A study on Fontaine's perfectoid rings and their algebraizations
（フォンテーニュのパーフェクトイド環とその代数化について）

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February 14, 2020

Acknowledgements

I would like to express my greatest thanks and appreciation to Professor Kazuhiro Fujiwara for his elaborate guidance and kind supports. I am also deeply grateful to Professor Kazuma Shimomoto for his guidance and useful discussions. I would also like to express my gratitude to Professor Ryo Takahashi for valuable comments and encouragement. I also appreciate Professor Osamu Iyama and Professor Hidekazu Furusho for helpful comments. I also thank my colleagues and my family.

Contents

Chapter 0

Introduction

0.1 Backgrounds

In [49], Scholze established a new framework to reduce certain problems in mixed characteristic to problems in positive characteristic. The basic concept of this originates in Fontaine-Wintenberger's theorem, which states that the absolute Galois groups of $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{F}_p((T))$ are canonically isomorphic. Let K and $K^\flat$ be the completion of the valuation fields $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{F}_p((T))(T^{1/p^\infty})$, respectively. The essential part of the proof for Fontaine-Wintenberger's theorem is to show that the absolute Galois groups of K and $K^\flat$ are canonically isomorphic. For the valuation subrings $K^\circ \subset K$ and $K^{\flat\circ} \subset K^\flat$, there is a canonical isomorphism

$$K^{\flat\circ} \cong \varprojlim_{\mathrm{Frob}} K^\circ/(p) \tag{0.1}$$

where each transition map $K^\circ/(p) \to K^\circ/(p)$ is the Frobenius endomorphism. In particular, $K^\flat$ can be constructed from K in a ring theoretic manner.

As a natural generalization of complete valuation fields appeared above, Scholze introduced a new class of complete valuation fields, called *perfectoid fields* (this is a special class of *deeply ramified fields*, which had already been treated in earlier works; cf. [10] and [21]). To any perfectoid field K, one can associate a perfectoid field $K^\flat$ of characteristic p such that (0.1) holds true ($K^\flat$ is called the *tilt* of K). Moreover, the absolute Galois groups of K and $K^\flat$ are canonically isomorphic ([49, Theorem 3.7]). The original aim of [49] is to generalize this result to a comparison of geometric objects over K and geometric objects over $K^\flat$ to give a new result on Deligne's weight-monodromy conjecture. First, Scholze defined a special class of uniform Banach K-algebras, called *perfectoid K-algebras*. Then he established an equivalence of categories, called the *tilting equivalence*, between the category of perfectoid K-algebras and the category of perfectoid $K^\flat$-algebras ([49, Theorem 5.2]). Furthermore, as a generalization of this result, he established a geometric framework by introducing *perfectoid spaces*. Like the association $K \mapsto K^\flat$, to any perfectoid space X over K one can associate a perfectoid space $X^\flat$ over $K^\flat$, and in fact it induces an equivalence of categories ($X^\flat$ is called the *tilt* of X).

Theorem 0.1.1 (Tilting equivalence for perfectoid spaces, [49, Proposition 6.17])**.** *The category of perfectoid spaces over K and the category of perfectoid spaces over $K^\flat$ are equivalent.*

Perfectoid K-algebras have another remarkable property, called the *almost purity*.

Theorem 0.1.2 (Almost purity for perfectoid K-algebras, [49, Theorem 7.9])**.** *Let R be a perfectoid K-algebra. Let S be a finite étale R-algebra. Then the complete integral closure S° of R° in S is almost finite étale over R°.*

The following is a consequence of the almost purity.

Theorem 0.1.3 ([49, Theorem 7.12])**.** *Let X be a perfectoid space over K with the tilt $X^\flat$ over $K^\flat$. Then tilting induces an equivalence of sites $X_{\text{ét}} \cong X^\flat_{\text{ét}}$.*

The theory of perfectoid spaces has many notable applications, which are not only to arithmetic geometry. In 2016, Yves André established *Perfectoid Abhyankar's Lemma* in [1] as a conceptual generalization of *Almost Purity Theorem* (see [49, Theorem 7.9]). Using his results, André proved the existence of big Cohen-Macaulay algebras in mixed characteristic in [2]. Perfectoid Abhyankar's Lemma is also called a ramified version of almost purity theorem. The assertion is the following.

Theorem 0.1.4 (André)**.** *Let $\mathcal{A}$ be a perfectoid K-algebra such that K is a perfectoid field with $p^{\frac{1}{p^n}} \in K^\circ$ for all $n > 0$. Fix a non-zerodivisor $g \in \mathcal{A}^\circ$. Assume that $g^{\frac{1}{p^n}} \in \mathcal{A}^\circ$ for all $n > 0$ and $\mathcal{B}'$ is a finite étale algebra over $\mathcal{A}[\frac{1}{g}]$. Denote by $\mathcal{B}^\circ$ the integral closure of $g^{-\frac{1}{p^\infty}}\mathcal{A}^\circ$ in $\mathcal{B}'$. Then the following statements hold.*

1. *The Frobenius endomorphism $Frob : \mathcal{B}^\circ/(p) \to \mathcal{B}^\circ/(p)$ is $(pg)^{\frac{1}{p^\infty}}$-almost surjective and it induces an injection $\mathcal{B}^\circ/(p^{\frac{1}{p}}) \hookrightarrow \mathcal{B}^\circ/(p)$.*

2. *The map $\mathcal{A}^\circ/(p^m) \to \mathcal{B}^\circ/(p^m)$ is $(pg)^{\frac{1}{p^\infty}}$-almost finite étale for all $m > 0$, and $\mathcal{A}^\circ[\frac{1}{g}] \to \mathcal{B}^\circ[\frac{1}{g}]$ is $(p)^{\frac{1}{p^\infty}}$-almost finite étale.*

0.2 Motivation and statement of results

Let A be a perfectoid K-algebra in Scholze's original sense (cf. [49, Definition 5.1]). Then by definition, A is a Banach K-algebra, and has a canonical integral structure $(A^\circ, (\varpi))$: it consists of an integrally closed K°-subalgebra $A^\circ \subset A$ and a principal ideal $(\varpi) \subset A^\circ$ such that the induced topology on A° is ϖ-adic, $A = A^\circ[\frac{1}{\varpi}]$, $p \in (\varpi^p)$, and $A^\circ/(\varpi^p)$ is semiperfect. Conversely, A can be naturally recovered from such a pair consisting of a K°-algebra and an ideal in it. Thus we can try to generalize the theory of perfectoid spaces by the following two ways:

(I) replacing (ϖ) with a finitely generated ideal that induces a coarser adic topology;

(II) weakening the completeness assumption on A° (e.g. to Henselianness or Zariskianness; see [20, **0**, §7]).

This thesis deals with several problems arising from each subject. Let us state the details below.

0.2.1 Big rings with $\mathfrak{m}$-adic topology

First we discuss (**I**). Let us begin with giving a typical example of a perfectoid ring, which plays a key role in the proof of the direct summand conjecture. Let R be a complete regular local ring of characteristic $(0, p)$ with $k := R/\mathfrak{m}$ perfect. Let $x_1, \ldots, x_d \in R$ be a regular system of parameters.

Set $R_\infty := R[x_1^{1/p^\infty}, x_2^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]$ (see Example 2.1.4 for the precise definition). Then the complete valued field $K := (W(k)[p^{1/p^\infty}])_p^\wedge[\frac{1}{p}]$ is a perfectoid field, and in the case when $x_1 = p$ (in particular R is unramified), $(R_\infty)_p^\wedge[\frac{1}{p}]$ equipped with a canonical topology is a perfectoid K-algebra. However, in the case when R is ramified, $p \in R$ cannot be a part of a regular system of parameters, and one may consider that a canonical topology on R_∞ should be defined by $\mathfrak{m}R_\infty$ rather than pR_∞. One of our aims is to make an interpretation of the adic ring $(R_\infty)_{\mathfrak{m}R_\infty}^\wedge$ in terms of the theory of perfectoid spaces.

Theorem A (Theorem 3.4.1)**.** *Let $(R, \mathfrak{m})$ and R_∞ be as above. Let $\widehat{R_\infty}$ be the $\mathfrak{m}R_\infty$-adic completion of R_∞ equipped with $\mathfrak{m}\widehat{R_\infty}$-adic topology. Then the valued triple $\mathbf{ZR}(\mathrm{Spf}(\widehat{R_\infty})^{\mathrm{rig}})$ is a perfectoid space.*

This result enables us to apply the theory of perfectoid spaces directly to a study of regular local rings with $\mathfrak{m}$-adic topology. For example, we believe that this approach can be applied to the absolute cohomological purity conjecture[1]. The main ingredients to prove Theorem A are the following:

- Fontaine's perfectoid rings ([16]);
- Fujiwara-Kato's enriched visualization functor $\mathbf{ZR}$ ([20]).

Let us explain them below.

In [16], Fontaine gave a natural definition of *perfectoid rings*, which is obtained by dropping the assumption on the base ring from the original definition of perfectoid K-algebras. Any perfectoid field is a perfectoid ring in Fontaine's sense, and a perfectoid ring (in Fontaine's sense) defined over a perfectoid field K is none other than a perfectoid K-algebra in Scholze's original sense. Moreover, one can generalize the notion of perfectoid spaces using Fontaine's perfectoid rings, but the resulting geometric object (which is an adic space in fact) amounts to a "perfectoid space" without a fixed base (indeed, such an adic space over a perfectoid field is actually a perfectoid space in Scholze's original sense). Thus we call it just a *perfectoid space* in the sequel (and in Theorem A).

By definition, a perfectoid space is an adic space. Originally, the category of adic spaces is introduced by Huber to give a generalization of rigid analytic varieties ([31]). An adic space has a structure that is finer than a locally ringed space: it is a datum $(X, \mathcal{O}_X, \mathcal{O}_X^+, \{v_x\}_{x\in X})$ consisting of a topological space X, two sheaves of topological rings $\mathcal{O}_X$ and $\mathcal{O}_X^+$ together with a morphism $\mathcal{O}_X^+ \hookrightarrow \mathcal{O}_X$ such that $(X, \mathcal{O}_X)$ and $(X, \mathcal{O}_X^+)$ are topologically locally ringed spaces, and a set of (semi)valuations $\{v_x\}_{x\in X}$ with an additional property. Such a quadruple is called a *valued triple*. In [20], Fujiwara and Kato established a natural framework to associate an arbitrary rigid space $\mathcal{X}$ to a valued triple $\mathbf{ZR}(\mathcal{X})$ functorially.

0.2.2 Tilting equivalence without a fixed base

In this thesis, we also develop the theory of Fontaine's perfectoid rings. As mentioned above, these topological rings induce perfectoid spaces without a fixed base. To establish the tilting equivalence in this framework, we introduce the notion of *distinguished ideal sheaves*. This globalizes Kedlaya-Liu's tilting equivalence, in which *primitive elements* of degree 1 plays a key role ([35, Theorem 3.6.5]).

[1] $(R_\infty)_p^\wedge$ is also applied to a study of purity for the Brauer groups in [9].

Let $\mathcal{C}$ be the category of perfectoid spaces X with $p \in \Gamma(X, \mathcal{O}_X)$ topologically nilpotent. Let $\mathcal{D}$ be the category of pairs $(Y, \mathcal{I})$ consisting of a perfectoid space Y of characteristic $p > 0$ and a distinguished ideal sheaf $\mathcal{I}$ on Y. We prove the following result.

Theorem B (Theorem 3.2.5)**.** *For any $X \in \mathrm{ob}(\mathcal{C})$, there exists unique (up to canonical isomorphisms) $(X^\flat, \mathcal{I}_X) \in \mathrm{ob}(\mathcal{D})$ with a functorial bijection*

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{Spa}(A, A^+), X) \cong \mathrm{Hom}_{\mathcal{D}}((\mathrm{Spa}(A^\flat, A^{+\flat}), \mathrm{Ker}(\theta_{A^+})^\sim), (X^\flat, \mathcal{I}_X))$$

for any perfectoid affinoid ring (A, A^+) with $p \in A^+$ topologically nilpotent. Moreover, the association $X \mapsto (X^\flat, \mathcal{I}_X)$ gives an equivalence of categories $\mathcal{C} \xrightarrow{\cong} \mathcal{D}$.

We call $X^\flat$ the *tilt* of X. We denote by $X_{\text{ét}}$ the étale site of X. As a generalization of Theorem 0.1.3, we prove the following. This is a consequence of the almost purity theorem for Fontaine's perfectoid rings.

Theorem C (Theorem 3.3.11)**.** *Let X be a perfectoid space and $X^\flat$ the tilt of X. Then tilting induces an equivalence of sites $X_{\text{ét}} \cong X^\flat_{\text{ét}}$.*

0.2.3 Decompletion of perfectoid rings

Next we discuss (**II**). When André announced the results in [2], not much was known for construction of big Cohen-Macaulay algebras (in mixed characterisitic) with distinguished properties. In 2017, it was proved that a big Cohen-Macaulay algebra constructed in [2] could be refined to be an *integral perfectoid* (cf. Definition 2.1.6) big Cohen-Macaulay algebra ([52]), but the following question (that had been asked earlier by P. Roberts in [46] and [47] implicitly) was open.

Question 0.2.1 (Roberts)**.** *Let $(R, \mathfrak{m})$ be a complete Noetherian local domain with its absolute integral closure R^+. Then does there exist an R-algebra B such that B is an almost Cohen-Macaulay R-algebra (cf. Definition A.0.2) and $R \subset B \subset R^+$?*

In André's construction, the obstruction to answer Question 0.2.1 lies in the process of taking p-adic completion: integrality of a ring extension is not preserved under completion in general. Namely, the difficulty arises when one applies the original version of Perfectoid Abhyankar's Lemma, because this theorem is valid only for perfectoid K-algebras, which are necessarily complete by definition. The secondary aim in this thesis is to weaken the p-adic completeness required in Perfectoid Abhyankar's Lemma. Using Zariskian rings, we establish a decompleted variant of this theorem as follows.

Theorem 0.2.2 (Nakazato-Shimomoto, Theorem 4.3.3)**.** *Let V be a Witt-perfect valuation ring of rank 1 admitting a compatible system of p-power roots $\{p^{\frac{1}{p^n}}\}_{n \geq 0}$. Let A be a p-torsion free flat V-algebra together with a non-zerodivisor $g \in A$ admitting a compatible system of p-power roots $\{g^{\frac{1}{p^n}}\}_{n \geq 0}$. Let $A[\frac{1}{pg}] \hookrightarrow B'$ be a finite étale extension, and B the integral closure of $g^{-\frac{1}{p^\infty}}A$ in B'. Suppose that*

1. *A is p-adically Zariskian and completely integrally closed in $A[\frac{1}{pg}]$;*

2. *A is $(pg)^{\frac{1}{p^\infty}}$-almost Witt-perfect that is integral over a Noetherian ring;*

3. (p, g) is a regular sequence on A.

Then the following assertions hold.

(a) *B is also $(pg)^{1/p^\infty}$-almost Witt-perfect.*

(b) *Assume that A is a normal ring that is torsion free and integral over a Noetherian normal domain. Then the induced map $A/(p^m) \to B/(p^m)$ is $(pg)^{\frac{1}{p^\infty}}$-almost finite étale for all $m > 0$.*

Here, *Witt-perfectness* is a condition that was introduced by Davis and Kedlaya ([11]) in a study of the Frobenius map on the Witt vectors of a ring in arbitrary characteristic. In our situation, roughly speaking, a Witt-perfect ring is viewed as an algebraization of the integral structure of a perfectoid ring (see Lemma 4.1.4 for details). While the proof of Pefectoid Abhyankar's Lemma requires the almost purity of perfectoid K-algebras as a crucial ingredient, fortunately the almost purity theorem for Witt-perfect rings is already established by Davis and Kedlaya ([11] and [12]).

Assuming p-adic completeness in our situation brings us to the original version of Perfectoid Abhyankar's Lemma. The idea of the proof of Theorem 0.2.2 is to reconstruct André's original proof in terms of Zariskian rings. Let us recall his strategy. In the case when g is the unity, Perfctoid Abhyankar's Lemma is none other than the almost purity theorem for perfectoid algebras, and so the essential part is to cope with singularity on the locus $g = 0$. For $\varepsilon > 0$, André considered the ring A_ε of bounded functions on the complement of ε-tubular neighborhood of $g = 0$. Then by the *Riemann's extension theorem* for perfectoid K-algebras ([1, Théorème 4.2.2]), he deduced that A is $(pg)^{1/p^\infty}$-almost equal to $\varprojlim_{\varepsilon > 0} A_\varepsilon$. To translate this step into our language, we establish an algebraic variant of the extension theorem as follows. It also plays a crucial role in our proof like the original version.

Theorem D (Theorem 4.2.1)**.** *Let A_0 be a ring with a non-zerodivisor ϖ that is ϖ-adically Zariskian and integral over a Noetherian ring. Let $g \in A_0$ be a non-zerodivisor. Let A^j be the Tate ring associated to $(A_0[\frac{\varpi^j}{g}], (\varpi))$ for every integer $j > 0$. Then we have an isomorphism of rings*

$$(A_0)^+_{A_0[\frac{1}{\varpi g}]} \xrightarrow{\cong} \varprojlim_j A^{j\circ}$$

where the transition map $A^{j+1\circ} \to A^{j\circ}$ is the natural one.

Notice that both $(A_0)^+_{A_0[\frac{1}{\varpi g}]}$ and $A^{j\circ}$ appear as integral structures of the same ring $A_0[\frac{1}{\varpi g}]$. To prove Theorem D, we describe these rings in terms of (semi)valuations of rank 1 on $A_0[\frac{1}{\varpi g}]$. The key ingredient is Proposition 1.6.13, which is closely related to a technique used in classical rigid geometry. This is also an analogue of the well-known fact in commutative algebra: an integrally closed domain A is the intersection of all valuation domains that lie between A and the field of fractions.

Let us return to Question 0.2.1. First of all, we remark that if R is of characteristic $p > 0$, then it is known that R^+ is a big Cohen-Macaulay R-algebra. This was proved by Hochster and Huneke (see [27], [28], [29], [33], [44] and [48]). If R has mixed characteristic of dimension 3, Heitmann proved that R^+ is a $(p)^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay R-algebra; see [24]. As an application of Theorem 0.2.2, Shimomoto extended Heitmann's result to the higher dimensional case, and gave a positive answer to Question 0.2.1 in a quite satisfactory form.

Theorem 0.2.3 (Shimomoto, Theorem A.0.5)**.** *Let $(R, \mathfrak{m})$ be a complete Noetherian local domain of mixed characteristic $p > 0$ with perfect residue field k. Let $p, x_2, \ldots, x_d$ be a system of parameters and let R^+ be the absolute integral closure of R. Then there exists an R-algebra T together with a nonzero element $g \in R$ such that the following hold:*

1. *T admits compatible systems of p-power roots $p^{\frac{1}{p^n}}, g^{\frac{1}{p^n}} \in T$ for all $n > 0$.*
2. *The Frobenius endomorphism $Frob : T/(p) \to T/(p)$ is surjective.*
3. *T is a $(pg)^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay normal domain with respect to $p, x_2, \ldots, x_d$ and $R \subset T \subset R^+$.*

Furthermore, one can arrange T such that $R[\frac{1}{pg}] \to T[\frac{1}{pg}]$ is a filtered colimit of finite étale $R[\frac{1}{pg}]$-algebras contained in $T[\frac{1}{pg}]$.

In other words, one can find an almost Cohen-Macaulay, Witt-perfect normal domain (its p-adic completion is integral perfectoid) between R and its absolute integral closure. Using Hochster's partial algebra modification and tilting, one can construct an integral perfectoid big Cohen-Macaulay R-algebra over T; see [52] for details.

0.3 Organization

In Chapter 1, we review or prove several fundamental results for later use, especially on integrality, semiperfect $\mathbb{F}_p$-algebras, and non-archimdean geometry. This chapter also contains some new results that are important for our purpose. In particular, Proposition 1.6.13 and Corollary 1.6.14 are applied to prove Theorem D, and Proposition 1.7.19 forms a part of the proof of Theorem A.

In Chapter 2, we develop the theory of Fontaine's perfectoid rings. The main goal of this section is to establish the tilting equivalence (Theorem 2.3.1) and the almost purity theorem (Theorem 2.4.1) for Fontaine's perfectoid rings (the proof of Theorem 2.4.1 will be completed in Chapter 3). We remark that most of the results in this chapter have already been proved in Gabber-Ramero's treatise ([22]).

In Chapter 3, as a globalization of the results in Chapter 2, we study perfectoid spaces without a fixed space. The first goal of this chapter is to establish the tilting equivalence for perfectoid spaces (Theorem 3.2.5) and an equivalence of étale sites under tilting (Theorem 3.3.11). As an example that this theory can be applied to, we then construct a Zariski-Riemann triple from a complete regular local ring, and prove that it is a perfectoid space (Theorem 3.4.1). Theorem B, C, and A correspond these results respectively.

In Chapter 4, we discuss decompletion of perfectoid rings, especially pefectoid K-algebras. We compare a class of "decompleted" perfectoid rings with Davis-Kedlaya's Witt-perfect rings, and then prove the main theorem in this section, Witt-perfect Abhyankar's lemma (Theorem 4.3.3). As a key step for this, we establish an algebraic variant of Riemann's extension theorem (Theorem 4.2.1, or Theorem D).

0.4 Convention and notations

- All rings are assumed to be commutative and contain the unity. Moreover, all ring homomorphisms are assumed to be unital.

- For a category $\mathcal{C}$, we denote by $\mathrm{ob}(\mathcal{C})$ (resp. $\mathrm{hom}(\mathcal{C})$) the class of objects (resp. morphisms) in $\mathcal{C}$.
- For a ring A, we denote by A-**Mod** (resp. A-**Alg**) the category of A-modules (resp. A-algebras).
- Unless otherwise stated, we denote by p a fixed prime number.
- For a ring A, we denote by $A[T^{1/p^\infty}]$ the direct limit $\varinjlim_{n\geq 0} A[T^{1/p^n}]$ where $A[T]$ is the polynomial ring and each $A[T^{1/p^n}]$ $(n \geq 1)$ is defined recursively as

$$A[T^{1/p^n}] = A[T^{1/p^{n-1}}][X]/(X^p - T^{1/p^{n-1}}) \ .$$

- For a ring A and an element $t \in A$, we denote by $\{t^{1/p^n}\}_{n\geq 0}$ a system of elements in A such that $(t^{1/p^{n+1}})^p = t^{1/p^n}$ for every $n \geq 0$, and call it a *compatible system of* p*-power roots* of t.
- For a ring A, and A-module M, and a non-zerodivisor $t \in A$ that admits a compatible system of p-power roots $\{t^{1/p^n}\}_{n\geq 0}$, we denote by $t^{-1/p^\infty}M$ the A-submodule of $M[\frac{1}{t}]$ defined as

$$t^{-1/p^\infty}M := \{x \in M[\tfrac{1}{t}] \mid t^{1/p^n}x \in M \text{ for every } n \geq 0\} \ .$$

- Unless otherwise stated, we mean by a *pair* a pair (A, I) consisting of a ring A and an ideal $I \subset A$. A *morphism of pairs* $(A, I) \to (B, J)$ is a ring homomorphism $f : A \to B$ such that $I^n \subset f^{-1}(J)$ for some $n > 0$.
- For a pair (A, I) such that I admits a system of ideals $\{I_n\}_{n\geq 0}$ in A with $I_0 = I$ and $I_{n+1}^p = I_n$ for every $n \geq 0$, we denote by I^{1/p^∞} the ideal $\bigcup_{n\geq 0} I_n$ in A.
- Let A be a ring, and M an A-module equipped with an A-linear topology defined by a descending filtration of A-submodules $\{F_\lambda\}_{\lambda\in\Lambda}$. We say that M is *Hausdorff complete* (resp. *Hausdorff*, resp. *complete*) if the natural map $M \to \varprojlim_\lambda M/F_\lambda$ is bijective (resp. injective, resp. surjective).
- For a pair (A, I) and an A-module M, we denote by $(M)^\wedge_I$ the I-adic completion of M. If I is a principal ideal generated by $t \in A$, then $(M)^\wedge_I$ is also denoted by $(M)^\wedge_t$.
- For a pair (A, I) and an A-module M, we denote by $M_{I\text{-tor}}$ the submodule of M consisting of all elements $x \in M$ such that $I^l x$ is the zero module for some $l > 0$.
- We say that a ring A is *normal* if for any prime ideal $\mathfrak{p} \subset A$ the localization $A_\mathfrak{p}$ is an integral domain that is integrally closed in its field of fractions.

Chapter 1

Preliminaries

In this chapter, we review or prove several fundamental results for later use, especially on integrality, semiperfect $\mathbb{F}_p$-algebras, and non-archimdean geometry. We remark that this chapter contains some new results that are important for our purpose. Especially, Proposition 1.6.13 and Corollary 1.6.14 are applied to prove the weak extension theorem (Theorem 4.2.1), and Proposition 1.7.19 forms a part of the proof of Theorem 3.4.1.

1.1 Almost mathematics

A *basic setup* is a pair $(V, \mathfrak{m})$ consisting of a ring V and an ideal $\mathfrak{m} \subset V$ such that $\mathfrak{m}^2 = \mathfrak{m}$. Let $(V, \mathfrak{m})$ be a basic setup. We say that a V-module M is $\mathfrak{m}$-*almost zero* if $\mathfrak{m}M = 0$. For a V-linear map $f : M \to N$, we say that f is $\mathfrak{m}$-*almost injective* (resp. $\mathfrak{m}$-*almost surjective*) if the kernel (resp. the cokernel) of f is $\mathfrak{m}$-almost zero. Moreover, we say that f is $\mathfrak{m}$-*almost isomorphic* if f is $\mathfrak{m}$-almost injective and $\mathfrak{m}$-almost surjective.

The full subcategory Σ of V-**Mod** consisting of all $\mathfrak{m}$-almost zero modules is a Serre subcategory. We denote by V^a-**Mod** the quotient category V-**Mod**$/\Sigma$, and by $\cdot^a$ the natural localization functor (i.e. for any V-module M, M^a is the same module regarded as an object of V^a-**Mod**).

Example 1.1.1.

1. Let V be an arbitrary ring and $\mathfrak{m} \subset V$ the unit ideal. Then $(V, \mathfrak{m})$ is a basic setup. With respect to this basic setup, the quotient category V^a-**Mod** is none other than V-**Mod**.

2. Let A_0 be a ring with a sequence of non-zerodivisors $\{t_n\}_{n \geq 0}$ such that A_0 is integrally closed in $A := A_0[\frac{1}{t_0}]$ and assume that for every $n \geq 0$ we have $t_{n+1}^p = t_n u_n$ for some unit $u_n \in A_0^\times$. Denote by I the ideal $\sqrt{(t_0)} \subset A_0$. Then the pair (A_0, I) is a basic setup. Let us observe it. Pick $x \in I$. Then we have $x^{p^m} = t_0 a$ for some $m > 0$ and $a \in A_0$. Thus, since $t_m^{p^m} = t_0 u$ for some unit $u \in A_0$, an equality $(\frac{x}{t_m})^{p^m} = au^{-1}$ holds in A. Hence x lies in $t_m A_0$, because A_0 is integrally closed in A. Consequently, we have $I = \varinjlim_n t_n A_0$. Therefore, we find that $I = I^p$. Moreover, I is flat over A_0.

Let $\widetilde{\mathfrak{m}}$ denote the V-module $\mathfrak{m} \otimes_V \mathfrak{m}$. For any V-modules M and N, there is a functorial identification

$$\mathrm{Hom}_{V^a\text{-}\mathbf{Mod}}(M^a, N^a) = \mathrm{Hom}_V(\widetilde{\mathfrak{m}} \otimes_V M, N) \tag{1.1}$$

by [21, §2.2.2]. Thus one can check that V^a-**Mod** is an abelian monoidal category with tensor products defined as

$$M^a \otimes N^a = (M \otimes_V N)^a \ .$$

Then an *almost V-algebra* is defined as a commutative unitary monoid in V^a-**Mod** (cf. [21, §2.2.6]). Thus one obtains the category of almost V-algebras V^a-**Alg** so that the localization functor $\cdot^a : V\text{-}\mathbf{Mod} \to V^a\text{-}\mathbf{Mod}$ restricts to a functor V-**Alg** $\to V^a$-**Alg**. Moreover, for any V-algebra R, the notion of R^a-module is naturally defined (cf. [21, §2.2.6 and §2.2.7]), and then $\cdot^a$ also gives a functor from R-**Mod** to the category of R^a-modules R^a-**Mod**.

For any V^a-module M, define $M_* := \mathrm{Hom}_{V^a\text{-}\mathbf{Mod}}(V^a, M)$. Then by (1.1), we obtain a functor

$$V^a\text{-}\mathbf{Mod} \to V\text{-}\mathbf{Mod},\ M \mapsto M_*$$

such that $M \cong (M_*)^a$. Moreover, for any V^a-algebra A, A_* naturally admits a structure as a V-algebra (cf. [21, §2.2.9]), and we obtain the functor V^a-**Alg** $\to V$-**Alg**, $A \mapsto A_*$. Then the isomorphism $A \cong (A_*)^a$ gives an isomorphism of almost V^a-algebras ([21, Proposition 2.2.14(i)]). Furthermore, these functors $\cdot_*$ are the right adjoints of the localization functors $\cdot^a$ ([21, Proposition 2.2.14(ii)]).

In the sequel, we always assume that $\widetilde{\mathfrak{m}}$ is a flat V-module. For example, this assumption is realized if $\mathfrak{m}$ is a filtered union of principal ideals ([21, Proposition 2.1.7]) like $\sqrt{(t_0)}$ appeared in Example 1.1.1 2.

Several ring theoretic notions are naturally extended to the framework of almost mathematics. Originally they are defined in the categories R^a-**Mod** and V^a-**Alg**, but can be translated into languages on R-**Mod** and V-**Alg**, as follows.

Definition 1.1.2 (cf. [21, Proposition 2.3.10 and Remark 2.4.12])**.** Fix a basic setup $(V, \mathfrak{m})$. Let A be a V-algebra, and M an A-module.

1. We say that M is $\mathfrak{m}$-*almost flat* (or M^a is flat), if $\mathrm{Tor}_i^A(M, N)$ is $\mathfrak{m}$-almost zero for every A-module N and every $i > 0$.

2. We say that M is $\mathfrak{m}$-*almost projective* (or M^a is almost projective), if $\mathrm{Ext}_A^i(M, N)$ is $\mathfrak{m}$-almost zero for every A-module N and every $i > 0$.

3. We say that M is $\mathfrak{m}$-*almost finitely generated* (or M^a is finitely generated), if for every $\varepsilon \in \mathfrak{m}$ there exists some finitely generated A-module M_ε together with an A-linear map $M_\varepsilon \to M$ whose cokernel is annihilated by ε.

4. We say that M is $\mathfrak{m}$-*almost finitely presented* (or M^a is finitely presented), if for every $\varepsilon \in \mathfrak{m}$ there exists some finitely presented A-module M_ε together with an A-linear map $M_\varepsilon \to M$ whose kernel and cokernel are annihilated by ε.

Definition 1.1.3. Fix a basic setup $(V, \mathfrak{m})$ and let $A \to B$ be an V-algebra homomorphism.

1. We say that $A \to B$ is $\mathfrak{m}$-*almost flat* (resp. $\mathfrak{m}$-*almost projective*) if B is an $\mathfrak{m}$-almost flat ($\mathfrak{m}$-almost projective) A-module.

2. We say that $A \to B$ is $\mathfrak{m}$-*almost unramified* (or $A^a \to B^a$ is unramified) if the multiplication map $B \otimes_A B \to B$ is $\mathfrak{m}$-almost projective.

3. We say that $A \to B$ is $\mathfrak{m}$-*almost étale* (or $A^a \to B^a$ is étale) if $A \to B$ is $\mathfrak{m}$-almost flat and $\mathfrak{m}$-almost unramified.

4. We say that $A \to B$ is $\mathfrak{m}$-*almost finite étale* (or $A^a \to B^a$ is finite étale) if $A \to B$ is $\mathfrak{m}$-almost étale and B is an $\mathfrak{m}$-almost finitely presented A-module.

With respect to a special basic setup, an A-linear map that is almost isomorphic induces an isomorphism of A-modules. We use the following lemma in the proof of Proposition 2.2.2.

Lemma 1.1.4. *Let A be a ring with a nonzero divisor $\pi \in A$ such that A admits a compatible system of p-power roots $\{\pi^{1/p^n}\}_{n\geq 0}$. Consider the basic setup $(A, (\pi)^{1/p^\infty})$ (see §0.4 for notation).*

1. *Let M be a π-torsion free A-module. Then we have an identification*
$$(M^a)_* = \pi^{-1/p^\infty} M$$
(see §0.4 for notation).

2. *Let $f : M \to N$ be an A-linear map that is $(\pi)^{1/p^\infty}$-almost isomorphic. Then the induced A-linear map $f' : ((M/M_{\pi\text{-tor}})^a)_* \to ((N/N_{\pi\text{-tor}})^a)_*$ is an isomorphism.*

Proof. 1: The A-linear map
$$M \to \mathrm{Hom}_A((\pi)^{1/p^\infty}, M),\ m \mapsto (x \mapsto xm)$$
is injective and becomes an isomorphism after base extension along $A \to A[\frac{1}{\pi}]$. Moreover, an element $f \in \mathrm{Hom}_{A[\frac{1}{\pi}]}(A[\frac{1}{\pi}], M[\frac{1}{\pi}])$ $(= M[\frac{1}{\pi}])$ come from $\mathrm{Hom}_A((\pi)^{1/p^\infty}, M)$ if and only if $\pi^{1/p^n} f(a) \in M$ for every $n \geq 0$ and every $a \in A$. Thus, since $\widetilde{\mathfrak{m}} \cong \mathfrak{m}$ for $\mathfrak{m} = (\pi)^{1/p^\infty}$, the assertion follows in view of (1.1).

2: Since the map $M/M_{\pi\text{-tor}} \to N/N_{\pi\text{-tor}}$ induced by f is also $(\pi)^{1/p^\infty}$-almost isomorphic, we may assume that M and N are π-torsion free. Then $(M^a)_* = \pi^{-1/p^\infty} M$ and $(N^a)_* = \pi^{-1/p^\infty} N$ by the assertion 1. Thus it is easily seen that f and f' are injective. Let us show that f' is surjective. Pick $x \in \pi^{-1/p^\infty} N$. Then for every $n \geq 0$, there exists some $y_n \in M$ such that $\pi^{1/p^n} x = f(y_n)$. Since f is injective, we find that $\pi^{-1/p^n} y_n = \pi^{-1/p^{n+1}} y_{n+1}$ for every $n \geq 0$. Thus, $y := \pi^{-1} y_0$ belongs to $\pi^{-1/p^\infty} M$ and satisfies $f'(y) = x$. Hence the assertion follows. □

As is easily checked, for a ring A and a non-zerodivisor $t \in A$, any flat A-module M is t-torsion free. The following lemma gives a variant of this statement. It is used for proving the almost purity theorem (cf. Lemma 2.4.2 1).

Lemma 1.1.5 ([49, Lemma 5.3(i)])**.** *Let $(V, \mathfrak{m})$ be a basic setup and let A_0 be a V-algebra. Assume that $\mathfrak{m}$ admits a sequence of elements $\{t_n\}_{n\geq 0}$ such that A_0 is t_0-torsion free, $\mathfrak{m}A_0$ is generated by the set of all $t_n \in A_0$ $(n \geq 0)$ and for every $n \geq 0$ we have $t_{n+1}^{k_n} = t_n u_n$ for some $k_n > 0$ and some unit $u_n \in A_0^\times$. Put $A := A_0[\frac{1}{t_0}]$. Let M_0 be an $\mathfrak{m}$-almost flat A_0-module. Then $(M_0^a)_*$ is t_0-torsion free.*

Proof. By the assumption on $\mathfrak{m}$ and A_0, any $\mathfrak{m}$-almost zero element in $\mathrm{Hom}_V(\mathfrak{m}, M_0)$ must be zero. Thus the argument in the proof of [49, Lemma 5.3(i)] still works under our setting. □

The following lemma is a generalization of a well-known fact on adically complete modules ([38, Theorem 8.4]). We use this for characterizing *almost perfectoid algebras* (cf. Lemma 1.3.19, Lemma 2.3.7, and Definition 2.3.8).

Lemma 1.1.6. *Let (A, I) be a pair with I finitely generated. Let $f : M \to N$ be a homomorphism between $A[T^{1/p^\infty}]$-modules. Suppose that M is I-adically complete, N is I-adically Hausdorff, and the homomorphism $\overline{f} : M/IM \to N/IN$ induced from f is $(T)^{\frac{1}{p^\infty}}$-almost surjective. Then f is also $(T)^{\frac{1}{p^\infty}}$-almost surjective.*

Proof. Pick $n \in N$ and $h \in \mathbb{Z}[\frac{1}{p}] \cap \mathbb{Q}_{>0}$. It suffice to find some $m \in M$ such that $f(m) = T^h n$. To carry out this, we prove the following key claim.

Claim 1.1.7. *There exists a sequence of elements $\{m_i\}_{i\geq 0}$ in M such that for every $i \geq 0$ we have $m_{i+1} \equiv m_i \mod I^{i+1}$ and*

$$T^h n = f(m_i) + x_i T^{h_i} n_i \tag{1.2}$$

for some $n_i \in N$, $x_i \in I^i$, and $h_i \in \mathbb{Z}[\frac{1}{p}] \cap \mathbb{Q}_{\geq 0}$.

