} \frac{|\Gamma_j(v) \cap \Gamma_k(v_0)|}{|\Gamma_j(v)|} = \frac{p^i_{j,k}}{p^0_{j,j}}. \tag{3.8}$$

(Pay attention to the subscripts in the numerator of the last member.) These constants are determined independently of a choice of v_0. Now, we show that the convolution $\circ$ is commutative and associative. The commutativity is deduced from the identities (c) and (f) in Proposition 2.7; Since

$$p^i_{j,k} p^0_{i,i} = p^k_{j,i} p^0_{k,k} = p^k_{i,j} p^0_{k,k} = p^j_{i,k} p^0_{j,j},$$

we obtain that

$$P^k_{i,j} = \frac{p^i_{j,k}}{p^0_{j,j}} = \frac{p^j_{i,k}}{p^0_{i,i}} = P^k_{j,i}.$$

This implies that $R_i \circ R_j = R_j \circ R_i$. For the associativity $(R_h \circ R_i) \circ R_j = R_h \circ (R_i \circ R_j)$, we use the identities (c), (e) and (f) in Proposition 2.7. By direct calculations, we find that

$$(R_h \circ R_i) \circ R_j = \sum_{k \in I} \left(\sum_{l \in I} P_{h,i}^l P_{l,j}^k \right) R_k,$$
$$R_h \circ (R_i \circ R_j) = \sum_{k \in I} \left(\sum_{l \in I} P_{i,j}^l P_{h,l}^k \right) R_k,$$

so it suffices to show that $\sum_{l \in I} P_{h,i}^l P_{l,j}^k = \sum_{l \in I} P_{i,j}^l P_{h,l}^k$ hold for all $k \in I$. In fact, for any $k \in I$, computations proceed as follows:

$$\begin{aligned}
\sum_{l \in I} P_{h,i}^l P_{l,j}^k &= \frac{1}{p_{i,i}^0 p_{j,j}^0} \sum_{l \in I} p_{j,k}^l p_{i,l}^h \\
&= \frac{1}{p_{i,i}^0 p_{j,j}^0} \sum_{l \in I} p_{i,j}^l p_{l,k}^h \\
&= \frac{1}{p_{i,i}^0 p_{j,j}^0} \sum_{l \in I} p_{j,i}^l p_{l,k}^h \\
&= \frac{1}{p_{i,i}^0 p_{j,j}^0} \sum_{l \in I} \frac{p_{j,l}^i p_{i,i}^0}{p_{l,l}^0} \cdot p_{l,k}^h \\
&= \sum_{l \in I} \frac{p_{j,l}^i}{p_{j,j}^0} \cdot \frac{p_{l,k}^h}{p_{l,l}^0} \\
&= \sum_{l \in I} P_{i,j}^l P_{h,l}^k.
\end{aligned}$$

The above arguments provides the following theorem.

Theorem 3.3. Let $X = (V, E)$ be a distance-regular graph and $v_0 \in V$ the base point. Then, $\mathcal{R}(X)$ becomes a hermitian discrete hypergroup. Moreover, the hypergroup structure of $\mathcal{R}(X)$ is independent of v_0.

There is another way to construct a hermitian discrete hypergroup from a distance-regular graph, which was also introduced by Wildberger [15]. This method is based on the "Bose–Mesner algebra," which is associated to an association scheme. Let $(Y, \mathcal{R})$ be an association scheme, where $\mathcal{R} = \{R_i\}_{i \in I}$ with an index set I. We define matrices $A^{(i)} = (A_{x,y}^{(i)})_{x,y \in Y} \in \mathrm{Mat}_Y(\mathbb{Z})$ for each $i \in I$ as

$$A_{x,y}^{(i)} = \begin{cases} 1 & ((x,y) \in R_i), \\ 0 & ((x,y) \notin R_i), \end{cases}$$

like (2.8). Then, a product $A^{(i)} A^{(j)}$ of two matrices can be written in a linear combination of $A^{(k)}$'s as

$$A^{(i)} A^{(j)} = \sum_{k \in I} p_{i,j}^k A^{(k)},$$

where $p^k_{i,j} = 0$ except for finitely many $k \in I$. Moreover, it can be verified that, for any i, $j \in I$, $A^{(i)}A^{(j)} = A^{(j)}A^{(i)}$ from the identity (c) in Proposition 2.7. Hence we have that the $\mathbb{C}$-vector space $\bigoplus_{k\in I} \mathbb{C}A^{(k)}$ turns into an associative and commutative $\mathbb{C}$-algebra with respect to the ordinary addition and multiplication of matrices. This $\mathbb{C}$-algebra is called Bose–Mesner algebra. Setting $C^{(i)} = (p^0_{i,i})^{-1}A^{(i)}$ for each $i \in I$, we have

$$\begin{aligned} C^{(i)}C^{(j)} &= \sum_{k\in I} \frac{p^k_{i,j}p^0_{k,k}}{p^0_{i,i}p^0_{j,j}} C^{(k)} \\ &= \sum_{k\in I} \frac{p^i_{j,k}}{p^0_{j,j}} C^{(k)} \end{aligned} \tag{3.9}$$

from (c) and (f) in Proposition 2.7. When $(Y, \mathcal{R}) = (V, \mathcal{R}(X))$ for some distance-regular graph $X = (V, E)$, comparing (3.8) and (3.9) yields that $\{C^{(i)}\}_{i\in I}$ has the same structure of a hermitian discrete hypergroup as that of $\mathcal{R}(X)$.

3.2 Examples of distance-regular graphs

Let us now see examples of distance-regular graphs. It seems that graph theorists usually consider only finite ones so that there are known many examples of finite distance-regular graphs. Some elementary examples and infinite ones will be introduced in this section. For more examples, see e.g. [4].

(i) Complete graphs

One of the simplest examples of distance-regular graphs are complete graphs. For $n \in \mathbb{N}$ with $n \geq 2$, let K_n denote the complete graph with n vertices. Then, $\mathrm{diam}(K_n) = 1$ and the intersection array of K_n is $(n-1; 1)$. The canonical partition of $V(K_n)\times V(K_n)$, where $V(K_n)$ denotes the vertex set of K_n, consists of two sets, the diagonal set R_0 and its complement R_1. The intersection numbers $p^k_{i,j}$ of $(V(K_n), \mathcal{R}(K_n))$ are given by

$$\begin{aligned} p^0_{0,0} &= 1, & p^1_{0,0} &= 0, \\ p^0_{0,1} = p^0_{1,0} &= 0, & p^1_{0,1} = p^1_{1,0} &= 1, \\ p^0_{1,1} &= n-1, & p^1_{1,1} &= n-2, \end{aligned}$$

so that we can compute the coefficients $P^k_{i,j} = p^i_{j,k}/p^0_{j,j}$ to be

$$\begin{aligned} P^0_{0,0} &= 1, & P^1_{0,0} &= 0, \\ P^0_{0,1} = P^0_{1,0} &= 0, & P^1_{0,1} = P^1_{1,0} &= 1, \\ P^0_{1,1} &= \frac{1}{n-1}, & P^1_{1,1} &= \frac{n-2}{n-1}. \end{aligned}$$

Figure 1: Infinite 2-regular tree

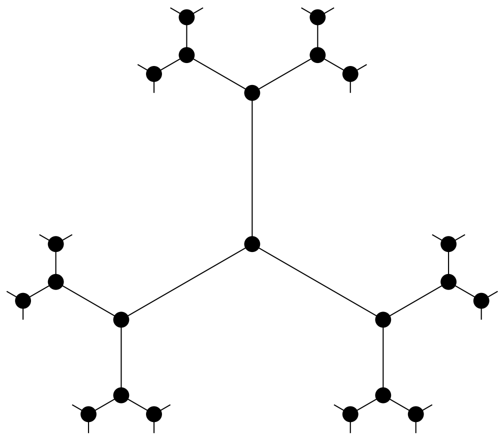

Figure 2: Infinite 3-regular tree

By these computations, the structure identity of $\mathcal{R}(K_n)$ turns out that

$$R_1 \circ R_1 = \frac{1}{n-1} R_0 + \frac{n-2}{n-1} R_1.$$

This hypergroup $\mathcal{R}(K_n)$ is isomorphic to a special case of $\mathbb{Z}_q(2)$, which is called the "q-deformation of $\mathbb{Z}/2\mathbb{Z}$." For details of q-deformation hypergroups, see [8], [13] for example.

(ii) Platonic solids

Consider the platonic solid with n faces (of course we only consider the cases when $n = 4$, 6, 8, 12, 20). Let V_n be the set of its vertices and E_n the set of its edges. The graph $\mathcal{S}_n = (V_n, E_n)$ is known to be distance-regular. As for the structure identities of $\mathcal{R}(\mathcal{S}_n)$, we refer to Wildberger's article [14].

(iii) Infinite regular trees

For each $n \in \mathbb{N}$ with $n \geq 2$, there exists a unique, but except for isomorphic ones, infinite n-regular connected graph without any cycles. (A *cycle* means a path from a vertex to itself which does not pass through the same vertex twice except for the initial and the terminal vertex.) We call such a graph the *infinite n-regular tree*. (The term "tree" means a graph without any cycles.) For example, the infinite 2-regular tree, 3-regular tree and 4-regular tree are partially drawn as in Figures 1, 2 and 3, respectively. Let $\mathcal{T}_n = (V_n, E_n)$ denote the infinite n-regular tree and take an arbitrary vertex $v_0 \in V_n$ as the base point.