Proof of Claim 1.1.7. We can take $0 \in M$ as a candidate for m_0. Pick an arbitrary $i \geq 0$ and suppose that (1.2) holds for this i. Then by assumption, there exist some $m'_{i+1} \in M$, $n_{i+1} \in N$, and $x \in I$ such that $T^{h_i/p} n_i = f(m'_{i+1}) + x n_{i+1}$. Hence we have

$$T^h n = f(m_i + x_i T^{h_i(1-p^{-1})} m'_{i+1}) + x x_i T^{h_i(1-p^{-1})} n_{i+1} \ .$$

Hence by putting $m_{i+1} := x_i T^{h_i(1-p^{-1})} m'_{i+1}$, $x_{i+1} := x x_i$, and $h_{i+1} := h_i(1 - p^{-1})$, we find that $m_{i+1} \equiv m_i \mod I^{i+1}$ and (1.2) holds for $i + 1$. Hence the assertion follows. □

Equip M and N with the I-adic topologies. Then, since M is complete, there is some $m \in M$ to which $\{m_i\}_{i\geq 0}$ in the claim converges. Then by (1.2), we have

$$f(m) \equiv f(m_i) \equiv T^h n \mod I^i N$$

for every $i \geq 0$. Hence $f(m) = T^h n$ because N is Hausdorff, as wanted. □

1.2 Integrality and almost integrality

Here we discuss the notion of integrality and almost integrality.

Definition 1.2.1. Let $A \subset B$ be a ring extension.

1. An element $b \in B$ is *integral* over A, if $\sum_{n=0}^{\infty} A \cdot b^n$ is a finitely generated A-submodule of B. The set of all elements, denoted as C, of B that are integral over A forms an A-subalgebra of B. If $A = C$, then A is called *integrally closed* in B.

2. An element $b \in B$ is *almost integral* over A, if $\sum_{n=0}^{\infty} A \cdot b^n$ is contained in a finitely generated A-submodule of B. The set of all elements, denoted as C, of B that are almost integral over A forms an A-subalgebra of B, which is called the *complete integral closure* of A in B. If $A = C$, then A is called *completely integrally closed* in B.

3. We say that A is *p-root closed* in B, if any element $b \in B$ that satisfies $b^p \in A$ belongs to A.

This definition can be extended to any ring homomorphism $A \to B$ in a natural way: Let A be a ring, B an A-algebra, and $b \in B$ an element. Then we say that b is *integral* (resp. *almost integral*) over A, if b is integral (resp. almost integral) over the image of A in B. Unlike integral closure, the complete integral closure of an integral domain in its field of fractions is not necessarily completely integrally closed in the same field of fractions; see [23] for such examples.

Notation: For a ring A and an A-algebra B, we denote by A^+_B (resp. A^*_B) the A-subalgebra of B consisting of all elements that are integral (resp. almost integral) over A.

Let us describe the relationship between A^+_B and A^*_B in a special case.

Lemma 1.2.2. *Let A_0 be a ring with a non-zerodivisor t. Let B_0 be a t-torsion free A_0-algebra such that the induced $A_0[\frac{1}{t}]$-algebra $B_0[\frac{1}{t}]$ is module-finite. Then we have $t(A_0)^*_{B_0[\frac{1}{t}]} \subset (A_0)^+_{B_0[\frac{1}{t}]}$.*

Proof. Put $A := A_0[\frac{1}{t}]$, $B := B_0[\frac{1}{t}]$ and $B'_0 := (A_0)^+_B$. Pick $b \in (A_0)^*_B$. Then there exists a finitely generated A_0-submodule $N_0 \subset B$ such that b^n belongs to N_0 for every $n > 0$. On the other hand, since B is module-finite over A, we have $B = B'_0[\frac{1}{t}]$. Thus, $t^l N_0$ is contained in B'_0 for some $l > 0$. In particular, $(tb)^l$ $(= t^l b^l)$ lies in B'_0 and therefore, so does tb by the definition of B'_0. Hence tb is integral over A_0, as desired. □

Next we observe the interrelation of the three notions of "integral closedness" given in Definition 1.2.1 in the following situation.

Lemma 1.2.3. *Let A_0 be a ring with a non-zerodivisor $t \in A_0$. Fix a prime number $p > 0$. Assume that A_0 admits a compatible system of p-power roots $\{t^{\frac{1}{p^n}}\}_{n \geq 0}$. Then the following assertions hold.*

1. *The following conditions are equivalent.*
   - (a) *A_0 is p-root closed in $A_0[\frac{1}{t}]$, and $t^{-1/p^\infty} A_0 = A_0$.*
   - (b) *A_0 is integrally closed in $A_0[\frac{1}{t}]$, and $t^{-1/p^\infty} A_0 = A_0$.*
   - (c) *A_0 is completely integrally closed in $A_0[\frac{1}{t}]$.*
2. *If A_0 is p-root closed in A, then $(A_0)^*_A = t^{-1/p^\infty} A_0$.*
3. *Assume that $p \in t^p A_0$. Then A_0 is p-root closed in $A_0[\frac{1}{t}]$ if and only if the kernel of the Frobenius map $\Phi_{A_0/(t^p)}$ is generated by $\bar{t} \in A_0/(t^p)$.*

Proof. 1: Since $(b) \Rightarrow (a)$ is clear, it is enough to prove the implications $(a) \Rightarrow (c)$ and $(c) \Rightarrow (b)$. We first assume (a). Pick $x \in A_0[\frac{1}{t}]$ that is almost integral over A_0. Then there is some $l > 0$ such that $t^l x^{p^n} \in A_0$ for every $n > 0$. Hence by assumption, we have $t^{l/p^n} x \in A_0$ for every $n \geq 0$. Therefore we find $x \in t^{-1/p^\infty} A_0 = A_0$, and so we obtain the implication $(a) \Rightarrow (c)$. Next we assume (c). Pick an element $y \in t^{-\frac{1}{p^\infty}} A_0$. Then for every $n > 0$, there exists $a_n \in A_0$ such that $t^{\frac{1}{p^n}} y = a_n$ and therefore, $y^n = t^{-1}(t^{\frac{1}{p^n}})^{p^n - n} a_n^n \in A_0[\frac{1}{t}]$ (here note that $(t^{\frac{1}{p^n}})^{p^n - n} a_n^n \in A_0$). Thus, it holds that $ty^n \in A_0$ for every $n > 0$. Hence we have $y \in A_0$ by assumption, which implies $t^{-1/p^\infty} A_0 = A_0$. Therefore the implication $(c) \Rightarrow (a)$ holds, as wanted.

2: $t^{-1/p^\infty}A_0$ is p-root closed in A. Hence by the assertion 1, $(t^{-1/p^\infty}A_0)^*_A = t^{-1/p^\infty}A_0$. On the other hand, we have $(A_0)^*_A = (t^{-1/p^\infty}A_0)^*_A$ because $t(t^{-1/p^\infty}A_0) \subset A_0$. Hence the assertion follows.

3: Assume that $\mathrm{Ker}(\Phi_{A_0/(t^p)})$ is generated by $\overline{t} \in A_0/(t^p)$. Pick $x \in A_0[\frac{1}{t}]$ for which $x^p \in A_0$. Let k be a positive integer for which $t^k x \in A_0$. Then we have $\Phi_{A_0/(t^p)}(\overline{t^k x}) = \overline{0}$, and therefore $t^k a \in tA_0$ by assumption. Hence it holds that $t^{k-1}x \in A_0$. Thus by induction we find $x \in A_0$, which implies that A_0 is p-root closed in $A_0[\frac{1}{t}]$. □

Notice that complete integral closedness has the following property like integral closedness.

Lemma 1.2.4. *Let $A \subset B \subset C$ be ring extensions. Assume that A is completely integrally closed in C. Then A is also completely integrally closed in B.*

Proof. For $b \in B$, assume that $\sum_{n=0}^{\infty} A \cdot b^n$ is contained in a finitely generated A-submodule of B. Then this property remains true when regarded as an A-submodule of C. So we have $b \in A$ by our assumption. □

Now we study the relation between the operation $(\,\cdot\,)^+_{\cdot}$ (or $(\,\cdot\,)^*_{\cdot}$) and Hausdorff completion.

Lemma 1.2.5. *Let A_0 be a ring with a non-zerodivisor t. Denote by $\widehat{A_0}$ the t-adic completion of A_0. Put $A := A_0[\frac{1}{t}]$, $A' := \widehat{A_0}[\frac{1}{t}]$, $A^+ := (A_0)^+_A$, and $A^\circ := (A_0)^*_A$.*

1. *Suppose that there exists some $c \geq 0$ for which $t^c A^+ \subset A_0$ (resp. $t^c A^\circ \subset A_0$). Then the following assertions hold.*
   - (a) *One has $t^c(\widehat{A_0})^+_{A'} \subset \widehat{A_0}$ (resp. $t^c(\widehat{A_0})^*_{A'} \subset \widehat{A_0}$).*
   - (b) *Denote by $\widehat{A^+}$ and $\widehat{A^\circ}$ the t-adic completions. Then the inclusion map $A_0 \hookrightarrow A^+$ (resp. $A_0 \hookrightarrow A^\circ$) induces an isomorphism $(\widehat{A_0})^+_{A'} \xrightarrow{\cong} \widehat{A^+}$ (resp. $(\widehat{A_0})^*_{A} \xrightarrow{\cong} \widehat{A^\circ}$).*
2. *If there exists some $c \geq 0$ for which $t^c(\widehat{A_0})^+_{A'} \subset \widehat{A_0}$ (resp. $t^c(\widehat{A_0})^*_{A'} \subset \widehat{A_0}$), then one has $t^c A^+ \subset A_0$ (resp. $t^c A^\circ \subset A_0$).*

To prove this, we verify a fundamental lemma.

Lemma 1.2.6. *Let A_0 be a ring and let $I_0 \subset A_0$ be a finitely generated ideal. Let $f_0 : N_0 \hookrightarrow M_0$ be an injective homomorphism between A_0-modules. Denote by $\widehat{M_0}$ and $\widehat{N_0}$ the I_0-adic completions of M_0 and N_0, respectively. Let $\widehat{f_0} : \widehat{N_0} \to \widehat{M_0}$ be the $\widehat{A_0}$-linear map induced by f_0. Assume that the cokernel of f_0 is annihilated by I_0^m for some $m \geq 0$. Then, $\widehat{f_0}$ is also injective and the cokernel of $\widehat{f_0}$ is annihilated by $I_0^m\widehat{A_0}$.*

Proof of Lemma 1.2.6. Consider the exact sequence: $0 \to N_0 \xrightarrow{f_0} M_0 \to L_0 \to 0$ with L_0 being the cokernel of f_0. By assumption, the A_0-module L_0 is killed by I_0^m. For an arbitrary $n > 0$, we have the induced exact sequence:

$$0 \to N_0/(I_0^n M_0 \cap N_0) \to M_0/I_0^n M_0 \to L_0/I_0^n L_0 \to 0. \tag{1.3}$$

Since $\varprojlim^1_n N_0/(I_0^n M_0 \cap N_0) \cong 0$ and $I_0^{m+n} M_0 \cap N_0 \subset I_0^n N_0$, the following sequence induced by (1.3) is exact:

$$0 \to \widehat{N_0} \to \widehat{M_0} \to \widehat{L_0} \cong L_0 \to 0,$$

where $\widehat{L_0}$ denotes the I_0-adic completion of L_0. In particular, the cokernel of $\widehat{f_0}$ is killed by $I_0^m\widehat{A_0}$. This yields the assertion. □

Moreover, we need the following result from [4]; see also [54, Tags 0BNR].

Lemma 1.2.7 (Beauville-Laszlo)**.** *Let A be a ring with a non-zerodivisor $t \in A$ and let $\widehat{A}$ be the t-adic completion. Then t is a non-zerodivisor of $\widehat{A}$ and one has the commutative diagram:*

$$\begin{array}{ccc} A_0 & \xrightarrow{\psi} & \widehat{A_0} \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\iota'} \\ A_0[\frac{1}{t}] & \xrightarrow[\psi_t]{} & \widehat{A_0}[\frac{1}{t}] \end{array} \tag{1.4}$$

that is cartesian. In other words, we have $A \cong A[\frac{1}{t}] \times_{\widehat{A}[\frac{1}{t}]} \widehat{A}$.

Now let us start to prove Lemma 1.2.5.

Proof of Lemma 1.2.5. We first prove the assertion 1. By assumption, $t^c A^+ \subset A_0$ (resp. $t^c A^\circ \subset A_0$) for some $c \geq 0$. Hence we have inclusions $\widehat{A_0} \subset (\widehat{A_0})^+_{A'} \subset \widehat{A^+}$ (resp. $\widehat{A_0} \subset (\widehat{A_0})^*_{A'} \subset \widehat{A^\circ}$). Thus, we also have inclusions $t^c(\widehat{A_0})^+_{A'} \subset t^c\widehat{A^+} \subset \widehat{A_0}$ (resp. $t^c(\widehat{A_0})^*_{A'} \subset t^c\widehat{A^\circ} \subset \widehat{A_0}$), which yields the assertion (a). In particular, $\{t^n\widehat{A_0}\}_{n\geq 1}$ gives a fundamental system of open neighborhoods of $0 \in (\widehat{A_0})^+_{A'}$ (resp. $0 \in (\widehat{A_0})^*_{A'}$). Hence $(\widehat{A_0})^+_{A'}$ (resp. $(\widehat{A_0})^*_{A'}$) is t-adically Hausdorff complete. Thus, by the universal property of completion (cf. [20, Proposition 7.1.9 in Chapter 0]), we obtain the A^+-linear map (resp. A°-linear map) $\widehat{A^+} \to (\widehat{A_0})^+_{A'}$ (resp. $\widehat{A^\circ} \to (\widehat{A_0})^*_{A'}$) and the composite $\widehat{A^+} \to (\widehat{A_0})^+_{A'} \hookrightarrow \widehat{A^+}$ (resp. $\widehat{A^\circ} \to (\widehat{A_0})^*_{A'} \hookrightarrow \widehat{A^\circ}$) is the identity. Therefore $(\widehat{A_0})^+_{A'} \hookrightarrow \widehat{A^+}$ (resp. $(\widehat{A_0})^*_{A'} \hookrightarrow \widehat{A^\circ}$) is an isomorphism, which yields the assertion (b).

Next we show the assertion 2. We consider the commutative diagram (4.3). Keeping the notation as above, assume that $\widehat{A_0}$ is integrally closed (resp. completely integrally closed) in $\widehat{A_0}[\frac{1}{t}]$. Pick an element $x \in A^+$ (resp. $y \in A^\circ$). Then one can check that $\psi_t(x) \in \widehat{A_0}[\frac{1}{t}]$ (resp. $\psi_t(y) \in \widehat{A_0}[\frac{1}{t}]$) is integral (resp. almost integral) over $\widehat{A_0}$, because the diagram (4.3) commutes. Hence by assumption, $\psi_t(t^c x)$ (resp. $\psi_t(t^c y)$) comes from $\widehat{A_0}$ for some $c \geq 0$. Thus, letting B_0 (resp. C_0) be the A_0-subalgebra of $A_0[\frac{1}{t}]$ generated by all elements of $t^c A^+$ (resp. $t^c A^\circ$), we find that the composite map $B_0 \hookrightarrow A_0[\frac{1}{t}] \xrightarrow{\psi_t} \widehat{A_0}[\frac{1}{t}]$ (resp. $C_0 \hookrightarrow A_0[\frac{1}{t}] \xrightarrow{\psi_t} \widehat{A_0}[\frac{1}{t}]$) factors through $\widehat{A_0}$. Therefore, Lemma 1.2.7 implies that $B_0 \subset A_0$ (resp. $C_0 \subset A_0$). Consequently, we have $t^c A^+ \subset A_0$ (resp. $t^c A^\circ \subset A$), as wanted. □

Corollary 1.2.8. *Keep the notation as in Lemma 1.2.5. Then $(A_0)^+_A = A_0$ (resp. $(A_0)^*_A = A_0$) if and only if $(\widehat{A_0})^+_{A'} = \widehat{A_0}$ (resp. $(\widehat{A_0})^*_{A'} = \widehat{A_0}$).*

Considering *Zariskization* (see Definition 1.2.9 below) instead of completion, we obtain a result that is similar to Lemma 1.2.5 1(b) without any assumption (Lemma 1.2.10).

Definition 1.2.9 (cf. [20, **0**, §7.3(b)] or [56, Definition 3.1])**.** Let (A, I) be a pair. Then we denote by A^{Zar}_I the localization $(1+I)^{-1}A$, and call it the *I-adic Zariskization* of A.

Lemma 1.2.10. *Let A_0 be a ring with a non-zerodivisor t. Put $A := A_0[\frac{1}{t}]$ and $A' := (A_0)^{Zar}_t[\frac{1}{t}]$. Then the inclusion map $A_0 \hookrightarrow (A_0)^+_A$ induces an isomorphism $((A_0)^{Zar}_t)^+_{A'} \xrightarrow{\cong} ((A_0)^+_A)^{Zar}_t$.*

Lemma 1.2.10 is a consequence of the following lemma.

Lemma 1.2.11. *Let $A \subset B$ be an integral ring extension. Let $I \subset A$ be an ideal. Then the following assertions hold.*

1. *The induced homomorphism $A_I^{Zar} \to B_{IB}^{Zar}$ is also integral.*
2. *Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be the filtered system of all module-finite A-subalgebras of B. Then we have a canonical isomorphism of rings $\varinjlim_\lambda (A_\lambda)_{IA_\lambda}^{Zar} \xrightarrow{\cong} B_{IB}^{Zar}$.*

Proof of Lemma 1.2.11. 1: Set $B' = B \otimes_A A_I^{Zar}$. Then the map $A_I^{Zar} \to B_{IB}^{Zar}$ is given as the composite of the integral homomorphism $A_I^{Zar} \to B'$ and the canonical B-algebra homomorphism $B' \to B_{IB}^{Zar}$. Moreover, since B' is IB'-adically Zariskian by [38, §9, Lemma 2], we have the B-algebra homomorphism $B_{IB}^{Zar} \to B'$. Since the composite $B_{IB}^{Zar} \to B' \to B_{IB}^{Zar}$ is the identity map by the universal property, the map $B' \to B_{IB}^{Zar}$ is surjective. Hence the assertion follows.

2: Since B is integral over A, we have $\varinjlim_\lambda A_\lambda = B$. For each $\lambda \in \Lambda$, the map $A_\lambda \hookrightarrow B$ induces the A_λ-algebra homomorphism $\varphi_\lambda : (A_\lambda)_{IA_\lambda}^{Zar} \to B_{IB}^{Zar}$. Hence we have the B-algebra homomorphism $\varphi : \varinjlim_\lambda (A_\lambda)_{IA_\lambda}^{Zar} \to B_{IB}^{Zar}$. Now for any $x \in IB$, there exists some $\lambda \in \Lambda$ such that $1 + x \in 1 + IA_\lambda$. Hence φ is injective. Set $C := \varinjlim_\lambda (A_\lambda)_{IA_\lambda}^{Zar}$. Then, since $A_I^{Zar} \to C$ is integral by the assertion 1, C is IC-adically Zariskian. Hence we obtain the B-algebra homomorphism $\psi : B_{IB}^{Zar} \to C_{IC}^{Zar}$, and the composite $\varphi \circ \psi$ is the identity map by the universal property. Therefore φ is surjective. Thus the assertion follows. $\square$

Proof of Lemma 1.2.10. Since integrality of a ring extension is preserved under localization, it suffices to show that $((A_0)_A^+)_t^{Zar} \cong (A_0)_A^+ \otimes_{A_0} (A_0)_t^{Zar}$. First, we have an isomorphism $\varinjlim_\lambda (A_\lambda)_t^{Zar} \xrightarrow{\cong} ((A_0)_A^+)_t^{Zar}$ by Lemma 1.2.11 2. Moreover, for each $\lambda \in \Lambda$, there exists some $m > 0$ for which $t^m A_\lambda \subset A_0$. Then, since $1 + t^{m+1}A_\lambda \subset 1 + tA_0$, we have $(A_\lambda)_t^{Zar} \cong (A_\lambda)_{t^{m+1}}^{Zar} \cong A_\lambda \otimes_{A_0} (A_0)_t^{Zar}$. Thus the assertion follows. $\square$

Lemma 1.2.12. *Let A be a normal domain with field of fractions* $\mathrm{Frac}(A)$ *and assume that* $\mathrm{Frac}(A) \hookrightarrow B$ *is an integral extension such that B is reduced. Denote by $C := A_B^+$ the integral closure of A in B. Then $C_\mathfrak{p}$ is a normal domain for any prime ideal $\mathfrak{p}$ of C.*

Proof. Notice that B can be written as the filtered colimit of finite integral subextensions $\mathrm{Frac}(A) \to B' \to B$. Without loss of generality, we may assume and do that $\mathrm{Frac}(A) \to B$ is a finite integral extension. Since $\mathrm{Frac}(A)$ is a field, B is a reduced Artinian ring, so that we can write $B = \Pi_{i=1}^m L_i$ with L_i being a field. Since $A \to C$ is torsion free and integral, we see that $\mathrm{Frac}(A) \otimes_A C$ is the total ring of fractions of C, which is just B. In other words, C has finitely many minimal prime ideals, because so does B. Then by [54, Tag 030C], C is a finite product of normal domains, which shows that $C_\mathfrak{p}$ is a normal domain for any prime ideal $\mathfrak{p} \subset C$. $\square$

1.3 Perfect rings and semiperfect rings

First we give definitions of (almost) semiperfect rings.

Definition 1.3.1 (Almost semiperfect ring)**.** Let R be an $\mathbb{F}_p$-algebra.

1. We say that R is *perfect* (resp. *semiperfect*), if the Frobenius endomorphism on R is bijective (resp. surjective).

2. Assume that R is a $\mathbb{Z}[T^{1/p^\infty}]$-algebra with a basic setup $(\mathbb{Z}[T^{1/p^\infty}], (T)^{1/p^\infty})$. We say that R is $(T)^{1/p^\infty}$-*almost semiperfect*, if the Frobenius endomorphism on R is $(T)^{1/p^\infty}$-almost surjective.

First we give several lemmas on (almost) semiperfect rings for later use. For an element t in a ring A_0, we shall say that A_0 is t-*adically Zariskian*, if t is contained in the Jacobson radical of A_0. Notice that for any ring, one can define a trivial $\mathbb{Z}[T^{1/p^\infty}]$-algebra structure by assigning the unity to each T^{1/p^n}. Over such a $\mathbb{Z}[T^{1/p^\infty}]$-algebra, $(T)^{1/p^\infty}$-almost semiperfectness is equivalent to semiperfectness.

Lemma 1.3.2. *Let A_0 be a ring with a non-zerodivisor t such that $p \in t^pA_0$. Assume that A_0 is t-adically Zariskian and $A_0/(t^p)$ is semiperfect. Then there exists a sequence $\{t_n\}_{n\geq 0}$ in A_0 such that $t_0 = t$ and for every $n \geq 0$, we have $t_{n+1}^p = t_n u_n$ for some unit $u_n \in A_0^\times$.*

Proof. We carry out the proof by induction. Put $t_0 := t$. Then, since $A_0/(t_0^p)$ is semiperfect, we find $a_0, b_0 \in A_0$ for which $a_0^p = t_0 + t_0^p b_0 = t_0(1 + t_0^{p-1}b_0)$. Here $1 + t_0^{p-1}b_0$ is a unit in A_0, because A_0 is t-adically Zariskian. Hence we can take $t_1 = a_0$. Next pick an integer $m > 0$ and assume that the assertion holds true for every $n \leq m-1$. Take $a_m, b_m \in A_0$ for which $t_m = a_m^p + t^p b_m$. Now $(t_m^{p^m}) = (t)$ as ideals by assumption and therefore, we have $c_m \in A$ for which $t = t_m c_m$. Then it holds that $a_m^p = t_m(1 + t^{p-1}b_m c_m)$ and $1 + t^{p-1}b_m c_m \in A_0^\times$. Hence we can take $t_{m+1} = a_m$, which completes the proof. □

For a surjective $\mathbb{F}_p$-algebra homomorphism $R \twoheadrightarrow S$ with S semiperfect, clearly the semiperfectness does not lift to R in general. On the other hand, in the situations we deal with later, the following assertion holds.

Lemma 1.3.3. *Let A_0 be a $\mathbb{Z}[T^{1/p^\infty}]$-algebra with a non-zerodivisor ϖ such that $p \in \varpi^p A_0$. Assume that ϖ admits a p-th root $\varpi^{\frac{1}{p}} \in A_0$. Then the following assertions hold.*

1. *$A_0/(\varpi)$ is $(T)^{1/p^\infty}$-almost semiperfect if and only if $A_0/(\varpi^p)$ is $(T)^{1/p^\infty}$-almost semiperfect.*
2. *Equip A_0 with the ϖ-adic topology and assume further that $(p) \subset A_0$ is closed and A_0 is Hausdorff complete. Then $A_0/(\varpi^p)$ is $(T)^{1/p^\infty}$-almost semiperfect if and only if $A_0/(p)$ is $(T)^{1/p^\infty}$-almost semiperfect.*

Proof. If $A_0/(\varpi^p)$ (resp. $A_0/(p)$) is $(T)^{1/p^\infty}$-almost semiperfect, then clearly so is $A_0/(\varpi)$ (resp. $A_0/(\varpi^p)$). Thus, it suffices to prove the inverse implications. Fix an arbitrary integer $n \geq 0$, and put $g^{1/p^k} := T^{1/p^k} \cdot 1 \in A_0$ for every $k \geq 0$. Assume that $A_0/(\varpi)$ is $(T)^{1/p^\infty}$-almost semiperfect. Pick an element $a \in A_0$, and put $a' := g^{\frac{1}{p^n}} a$. Then by assumption, there exist some $a_0, b_0 \in A_0$ such that $g^{\frac{p-1}{p^{n+1}}} a = a_0^p + \varpi b_0$. Multiplying both sides by $g^{\frac{1}{p^{n+1}}}$, we obtain $a' = g^{\frac{1}{p^{n+1}}} a_0^p + \varpi(g^{\frac{1}{p^{n+1}}} b_0)$. Similarly, we can find some $a_1, b_1 \in A_0$ such that $g^{\frac{1}{p^{n+1}}} b_0 = g^{\frac{1}{p^{n+2}}} a_1^p + \varpi(g^{\frac{1}{p^{n+2}}} b_1)$. This procedure yields the following assertion: if a system of elements $a_0, \ldots, a_m \in A_0$ satisfies $a' \equiv \sum_{i=0}^m g^{\frac{1}{p^{n+i+1}}} a_i^p \varpi^i \mod (g^{\frac{1}{p^{n+m+1}}} \varpi^{m+1})$, then there exists some $a_{n+1} \in A_0$ for which $a' \equiv \sum_{i=0}^{m+1} g^{\frac{1}{p^{n+i+1}}} a_i^p \varpi^i \mod (g^{\frac{1}{p^{n+m+2}}} \varpi^{m+2})$. Hence by axiom of choice, we obtain a sequence $\{a_m\}_{m\geq 0}$ in A_0 such that $a' \equiv \sum_{i=0}^m g^{\frac{1}{p^{n+i+1}}} a_i^p \varpi^i \mod (g^{\frac{1}{p^{n+m+1}}} \varpi^{m+1})$ for every $m \geq 0$. In particular, we have

$$a' \equiv \sum_{i=0}^{p-1} g^{\frac{1}{p^{n+i+1}}} a_i^p \varpi^i \equiv \left(\sum_{i=0}^{p-1} g^{\frac{1}{p^{n+i+2}}} a_i \varpi^{\frac{i}{p}}\right)^p \mod (\varpi^p),$$

which yields 1. To prove 2, we equip A_0 with the ϖ-adic topology, and assume further that $(p) \subset A_0$ is closed and A_0 is Hausdorff complete. Set $b_m := \sum_{i=0}^m g^{\frac{1}{p^{n+i+2}}} a_i \varpi^{\frac{i}{p}}$ $(m \geq 0)$ and $b := \lim_{m \to \infty} b_m \in A_0$. Then $\sum_{i=0}^m g^{\frac{1}{n+i+1}} a_i^p \varpi^i - b_m^p \in (p)$ for every m. Hence it follows that

$$a' - b^p = \lim_{m \to \infty} \sum_{i=0}^m g^{\frac{1}{n+i+1}} a_i^p \varpi^i - \lim_{m \to \infty} b_m^p = \lim_{m \to \infty} \Big(\sum_{i=0}^m g^{\frac{1}{n+i+1}} a_i^p \varpi^i - b_m^p \Big) \in (p),$$

because $(p) \subset A_0$ is a closed ideal and so 2 follows. □

Corollary 1.3.4. *Let A_0 be a ring with a non-zerodivisor ϖ such that $p \in \varpi^p A_0$ and A_0 is integrally closed in $A_0[\frac{1}{\varpi}]$. Let $g \in A_0$ be an element. Assume that ϖ and g admit compatible systems of p-power roots $\{\varpi^{1/p^n}\}$ and $\{g^{1/p^n}\}$ in A_0, respectively. Then the following conditions are equivalent.*

- (a) *The Frobenius endomorphism on $A_0/(\varpi^p)$ is $(g)^{1/p^\infty}$-almost surjective.*
- (b) *The Frobenius endomorphism on $A_0/(\varpi^p)$ is $(\varpi g)^{1/p^\infty}$-almost surjective.*

Proof. $(a) \Rightarrow (b)$ is clear. To show the converse, we assume (b). Fix an arbitrary integer $n \geq 0$, and put $\varpi^{-1/p^k} := (\varpi^{1/p^k})^{-1}$ for every $k \geq 1$. Pick $a \in A_0$. Then there exist some $b, c \in A_0$ such that $\varpi^{1/p^n} g^{1/p^n} a = b^p + \varpi^p c$ and therefore, $g^{1/p^n} a = (\varpi^{-\frac{1}{p^{n+1}}} b)^p + \varpi^{-1/p^n} \varpi^p c$. Here $\varpi^{-1/p^n} \varpi^p c$ belongs to ϖA_0. Moreover, $\varpi^{-\frac{1}{p^{n+1}}} b$ belongs to A_0, because $(\varpi^{-\frac{1}{p^{n+1}}} b)^p \in A_0$ and A_0 is integrally closed in $A_0[\frac{1}{\varpi}]$. Thus we see that the Frobenius endomorphism on $A_0/(\varpi)$ is $(g)^{1/p^\infty}$-almost surjective and Lemma 1.3.3 implies (a), as wanted. □

Lemma 1.3.3 also implies the following fact. Here notice that A^+ and A° are essentially different in general.

Lemma 1.3.5. *Let A be a Tate ring with a topologically nilpotent unit $t \in A$. Let A^+ be a $\mathbb{Z}[T^{1/p^\infty}]$-algebra that is an open and integrally closed subring of A. Put $\varpi = t^p$ and assume that $p \in \varpi^p A^+$. Then the following conditions are equivalent.*

- (a) *$A^\circ/(\varpi^p)$ is $(T)^{1/p^\infty}$-almost semiperfect.*
- (b) *$A^+/(\varpi^p)$ is $(T)^{1/p^\infty}$-almost semiperfect.*

Proof. Fix an arbitrary integer $n \geq 0$ and put $g^{1/p^n} := T^{1/p^n} \cdot 1 \in A^+$. Assume that $A^\circ/(\varpi^p)$ is $(T)^{1/p^\infty}$-almost semiperfect. Pick $a \in A^+$. Then $g^{1/p^n} a = b^p + \varpi^p c$ for some $b, c \in A^\circ$. As $\varpi c \in A^+$, we find that $b^p \in A^+$ and thus $b \in A^+$. Hence $A^+/(\varpi)$ is $(T)^{1/p^\infty}$-almost semiperfect, which implies that $A^+/(\varpi^p)$ is also $(T)^{1/p^\infty}$-almost semiperfect in view of Lemma 1.3.3. Hence $(a) \Rightarrow (b)$ holds. To show the converse, assume that $A^+/(\varpi^p)$ is $(T)^{1/p^\infty}$-almost semiperfect. Pick $d \in A^\circ$. Then $\varpi d \in A^+$ and thus $g^{1/p^n} \varpi d = e^p + \varpi^p f$ for some $e, f \in A^+$. Then $g^{1/p^n} d = (\frac{e}{t})^p + \varpi^{p-1} f$. Since A° is integrally closed in A, we obtain $\frac{e}{t} \in A^\circ$. Hence $A^\circ/(\varpi)$ is $(T)^{1/p^\infty}$-almost semiperfect. Therefore, $A^\circ/(\varpi^p)$ is semiperfect in view of Lemma 1.3.3, as required. □

By definition, the following lemma follows immediately.

Lemma 1.3.6. *Let R be a perfect $\mathbb{F}_p$-algebra. Let R_0 be a subring of R. Then, R_0 is p-root closed in R if and only if R_0 is perfect.*

Finally we give a remark on torsion elements in perfect rings.

Lemma 1.3.7. *Let R be a perfect $\mathbb{F}_p$-algebra and $t \in R$ be any element. Then $R_{t\text{-tor}}$ is $(t)^{1/p^\infty}$-almost zero. In particular, R has bounded t-torsion.*

Proof. Pick $x \in R_{t\text{-tor}}$. Then $t^l x = 0$ for some $l > 0$. Since E is perfect, $t^{l/p^n} x^{l/p^n} = 0$ for any $n \geq 0$. Hence $t^{l/p^n} x = t^{l/p^n} x^{l/p^n} (x^{l/p^n})^{p^n - l} = 0$. Hence the assertion follows. $\square$

1.3.1 Direct perfection and inverse perfection

Here we review two procedures of *perfection* of $\mathbb{F}_p$-algebras.

Definition 1.3.8 (Direct perfection)**.** Let R be an $\mathbb{F}_p$-algebra. Let $R_n := R$ for every $n \geq 0$ and $\varphi_n : R_n \to R_{n+1}$ be the Frobenius map. The *direct perfection* of R is the direct limit $\varinjlim_n R_n$. We denote by $\varphi_R : R \to \varinjlim_n R_n$ the composite $R \xrightarrow{\cong} R_0 \to \varinjlim_n R_n$.

Definition 1.3.9 (Inverse perfection)**.** Let R be an $\mathbb{F}_p$-algebra. The *inverse perfection* of R is an inverse limit $\varprojlim_{\mathrm{Frob}} R$ in which each transition map is the Frobenius map

$$R \to R,\ r \mapsto r^p \ .$$

The *first projection* of the inverse perfection $\varprojlim_{\mathrm{Frob}} R$ is the ring homomorphism

$$\mathrm{pr}_0 : \varprojlim_{\mathrm{Frob}} R \to R,\ (\ldots, r_1, r_0) \mapsto r_0 \ .$$

It is easily seen that the direct perfection and the inverse perfection of an $\mathbb{F}_p$-algebra are perfect. First we refer to some categorical meaning of these procedures of perfection.

Lemma 1.3.10. *Let $\mathcal{C}$ be the category of $\mathbb{F}_p$-algebras and $\mathcal{C}^{\mathrm{perf}}$ the category of perfect $\mathbb{F}_p$-algebras. Let $F : \mathcal{C} \to \mathcal{C}^{\mathrm{perf}}$ (resp. $G : \mathcal{C} \to \mathcal{C}^{\mathrm{perf}}$) be the functor that associates to $R \in \mathrm{ob}(\mathcal{C})$ the direct perfection $\varinjlim_{\mathrm{Frob}} R$ (resp. the inverse perfection $\varprojlim_{\mathrm{Frob}} R$). Then F (resp. G) is a left adjoint (resp. a right adjoint) of the forgetful functor $\iota : \mathcal{C}^{\mathrm{perf}} \to \mathcal{C}$.*

Proof. Let R be an $\mathbb{F}_p$-algebra and S a perfect $\mathbb{F}_p$-algebra. Let $\varphi_R : R \to F(R)$ and $\varphi_S : S \to F(S)$ be as in Definition 1.3.8. Then φ_S is an isomorphism, and for any ring homomorphism $R \to S$, the diagram:

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow{\scriptstyle \varphi_R} & & \downarrow{\scriptstyle \varphi_S} \\ F(R) & \longrightarrow & F(S) \end{array}$$

commutes. Thus the map

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{perf}}}(F(R), S) \to \mathrm{Hom}_{\mathcal{C}}(R, S),\ f \mapsto f \circ \varphi_R$$

admits the inverse map

$$\mathrm{Hom}_{\mathcal{C}}(R, S) \to \mathrm{Hom}_{\mathcal{C}^{\mathrm{perf}}}(F(R), S),\ g \mapsto \varphi_S^{-1} \circ F(g) \ .$$

Hence F is a left adjoint of ι. Similarly, one can show that G is right adjoint of ι using the first projection map pr_0 (cf. Definition 1.3.9). $\square$

Corollary 1.3.11. *Let the notation be as in Lemma 1.3.10. Let R be a perfect $\mathbb{F}_p$-algebra. Let S and T be perfect R-algebras. Then we have $F(S \otimes_R T) = F(S) \otimes_{F(R)} F(T)$. In particular, $S \otimes_R T$ is perfect.*

Proof. By Lemma 1.3.10, the functor F is right exact. Hence the diagram of morphisms in $\mathcal{C}^{\mathrm{perf}}$:

$$\begin{array}{ccc} F(R) & \longrightarrow & F(T) \\ \downarrow & & \downarrow \\ F(S) & \longrightarrow & F(S \otimes_R T) \end{array}$$

is cocartesian. Since the forgetful functor $\iota : \mathcal{C}^{\mathrm{perf}} \to \mathcal{C}$ is also right exact by Lemma 1.3.10 again, the assertion follows. □

We often use the following property of inverse perfection.

Lemma 1.3.12. *Let R be a semi-perfect $\mathbb{F}_p$-algebra, and $\varprojlim_{\mathrm{Frob}} R$ the inverse perfection of R. Then, $a \in \varprojlim_{\mathrm{Frob}} R$ is invertible if and only if $\mathrm{pr}_0(a) \in R$ is invertible.*

Proof. If a is invertible, clearly $\mathrm{pr}_0(a)$ is invertible. To verify the converse, we assume that $\mathrm{pr}_0(a) \in R^\times$. Since R is semi-perfect, there exists $b \in \varprojlim_{\mathrm{Frob}} R$ such that $\mathrm{pr}_0(a)\mathrm{pr}_0(b) = 1$. Thus it suffices to show that $\mathrm{pr}_0^{-1}(\{1\}) \subset (\varprojlim_{\mathrm{Frob}} R)^\times$. Consider the discrete topology on R, and the inverse limit topology on $\varprojlim_{\mathrm{Frob}} R$. Then $\varprojlim_{\mathrm{Frob}} R$ is complete under this topology. Moreover, $\mathrm{Ker}(\mathrm{pr}_0)$ consists of topological nilpotent elements. Thus $\mathrm{pr}_0^{-1}(\{1\}) = 1 + \mathrm{Ker}(\mathrm{pr}_0)$ is contained in $(\varprojlim_{\mathrm{Frob}} R)^\times$, as desired. □

Inverse perfection is useful for a study of Witt vectors. Let us observe it.

Lemma 1.3.13. *Let $\mathcal{O}$ be a topological ring whose topology is defined by an ideal $J \subset \mathcal{O}$. Assume that $p \in J$ and $\mathcal{O}$ is Hausdorff complete. Then there exists a unique multiplicative map $\varphi : \varprojlim_{\mathrm{Frob}} \mathcal{O}/J \to \mathcal{O}$ such that the diagram*

$$\begin{array}{ccc} \varprojlim_{\mathrm{Frob}} \mathcal{O}/J & \xrightarrow{\ \varphi\ } & \mathcal{O} \\ & \searrow^{\mathrm{pr}_0} & \downarrow{\scriptstyle \mathrm{mod}\ J} \\ & & \mathcal{O}/J \end{array}$$

commutes. Explicitly, φ is defined as

$$\varphi((\ldots, \overline{a_2}, \overline{a_1}, \overline{a_0})) = \lim_{n \to \infty} a_n^{p^n} .$$

Proof. Since $\mathcal{O}$ is J-adically Hausdorff complete, it suffices to show the following claim.

Claim 1.3.14. *Let n be a nonnegative integer. Let φ_n be a multiplicative map*

$$\varprojlim_{\mathrm{Frob}} \mathcal{O}/J \to \mathcal{O}/J^n,\ (\ldots, \overline{a_2}, \overline{a_1}, \overline{a_0}) \mapsto a_n^{p^n} \bmod J^n .$$

Then there exists a unique multiplicative map $\varphi_{n+1} : \varprojlim_{\mathrm{Frob}} \mathcal{O}/J \to \mathcal{O}/J^{n+1}$ *such that the diagram*

$$\begin{array}{ccc} \varprojlim_{\mathrm{Frob}} \mathcal{O}/J & \xrightarrow{\varphi_{n+1}} & \mathcal{O}/J^{n+1} \\ & \searrow^{\varphi_n} & \downarrow \\ & & \mathcal{O}/J^n \end{array}$$

commutes. Moreover, φ_{n+1} *is defined as*

$$\varphi_{n+1}((\ldots, \overline{a_2}, \overline{a_1}, \overline{a_0})) = a_{n+1}^{p^{n+1}} \mod J^{n+1} \ . \tag{1.5}$$

Proof of Claim 1.3.14. (1.5) gives a multiplicative map $\varphi_{n+1} : \varprojlim_{\mathrm{Frob}} \mathcal{O}/J \to \mathcal{O}/J^{n+1}$ with the desired property. Let us show the uniqueness. Let $\varphi'_{n+1} : \varprojlim_{\mathrm{Frob}} \mathcal{O}/J \to \mathcal{O}/J^{n+1}$ be a multiplicative map with the desired property. Pick an element $a := (\ldots, \overline{a_2}, \overline{a_1}, \overline{a_0}) \in \varprojlim_{\mathrm{Frob}} \mathcal{O}/J$. Then we have $\varphi'_{n+1}(a) = \varphi'_{n+1}(a^{1/p})^p$. Let $b_{n+1} \in \mathcal{O}$ be a representative of $\varphi'_{n+1}(a^{1/p}) \in \mathcal{O}/J^{n+1}$. Then by assumption,

$$b_{n+1} \mod J^n = \varphi_n(a^{\frac{1}{p}}) = a_{n+1}^{p^n} \mod J^n \ .$$

Hence by Claim 1.3.15 below, we have

$$\varphi'_{n+1}(a) = b_{n+1}^p \mod J^{n+1} = a_{n+1}^{p^{n+1}} \mod J^{n+1},$$

as wanted. □

Claim 1.3.15. $a \equiv b \mod J^n \Rightarrow a^p \equiv b^p \mod J^{n+1}$.

Proof of Claim 1.3.15. If $a = b + x_n$ for some $x_n \in J^n$, then $a^p = b^p + p\sum_{i=1}^{p-1} c_i b^{p-i} x_n^i + x_n^p$. Thus we have $a^p - b^p \in J^{n+1}$ because $p \in J$. □

□

Corollary 1.3.16. *Let* $\mathcal{O}$ *be a topological ring whose topology is defined by an ideal* $J \subset \mathcal{O}$. *Assume that* $p \in J$ *and* $\mathcal{O}$ *is Hausdorff complete. Then there exists a unique continuous ring homomorphism* $\theta : W(\varprojlim_{\mathrm{Frob}} \mathcal{O}/J) \to \mathcal{O}$ *such that the following diagram commutes:*

$$\begin{array}{ccc} W(\varprojlim_{\mathrm{Frob}} \mathcal{O}/J) & \xrightarrow{\theta} & \mathcal{O} \\ \omega_0 \downarrow & & \downarrow \mod J \\ \varprojlim_{\mathrm{Frob}} \mathcal{O}/J & \xrightarrow{\mathrm{pr}_0} & \mathcal{O}/J \ . \end{array}$$

Moreover, for $a = (\ldots, \bar{a}_1, \bar{a}_2) \in \varprojlim_{\mathrm{Frob}} \mathcal{O}/J$, *it holds that*

$$\theta([a]) = \lim_{n \to \infty} a_n^{p^n} \ .$$

Remark 1.3.17. Note that every $a \in \varprojlim_{\mathrm{Frob}} \mathcal{O}/J$ can be expressed as

$$a = (\ldots, \ \overline{\theta([a^{p^{-2}}])}, \ \overline{\theta([a^{p^{-1}}])}, \ \overline{\theta([a])}) \ .$$

Using θ, we can show that the structure of $\varprojlim_{\mathrm{Frob}} \mathcal{O}/J$ depends only on the topology on $\mathcal{O}$.

Corollary 1.3.18. *Let the notations and the assumption be as in Corollary 1.3.16. Then $\varprojlim_{\mathrm{Frob}} \mathcal{O}/J$ is independent of the choice of J.*

Proof. Let $J' \subset \mathcal{O}$ be an ideal that contains p and defines the topology on $\mathcal{O}$. Then we can consider the two ring homomorphisms

$$f : \varprojlim_{\mathrm{Frob}} \mathcal{O}/J \to \varprojlim_{\mathrm{Frob}} \mathcal{O}/J', \ a \mapsto (\ldots, \overline{\theta([a^{1/p}])}, \overline{\theta([a])})$$

and

$$f' : \varprojlim_{\mathrm{Frob}} \mathcal{O}/J' \to \varprojlim_{\mathrm{Frob}} \mathcal{O}/J, \ a' \mapsto (\ldots, \overline{\theta([a'^{1/p}])}, \overline{\theta([a'])}) \ ,$$

and then each one defines the inverse map of the other one. Hence the assertion follows. □

The map θ is especially important in the case when $\mathcal{O}/(p)$ is semiperfect.

Lemma 1.3.19. *Let the notations and the assumption be as in Corollary 1.3.16. Suppose that $\mathcal{O}/(p)$ is $(T)^{1/p^\infty}$-almost semiperfect. Then θ is $(T)^{1/p^\infty}$-almost surjective.*

Proof. By assumption, the map $\overline{\theta} : \varprojlim_{\mathrm{Frob}} \mathcal{O}/J \to \mathcal{O}/(p)$ induced by θ is $(T)^{1/p^\infty}$-almost surjective. Moreover, $W(\varprojlim_{\mathrm{Frob}} \mathcal{O}/J)$ is p-adically complete and $\mathcal{O}$ is p-adically Hausdorff. Thus the assertion follows from Lemma 1.1.6. □

1.4 Topologocal rings

1.4.1 f-adic rings

Let us begin with recalling basic terms on topological rings.

Definition 1.4.1. Let A be a topological ring.

1. We say that a subset $S \subset A$ is *bounded* if for every open neighborhood U of $0 \in A$ there exists some open neighborhood V of $0 \in A$ such that $V \cdot S \subset U$.

2. We say that an element $a \in A$ is *powerbounded* if the subset $\{a^n\}_{n \geq 1}$ of A is bounded.

3. We say that an element $a \in A$ is *topologically nilpotent* if the sequence $\{a^n\}_{n \geq 1}$ in A converges to $0 \in A$.

We denote by A° (resp. $A^{\circ\circ}$) the subset consisting of all powerbounded (resp. topologically nilpotent) elements in A (notice that A° forms a subring of A, and $A^{\circ\circ}$ is an ideal in A°).

In [30], Huber introduced an important class of topological rings, called *f-adic rings*. These topological rings give a generalization of classical *Tate algebras* (cf. [7]).

Definition 1.4.2 (f-adic rings)**.** An *f-adic ring* is a topological ring A that contains an open subring $A_0 \subset A$ together with a finitely generated ideal $I_0 \subset A_0$ such that the topology on A_0 induced from A is I_0-adic. A_0 is called a *ring of definition* of A and I_0 is called an *ideal of definition* of A and the pair (A_0, I_0) is called a *pair of definition* of A.

We will study a special class of f-adic rings called *Tate rings* (cf. Definition 1.4.4). Before going further, we give a remark on perfectness of an f-adic ring.

Lemma 1.4.3. *Let A be a perfect f-adic ring of characteristic $p > 0$. Assume that A has a ring of definition that is p-root closed in A. Then the Hausdorff completion of A is also perfect.*

Proof. Let (A_0, I_0) be a pair of definition of A_0 such that A_0 is p-root closed in A. Put $A' := \varprojlim_n A/I_0^n$. Let $r > 0$ be an integer for which I_0 is generated by r elements of A_0. Let $\Psi : A \to A$ is the inverse map of the Frobenius map. Then, since A_0 is p-root closed in A, it holds that

$$a_1 \equiv a_2 \mod I_0^{npr} \Rightarrow \Psi^{-1}(a_1) \equiv \Psi^{-1}(a_2) \mod I_0^n$$

for $a_1, a_2 \in A$ and every $n \geq 1$. Hence we obtain a map

$$A' \to A',\ (\overline{a_n})_{n\geq 1} \mapsto (\overline{\Psi^{-1}(a_{npr})})_{n\geq 1}\ ,$$

which is the inverse map of the Frobenius map on A'. □

1.4.2 Tate rings

First let us recall the definition.

Definition 1.4.4 (Tate rings, cf. [30, Proposition 1.5])**.** A *Tate ring* is an f-adic ring A that satisfies either one of the following equivalent conditions:

1. A has a topologically nilpotent unit;
2. A admits a pair of definition (A_0, I_0) such that I_0 is a principal ideal and $I_0 A = A$.

For a Tate ring A, we call a topologically nilpotent unit $t \in A$ a *pseudo-uniformizer* of A.

Any Tate ring comes from a pair of a ring and a non-zerodivisor in it as follows.

Lemma 1.4.5. *The following assertions hold:*

1. *Let A_0 be a ring with a non-zerodivisor t, and put $A := A_0[\frac{1}{t}]$. Equip A with the linear topology defined by $\{t^n A_0\}_{n\geq 1}$. Then A is equipped with the structure as a Tate ring with a ring of definition A_0 and a pseudo-uniformizer $t \in A_0$.*
2. *Conversely, let A be a Tate ring with a ring of definition A_0 and a pseudo-uniformizer $t \in A_0$. Then one has $A = A_0[\frac{1}{t}]$ (as a ring).*

Proof. 1 is easy to check. 2 is due to [30, Lemma 1.5]. □

One can define a canonical topology on a finitely generated module over a Tate ring.

Lemma 1.4.6. *Let A be a Tate ring and let M be a finitely generated A-module. Take a ring of definition $A_0 \subset A$, a pseudo-uniformizer $t \in A_0$ and a finite generating set S of M over A. Let $M_0 \subset M$ be the A_0-submodule generated by S. Equip M with the linear topology defined by $\{t^n M_0\}_{n>0}$.*

1. *The topology on M is independent of the choices of A_0, t and S.*

2. *For every finitely generated A_0-submodule N_0 of M such that $M = N_0[\frac{1}{t}]$, the induced topology on N_0 coincides with the t-adic topology.*

3. *Let $f : M \to N$ be a homomorphism of A-modules, where N is finitely generated. Equip N with the topology defined above. Then f is continuous.*

Proof. Since $M = \bigcup_{n>0}(M_0 : (t^n))$, for every finitely generated A_0-submodule N_0 of M, there exists some $c > 0$ such that $t^c N_0 \subset M_0$. Hence 2 and 3 are easy to see. To show 1, let us consider another data (A'_0, t', S', M'_0). Pick an integer $m > 0$. Then it suffices to check that there exists some $m' > 0$ for which $t'^{m'} M'_0 \subset t^m M_0$ holds. Let $N'_0 \subset M$ be the A'_0-submodule generated by S. Then $t'^{c_1} M'_0 \subset N'_0$ for some $c_1 > 0$, as M'_0 is finitely generated. Meanwhile, since $t^m A_0 \subset A$ is open, there exists some $c_2 > 0$ such that $t'^{c_2} A'_0 \subset t^m A_0$ and so $t'^{c_2} N'_0 \subset t^m M_0$. Hence by putting $m' := c_1 + c_2$, we obtain $t'^{m'} M'_0 \subset t^m M_0$, as wanted. □

Notice that one can set $M = A$ in Lemma 1.4.6, and the resulting topology on A coincides with the original one. Now we can give a canonical Tate ring structure to any module-finite algebra extension of a Tate ring.

Lemma 1.4.7. *Let A be a Tate ring and let B be a module-finite A-algebra. Equip B with the topology as in Lemma 1.4.6. Then B is equipped with the structure as a Tate ring with the following property:*

- *for every ring of definition A_0 and every pseudo-uniformizer $t \in A_0$ of A, there exists a ring of definition B_0 of B such that B_0 is integral over A_0 and $t \in B_0$ is a pseudo-uniformizer of B.*

Proof. Take a system of generators $x_1, \ldots, x_r$ of the A-module B. Multiplying each x_i by a power of t if necessary, we may assume that they are integral over A_0. Let $B_0 \subset B$ be an A_0-subalgebra generated by $x_1, \ldots, x_r$. As $B = B_0[\frac{1}{t}]$, we can introduce a Tate ring structure into B with a ring of definition B_0 and a pseudo-uniformizer $t \in B_0$ as in Lemma 1.4.5 1. Meanwhile, since each x_i is integral over A_0, B_0 is a module-finite A_0-algebra. Hence the topology on B coincides with the one defined by setting $M = B$ in Lemma 1.4.6. □

If a finitely generated module M over a Tate ring admits a structure of a finitely generated module over another Tate ring, then one can consider two canonical topologies on M. In the following situation, these topologies coincide.

Lemma 1.4.8. *Let A be a Tate ring, and let B be a module-finite A-algebra. Equip B with the canonical structure as a Tate ring as in Lemma 1.4.7. Let M be a finitely generated B-module. Then the following two topologies:*

- *the canonical topology on M as a finitely generated A-module;*

- *the canonical topology on M as a finitely generated B-module;*

coincide.

Proof. Let A_0 be a ring of definition of A and $t \in A_0$ a pseudo-uniformizer of A. Then we can take a ring of definition B_0 of B that is finitely generated over A_0 and satisfies $B = B_0[\frac{1}{t}]$. Let M_0 be a finitely generated B_0-submodule of M such that $M = M_0[\frac{1}{t}]$. Then, also as an A_0-module, M_0 is finitely generated and satisfies $M = M_0[\frac{1}{t}]$. Hence the assertion follows. □

Next let us consider a base extension of a module-finite algebra over a Tate ring. Then one may define two types of canonical topologies on it, but they are the same.

Lemma 1.4.9. *Let A be a Tate ring, and $(A_0, (t))$ a pair of definition of A. Let $A \to A'$ be a continuous ring map between Tate rings. Let B be a module-fintie A-algebra, and set $B' := B \otimes_A A'$. Equip B (resp. B') with the canonical topology by regarding it as a module-finite A-algebra (resp. A'-algebra). Let B_0 be a ring of definition of B that is an A_0-subalgebra of B, and A'_0 be a ring of definition of A' that is an A_0-subalgebra of A'. Let B'_0 be the image of the natural map $B_0 \otimes_{A_0} A'_0 \to B'$. Then $(B'_0, (t))$ is a pair of definition of B'.*

Proof. Since $A = A_0[\frac{1}{t}]$, $A' = A'_0[\frac{1}{t}]$, and $B = B_0[\frac{1}{t}]$, we have $B' = B'_0[\frac{1}{t}]$. Thus by Lemma 1.4.5 1, it suffices to check that $\{t^n B'_0\}_{n\geq 1}$ forms a fundamental system of neighbourhoods of $0 \in B'$. In view of the definition of B'_0, we may assume that B_0 is generated by finite elements $x_1, \dots, x_s \in B_0$ as an A_0-submodule of B (such a ring of definition exists by the proof of Lemma 1.4.7). Put $x'_i := x_i \otimes_A 1_{A'}$ for $i = 1, \dots, s$. Then B'_0 is generated by $x'_1, \dots, x'_s$ as an A'_0-submodule of B'. Thus, since $B' = B'_0[\frac{1}{t}]$, the assertion follows. □

Next we discuss *uniformity* of Tate rings.

Definition 1.4.10 (Preuniform Tate rings)**.** Let A be a Tate ring. We say that A is *preuniform* if A° is bounded. We say that A is *uniform* if it is preuniform and Hausdorff complete.

For a perfect Tate ring, there are several characterizations of preuniformity.

Proposition 1.4.11. *Let R be a perfect Tate ring of characteristic p. Then the following conditions are equivalent.*

- (a) *R is preuniform.*
- (b) *The Frobenius map on R is open.*
- (c) *R has a ring of definition R_0 that is p-root closed in R.*
- (d) *The Hausdorff completion of R is perfect.*

Proof. First we show $(a) \Rightarrow (b)$. Consider the inverse Frobenius map $\Psi : R \to R,\ x \mapsto x^{1/p}$. Let R_0 be a ring of definition of R, and $tR_0 \subset R_0$ an ideal of definition. Assume that R is uniform. Then one may assume that $R_0 = R^\circ$. Here $\Psi(R^\circ) \subset R^\circ$ and thus $\Psi(t^{np}R^\circ) \subset t^n R^\circ$ for every $n \geq 0$. Hence Ψ is continuous. We then verify the converse $(b) \Rightarrow (a)$. Assume that Ψ is continuous. Then there exists $N > 0$ such that $t^N \Psi(R_0) \subset \Psi(t^{Np}R_0) \subset R_0$. Hence it holds that for every $y \in R$

$$y \in R_0 \Rightarrow t^N y^{1/p} \in R_0\ . \tag{1.6}$$

Pick $x \in R^\circ$. Then there exists $d > 0$ such that $t^d x^{p^n} \in R_0$ for every $n \geq 1$. By applying (1.6) recursively, one obtains $t^{Na_n} x \in R_0$ where $a_n = N(\sum_{i=0}^{n-1} 1/p^i) + d/p^n$. Since $\lim_{n\to\infty} a_n = \frac{Np}{p-1}$, a constant $c > \frac{Np}{p-1}$ satisfies $t^c x \in R_0$ ($\forall x \in R^\circ$). Therefore R° is bounded, i.e., R is uniform. To see $(a) \Rightarrow (c)$, it suffices to take A^+ as a ring of definition of A. $(c) \Rightarrow (d)$ is due to Lemma 1.4.3. Finally, we show $(d) \Rightarrow (a)$. Let $\widehat{R}$ be th e Hausdorff completion of R. Then by Banach's open mapping theorem, the Frobenius map on $\widehat{R}$ is open. Then by the implication $(b) \Rightarrow (a)$, $\widehat{R}$ is uniform. Hence by Lemma 1.2.5, R is preuniform. □

Let us recall the following fact.

Lemma 1.4.12. *A Hausdorff preuniform Tate ring is reduced. In particular, a uniform Tate ring is reduced.*

Proof. Let A be a Hausdorff preuniform Tate ring. Then, since A° is bounded, we can take A° as a ring of definition of A. Pick a pseudo-uniformizer $t \in A^\circ$ of A. Then A° is t-adically Hausdorff because A is Hausdorff. Let $f \in A$ be such that $f^k = 0$ for some $k > 0$. Then for an integer $n > 0$, $t^{-n} f$ is nilpotent, and therefore we get $t^{-n} f \in A^\circ$. Hence $f \in t^n A^\circ$. Since A° is t-adically Hausdorff and n is arbitrary, it follows that $f = 0$. □

Finally, we define uniformity also for pairs that induce Tate rings.

Definition 1.4.13 (Preuniform pairs)**.** Let (A_0, I_0) be a pair of a ring A_0 and an ideal $I_0 \subset A_0$.

1. We say that (A_0, I_0) is *preuniform*, if A_0 has a non-zerodivisor t with the following property:
   - $I_0 = tA_0$, and there exits some $c > 0$ for which $t^c (A_0)^+_{A_0[\frac{1}{t}]} \subset A_0$.
2. We say that (A_0, I_0) is *uniform*, if it is preuniform and A_0 is I_0-adically Hausdorff complete.

The above definition is derived from the following fact, which is easy to prove.

Lemma 1.4.14. *Let A_0 be a ring with a non-zerodivisor $t \in A_0$ and let A be the Tate ring associated to $(A_0, (t))$. Then A is preuniform (resp. uniform) if and only if the pair $(A_0, (t))$ is preuniform (resp. uniform).*

1.4.3 Admissible rings

Here we recall another class of topological rings, which appears in formal geometry. Notice that these topological rings are required to be *Hausdorff complete*.

Definition 1.4.15 (Admissible rings and adic rings)**.**

1. An *admissible ring* is a Hausdorff complete topological ring A such that the topology is defined by a descending filtration $\{F_\lambda\}_{\lambda \in \Lambda}$ of ideals, and A admits an ideal I with the following properties:
   (a) I is open;
   (b) I is topologically nilpotent (i.e. for any $\lambda \in \Lambda$ there exists $n \geq 0$ such that $I^n \subset F^\lambda$).

   We call I an *ideal of definition* of A.
2. An admissible ring A is said to be *of finite ideal type*, if A admits an ideal of definition that is finitely generated.
3. An *adic ring* is an admissible ring A whose topology is I-adic for some ideal $I \subset A$.

1.5 Norms and Banach rings

1.5.1 Non-archimedean seminorms

Here we review several basic facts on non-archimedean seminorms. Our basic references are [7, Chapter 1], [20, Chapter 2, Appendix C] and [35, Chapter 2].

Definition 1.5.1. Let A be a ring.

1. A function $\alpha : A \to \mathbb{R}_{\geq 0}$ is called a (*non-archimedean*) *seminorm* on A, if it satisfies the following conditions.

   (a) $\alpha(0) = 0$.

   (b) $\alpha(f - g) \leq \max\{\alpha(f), \alpha(g)\}$ for every $f, g \in A$.

   (c) $\alpha(1) \leq 1$.

   (d) $\alpha(fg) \leq \alpha(f)\alpha(g)$ for every $f, g \in A$.

   A seminorm α on A is called a *norm*, if it satisfies the following condition.

   (a') For $f \in A$, one has $\alpha(f) = 0$ if and only if $f = 0$.

2. Let α and α' be seminorms on A. We say that α and α' are *equivalent* (or α is *equivalent* to α'), if there exist real numbers $C, C' > 0$ such that one has

   $$\alpha'(f) \leq C\alpha(f) \leq C'\alpha'(f)$$

   for every $f \in A$. We mean by $\alpha \sim \alpha'$ that α is equivalent to α'.

3. Let α be a seminorm on A. We say that $f \in A$ is *powermultiplicative* with respect to α, if $\alpha(f^n) = \alpha(f)^n$ for every $n > 0$. We say that $f \in A$ is *multiplicative* with respect to α, if $\alpha(fg) = \alpha(f)\alpha(g)$ for every $g \in A$. We say that α is *powermultiplicative*, if any $f \in A$ is powermultiplicative with respect to α. We say that α is *multiplicative*, if any $f \in A$ is multiplicative with respect to α.

Here we list some basic facts on seminorms.

Lemma 1.5.2. *Let A be a ring and let α, α', and α'' be seminorms on A.*

1. *If $\alpha \sim \alpha'$ and $\alpha' \sim \alpha''$, then $\alpha \sim \alpha''$.*

2. *If α and α' are powermultiplicative and $\alpha \sim \alpha'$, then $\alpha = \alpha'$.*

3. *Suppose that α is not identically zero. Then $\alpha(1) = 1$. Moreover, for a unit $u \in A$, u is multiplicative with respect to α if and only if $\alpha(u^{-1}) = \alpha(u)^{-1}$.*

Proof. The assertions 1 and 2 are easy to check. We verify 3. By definition we have $\alpha(1) = 0$ or $\alpha(1) = 1$, and thus $\alpha(1) = 1$ by assumption. Pick a unit $u \in A$. Since $1 = \alpha(1) = \alpha(u \cdot u^{-1})$, the multiplicativity of u implies $\alpha(u)^{-1} = \alpha(u^{-1})$. Now we assume $\alpha(u)^{-1} = \alpha(u^{-1})$ conversely. Then for every $f \in A$, we have $\alpha(f) = \alpha(u^{-1}uf) \leq \alpha(u^{-1}) \cdot \alpha(uf) = \alpha(u)^{-1}\alpha(uf)$. Therefore $\alpha(u) \cdot \alpha(f) \leq \alpha(uf)$. Combining it with $\alpha(uf) \leq \alpha(u) \cdot \alpha(f)$, we obtain $\alpha(uf) = \alpha(u) \cdot \alpha(f)$ as wanted. $\square$

Let A be a ring, and α a seminorm on A. We put $F_r := \{f \in A \mid \alpha(f) < r\}$ for every positive real number r. Then F_1 forms a subring of A, and each F_r forms an F_1-submodule of A. Hence one can define the linear topology on A such that the filtration $\{F_r\}_{r>0}$ forms a fundamental system of open neighbourhoods of $0 \in A$. Moreover the ring A equipped with this topology is a topological ring. For this topological ring A, a subset $S \subset A$ is bounded (Definition 1.4.1) if and only if $S \subset F_r$ for some $r > 0$. If two seminorms α and α' on A are equivalent, then they define the same topology on A (but the converse does not hold in general). The following class of seminorms are useful for characterization of Tate rings.

Definition 1.5.3. Let A_0 be a ring with a non-zerodivisor $t \in A_0$. We say that a seminorm α on $A_0[\frac{1}{t}]$ is *associated to* $(A_0, (t))$, if there exists a real number $c > 1$ such that γ is equivalent to the seminorm $\gamma_{A_0,(t),c}$ defined by

$$\gamma_{A_0,(t),c}(f) := \begin{cases} c^{\min\{m \in \mathbb{Z} \mid t^m f \in A_0\}} & (f \notin \cap_{n=0}^{\infty} t^n A_0) \\ 0 & (f \in \cap_{n=0}^{\infty} t^n A_0) \end{cases} .$$

Lemma 1.5.4. *Let A be a topological ring. Let $t \in A$ be an element, and A_0 a subring of A. Then the following conditions are equivalent.*

(*a*) *A is a Tate ring with a ring of definition A_0 and a pseudo-uniformizer $t \in A_0$.*

(*b*) *$t \in A$ is a unit, and there exists some seminorm α on A with the following properties:*

- *the topology on A is induced by α;*
- *$\alpha(t^n) < 1$ for some $n > 0$, and $A_0 = \{f \in A \mid \alpha(f) \leq 1\}$.*

Proof. To see (*a*) $\Rightarrow$ (*b*), it is enough to consider the seminorm $\gamma_{A_0,(t),2}$. The converse is easy to check. □

For a ring A and a seminorm α on A, we define a function $\alpha_{\mathrm{sp}} : A \to \mathbb{R}_{\geq 0}$ by

$$\alpha_{\mathrm{sp}}(f) := \inf_{n \geq 1} \alpha(f^n)^{\frac{1}{n}} \quad (f \in A) .$$

α_{sp} has the following properties.

Lemma 1.5.5 ([7, §1.3.2])**.** *Let A be a ring, and α a seminorm on A.*

1. *$\alpha_{\mathrm{sp}}(f) = \lim_{n \to \infty} \alpha(f^n)^{\frac{1}{n}}$ for every $f \in A$, and α_{sp} is a powermultiplicative seminorm on A.*
2. *If $f \in A$ is multiplicative with respect to α, then f is also multiplicative with respect to α_{sp}.*

We call α_{sp} the *spectral seminorm* associated to α. Using spectral seminorms, we obtain the following characterization of preuniformity.

Lemma 1.5.6. *Let A_0 be a ring with a non-zerodivisor $t \in A_0$.*

1. *$t \in A_0[\frac{1}{t}]$ is multiplicative with respect to the seminorm $\alpha_{A_0,(t),c}$ for every $c > 1$.*
2. *The following conditions are equivalent.*

(a) $(A_0, (t))$ *is preuniform.*

(b) *For any seminorm* α *associated to* $(A_0, (t))$*, one has* $\alpha \sim \alpha_{\mathrm{sp}}$ *(or, equivalently,* α *is equivalent to a powermultiplicative seminorm).*

(c) *For some seminorm* α *associated to* $(A_0, (t))$*, one has* $\alpha \sim \alpha_{\mathrm{sp}}$*.*

Proof. The assertion 1 is clear. Let us prove 2. To see $(b) \Rightarrow (a)$, it is enough to consider the topology induced by a powermultiplicative seminorm. Here we show $(a) \Rightarrow (b)$. Since A° is bounded with respect to (the topology induced by) α, we may assume that $\alpha = \alpha_{A^\circ,(t),2}$. Then t is multiplicative with respect to α and α_{sp}. Thus, it suffices to show the existence of some constants $C, C' > 0$ such that $C\alpha(a) \leq \alpha_{\mathrm{sp}}(a) \leq C'\alpha(a)$ for an arbitrary $a \in A^\circ \setminus tA^\circ$. Now it holds that $a^n \notin t^n A^\circ$ for an arbitrary $n > 0$; otherwise, $(t^{-1}a)^l$ would belong to A° for some $l > 0$, which implies that $a \in tA^\circ$ as A° is integrally closed. Hence $\alpha(t) < \alpha(a^n)^{\frac{1}{n}}$, and therefore $\alpha(t) \leq \alpha_{\mathrm{sp}}(a) \leq 1$. Thus we have $2^{-1}\alpha(a) \leq \alpha_{\mathrm{sp}}(a) \leq \alpha(a)$, as wanted. □

Moreover, if A_0 is a valuation ring V and $V[\frac{1}{t}]$ is a field, then α_{sp} is a multiplicative norm.

Lemma 1.5.7. *Let* V *be a valuation ring that contains an element* $t \in V$ *for which* V *is* t*-adically Hausdorff. Let* $\alpha : V[\frac{1}{t}] \to \mathbb{R}_{\geq 0}$ *be a seminorm associated to* $(V, (t))$*. Then the spectral seminorm* α_{sp} *is multiplicative.*

Proof. We denote by K the field of fractions $V[\frac{1}{t}]$. In view of Lemma 1.5.2 and Lemma 1.5.6 2, we may assume that $\alpha = \alpha_{V,(t),c}$ for a real number $c > 1$. It suffices to prove that any $f \in K$ is multiplicative with respect to α_{sp}. As clearly $0 \in K$ is multiplicative, we assume that $f \neq 0$. By Lemma 1.5.2 3, we are reduced to showing that $\alpha(f^{-1})_{\mathrm{sp}} = \alpha_{\mathrm{sp}}(f)^{-1}$. Pick an arbitrary $g \in K^\times$. We put $\lambda(g) := \min\{m \in \mathbb{Z} \mid t^m g \in V\}$. Since V is a valuation ring, we have $t^{-(\lambda(g)-1)}V \subset gV \subset t^{-\lambda(g)}V$. It implies that $t^{\lambda(g)}V \subset g^{-1}V \subset t^{\lambda(g)-1}V$. Therefore, we find that

$$\alpha(g)^{-1} = c^{-\lambda(g)} \leq \alpha(g^{-1}) \leq c^{-(\lambda(g)-1)} = c\alpha(g)^{-1} .$$

Thus we have $(\alpha(f^n)^{\frac{1}{n}})^{-1} \leq \alpha(f^{-n})^{\frac{1}{n}} \leq c^{\frac{1}{n}}(\alpha(f^n)^{\frac{1}{n}})^{-1}$ for every $n > 0$. Taking the limits, we obtain $\alpha_{\mathrm{sp}}(f)^{-1} = \alpha_{\mathrm{sp}}(f^{-1})$ as wanted. □

Remark 1.5.8. In the situation of Lemma 1.5.7, V_K^* is a valuation ring of rank 1 and the multiplicative norm α_{sp} is a valuation corresponding to V_K^*.

Finally we recall the definition of *Banach rings*.

Definition 1.5.9 (Banach rings)**.**

1. A *Banach ring* is a ring R equipped with a norm α such that R is complete with respect to the topology defined by α.

2. Let R and S be Banach rings, and denote by α_R and α_S the norms on R and S respectively. We say that a ring homomorphism $\varphi : R \to S$ is *bounded*, if one has $\alpha_S(\varphi(f)) \leq \alpha_R(f)$ for every $f \in R$ (notice that this property is closed with respect to composition).

3. Let R be a Banach ring. A *Banach* R*-algebra* is a Banach ring S equipped with a bounded homomorphism $\varphi : R \to S$.

4. Let A_0 be a ring with a non-zerodivisor $t \in A_0$ that is t-adically Hausdorff complete. We say that a Banach ring R is *associated to* $(A_0, (t))$, if the underlying ring R is equal to $A_0[\frac{1}{t}]$ and the norm on R is associated to $(A_0, (t))$.

From now on, we view a Banach ring also as a (Hausdorff complete) topological ring by considering the topology defined by the norm. Then we can say that a Banach ring associated to $(A_0, (t))$ (in Definition 1.5.9 3) and the Tate ring associated to $(A_0, (t))$ have the same topological ring structure.

1.5.2 α-stability

Here we review several results on seminorms on Witt rings established by Kedlaya ([34]). First let us recall the following definition.

Definition 1.5.10. Let R be a perfect $\mathbb{F}_p$-algebra. Let α be a seminorm on R whose image is contained in $[0, 1]$.

1. We denote by $\lambda(\alpha)$ the function $W(R) \to \mathbb{R}$ defined as

$$\lambda(\alpha)\left(\sum_{i=0}^{\infty} p^i[\xi_i]\right) = \max_{i\geq 0}\{p^{-i}\alpha(\xi_i)\} \ .$$

2. We denote by $\Lambda(\alpha)$ the function $W(R) \to \mathbb{R}$ defined as

$$\Lambda(\alpha)\left(\sum_{i=0}^{\infty} p^i[\xi_i]\right) = \max_{i\geq 0}\{\alpha(\xi_i)\} \ .$$

By [34, Lemma 4.1], $\lambda(\alpha)$ is a seminorm, and if α is powermultiplicative (resp. multiplicative), then $\lambda(\alpha)$ is also powermultiplicative (resp. multiplicative). As a corollary, one can show this assertion also for $\Lambda(\alpha)$ ([34, Corollary 4.2]).

Next we recall the notion of *α-stability*, which plays an important role in the process of *untilting* for perfectoid rings (cf. §2.1.3).

Definition 1.5.11. Let R be a perfect $\mathbb{F}_p$-algebra, and α a powermultiplicative seminorm on R whose image is contained in $[0, 1]$. We say that $x = \sum_{i=0}^{\infty} p^i[\xi_i] \in W(R)$ is *α-stable*, if $\alpha(\xi_i) = 0$ for every $i \geq 0$ or $p^{-i}\alpha(x_i) < \alpha(\xi_0)$ for every $i \geq 1$. We say that x is *strictly α-stable*, if $\alpha(\xi_i) \leq \alpha(\xi_0)$ for every $i \geq 1$.

The following fact is especially important.

Proposition 1.5.12 ([34, Theorem 5.11(b)])**.** *Let R be a uniform Banach ring that is a perfect $\mathbb{F}_p$-algebra, and α the spectral norm on R. Let $z = \sum_{i=0}^{\infty} p^i[\zeta_i] \in W(R^\circ)$ be an element such that $\alpha(\zeta_0) \leq p^{-1}$ and $\zeta_1 \in (R^\circ)^\times$. Then, for every α-stable element $y \in W(R^\circ)$, we have*

$$\overline{\lambda(\alpha)}(y \mod z) = \lambda(\alpha)(y) \ .$$

The element z in Proposition 1.5.12 is called a *distinguished element* (cf. §2.1.2). Such an element also has the following property.