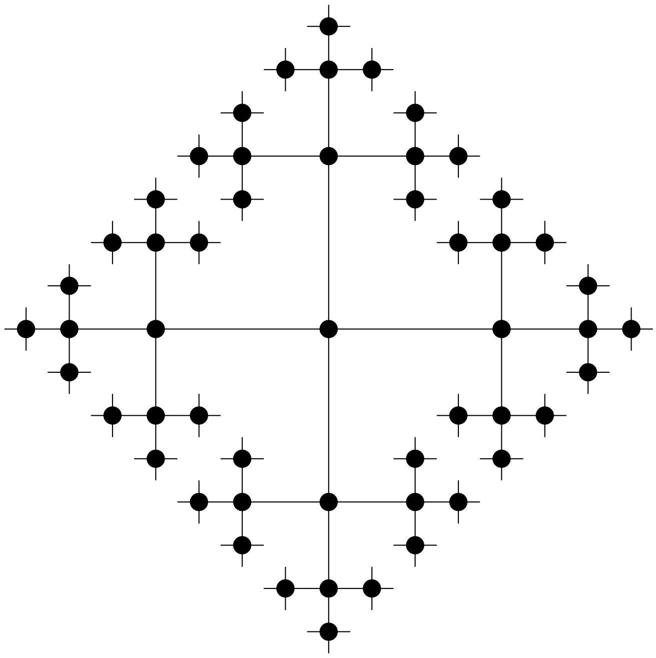

Figure 3: Infinite 4-regular tree

We remark that $\mathcal{T}_n$ can be realized as a "Cayley graph" when n is even. Some Cayley graphs appear here and in the following sections, so we recall the definition.

Definition 3.4. Let G be a group and Ω be a subset of $G \setminus \{1_G\}$ satisfying

$$g \in \Omega \Rightarrow g^{-1} \in \Omega. \tag{3.10}$$

We define a graph $X = (V, E)$ as follows:

- $V = G$.
- $E = \big\{\{g, h\} \ ; \ g, h \in G, g^{-1}h \in \Omega\big\}$.

This graph X is called the *Cayley graph* and denoted by $\mathrm{Cay}\,(G, \Omega)$.

A Cayley graph $\mathrm{Cay}\,(G, \Omega)$ must be $|\Omega|$-regular and vertex-transitive. It becomes connected if and only if Ω generates G.

For $m \in \mathbb{N}$, we let F_m be the free group generated by m symbols $g_1, g_2, \cdots, g_m$ and $\Omega_m = \{g_1^{\pm 1}, g_2^{\pm 1}, \cdots, g_m^{\pm 1}\} \subset F_m$. The Cayley graph $\mathrm{Cay}\,(F_m, \Omega_m)$ is classically known to be isomorphic to the infinite $2m$-regular tree $\mathcal{T}_{2m}$.

Now, we show that $\mathcal{T}_n$ is a distance-regular graph to prove the claim (a) of Theorem 1.2. It will be proved in Proposition 3.6. Lemma 3.5, Proposition 3.6, 3.7 and Corollary 3.8 are new results obtained in [6]. The following lemma gives some significant properties of the infinite regular trees and is useful before we prove that $\mathcal{T}_n$ is a distance-regular graph.

Lemma 3.5. Let $X = (V, E)$ be an infinite connected graph all of whose vertices have finite degrees.

(a) For every $v \in V$, we have $e(v) = \infty$.

(b) Suppose that X has no cycles. Then, for any v, $w \in V$, there exists a unique geodesic from v to w.

Proof. The claim (a) follows from the proof of Proposition 3.1.

We now give a proof of the claim (b). This is clear when $v = w$, so that we may assume that $d\,(v, w) > 0$. Suppose that there would exist two distinct geodesics

$$v = v_0 \to v_1 \to \cdots \to v_{i-1} \to v_i = w,$$
$$v = w_0 \to w_1 \to \cdots \to w_{i-1} \to w_i = w$$

from v to w, where $i = d\,(v, w)$. (There exist at least one geodesics from v to w since X is connected.) Then, we could find at least one $j \in \{1, 2, \cdots, i-1\}$ such that $v_j \neq w_j$. Letting $k^* = \max\{k \in \{0, 1, \cdots, j-1\}\ ;\ v_k = w_k\}$ and $k^{**} = \min\{k \in \{j+1, j+2, \cdots, i\}\ ;\ v_k = w_k\}$ for this j, we would obtain a cycle

$$v_{k^*} \to v_{k^*+1} \to \cdots \to v_{k^{**}-1} \to v_{k^{**}} = w_{k^{**}}$$
$$\to w_{k^{**}-1} \to \cdots \to w_{k^*+1} \to w_{k^*} = v_{k^*},$$

which contradicts the assumption that X has no cycles. □

By using this lemma, we can prove that $\mathcal{T}_n$ is distance-regular.

Proposition 3.6. Let $n \in \mathbb{N}$ with $n \geq 2$. Then the infinite n-regular tree $\mathcal{T}_n$ is a distance-regular graph with the intersection array $(n, n-1, n-1, n-1, \cdots; 1, 1, 1, 1, \cdots)$ (i.e. $b_0 = n$, $b_1 = b_2 = b_3 = \cdots = n-1$, $c_1 = c_2 = c_3 = \cdots = 1$).

Proof. We calculate the entries b_i's and c_i's of the intersection array of $\mathcal{T}_n$. Since $\mathcal{T}_n$ is n-regular, we have $b_0 = n$.

Now we take arbitrary two vertices v, $w \in V_n$ of distance $i \geq 1$. We are going to show that $|\Gamma_{i+1}(v) \cap \Gamma_1(w)| = n - 1$ and $|\Gamma_{i-1}(v) \cap \Gamma_1(w)| = 1$.

By Lemma 3.5 (b), there exists a unique geodesic $v = v_0 \to v_1 \to \cdots \to v_{i-1} \to v_i = w$ from v to w. Since $v_{i-1} \in \Gamma_{i-1}(v) \cap \Gamma_1(w)$, we have $|\Gamma_{i-1}(v) \cap \Gamma_1(w)| \geq 1$. If $\Gamma_{i-1}(v) \cap \Gamma_1(w)$ would possess another vertex w_{i-1}, there would exist the second geodesic $v = w_0 \to w_1 \to \cdots \to w_{i-1} \to w_i = w$ from v to w. This contradicts the uniqueness of the geodesic from v to w, so we find that $|\Gamma_{i-1}(v) \cap \Gamma_1(w)| = 1$.

Next, we check that $\Gamma_i(v) \cap \Gamma_1(w) = \varnothing$. If there would exist a vertex $u \in \Gamma_i(v) \cap \Gamma_1(w)$, we would find a geodesic $v = u_0 \to u_1 \to \cdots \to u_{i-1} \to u_i = u$ from v to u and a geodesic $v = v_0 \to v_1 \to \cdots \to v_{i-1} \to v_i = w$ from v to w. Letting $k^* = \max\{k \in \{0, 1, \cdots, i-1\}\,;\, u_k = v_k\}$, we would obtain a cycle

$$\begin{aligned} &u_{k^*} \to u_{k^*+1} \to \cdots \to u_{i-1} \to u_i = u \\ &\qquad\qquad\qquad \to w = v_i \to v_{i-1} \to \cdots \to v_{k^*+1} \to v_{k^*} = u_{k^*}, \end{aligned}$$

which contradicts that $\mathcal{T}_n$ has no cycles. Therefore, it turns out that $\Gamma_i(v) \cap \Gamma_1(w) = \varnothing$.

We find that $\Gamma_1(w)$ is partitioned into mutually disjoint three subsets, that is,

$$\Gamma_1(w) = (\Gamma_{i-1}(v) \cap \Gamma_1(w)) \sqcup (\Gamma_i(v) \cap \Gamma_1(w)) \sqcup (\Gamma_{i+1}(v) \cap \Gamma_1(w)).$$

Hence we have

$$\begin{aligned} |\Gamma_{i+1}(v) \cap \Gamma_1(w)| &= |\Gamma_1(w)| - |\Gamma_{i-1}(v) \cap \Gamma_1(w)| - |\Gamma_i(v) \cap \Gamma_1(w)| \\ &= n - 1. \end{aligned}$$

These computations work for any v and w with $d(v, w) = i$, so that we obtain that $b_i = n - 1$ and $c_i = 1$. □

To obtain the structure identities of $\mathcal{R}(\mathcal{T}_n)$, we have to compute the intersection numbers $p_{i,j}^k$.

Proposition 3.7. Let $n \in \mathbb{N}$ with $n \geq 2$ and $p_{i,j}^k$ be the intersection numbers of the association scheme $(V_n, \mathcal{R}(\mathcal{T}_n))$.

(a) For any $i \in \mathbb{N}$, we have $p_{i,i}^0 = n(n-1)^{i-1}$.

(b) For any i, j, $k \in \mathbb{N}$, we can get the following:

$$p_{i,j}^k = \begin{cases} (n-1)^i & (j = i + k), \\ (n-2)(n-1)^{i-h-1} & (j = i + k - 2h, 0 < h < \min\{i, k\}), \\ (n-1)^{i-\min\{i,k\}} & (j = |i - k|), \\ 0 & (\text{otherwise}). \end{cases} \tag{3.11}$$

Proof. Take an arbitrary vertex $v_0 \in V_n$ as a base point. First, we compute $p_{i,i}^0$.