Lemma 1.5.13 ([34, Lemma 5.16]). *Let the notation and hypotheses be as in Proposition 1.5.12. Let $x \in R^\circ$, and let $\varepsilon > 0$ be an integer. Then there exists $y_\varepsilon = \sum_{i=0}^{\infty} p^i[\eta_{\varepsilon,i}] \in W(R^\circ)$ such that $x \equiv y_\varepsilon \mod (z)$ and for every multiplicative seminorm γ on R° bounded above by α we have*

$$\gamma(\eta_{\varepsilon,i}) \quad \leq \quad \max\{\gamma(\eta_{\varepsilon,0}),\ \varepsilon\} \quad (\forall i \geq 1)\ .$$

Proof. Let w be the inverse element of $p^{-1}(z - [\omega_0(z)])$. Put $x_0 := x$, and for $i \geq 0$, define

$$x_{i+1} = x_i - p^{-1}w(x_i - [\omega_0(x_i)])z$$

recursively. Note that by definition $x_{i+1} = [\omega_0(x_i)] - p^{-1}w(x_i - [\omega_0(x_i)])[\zeta_0]$ for $i \geq 0$. We set $x_i = \sum_{j=0}^{\infty} p^j[\xi_{ij}]$ for $i \geq 0$.

Take a multiplicative seminorm γ on R° bounded above by α. Let N be the least non-negative integer for which $\gamma(\xi_{N0}) > \gamma(\zeta_0)^{N+1}$ if such integer exists, or ∞ otherwise. Then the proposition follows from the following claims.

Claim 1.5.14. *If $i \leq N$, then $\Lambda(\gamma)(x_i) \leq \gamma(\zeta_0)^i$.*

Claim 1.5.15. *If $N \neq \infty$ and $i \geq N$, then $\gamma(\xi_{i+1,0}) = \Lambda(\gamma)(x_{i+1}) = \gamma(\xi_{N0})$.*

Indeed, let n be the least non-negative integer such that $\gamma(\zeta_0)^n \leq \varepsilon$, and put $y_\varepsilon = \sum_{i=0}^{\infty} p^i[\eta_{\varepsilon,i}] = x_n$. If $n \leq N$, we have $\Lambda(\lambda)(y) \leq \gamma(\zeta_0)^n \leq \varepsilon$ by Claim 1.5.14. Hence $\gamma(\eta_{\varepsilon,i}) \leq \varepsilon$ for every $i \geq 0$. If $n > N$, we have $\Lambda(\gamma)(y_\varepsilon) = \gamma(\eta_{\varepsilon,0})$ by Claim 1.5.15. Hence $\gamma(\eta_{\varepsilon,i}) \leq \gamma(\eta_{\varepsilon,0})$ for every $i \geq 0$, which completes the proof. □

Proof of Claim 1.5.14. The case where $i = 0$ is obvious. Assume that $\Lambda(\gamma)(x_i) \leq \gamma(\zeta_0)^i$ for some $0 \leq i < N$. Then we have

$$\begin{aligned}\Lambda(\gamma)(x_{i+1}) &= \Lambda(\gamma)([\xi_i] - p^{-1}w(x_i - [\xi_{i0}])[\zeta_0]) \\ &\leq \max\{\gamma(\xi_i),\ \Lambda(\gamma)(x_i)\gamma(\zeta_0)\} \\ &\leq \gamma(\zeta_0)^{i+1}.\end{aligned}$$

This completes the induction. □

Proof of Claim 1.5.15. We prove it by induction. First we show that $\Lambda(\gamma)(x_{N+1}) = \gamma(\xi_{N0})$. Now we have

$$\Lambda(\gamma)(x_{N+1} - [\xi_{N0}]) = \Lambda(\gamma)((x_N - [\xi_{N0}])[\zeta_0]) \leq \Lambda(\gamma)(x_N)\gamma([\zeta_0]) \leq \gamma(\zeta_0)^{N+1}$$

by Claim 1.5.14. Meanwhile, $\Lambda(\gamma)([\xi_{N0}]) = \gamma(\xi_{N0}) > \gamma(\zeta_0)^{N+1}$ by the definition of N. Hence $\Lambda(\gamma)(x_{N+1}) = \gamma(\xi_{N0})$. Moreover, $\xi_{N+1,0} = \xi_{N0} - u\xi_{N1}\zeta_0$ for some $u \in (R^\circ)^\times$. Here by Claim 1.5.14,

$$\gamma(u\xi_{N1}\zeta_0) \leq \gamma(\xi_{N1})\gamma(\zeta_0) \leq \Lambda(\gamma)(x_N)\gamma(\zeta_0) \leq \gamma(\zeta_0)^{N+1} < \gamma(\xi_{N0})\ .$$

Thus we obtain $\gamma(\xi_{N+1,0}) = \gamma(\xi_{N0}) = \Lambda(\gamma)(x_{N+1})$, as desired.

Next assume that $\gamma(\xi_{i+1,0}) = \Lambda(\gamma)(x_{i+1}) = \gamma(\xi_{N0})$ for some $i \geq N$. We show that $\gamma(\xi_{i+2,0}) = \Lambda(\gamma)(x_{i+2}) = \gamma(\xi_{N0})$. Now $\xi_{i+2,0} = \xi_{i+1,0} - u\xi_{i+1,1}\zeta_0$ for some $u \in (R^\circ)^\times$. Here, by assumption,

$$\gamma(u\xi_{i+1,1}\zeta_0) \leq \gamma(\xi_{i+1,1})\gamma(\zeta_0) \leq \gamma(\xi_{i+1,0})\gamma(\zeta_0) < \gamma(\xi_{i+1,0})\ .$$

Thus $\gamma(\xi_{i+2,0}) = \gamma(\xi_{i+1,0}) = \gamma(\xi_{N0})$. Moreover,

$$\begin{aligned}\Lambda(\gamma)(x_{i+2} - [\xi_{i+1,0}]) &= \Lambda(\gamma)((x_{i+1} - [\xi_{i+1,0}])[\zeta_0]) \\ &\leq \Lambda(\gamma)(x_{i+1})\gamma([\zeta_0]) \\ &< \gamma(\xi_{i+1,0})\end{aligned}$$

by assumption. Hence $\Lambda(\gamma)(x_{i+2}) = \gamma(\xi_{i+1,0})$. Thus we obtain $\gamma(\xi_{i+2}) = \Lambda(\gamma)(x_{i+2}) = \gamma(\xi_{N0})$, which completes the induction. □

1.6 Adic spaces

1.6.1 Adic spectra

Here we recall several definitions and basic facts on Huber's adic spectrum. Our basic references are [30] and [31].

Let us begin with recalling the notion of *continuous semivaluations*. For a totally ordered abelian group Γ, we extend a totally ordered monoid structure to a union $\Gamma \cup \{\infty\}$ by $x + \infty = \infty + x = \infty$ and $x \leq \infty$ for every $x \in \Gamma \cup \{\infty\}$.

Definition 1.6.1 (Continuous semivaluations)**.**

1. Let A be a ring. A *semivaluation* on A is a map $v : A \to \Gamma \cup \{\infty\}$ (where Γ is a totally ordered abelian group) with the following properties:
   (a) $v(1) = 0$, and $v(0) = \infty$;
   (b) $v(xy) = v(x) + v(y)$ for every $x, y \in A$;
   (c) $v(x + y) \geq \min\{v(x), v(y)\}$ for every $x, y \in A$.
2. Let A be a topological ring. We say that a semivaluation $v : A \to \Gamma \cup \{\infty\}$ is *continuous* if for every $\gamma \in \Gamma_v$ there exists a neighbourhood U of 0 in A such that $v(f) > \gamma$ for every $f \in U$.

For a semivaluation $v : A \to \Gamma \cup \{\infty\}$, we denote by Γ_v the subgroup of Γ generated by $\mathrm{Im}(v) \setminus \{\infty\}$. Two semivaluations $v : A \to \Gamma \cup \{\infty\}$ and $v' : A \to \Gamma' \cup \{\infty\}$ are said to be *equivalent*, if the following equivalent conditions are satisfied:

1. For every $x, y \in A$, $v(x) \leq v(y)$ if and only if $v'(x) \leq v'(y)$.
2. There exists a semivaluation $w : A \to \Gamma_w \cup \{\infty\}$ together with injective ordered monoid homomorphisms $h : \Gamma_w \cup \{\infty\} \to \Gamma \cup \{\infty\}$ and $h' : \Gamma_w \cup \{\infty\} \to \Gamma' \cup \{\infty\}$ such that $\mathrm{Im}(h|_{\Gamma_w}) \subset \Gamma_v$, $\mathrm{Im}(h'|_{\Gamma_w}) \subset \Gamma_{v'}$, $v = h \circ w$, and $v' = h' \circ w$.

Remark 1.6.2. Let A be a ring and $v : A \to \Gamma \cup \{\infty\}$ a semivaluation. Then $\mathfrak{p} := v^{-1}(\{\infty\})$ is a prime ideal in A, and v naturally extends to the valuation $\mathrm{Frac}(A/\mathfrak{p}) \to \Gamma \cup \{\infty\}$. Let V_v be the corresponding valuation ring. Then for a semivaluation v' that is equivalent to v, we have $V_v = V_{v'}$. We define the *height* (or *rank*) of v as that of V_v.

Huber's adic spectrum is defined for f-adic rings together with some integral structure. They are represented as *affinoid rings* defined below.

Definition 1.6.3 (Affinoid rings)**.** An *affinoid ring* is a pair (A, A^+) consisting of an f-adic ring A and a subring $A^+ \subset A$ that is open, integrally closed, and contained in A°. A *morphism* of affinoid rings $(A, A^+) \to (B, B^+)$ is a continuous ring homomorphism $f : A \to B$ such that $f(A^+) \subset B^+$. We say that an affinoid ring (A, A^+) is *complete* (resp. *Hausdorff*) if A is complete (resp. Hausdorff).

Let (A, A^+) be an affinoid ring. If a semivaluation v on A is continuous, then so is every semivaluation that is equivalent to v. The property $v(a) \geq 0$ $(\forall a \in A^+)$ is also preserved under equivalence of semivaluations. Thus one can define the following topological space.

Definition 1.6.4 (Adic spectra)**.** For an affinoid ring (A, A^+), the *adic spectrum* $\mathrm{Spa}(A, A^+)$ is the topological space defined as follows:

- the underlying set consists of all equivalence classes of continuous semivaluations v on A such that $v(a) \geq 0$ for every $a \in A^+$;
- the topology is generated by the family of subsets $\{U_{a,b}\}_{a,b\in A}$ where

$$U_{a,b} = \{v \in \mathrm{Spa}(A, A^+) \mid v(a) \geq v(b) \neq \infty\} \ .$$

Let $x \in \mathrm{Spa}(A, A^+)$. Then all representatives of x define the same valuation ring V_x as in Remark 1.6.2. Let $K_x = \mathrm{Frac}(V_x)$. In the sequel, we denote by v_x the semivaluation

$$A \to K_x \to (K_x^\times / V_x^\times) \cup \{\infty\}$$

where $A \to K_x$ is the natural map and $K_x \to (K_x^\times / V_x^\times) \cup \{\infty\}$ is the valuation associated to V_x (cf. [20, **0**, §6.2(b)]).

Rational localization and structure presheaves

First we recall *rational localization.* This notion is defined for *rational subsets* of an adic spectrum.

Definition 1.6.5 (Rational subsets)**.** Let (A, A^+) be an affinoid ring and $X = \mathrm{Spa}(A, A^+)$. A *rational subset of* X is a subset of X of the form

$$X(\tfrac{f_1,\ldots,f_n}{f_0}) := \{x \in X \mid v_x(f_i) \geq v_x(f_0) \neq \infty \ (i = 1, \ldots, n)\}$$

for some $f_0, \ldots, f_n \in A$ such that $(f_0, \ldots, f_n)A$ is the unit ideal.

Proposition 1.6.6 ([31, Lemma 1.5])**.** *Let (A, A^+) be an affinoid ring, and $U \subset \mathrm{Spa}(A, A^+)$ a rational subset. Then there exists a morphism h from (A, A^+) to a Hausdorff complete affinoid ring (B, B^+) such that* $\mathrm{Im}(\mathrm{Spa}(h)) \subset U$ *with the following universal property:*

- *for every morphism f from (A, A^+) to a Hausdorff complete affinoid ring (B', B'^+) such that* $\mathrm{Im}(\mathrm{Spa}(f)) \subset U$*, there exists a unique morphism $g : (B, B^+) \to (B', B'^+)$ such that $f = g \circ h$.*

Such affinoid ring (B, B^+) as in Proposition 1.6.6 is unique up to canonical isomorphisms. We call $h : (A, A^+) \to (B, B^+)$ in Proposition 1.6.6 the *rational localization* corresponding to U. Using this notion, one can define structure presheaves on an adic spectrum.

Proposition 1.6.7 ([31], Proposition 1.6)**.** *Let (A, A^+) be an affinoid ring. Let $U \subset \mathrm{Spa}(A, A^+)$ be a rational subset, and $(A, A^+) \to (F_A(U), F_A(U)^+)$ the corresponding rational localization. Then* $\mathrm{Spa}(F_A(U), F_A(U)^+) \cong U$.

Let (A, A^+) be an affinoid ring. For $X = \mathrm{Spa}(A, A^+)$, the presheaf $\mathcal{O}_X$ on X is defined as

$$\mathcal{O}_X(U) = \varprojlim_{W \subset U} F_A(W)$$

where W runs over all rational open subsets contained in U. Now pick any $x \in X$. Then in view of Proposition 1.6.7, for any rational open neighborhood W of x, the corresponding semivaluation $v_x : A \to \Gamma_x \cup \{\infty\}$ extends uniquely to a continuous semivaluation $v_{W,x} : \mathcal{O}_X(W) \to \Gamma_x \cup \{\infty\}$. Hence v_x also extends uniquely to a semivaluation $\widetilde{v}_x : \mathcal{O}_{X,x} \to \Gamma_x \cup \{\infty\}$. The presheaf $\mathcal{O}_X^+$ on X is defined as

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) \mid \widetilde{v}_x(f) \geq 0 \text{ for every } x \in U\} \ .$$

Notice that uniformity of perfect Tate rings is preserved under rational localization.

Proposition 1.6.8. *Let (R, R^+) be an affinoid ring of characteristic $p > 0$, $U \subset \mathrm{Spa}(R, R^+)$ a rational subset, and $h : (R, R^+) \to (S, S^+)$ the rational localization corresponding to U. If R is a perfect uniform Tate ring, then so is S.*

Proof. Since R is a Tate ring, S is also a Tate ring. We show that S is perfect and uniform. Since R is a perfect uniform Tate ring, the inverse Frobenius Ψ_R on (R, R^+) is continuous (Proposition 1.4.11). Hence one can check that $h' = h \circ \Psi_R$ also defines the rational localization corresponding to U, because $\mathrm{Spa}(\Psi_R)$ is identity and thus does not change any topological structure. Therefore the Frobenius homomorphism Φ_S on (S, S^+), which satisfies $h = \Phi_S \circ h'$, is an isomorphism. Hence S is perfect, and uniform by Lemma 1.4.11. □

Lemma 1.5.13 can be naturally extended to the following lemma. The proof is similar.

Lemma 1.6.9. *Let R be a perfect uniform Tate ring. Let $z = \sum_{i=0}^{\infty} p^i [\zeta_i] \in W(R^+)$ be an element such that ζ_0 is topologically nilpotent and $\zeta_1 \in (R^+)^\times$. Let $x \in W(R^+)$, and let $c > 0$ be an integer. Then there exists $y_c = \sum_{i=0}^{\infty} p^i [\eta_{c,i}] \in W(R^+)$ such that $x \equiv y_c \mod (z)$ and for every $x \in \mathrm{Spa}(R, R^+)$ we have*

$$v(\eta_{c,i}) \geq \min\{v(\eta_{c,0}), v(t^c)\} \quad (\forall i \geq 1) \ .$$

1.6.2 Adic spaces

First we recall *valued triples.*

Definition 1.6.10 (Valued triple)**.**

1. A *valued triple* is a datum $(X, \mathcal{O}_X^+, \mathcal{O}_X, \{v_x\}_{x \in X})$ consisting of
   - a topological space X;
   - sheaves of topological rings $\mathcal{O}_X^+$ and $\mathcal{O}_X$ on X together with a morphism $\iota : \mathcal{O}_X^+ \hookrightarrow \mathcal{O}_X$ such that
     (a) ι maps $\mathcal{O}_X^+$ isomorphically onto an open subsheaf of $\mathcal{O}_X$, and

(b) $(X, \mathcal{O}_X^+)$ and $(X, \mathcal{O}_X)$ are locally ringed spaces;

- a set $\{v_x\}_{x\in X}$ consisting of continuous semivaluaions $v_x : \mathcal{O}_{X,x} \to \Gamma_x \cup \{\infty\}$ $(x \in X)$ such that for any $x \in X$, $\mathcal{O}_{X,x}^+ = \{s \in \mathcal{O}_{X,x} \mid v_x(s) \geq 0\}$, and $\{s \in \mathcal{O}_{X,x} \mid v_x(s) > 0\}$ forms the maximal ideal of $\mathcal{O}_{X,x}^+$.

2. A morphism of valued triples $(X, \mathcal{O}_X^+, \mathcal{O}_X, \{v_x\}_{x\in X}) \to (Y, \mathcal{O}_Y^+, \mathcal{O}_Y, \{v_y\}_{y\in Y})$ is a morphism of topologically locally ringed spaces $(\varphi, h) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ with the following properties:
   - $h(\varphi^{-1}\mathcal{O}_Y^+) \subset \mathcal{O}_X^+$, and the induced morphism $(\varphi, h) : (X, \mathcal{O}_X^+) \to (Y, \mathcal{O}_Y^+)$ is also a morphism of topologically locally ringed spaces;
   - for any $x \in X$, $v_{\varphi(x)}$ is equivalent to $v_x \circ h_x$.

Example 1.6.11. Let (A, A^+) be an affinoid ring, and X the associated adic spectrum $\mathrm{Spa}(A, A^+)$. Suppose that $\mathcal{O}_X$ is a sheaf (for example, this assumption is realized if A has a ring of definition that is Noetherian; cf. [31, Theorem 2.2]). Then $\mathcal{O}_X^+$ is also a sheaf, and the datum $(X, \mathcal{O}_X^+, \mathcal{O}_X, \{v_x\}_{x\in X})$ is a valued triple. In this case, we call $(X, \mathcal{O}_X^+, \mathcal{O}_X, \{v_x\}_{x\in X})$ the *adic space associated to* (A, A^+).

Definition 1.6.12 (Adic spaces)**.**

1. An *affinoid adic space* is a valued triple that is isomorphic to the adic space associated to an affinoid ring (see Example 1.6.11).

2. An *adic space* is a valued triple that is locally an affinoid adic space. A morphism of adic spaces is a morphism of valued triples.

1.6.3 Study of integral structures

Here we establish the following result. It plays a key role in the proof of the weak extension theorem (Theorem 4.2.1).

Proposition 1.6.13. *Let (A, A^+) be an affinoid Tate ring, and $(A_0, (t))$ a pair of definition of A. Let $s \in A_0$ be an element such that $t \in sA_0$. Let U be the subspace of* $\mathrm{Spa}(A, A^+)$ *defined as*

$$U := \left\{ x \in \mathrm{Spa}(A, A^+) \;\middle|\; v_{\widetilde{x}}(s) > 0 \;\; \textit{for the maximal generalization } \widetilde{x} \textit{ of } x \right\}.$$

Suppose that A_0 is s-adically Zariskian and integral over a Noetherian ring. Then we have

$$(A_0)_A^+ = \left\{ a \in A \;\middle|\; v_x(a) \geq 0 \textit{ for any } x \in [U] \right\} \tag{1.7}$$

(where $[U]$ denotes the maximal Hausdorff quotient of U).

Proof. Since the containment $\subset$ is clear, let us prove the reverse containment $\supset$. Let $a \in A$ be such that $v_x(a) \geq 0$, where $x \in [\mathrm{Spa}(A, A^+)]$ satisfies $v_x(s) \geq 0$. Let $A_0[a^{-1}]$ be the subring of the localization A_a which is generated by a^{-1} over A_0.[1] Consider the ring extension $A_0[a^{-1}] \subset A_a$. First suppose that a^{-1} is a unit in $A_0[\frac{1}{a}]$. Then we can write

$$a = \frac{c_0}{a^{n-1}} + \frac{c_1}{a^{n-2}} + \cdots + c_{n-1}$$

[1] Notice that $D[\frac{1}{y}]$ is *not* the localization of A with respect to the multiplicative system $\{y^n\}_{n\geq 0}$.

for $c_i \in A_0$. Then we have $a^n - c_{n-1}a^{n-1} - \cdots - c_0 = 0$. Hence $a \in A_0[a^{-1}]$ is integral over A_0.

Thus it suffices to show that $a^{-1} \in A_0[a^{-1}]$ is a unit. We suppose that it is not so and derive a contradiction. We may assume that a is not nilpotent. Since the element $s \in A_0$ is in the Jacobson radical by assumption, there exists a chain of ideals $\mathfrak{p} \subset \mathfrak{m} \subset A_0[a^{-1}]$ such that $\mathfrak{p}$ is minimal and $s, a^{-1} \in \mathfrak{m}$. On the other hand, there exists a Noetherian subring $R \subset A_0$ such that $R[a^{-1}] \subset A_0[a^{-1}]$ is an integral extension. Notice that $R[a^{-1}]$ is also Noetherian by Hilbert's Basis Theorem.

Then, one can find a valuation ring $A_0[a^{-1}]/\mathfrak{p} \subset V \subset \mathrm{Frac}(A_0[a^{-1}]/\mathfrak{p})$ such that the center (the maximal ideal) of V contains a^{-1} and the Krull dimension of V is 1, as follows. Let $\mathfrak{n} := \mathfrak{m} \cap R[a^{-1}]$ and $\mathfrak{q} := \mathfrak{p} \cap R[a^{-1}]$. Then we have a Noetherian subdomain $R[a^{-1}]/\mathfrak{q} \subset \mathrm{Frac}(R[a^{-1}]/\mathfrak{q})$. By [55, Theorem 6.3.2 and Theorem 6.3.3], there is a Noetherian valuation ring $V_{\mathfrak{n}}$ such that $R[a^{-1}]/\mathfrak{q} \subset V_{\mathfrak{n}} \subset \mathrm{Frac}(R[a^{-1}]/\mathfrak{q})$ and the center of $V_{\mathfrak{n}}$ contains $\overline{\mathfrak{n}} \subset R[a^{-1}]/\mathfrak{q}$. We have the commutative diagram:

$$\begin{array}{ccc} \mathrm{Frac}(R[a^{-1}]/\mathfrak{q}) & \longrightarrow & \mathrm{Frac}(A_0[a^{-1}]/\mathfrak{p}) \\ \uparrow & & \uparrow \\ V_{\mathfrak{n}} & \longrightarrow & V \end{array}$$

where V is defined as the localization of the integral closure of $V_{\mathfrak{n}}$ in $\mathrm{Frac}(A_0[a^{-1}]/\mathfrak{p})$ at the maximal ideal containing $\overline{\mathfrak{m}}$. Hence V is a valuation ring of Krull dimension 1. Since $\mathfrak{p} \subset A_0[a^{-1}]$ is a minimal prime ideal and $t \in A_0[a^{-1}]$ is a non-zerodivisor, we have $t \notin \mathfrak{p}$. Hence the composite $A_0 \to A_0[a^{-1}] \to V$ extends to the map $A \to A_a \to \mathrm{Frac}(V)$, which induces a continuous semivalation $v : A \to \mathbb{R} \cup \{\infty\}$ with $v(s) > 0$ because $s \in \mathfrak{m}$.

By our assumption, we have $v(a) \geq 0$. Since $a^{-1} \in V$ is in the center, we know $v(a^{-1}) > 0$. However, these facts are not compatible with $v(a) + v(a^{-1}) = v(a \cdot a^{-1}) = 0$ and thus, $a^{-1} \in A_0[a^{-1}]$ must be a unit, as desired. ☐

Corollary 1.6.14. *Let A_0 be a ring that is integral over a Noetherian ring, and $t \in A_0$ a non-zerodivisor. Let A be the Tate ring associated to $(A_0, (t))$. Then we have*

$$(A_0)^+_A = A^\circ = \left\{ a \in A \;\middle|\; v_x(a) \geq 0 \textit{ for any } x \in [\mathrm{Spa}(A, (A_0)^+_A)] \right\}.$$

In particular, an element $a \in A_0[\frac{1}{t}]$ is integral over A_0 if and only if it is almost integral over A_0.

Proof. Set $X = \mathrm{Spa}(A, (A_0)^+_A)$. Since we know that

$$(A_0)^+_A \subset A^\circ \subset \left\{ a \in A \;\middle|\; v_x(a) \geq 0 \text{ for any } x \in [X] \right\},$$

it suffices to show the reverse inclusion. Pick $c \in A$ such that $v_x(c) \geq 0$ for any $x \in [X]$. By assumption, there exists a Noetherian subring $R \subset A_0$ such that $t \in R$ and the filtered system $\{R_\lambda\}_{\lambda \in \Lambda}$ of all module-finite R-subalgebras in A_0 satisfies $A_0 = \varinjlim_\lambda R_\lambda$. Then by Lemma 1.2.11, $A_0' := \varinjlim_\lambda (R_\lambda)^{Zar}_t$ is integral over a Noetherian ring R^{Zar}_t. Let A' be the Tate ring associated to $(A_0', (t))$, and $X' = \mathrm{Spa}(A', (A_0')^+_{A'})$. Then Proposition 1.6.13 implies that

$$(A_0')^+_{A'} = (A')^\circ = \left\{ a \in A' \;\middle|\; v_{x'}(a) \geq 0 \text{ for any } x' \in [X'] \right\}.$$

Moreover, for the continuous ring map $\psi : A \to A'$, we have $v(\psi(c))_{x'} \geq 0$ for any $x' \in X'$ by assumption. Thus we find that $\psi(c) \in (A'_0)^+_{A'}$. On the other hand, $A'_0 \cong (A_0)^{Zar}_t$ by Lemma 1.2.11 and hence we have

$$((A_0)^+_A)^{Zar}_t \cong (A'_0)^+_{A'}$$

by Lemma 1.2.10. Since the map $(A_0)^+_A \to ((A_0)^+_A)^{Zar}_t$ becomes an isomorphism after t-adic completion, one can deduce from Beauville-Laszlo's lemma (Lemma 1.2.7) that the diagram of ring homomorphisms

$$\begin{array}{ccc} (A_0)^+_A & \longrightarrow & ((A_0)^+_A)^{Zar}_t \\ \downarrow & & \downarrow \\ A & \xrightarrow[\psi]{} & A' \end{array}$$

is cartesian. Thus we obtain $c \in (A_0)^+_A$, as wanted. $\square$

1.7 Rigid geometry

The goal of this section is to prove Proposition 1.7.19 in §1.7.6, which describes the relationship between adic spaces and rigid spaces. The other subsections are devoted to recalling several definitions and basic facts on rigid geometry that we need. Our basic reference is [20].

1.7.1 Formal schemes

First let us recall the definition of formal schemes and adic formal schemes of finite ideal type.

Definition 1.7.1 (Formal schemes)**.**

1. For an admissible ring A (cf. Definition 1.4.3), the *formal spectrum* $\operatorname{Spf} A$ is a topologically locally ringed space $(X, \mathcal{O}_X)$ defined as follows:
   - the underlying topological space X is the set of all open prime ideals of A endowed with the subspace topology induced by $\operatorname{Spec} A$;
   - the structure sheaf $\mathcal{O}_X$ is the sheaf of topological rings $\varprojlim_I \widetilde{A/I}|_X$, where I runs through all ideals of definition of A.
2. An *affine formal scheme* is a topologically locally ringed space that is isomorphic to $\operatorname{Spf} A$ for an admissible ring A.
3. A *formal scheme* is a topologically locally ringed space that is locally isomorphic to an affine formal scheme. A morphism of formal schemes is a morphism of topologically locally ringed spaces.

Definition 1.7.2 (Adic formal schemes of finite ideal type)**.**

1. A formal scheme X is said to be *adic* if it admits an affine open covering $X = \bigcup_{i \in I} U_i$ such that each U_i is isomorphic to $\operatorname{Spf} A$ for an adic ring A_i (cf. Definition 1.4.3).

2. An adic formal scheme X is said to be *of finite ideal type* if it admits an affine open covering $X = \bigcup_{i\in I} U_i$ such that each U_i is isomorphic to $\mathrm{Spf}\, A_i$ for an adic ring A_i of finite ideal type (cf. Definition 1.4.3).

The notion of ideals of definition (of an admissible ring) is globalized as follows.

Definition 1.7.3 (Ideals of definition)**.**

1. Let A be an admissible ring, and $J \subset A$ an open ideal. Then we denote by J^{Δ} the sheaf $\varprojlim_{I\subset J} \widetilde{J/I}$ on $X := \mathrm{Spf}\, A$, where I runs through all ideals of definition contained in J (in fact, J^{Δ} is a sheaf of ideals of $\mathcal{O}_X$).

2. Let X be a formal scheme. An *ideal of definition* of X is an ideal sheaf $\mathcal{I}$ of $\mathcal{O}_X$ with the following property:

   - there exists an affine open covering $X = \bigcup_{i\in I} U_i$ such that each isomorphism $(f_i, \varphi_i) : U_i \xrightarrow{\cong} \mathrm{Spf}\, A_i$ induces an isomorphism $f_i^{-1}(I_i^{\Delta}) \cong \mathcal{I}|_{U_i}$ for an ideal of definition I_i of A_i.

   An ideal of definition $\mathcal{I}$ of X is said to be *of finite type* if it is of finite type as an $\mathcal{O}_X$-module.

Similarly, the notion of adic morphisms of adic rings (cf. [20, **I**, Definition 1.1.5]) is also globalized.

Definition 1.7.4 (Adic morphisms)**.** A morphism $f : X \to Y$ of adic formal schemes of finite ideal type is said to be *adic* if there exists an open covering $Y = \bigcup_{j\in J} V_j$ with the following property:

- for each $j \in J$, there exists an ideal of definition $\mathcal{I}_j$ of V_j such that the ideal $\mathcal{I}_j\mathcal{O}_{f^{-1}(V_j)}$ is an ideal of definition of the open formal subscheme $f^{-1}(V_j) \subset X$.

1.7.2 Admissible blow-ups

First let us recall the definition of admissible ideals (cf. [20, **I**, Definition 3.7.4]).

Definition 1.7.5 (Admissible ideals)**.** Let X be an adic formal scheme of finite ideal type. An *admissible ideal* of X is an ideal sheaf $\mathcal{J}$ of $\mathcal{O}_X$ that satisfies the following conditions:

1. $\mathcal{J}$ is complete (cf. [20, **I**, §3.1(a)]);

2. for any open subset $U \subset X$ considered as an open formal subscheme and for any ideal of definition $\mathcal{I}$ of finite type of U, the sheaf $(\mathcal{J}|_U)/\mathcal{I}(\mathcal{J}|_U)$ is a quasi-coherent sheaf on the scheme $(U, \mathcal{O}_U/\mathcal{I})$;

3. $\mathcal{J}$ is of finite type as an $\mathcal{O}_X$-module;

4. $\mathcal{J}$ contains locally an ideal of definition.

An *admissible blow-up* is defined as follows. Let X be an adic formal scheme of finite ideal type such that there exists an ideal of definition of finite type $\mathcal{I} \subset \mathcal{O}_X$. For each $k \geq 0$, we denote by X_k the closed subscheme $(X, \mathcal{O}_X/\mathcal{I}^{k+1})$ of X. Let $\mathcal{J} \subset \mathcal{O}_X$ be an admissible ideal. Then we can consider the projective X_k-scheme

$$X'_k = \mathbf{Proj}(\oplus_{n>0}(\mathcal{J}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X_k})) \to X_k\ . \tag{1.8}$$

Here, for any $k \leq l$, we have a natural closed immersion $X'_k \hookrightarrow X'_l$ that is compatible with the immersion $X_k \hookrightarrow X_l$. In this situation, the direct limit $\varinjlim_{k \geq 0} X'_k$ is an adic formal scheme of finite ideal type, and (1.8) induces the adic morphism

$$\varinjlim_{k \geq 0} \mathbf{Proj}(\oplus_{n>0}(\mathcal{J}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X_k})) \xrightarrow{\pi} X \ . \tag{1.9}$$

Definition 1.7.6 (Admissible blow-ups)**.** Let X be an adic formal scheme of finite ideal type, and $\mathcal{J} \subset \mathcal{O}_X$ an admissible ideal. An *admissible blow-up along* $\mathcal{J}$ is an adic morphism $\pi : X' \to X$ of adic formal schemes of finite type that is locally isomorphic to a morphism of the form (1.9).

Notice that the notion of *quasi-compactness* and *quasi-separatedness* is defined for morphisms of formal schemes as it is defined for morphisms of schemes (cf. [20, **I**, §1.6]). We say that a formal scheme X is *coherent* if the morphism $X \to \operatorname{Spec} \mathbb{Z}$ is quasi-compact and quasi-separated. For example, any affine formal scheme is coherent. Any coherent adic formal scheme of finite ideal type has an ideal of definition of finite type ([20, **I**, Corollary 3.7.12]).

Definition 1.7.7. Let X be a coherent adic formal scheme of finite ideal type.

1. We denote by $\mathbf{BL}_X$ the category defined as follows.
   - $\mathrm{ob}(\mathbf{BL}_X)$ consists of all admissible blow-ups of X.
   - For objects $\pi : X' \to X$ and $\pi' : X'' \to X$ of $\mathbf{BL}_X$, a morphism $\pi' \to \pi$ is defined as a morphism of formal schemes $X'' \to X'$ over X.
2. We denote by AId_X the set of all admissible ideals of X equipped with the following preorder: $\mathcal{J} \leq \mathcal{J}'$ if there exists an admissible ideal $\mathcal{J}'$ such that $\mathcal{J} = \mathcal{J}'\mathcal{J}''$.

Proposition 1.7.8 ([20, **II**, Proposition 1.3.1])**.**

1. *The category* $\mathbf{BL}_X$ *is cofiltered, and the identity map* $\mathbf{Id}_X$ *gives the final object.*
2. $\mathrm{AId}^{\mathrm{op}}$ *is a directed set, and the functor* $\mathrm{AId}_X \to \mathbf{BL}_X$ *that associates to* $\mathcal{J}$ *the admissible blow-up along* $\mathcal{J}$ *is cofinal.*

1.7.3 Coherent rigid spaces

Let us recall the category of coherent rigid spaces ([20, **II**, Definition 2.1.1]). In view of Raynaud's discovery ([45]), this category is defined as the quotient of the category of coherent adic formal schemes of finite ideal type modulo admissible blow-ups. We denote by $\mathbf{AcCFs}^*$ the category of adic coherent formal schemes of finite ideal type with adic morphisms.

Definition 1.7.9 (Coherent rigid spaces)**.**

1. The category of *coherent rigid spaces* is the category $\mathbf{CRf}$ defined as follows.
   - The objects of $\mathbf{CRf}$ are the same as those of the category $\mathbf{AcCFs}^*$. For any $X \in \mathrm{ob}(\mathbf{AcCFs}^*)$, we denote by X^{rig} the same object regarded as an object of $\mathbf{CRf}$.
   - For any $X, Y \in \mathrm{ob}(\mathbf{AcCFs}^*)$, $\mathrm{Hom}_{\mathbf{CRf}}(X^{\mathrm{rig}}, Y^{\mathrm{rig}})$ is the colimit $\varinjlim F_{(X,Y)}$ where $F_{(X,Y)}$ is the functor $\mathbf{BL}_X^{\mathrm{op}} \to \mathbf{Sets}$ that associates to $\pi : X' \to X$ the set $\mathrm{Hom}_{\mathbf{AcCFs}^*}(X', Y)$.

An object of the category **CRf** is called a *coherent rigid space*.

2. Let $\mathcal{X}$ be a coherent rigid space. A *formal model* of $\mathcal{X}$ is a couple (X, ϕ) consisting of $X \in \mathrm{ob}(\mathbf{AcCFs}^*)$ and an isomorphism $\phi : X^{\mathrm{rig}} \xrightarrow{\cong} \mathcal{X}$ in **CRf**.

By definition, a morphism $\varphi \in \mathrm{Hom}_{\mathbf{CRf}}(X^{\mathrm{rig}}, Y^{\mathrm{rig}})$ is given by a diagram of adic morphisms

$$\begin{array}{ccc} & X' & \\ {\scriptstyle\pi}\swarrow & & \searrow \\ X & & Y \end{array}$$

where $\pi : X' \to X$ is an admissible blow-up.

General rigid spaces are defined by patching coherent rigid spaces (cf. [20, **II**, §2.2(c)]).

1.7.4 Visualization

First we recall the definition of Zariski-Riemann spaces and the structure sheaves.

Definition 1.7.10 (Zariski-Riemann spaces)**.** Let $\mathcal{X}$ be a coherent rigid space, and (X, ϕ) a formal model of X. Let S_X be the functor from $\mathbf{BL}_X$ to the category of locally ringed spaces that associates to $\pi : X' \to X$ the underlying locally ringed space X'.

1. We define the locally ringed space $(\langle\mathcal{X}\rangle, \mathcal{O}_{\mathcal{X}}^{\mathrm{int}})$ as the limit

$$(\langle\mathcal{X}\rangle, \mathcal{O}_{\mathcal{X}}^{\mathrm{int}}) = \varprojlim S_X$$

(cf. Proposition 1.7.8). The topological space $\langle\mathcal{X}\rangle$ is called the *Zariski-Riemann space* associated to $\mathcal{X}$, and the sheaf of local rings $\mathcal{O}_{\mathcal{X}}^{\mathrm{int}}$ is called the *integral structure sheaf* of $\langle\mathcal{X}\rangle$.

2. The projection map $\langle\mathcal{X}\rangle \to X$ of locally ringed spaces is called the *specialization map* and denoted by $\mathrm{sp}_X : \langle\mathcal{X}\rangle \to X$.

Definition 1.7.11 (Rigid structure sheaves)**.** Let $\mathcal{X}$ be a coherent rigid space, $\langle\mathcal{X}\rangle$ the associated Zariski-Riemann space, and $\mathcal{O}_{\mathcal{X}}^{\mathrm{int}}$ the integral structure sheaf on $\langle\mathcal{X}\rangle$.

1. An *ideal of definition of finite type* is an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}^{\mathrm{int}}$ with the following property:

   - there exists a formal model (X, ϕ) of $\mathcal{X}$ and an ideal of definition $\mathcal{I}_X$ of finite type (of X) such that $\mathcal{I} = (\mathrm{sp}_X^{-1}\mathcal{I}_X)\mathcal{O}_{\mathcal{X}}^{\mathrm{int}}$.

2. The *rigid structure sheaf* of $\mathcal{X}$ is the sheaf of rings $\mathcal{O}_{\mathcal{X}}$ on $\langle\mathcal{X}\rangle$ defined as

$$\mathcal{O}_{\mathcal{X}} = \varinjlim_{n>0} \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}^{\mathrm{int}}}(\mathcal{I}^n, \mathcal{O}_{\mathcal{X}}^{\mathrm{int}}) \ ,$$

where $\mathcal{I}$ is an ideal of definition of finite type (this definition is independent of the choice of $\mathcal{I}$).

Let $\mathcal{X}$ be a coherent rigid space and $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}^{\mathrm{int}}$ an ideal of definition of finite type. Then one can equip $\mathcal{O}_{\mathcal{X}}^{\mathrm{int}}$ with the structure as a sheaf of topological rings such that $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}^{\mathrm{int}}$ is an open ideal sheaf and for any $x \in \langle \mathcal{X} \rangle$ the topology on $\mathcal{O}_{\mathcal{X},x}^{\mathrm{int}}$ is $\mathcal{I}_x$-adic. Moreover, it induces the structure on $\mathcal{O}_{\mathcal{X}}$ as a sheaf of topological rings such that $\mathcal{O}_{\mathcal{X}}^{\mathrm{int}}$ is canonically isomorphic to an open subsheaf of $\mathcal{O}_{\mathcal{X}}$.

By the property of admissible blow-ups, the following fact follows.

Proposition 1.7.12 ([20, **II**, Corollary 3.2.7 and 3.2.8])**.** *Let $\mathcal{X}$ be a coherent rigid space, and $x \in \langle \mathcal{X} \rangle$ a point. Let $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}^{\mathrm{int}}$ be an ideal of definition of finite type. Set $V_x = \mathcal{O}_{\mathcal{X},x}^{\mathrm{int}} / \cap_{n \geq 1} \mathcal{I}_x^n$. Then the following assertions hold.*

1. *$\mathcal{I}_x$ is a principal ideal generated by a non-zerodivisor $a \in \mathcal{O}_{\mathcal{X},x}^{\mathrm{int}}$.*
2. *$\mathcal{O}_{\mathcal{X},x} = \mathcal{O}_{\mathcal{X},x}^{\mathrm{int}}[\frac{1}{a}]$ is a local ring, and V_x is a valuation ring of the residue field of $\mathcal{O}_{\mathcal{X},x}$ that is $\overline{a}$-adically Hausdorff and Henselian.*
3. *$\mathcal{O}_{\mathcal{X},x}^{\mathrm{int}} = \{ f \in \mathcal{O}_{\mathcal{X},x} \mid f \mod \mathfrak{m}_{\mathcal{O}_{\mathcal{X},x}} \in V_x \}$.*

Thus we obtain the following valued triple.

Definition 1.7.13 (Zariski-Riemann triples)**.** Let $\mathcal{X}$ be a coherent rigid space. The *Zariski-Riemann triple* associated to $\langle \mathcal{X} \rangle$ is the valued triple

$$\mathbf{ZR}(\mathcal{X}) = (\langle \mathcal{X} \rangle, \mathcal{O}_{\mathcal{X}}^{\mathrm{int}}, \mathcal{O}_{\mathcal{X}}, \{v_x\}_{x \in X})$$

where v_x is a continuous semivaluation on $\mathcal{O}_{\mathcal{X},x}$ corresponding to V_x given in Proposition 1.7.12.

Also for general rigid spaces, the associated Zariski-Riemann triples are defined (cf. [20, **II**, §3]).

Rigid points

Let us recall the notion of *points* in rigid geometry.

Definition 1.7.14 (Rigid points, cf. [20, **II**, Proposition 3.3.3])**.**

1. Let $\mathcal{X}$ be a coherent rigid space. A *rigid point* of $\mathcal{X}$ is a morphism of locally ringed spaces
$$\alpha : \operatorname{Spf} V \to (\langle \mathcal{X} \rangle, \mathcal{O}_{\mathcal{X}}^{\mathrm{int}})$$
where V is a valuation ring that is a-adically Hausdorff complete for some $a \in V \setminus \{0\}$.
2. Let X be an adic formal scheme of finite ideal type. A *rigid point* of X is a morphism of formal schemes
$$\alpha : \operatorname{Spf} V \to X$$
where V is a valuation ring that is a-adically Hausdorff complete for some $a \in V \setminus \{0\}$.
3. Let $\mathcal{X}$ and X be as above. Two rigid points $\alpha : \operatorname{Spf} V \to \langle \mathcal{X} \rangle$ and $\beta : \operatorname{Spf} W \to \langle \mathcal{X} \rangle$ (resp. $\alpha' : \operatorname{Spf} V \to X$ and $\beta' : \operatorname{Spf} W \to X$) of $\mathcal{X}$ (resp. X) are said to be *isomorphic* if there exists an isomorphism of formal schemes $f : \operatorname{Spf} V \xrightarrow{\cong} \operatorname{Spf} W$ such that $\alpha = \beta \circ f$ (resp. $\alpha' = \beta' \circ f$).

Rigid points have the following notable properties.

Proposition 1.7.15 ([20, **II**, Proposition 3.3.4]). *Let $\mathscr{X}$ be a coherent rigid space, and $x \in \langle \mathscr{X} \rangle$ be any point.*

1. *There exists a rigid point of the form*

$$\alpha_x : \operatorname{Spf} \widehat{V_x} \to \langle \mathscr{X} \rangle, \ \mathfrak{m}_{\widehat{V_x}} \mapsto x,$$

where $\widehat{V_x}$ is the a-adic completion of V_x (see Proposition 1.7.12 for the notation) such that the induced map of stalks at x is the canonical map

$$A_x = \mathcal{O}^{\mathrm{int}}_{\mathscr{X},x} \to \widehat{V_x} \ .$$

2. *Let $\alpha : \operatorname{Spf} V \to \langle \mathscr{X} \rangle$ be a rigid point such that $\alpha(\mathfrak{m}_V) = x$. Then there exists uniquely an injective homomorphism $\widehat{V_x} \hookrightarrow V$ such that V dominates $\widehat{V_x}$ and the diagram*

$$\begin{array}{ccc} \operatorname{Spf} V & \xrightarrow{\alpha} & \langle \mathscr{X} \rangle \\ \downarrow & \nearrow_{\alpha_x} & \\ \operatorname{Spf} \widehat{V_x} & & \end{array}$$

commutes.

Proposition 1.7.16 ([20, **II**, Proposition 3.3.6(1)]). *Let $C_{\mathscr{X}}$ denote the isomorphism classes of of rigid points of $\mathscr{X}$, and C_X denote the isomorphism classes of rigid points of X. Then the map*

$$C_{\mathscr{X}} \to C_X, \ \alpha \mapsto \operatorname{sp}_X \circ \alpha$$

is bijective.

1.7.5 Affinoids and affinoid subdomains

Definition 1.7.17 (Affinoids and affinoid subdomains).

1. A coherent rigid space $\mathscr{X}$ is called an *affinoid* if there exists a formal model (X, ϕ) of $\mathscr{X}$ with X an affine formal scheme.

2. For an affinoid $\mathscr{X}$, an affinoid open subspace of $\mathscr{X}$ (cf. [20, **II**, Definition 6.1.2]) is called an *affinoid subdomain* of $\mathscr{X}$.

Example 1.7.18 (Rational subdomains). Let $\mathscr{X} = (\operatorname{Spf} A)^{\mathrm{rig}}$ be an affinoid, and $I = (t) \subset A$ an ideal of definition. Take elements $f_0, \ldots, f_m \in A$ such that $(f_0, \ldots, f_m)A[\frac{1}{t}]$ is the unit ideal. Set $J := (f_0, \ldots, f_m)A_0$ and $\mathscr{J} := J^{\Delta}$. Take the admissible blow-up $X' \to X$ along $\mathscr{J}$, and let W be the affine part of X', where the ideal $\mathscr{J}X'$ is generated by f_0. Then the associated rigid space W^{rig} is an affinoid subdomain of $\mathscr{X}$ denoted by $\mathscr{X}(\frac{f_1}{f_0}, \ldots, \frac{f_m}{f_0})$, and called a *rational subdomain*. Let $A[\frac{f_1}{f_0}, \ldots, \frac{f_m}{f_0}]$ be the A-subalgebra of the localization A_{f_0} generated by $\frac{f_1}{f_0}, \ldots, \frac{f_m}{f_0}$. We define a ring A_W as the t-adic completion of $A[\frac{f_1}{f_0}, \ldots, \frac{f_m}{f_0}]$. Then we have $W = \operatorname{Spf} A_W$.

1.7.6 Relation with adic spaces

Here we establish the following result. It is applied to the proof of one of our main results (Theorem 3.4.1).

Proposition 1.7.19. *Let (A, A^+) be a complete affinoid ring such that* $\mathrm{Spa}(A, A^+)$ *is an adic space. Let (A_0, I_0) be a pair of definition of A. Put $X := \mathrm{Spf}(A_0)$, $\mathcal{X} := X^{\mathrm{rig}}$, and $Z := \mathrm{Spa}(A, A^+)$. Then there exist a continuous map $\varphi : \mathrm{Spa}(A, A^+) \to \langle \mathcal{X} \rangle$ and an isomorphism of sheaves $\mathcal{O}_{\mathcal{X}} \xrightarrow{\cong} \varphi_* \mathcal{O}_Z$.*

Proof. First we construct a continuous map

$$\varphi : |Z| \to |\langle \mathcal{X} \rangle| \ .$$

Pick $z \in Z$. Then we have a corresponding semivalutaion $v_z : A \to \Gamma_z \cup \{\infty\}$. Put $\mathfrak{p}_z := v_z^{-1}(\{\infty\})$. Then v_z factors through the valuation $\widetilde{v}_z : \mathrm{Frac}(A/\mathfrak{p}_z) \to \Gamma_z \cup \{\infty\}$. Let V_z be the valuation ring of $\mathrm{Frac}(A/\mathfrak{p}_z)$ corresponding to $\widetilde{v}_z$, and $\widehat{V_z}$ the completion. Then we have a continuous ring homomorphism $A_0 \to \widehat{V_z}$. Hence we obtain rigid points $\mathrm{Spf}\, \widehat{V_z} \to \mathrm{Spf}\,(A_0)$ and $\alpha_z : \mathrm{Spf}\, \widehat{V_z} \to \langle \mathcal{X} \rangle$. Let $\mathfrak{m}_z$ be the maximal ideal of $\widehat{V_z}$. Then we obtain a map

$$\varphi : |Z| \to |\langle \mathcal{X} \rangle|,\ z \mapsto \alpha_z(\mathfrak{m}_z) \ .$$

Let us construct an isomorphism $\mathcal{O}_{\mathcal{X}} \cong \varphi_* \mathcal{O}_Z$. We may assume that $I_0 = tA_0$ for some $t \in A_0$. Consider a presheaf $\mathcal{F}$ on $\langle \mathcal{X} \rangle$ defined by

$$V \mapsto (\varinjlim_{\mathcal{J} \in \mathrm{AId}_X} (\varinjlim_{\mathrm{sp}_{X_{\mathcal{J}}}(V) \subset W} \mathcal{O}_{X_{\mathcal{J}}}(W)))[\tfrac{1}{t}] \ .$$

Notice that the sheafification of $\mathcal{F}$ is isomorphic to $\mathcal{O}_{\mathcal{X}}$. Since $\mathcal{O}_Z$ is a sheaf, it suffices to find an open basis $\{U_\lambda\}_{\lambda \in \Lambda}$ of $\langle \mathcal{X} \rangle$ and a family of isomorphisms $\{h_\lambda : \mathcal{F}(U_\lambda) \to \varphi_* \mathcal{O}_Z(U_\lambda)\}_{\lambda \in \Lambda}$ with the following property: for any $\lambda \in \Lambda$, any $x \in U_\lambda$, and any open neigborhood V of x contained in U_λ, there exists some $\lambda' \in \Lambda$ such that $U_{\lambda'} \subset V$, and h_λ and $h_{\lambda'}$ commute with the restriction maps. Let us construct an isomorphism $h_{\mathcal{W}} : \mathcal{F}(\langle \mathcal{W} \rangle) \to \varphi_* \mathcal{O}_Z(\langle \mathcal{W} \rangle)$ for any rational subdomain $\mathcal{W}$ of $\mathcal{X}$ with the desired property. Notice that for any distinguished formal model W of $\mathcal{W}$, we have $\mathcal{F}(\langle \mathcal{W} \rangle) = \varinjlim_{\mathcal{K} \in \mathrm{AId}_W} \Gamma(W_{\mathcal{K}}, \mathcal{O}_{W_{\mathcal{K}}})[\frac{1}{t}]$ (cf. [20, **II**, Definition 2.1.8 and Proposition 3.1.5]). Now we fix a rational subdomain $\mathcal{W}$, its formal model W, and the ring $(A_0)_W$ as in Example 1.7.18. We want to construct an isomorphism

$$\Gamma(W_{\mathcal{K}}, \mathcal{O}_{W_{\mathcal{K}}})[\tfrac{1}{t}] \xrightarrow{\cong} \Gamma(\langle \mathcal{W} \rangle, \varphi_* \mathcal{O}_Z)$$

such that the diagram

$$\begin{array}{ccc} \Gamma(W, \mathcal{O}_{X'}) & \longrightarrow & \Gamma(W_{\mathcal{K}}, \mathcal{O}_{W_{\mathcal{K}}})[\frac{1}{t}] \\ & \searrow & \downarrow \cong \\ & & \Gamma(\langle \mathcal{W} \rangle, \varphi_* \mathcal{O}_Z) \end{array}$$

commutes. Pick $g_1, \ldots, g_l \in (A_0)_W$ such that $(g_1, \ldots, g_l)(A_0)_W[\frac{1}{t}]$ is the unit ideal. Set $K = (g_1, \ldots, g_l)(A_0)_W$ and $\mathcal{K} = K^{\Delta}$. Let $W_{\mathcal{K}}^{(j)}$ be the affine part of $W_{\mathcal{K}}$, where the ideal $\mathcal{K}\mathcal{O}_{W_{\mathcal{K}}}$ is generated by g_j. Then we have the exact sequence

$$0 \to \Gamma(W_{\mathcal{K}}, \mathcal{O}_{W_{\mathcal{K}}}) \to \prod_j \Gamma(W_{\mathcal{K}}^{(j)}, \mathcal{O}_{W_{\mathcal{K}}}) \to \prod_{j,j'} \Gamma(W_{\mathcal{K}}^{(j)} \cap W_{\mathcal{K}}^{(j')}, \mathcal{O}_{W_{\mathcal{K}}}) \ .$$

Here each restriction map $\Gamma(W_{\mathcal{K}}^{(j)}, \mathcal{O}_{W_{\mathcal{K}}}) \to \Gamma(W_{\mathcal{K}}^{(j)} \cap W_{\mathcal{K}}^{(j')}, \mathcal{O}_{W_{\mathcal{K}}})$ is given by taking t-adic completion for the $(A_0)_W$-algebra homomorphism

$$(A_0)_W[\tfrac{g_1}{g_j}, \dots, \tfrac{g_l}{g_j}] \to (A_0)_W[\tfrac{g_1}{g_j}, \dots, \tfrac{g_l}{g_j}, \tfrac{g_1}{g_{j'}}, \dots, \tfrac{g_l}{g_{j'}}] \ ,$$

which becomes the restriction map

$$\Gamma(\varphi^{-1}(\langle (W_{\mathcal{K}}^{(j)})^{\mathrm{rig}} \rangle), \mathcal{O}_Z) \to \Gamma(\varphi^{-1}(\langle (W_{\mathcal{K}}^{(j)})^{\mathrm{rig}} \rangle) \cap \varphi^{-1}(\langle (W_{\mathcal{K}}^{(j')})^{\mathrm{rig}} \rangle), \mathcal{O}_Z)$$

after inverting t. Thus, we have the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Gamma(W_{\mathcal{K}}, \mathcal{O}_{W_{\mathcal{K}}})[\frac{1}{t}] & \longrightarrow & \prod_j \Gamma(W_{\mathcal{K}}^{(j)}, \mathcal{O}_{W_{\mathcal{K}}})[\frac{1}{t}] & \longrightarrow & \prod_{j,j'} \Gamma(W_{\mathcal{K}}^{(j)} \cap W_{\mathcal{K}}^{(j')}, \mathcal{O}_{W_{\mathcal{K}}})[\frac{1}{t}] \\
 & & & & \cong \downarrow & & \downarrow \cong \\
0 & \longrightarrow & \Gamma(\langle (W_{\mathcal{K}})^{\mathrm{rig}} \rangle, \varphi_* \mathcal{O}_Z) & \longrightarrow & \prod_j \Gamma(\langle (W_{\mathcal{K}}^{(j)})^{\mathrm{rig}} \rangle, \varphi_* \mathcal{O}_Z) & \longrightarrow & \prod_{j,j'} \Gamma(\langle (W_{\mathcal{K}}^{(j)})^{\mathrm{rig}} \rangle \cap \langle (W_{\mathcal{K}}^{(j')})^{\mathrm{rig}} \rangle, \varphi_* \mathcal{O}_Z)
\end{array}$$

where the vertical maps are isomorphisms. Hence we obtain the desired isomorphism

$$\Gamma(W_{\mathcal{K}}, \mathcal{O}_{W_{\mathcal{K}}})[\tfrac{1}{t}] \xrightarrow{\cong} \Gamma(\langle (W_{\mathcal{K}})^{\mathrm{rig}} \rangle, \varphi_* \mathcal{O}_Z) \ .$$

Hence we obtain isomorphisms $h_{\mathcal{W}} : \mathcal{F}(\langle \mathcal{W} \rangle) \to \varphi_* \mathcal{O}_Z(\langle \mathcal{W} \rangle)$ and $h_{\mathcal{W},\mathcal{K},j} : \mathcal{F}(\langle (W_{\mathcal{K}}^{(j)})^{\mathrm{rig}} \rangle) \to \varphi_* \mathcal{O}_Z(\langle (W_{\mathcal{K}}^{(j)})^{\mathrm{rig}} \rangle)$ $(j = 1, \dots, l)$ such that the diagram

$$\begin{array}{ccc}
\mathcal{F}(\langle \mathcal{W} \rangle) & \longrightarrow & \mathcal{F}(\langle (W_{\mathcal{K}}^{(j)})^{\mathrm{rig}} \rangle) \\
{\scriptstyle h_{\mathcal{W}}} \downarrow & & \downarrow {\scriptstyle h_{\mathcal{W},\mathcal{K},j}} \\
\varphi_* \mathcal{O}_Z(\langle \mathcal{W} \rangle) & \longrightarrow & \varphi_* \mathcal{O}_Z(\langle (W_{\mathcal{K}}^{(j)})^{\mathrm{rig}} \rangle)
\end{array}$$

commutes. Since open coverings of $\langle \mathcal{W} \rangle$ of the form $\langle \mathcal{W} \rangle = \bigcup_{j=1}^{l} \langle (W_{\mathcal{K}}^{(j)})^{\mathrm{rig}} \rangle$ are cofinal, each $(W_{\mathcal{K}}^{(j)})^{\mathrm{rig}}$ is a rational subdomain of $\mathcal{X}$, and $h_{(W_{\mathcal{K}}^{(j)})^{\mathrm{rig}}} = h_{\mathcal{W},\mathcal{K},j}$, the family of isomorphisms $\{h_{\mathcal{W}}\}$ has the desired property. □

1.8 Finite étale extension

1.8.1 Trace maps

Let A be a ring, B a finite projective A-algebra, and $B^* := \mathrm{Hom}_A(B, A)$. Then the A-linear map

$$\omega : B \otimes_A B^* \to \mathrm{End}_A(B), \ b \otimes \varphi \mapsto (x \mapsto b\varphi(x))$$

is an isomorpshim. Thus one gets the trace map $\mathrm{Tr}_{B/A} : B \to A$ as the composite

$$B \to \mathrm{End}_A(B) \xrightarrow{\omega^{-1}} B \otimes_A B^* \xrightarrow{\mathrm{ev}} A$$

where $\mathrm{ev} : B \otimes_A B^* \to A$ is the evaluation map $b \otimes \varphi \mapsto \varphi(b)$. Let us recall the following well known fact.

Theorem 1.8.1 ([17, Corollary 4.6.8]). *Let A be a ring, and B a finite étale A-algebra. Then the A-linear map*

$$\tau : B \to B^*,\ b \mapsto \mathrm{Tr}_{B/A}(b\ \cdot\ ) \tag{1.10}$$

is an isomorphism.

We use the following proposition in the proof of the almost purity theorem (see Proposition 2.4.3).

Proposition 1.8.2. *Let A be a ring and B a finite étale A-algebra. Let $\mu : B \otimes_A B \to B$ be the multiplication map and let $e \in B \otimes_A B$ be such that $e^2 = e$, $\mu(e) = 1$, and $e(\mathrm{Ker}(\mu)) = 0$. Then $\omega((\mathbf{Id}_B \otimes \tau)(e)) = \mathbf{Id}_B$. In particular, if $e = \sum_{i=1}^{n} b_i \otimes b_i'$, then $\sum_{i=1}^{n} b_i \mathrm{Tr}_{B/A}(b_i' x) = x$ for every $x \in B$.*

Proof. Since $e(1 \otimes x - x \otimes 1) = 0$, we have

$$\begin{aligned} \sum_{i=1}^{n} b_i \mathrm{Tr}_{B/A}(b_i' x) &= \omega((\mathbf{Id}_B \otimes \tau)((1 \otimes x)e))(1) = \omega((\mathbf{Id}_B \otimes \tau)((x \otimes 1)e))(1) \\ &= \sum_{i=1}^{n} x b_i \mathrm{Tr}_{B/A}(b_i') = x(\sum_{i=1}^{n} b_i \mathrm{Tr}_{B/A}(b_i'))\ . \end{aligned}$$

Hence it suffices to check that $c := \sum_{i=1}^{n} b_i \mathrm{Tr}_{B/A}(b_i')$ is equal to 1. For every $x \in B$, we have

$$\begin{aligned} \mathrm{Tr}_{B/A}(cx) &= \mathrm{ev} \circ \omega^{-1}((xc\ \cdot\ )) = \mathrm{ev}(\sum_{i=1}^{n} x b_i \otimes \mathrm{Tr}_{B/A}(\ \cdot\ b_i')) \\ &= \sum_{i=1}^{n} \mathrm{Tr}_{B/A}(x b_i b_i') = \mathrm{Tr}_{B/A}(x\mu(e)) = \mathrm{Tr}_{B/A}(x)\ . \end{aligned}$$

Hence $c = 1$ by Theorem 1.8.1. □

1.8.2 Hausdorff completeness

Proposition 1.8.3. *Let A be a ring and let B be a finite étale A-algebra. Let $A_0 \subset A$ be a subring with an ideal $I_0 \subset A_0$. Let B_0 be an A_0-subalgebra of B such that $B = \bigcup_{n \geq 1}(B_0 : I_0^n)$. Assume that there exists an integer $c > 0$ such that $\mathrm{Tr}_{B/A}(tm) \in A_0$ for every $t \in I_0^c$ and every $m \in B_0$.*

1. *If A_0 is I_0-adically Hausdorff, then so is B_0.*
2. *If A_0 is I_0-adically complete, then so is B_0.*

Proof. Let us prove 1. Pick $m \in \bigcap_{n=0}^{\infty} I_0^n B_0$. Since (1.10) is injective, it suffices to show that $\mathrm{Tr}_{B/A}(mx) = 0$ for an arbitrary element $x \in B$. Take $l > 0$ for which $I_0^l x \subset B_0$. Then for every $n > 0$, we have $m \in I_0^{n+c+l} B_0$ and thus, there exist $t_{n,i} \in I_0^n$, $u_{n,i} \in I_0^{c+l}$, and $m_{n,i} \in B_0$ $(i = 1, \ldots, r)$ such that $m = \sum_{i=1}^{r} t_{n,i} u_{n,i} m_{n,i}$. Hence

$$\mathrm{Tr}_{B/A}(mx) = \sum_{i=1}^{r} t_{n,i} \mathrm{Tr}_{B/A}(u_{n,i} x m_{n,i}) \in I_0^n A_0$$

for every $n > 0$. Thus, since A_0 is I_0-adically Hausdorff, we have $\mathrm{Tr}_{B/A}(mx) = 0$, as desired.

Next we prove 2. Since B is a finite projective A-module, there exist A-homomorphisms $s : B \to A^{\oplus d}$ and $\pi : A^{\oplus d} \to B$ such that $\pi \circ s$ is the identity. Let us equip A (resp. $A^{\oplus d}$, resp. B) with the topology such that $\{I_0^n A_0\}_{n\geq 1}$ (resp. $\{I_0^n A_0^{\oplus d}\}_{n\geq 1}$, resp. $\{I_0^n B_0\}_{n\geq 1}$) forms a system of fundamental open neighborhoods of 0. We consider these topologies in what follows. Since (1.10) is surjective, there exist $b_1, \ldots, b_d \in B$ such that $s(x) = (\mathrm{Tr}_{B/A}(b_1 x), \ldots, \mathrm{Tr}_{B/A}(b_d x))$ for every $x \in B$. Let $\{m_n\}_{n\geq 1}$ be a Cauchy sequence in B_0 with respect to the I_0-adic topology. Then for each $i = 1, \ldots, d$, $\{\mathrm{Tr}_{B/A}(b_i m_n)\}_{n\geq 1}$ forms a Cauchy sequence in A. Hence by assumption, each $\{\mathrm{Tr}_{B/A}(b_i m_n)\}_{n\geq 1}$ converges to some $a_i \in A$. Then $\{s(m_n)\}_{n\geq 1}$ converges to $(a_1, \ldots, a_d) \in A^{\oplus d}$. Thus, $\{m_n\}_{n\geq 1} = \{\pi(s(m_n))\}_{n\geq 1}$ converges to $\pi((a_1, \ldots, a_d)) \in B$. Since $\pi((a_1, \ldots, a_d)) - m_n \in B_0$ for $n \gg 0$, we have $\pi((a_1, \ldots, a_d)) \in B_0$. Hence the assertion follows. $\square$

Corollary 1.8.4. *Let A_0 be a ring with a non-zerodivisor t and put $A = A_0[\frac{1}{t}]$. Let B be a finite étale A-algebra. Let $B_0 \subset B$ be an A_0-subalgebra for which $B = B_0[\frac{1}{t}]$. Assume that there exists some $l > 0$ such that $\mathrm{Tr}_{B/A}(t^l b) \in A_0$ for every $b \in B_0$. Denote by $\widehat{A_0}$ and $\widehat{B_0}$ the t-adic completions of A_0 and B_0, respectively. Then the natural A_0-algebra homomorphism $B_0 \otimes_{A_0} \widehat{A_0} \to \widehat{B_0}$ induces an isomorphism:*

$$(B_0 \otimes_{A_0} \widehat{A_0})/(0)^{t-\mathrm{sat}} \xrightarrow{\cong} \widehat{B_0} \tag{1.11}$$

(where $(0)^{t\text{-sat}}$ denotes the (t)-saturation of the ideal $(0) \subset B_0 \otimes_{A_0} \widehat{A_0}$). In particular, the natural A-algebra homomorphism

$$(B_0 \otimes_{A_0} \widehat{A_0})[\tfrac{1}{t}] \to (\widehat{B_0})[\tfrac{1}{t}]$$

is an isomorphism.

Proof. Since $\widehat{B_0}$ is t-torsion free, the map $\varphi : B_0 \otimes_{A_0} \widehat{A_0} \to \widehat{B_0}$ induces a commutative diagram:

$$\begin{array}{ccc} B_0 \otimes_{A_0} \widehat{A_0} & \xrightarrow{\pi} & (B_0 \otimes_{A_0} \widehat{A_0})/(0)^{t-\mathrm{sat}} \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \widetilde{\varphi}} \\ & & \widehat{B_0} \end{array}$$

where π is the canonical projection map. We prove that $\widetilde{\varphi}$ is an isomorphism. First we show that $(B_0 \otimes_{A_0} \widehat{A_0})/(0)^{t-\mathrm{sat}}$ is t-adically Hausdorff complete. Let us apply Proposition 1.8.3 by setting $I_0 = (t)$. Put $A' := (\widehat{A_0})[\frac{1}{t}]$. Notice that $(B_0 \otimes_{A_0} \widehat{A_0})/(0)^{t-\mathrm{sat}}$ is isomorphic to the $\widehat{A_0}$-subalgebra $C_0 \subset (B_0 \otimes_{A_0} \widehat{A_0})[\frac{1}{t}]$ that is the image of $B_0 \otimes_{A_0} \widehat{A_0} \to (B_0 \otimes_{A_0} \widehat{A_0})[\frac{1}{t}]$, and $(B_0 \otimes_{A_0} \widehat{A_0})[\frac{1}{t}]$ is identified with the finite étale A'-algebra $B \otimes_A A'$. Since we have

$$\mathrm{Tr}_{B\otimes_A A'/A'}(b \otimes_A 1_{A'}) = \mathrm{Tr}_{B/A}(b) \otimes_A 1_{A'} \ (\forall b \in B),$$

it follows that every element $c \in C_0$ satisfies $\mathrm{Tr}_{B\otimes_A A'/A'}(t^l c) \in \widehat{A_0}$. Hence C_0 is t-adically Hausdorff complete by Proposition 1.8.3, and therefore so is $(B_0 \otimes_{A_0} \widehat{A_0})/(0)^{t-\mathrm{sat}}$, as wanted. Now note that the t-adic completion $\widehat{B_0 \otimes_{A_0} \widehat{A_0}}$ of $B_0 \otimes_{A_0} \widehat{A_0}$ is naturally isomorphic to $\widehat{B_0}$. Then by the universality of completion [20, Proposition 7.1.9 in Chapter 0], the isomorphism $\widehat{B_0 \otimes_{A_0} \widehat{A_0}} \xrightarrow{\cong} \widehat{B_0}$ factors as

$$\widehat{B_0 \otimes_{A_0} \widehat{A_0}} \xrightarrow{\widehat{\pi}} (B_0 \otimes_{A_0} \widehat{A_0})/(0)^{t-\mathrm{sat}} \xrightarrow{\widetilde{\varphi}} \widehat{B_0},$$

where the map $\widehat{\pi}$ is surjective, because π is so. Since $\widetilde{\varphi} \circ \widehat{\pi}$ is an isomorphism, it follows that $\widehat{\pi}$ is an isomorphism. Thus, $\widetilde{\varphi}$ is also an isomorphism. Finally, It readily follows that $(B_0 \otimes_{A_0} \widehat{A_0})[\frac{1}{t}] \to (\widehat{B_0})[\frac{1}{t}]$ is an isomorphism. □

1.8.3 Studies on preuniform pairs

Here we establish several basic properties of preuniform pairs. We especially investigate the following condition.

Definition 1.8.5. Let $f_0 : (A_0, I_0) \to (B_0, J_0)$ be a morphism of pairs. Then we say that f_0 satisfies the condition $(*)$, if $I_0 = tA_0$ and $J_0 = tB_0$ for some $t \in A_0$ and f_0 satisfies the following axioms:

(a) (A_0, I_0) is preuniform.

(b) The ring map $A_0[\frac{1}{t}] \to B_0[\frac{1}{t}]$ induced by f_0 is finite étale.

(c) B_0 is t-torsion free, and $B_0 \subset (A_0)^*_{B_0[\frac{1}{t}]}$.

Example 1.8.6. Let V be a valuation ring with a non-zero element $t \in V$ for which V is t-adically Hausdorff. Set $K := \mathrm{Frac}(V)$. Let L be a finite separable extension of K. Let W be the integral closure of V in L. Then the morphism $(V, (t)) \to (W, (t))$ satisfies $(*)$.

A morphism $(A_0, (t)) \to (B_0, (t))$ satisfying $(*)$ has the following good properties.

Proposition 1.8.7. *Let $f_0 : (A_0, (t)) \to (B_0, (t))$ be a morphism of pairs that satisfies $(*)$. Put $A := A_0[\frac{1}{t}]$ and $B := B_0[\frac{1}{t}]$. Then the following assertions hold.*

1. *There exists an integer $c > 0$ such that $\mathrm{Tr}_{B/A}(t^c B_0) \subset A_0$.*
2. *There exist an integer $l > 0$, a finite free A_0-module F, and A_0-homomorphisms $B_0 \to F \to B_0$ whose composition is multiplication by t^l. In particular, $t^l B_0$ is contained in a finitely generated A_0-submodule of B_0.*
3. *One has $(A_0)^*_B = (B_0)^*_B$.*
4. *The pair $(B_0, (t))$ is preuniform.*

To prove this, we need the following lemma.

Lemma 1.8.8. *Let A_0 be a ring with a non-zerodivisor t, and put $A = A_0[\frac{1}{t}]$. Let B be a finite étale A-algebra. Assume that A_0 is integrally closed in A. Then for every integral element $b \in B$ over A, $\mathrm{Tr}_{B/A}(b)$ belongs to A_0.*

Proof of Lemma 1.8.8. A proof of the lemma in the special case that t is a prime number is given in [12, Lemma 2.6], but the same proof is valid for the general case. □

Here notice the following fact.

Lemma 1.8.9. *Keep the notation as in Proposition 1.8.7. Put* $A^+ := (A_0)^+_A$ *and* $B^+ := (A_0)^+_B$*. Let* $f : A \to B$ *be the ring map induced by* f_0*, and let* $f^+ : A^+ \to B^+$ *be the ring map such that* $f|_{A^+}$ *factors through* f^+*. Then* $B = B^+[\frac{1}{t}]$*, and the morphism* $f^+ : (A^+, (t)) \to (B^+, (t))$ *satisfies* $(*)$*.*

Proof of Lemma 1.8.9. It is clear because B is integral over A and $A = A_0[\frac{1}{t}]$. □

Now let us start to prove Proposition 1.8.7.

Proof of Proposition 1.8.7. We define the morphism $f^+ : (A^+, (t)) \to (B^+, (t))$ as in Lemma 1.8.9. Then by Lemma 1.8.8, we have $\mathrm{Tr}_{B/A}(B^+) \subset A^+$. Thus, since $(A_0, (t))$ is preuniform, there exists some $c_1 > 0$ such that $\mathrm{Tr}_{B/A}(t^{c_1}B^+) \subset A_0$. Meanwhile, we have $tB_0 \subset t(A_0)^*_B \subset B^+$ by Lemma 1.2.2. Thus it holds that $\mathrm{Tr}_{B/A}(t^{c_1+1}B_0) \subset A_0$, which yields the assertion 1.

Let us prove 2. Since B is finite projective over A, we have A-homomorphisms $\pi : A^{\oplus d} \to B$ and $s : B \to A^{\oplus d}$ such that $\pi \circ s$ is the identity. Now since (1.10) is surjective, there exist $b_1, \dots, b_d \in B$ such that $s(x) = (\mathrm{Tr}_{B/A}(b_1x), \dots, \mathrm{Tr}_{B/A}(b_dx))$ for every $x \in B$. Hence by 1, there exists $l_1 > 0$ such that $(t^{l_1}s)|_{B_0}$ factors through an A_0-homomorphism $s_{l_1} : B_0 \to A_0^{\oplus d}$. Now there also exists an integer $l_2 > 0$ such that $(t^{l_2}\pi)|_{A_0^{\oplus d}}$ factors through an A_0-homomorphism $\pi_{l_2} : A_0^{\oplus d} \to B_0$. Then $\pi_{l_2} \circ s_{l_1}$ is multiplication by $t^{l_1+l_2}$, which yields the claim.

Next we prove 3. The containment $(A_0)^*_B \subset (B_0)^*_B$ is easy to see. Let us show the reverse inclusion. Pick an element $b \in B$ and assume that b is almost integral over B_0. Then, since $B = B_0[\frac{1}{t}]$, there exists some $m > 0$ such that $t^m(\sum_{n=0}^{\infty} B_0 \cdot b^n) \subset B_0$. Hence by the assertion 2, $t^{l+m}(\sum_{n=0}^{\infty} B_0 \cdot b^n)$ is contained in a finitely generated A_0-submodule of B_0. Therefore, $\sum_{n=0}^{\infty} A_0 \cdot b^n$ is contained in a finitely generated A_0-submodule of B. Hence $b \in B$ is almost integral over A_0, as desired.