We have $p_{1,1}^0 = |\Gamma_1(v_0)| = \deg(v_0) = n$, so the desired counting is obtained when $i = 1$. Now, we show that $p_{i,i}^0 = n(n-1)^{i-1}$ by induction on $i \in \mathbb{N}$. We assume that $p_{i-1,i-1}^0 = n(n-1)^{i-2}$ for some $i \geq 2$. Take

$w \in \Gamma_{i-1}(v_0)$ arbitrarily. Then, by Proposition 3.6, just one vertex belongs to $\Gamma_{i-2}(v_0)$ among n vertices adjacent to w, and the other $n-1$ vertices belong to $\Gamma_i(v_0)$. Thus we get $|\Gamma_i(v_0) \cap \Gamma_1(w)| = n-1$. If we take another vertex $w' \in \Gamma_{i-1}(v_0)$, it turns out that $\Gamma_i(v_0) \cap \Gamma_1(w)$ and $\Gamma_i(v_0) \cap \Gamma_1(w')$ must be disjoint. Hence we obtain

$$\begin{aligned}
p^0_{i,i} = |\Gamma_i(v_0)| &= \left| \Gamma_i(v_0) \cap \bigcup_{w \in \Gamma_{i-1}(v_0)} \Gamma_1(w) \right| \\
&= \left| \bigcup_{w \in \Gamma_{i-1}(v_0)} (\Gamma_i(v_0) \cap \Gamma_1(w)) \right| \\
&= \sum_{w \in \Gamma_{i-1}(v_0)} |\Gamma_i(v_0) \cap \Gamma_1(w)| \\
&= |\Gamma_i(v_0) \cap \Gamma_1(w)| \cdot p^0_{i-1,i-1} \\
&= n(n-1)^{i-1}
\end{aligned}$$

from the induction hypothesis.

Secondly, we show (3.11). We now determine the cases when $p^k_{i,j} = |\Gamma_i(v) \cap \Gamma_j(v_0)| = 0$ for $v \in \Gamma_k(v_0)$. We note that there exists a one-to-one correspondence between V_n and the set of all geodesics on $\mathcal{T}_n$ starting at v_0, which is derived from Lemma 3.5 (b). Suppose that $w \in \Gamma_i(v) \cap \Gamma_j(v_0)$. Then, we can take the unique geodesic

$$v = u_0 \to u_1 \to \cdots \to u_{k-1} \to u_k = v_0$$

from v to v_0 and also the unique geodesic

$$v = w_0 \to w_1 \to \cdots \to w_{i-1} \to w_i = w$$

from v to w.

If $u_0 = w_0$, $u_1 = w_1$, $\cdots$, $u_h = w_h$ and $u_{h+1} \neq w_{h+1}$ hold for some $0 \leq h < \min\{i, k\}$, then the path

$$\begin{aligned}
v_0 = u_k \to u_{k-1} \to \cdots \to u_{h+1} \to u_h = w_h& \\
\to w_{h+1} \to \cdots \to w_{i-1} \to w_i = w&
\end{aligned}$$

must be the geodesic from v_0 to w. In these cases, we have $j = d\,(v_0, w) = i + k - 2h$. (Remark that we always have $u_0 = w_0 = v$.)

Similarly, if $u_0 = w_0$, $u_1 = w_1$, $\cdots$, $u_{\min\{i,k\}} = w_{\min\{i,k\}}$, then the path

$$\begin{aligned}
v_0 = u_k \to u_{k-1} \to \cdots \to u_{\min\{i,k\}+1} \to u_{\min\{i,k\}} = w_{\min\{i,k\}}& \\
\to w_{\min\{i,k\}+1} \to \cdots \to w_{i-1} \to w_i = w&
\end{aligned}$$

must be the geodesic from v_0 to w, and we get $j = d(v_0, w) = i + k - 2\min\{i, k\} = |i - k|$. Therefore, we find that $w \in \Gamma_i(v) \cap \Gamma_j(v_0)$ for $v \in \Gamma_k(v_0)$ only if $j = i + k - 2h$ for some $h \in \mathbb{N}_0$ such that $0 \le h \le \min\{i, k\}$. In other words, when $v \in \Gamma_k(v_0)$, we obtain that $p^k_{i,j} = |\Gamma_i(v) \cap \Gamma_j(v_0)| = 0$ unless $j = i + k - 2h$ for some $h \in \mathbb{N}_0$ such that $0 \le h \le \min\{i, k\}$.

Next, we calculate $|\Gamma_i(v) \cap \Gamma_{i+k-2h}(v_0)|$ for $h = 0, 1, \cdots, \min\{i, k\}$. Let the path

$$v = u_0 \to u_1 \to \cdots \to u_{k-1} \to u_k = v_0$$

be the unique geodesic from v to v_0. Then, by the above argument, we find that $|\Gamma_i(v) \cap \Gamma_{i+k-2h}(v_0)|$ is equal to the number of geodesics

$$v = w_0 \to w_1 \to \cdots \to w_{i-1} \to w_i = w \tag{3.12}$$

with $(u_0 = w_0,)$ $u_1 = w_1, \cdots, u_h = w_h$ and $u_{h+1} \neq w_{h+1}$ when $h < \min\{i, k\}$.

When $0 < h < \min\{i, k\}$, each of such geodesics satisfies neither $w_{h+1} = u_{h+1}$ nor $w_{h+1} = w_{h-1}$. (It requires the latter condition that the path (3.12) is a geodesic. Since $d(v, u_{h+1}) = h + 1 \neq h - 1 = d(v, w_{h-1})$, we remark that $u_{h+1} \neq w_{h-1}$.) Thus the number of possible vertices as w_{h+1} is equal to $n - 2$. Furthermore, each of $w_{h+2}, \cdots, w_i(= w)$ can be chosen from exactly $n - 1$ candidates. (The candidates of $w_{h+2}, \cdots, w_i$ exclude a vertex which coincides with $w_h, \cdots, w_{i-2}$, respectively.) These observations yield that

$$\begin{aligned}
&|\Gamma_i(v) \cap \Gamma_{i+k-2h}(v_0)| \\
=&\left|\left\{(w_{h+1}, \cdots, w_i) \in V_n^{i-h}\,;\, w_{h+1} \neq u_{h+1}, w_{h+1} \neq w_{h-1}, w_{h+2} \neq w_h, \cdots, w_i \neq w_{i-2}\right\}\right| \\
=&(n-2)(n-1)^{i-h-1}
\end{aligned}$$

for $0 < h < \min\{i, k\}$.

When $h = 0$, the number of candidates of w_1 changes into $n - 1$ since we can arbitrarily take $w_1 \in \Gamma_1(w_0)$ except for u_1. Thus we obtain that

$$\begin{aligned}
&|\Gamma_i(v) \cap \Gamma_{i+k}(v_0)| \\
=&\left|\left\{(w_1, \cdots, w_i) \in V_n^i\,;\, w_1 \neq u_1, w_2 \neq w_0, \cdots, w_i \neq w_{i-2}\right\}\right| \\
=&(n-1)^i.
\end{aligned}$$

When $i \le k$ and $h = \min\{i, k\} = i$, all of $w_1, w_2, \cdots, w_i$ are uniquely determined, and we have that

$$\left|\Gamma_i(v) \cap \Gamma_{|i-k|}(v_0)\right| = 1 = (n-1)^{i-\min\{i,k\}}.$$

When $i > k$ and $h = \min\{i, k\} = k$, the first k vertices $w_1, w_2, \cdots, w_k$ are uniquely determined and each of $w_{k+1}, \cdots, w_i$ can be chosen from exactly

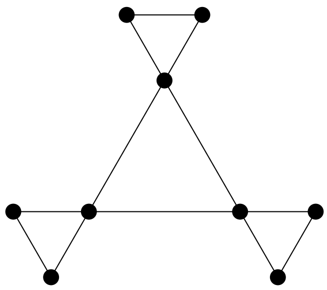

Figure 4: Construction of X_1

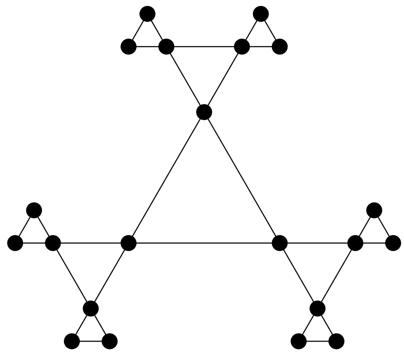

Figure 5: Construction of X_2

$n-1$ candidates. (The candidates of $w_{k+1}, \cdots, w_i$ exclude a vertex which coincides with $w_{k-1}, \cdots, w_{i-2}$, respectively.) Thus we obtain that

$$\begin{aligned}
&\left|\Gamma_i(v) \cap \Gamma_{|i-k|}(v_0)\right| \\
=&\left|\left\{(w_{k+1}, \cdots, w_i) \in V_n^{i-k}\, ;\, w_{k+1} \neq w_{k-1}, w_{k+2} \neq w_k, \cdots, w_i \neq w_{i-2}\right\}\right| \\
=&(n-1)^{i-k} \\
=&(n-1)^{i-\min\{i,k\}}.
\end{aligned}$$

□

The coefficients $P_{i,j}^k$ of the structure identities of $\mathcal{R}(\mathcal{T}_n)$ can be easily computed from Proposition 3.6, so that we obtain the following.