Finally we prove 4. By the assertion 3, we have $(B_0)^+_B \subset (B_0)^*_B \subset (A_0)^*_B$. Hence in view Lemma 1.2.2, we have $t(B_0)^+_B \subset t(A_0)^*_B \subset (A_0)^+_B$. On the other hand, by Lemma 1.8.9 and the assertion 2, there exists some $l' > 0$ such that $t^{l'}(A_0)^+_B$ is contained in a finitely generated $(A_0)^+_A$-submodule N_0 of $(A_0)^+_B$. Moreover, since $(A_0, (t))$ is preuniform and $B = B_0[\frac{1}{t}]$, we have $t^{c'}N_0 \subset B_0$ for some $c' > 0$. Thus, $t^{c'+l'+1}(B_0)^+_B$ is contained in B_0. This yields the assertion. □

Corollary 1.8.10. *Let A be a preuniform Tate ring, and $(A_0, (t))$ a pair of definition. Let $f : A \to B$ be a finite étale ring map. Equip B with the canonical structure as a Tate ring (cf. Lemma 1.4.7). Then the following assertions hold.*

1. *B is also preuniform. In particular, $(A_0)^+_B$ and $(A_0)^*_B$ are rings of definition of B.*

2. *The morphism $(A^\circ, (t)) \to (B^\circ, (t))$ satisfies $(*)$.*

3. *$B^\circ = (A^\circ)^*_B$.*

4. *If f is injective, then $f^{-1}(B^\circ) = A^\circ$.*

5. *If A is uniform, then so is B.*

Proof. The assertions 1, 2, and 3 are immediate consequences of Lemma 1.4.7 and Proposition 1.8.7. The assertion 5 follows from 1 and Proposition 1.8.3. Let us prove 4. Assume that f is injective. Since the containment $A^\circ \subset f^{-1}(B^\circ)$ is clear, it suffices to show the reverse inclusion. Pick $a \in A$ for which $f(a) \in B^\circ$. Then, since $B^\circ = (A^\circ)^*_B$ by the assertion 3, there is some $l > 0$ such that $t^l a^n \in A^\circ$ for every $n > 0$. Hence we have $a \in (A^\circ)^*_A = A^\circ$, as wanted. □

In the next proposition, we show that the condition $(*)$ is stable under completion. This property is quite important for a study on decompletion of perfectoid rings (cf. Chapter 4). The proposition itself can be interpreted as a practical form of Corollary 1.8.4.

Proposition 1.8.11. *Keep the notation as in Proposition 1.8.7. Denote by $\widehat{A_0}$ and $\widehat{B_0}$ the t-adic completions of A_0 and B_0, respectively. Let $\widehat{f_0} : (\widehat{A_0}, (t)) \to (\widehat{B_0}, (t))$ be the morphism of pairs induced by f_0. Put $A' := \widehat{A_0}[\frac{1}{t}]$ and $B' := \widehat{B_0}[\frac{1}{t}]$. Then the following assertions hold.*

1. *The natural A'-algebra homomorphism $B \otimes_A A' \to B'$ is an isomorphism.*
2. *$\widehat{f_0} : (\widehat{A_0}, (t)) \to (\widehat{B_0}, (t))$ also satisfies $(*)$.*

Proof. 1 is a consequence of Corollary 1.8.4 and Proposition 1.8.7 1. Let us prove 2. By Lemma 1.2.5, the morphism $\widehat{f_0} : (\widehat{A_0}, (t)) \to (\widehat{B_0}, (t))$ satisfies (a) in Definition 1.8.5. Moreover, as a corollary of the assertion 1, one finds that $\widehat{f_0}$ also satisfies (b). Hence the remaining part is to show that $\widehat{B_0}$ is contained in $(\widehat{A_0})^*_{B'}$. To carry out this, we take a finitely generated A_0-submodule $N_0 \subset B_0$ such that $t^l B_0 \subset N_0$ for some $l > 0$ by applying Proposition 1.8.7 2. Denote by $\widehat{N_0}$ the t-adic completion of N_0. Then by Lemma 1.2.6, $\widehat{N_0}$ is viewed as an $\widehat{A_0}$-submodule of $\widehat{B_0}$ such that

$$t^l \widehat{B_0} \subset \widehat{N_0}. \tag{1.12}$$

By applying the topological Nakayama's lemma [38, Theorem 8.4], one finds that

$$\widehat{N_0} \text{ is a finitely generated } \widehat{A_0}\text{-module}, \tag{1.13}$$

because $\widehat{N_0}/t\widehat{N_0} \cong N_0/tN_0$ is finitely generated over $\widehat{A_0}/t\widehat{A_0} \cong A_0/tA_0$. Combining (1.12) and (1.13) together, we conclude that $\widehat{B_0}$ is contained in a finitely generated $\widehat{A_0}$-submodule of B'. In particular it holds that $\widehat{B_0} \subset (\widehat{A_0})^*_{B'}$, as wanted. □

Corollary 1.8.12. *Let $(A_0, (t))$ be a preuniform pair, and A the associated Tate ring. Let $\widehat{A_0}$ be the t-adic completion of A_0, and $\mathcal{A}$ the Tate ring associated to $(\widehat{A_0}, (t))$. Let B be a finite étale A-algebra, and denote by $\mathcal{B}$ the finite étale $\mathcal{A}$-algebra $B \otimes_A \mathcal{A}$. Equip B and $\mathcal{B}$ with the canonical structure as a Tate ring (cf. Lemma 1.4.7) respectively. Then the following assertions hold.*

1. *Let $\widehat{B^\circ}$ be the t-adic completion of B°. Then the natural ring map $\varphi : \mathcal{B} \to \widehat{B^\circ}[\frac{1}{t}]$ is an isomorphism, and induces an isomorphism $\mathcal{B}^\circ \xrightarrow{\cong} \widehat{B^\circ}$.*
2. *For the natural map $\psi : B \to \mathcal{B}$, it holds that $\psi^{-1}(\mathcal{B}^\circ) = B^\circ$.*

Proof. We first prove 1. Let $\widehat{A^\circ}$ be the t-adic completion of A°. Put $A' := \widehat{A^\circ}[\frac{1}{t}]$ and $B' := \widehat{B^\circ}[\frac{1}{t}]$. Then the natural ring map $\mathcal{A} \to A'$ is an isomorphism by Lemma 1.2.6, and it induces $\widehat{A^\circ} \xrightarrow{\cong} \mathcal{A}^\circ$

by Corollary 1.2.8. Since the morphism $(A^\circ, (t)) \to (B^\circ, (t))$ satisfies $(*)$ by Corollary 1.8.10 2, φ is an isomorphism by Proposition 1.8.11 1. Thus, φ induces an isomorphism of $\widehat{A^\circ}$-algebras $(\mathcal{A}^\circ)^*_{\mathcal{B}} \xrightarrow{\cong} (\widehat{A^\circ})^*_{B'}$. Therefore it suffices to show that

$$\mathcal{B}^\circ = (\mathcal{A}^\circ)^*_{\mathcal{B}} \text{ and } \widehat{B^\circ} = (\widehat{B^\circ})^*_{B'} = (\widehat{A^\circ})^*_{B'} \ . \tag{1.14}$$

By Proposition 1.8.11 2, the morphism $(\widehat{A^\circ}, (t)) \to (\widehat{B^\circ}, (t))$ also satisfies $(*)$. Hence (1.14) follows from Proposition 1.8.7 3 and Corollary 1.8.10 3, as wanted.

Let us prove the assrtion 2. Let B'_0 be the subring $\psi^{-1}(\mathcal{B}^\circ)$ of B. By Corollary 1.8.10 3, we have $B^\circ = (A^\circ)^*_B$ and $\mathcal{B}^\circ = (\mathcal{A}^\circ)^*_{\mathcal{B}} = (\widehat{A^\circ})^*_{\mathcal{B}}$. On the other hand, we have

$$\psi((A^\circ)^*_B) \subset (A^\circ)^*_{\mathcal{B}} \subset (\widehat{A^\circ})^*_{\mathcal{B}} \ .$$

Thus it holds that $B^\circ \subset B'_0$. We then prove the reverse inclusion. By the assertion 1, we have the commutative diagram:

$$\begin{array}{ccc} \mathcal{B}^\circ & \xrightarrow{\cong} & \widehat{B^\circ} \\ \downarrow & & \downarrow \\ \mathcal{B} & \xrightarrow[\varphi]{\cong} & \widehat{B^\circ}[\frac{1}{t}] \end{array}$$

where the vertical arrows denote the inclusion maps. Hence the map $(\varphi \circ \psi)|_{B'_0}$ factors through $\widehat{B^\circ}$. Thus by Lemma 1.2.7, we have $B'_0 \subset B^\circ$, as wanted. □

Lemma 1.8.13. *Let A be a Tate ring and let $A \hookrightarrow B$ be a Galois extension with Galois group G. Equip B with the canonical structure as a Tate ring as in Lemma 1.4.7. Then the action of G preserves B°. Moreover, if further A is preuniform, then $(B^\circ)^G = A^\circ$.*

Proof. Let A_0 be a ring of definition of A and let $t \in A_0$ be a pseudo-uniformizer of A. As in the proof of Lemma 1.4.7, we can take a ring of definition B_0 of B that is finitely generated as an A_0-module and satisfies $B = B_0[\frac{1}{t}]$. Pick $b \in B^\circ$ and $\sigma \in G$ arbitrarily. Then there is some $l > 0$ such that $t^l b^n \in B_0$ and therefore, $t^l \sigma(b)^n \in \sigma(B_0)$ for every $n > 0$. Meanwhile, since $\sigma(B_0)$ is also finitely generated as an A_0-module, we have $t^{l'}\sigma(B_0) \subset B_0$ for some $l' > 0$. Hence $\sigma(b)$ is also almost integral over B_0. Thus, the action of G preserves B°. If further A is preuniform, then we have

$$(B^\circ)^G = B^G \cap B^\circ = A \cap B^\circ = A^\circ$$

by Corollary 1.8.10 4, as wanted. □

Chapter 2

Perfectoid rings

In this chapter, we develop the theory of Fontaine's perfectoid rings. In §2.1, we recall the definition of Fontaine's perfectoid rings and discuss their *tilts*. In §2.2, we prove the existence of fiber coproducts in the category of perfectoid rings. In §2.3, we establish the tilting equivalence for perfectoid rings (see Theorem 2.3.1). In §2.4, we state the almost purity theorem for perfectoid rings (Theorem 2.4.1) and prove it in some special cases (the proof of Theorem 2.4.1 will be completed in Chapter 3).

2.1 Fontaine's perfectoid rings

Here we discuss basic properties of Fontaine's perfectoid rings. Fontaine's perfectoid ring is characterized as a uniform Tate ring with a special pseudo-uniformizer described below.

Definition 2.1.1 (Perfectoid pseudo-uniformizer)**.** Let A be a uniform Tate ring.

1. We say that a pseudo-uniformizer ϖ is *quasi-perfectoid*, if $p \in \varpi^p A^\circ$ and there exists a system of elements $\{\varpi_n\}_{n \geq 0}$ in A° such that $\varpi_0 = \varpi$ and $\varpi_{n+1}^p \equiv \varpi_n \mod \varpi^p A^\circ$ for every $n \geq 0$.

2. We say that a pseudo-uniformizer ϖ is *perfectoid*, if $p \in \varpi^p A^\circ$ and $A^\circ/(\varpi^p)$ is semiperfect.

Our definition of perfectoid rings and fields is the following.

Definition 2.1.2 (Fontaine's perfectoid rings)**.** A *perfectoid ring* (resp. *quasi-perfectoid ring*) is a uniform Tate ring with a perfectoid pseudo-uniformizer (resp. a quasi-perfectoid pseudo-uniformizer). A *perfectoid field* is a perfectoid ring that is a field and whose associated norm is equivalent to a multiplicative norm.

We say that an affinoid ring (A, A^+) is *perfectoid* if A is a perfectoid ring.

Example 2.1.3.

1. For a uniform Tate ring A of characteristic p, A is a perfectoid ring if and only if A is perfect.

2. Let K be the Hausdorff completion of the Tate ring associated to $(\mathbb{Z}_p[p^{1/p^\infty}], (p))$. Then K is a perfectoid field with a perfectoid pseudo-uniformizer $p^{1/p}$.

3. Let F be the Hausdorff completion of the Tate ring associated to $(\mathbb{F}_p[[T]][T^{1/p^\infty}], (T))$. Then F is a perfectoid field with a perfectoid pseudo-uniformizer $T^{1/p}$.

4. Any uniform Tate ring defined over a perfectoid ring is a quasi-perfectoid ring. For example, the Hausdorff completion of the Tate ring associated to $(\mathbb{Z}_p[p^{1/p^\infty}][T], (p))$ is a quasi-perfectoid ring.

Example 2.1.4. We exhibit an example of a perfectoid ring that does not fit into the original definition of perfectoid K-algebras by Scholze. Let $(R, \mathfrak{m})$ be a complete regular local ring of characteristic $(0, p)$ with $R/\mathfrak{m}$ perfect. Let $x_1, \ldots x_d$ be a regular system of parameters of R. Let R^+ be the absolute integral closure of R (i.e. the integral closure of R in the algebraic closure of the field of fractions $\mathrm{Frac}(R)$). For each $i = 1, \ldots, d$, let $\{x_i^{1/p^n}\}_{n \geq 0}$ be a compatible system of p-power roots of x_i in R^+ (see §0.4 for notation). Let R_n be the R-subalgebra $R[x_1^{1/p^n}, \ldots, x_d^{1/p^n}]$ of R^+ for an integer $n \geq 0$, and set $R_\infty := \varinjlim_n R_n$. For each $i = 1, \ldots, d$, let $R_{\infty,i}$ be the R_∞-subalgebra $R_\infty[\frac{x_1}{x_i}, \ldots, \frac{x_d}{x_i}]$ of $R_\infty[\frac{1}{x_i}]$. Let $\widehat{R_\infty}$ and $\widehat{R_{\infty,i}}$ be the $\mathfrak{m}$-adic completions. Let $S_{\infty,i}$ be the Tate ring associated to $(\widehat{R_{\infty,i}}, (x_i))$. Then $S_{\infty,i}$ is a perfectoid ring. Let us observe it below.

Denote by $R_n[(\frac{x_1}{x_i})^{1/p^n}, \ldots, (\frac{x_d}{x_i})^{1/p^n}]$ the R_n-subalgebra $R_n[({x_i}^{1/p^n})^{-1} x_1^{1/p^n}, \ldots, ({x_i}^{1/p^n})^{-1} x_1^{1/p^n}]$ of $R_n[(x_i^{1/p^n})^{-1}] = R_n[\frac{1}{x_i}]$ and set $R_\infty[(\frac{x_1}{x_i})^{1/p^\infty}, \ldots, (\frac{x_d}{x_i})^{1/p^\infty}] := \varinjlim_n R_n[(\frac{x_1}{x_i})^{1/p^n}, \ldots, (\frac{x_d}{x_i})^{1/p^n}]$. Since R_n is a regular local ring with a regular system of parameters $x_1^{1/p^n}, \ldots, x_d^{1/p^n}$, one can easily show that $R_n[(\frac{x_1}{x_i})^{1/p^n}, \ldots, (\frac{x_d}{x_i})^{1/p^n}]$ is completely integrally closed in $R_n[\frac{1}{x_i}]$. Hence the direct limit $R_\infty[(\frac{x_1}{x_i})^{1/p^\infty}, \ldots, (\frac{x_d}{x_i})^{1/p^\infty}]$ is completely integrally closed in $R_\infty[\frac{1}{x_i}]$. Thus, since we have

$$x_i(R_\infty[(\tfrac{x_1}{x_i})^{\frac{1}{p^\infty}}, \ldots, (\tfrac{x_d}{x_i})^{\frac{1}{p^\infty}}]) \subset R_{\infty,i} \ ,$$

the pair $(R_{\infty,i}, (x_i))$ is preuniform and so is $S_{\infty,i}$. Moreover, since

$$S^\circ_{\infty,i}/(x_i) \cong R_\infty[(\tfrac{x_1}{x_i})^{\frac{1}{p^\infty}}, \ldots, (\tfrac{x_d}{x_i})^{\frac{1}{p^\infty}}]/(x_i) = R_\infty[(\tfrac{x_1}{x_i})^{\frac{1}{p^\infty}}, \ldots, (\tfrac{x_d}{x_i})^{\frac{1}{p^\infty}}]/\mathfrak{m}(R_\infty[(\tfrac{x_1}{x_i})^{\frac{1}{p^\infty}}, \ldots, (\tfrac{x_d}{x_i})^{\frac{1}{p^\infty}}]) \ ,$$

$S^\circ_{\infty,i}$ is semiperfect. Hence $S_{\infty,i}$ is a perfectoid ring with a perfectoid pseudo-uniformizer $x_i^{1/p}$.

Remark 2.1.5. Let F be the perfectoid field as in Example 2.1.3 3. Let R be a perfectoid ring of characteristic p with a pseudo-uniformizer ϖ. Then we have the ring homomorphism

$$\mathbb{F}_p[[T]][T^{\frac{1}{p^\infty}}] \to R^\circ, \ T^{\frac{1}{p^n}} \mapsto \varpi^{\frac{1}{p^n}} \ ,$$

which induces a continuous ring homomorphism $F \to R$. In particular, any perfectoid ring of characteristic p can be regarded as a perfectoid K-algebra in Scholze's original sense (where $K = F$, cf. [49, Definition 5.1]).

The following class of rings is used for characterization of integral structures of (Fontaine's) perfectoid rings.

Definition 2.1.6 (Integral perfectoid rings, [39])**.** An *integral perfectoid ring* is a ring A that has a non-zerodivisor ϖ with the following properties:

1. A is ϖ-adically Hausdorff complete;

2. $p \in \varpi^p A$;

3. the ring homomorphism

$$A/(\varpi) \to A/(\varpi^p),\ \overline{a} \mapsto \overline{a}^p$$

is an isomorphism.

ϖ is called a *perfectoid pseudo-uniformizer* of A.

Lemma 2.1.7 ([39, Definition 2.1])**.** *For a Tate ring A, the following conditions are equivalent.*

1. *A is a perfectoid ring.*

2. *There exists a pair of definition $(A_0, (\varpi))$ of A such that A_0 is an integral perfectoid ring with ϖ a perfectoid pseudo-uniformizer.*

3. *A is uniform, and for any open and integrlly closed subring A^+ of A° there exists a psedo-uniformizer ϖ of A such that A^+ is an integral perfectoid ring with ϖ a perfectoid pseudo-uniformizer.*

A quasi-perefctoid pseudo-uniformizer can be refined as follows.

Lemma 2.1.8. *Let (A, A^+) be a quasi-perfectoid affinoid ring. Then there exists a system of elements $\{\varpi_n\}_{n\geq 0}$ in A^+ such that $p \in \varpi_0^p A^+$ and $\varpi_{n+1}^p \equiv \varpi_n \mod \varpi_0^p A^+$ for every $n \geq 0$.*

Proof. Let $\varpi \in A^+$ be a quasi-perfectoid pseudo-uniformizer. Then, $p \in \varpi^p A^\circ$ and we have a system of elements $\{\varpi_n\}_{n\geq 0}$ in A° such that $\varpi_0 = \varpi$ and $\varpi_{n+1}^p \equiv \varpi_n \mod \varpi^p A^\circ$ for every $n \geq 0$. Since A° is ϖ-adically Hausdorff complete, we find that $\varpi = \varpi_0^p u$ for some unit $u \in A^\circ$. In particular, $\varpi^p A^\circ = \varpi_1^p \varpi^{p-1} A^\circ \subset \varpi_1^p A^+$. Hence by letting $\varpi_n' = \varpi_{n+1}$ for every $n \geq 0$, we obtain the sequence of elements $\{\varpi_n'\}_{n\geq 0}$ with the desired property. □

Similarly, one can replace a perfectoid pseudo-uniformizer so that the following statement holds.

Lemma 2.1.9. *Let (A, A^+) be a perfectoid affinoid ring. Then the following assertions hold.*

1. *There exists a pseudo-uniformizer $\varpi' \in A$ such that $p \in \varpi'^p A^+$ and $A^+/(\varpi'^p)$ is semiperfect.*

2. *Assume further that $p \in A$ is a topologically nilpotent unit. Then $A^+/(p)$ is semiperfect, and there exists some $t \in A^+$ such that $p = t^p u$ for some unit $u \in A^+$. Moreover, there exists some $s \in A^+$ such that $s^p \equiv p \mod p^2 A^+$.*

Proof. 1: Let $\varpi \in A^\circ$ be a perfectoid pseudo-uniformizer of A. Then by Lemma 1.3.2, we find some $\varpi_1 \in A^\circ$ such that $\varpi = \varpi_1^p u_1$ for some unit $u_1 \in A^\circ$ (notice that ϖ_1 is a topologically nilpotent unit in A). Hence $p \in \varpi_1^p \varpi^{p-1} A^\circ$ and since $\varpi^{p-1} A^\circ \subset A^+$, we have $p \in \varpi_1^p A^+$. Moreover, since $A^\circ/(\varpi_1^p)$ is semiperfect, $A^+/(\varpi_1^p)$ is also semiperfect by Lemma 1.3.5. So letting $\varpi' := \varpi_1$ completes 1.

2: Next we assume further that $p \in A$ is a topologically nilpotent unit. Then, since $(p) \subset A^+$ is closed with respect to the ϖ_1-adic topology, $A^+/(p)$ is semiperfect in view of Lemma 1.3.3 2. To prove the existence of $t \in A^+$ and a unit $u \in A^+$ such that $p = t^p u$, we take $a \in A^+$ for which $p = \varpi_1^p a$. Then we have $b, c \in A^+$ that satisfy $a = b^p + pc$ and so $p(1 - \varpi_1^p c) = (\varpi_1 b)^p$. Thus, since $u := 1 - \varpi_1^p c \in A^+$ is a unit, it suffices to take $t = \varpi_1 b$. Finally, let us prove the existence of $s \in A^+$ as in the assertion. Take $d, e \in A^+$ such that $u = d^p + pe$. Then $p = t^p(d^p + pe)$ and therefore, $p = (td)^p + pt^p e$. Here notice that $t^p = pu^{-1} \in pA^+$. Thus we have $p \equiv (td)^p \mod p^2 A^+$, as wanted. □

2.1.1 Tilting

Here we study *tilt* of quasi-perfectoid rings.

Definition 2.1.10. Let (A, A^+) be an affinoid ring where A is a quasi-perfectoid ring. Let $\varpi \in A$ be a perfectoid pseudo-uniformizer such that $p \in \varpi^p A^+$ (cf. Lemma 2.1.9), and $(A^+)^\flat$ the inverse inverse perfection of $A^+/(\varpi^p)$ (cf. Definition 1.3.9).

By Proposition 1.3.16, there exists a unique continuous ring homomorphism $\theta_{A^+,\varpi} : W((A^+)^\flat) \to A^+$ such that the following diagram commutes:

$$\begin{array}{ccc} W((A^+)^\flat) & \xrightarrow{\theta_{A^+,\varpi}} & A^+ \\ {\scriptstyle \omega_0}\downarrow & & \downarrow{\scriptstyle \bmod (\varpi^p)} \\ (A^+)^\flat & \xrightarrow{\mathrm{pr}_0} & A^+/(\varpi^p) \ . \end{array}$$

Lemma 2.1.11. *For subrings $A^+ \subset B^+ \subset A^\circ$ that are open and integrally closed in A, there exists a natural injection $j : (A^+)^\flat \to (B^+)^\flat$.*

Proof. Define a ring homomorphism $j : (A^+)^\flat \to (B^+)^\flat$ as

$$j(a) = (\ldots, \overline{\theta_{A^+}([a^{1/p}])}, \overline{\theta_{A^+}([a])}) \quad (a \in (A^+)^\flat) \ .$$

If $j(a) = 0$, $a \in (\varpi)^{p^n}$ for every $n \geq 0$. Hence $a = 0$. □

Using θ, we can deduce an essential property of a quasi-perfectoid ring.

Lemma 2.1.12. *The following assertions hold.*

1. *Let A be a quasi-perfectoid ring. Then there exists a pseudo-uniformizer $\varpi \in A$ that admits a compatible system of p-power roots $\{\varpi^{1/p^n}\}_{n \geq 0}$ in A.*

2. *Let (A, A^+) be a perfectoid affinoid ring. Then there exists a pseudo-uniformizer $\varpi \in A$ that admits a compatible system of p-power roots $\{\varpi^{1/p^n}\}_{n \geq 0}$ in A such that $p \in \varpi^p A^+$ and $A^+/(\varpi^p)$ is semiperfect.*

Proof. Let (A, A^+) be a quasi-perfectoid affinoid ring. Then by Lemma 2.1.8, we have a system of elements $\{\varpi_n\}_{n \geq 0}$ in A^+ such that $p \in \varpi_0^p A^+$ and $\varpi_{n+1}^p \equiv \varpi_n \mod \varpi_0^p A^+$ for every $n \geq 0$. Consider the map $\mathrm{pr}_0 : (A^+)^\flat \to A^+/(\varpi_0^p)$. Then we can find an element $\varpi^\flat \in \mathrm{pr}_0^{-1}(\{\varpi_0 \mod (\varpi_0^p)\})$. Then $\varpi := \theta_{A^+}([\varpi^\flat])$ admits a compatible system of p-power roots $\{\varpi^{1/p^n}\}_{n \geq 1}$, and satisfies $\varpi \equiv \varpi_0 \mod \varpi_0^p A^+$ by Remark 1.3.17. Then, since $\varpi = \varpi_0 u$ for some unit $u \in A^+$, ϖ is a pseudo-uniformizer. Hence the assertion 1 follows. Suppose further that A is a pefectoid ring. Then by Lemma 2.1.9 1, we may assume that $A^+/(\varpi_0^p)$ is semiperfect. Hence $A^+/(\varpi^p)$ is also semiperfect because $\varpi = \varpi_0 u$, which yields the assertion 2. □

Corollary 2.1.13. *Let A be a quasi-perfectoid ring. Consider the basic setup $(A^\circ, A^{\circ\circ})$. Then the canonical ring homomorphism $A^\circ \to (A^{\circ a})_*$ is an isomorphism.*

Proof. By Lemma 2.1.12, we can find a pseudo-uniformizer $\varpi \in A$ that admits a compatible system of p-power roots $\{\varpi^{1/p^n}\}_{n \geq 0}$. Hence we have $(A^{\circ a})_* = \varpi^{-\frac{1}{p^\infty}} A^\circ$ by Lemma 1.1.4 1. Thus, since $A^\circ = \varpi^{-\frac{1}{p^\infty}} A^\circ$ by Lemma 1.2.3, the assertion follows. □

Let us construct a uniform Tate ring from $(A^+)^\flat$. First we show the following.

Lemma 2.1.14. *Let ϖ be as in Lemma 2.1.12. Put $\varpi^\flat := (\dots, \overline{\varpi^{1/p^2}}, \overline{\varpi^{1/p}}, \overline{\varpi})$. Then $\varpi^\flat \in (A^+)^\flat$ is a non-zerodivisor.*

Proof. Pick $x \in (A^+)^\flat$ such that $\varpi^\flat x = 0$. Then we have $\theta_{A^+}([\varpi^\flat x]) = \varpi u(\theta_{A^+}([x])) = 0$ for some unit $u \in A^+$. Hence $\theta_{A^+}([x]) = 0$. Thus, since A is reduced by Lemma 1.4.12, we have $\theta_{A^+}([x^{1/p^n}]) = 0$ for every $n > 0$. Therefore $x = 0$ by Remark 1.3.17, as wanted. □

The main goal in this subsection is to prove the following statement.

Proposition 2.1.15. *Let $A^\flat$ be the Tate ring associated to $((A^+)^\flat, (\varpi^\flat))$. Let $\cdot^\sharp : A^\flat \to A$ be the multiplicative map induced by $\theta_{A^+}([\,\cdot\,]) : (A^+)^\flat \to A^+$. Then the following assertions hold.*

1. *$A^\flat$ is independent of the choice of ϖ.*
2. *$A^\flat$ is a perfectoid ring.*
3. *$\cdot^\sharp : A^\flat \to A$ is continuous.*

To prove this, we need to study basic properties of $\varpi^\flat$.

Lemma 2.1.16. *Let (A, A^+) be a quasi-perfectoid affinoid ring and $\varpi \in A$ a pseudo-uniformizer that admits a compatible system of p-power roots $\{\varpi^{1/p^n}\}_{n \geq 1}$ in A. Then the following assertions hold.*

1 *For every $\xi \in (A^+)^\flat$ and every $m > 0$, it holds that*

$$\xi \in (\varpi^\flat)^m (A^+)^\flat \iff \theta_{A^+}([\xi]) \in (\varpi)^m A^+.$$

2 *$(A^+)^\flat$ is $\varpi^\flat$-adically Hausdorff complete.*

Proof. 1: Suppose that $\xi \in (\varpi^\flat)^m$. Then, since the map $\theta_{A^+}([\,\cdot\,])$ is multiplicative, clearly $\theta_{A^+}([\xi]) \in (\varpi)^m$. Conversely, suppose that $\theta_{A^+}([\xi]) \in (\varpi)^m$. Put $x_n := \theta_{A^+}([\xi^{1/p^n}])$ for every $n \geq 0$. Then by assumption, we have $x_n^{p^n} \in \varpi^m A^+$. Hence $\varpi^{-m} x_n^{p^n} \in A$ lies in A^+, and therefore so does $(\varpi^{1/p^n})^{-m} x_n$ because A^+ is integrally closed in A. Then by putting $a_n := (\varpi^{1/p^n})^{-m} x_n$ for every $n \geq 0$, we have $a_{n+1}^p = a_n$ and $x_n = (\varpi^{1/p^n})^m a_n$. Thus we obtain an element $\alpha := (\dots, \overline{a_2}, \overline{a_1}, \overline{a_0}) \in (A^+)^\flat$ such that $\xi = (\varpi^\flat)^m \alpha$, as wanted.

2: By the assertion 1, the kernel of the ring homomorphism

$$\mathrm{pr}_{\varpi,n} : \varprojlim_{\mathrm{Frob}} A^+/(\varpi^p) \to A^+/(\varpi^p)\, (\dots, \overline{x_2}, \overline{x_1}, \overline{x_0}) \mapsto \overline{x_n}$$

is generated by $(\varpi^\flat)^{np}$. Hence the $\varpi^\flat$-adic topology on $\varprojlim_{\mathrm{Frob}} A^+/(\varpi^p)$ coincides with the limit topology. Thus $(A^+)^\flat$ is $\varpi^\flat$-adically Hausdorff complete. □

Now we can prove Proposition 2.1.15.

Proof of Proposition 2.1.15. 1: Let ϖ and ϖ' be pseudo-uniformizers of A that admit compatible systems of p-power roots. Then $\varpi^c \in \varpi' A^+$ for some $c > 0$. Hence by Lemma 2.1.16 1, we have $(\varpi^c)^\flat \in (\varpi')^\flat (A^+)^\flat$. Since $(\varpi^c)^\flat = (\varpi^\flat)^c$ by definition, we obtain $(\varpi^\flat)^c \in (\varpi')^\flat (A^+)^\flat$. Similarly, we have $((\varpi')^\flat)^{c'} \in \varpi^\flat (A^+)^\flat$ for some $c' > 0$. Thus the assertion follows.

2: Since $(A^+)^\flat[\frac{1}{\varpi^\flat}]$ is a perfect $\mathbb{F}_p$-algebra, it suffices to show that $A^\flat$ is uniform. By Lemma 2.1.16 2, $A^\flat$ is Hausdorff complete. Let us prove that $A^\flat$ admits a power-multiplicative norm that is compatible with the topology on $A^\flat$ (cf. Lemma 1.5.6). Let $\alpha : A \to \mathbb{R}_{\geq 0}$ be the spectral norm associated to $\gamma_{A^+,(\varpi),2}$ (cf. Definition 1.5.3). Then we obtain the composite

$$\alpha^\flat : (A^+)^\flat \xrightarrow{\theta_{A^+}([\,\cdot\,])} A^+ \xrightarrow{\alpha|_{A^+}} \mathbb{R} \ ,$$

which is a power-multiplicative norm on $(A^+)^\flat$. Moreover, since $\varpi^\flat$ is multiplicative with respect to $\alpha^\flat$, this norm naturally extends to $A^\flat$. Then we have $\alpha^\flat = \alpha(\,\cdot^\sharp)$. Moreover, by Lemma 2.1.16 1, we also have $\gamma_{(A^+)^\flat,(\varpi^\flat),2} = \gamma_{A^+,(\varpi),2}(\,\cdot^\sharp)$. Thus, since $\alpha \sim \gamma_{A^+,(\varpi),2}$ by Lemma 1.5.6, we obtain $\alpha^\flat \sim \gamma_{(A^+)^\flat,(\varpi^\flat),2}$ as wanted.

3: This assertion immediately follows from Lemma 2.1.16 1. $\square$

Let us check the functoriality.

Lemma 2.1.17. *Let (A, A^+) and (B, B^+) be perfectoid affinoid algebras, and $f : (A, A^+) \to (B, B^+)$ a continuous ring homomorphism. Then there exists a natural continuous ring homomorphism $(f^+)^\flat : (A^+)^\flat \to (B^+)^\flat$ such that the resulting diagram*

$$\begin{array}{ccc} W((A^+)^\flat) & \xrightarrow{W((f^+)^\flat)} & W((B^+)^\flat) \\ \theta_{A^+}\downarrow & & \downarrow\theta_{B^+} \\ A^+ & \xrightarrow[f^+]{} & B^+ \end{array}$$

commutes.

Proof. We define a ring homomorphism $(f^+)^\flat : (A^+)^\flat \to (B^+)^\flat$ by

$$a \mapsto (\ldots, \overline{f(\theta_{A^+}([a^{1/p}]))}, \overline{f(\theta_{A^+}([a]))}) \ .$$

Then for every $a \in (A^+)^\flat$, one has $\theta_{B^+}([(f^+)^\flat(a)]) = \theta_{A^+}([a])$. Hence the diagram commutes. Thus, by Lemma 2.1.16 3 and the continuity of f^+, $(f^+)^\flat$ is continuous. $\square$

2.1.2 Distinguished elements

Here we study *distinguished elements.*

Definition 2.1.18 (Distinguished elements)**.** Let (R, R^+) be an affinoid ring where R is a perfectoid ring of characteristic p. An element $x = \sum_{n=0}^{\infty}[\xi_n]p^n \in W(R^+)$ is said to be *distinguished*, if ξ_0 is topologically nilpotent and $\xi_1 \in R^+$ is invertible.

First notice that any distinguished element is a non-zerodivisor.

Lemma 2.1.19. *Let (R, R^+) be an affinoid ring where R is a perfectoid ring of characteristic p. Let $z = \sum_{n=0}^{\infty}[\zeta_n]p^n \in W(R^+)$ be a distinguished element. Then z is a non-zerodivisor in $W(R^+)$.*

Proof. Pick $x = \sum_{n=0}^{\infty}[\xi_n]p^n \in W(R^+)$ such that $zx = 0$. Since $W(R^+)$ is p-torsion free, it suffices to show that $\xi_0 = 0$. By assumption, we have $\zeta_0\xi_0 = 0$ and $\zeta_0\xi_1 + \zeta_1\xi_0 = 0$. Hence

$$0 = \xi_0(\zeta_0\xi_1 + \zeta_1\xi_0) = \zeta_1\xi_0^2 \ ,$$

which implies $\xi^2 = 0$ because $\zeta_1 \in R^+$ is invertible. Thus by perfectness of R, we obtain $\xi_0 = 0$ as wanted. □

Lemma 2.1.20. *Let (A, A^+) be a quasi-perfectoid affinoid ring. Suppose that* $\mathrm{Ker}(\theta_{A^+})$ *contains a distinguished element z. Then* $\mathrm{Ker}(\theta_{A^+})$ *is a principal ideal generated by z.*

Proof. We write $z = \sum_{i=0}^{\infty} p^i[\zeta_i]$. Pick an element $x = \sum_{i=0}^{\infty} p^i[\xi_i] \in \mathrm{Ker}(\theta_{A^+})$. Let α be a power-multiplicative norm on $A^\flat$ that induces the topology on $A^\flat$ and satisfies $\alpha(\zeta_0) = p^{-1}$. Then by [34, Lemma 5.6], there exists a strictly α-stable element $y = \sum_{i=0}^{\infty} p^i[\eta_i] \in W((A^+)^\flat)$ congruent to x modulo (z). It suffices to show that $\alpha(\eta_0) = 0$. Since x and z belong to $\mathrm{Ker}(\theta_{A^+})$, so does y. Hence we have

$$\theta_{A^+}([\eta_0]) = -\sum_{i=1}^{\infty} p^i\theta_{A^+}([\eta_i]) \ . \tag{2.1}$$

Moreover, by Lemma 2.1.16 1, there exists a norm β on A such that $\beta(p) < 1$ and

$$\beta(\theta_{A^+}([\,\cdot\,])) = \alpha(\,\cdot\,) \ .$$

Then we have

$$\beta(\theta_{A^+}([\eta_0])) \geq \beta(\theta_{A^+}([\eta_i])) \quad (\forall i \geq 1)$$

by strict α-stability of y. Thus, if $\theta_{A^+}([\eta_0]) \neq 0$, then we have $\beta(\sum_{i=1}^{\infty} p^i\theta_{A^+}([\eta_i])) < \beta(\theta_{A^+}([\eta_0]))$ and it contradicts to (2.1). Hence the assertion follows. □

Lemma 2.1.21. *Let (A, A^+) be a perfectoid affinoid ring. Then* $\mathrm{Ker}(\theta_{A^+})$ *has a distinguished element.*

Proof. By Lemma 2.1.12, one can take a pseudo-uniformizer $\varpi \in A$ that admits a compatible system of p-power roots $\{\varpi^{1/p^n}\}_{n\geq 1}$ in A such that $p \in \varpi^p A^+$. Then, since θ_{A^+} is surjective by Lemma 1.3.19, there exists $x = \sum_{i=0}^{\infty} p^i[\xi_i] \in W((A^+)^\flat)$ such that $p = \theta_{A^+}([\varpi^\flat]^p x)$. Hence $z := -[\varpi^\flat]^p x + p$ belongs to $\mathrm{Ker}(\theta_{A^+})$. Here

$$z \equiv [-(\varpi^\flat)^p\xi_0] + p[1 - (\varpi^\flat)^p\xi_1] \mod (p^2)$$

and $\varpi^\flat$ is topologically nilpotent. Hence z is a distinguished element, as wanted. □

2.1.3 Untilting

Let (R, R^+) be a perfectoid affinoid ring of characteristic $p > 0$. Let $z = \sum_{i=0}^{\infty} p^i[\zeta_i] \in W(R^+)$ be a distinguished element. Put $I := zW(R^+)$ and $(R^+)^{\sharp,I} := W(R^+)/I$. Let $t \in R$ be a topological nilpotent unit. Let α be a power-multiplicative norm on R such that $\alpha(t)\alpha(t^{-1}) = 1$ and $\alpha(\zeta_0) \leq p^{-1}$. Then $\lambda(\alpha)$ is a power-multiplicative norm on $W(R^+)$ by [34, Lemma 4.1], and induces the quotient seminorm $\overline{\lambda(\alpha)}$ on $(R^+)^{\sharp,I}$.