Corollary 3.8. Let $n \in \mathbb{N}$ with $n \geq 2$, i, $j \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then, the coefficients $P_{i,j}^k$ of structure identities of $\mathcal{R}(\mathcal{T}_n)$ are given by

$$P_{i,j}^k = \begin{cases} (n-1)/n & (k = i+j), \\ (n-2)/n(n-1)^k & (k = i+j-2h, 0 < h < \min\{i,j\}), \\ 1/n(n-1)^{\min\{i,j\}-1} & (k = |i-j|), \\ 0 & (\text{otherwise}). \end{cases}$$

(iv) Linked triangles

Here, we see an example of infinite distance-regular graphs with cycles; Such an example can be constructed by linking triangular graphs inductively.

As the beginning step, we let $X_0 = K_3 = (V_0, E_0)$. We make three copies of K_3 "linked" to X_0 to share distinct vertices. Then, we have a graph with nine vertices and nine edges. (See Figure 4.) Let it be denoted by X_1. Next, we make copies of K_3 linked to each vertex of $X_1 = (V_1, E_1)$ that is not yet linked with another triangle. We need six copies of K_3 in this step to obtain the graph $X_2 = (V_2, E_2)$ with twenty-one vertices and thirty edges. (See Figure 5.) Repeating this process, we have an ascending sequence $\{X_n = (V_n, E_n)\}_{n\in\mathbb{N}_0}$ of finite graphs. We let an infinite graph $X_\infty = (V_\infty, E_\infty)$ be defined by the union of all of X_n: It consists of the vertex set $V_\infty = \bigcup_{n\in\mathbb{N}_0} V_n$ and the edge set $E_\infty = \bigcup_{n\in\mathbb{N}_0} E_n$. In this thesis, we call the graph X_∞ the "linked-triangle graph."

We shall show that X_∞ is a distance-regular graph. Preparatory to the proof, we label each element of V_∞ with a three-letter word. Consider the set W of non-empty words of finite length composed of the three letters a, b and c, in which every two consequent letters differ. We regard W as a vertex set, and two words $v = l_1 l_2 \cdots l_m$, $w = l'_1 l'_2 \cdots l'_n \in W$, where l_1, l_2, $\cdots$, l_m, l'_1, l'_2, $\cdots$, $l'_n \in \{a, b, c\}$, are said to be adjacent if and only if one of the following is satisfied:

- It holds that $|m - n| = 1$ and that $l_1 = l'_1$, $l_2 = l'_2$, $\cdots$, $l_{\min\{m,n\}} = l'_{\min\{m,n\}}$.
- It holds that $m = n$ and that $l_1 = l'_1$, $l_2 = l'_2$, $\cdots$, $l_{m-1} = l'_{m-1}$, $l_m \neq l'_m$.

Then, we have an infinite graph isomorphic to X_∞, so that we regard every vertex of X_∞ as labeled by an element of W. We identify the vertex set V_∞ and the word set W in the following arguments. In what follows, we use the notation $\ell(v)$ for the length of the word $v \in W = V_\infty$.

To show that X_∞ is distance-regular, we have to grasp the basic properties of X_∞ in the following lemma. This lemma is a new result obtained in [6].

Lemma 3.9. (a) The linked-triangle graph X_∞ admits no cycles of length four or greater.

(b) The linked-triangle graph X_∞ is connected, and the geodesic from v to w in X_∞ is unique for any v, $w \in V_\infty$.

Proof. To prove the claim (a), we suppose that X_∞ would contain a cycle

$$v_0 \to v_1 \to v_2 \to \cdots \to v_{L-1} \to v_0$$

of length $L \geq 4$ with $v_0 = l_1 l_2 \cdots l_n$, where $n \in \mathbb{N}$ and $l_1, l_2, \cdots, l_n \in \{a, b, c\}$. Changing the initial vertex if necessary, we may assume that $\ell(v_0) \geq \ell(v_i)$ for $i = 1, 2, \cdots, L-1$. If $n = 1$, then all of v_i's are distinct and belong to $\{a, b, c\}$, but this is impossible. If $n \geq 2$, then, since both v_1 and v_{L-1} are adjacent to v_0, we find that one of them should be $l_1 l_2 \cdots l_{n-1}$ and that the other $l_1 l_2 \cdots l_{n-1} l'_n$ for $l'_n \neq l_n$, l_{n-1}. Changing the direction of the cycle if necessary, we may assume that $v_1 = l_1 l_2 \cdots l_{n-1}$ and $v_{L-1} = l_1 l_2 \cdots l_{n-1} l'_n$. Since v_{L-1} is adjacent to v_0, v_1 and v_{L-2}, and $v_{L-2} \neq v_0$, v_1, we would obtain that $\ell(v_{L-2}) = \ell(v_{L-1}) + 1 > \ell(v_0)$. This is a contradiction to the maximality of $\ell(v_0)$, so that we obtain the claim (a).

To show the claim (b), we let the word representations of v and w be denoted by $v = l_1 l_2 \cdots l_m$ and $w = l'_1 l'_2 \cdots l'_n$, respectively, where $m, n \in \mathbb{N}$, and $l_1, l_2, \cdots, l_m, l'_1, l'_2, \cdots, l'_n \in \{a, b, c\}$. In addition, set

$$k^* = \begin{cases} \min\{1 \le k \le \min\{m, n\} \ ; \ l_k \ne l'_k\} & (l_k \ne l'_k \text{ for some } 1 \le k \le \min\{m, n\}), \\ \min\{m, n\} + 1 & (l_k = l'_k \text{ for } 1 \le k \le \min\{m, n\}). \end{cases}$$

When $k^* = 1$, one can take a path

$$\begin{aligned} v = l_1 l_2 \cdots l_m \to l_1 l_2 \cdots l_{m-1} \to \cdots \to l_1 l_2 \to l_1 \\ \to l'_1 \to l'_1 l'_2 \to \cdots \to l'_1 l'_2 \cdots l'_{n-1} \to l'_1 l'_2 \cdots l'_n = w \end{aligned}$$

from v to w, and when $k^* \ge 2$, one can take a path

$$\begin{aligned} v &= l_1 l_2 \cdots l_m \to l_1 l_2 \cdots l_{m-1} \to \cdots \to l_1 l_2 \cdots l_{k^*} \to l_1 l_2 \cdots l_{k^*-1} \\ &= l'_1 l'_2 \cdots l'_{k^*-1} \to l'_1 l'_2 \cdots l'_{k^*} \to \cdots \to l'_1 l'_2 \cdots l'_{n-1} \to l'_1 l'_2 \cdots l'_n = w \end{aligned}$$

from v to w. Therefore, X_∞ is connected, so that there exists at least one geodesic from v to w.

Two distinct geodesics from v to w would allow us to find a cycle of length four or greater, whose existence contradicts to the claim (a) of Lemma 3.9. $\square$

We can prove that X_∞ is distance-regular by an argument similar to that for Lemma 3.5. This proposition is a new result obtained in [6].

Proposition 3.10. The linked-triangle graph X_∞ is a distance-regular graph with the intersection array $(4, 2, 2, 2, \cdots ; 1, 1, 1, 1, \cdots)$ (i.e. $b_0 = 4$, $b_1 = b_2 = b_3 = \cdots = 2$, $c_1 = c_2 = c_3 = \cdots = 1$).

Proof. Since every vertex in V_∞ possesses exactly four neighborhoods, we have $b_0 = 4$.

We show that $c_i = 1$ for any $i \in \mathbb{N}$. Take an $i \in \mathbb{N}$ and two vertices v, $w \in V_\infty$ with $d(v, w) = i$ arbitrarily. Then, there exists a unique geodesic

$$v = v_0 \to v_1 \to \cdots \to v_{i-1} \to v_i = w$$

from v to w. Noting that $v_{i-1} \in \Gamma_{i-1}(v) \cap \Gamma_1(w)$, we find that $|\Gamma_{i-1}(v) \cap \Gamma_1(w)| \ge 1$. On the other hand, the uniqueness of the geodesic ensures that $|\Gamma_{i-1}(v) \cap \Gamma_1(w)| \le 1$, so we have $|\Gamma_{i-1}(v) \cap \Gamma_1(w)| = 1$. This means that $c_i = 1$.

Next, we shall compute $|\Gamma_{i+1}(v) \cap \Gamma_1(w)|$ for $i \in \mathbb{N}$ and v, $w \in V_\infty$ with $d(v, w) = i$. Take the unique geodesic

$$v = v_0 \to v_1 \to \cdots \to v_{i-1} \to v_i = w$$

from v to w. It follows that $v_{i-1} \notin \Gamma_{i+1}(v) \cap \Gamma_1(w)$ from the previous argument. Noting that v_{i-1} and w are adjacent and that there exists just one 3-cycle containing v_{i-1} and w, we find by considering the word representations that there exists exactly one vertex w' that is adjacent to both v_{i-1} and w. The path $v \to v_1 \to \cdots \to v_{i-1} \to w'$ should be the geodesic from v to w', so that w' belongs to $\Gamma_i(v) \cap \Gamma_1(w)$. The other two neighborhoods of w belong to neither $\Gamma_{i-1}(v)$ nor $\Gamma_i(v)$, hence we find that they must belong to $\Gamma_{i+1}(v) \cap \Gamma_1(w)$ and $|\Gamma_{i+1}(v) \cap \Gamma_1(w)| = 2$. This implies that $b_i = 2$. $\square$

Let us determine the coefficients of the structure identities of $\mathcal{R}(X_\infty)$. We can use the word representations to calculate the distance between two vertices of X_∞. This Lemma 3.11, the following Proposition 3.12 and Corollary 3.13 are new results obtained in [6].

Lemma 3.11. Let v, $w \in V_\infty$ and $v = l_1 l_2 \cdots l_m$ and $w = l'_1 l'_2 \cdots l'_n$ be their word representations, where m, $n \in \mathbb{N}$ and l_1, l_2, $\cdots$, l_m, l'_1, l'_2, $\cdots$, $l'_n \in \{a, b, c\}$. Then,

$$d\,(v, w) = \begin{cases} m + n - 2k^* + 1 & (l_i \neq l'_i \text{ for some } 1 \leq i \leq \min\{m, n\}), \\ |m - n| & (l_i = l'_i \text{ for } 1 \leq i \leq \min\{m, n\}), \end{cases} \tag{3.13}$$

where $k^* = \min\{1 \leq k \leq \min\{m, n\}\,;\, l_k \neq l'_k\}$ in the case when $l_i \neq l'_i$ for some $1 \leq i \leq \min\{m, n\}$.

Proof. It is obvious that (3.13) holds when $v = w$, so we may assume that $v \neq w$. The unique geodesic from v to w is given by

$$\begin{aligned} v = l_1 l_2 \cdots l_m \to l_1 l_2 \cdots l_{m-1} \to \cdots \to l_1 l_2 \cdots l_{k^*-1} l_{k^*} = l'_1 l'_2 \cdots l'_{k^*-1} l_{k^*} \\ \to l'_1 l'_2 \cdots l'_{k^*-1} l'_{k^*} \to \cdots \to l'_1 l'_2 \cdots l'_{n-1} \to l'_1 l'_2 \cdots l'_n = w \end{aligned}$$

if $l_i \neq l'_i$ for some $1 \leq i \leq \min\{m, n\}$ and by

$$v = l_1 l_2 \cdots l_m \to l_1 l_2 \cdots l_{m-1} \to \cdots \to l_1 l_2 \cdots l_n = l'_1 l'_2 \cdots l'_n = w$$

if $l_i = l'_i$ for $i = 1, \cdots, \min\{m, n\}$. The desired evaluations are obtained from these observations. $\square$

Proposition 3.12. Let $p^k_{i,j}$ be the intersection numbers of the association scheme $(V_\infty, \mathcal{R}(X_\infty))$.

(a) For any $i \in \mathbb{N}$, we have $p^0_{i,i} = 2^{i+1}$.

(b) For i, j, $k \in \mathbb{N}$, we can get the following:

$$p^k_{i,j} = \begin{cases} 2^{\max\{i-k,0\}} & (j = |i - k|), \\ 2^{\max\{i-k,0\}+h-1} & (j = |i - k| + 2h - 1, 1 \leq h \leq \min\{i, k\}), \\ 2^i & (j = i + k), \\ 0 & (\text{otherwise}). \end{cases} \tag{3.14}$$

Proof. We note that $p^0_{i,i} = |\Gamma_i(a)|$. There are two ways to get a vertex in $\Gamma_i(a)$; The first one is to add i letters $l_1, \cdots, l_i$ to the tail of a. The second one is to add $i-1$ letters to the tail after changing a into another letter, b or c. The numbers of vertices provided by each way are both 2^i. Thus we have $p^0_{i,i} = 2^i + 2^i = 2^{i+1}$.

To show the assertion (b), take two vertices $v, w \in V_\infty$ with $d(v, w) = k$. Since X_∞ is a distance-regular graph, we may assume that the word representations of v and w are given by $v = al_1l_2 \cdots l_k$ and $w = a$, respectively, where $l_1, l_2, \cdots, l_k \in \{a, b, c\}$.

We now classify the vertices belonging to $\Gamma_i(v)$ by distance from $w = a$. We implicitly use Lemma 3.11 in the following arguments.

First, we consider the case when $i \le k$. There are three ways to get a vertex in $\Gamma_i(v)$; The first one is to delete the last i letters $l_k, \cdots, l_{k-i+1}$ from v. The second one is to add i letters $l'_{k+1}, \cdots, l'_{k+i}$ to the tail of v. The third one is, for $1 \le h \le i$, to add $h-1$ letters $l'_{k-i+h+1}, \cdots, l'_{k-i+2h-1}$ to the tail after deleting the last $i-h$ letters $l_k, \cdots, l_{k-i+h+1}$ from v and changing the last letter l_{k-i+h} of the remaining word into the other letter l'_{k-i+h}. The number of vertices provided by each way are found to be 1, 2^i and 2^{h-1}, and the distances between a and a vertex provided by each way are $k-i$, $k+i$ and $k-i+2h-1$, respectively.

Next, we consider the case when $i > k$. In this case, there are three ways to get a vertex in $\Gamma_i(v)$ given as follows; The first one is to add $i-k-1$ letters $l'_1, \cdots, l'_{i-k-1}$ to the tail of v after deleting the last k letters $l_k, \cdots, l_1$ from v and changing the remaining letter a into the other letter l'_0. The second one is to add i letters $l'_{k+1}, \cdots, l'_{k+i}$ to the tail of v. The third one is, for $1 \le h \le k$, to add $i-k+h-1$ letters $l'_{h+1}, \cdots, l'_{i-k+2h-1}$ to the tail after deleting the last $k-h$ letters $l_k, \cdots, l_{h+1}$ from v and changing the last letter l_h of the remaining word into the other letter l'_h. The numbers of vertices provided by each way are found to be 2^{i-k}, 2^i and $2^{i-k+h-1}$ (note that there are two candidates for l'_0 in the explanation of the first way), and the distances between a and a vertex provided by each way are $i-k$, $i+k$ and $i-k+2h-1$, respectively. The conclusion (3.14) can be deduced from these observations. $\square$

By Proposition 3.12, we can compute the structure identities of $\mathcal{R}(X_\infty)$.

Corollary 3.13. Let $i, j \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then, the coefficients $P^k_{i,j}$ of structure identities of $\mathcal{R}(X_\infty)$ are given by

$$P^k_{i,j} = \begin{cases} 1/2 & (k = i+j), \\ 1/2^{\min\{i,j\}-h+2} & (k = |i-j| + 2h - 1, 1 \le h \le \min\{i,j\}), \\ 1/2^{\min\{i,j\}+1} & (k = |i-j|), \\ 0 & (\text{otherwise}). \end{cases}$$

4 Non-distance-regular graphs producing a hypergroup

In the previous section, we certified that a random walk on any distance-regular graph produces a hermitian discrete hypergroup. However, we should note that a random walk on a certain non-distance-regular graph produces a hermitian discrete hypergroup. We are going to see several examples of non-distance-regular graphs on which a random walk produces a hermitian discrete hypergroup. It is a (graph theoretical) problem to determine pairs of a graph $X = (V, E)$ and a base point $v_0 \in V$ which make $(\mathcal{R}(X), \circ_{v_0})$ a hermitian discrete hypergroup.

4.1 Fundamental observations

For simplicity, we set some jargons for graphs.

Definition 4.1. Let $X = (V, E)$ be a connected graph all of whose vertices have finite degrees and $v_0 \in V$.

(a) The given graph X is said to satisfy the *self-centered condition* if X is either infinite or self-centered.

(b) The pair (X, v_0) is said to be *hypergroup productive* if $\mathcal{R}(X)$ becomes a hermitian discrete hypergroup with respect to the convolution $\circ_{v_0}$. The graph X is said to be *hypergroup productive* if (X, v_0) is hypergroup productive pair for any $v_0 \in V$.

It immediately follows from Proposition 3.1 that the convolution $\circ_{v_0}$ on $\mathcal{R}(X)$ is well-defined if and only if X satisfies the self-centered condition. We proved in Section 3.1 that every distance-regular graph is a hypergroup productive graph.

When $X = (V, E)$ is a distance-regular graph, structure identities of $(\mathcal{R}(X), \circ_{v_0})$ are independent of a choice of the base point v_0. On the other hand, structure identities of $(\mathcal{R}(X), \circ_{v_0})$ sometimes depend on a choice of the base point v_0 when X is not distance-regular (see Section 4.2). The following proposition shows a sufficient condition for that two choices of the base point give the same convolution on $\mathcal{R}(X)$.

Proposition 4.2. Let $X = (V, E)$ be a graph satisfying the self-centered condition and v_0, $v_1 \in V$. Suppose that there exists an automorphism φ of X such that $\varphi(v_0) = v_1$. Then, $\circ_{v_0} = \circ_{v_1}$, that is, $S \circ_{v_0} S' = S \circ_{v_1} S'$ holds for any S, $S' \in \mathbb{C}\mathcal{R}(X)$. In particular, if X is vertex-transitive, then $\circ_{v_0} = \circ_{v_1}$ holds for any v_0, $v_1 \in V$.