Proposition 2.1.22. $\overline{\lambda(\alpha)}$ *is a power-multiplicative norm.*

Proof. We first show that $\overline{\lambda(\alpha)}$ is a norm. Take an element $\overline{x} \in A^+$. By [34, Lemma 5.5], we may assume that $x = \sum_{i=0}^{\infty} p^i[\xi_i]$ is strictly α-stable. In this situation, one has $\overline{\lambda(\alpha)}(\overline{x}) = \alpha(\xi_0)$ by Proposition 1.5.12. Indeed, since x is α-stable, one has

$$\alpha(\xi_0) \leq \lambda(\alpha)(y)$$

for every $y \in W(R^\circ)$ such that $y \equiv \xi_0 \mod zW(R^\circ)$. Hence $\alpha(\xi_0) \leq \lambda(\alpha)(y')$ for every $y' \in W(R^+)$ such that $y' \equiv \xi_0 \mod I$, which implies $\overline{\lambda(\alpha)}(\overline{x}) = \alpha(\xi_0)$.

Assume that $\overline{\lambda(\alpha)}(\overline{x}) = 0$. Then $\alpha(\xi_0) = 0$. Since x is α-stable and α is a norm, we have $x = 0$. Thus $\overline{\lambda}(\alpha)$ is a norm.

Hence it is enough to show that x^n is strictly α-stable for every $n \geq 1$. We carry out this by induction. If $n = 1$, the statement is obvious. Let $n \geq 2$, and assume that x^{n-1} is strictly α-stable. Put $x^{n-1} = \sum_{i=0}^{\infty} p^i[\xi'_i]$ and $x^n = \sum_{i=0}^{\infty} p^i[\xi''_i]$. Take an arbitrary integer $k \geq 1$. Let us show $\alpha(\xi''_k) \leq \alpha(\xi''_0)$. Since $\alpha(\xi''_k) \leq \Lambda(\alpha)(\sum_{i=0}^{k} p^i[\xi''_i])$ and

$$\sum_{i=0}^{k} p^i[\xi''_i] \equiv \sum_{i=0}^{k} p^i \left(\sum_{j=0}^{i} [\xi_j \xi'_{i-j}] \right) \mod (p^{k+1}) \ ,$$

we have

$$\alpha(\xi''_k) \leq \Lambda(\alpha) \left(\sum_{i=0}^{k} p^i \left(\sum_{j=0}^{i} [\xi_j \xi'_{i-j}] \right) \right) \leq \max_{i,j} \{ \alpha(\xi_i \xi'_j) \} \ .$$

Moreover, we have $\alpha(\xi_i \xi'_j) \leq \alpha(\xi_0)\alpha(\xi'_0)$ $(0 \leq \forall i, j \leq k)$ by assumption, and $\alpha(\xi_0)\alpha(\xi'_0) = \alpha({\xi_0}^n) = \alpha(\xi''_0)$ because α is power-multiplicative. Thus $\alpha(\xi''_k) \leq \alpha(\xi''_0)$, as desired. ☐

We denote $(R^+)^{\sharp, I}$ by $(R^+)^\sharp$ for short. Define a map

$$\cdot^\sharp : R^+ \to (R^+)^\sharp, \ x \mapsto \overline{[x]} \ ,$$

where $\overline{[x]}$ means an element $[x] \mod I \in W(R^+)/I$. Note that $t^\sharp$ is a non-zerodivisor. We denote by $R^{\sharp, I}$ (or $R^\sharp$ for short) or $(R, I)^\sharp$ the Tate ring associated to $((R^+)^\sharp, (t^\sharp))$, and by $(R, R^+)^{\sharp, I}$ (or $(R, R^+)^\sharp$ for short) or $((R, R^+), I)^\sharp$ the affinoid ring $(R^\sharp, (R^+)^\sharp)$. Then by the above discussions, we obtain the following.

Corollary 2.1.23. $(R, I)^\sharp$ *(resp.* $((R, R^+), I)^\sharp$*) is a perfectoid ring (resp. a perfectoid affinoid ring).*

2.2 Fiber coproducts

Lemma 2.2.1. *Let* (R, R^+) *be a perfectoid affinoid ring of characteristic* $p > 0$*. Let* (S, S^+) *and* (T, T^+) *be perfectoid affinoid algebras over* (R, R^+)*.*

1. *Let* U_{00} *be the image of* $S^+ \otimes_{R^+} T^+$ *in* $S \otimes_R T$*. Then* U_{00} *is p-root closed in* $S \otimes_R T$*.*

2. $S \widehat{\otimes}_R T$ *is perfect and uniform.*

3. Let U_0 be the t-adic completion of U_{00}. Then $U_0 \hookrightarrow U^\circ$ is $R^{\circ\circ}$-almost isomorphic.

Proof of Lemma 2.2.1. 1: Pick $f \in D_{00}$. Then we have a presentation $f = \sum_{i=0}^{r} s_i \otimes t_i$ where $s_i \in S^+$ and $t_i \in T^+$. Since D is perfect, $f^{1/p} = \sum_{i=1}^{r} s_i^{1/p} \otimes t_i^{1/p}$. Here $s_i^{1/p} \in S^+$ and $t_i^{1/p} \in T^+$ because S^+ (resp. T^+) is integrally closed in S (resp. T). Hence we have $f^{1/p} \in D_{00}$, as wanted.

2: Since D_{00} is p-root closed in $S \otimes_R T$, the Hausdorff completion D is perfect and uniform by Proposition 1.4.11.

3: By Lemma 1.3.11, Lemma 1.3.6, Lemma 1.4.3, and the assertion 1, U_0 is a perfect $\mathbb{F}_p$-algebra. Hence U_0 is p-root closed in U by Lemma 1.3.6 and the assertion 2. Hence $U_0 \hookrightarrow U^\circ$ is t^{1/p^∞}-almost isomorphic by Lemma 1.2.3 3.

□

Let A be a perfectoid ring (such that $p \in A^{\circ\circ}$). Let B and C be perfectoid A-algebras. Put $R := A^\flat$, $S := B^\flat$, and $T := C^\flat$. Let $z \in W(A^{\flat\circ})$ be such that $\mathrm{Ker}(\theta_{A^\circ}) = zW(A^{\flat\circ})$. Let $t \in R$ be a pseudo-uniformizer, and put $\varpi := t^\sharp \in A$. Put $D_{00} := \mathrm{Im}(B^+ \otimes_{A^+} C^+ \to B \otimes_A C)$ (resp. $U_{00} := \mathrm{Im}(S^+ \otimes_{R^+} T^+) \to S \otimes_R T$. Equip $B \otimes_A C$ (resp. $S \otimes_R T$) with the linear topology defined by $\{\varpi^n D_{00}\}_{n \geq 1}$ (resp. $\{t^n U_{00}\}_{n \geq 1}$). Let D (resp. U) be the Hausdorff completion of $B \otimes_A C$ (resp. $S \otimes_R T$). Let $D_0 := (D_{00})^\wedge_{(\varpi)} \subset D$ (resp. $U_0 := (U_{00})^\wedge_{(t)} \subset U$). Let U^+ be the integral closure of U_0 in U.

Proposition 2.2.2. *We have an isomorphism of affinoid rings $(D, D^+) \cong (U, U^+)^{\sharp,(z)}$. In particular, $B \widehat{\otimes}_A C$ is a perfectoid ring.*

To see this, we prove the following.

Proposition 2.2.3. *There is an A^+-algebra homomorphism $(B^+ \otimes_{A^+} C^+)^\wedge_{(\varpi)} \to W(U^+)/(z)$ whose kernel and cokernel are killed by $A^{\circ\circ}$.*

Proof. Applying the functor $W(\,\cdot\,)$ to the diagram

$$\begin{array}{ccccc} R^+ & \longrightarrow & T^+ & & \\ \downarrow & & \downarrow & & \\ S^+ & \longrightarrow & S^+ \otimes_{R^+} T^+ & \longrightarrow & U^+ \end{array}$$

yields ring homomorphisms

$$W(S^+) \otimes_{W(R^+)} W(T^+) \to W(S^+ \otimes_{R^+} T^+) \to W(U^+)\ . \tag{2.2}$$

Since $W(S^+) \otimes_{W(R^+)} W(T^+)/(z) = B^+ \otimes_{A^+} C^+$ by the tilting equivalence, the composite (2.2) induces a ring homomorphism $B^+ \otimes_{A^+} C^+ \to W(U^+)/(z)$. Applying ϖ-adic completion to this, we obtain a ring homomorphism $f : (B^+ \otimes_{A^+} C^+)^\wedge_\varpi \to W(U^+)/(z)$. Let us show that f is $(\varpi)^{1/p^\infty}$-almost isomorphic. For this, it suffices to construct a commutative diagram of ring homomorphisms

$$\begin{array}{ccc} ((W(S^+) \otimes_{W(R^+)} W(T^+))^\wedge_p)^\wedge_{[t]}/(z) & \xrightarrow{\ \overline{\alpha}\ } & B^+ \widehat{\otimes}_{A^+} C^+ \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ W(S^+ \otimes_{R^+} T^+)^\wedge_{[t]}/(z) & \xrightarrow[\ \overline{\beta}\ ]{} & W(U^+)/(z) \end{array}$$

where g, $\overline{\alpha}$, and $\overline{\beta}$ are $([t])^{1/p^\infty}$-almost isomorphic. Applying $[t]$-adic completion to the projection map $W(S^+)\otimes_{W(R^+)} W(T^+) \twoheadrightarrow W(S^+)\otimes_{W(R^+)} W(T^+)/(z)$, we obtain a surjective homomorphism

$$(W(S^+)\otimes_{W(R^+)} W(T^+))^{\wedge}_{[t]} \twoheadrightarrow B^+\widehat{\otimes}_{A^+}C^+ \ . \tag{2.3}$$

Since $B^+\widehat{\otimes}_{A^+}C^+$ is also p-adically Hausdorff complete by [19, Lemma 2.1.1], (2.3) factors through a surjective homomorphism $\alpha : ((W(S^+)\otimes_{W(R^+)} W(T^+))^{\wedge}_{p})^{\wedge}_{[t]} \twoheadrightarrow B^+\widehat{\otimes}_{A^+}C^+$, which induces a ring homomorphism $\overline{\alpha} : ((W(S^+)\otimes_{W(R^+)} W(T^+))^{\wedge}_{p})^{\wedge}_{[t]}/(z) \twoheadrightarrow B^+\widehat{\otimes}_{A^+}C^+$. Moreover, applying p-adic completion and $[t]$-adic completion to the homomorphisms (2.2), we obtain homomorphisms

$$((W(S^+)\otimes_{W(R^+)} W(T^+))^{\wedge}_{p})^{\wedge}_{[t]} \to W((S^+\otimes_{R^+}T^+)^{\wedge}_{t}) \xrightarrow{\beta} W(U^+) \ , \tag{2.4}$$

where the first one is an isomorphism because $(W(S^+)\otimes_{W(R^+)}W(T^+))^{\wedge}_{p} \cong W(S^+\otimes_{R^+}T^+)$. Hence it induces an isomorphism $g : ((W(S^+)\otimes_{W(R^+)}W(T^+))^{\wedge}_{p})^{\wedge}_{[t]}/(z) \xrightarrow{\cong} W((S^+\otimes_{R^+}T^+)^{\wedge}_{t})/(z)$. Moreover, the second homomorphism β induces a homomorphism $\overline{\beta} : W((S^+\otimes_{R^+}T^+)^{\wedge}_{t})/(z) \to W(U^+)/(z)$. By the construction, we find that $f\circ\overline{\alpha} = \overline{\beta}\circ g$. Thus we are reduced to showing the following claim.

Claim 2.2.4. *$\overline{\beta}$ is $(\varpi)^{\frac{1}{p^\infty}}$-almost isomorphic.*

Proof of Claim 2.2.4. By Lemma 2.2.1 3, the map $\varphi : S^+\widehat{\otimes}_{R^+}T^+ \to U^+$ is $(t)^{1/p^\infty}$-almost isomorphic. Hence the maps β and $\overline{\beta}$ are $([t])^{1/p^\infty}$-almost surjective. Let us show that $\overline{\beta}$ is also $([t])^{1/p^\infty}$-almost injective. Consider the diagram with exact laws:

$$\begin{array}{ccccccccc}
0 & \to & zW(S^+\widehat{\otimes}_{R^+}T^+) & \to & W(S^+\widehat{\otimes}_{R^+}T^+) & \to & W(S^+\widehat{\otimes}_{R^+}T^+)/(z) & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \overline{\beta}} & & \\
0 & \to & zW(U^+) & \to & W(U^+) & \to & W(U^+)/(z) & \to & 0 \ .
\end{array}$$

Then by snake lemma, we obtain the short exact sequence

$$\mathrm{Ker}(\beta) \to \mathrm{Ker}(\overline{\beta}) \to zW(U^+)/z(\mathrm{Im}(\beta)) \ .$$

Since β is $([t])^{\frac{1}{p^\infty}}$-almost surjective, $zW(U^+)/z(\mathrm{Im}(\beta))$ is $([t])^{\frac{1}{p^\infty}}$-almost zero. On the other hand, since

$$\mathrm{Ker}(\beta) = \{\sum_{i=0}^{\infty} p^i[\xi_i] \in W(S^+\widehat{\otimes}_{R^+}T^+) \mid \xi_i \in \mathrm{Ker}(\varphi)\ (\forall i \geq 0)\} \ ,$$

$\mathrm{Ker}(\beta)$ is also $([t])^{\frac{1}{p^\infty}}$-almost zero. Thus $\mathrm{Ker}(\overline{\beta})$ is $([t])^{1/p^\infty}$-almost zero, as wanted. □

□

Proof of Proposition 2.2.2. In view of Lemma 1.1.4 and Proposition 2.2.3, it suffice to show that the homomorphism $(B^+\otimes_{A^+}C^+)^{\wedge}_{\varpi} \to D_0$ induced by $\Psi : B^+\otimes_{A^+}C^+ \to D_{00}$ is $(\varpi)^{1/p^\infty}$-almost isomorphic. Put $I = \bigcap_{n=0}^{\infty}\varpi^n(B^+\otimes_{A^+}C^+)$ and $J = \bigcap_{n=0}^{\infty}\varpi^n D_{00}$. It is enough to show that $(B^+\otimes_{A^+}C^+)/I \twoheadrightarrow D_{00}/J$ is $(\varpi)^{1/p^\infty}$-almost injective. Pick $f \in B^+\otimes_{A^+}C^+$ such that $\Psi(f) \in J$. Then, since $\Psi(I) = J$, we have $f \in \mathrm{Ker}(\Psi) + I$. Here notice that $\mathrm{Ker}(\Psi) = (B^+\otimes_{A^+}C^+)_{\varpi\text{-tor}}$. On the other hand, since $(B^+ \otimes_{A^+} C^+)^\wedge_\varpi \to W(U^+)/(z)$ is $(\varpi)^{1/p^\infty}$-almost isomorphic by Proposition 2.2.3 and $W(U^+)/(z)$ is ϖ-torsion free, $((B^+ \otimes_{A^+} C^+)^\wedge_\varpi)_{\varpi\text{-tor}}$ is $(\varpi)^{1/p^\infty}$-almost zero. Therefore $\varpi^{1/p^n}(B^+ \otimes_{A^+} C^+)_{\varpi\text{-tor}} \subset I$ for every $n > 0$. Thus $\varpi^{1/p^n} f \in I$ for every $n > 0$, which implies that $(B^+ \otimes_{A^+} C^+)/I \twoheadrightarrow D_{00}/J$ is $(\varpi)^{1/p^\infty}$-almost injective, as wanted. □

2.3 Tilting equivalences

We define categories $\mathscr{A}^\triangleright, \mathscr{R}^\triangleright, \mathscr{A}$, and $\mathscr{R}$ as follows.

- $\mathrm{ob}(\mathscr{A}^\triangleright)$ consists of perfectoid rings A with $p \in A$ topologically nilpotent; the morphisms are continuous ring homomorphisms.
- $\mathrm{ob}(\mathscr{R}^\triangleright)$ consists of pairs $\{(R, I)\}$ where R is a perfectoid ring of characteristic p and I is an ideal in $W(R^\circ)$ which is generated by a distinguished element; a morphism $(R, I) \to (S, J)$ in $\mathscr{R}$ is a continuous ring homomorphism $\varphi : R \to S$ such that the restriction map $\varphi|_{R^\circ} : R^\circ \to S^\circ$ satisfies $W(\varphi|_{R^\circ})(I) \subset J$.
- $\mathrm{ob}(\mathscr{A})$ consists of perfectoid affinoid rings (A, A^+) with $p \in A$ topologically nilpotent; the morphisms are continuous affinoid ring homomorphisms.
- $\mathrm{ob}(\mathscr{R})$ consists of pairs $\{((R, R^+), I)\}$ where (R, R^+) is a perfectoid affinoid ring of characteristic p, and I is an ideal in $W(R^+)$ which is generated by a distinguished element; a morphism $((R, R^+), I) \to ((S, S^+), J)$ in $\mathscr{R}$ is a continuous affinoid ring homomorphism $\varphi : (R, R^+) \to (S, S^+)$ such that the underlying map $\varphi|_{R^+} : R^+ \to S^+$ satisfies $W(\varphi|_{R^+})(I) \subset J$.

By Proposition 2.1.15, Lemma 2.1.17, and Lemma 2.1.21, the association $A \mapsto (A^\flat,\ \mathrm{Ker}(\theta_{A^\circ}))$ (resp. $(A, A^+) \mapsto ((A^\flat, (A^+)^\flat),\ \mathrm{Ker}(\theta_{A^+})))$ defines a functor from $\mathscr{A}^\triangleright$ to $\mathscr{R}^\triangleright$ (resp. from $\mathscr{A}$ to $\mathscr{R}$). Moreover, by Corollary 2.1.23, the association $(R, I) \mapsto (R, I)^\sharp$ (resp. $((R, R^+), I) \mapsto ((R, R^+), I)^\sharp$) defines a functor from $\mathscr{R}^\triangleright$ to $\mathscr{A}^\triangleright$ (resp. from $\mathscr{R}$ to $\mathscr{A}$).

Theorem 2.3.1. *The associations*

$$A \mapsto (A^\flat,\ \mathrm{Ker}(\theta_{A^\circ})) \quad \text{and} \quad (R, I) \mapsto (R, I)^\sharp$$

give an equivalence of categories between $\mathscr{A}^\triangleright$ and $\mathscr{R}^\triangleright$. Moreover, the associations

$$(A, A^+) \mapsto ((A^\flat, (A^+)^\flat),\ \mathrm{Ker}(\theta_{A^\circ})), \qquad ((R, R^+), I) \mapsto ((R, R^+), I)^\sharp$$

give an equivalence of categories between $\mathscr{A}$ and $\mathscr{R}$.

To prove this theorem, it suffices to check the following statements.

Lemma 2.3.2. *Let $(A, A^+) \in \mathrm{ob}(\mathscr{A})$ and $\{(R, R^+), I\} \in \mathrm{ob}(\mathscr{R})$.*

1. *The ring isomorphism $\overline{\theta}_{A^+} : ((A^+)^\flat)^\sharp \to A^+$ induced from θ_{A^+} is a homeomorphism.*
2. *The map*

$$\varphi_{R^+} : R^+ \to ((R^+)^\sharp)^\flat,\ r \mapsto (\ldots, \overline{(r^{1/p})^\sharp}, \overline{r^\sharp})$$

is an isomorphism of topological rings. Moreover, in the induced isomorphism $W(((R^+)^\sharp)^\flat) \to W(R^+)$, $\mathrm{Ker}(\theta_{(R^+)^\sharp})$ corresponds to I.

Proof. 1 follows immediately from $\theta'_{A^+}((\varpi^\flat)^\sharp) = \varpi u$ for some unit $u \in A^+$. Let us prove 2. Since R^+ is t-adically complete, one has an isomorphism

$$R^+ \to \varprojlim R^+/t^p R^+,\ r \mapsto (\ldots, \overline{r^{1/p}}, \overline{r})\ .$$

Moreover, the isomorphism $R^+/t^p R^+ \simeq W(R^+)/(I + [t]^p W(R^+))$ induces an isomorphism

$$\varprojlim R^+/t^p R^+ \to ((R^+)^\sharp)^\flat,\ (\ldots, \overline{r_1}, \overline{r_0}) \mapsto (\ldots, \overline{[r_1]}, \overline{[r_0]})\ .$$

Since φ_{R^+} is obtained by composition of these maps, it is a ring isomorphism. Moreover, under this map $(t^\sharp)^\flat$ corresponds to t. Hence φ_{R^+} is also a homeomorphism.

Next we investigate the kernel of $\theta_{(R^+)^\sharp} : W(((R^+)^\sharp)^\flat) \to (R^+)^\sharp$. For $r \in R^+$, we have

$$\theta_{(R^+)^\sharp} \circ W(\varphi)([r]) = \theta_{(R^+)^\sharp}([(\ldots, \overline{(r^\sharp)^{1/p}}, \overline{r^\sharp})]) = r^\sharp = [r] \mod I\ .$$

Therefore

$$\theta_{(R^+)^\sharp} \circ W(\varphi)(\sum_{i=0}^{\infty} p^i [r_i]) = \sum_{i=0}^{\infty} p^i [r_i] \mod I\ .$$

Hence $\mathrm{Ker}(\theta_{(R^+)^\sharp} \circ W(\varphi_{R^+})) = I$, and thus $\mathrm{Ker}(\theta_{(R^+)^\sharp}) = W(\varphi_{R^+})(I)$. □

Corollary 2.3.3. *Let $\mathscr{A}^\rhd_{\mathbf{f}}$ (resp. $\mathscr{R}^\rhd_{\mathbf{f}}$) be a full subcategory of $\mathscr{A}^\rhd$ (resp. $\mathscr{R}^\rhd$) whose objects are perfectoid fields. Then the categorical equivalence in Theoerm 2.3.1 induces an equivalence of categories between $\mathscr{A}^\rhd_{\mathbf{f}}$ and $\mathscr{R}^\rhd_{\mathbf{f}}$.*

Proof. Let K and (F, I) be objects of $\mathscr{A}_{\mathbf{f}}$ and $\mathscr{R}_{\mathbf{f}}$ respectively. Let $\varpi \in K$ (resp. $t \in F$) be a perfectoid pseudo-uniformizer that admits a compatible system of p-power roots. Then by Lemma 2.1.16 3 (resp. Proposition 1.5.12), $\gamma_{(K^\circ)^\flat,(\varpi^\flat),2}$ (resp. $\gamma_{(F^\circ)^\sharp,(t^\sharp),2}$) is equivalent to a multiplicative norm. Thus it suffices to check that $K^\flat$ (resp. $F^\sharp$) is a field.

First we verify that $K^\flat = (K^\circ)^\flat[(\varpi^\flat)^{-1}]$ is a field. Pick a nonzero element $s \in (K^\circ)^\flat$. Then $\theta_{K^\circ}([s^{1/p^n}]) \neq 0$ for every $n \geq 0$. Take an integer $d > 0$ such that $t_0 := \varpi^d \theta_{K^\circ}([s])^{-1} \in K^\circ$. Then $t_n := \varpi^{d/p^n} \theta_{K^\circ}([s^{1/p^n}])^{-1}$ satisfies $t_n^{p^n} = t_0$. Hence we obtain an element $t := (\ldots, \overline{t_1}, \overline{t_0}) \in (K^\circ)^\flat$ that satisfies $st = (\varpi^\flat)^d$. Hence the assertion follows.

Next we verify that $F^\sharp = (W(F^\circ)/I)[(t^\sharp)^{-1}]$ is a field. Let α be the spectral norm associated to $\gamma_{F^\circ,(t),2}$. Pick a non-zero element $x = \sum_{i=0}^\infty p^i \overline{[\xi_i]} \in W(F^\circ)/I$. We may assume that $\sum_{i=0}^\infty p^i [\xi_i]$ is α-stable. In this situation, $x = \overline{[\xi_0]} u$ where $u = \sum_{i=0}^\infty p^i \overline{[\xi_i/\xi_0]} \in (F^\circ)^\sharp$ is an invertible element. Take an integer $e > 0$ such that $t^e \xi_0^{-1} \in F^\circ$. Then $y := \overline{[t^e \xi_0^{-1}]} u \in (F^\circ)^\sharp$ satisfies $xy = (t^\sharp)^e$. Hence the assertion follows. □

The functor ♮

First notice the following.

Lemma 2.3.4. *Let A be a perfectoid ring. Let B be a uniform Tate ring together with a continuous ring homomorphism $A \to B$. Then $\mathrm{Ker}(\theta_{B^+})$ is a distinguished ideal in $W(B^+)$.*

Proof. $\mathrm{Ker}(\theta_{A^+})$ has a distinguished element z by Lemma 2.1.21. The image of z in $W(B^+)$ is also distinguished. Hence $\mathrm{Ker}(\theta_{B^+}) = zW(B^+)$ by Lemma 2.1.20. □

Thus we can define the functor $\natural$.

Definition 2.3.5. Let A be a perfectoid ring, and B a quasi-perfectoid A-algebra. Then we define $B^{\natural}$ as the perfectoid ring $(B^{\flat})^{\sharp, I}$ where $I = \mathrm{Ker}(\theta_{B^+})$.

$\natural$ gives a right adjoint functor.

Lemma 2.3.6. *The association $B \mapsto B^{\natural}$ gives a right adjoint functor of the forgetful functor $\iota : \mathscr{A}/A \to \mathscr{A}'/A$.*

Proof. Let $B \in \mathrm{ob}(\mathscr{A}/A)$ and $C \in \mathrm{ob}(\mathscr{A}'/A)$. Then by Lemma 2.1.17, we have the following commutative diagram:

$$\begin{array}{ccc} B^{\natural} & \xrightarrow{f^{\natural}} & C^{\natural} \\ {\scriptstyle \iota_B}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \iota_C} \\ B & \xrightarrow[f]{} & C \end{array}$$

where ι_B is an isomorphism and ι_C is a monomorphism. Hence the map

$$\mathrm{Hom}_{\mathscr{A}'/A}(B, C) \to \mathrm{Hom}_{\mathscr{A}/A}(B, C^{\natural}),\ f \mapsto (f^{\flat})^{\sharp} \circ \iota_B^{-1}$$

admits the inverse map

$$\mathrm{Hom}_{\mathscr{A}/A}(B, C^{\natural}) \to \mathrm{Hom}_{\mathscr{A}'/A}(B, C),\ g \mapsto \iota_C \circ g\ .$$

Thus the assertion follows. □

2.3.1 Almost perfectoid algebras

In the sequel, we use the following notation: for a Hausdorff complete Tate ring A that admits a pair of definition $(A_0, (t))$, we denote by $A\langle T^{\frac{1}{p^\infty}}\rangle$ the Hausdorff completion of the Tate ring associated to $(A_0[T^{1/p^\infty}], (t))$.

Lemma 2.3.7. *Let A be a perfectoid ring and $\varpi \in A$ a pseudo-uniformizer. Let B be a uniform $A\langle T^{1/p^\infty}\rangle$-algebra. Then the following conditions are equivalent.*

1. *$B/(p)$ is $(T)^{1/p^\infty}$-almost semiperfect.*
2. *$\iota : B^{\natural} \to B$ is $(T)^{1/p^\infty}$-almost bijective.*

Proof. 1 ⇒ 2: By our assumption and Lemma 1.3.19, the map $\theta : W((B^\circ)^\flat) \to B^\circ$ is $(T)^{1/p^\infty}$-almost surjective. Hence the assertion follows.

2 ⇒ 1: We have the commutative diagram:

$$\begin{array}{ccc} (B^\circ)^\flat/([\varpi^\flat]) & \xrightarrow{\bar{\iota}} & B^\circ/(p) \\ {\scriptstyle \mathrm{Frob}}\downarrow & & \downarrow{\scriptstyle \mathrm{Frob}} \\ (B^\circ)^\flat/([\varpi^\flat]) & \xrightarrow[\bar{\iota}]{} & B^\circ/(p)\ . \end{array}$$

Thus, since ι is $(T)^{1/p^\infty}$-almost surjective and $(B^\circ)^\flat/([\varpi^\flat])$ is semiperfect, the Frobenius map on $B^\circ/(p)$ is also $(T)^{1/p^\infty}$-almost surjective. □

Now we can extend André's almost perfectoid algebras (cf. Definition 3.5.2 and Proposition 3.5.4 in [1]).

Definition 2.3.8 (Almost perfectoid $R\langle T^{\frac{1}{p^\infty}}\rangle$-algebra)**.** Let R be a perfectoid ring and let A be a uniform Banach $R\langle T^{\frac{1}{p^\infty}}\rangle$-algebra. Let $\mathfrak{m} := T^{\frac{1}{p^\infty}} R^{\circ\circ} R^{\circ}\langle T^{\frac{1}{p^\infty}}\rangle$ be an ideal of $R^{\circ}\langle T^{\frac{1}{p^\infty}}\rangle$. We say that A is an *almost perfectoid* $R\langle T^{\frac{1}{p^\infty}}\rangle$*-algebra*, if the Frobenius endomorphism on $A^{\circ}/(\varpi)$ is $\mathfrak{m}$-almost surjective.

2.3.2 Correspondence of rational subsets

Proposition 2.3.9. *Let (A, A^+) be a perfectoid affinoid ring. Let $U \subset \mathrm{Spa}(A, A^+)$ be a rational subset, and $h : (A, A^+) \to (B, B^+)$ the corresponding rational localization. Then (B, B^+) is isomorphic to $((S, S^+), I)^\sharp$ as an affinoid ring over (A, A^+). In particular, B is a perfectoid ring.*

Proof. Let $U = \{v \in \mathrm{Spa}(A, A^+) \mid v(f_i) \geq v(f_0) \neq \infty,\ i = 1, \ldots, n\}$. First we construct an isomorphism

$$B \cong A\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle / I^- \ .$$

where I is an ideal in $A\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle$ generated by $\{f_0^{1/p^k} T_i^{1/p^k} - f_i^{1/p^k}\}_{1\leq i\leq n, k\geq 1}$. Here $A\langle T_1, \ldots, T_n\rangle$ is the completion of a polynomial ring $A[T_1, \ldots, T_n]$ with respect to the topology defined by $\{t^n A^{\circ}[T_1, \ldots, T_n]\}_{n>0}$. Recall the definition of B and $A\langle T_1, \ldots, T_n\rangle$. Consider the ring A_g and its A°-subalgebra $A' = A^{\circ}[f_1/g, \ldots, f_n/g]$. Equip A_g with the topology defined by $\{\varpi^n A'\}_n$.

Consider an A-algebra homomorphism

$$f : A[T_1^{\frac{1}{p^\infty}}, \ldots, T_n^{\frac{1}{p^\infty}}] \to A_{f_0},\ T_i^{\frac{1}{p^k}} \mapsto \frac{f_i^{1/p^k}}{f_0^{1/p^k}} \ .$$

Then $f(\varpi^m A^{\circ}[T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}]) \subset \varpi^m A^{\circ}[(f_1/f_0)^{1/p^\infty}, \ldots, (f_n/f_0)^{1/p^\infty}]$. Hence f is continuous. Thus f extends to a continuous A-algebra homomorphism

$$\widehat{f} : A\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle \to B \ ,$$

which induces a continuous A-algebra homomorphism

$$\overline{\widehat{f}} : A\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle / I^- \to B \ .$$

Since $\overline{f_i}/\overline{f_0} = \overline{T_i} \in (A\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle / I^-)^{\circ}$, one obtains a continuous A-algebra homomorphism

$$F_A(U) \to A\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle / I^-$$

by the universality of localization. Moreover, since $\overline{T}_i = \overline{f_i}/\overline{f_0}$ $(i = 1, \ldots, n)$, the composite

$$A\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle / I^- \to A\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle / I^-$$

is identity. Since f is surjective and $f(\varpi^m A^{\circ}[T_i^{1/p^\infty}]) \supset \varpi^m A^{\circ}[\frac{f_i}{f_0}]$, $\overline{\widehat{f}}$ is surjective. Hence $\overline{\widehat{f}}$ is an isomorphism of topological rings.

Like the above argument, one obtains a surjective and continuous R-algebra homomorphism

$$\psi : R\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle \to S,\ X_i^{\frac{1}{p^k}} \mapsto \left(\frac{\varphi_i}{\varphi_0}\right)^{\frac{1}{p^k}}$$

(notice that $R\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle$ is perfectoid and $(R\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle)^\sharp \cong A\langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}\rangle$). Then $\mathrm{Ker}(\psi)$ is the closure of the ideal generated by $\varphi_0^{1/p^k} X_i^{1/p^k} - \varphi_i^{1/p^k}$ $(i = 1, \dots, n,\ k \geq 1)$. Moreover, restricting ψ to $R^\circ\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle$ yields a surjective homomorphism

$$\psi_0 : R^\circ\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle \to R^\circ\left\langle \left(\frac{\varphi_1}{\varphi_0}\right)^{1/p^\infty}, \dots, \left(\frac{\varphi_n}{\varphi_0}\right)^{1/p^\infty} \right\rangle,\ X_i^{\frac{1}{p^k}} \mapsto \left(\frac{\varphi_i}{\varphi_0}\right)^{\frac{1}{p^k}} .$$

Hence we get a continuous ring homomorphism

$$g : (W(R^\circ\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle)/(z))\left[\frac{1}{t^\sharp}\right] \to (W(S^\circ)/(z))\left[\frac{1}{t^\sharp}\right],\ \frac{\overline{x}}{(t^\sharp)^d} \mapsto \frac{\overline{W(\psi_0)(x)}}{(t^\sharp)^d} .$$

Since S is uniform, there exists $l > 0$ such that $t^l S^\circ \subset R^\circ\langle(\frac{\varphi_1}{\varphi_0})^{1/p^\infty}, \dots, (\frac{\varphi_n}{\varphi_0})^{1/p^\infty}\rangle$. Hence g is surjective. We show that

$$\mathrm{Ker}(g) = I^- .$$

Recall that I is the ideal of $(W(R^\circ\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle)/(z))_{t^\sharp}$ generated by $(\varphi_0^{1/p^k})^\sharp (X_i^{1/p^k})^\sharp - (\varphi_i^{1/p^k})^\sharp$ for every $i = 1, \dots, n$ and $k \geq 0$. The inclusion $\supset$ is clear. Let us prove the inverse inclusion. For an element $\frac{\overline{x}}{(t^\sharp)^d} \in (W(R^\circ\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle)/(z))_{t^\sharp}$, we have

$$\begin{aligned}
\frac{\overline{x}}{(t^\sharp)^d} \in \mathrm{Ker}(g) \quad &\Leftrightarrow \quad W(\psi_0)(x) \in zW(S^\circ) \\
&\Leftrightarrow \quad W(\psi_0)([t]^l x) \in zW(R^\circ\langle(\varphi_1/\varphi_0)^{1/p^\infty}, \dots, (\varphi_n/\varphi_0)^{1/p^\infty}\rangle) \text{ for some } l > 0 \\
&\Leftrightarrow \quad \text{there exists } x' = \sum_{i=0}^{\infty} p^i [\xi'_i] \text{ such that } \psi_0(\xi'_i) = 0 \ (\forall i \geq 0) \text{ and} \\
&\qquad \frac{\overline{x}}{(t^\sharp)^d} = \frac{\overline{x'}}{(t^\sharp)^{d+l}} \text{ for some } l > 0 .
\end{aligned}$$

Thus, it suffices to show that any $\xi \in \mathrm{Ker}\,\psi_0$ satisfies $\xi^\sharp \in I^-$. Put $\xi = \sum_{i,k} r_{ik}(\varphi_0^{1/p^k} X_i^{1/p^k} - \varphi_i^{1/p^k})$ where $r_{ik} \in R^\circ$. Consider the elements x_n $(n \geq 0)$ in $W(R^\circ\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle$ defined by

$$x_n := \sum_{i,k} [r_{ik}^{\frac{1}{p^n}}]([\varphi_0^{\frac{1}{p^{n+k}}}][X_i^{\frac{1}{p^{n+k}}}] - [\varphi_i^{\frac{1}{p^{n+k}}}]) .$$

Then, since $x_n \equiv [\xi^{1/p^n}] \mod (p)$ for every n, we have $\overline{x_n}^{p^n} \to \xi^\sharp$ $(n \to \infty)$ by Claim 1.3.15. Hence $\xi \in I^-$, as wanted. □

Corollary 2.3.10. *Let (A, A^+) be a perfectoid affinoid ring. Let $k(x)$ be the residue field of a point $x \in \mathrm{Spa}(A, A^+)$. Then the Hausdorff completion $\widehat{k(x)}$ is a perfectoid field.*

Proof. Set $X = \mathrm{Spa}(A, A^+)$. Let $\varpi \in A$ be a pseudo-uniformizer such that $A^+/(\varpi^p)$ is a semiperfect $\mathbb{F}_p$-algebra. In view of Proposition 2.3.9 and the isomorphism $(\varinjlim_{x\in U} \mathcal{O}_X^+(U))/(\varpi^p) \cong \varinjlim_{x\in U}(\mathcal{O}_X^+(U)/(\varpi^p))$, we find that $k(x)^+/(\varpi^p)$ is semiperfect and therefore so is $k(x)^\circ/(\varpi^p)$ by Lemma 1.3.5. Thus by Lemma 1.5.7, $\widehat{k(x)}$ is a perfectoid field. □

Moreover, we find the following.