Proof. Since the convolutions $\circ_{v_0}$ and $\circ_{v_1}$ are bilinear, it suffices to show that $R_i \circ_{v_0} R_j = R_i \circ_{v_1} R_j$ for every i, $j \in I$.

Let $R_i \circ_{v_0} R_j = \sum_{k\in I} P_{i,j}^k R_k$ and $R_i \circ_{v_1} R_j = \sum_{k\in I} Q_{i,j}^k R_k$ for given i, $j \in I$. Since the automorphism φ of X preserves the distance on X, that is, $d(\varphi(v), \varphi(w)) = d(v, w)$ holds for any v, $w \in V$, we have

$$\begin{aligned} Q_{i,j}^k &= \frac{1}{|\Gamma_i(v_1)|} \sum_{v\in\Gamma_i(v_1)} \frac{|\Gamma_j(v) \cap \Gamma_k(v_1)|}{|\Gamma_j(v)|} \\ &= \frac{1}{|\Gamma_i(\varphi(v_0))|} \sum_{w\in\Gamma_i(v_0)} \frac{|\Gamma_j(\varphi(w)) \cap \Gamma_k(\varphi(v_0))|}{|\Gamma_j(\varphi(w))|} \\ &= \frac{1}{|\Gamma_i(v_0)|} \sum_{w\in\Gamma_i(v_0)} \frac{|\Gamma_j(w) \cap \Gamma_k(v_0)|}{|\Gamma_j(w)|} = P_{i,j}^k. \end{aligned}$$

□

It immediately follows from Proposition 4.2 that the hypergroup productivity is preserved by an automorphism in the sense of the following.

Corollary 4.3. Let $X = (V, E)$ be a graph satisfying the self-centered condition and v_0, $v_1 \in V$. Suppose that there exists an automorphism φ of X such that $\varphi(v_0) = v_1$. Then, (X, v_0) is a hypergroup productive pair if and only if so is (X, v_1).

A Cayley graph is vertex-transitive, so it turns to a hypergroup productive graph if it admits a hypergroup productive pair. Moreover, in our context, at most one hypergroup structure can be introduced to the canonical partition of a Cayley graph.

To show that a given pair (X, v_0) is hypergroup productive, where X is a graph satisfying the self-centered condition and $v_0 \in V$, we have to show the commutativity and the associativity of the convolution $\circ_{v_0}$, as we saw in Section 3.1. In particular, the associativity $(R_h \circ_{v_0} R_i) \circ_{v_0} R_j = R_h \circ_{v_0} (R_i \circ_{v_0} R_j)$ should be certificated for every h, i, $j \in I$, but it can be reduced to an easier case.

Proposition 4.4. Let $X = (V, E)$ be a graph satisfying the self-centered condition and $v_0 \in V$. Assume that the convolution $\circ_{v_0}$ satisfies the following two identities for any i, $j \in I$:

$$\begin{aligned} R_i \circ_{v_0} R_j &= R_j \circ_{v_0} R_i, \\ (R_1 \circ_{v_0} R_i) \circ_{v_0} R_j &= R_1 \circ_{v_0} (R_i \circ_{v_0} R_j). \end{aligned}$$

Then, $\circ_{v_0}$ is commutative and associative on $\mathbb{C}\mathcal{R}(X)$, so that (X, v_0) is a hypergroup productive pair.

Proof. In this proof, we simply write $\circ$ instead of $\circ_{v_0}$.

Since the convolution $\circ$ is commutative on $\mathcal{R}(X) \times \mathcal{R}(X)$ and bilinear on $\mathbb{C}\mathcal{R}(X) \times \mathbb{C}\mathcal{R}(X)$, the commutativity extends to the whole $\mathbb{C}\mathcal{R}(X) \times \mathbb{C}\mathcal{R}(X)$.

We will check the associativity of $\circ$. If the given convolution $\circ$ is associative on $\mathcal{R}(X)$, that is, $(R_h \circ R_i) \circ R_j = R_h \circ (R_i \circ R_j)$ holds for any h, i, $j \in I$, then the associativity of $\circ$ extends to the whole $\mathbb{C}\mathcal{R}(X)$ from the bilinearity of $\circ$. Hence what we have to show is that

$$(R_h \circ R_i) \circ R_j = R_h \circ (R_i \circ R_j) \tag{4.1}$$

for any h, i, $j \in I$.

When $h = 0$, (4.1) is immediately obtained from the fact that R_0 is the neutral element of $\mathbb{C}\mathcal{R}(X)$. Thus, by our assumption, we have the conclusion when $\mathrm{diam}(X) = 1$. We assume that $\mathrm{diam}(X) \geq 2$. To show (4.1) by induction on h, suppose that $h \in I$ with $h \geq 2$ and the claim is true in the cases up to $h - 1$. By (3.4), we find that

$$R_h = \frac{1}{P^h_{1,h-1}} \left(R_1 \circ R_{h-1} - P^{h-2}_{1,h-1} R_{h-2} - P^{h-1}_{1,h-1} R_{h-1} \right).$$

(Note that it follows that $P^h_{1,h-1} > 0$ from the connectivity of X.) By using this identity, we get

$$\begin{aligned} &(R_h \circ R_i) \circ R_j \\ =&\frac{1}{P^h_{1,h-1}} \left[((R_1 \circ R_{h-1}) \circ R_i) \circ R_j \right] \\ &- \frac{P^{h-2}_{1,h-1}}{P^h_{1,h-1}} \left[(R_{h-2} \circ R_i) \circ R_j \right] - \frac{P^{h-1}_{1,h-1}}{P^h_{1,h-1}} \left[(R_{h-1} \circ R_i) \circ R_j \right]. \end{aligned}$$

The induction hypothesis gives that

$$(R_{h-1} \circ R_i) \circ R_j = R_{h-1} \circ (R_i \circ R_j), \tag{4.2}$$
$$(R_{h-2} \circ R_i) \circ R_j = R_{h-2} \circ (R_i \circ R_j), \tag{4.3}$$

and the induction basis gives that

$$(R_1 \circ R_{h-1}) \circ R_i = R_1 \circ (R_{h-1} \circ R_i).$$

Therefore, appealing to the induction basis, the induction hypothesis and the bilinearity of $\circ$, we have that

$$\begin{aligned}
&((R_1 \circ R_{h-1}) \circ R_i) \circ R_j \\
=&(R_1 \circ (R_{h-1} \circ R_i)) \circ R_j \\
=&\sum_{k=0}^{h+i-1} P^k_{h-1,i}((R_1 \circ R_k) \circ R_j) \\
=&\sum_{k=0}^{h+i-1} P^k_{h-1,i}(R_1 \circ (R_k \circ R_j)) \\
=&R_1 \circ ((R_{h-1} \circ R_i) \circ R_j) \\
=&R_1 \circ (R_{h-1} \circ (R_i \circ R_j)) \\
=&\sum_{k=0}^{i+j} P^k_{i,j}(R_1 \circ (R_{h-1} \circ R_k)) \\
=&\sum_{k=0}^{i+j} P^k_{i,j}((R_1 \circ R_{h-1}) \circ R_k) \\
=&(R_1 \circ R_{h-1}) \circ (R_i \circ R_j).
\end{aligned} \tag{4.4}$$

These identities (4.2), (4.3) and (4.4) make the end of the proof with

$$\begin{aligned}
&\frac{1}{P^h_{1,h-1}}\left[((R_1 \circ R_{h-1}) \circ R_i) \circ R_j\right] \\
&-\frac{P^{h-2}_{1,h-1}}{P^h_{1,h-1}}\left[(R_{h-2} \circ R_i) \circ R_j\right] - \frac{P^{h-1}_{1,h-1}}{P^h_{1,h-1}}\left[(R_{h-1} \circ R_i) \circ R_j\right] \\
=&\frac{1}{P^h_{1,h-1}}\left[(R_1 \circ R_{h-1}) \circ (R_i \circ R_j)\right] \\
&-\frac{P^{h-2}_{1,h-1}}{P^h_{1,h-1}}\left[R_{h-2} \circ (R_i \circ R_j)\right] - \frac{P^{h-1}_{1,h-1}}{P^h_{1,h-1}}\left[R_{h-1} \circ (R_i \circ R_j)\right] \\
=&\left[\frac{1}{P^h_{1,h-1}}(R_1 \circ R_{h-1} - P^{h-2}_{1,h-1}R_{h-2} - P^{h-1}_{1,h-1}R_{h-1})\right] \circ (R_i \circ R_j) \\
=&R_h \circ (R_i \circ R_j).
\end{aligned}$$

□

4.2 Examples of hypergroup productive graphs

We see some examples of non-distance-regular hypergroup productive graphs in this section. One can check the associativity by direct calculations for each case, so we shall only give the structure identities and omit the proof of associativity.

(i) Prisms

The prism graphs are the simplest examples of non-distance-regular graphs, on which a random walk produces a hermitian discrete hypergroup. Consider the n-gonal prism for $n \geq 3$ and let V_n denote the set of its vertices and E_n the set of its edges. The graph $\mathcal{P}_n = (V_n, E_n)$ is distance-regular if and only if $n = 4$ ($\mathcal{P}_4 \cong \mathcal{S}_6$), but any $n \geq 3$ allows $\mathcal{P}_n$ to produce a hermitian discrete hypergroup.

The n-gonal prism graph $\mathcal{P}_n$ can be realized as a Cayley graph; $\mathcal{P}_n = \mathrm{Cay}\left(\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \{(\overline{\pm 1}, \overline{0}), (\overline{0}, \overline{1})\}\right)$, where $\overline{a}$ denotes the residue class of $a \in \mathbb{Z}$. Hence, by Proposition 4.2, the hypergroup structure $(\mathcal{R}(\mathcal{P}_n), \circ_{v_0})$ is independent of a choice of the base point v_0.

It is the first attempt to reveal the structure identities of $\mathcal{R}(\mathcal{P}_n)$ in a general form. The structure identities of $\mathcal{R}(\mathcal{P}_n)$ are computed to be, if $n = 2m + 1$ with $m \geq 2$,

$$\begin{aligned} R_1 \circ R_j &= R_j \circ R_1 \\ &= \frac{3 - \delta_{j,1} + \delta_{j,m+1}}{6} R_{j-1} + \frac{\delta_{j,m}}{6} R_j + \frac{3 + \delta_{j,1} - \delta_{j,m} - \delta_{j,m+1}}{6} R_{j+1} \\ &\qquad (1 \leq j \leq m+1), \end{aligned}$$

$$\begin{aligned} R_i \circ R_j &= R_j \circ R_i \\ &= \frac{3 - \delta_{i,j}}{8} R_{|i-j|} + \frac{1 + \delta_{i,j}}{8} R_{|i-j|+2} + \frac{1}{8} R_{i+j-2} + \frac{3}{8} R_{i+j} \\ &\qquad (2 \leq i, j \leq m-1, i+j \leq m), \end{aligned} \tag{4.5}$$

$$\begin{aligned} R_i \circ R_j &= R_j \circ R_i \\ &= \frac{3 - \delta_{i,j}}{8} R_{|i-j|} + \frac{1 + \delta_{i,j}}{8} R_{|i-j|+2} + \frac{1}{8} R_{m-1} + \frac{1}{8} R_m + \frac{1}{4} R_{m+1} \\ &\qquad (2 \leq i, j \leq m-1, m+1 \leq i+j \leq m+2), \end{aligned} \tag{4.6}$$

$$\begin{aligned} R_i \circ R_j &= R_j \circ R_i \\ &= \frac{3 - \delta_{i,j}}{8} R_{|i-j|} + \frac{1 + \delta_{i,j}}{8} R_{|i-j|+2} + \frac{1}{8} R_{2m-i-j+1} + \frac{3}{8} R_{2m-i-j+3} \\ &\qquad (2 \leq i, j \leq m-1, i+j \geq m+3), \end{aligned} \tag{4.7}$$

$$R_i \circ R_m = R_m \circ R_i$$
$$= \frac{3}{8} R_{m-i} + \frac{1}{8} R_{m-i+1} + \frac{1+\delta_{i,2}}{8} R_{m-i+2} + \frac{3-\delta_{i,2}}{8} R_{m-i+3} \quad (2 \le i \le m-1), \quad (4.8)$$

$$R_m \circ R_m = \frac{1}{4} R_0 + \frac{1}{8} R_1 + \frac{2+\delta_{m,2}}{8} R_2 + \frac{3-\delta_{m,2}}{8} R_3,$$

$$R_i \circ R_{m+1} = R_{m+1} \circ R_i = \frac{3+\delta_{i,1}}{6} R_{m-i+1} + \frac{3-\delta_{i,1}}{6} R_{m-i+2} \quad (1 \le i \le m+1),$$

and to be, if $n = 2m$ with $m \ge 2$,

$$R_1 \circ R_j = R_j \circ R_1$$
$$= \frac{3-\delta_{j,1}+\delta_{j,m}+3\delta_{j,m+1}}{6} R_{j-1} + \frac{3+\delta_{j,1}-\delta_{j,m}-3\delta_{j,m+1}}{6} R_{j+1} \quad (1 \le j \le m+1),$$

$$R_i \circ R_j = R_j \circ R_i$$
$$= \frac{3-\delta_{i,j}}{8} R_{|i-j|} + \frac{1+\delta_{i,j}}{8} R_{|i-j|+2} + \frac{1+\delta_{i+j,m+1}}{8} R_{i+j-2} + \frac{3-\delta_{i+j,m+1}}{8} R_{i+j} \quad (2 \le i,j \le m-1, i+j \le m+1), \quad (4.9)$$

$$R_i \circ R_j = R_j \circ R_i$$
$$= \frac{3-\delta_{i,j}}{8} R_{|i-j|} + \frac{1+\delta_{i,j}}{8} R_{|i-j|+2} + \frac{1}{8} R_{2m-i-j} + \frac{3}{8} R_{2m-i-j+2} \quad (2 \le i,j \le m-1, i+j \le m+2), \quad (4.10)$$

$$R_i \circ R_m = R_m \circ R_i = \frac{1}{2} R_{m-i} + \frac{1}{2} R_{m-i+2} \quad (2 \le i \le m-1), \quad (4.11)$$

$$R_m \circ R_m = \frac{1}{3} R_0 + \frac{2}{3} R_2$$

$$R_i \circ R_{m+1} = R_{m+1} \circ R_i = R_{m-i+1} \quad (1 \le i \le m+1),$$

The identities for $R_i \circ R_j$ with $2 \leq i \leq m-1$ or with $2 \leq j \leq m-1$ in the above (i.e. (4.5) – (4.8) and (4.9) – (4.11)) should be omitted when $m = 2$. If $n = 3$, the structure identities of $\mathcal{R}(\mathcal{P}_3)$, given as follows, are slightly different from the above ones;

$$\begin{aligned} R_1 \circ R_1 &= \frac{1}{3}R_0 + \frac{2}{9}R_1 + \frac{4}{9}R_2, \\ R_1 \circ R_2 = R_2 \circ R_1 &= \frac{2}{3}R_1 + \frac{1}{3}R_2, \\ R_2 \circ R_2 &= \frac{1}{2}R_0 + \frac{1}{2}R_1. \end{aligned}$$

We note that there can be constructed a finite hermitian discrete hypergroup of arbitrarily large order from a prism graph.

(ii) Finite regular graphs producing two kinds of structure identities

Thus far every example of hypergroup productive graphs have induced a single structure of hermitian discrete hypergroups in its canonical partition, whereas there exist hypergroup productive graphs which may produce two (or more) structures of hermitian discrete hypergroups. Here, we see two such graphs, which are drawn as in Figures 6 and 7. The latter one can be realized as the line graph of the triangular prism $\mathcal{P}_3$. (The definition of the line graph refers to [2] etc.) Both of two graphs are of diameter two, and neither one is found to be vertex-transitive from Proposition 4.2.

Let the graph drawn in Figure 6 be denoted by X_1 and the other one X_2. The structure identities of $\mathcal{R}(X_1)$ are given by

$$\begin{aligned} R_1 \circ_{v_0} R_1 &= \frac{1}{4}R_0 + \frac{1}{4}R_1 + \frac{1}{2}R_2, \\ R_1 \circ_{v_0} R_2 = R_2 \circ_{v_0} R_1 &= R_1, \\ R_2 \circ_{v_0} R_2 &= \frac{1}{2}R_0 + \frac{1}{2}R_2 \end{aligned}$$

if the base point v_0 is chosen from filled vertices in Figure 6 and

$$\begin{aligned} R_1 \circ_{v_0} R_1 &= \frac{1}{4}R_0 + \frac{3}{8}R_1 + \frac{3}{8}R_2, \\ R_1 \circ_{v_0} R_2 = R_2 \circ_{v_0} R_1 &= \frac{3}{4}R_1 + \frac{1}{4}R_2, \\ R_2 \circ_{v_0} R_2 &= \frac{1}{2}R_0 + \frac{1}{2}R_1 \end{aligned}$$

if the base point v_0 is chosen from blank vertices in the same figure. On the

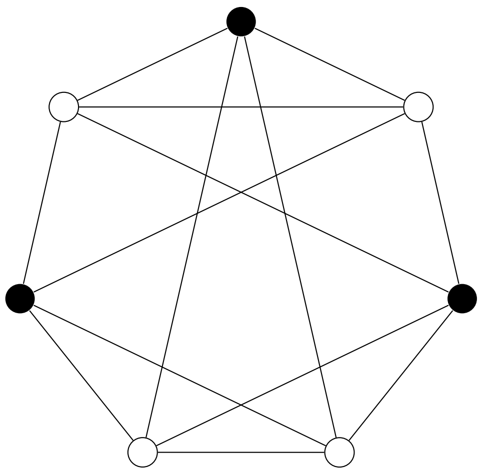

Figure 6: A 4-regular graph which produces two kinds of structure identities

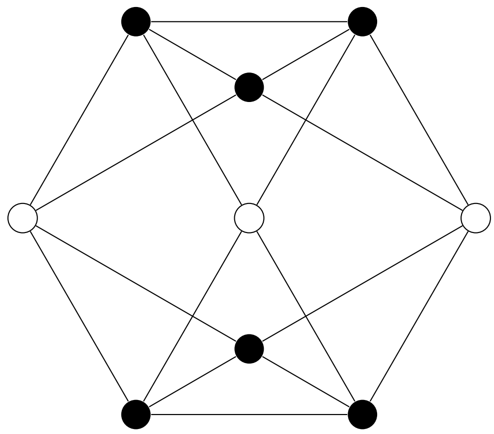

Figure 7: The line graph of $\mathcal{P}_3$

other hand, the structure identities of $\mathcal{R}(X_2)$ are computed to be

$$\begin{aligned}
R_1 \circ_{v_0} R_1 &= \frac{1}{4}R_0 + \frac{3}{8}R_1 + \frac{3}{8}R_2, \\
R_1 \circ_{v_0} R_2 = R_2 \circ_{v_0} R_1 &= \frac{3}{8}R_1 + \frac{5}{8}R_2, \\
R_2 \circ_{v_0} R_2 &= \frac{1}{4}R_0 + \frac{5}{8}R_1 + \frac{1}{8}R_2
\end{aligned}$$

if the base point v_0 chosen from filled vertices in Figure 7 and

$$\begin{aligned}
R_1 \circ_{v_0} R_1 &= \frac{1}{4}R_0 + \frac{1}{4}R_1 + \frac{1}{2}R_2, \\
R_1 \circ_{v_0} R_2 = R_2 \circ_{v_0} R_1 &= \frac{1}{2}R_1 + \frac{1}{2}R_2, \\
R_2 \circ_{v_0} R_2 &= \frac{1}{4}R_0 + \frac{1}{2}R_1 + \frac{1}{4}R_2
\end{aligned}$$

if the base point v_0 chosen from blank vertices in the same figure.

These facts are new results obtained in [6].

(iii) Complete bipartite graphs

Let $m,\ n \in \mathbb{N}$ with $m,\ n \geq 2$. The *complete bipartite graph* $K_{m,n} = (V_{m,n}, E_{m,n})$ is defined as follows:

- $V_{m,n} = \{u_1, u_2, \cdots, u_m, w_1, w_2, \cdots, w_n\} \quad (|V_{m,n}| = m + n)$.
- $E_{m,n} = \{\{u_\alpha, w_\beta\}\ ;\ 1 \leq \alpha \leq m, 1 \leq \beta \leq n\}$.

The complete bipartite graph $K_{m,n}$ is distance-regular if and only if $m = n$ and is of diameter two. We here meet an intriguing example, which is a non-regular hypergroup productive graph. This is a new example introduced in [6].

We can compute the structure identities of $\mathcal{R}(K_{m,n})$ to be

$$\begin{aligned} R_1 \circ_{v_0} R_1 &= \frac{1}{m} R_0 + \frac{m-1}{m} R_2, \\ R_1 \circ_{v_0} R_2 &= R_2 \circ_{v_0} R_1 = R_1, \\ R_2 \circ_{v_0} R_2 &= \frac{1}{m-1} R_0 + \frac{m-2}{m-1} R_2 \end{aligned}$$

if the base point v_0 is chosen from u_α's and to be

$$\begin{aligned} R_1 \circ_{v_0} R_1 &= \frac{1}{n} R_0 + \frac{n-1}{n} R_2, \\ R_1 \circ_{v_0} R_2 &= R_2 \circ_{v_0} R_1 = R_1, \\ R_2 \circ_{v_0} R_2 &= \frac{1}{n-1} R_0 + \frac{n-2}{n-1} R_2 \end{aligned}$$

if the base point v_0 is chosen from w_β's. Needless to say, these identities completely coincide when $m = n$.

(iv) Infinite ladder graph

Here an example of infinite hypergroup productive graphs, which can be drawn like a ladder as in Figure 8 will be introduced. This is a new example of hypergroup productive graph obtained in [6]. More precisely, we consider the Cayley graph $\mathcal{L} = \mathrm{Cay}\big(\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}), \{(\pm 1, \overline{0}), (0, \overline{1})\}\big)$ in this part. (As mentioned in Theorem 1.2, the symbols $\overline{0}$ and $\overline{1}$ denote the residue classes of 0 and 1 modulo 2, respectively.) Since Cayley graphs are vertex-transitive, Proposition 4.2 allows one to assume that $v_0 = (0, \overline{0})$ is the base point. The structure identities of $\mathcal{R}(\mathcal{L})$ can be computed to be

$$\begin{aligned} R_1 \circ R_1 &= \frac{1}{3} R_0 + \frac{2}{3} R_2, \\ R_1 \circ R_i = R_i \circ R_1 &= \frac{1}{2} R_{i-1} + \frac{1}{2} R_{i+1} \quad (i \geq 2), \\ R_i \circ R_i &= \frac{1}{4} R_0 + \frac{1}{4} R_2 + \frac{1}{8} R_{2i-2} + \frac{3}{8} R_{2i} \quad (i \geq 2), \\ R_i \circ R_j &= \frac{3}{8} R_{|i-j|} + \frac{1}{8} R_{|i-j|+2} + \frac{1}{8} R_{i+j-2} + \frac{3}{8} R_{i+j} \quad (i, j \geq 2, i \neq j). \end{aligned}$$

To prove the claim (b) of Theorem 1.2, the associativity remains to be checked. It can be shown by elementary calculations (and Proposition 4.4).

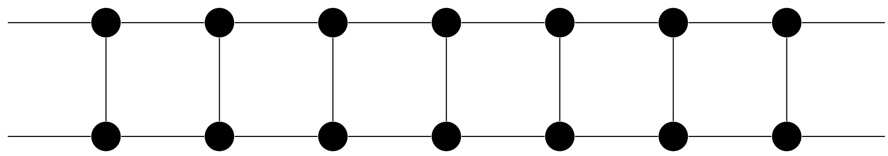

Figure 8: Infinite ladder graph

Remark. One can find that the Cayley graph $\mathrm{Cay}\left(\mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}), \{(\pm 1, \overline{0}), (0, \overline{\pm 1})\}\right)$ admits no hypergroup productive pairs when $n \geq 3$. In addition, it turns out that the Cayley graph $\mathrm{Cay}\,(\mathbb{Z} \oplus \mathbb{Z}, \{(\pm 1, 0), (0, \pm 1)\})$, which is actually expected to be hypergroup productive, also admits no hypergroup productive pairs. (This graph can be drawn as the square lattice in the Euclidean plane.) There are no known examples of graphs that admit a hypergroup productive pair but fail to be hypergroup productive graphs.

Remark. Hypergroups derived from a random walk on an infinite graph can be realized as polynomial hypergroups since $\mathrm{supp}\,(R_1 \circ R_i) \subset \{i-1, i, i+1\}$.

In particular, the hypergroup derived from a random walk on $\mathcal{T}_2$ coincides with the hypergroup which can be constructed by the Chebyshev polynomials of the first kind, so that the hypergroup $\mathcal{R}(\mathcal{T}_2)$ is called the Chebyshev hypergroup of the first kind. Tsurii [13] elucidated several properties of the Chebyshev hypergroup of the first kind. (He also investigated the Chebyshev hypergroup of the second kind in [13].)

For details of the polynomial hypergroups, see [3, Chapter 3] or [10] for example.

References

[1] R. A. Bailey, *Association schemes: Designed experiments, algebra and combinatorics*, Cambridge Univ. Press, 2004.

[2] N. Biggs, *Algebraic graph theory* (*second edition*), Cambridge Univ. Press, 1993.

[3] W. R. Bloom and H. Heyer, *Harmonic analysis of probability measures on hypergroups*, Walter de Gruyter & Co., 1995.

[4] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-regular graphs*, Springer-Verlag, 1989.

[5] C. F. Dunkl, *The measure algebra of a locally compact hypergroup*, Trans. Amer. Math. Soc. **179** (1973), 331–348.

[6] T. Ikkai and Y. Sawada, *Hypergroups derived from random walks on some infinite graphs*, Monatsh. Math. **189** (2) (2019), 321–353.

[7] R. I. Jewett, *Spaces with an abstract convolution of measures*, Adv. in Math. (Springer) **18** (1) (1975), 1–101.

[8] S. Kawakami, T. Tsurii and S. Yamanaka, *Deformations of finite hypergroups*, Sci. Math. Jpn. (in Editione Electronica) **28** (2015), 2015-21 (published online).

[9] R. Lasser, *Discrete commutative hypergroups*, `https://pdfs.semanticscholar.org/753f/a34b23835361a5a443312422dd18f37adbea.pdf`

[10] R. Lasser, *Orthogonal polynomials and hypergroups*, Rend. Mat. Appl. (7) **3** (1983), 185–209.

[11] Y. Matsuzawa, H. Ohno, A. Suzuki, T. Tsurii and S. Yamanaka, *Noncommutative hypergroup of order five*, J. Algebra Appl., to appear.

[12] R. Spector, *Mesures invariantes sur les hypergroupes*, Trans. Amer. Math. Soc. **239** (1978), 147–165.

[13] T. Tsurii, *Deformations of the Chebyshev hypergroups*, Sci. Math. Jpn. (in Editione Electronica) **28** (2015), 2015-54 (published online).

[14] N. J. Wildberger, *Hypergroups associated to random walks on Platonic solids*, Preprint, Univ. of NSW, 1994.

[15] N. J. Wildberger, *Finite commutative hypergroups and applications from group theory to conformal field theory*, Contemp. Math. **183** (1995), 413–434.

[16] N. J. Wildberger, *Strong hypergroups of order three*, J. of Pure Appl. Algebra **174** (2002), 95–115.