Lemma 2.3.11. *For the rational localization $\varphi : (A, A^+) \to (A_W, A_W^+)$ corresponding to a rational subset $W \subset \mathrm{Spa}(A, A^+)$, $\varphi^\flat : (A^\flat, (A^+)^\flat) \to (A_W^\flat, (A_W^+)^\flat)$ is the rational localization corresponding to W^+.*

Proof. It follows from Theorem 2.3.1, Proposition 2.3.9, and universal property of rational localization. □

Let us prove the following.

Proposition 2.3.12. *Let $((R, R^+), I) \in \mathrm{ob}(\mathcal{R})$ and $(A, A^+) := ((R, R^+), I)^\sharp$. Consider the map*

$$\cdot^\flat : \mathrm{Spa}(A, A^+) \to \mathrm{Spa}(R, R^+),\ [v] \mapsto [v \circ \cdot^\sharp]\ .$$

Then $\cdot^\flat$ is a homeomorphism. Moreover, any rational subset of $\mathrm{Spa}(A, A^+)$ *is given as the preimage of a rational subset of* $\mathrm{Spa}(R, R^+)$.

First we show the latter part of this proposition.

Lemma 2.3.13. *Keep the notation as in Proposition 2.3.12. Let $t \in R$ be a pseudo-uniformizer, $\varpi := t^\sharp \in A$, and $X := \mathrm{Spa}(A, A^+)$. Let $f_0, \dots, f_n \in A^+$ be elements such that $(f_0, \dots, f_n)A = A$. Let $N > 0$ be an integer with the following property:*

- *there exist $h_0, \dots, h_n \in A$ such that $\sum_{i=0}^n h_i f_i = 1$ and $\varpi^N h_i \in A^+$ for every $i = 0, \dots, n$.*

Then the following assertions hold.

1. *For each $i = 0, \dots, n$, there exists a presentation $f_i = \sum_{j=0}^\infty p^j \varphi_{ij}^\sharp$ of f_i such that*

$$v_x(\varphi_{ij}^\sharp) \geq \min\{v_x(\varphi_{i0}^\sharp),\ v_x(\varpi^{N+1}))\} \qquad (\forall j \geq 0)\ .$$

2. *For $\{\varphi_{ij}\}$ in the assertion 1 and for every $x \in X$, we have*

$$\min\{v_x(f_i), v_x(\varpi^{N+1})\} = \min\{v_x(\varphi_{i0}^\sharp), v_x(\varpi^{N+1})\} \quad (i = 0, \dots, n)\ .$$

3. *For $\{\varphi_{ij}\}$ in the assertion 1, we have*

$$X\left(\frac{f_1, \dots, f_n}{f_0}\right) = X\left(\frac{\varphi_{10}^\sharp, \dots, \varphi_{n0}^\sharp, \varpi^{N+1}}{\varphi_{00}^\sharp}\right)\ .$$

Proof. 1 is due to Lemma 1.6.9. Let us prove 2. Pick $x \in X$. In view of the assertion 1, for a sufficiently large $k \geq 0$ we have

$$v_x(f_i - \varphi_{i0}^\sharp) = v_x\left(\sum_{j=1}^{k} p^j \varphi_{ij}^\sharp + p^{k+1} \sum_{j=0}^{\infty} p^j \varphi_{i,k+1+j}^\sharp\right) > \min\{v_x(\varphi_{i0}^\sharp),\ v_x(\varpi^{N+1}))\} \ .$$

Thus, if $v_x(\varphi_{i0}^\sharp) \leq v_x(\varpi^{N+1})$, then $v_x(f_i) = v_x(\varphi_{i0}^\sharp)$. Moreover, if $v_x(\varphi_{i0}^\sharp) \geq v_x(\varpi^{N+1})$, then $v_x(f_i) \geq v_x(\varpi^{N+1})$. Hence the assertion 2 follows. Finally, we prove 3. Set $U := X(\frac{f_1,\ldots,f_n}{f_0})$ and $V := X(\frac{\varphi_{10}^\sharp,\ldots,\varphi_{n0}^\sharp,\varpi^{N+1}}{\varphi_{00}^\sharp})$. Then for every $x \in U$, we have

$$v_x(\varpi^N) = v_x\left(\sum_{i=0}^{n} \varpi^N h_i f_i\right) \geq \min_{0 \leq i \leq n} \{v_x(\varpi^N h_i) + v_x(f_i)\} \geq v_x(f_0) \ .$$

In particular, we have

$$v_x(f_0) < v_x(\varpi^{N+1}) \qquad (\forall x \in U) \ . \tag{2.5}$$

The assertion 2 and (2.5) imply that $v_x(f_0) = v_x(\varphi_{00}^\sharp)$ and $v_x(\varphi_{00}^\sharp) < v_x(\varpi^{N+1})$ for every $x \in U \cup V$. Pick $x \in U$ and $i \in \{1, \ldots, n\}$ arbitrarily. If $v_x(f_i) < v_x(\varpi^{N+1})$, then the assertion 2 implies that $v_x(f_i) = v_x(\varphi_{i0}^\sharp)$ and thus $v_x(\varphi_{i0}^\sharp) \geq v_x(\varphi_{00}^\sharp)$. If $v_x(f_i) \geq v_x(\varpi^{N+1})$, the assertion 2 implies that $v_x(\varphi_{i0}^\sharp) \geq v_x(\varpi^{N+1})$ and thus $v_x(\varphi_{i0}^\sharp) \geq v_x(\varphi_{00}^\sharp)$. Therefore $U \subset V$. Similarly, one can show that $V \subset U$. Thus we obtain $U = V$, as wanted. □

In view of Lemma 2.3.13, Proposition 2.3.12 follows from the following lemma.

Lemma 2.3.14. *The map in Proposition 2.3.12 is bijective.*

Proof. Put $X = \mathrm{Spa}(A, A^+)$, and $Y = \mathrm{Spa}(R, R^+)$. Since $\mathrm{Spa}(A, A^+)$ is T_0 and rational subsets form a basis of $\mathrm{Spa}(A, A^+)$, $\cdot^\flat$ is injective by Proposition 2.3.13.

We show that $\cdot^\flat$ is surjective. Pick $v_y \in Y$. Consider $k(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y$. Let $\widehat{k(y)}$ be the Hausdorff completion of $k(y)$. Then $\widehat{k(y)}$ is a perfectoid field by Corollary 2.3.10. Put $L = \widehat{k(y)}$. Let $z \in W(R^\circ)$ be a distinguished element for which $I = (zW(R^\circ)) \cap W(R^+)$. Put $F = (L, zW(L^\circ))^\sharp$. Let $\widetilde{v_y}$ be the maximal generalization of $v_y \in \mathrm{Spa}(\widehat{k(y)}, \widehat{k(y)^+})$, and w be a rank 1 valuation induced by the multiplicative norm on F. By Corollary 2.3.3, there exists an injective ordered group homomorphism $\Gamma_{\widetilde{v_y}} \to \Gamma_w$ such that the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{\ \sharp\ } & F \\ {\scriptstyle \widetilde{v_y}}\downarrow & & \downarrow{\scriptstyle w} \\ \Gamma_{\widetilde{v_y}} \cup \{\infty\} & \longrightarrow & \Gamma_w \cup \{\infty\} \ . \end{array}$$

Moreover, since $L^\circ/(t) \cong F^\circ/(\varpi)$, one has $L^\circ/\mathfrak{m}_L \cong F^\circ/\mathfrak{m}_F$. Therefore valuation subrings of $L^\circ/\mathfrak{m}_L$ correspond those of $F^\circ/\mathfrak{m}_F$ under this isomorphism. Hence there exists $w_x \in \mathrm{Spa}(F, F^\circ)$ such that $v_y \in \mathrm{Spa}(L, L^\circ)$ is equal to $(w_x)^\flat$. Thus the composition of the morphism $A \to F$ (induced by $R \to L$) and w_x, which defines an element of $\mathrm{Spa}(R, R^+)$, is sent to $v_y \in \mathrm{Spa}(A, A^+)$ by $\cdot^\flat$. This yields the assertion. □

2.4 Almost purity

The following theorem describes one of the most remarkable features of perfectoid rings.

Theorem 2.4.1. *Let A be a perfectoid ring.*

1. *Let B be a finite étale A-algebra equipped with the canonical topology as in Lemma 1.4.7. Then B is also a perfectoid ring, and $A^\circ \to B^\circ$ is almost finite étale.*

2. *The functor*
$$(A^{\circ a})_{\text{fét}} \to A_{\text{fét}},\ C_0 \mapsto (C_0)_*[\tfrac{1}{\varpi}]$$
gives an equivalence of categories. Moreover, the association $B \mapsto B^{\circ a}$ gives an inverse functor.

2.4.1 Construction of a functor $R_{\text{fét}} \to (R^\sharp)_{\text{fét}}$

We want to construct a functor
$$(A^{\circ a})_{\text{fét}} \to A_{\text{fét}}\ .$$

Lemma 2.4.2. *Let A be a perfectoid ring, and $\varpi \in A$ a perfectoid pseudo-uniformizer. Let C_0 be a finite étale $A^{\circ a}$-algebra. Then the following assertions hold.*

1. *$(C_0)_*$ is ϖ-torsion free.*

2. *Let C be the Tate ring associated to $((C_0)_*, (\varpi))$. Then C is a perfectoid ring with $C^\circ = (C_0)_*$, and $A \to C$ is finite étale.*

3. *The canonical topology on $(C_0)_*[\frac{1}{\varpi}]$ as a finitely generated A-module (cf. Lemma 1.4.6) coincides with the topology on C.*

Proof. Since $(C_0)_*$ is an almost flat A_0-module, it is ϖ-torsion free by Lemma 1.1.5. Let us show that C is perfectoid. For an $\mathbb{F}_p$-algebra R, we denote by Φ_R the Frobenius endomorphism on R, and by $R_{(1)}$ the ring R itself equipped with an R-algebra structure induced by $\Phi_R : R \to R_{(1)}$. Since the induced morphism $(A^\circ/(\varpi^p))^a \to ((C_0)_*/(\varpi^p))^a$ is étale by Lemma [21, Lemma 3.1.2], the diagram

$$\begin{array}{ccc}
(A^\circ/(\varpi^p))^a & \longrightarrow & ((C_0)_*/(\varpi^p))^a \\
\downarrow & & \downarrow \\
((A^\circ/(\varpi^p))_{(1)})^a & \longrightarrow & (((C_0)_*/(\varpi^p))_{(1)})^a
\end{array}$$

is cocartesian by [21, Theorem 3.5.13]. Hence the Frobenius map on $(C_0)_*/(\varpi^p)$ is almost surjective. Moreover, the kernel is almost isomorphic to

$$(\mathrm{Ann}_{A^\circ/(\varpi^p)}(A^\circ/(\varpi^p))_{(1)})((C_0)_*/(\varpi^p)) = \varpi((C_0)_*/(\varpi^p))\ .$$

Let us show that $\mathrm{Ker}(\Phi_{(C_0)_*/(\varpi^p)}) = (\varpi)$. Pick $x \in (C_0)_*$ for which $x^p \in \varpi^p C_*$. Then $\varepsilon x \in \varpi(C_0)_*$ for every $\varepsilon \in R^{\circ\circ}$. Thus we have $\varpi^{-1}x \in \varpi^{-1/p^\infty}(C_0)_* = (C_0)_*$, and so $x \in \varpi(C_0)_*$. Therefore $\mathrm{Ker}(\Phi_{(C_0)_*/(\varpi^p)}) = (\varpi)$. Hence by Lemma 1.2.3, $(C_0)_*$ is completely integrally closed in $(C_0)_*[\frac{1}{\varpi}]$.

Thus, C is a uniform Tate ring with $C^\circ = (C_0)_*$, and the Frobenius map on $(C_0)_*/(\varpi^p)$ is surjective by Lemma 1.3.4. Moreover, C is finite étale over A by [21, Lemma 3.1.2]. Hence the assertion 2 follows.

3: Equip the finitely generated A-module $(C_0)_*[\frac{1}{\varpi}]$ with the canonical topology as in Lemma 1.4.6. Put $C'_0 := (A^\circ)^*_{(C_0)_*[\frac{1}{\varpi}]}$. Then by Corollary 1.8.10 1, $(C_0)_*[\frac{1}{\varpi}]$ coincides with the Tate ring associated to $(C'_0, (\varpi))$. Thus it suffices to show that $C'_0 = (C_0)_*$. Now since $(C_0)_*$ is almost finitely generated over A°, $(C_0)_* \subset C'_0$. Therefore the morphism $(A^\circ, (\varpi)) \to ((C_0)_*, (\varpi))$ satisfies $(*)$. Hence by Proposition 1.8.7 3, we have

$$(A^\circ)^*_{(C_0)_*[\frac{1}{\varpi}]} = ((C_0)_*)^*_{(C_0)_*[\frac{1}{\varpi}]} = (C_0)_* \ ,$$

as wanted. □

2.4.2 Proof in special cases

Proposition 2.4.3. *Let R be a perfectoid ring of characteristic p. Consider the basic setup $(R^\circ, R^{\circ\circ})$. Let S be a finite étale R-algebra. Equip S with the canonical topology as in Lemma 1.4.6. Then S is a perfectoid ring, and $S^{\circ a}$ is finite étale over $R^{\circ a}$.*

Proof. By Corollary 1.8.10 and [21, Theorem 3.5.13], S is a perfectoid ring. Next we show that S° is uniformly almost finite generated and almost projective with respect to $(R^\circ, R^{\circ\circ})$. For this, it suffices to prove that for any $\varepsilon \in R^{\circ\circ}$ there exist R°-homomorphisms $u_\varepsilon : S^\circ \to (R^\circ)^n$ and $v_\varepsilon : (R^\circ)^n \to S^\circ$ of which the composition is equal to multiplication by ε. Since S is unramified over R, there exists $e \in S \otimes S$ and $e^2 = e$, $\mu(e) = 1$ and $e(\mathrm{Ker}(\mu)) = (0)$. Now there exists $N > 0$ such that $t^N e$ is contained in the image of $S^\circ \otimes_{R^\circ} S^\circ$ in $S \otimes_R S$, and thus so is $t^{N/t^m} e$ for every $m > 0$ because $e^2 = e$. Let $\varepsilon e = \sum_i^n a_{\varepsilon,i} \otimes b_{\varepsilon,i}$. Consider two homomorphisms

$$S^\circ \to (R^\circ)^n, \ s \mapsto (\mathrm{Tr}_{R/S}(sb_{\varepsilon,1}), \ldots, \mathrm{Tr}_{S/R}(sb_{\varepsilon,n}))$$

and

$$(R^\circ)^n \to S^\circ, \ (r_1, \ldots, r_n) \mapsto \sum_{i=1}^n a_{\varepsilon,i} r_i$$

Hence it suffices to show that $\sum_{i=1}^n a_{\varepsilon,i} \mathrm{Tr}_{S/R}(sb_{\varepsilon,i}) = \varepsilon s$ for every $s \in S$. We know that $\omega((\mathbf{Id}_S \otimes \tau)(e)) = \mathbf{Id}_S$ by Proposition 1.8.2. Hence $\omega((\mathbf{Id}_S \otimes \tau)(\varepsilon e)) = \varepsilon \mathbf{Id}_S$, as wanted. □

Corollary 2.4.4. *Let R be a perfectoid ring of characteristic $p > 0$. Then the functor*

$$\Phi : (R^{\circ a})_{\text{fét}} \to R_{\text{fét}}, \ S_0 \mapsto S'$$

established in Lemma 2.4.2 gives an equivalence of categories.

Proof. By Proposition 2.4.3, we have a functor

$$\Psi : R_{\text{fét}} \to (R^{\circ a})_{\text{fét}}, \ S \mapsto S^{\circ a} \ .$$

Let us show that Ψ gives a quasi-inverse of Φ. First we show that $(\Phi \circ \Psi)(S) \cong S$. Put $S' := (\Phi \circ \Psi)(S)$. Then $S' = \Phi(S^{\circ a}) = (S^{\circ a})_*[\frac{1}{\varpi}]$ as a ring and $S'^\circ \cong (S^{\circ a})_*$ by Lemma 2.4.2. Moreover, by Lemma 2.1.13, $S^\circ \to (S^{\circ a})_*$ is an isomorphism. Hence we have a functorial isomorphism $S \xrightarrow{\cong} S'$ of topological rings. Next we construct a functorial isomorphism $(\Psi \circ \Phi)(S_0) \xrightarrow{\cong} S_0$. By Lemma 2.4.2 again, we have $(\Psi \circ \Phi)(S_0) = (S_0)^a_*$. Hence the canonical isomorphism $S_0 \xrightarrow{\cong} (S_0)^a_*$ is the desired one. $\square$

Corollary 2.4.5. *Let the notation and hypotheses be as in Proposition 2.4.3. Then $(R^{\circ\sharp})^a \to (S^{\circ\sharp})^a$ is finite étale. In particular, $R^\sharp \to S^\sharp$ is finite étale.*

Proof. There exists a unique (up to canonical isomorphisms) finite étale $(R^{\circ\sharp})^a$-algebra C_0 such that $((C_0)_*/(\varpi))^a \cong (S^{\circ\sharp}/(\varpi))^a$. Since the functor $\cdot^a$ admits a left adjoint functor by [21, Proposition 2.2.14(i)], we have $((C_0)^\flat_*)^a \cong ((S^\circ)^{\sharp\flat})^a$. Hence $(C_0)^\flat_*[\frac{1}{t}] \cong S$ and so $(C_0)_* \cong S^{\circ\sharp}$. Hence $(R^{\circ\sharp})^a \to (S^{\circ\sharp})^a$ is finite étale. $\square$

By [21, Theorem 5.3.27], in the situation of Lemma 2.4.2, the association $C \mapsto C \otimes_{A^{\circ a}} (A^\circ/(\varpi))^a$ induces an equivalence of categories $(A^{\circ a})_{\text{fét}} \cong ((A^\circ/(\varpi))^a)_{\text{fét}}$. Moreover the following statement holds.

Corollary 2.4.6. *Let the notation and hypotheses be as in Proposition 2.4.3. Let $t \in R$ be a pseudo-uniformizer. Let $A := R^\sharp$ and $\varpi := t^\sharp \in A$. Then the essential image of the functor*

$$R_{\text{fét}} \cong (R^{\circ a})_{\text{fét}} \cong ((R^\circ/(t))^a)_{\text{fét}} \cong ((A^\circ/(\varpi))^a)_{\text{fét}} \cong (A^{\circ a})_{\text{fét}} \to A_{\text{fét}}$$

consists of all finite étale A-algebras B such that B equipped with the canonical topology is a perfectoid ring and $B^{\circ a}$ is finite étale over $A^{\circ a}$.

Proof. Let $A_{\text{sfét}}$ be the full subcategory of $A_{\text{fét}}$ consisting of finite étale A-algebras B such that B equipped with the canonical topology is a perfectoid ring and $B^{\circ a}$ is finite étale over $A^{\circ a}$. We obtain the functor $(A^{\circ a})_{\text{fét}} \to A_{\text{sfét}}$. By the argument in the proof of Corollary 2.4.4, we find that it is an equivalence of categories. Hence the assertion follows. $\square$

In the case when A is a perfectoid field, we can apply Scholze's result directly.

Theorem 2.4.7 ([49, Theorem 3.7], [35, Theorem 3.5.3])**.** *Let K be a perfectoid field. Then we have an equivalence of categories $K_{\text{fét}} \cong K^\flat_{\text{fét}}$.*

Chapter 3

Perfectoid spaces

In this chapter, as a globalization of the results in Chapter 2, we study perfectoid spaces without a fixed space. In §3.1, we show that for any perfectoid affinoid ring (A, A^+) the structure presheaves of $\mathrm{Spa}(A, A^+)$ are sheaves, and then define *perfectoid spaces* (without a fixed base). We establish the tilting equivalence for perfectoid spaces in §3.2 (Theorem 3.2.5), and then prove an equivalence of étale sites under tilting in §3.3 (Theorem 3.3.11). In §3.4, as an example that this theory can be applied to, we then construct a Zariski-Riemann triple from a complete regular local ring, and prove that it is a perfectoid space (Theorem 3.4.1).

3.1 Acyclicity

In this section, we observe that the following statement holds.

Proposition 3.1.1 (cf. [49, Proposition 6.14])**.** *Let (A, A^+) be a perfectoid affinoid ring, and $X = \mathrm{Spa}(A, A^+)$. Then for any rational open covering $X = \bigcup_{i\in I} U_i$ with I finite, the sequence*

$$0 \to \mathcal{O}_X(X)^\circ \xrightarrow{d_0^\circ} \prod_i \mathcal{O}_X(U_i)^\circ \xrightarrow{d_1^\circ} \prod_{i,j} \mathcal{O}_X(U_i \cap U_j)^\circ \to \cdots \tag{3.1}$$

is $A^{\circ\circ}$-almost exact. In particular, $\mathcal{O}_X$ is a sheaf, and $H^i(X, \mathcal{O}_X^+)$ is $A^{\circ\circ}$-almost zero for every $i > 0$.

Throughout this section, we denote by F the perfectoid field appeared in Example 2.1.3 2. Recall that any perfectoid affinoid ring (R, R^+) of characteristic p can be regarded as a perfectoid affinoid algebra over (F, F°) (cf. Remark 2.1.5).

Scholze proved [49, Proposition 6.14] by introducing the following notion and reducing the statement to classical results on affinoid F-algebras.

Definition 3.1.2 (p-finite affinoid rings)**.** Let (R, R^+) be an affinoid ring where R is an $\mathbb{F}_p$-algebra.

1. The *direct perfection* of (R, R^+) is the direct limit of the filtered system of affinoid rings $\{(R_n, R_n^+)\}_{n\geq 0}$ defined as follows: for every $n \geq 0$, $(R_n, R_n^+) = (R, R^+)$ and the transition map $(R_n, R_n^+) \to (R_{n+1}, R_{n+1}^+)$ is induced by the Frobenius endomorphism on R.

2. (R, R^+) is said to be *p-finite*, if (R, R^+) is the Hausdorff completion of the direct perfection of a reduced affinoid F-algebra (S, S^+) of topologically of finite type.

The following lemma is essential for the reduction.

Lemma 3.1.3 ([49, Lemma 6.13]). *Let (R, R^+) be a perfectoid affinoid ring of characteristic p. Then there exists a direct system of affinoid rings $\{(R_\lambda, R_\lambda^+)\}_{\lambda \in \Lambda}$ of characteristic p such that each (R_λ, R_λ^+) is a p-finite perfectoid ring and (R, R^+) is the Hausdorff completion of the direct limit $\varinjlim_\lambda (R_\lambda, R_\lambda^+)$.*

By [49, Proposition 6.14] and Remark 2.1.5, we can prove Proposition 3.1.1.

Proof of Proposition 3.1.1. Set $(R, R^+) = (A^\flat, (A^+)^\flat)$, $Y = \mathrm{Spa}(R, R^+)$, and $V_i = U_i^\flat$ for $i \in I$ (cf. Proposition 2.3.12). Then by [49, Proposition 6.14] and Remark 2.1.5, the sequence

$$0 \to \mathcal{O}_Y(Y)^\circ \xrightarrow{d_0^\circ} \prod_i \mathcal{O}_Y(V_i)^\circ \xrightarrow{d_1^\circ} \prod_{i,j} \mathcal{O}_Y(V_i \cap V_j)^\circ \to \cdots$$

is almost exact with respect to the basic setup $(F^\circ, (T)^{1/p^\infty})$. Hence by flatness and the isomorphisms $\mathcal{O}_X(W)^\circ/(\varpi^p) \cong \mathcal{O}_Y(W^\flat)^\circ/(t^p)$ (where $W \subset X$ is a rational subset), the sequence

$$0 \to \mathcal{O}_X(X)^\circ/(\varpi^p) \to \prod_i \mathcal{O}_X(U_i)^\circ/(\varpi^p) \to \prod_{i,j} \mathcal{O}_X(U_i \cap U_j)^\circ/(\varpi^p) \to \cdots$$

is $A^{\circ\circ}$-almost exact. Thus, since $\mathcal{O}_X(W)^\circ$ is ϖ-adically Hausdorff complete and ϖ-torsion free, (3.1) is also $A^{\circ\circ}$-almost exact. □

In particular, X is an adic space. Now we can define *perfectoid spaces*.

Definition 3.1.4. An *affinoid perfectoid space* is an adic space which is isomorphic to the adic space associated with a perfectoid affinoid ring. A *perfectoid space* is an adic space that is locally an affinoid perfectoid space. A morphism of perfectoid spaces is a morphism of adic spaces.

Lemma 3.1.5. *Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of perfectoid spaces. Assume that f is an open immersion. Then the morphism $X \times_Y Z \to Z$ is also an open immersion.*

Proof. We may assume that $Y = \mathrm{Spa}(A, A^+)$ and $X = \mathrm{Spa}(A_U, A_U^+)$ where the map $X \to Y$ is induced by the rational localization $(A, A^+) \to (A_U, A_U^+)$ corresponding a rational subset $U = Y\langle \frac{f_1, \ldots, f_r}{g} \rangle \subset Y$. Moreover we may assume that $Z = \mathrm{Spa}(B, B^+)$.

Let C be the completion of $A_U \otimes_A B$, and C^+ the completion (the topological closure) of the integral closure of $A_U^+ \otimes_{A^+} B^+$ in C. Then $X \times_Y Z$ is given as $\mathrm{Spa}(C, C^+)$. Moreover,

$$A_U^+ \otimes_{A^+} B^+ = (A^+\widehat{[\tfrac{f_1}{g}, \ldots, \tfrac{f_r}{g}]})_{A_g}^+ \otimes_{A^+} B^+ \ .$$

We observe that C^+ is equal to $((B^+[\frac{f_1}{g}, \ldots, \frac{f_r}{g}])_{B_g}^+)^\wedge$.

$$(B^+[\tfrac{f_1}{g}, \ldots, \tfrac{f_r}{g}])_{B_g}^+ \hookrightarrow A_g \otimes_A B$$

induces

$$((B^+[\tfrac{f_1}{g}, \ldots, \tfrac{f_r}{g}])_{B_g}^+)^\wedge \hookrightarrow (A_g \otimes_A B)^\wedge = (\widehat{A_g} \otimes_A B)^\wedge = D \ .$$

Moreover, the image of $A_U^+ \otimes_{A^+} B^+ \to C$ forms a dense subring of $((B^+[\frac{f_1}{g}, \ldots, \frac{f_r}{g}])_{B_g}^+)^\wedge$. Hence after taking completion, they agree (i.e. $C^+ = ((B^+[\frac{f_1}{g}, \ldots, \frac{f_r}{g}])_{B_g}^+)^\wedge$). □

3.2 Tilting equivalence of perfectoid spaces

3.2.1 Distinguished ideal sheaves

Here we introduce the notion of *distinguished ideal sheaves*. We use the following notation: for a topological space X and a sheaf of $\mathbb{F}_p$-algebra $\mathcal{F}$ on X, we denote by $\mathcal{WF}$ the sheaf of rings defined by $\mathcal{WF}(U) = W(\mathcal{F}(U))$.

Definition 3.2.1. Let (R, R^+) be a perfectoid affinoid ring of characteristic p. Let $I = (z) \subset W(R^+)$ be a distinguished ideal. Put $X := \mathrm{Spa}(R, R^+)$. Then we define an ideal sheaf $I^\sim$ of $\mathcal{WO}_X^+$ as the image of the morphism $\varphi_z : \mathcal{WO}_X^+ \to \mathcal{WO}_X^+$ given by

$$\varphi_z(U) : \mathcal{WO}_X^+(U) \to \mathcal{WO}_X^+(U),\ f \mapsto z|_U f\ .$$

Notice that for any rational subset $V \subset X$, $\mathcal{O}_X^+(V)$ is a z-torsion free R^+-algebra by Proposition 1.6.8 and Lemma 2.1.19. Hence $\mathcal{WO}_X^+(U) = zW(\mathcal{O}_X^+(U))$ for every open subset $U \subset X$.

Let us define distinguished ideal sheaves.

Definition 3.2.2 (Distinguished ideal sheaves)**.** Let X be a perfectoid space such that $\Gamma(X, \mathcal{O}_X)$ is an $\mathbb{F}_p$-algebra. We say that an ideal sheaf $\mathcal{I}$ of $\mathcal{WO}_X^+$ is *distinguished*, if there exists an affinoid perfectoid open covering $X = \bigcup_{i \in I} U_i$ such that each $(f_i, \psi_i) : U_i \xrightarrow{\cong} \mathrm{Spa}(R_i, R_i^+)$ induces an isomorphism $f_i^{-1}(I_i^\sim) \cong \mathcal{I}|_{U_i}$ for some distinguished ideal $I_i \subset W(R_i^+)$.

The following lemma is easy to check.

Lemma 3.2.3. *Let $\varphi : (R, R^+) \to (S, S^+)$ be a morphism of perfectoid affinoid rings of characteristic p. Let $I = (z) \subset W(R^+)$ be a distinguished ideal. Let $X = \mathrm{Spa}(R, R^+)$ and $Y = \mathrm{Spa}(S, S^+)$, and consider the induced morphism of ringed spaces $(Y, \mathcal{WO}_Y^+) \to (X, \mathcal{WO}_X^+)$. Then we have $I^\sim \mathcal{WO}_Y^+ = IW(S^+)^\sim$.*

Thus the following statement holds.

Lemma 3.2.4. *Let $(f, \varphi^+, \varphi) : X \to Y$ be a morphism of perfectoid spaces of characteristic p. Let $\mathcal{I}$ be a distinguished ideal sheaf on Y. Then the ideal sheaf $\mathcal{IWO}_X^+$ of $\mathcal{WO}_X^+$ is also distinguished.*

Proof. Take an affinoid perfectoid open covering $Y = \bigcup_{j \in J} U_j$ such that for each $j \in J$ there exists $((R_j, R_j^+), I_j) \in \mathrm{ob}(\mathscr{R})$ and $(g_j, \psi_j) : U_j \xrightarrow{\cong} \mathrm{Spa}(R_j, R_j^+)$ that induces $g_j^{-1}(I_j^\sim) \cong \mathcal{I}|_{U_j}$. Take an affinoid perfectoid open covering $f^{-1}(U_j) = \bigcup_{k \in K_j} V_{jk}$. Let $f_{jk} : V_{jk} \to U_j$ be a continuous map such that the diagram

$$\begin{array}{ccc} V_{jk} & \overset{\hookrightarrow}{\longrightarrow} & X \\ {\scriptstyle f_{jk}}\downarrow & & \downarrow{\scriptstyle f} \\ U_j & \underset{\hookrightarrow}{\longrightarrow} & Y \end{array}$$

commutes. Then φ^+ induces the morphism of sheaves $\varphi_{jk}^+ : f_{jk}^{-1}\mathcal{O}_{U_j}^+ \to \mathcal{O}_{V_{jk}}^+$ and hence the morphism of ringed spaces $(f_{jk}, \mathcal{W}(\varphi_{jk}^+)) : (V_{jk}, \mathcal{WO}_{V_{jk}}^+) \to (U_j, \mathcal{WO}_{U_j}^+)$. Then by Lemma 3.2.3, the ideal sheaf $\mathcal{I}|_{U_j}\mathcal{WO}_{V_{jk}}^+$ is distinguished. Thus, since $(\mathcal{IWO}_X^+)|_{V_{jk}} \cong \mathcal{I}|_{U_j}\mathcal{WO}_{V_{jk}}^+$, the assertion follows. $\square$

3.2.2 Category of pairs of a perfectoid space and a distinguished ideal sheaf

Let $\mathcal{C}$ be the category of perfectoid spaces X such that $p \in \Gamma(X, \mathcal{O}_X)$ is topologically nilpotent. Let $\mathcal{D}$ be the category of pairs $(X, \mathcal{I})$ consisting of a perfectoid space X of characteristic p and a distinguished ideal sheaf $\mathcal{I}$ on X. Morphisms in $\mathcal{D}$ are given as follows. Let $(f, \varphi, \varphi_+) : (X, \mathcal{O}_X, \mathcal{O}_X^+) \to (Y, \mathcal{O}_Y, \mathcal{O}_Y^+)$ be a morphism of perfectoid spaces. Then we obtain the morphism of sheaves $\mathcal{W}(\varphi_+)' : f^{-1}\mathcal{W}\mathcal{O}_Y^+ \to \mathcal{W}\mathcal{O}_X^+$. We define a morphism $(X, \mathcal{I}) \to (Y, \mathcal{J})$ in $\mathcal{D}$ as a morphism of perfectoid spaces $(f, \varphi, \varphi^+) : X \to Y$ such that $\mathcal{I} = \mathcal{J}\mathcal{W}\mathcal{O}_X^+$.

We say that a morphism in $\mathcal{D}$ is an *open immersion* if the underlying morphism of perfectoid spaces is an open immersion. Notice that for any morphisms $(X, \mathcal{I}) \to (Z, \mathcal{K}) \leftarrow (Y, \mathcal{J})$ in $\mathcal{D}$ the pair $(X \times_Z Y, \mathcal{K}\mathcal{W}\mathcal{O}^+_{X \times_Z Y}) \in \mathrm{ob}(\mathcal{D})$ gives the fiber product. Moreover, any base extension of an open immersion is again an open immersion (Lemma 3.1.5).

We will establish the following result and in particular an equivalence of categories $\mathcal{C} \xrightarrow{\cong} \mathcal{D}$.

Theorem 3.2.5. *The following assertions hold.*

1. *For any $X \in \mathrm{ob}(\mathcal{C})$, there exists unique (up to canonical isomorphisms) $(X^\flat, \mathcal{I}_X) \in \mathrm{ob}(\mathcal{D})$ with a functorial bijection*

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{Spa}(A, A^+), X) \cong \mathrm{Hom}_{\mathcal{D}}((\mathrm{Spa}(A^\flat, (A^+)^\flat), \mathrm{Ker}(\theta_{A^+})^\sim), (X^\flat, \mathcal{I}_X))$$

   for any $(A, A^+) \in \mathrm{ob}(\mathscr{A})$.

2. *The association $X \mapsto (X^\flat, \mathcal{I}_X)$ induces a functor $\cdot^\flat : \mathcal{C} \to \mathcal{D}$ with the following properties:*
   - *$\cdot^\flat$ is an equivalence of categories;*
   - *an open immersion corresponds to an open immersion;*
   - *for a morhism $\varphi : (B, B^+) \to (C, C^+)$ of perfectoid affinoid algebras over (A, A^+), one has $(\mathrm{Spa}(\varphi))^\flat = \mathrm{Spa}(\varphi^\flat)$.*

We call $X^\flat$ the tilt of X.

Proof. First let us observe that the assertion 1 in Theorem 3.2.5 implies that the association $X \mapsto (X^\flat, \mathcal{I}_X)$ defines a fully faithful functor $\mathcal{C} \to \mathcal{D}$. Let $\mathcal{C}_0$ and $\mathcal{D}_0$ be the full subcategories of $\mathcal{C}$ and $\mathcal{D}$ with

$$\mathrm{ob}(\mathcal{C}_0) = \{X \in \mathrm{ob}(\mathcal{C}) \mid X \cong \mathrm{Spa}(A, A^+) \text{ for some } (A, A^+) \in \mathrm{ob}(\mathscr{A})\}$$

and

$$\mathrm{ob}(\mathcal{D}_0) = \{(X, \mathcal{I}) \in \mathrm{ob}(\mathcal{D}) \mid (X, \mathcal{I}) \cong (\mathrm{Spa}(R, R^+), I^\sim) \text{ for some } ((R, R^+), I) \in \mathrm{ob}(\mathscr{R})\} .$$

To $X \in \mathrm{ob}(\mathcal{C})$, we assign the functor

$$F'_X : \mathcal{C}_0 \to \mathbf{Sets},\ \mathrm{Spa}(A, A^+) \mapsto \mathrm{Hom}_{\mathcal{C}}(\mathrm{Spa}(A, A^+), X) .$$

Similarly, to $(X, \mathcal{I}) \in \mathrm{ob}(\mathcal{D})$, we assign the functor

$$G'_{(X,\mathcal{I})} : \mathcal{D}_0 \to \mathbf{Sets},\ (\mathrm{Spa}(R, R^+), I^\sim) \mapsto \mathrm{Hom}_{\mathcal{D}}((\mathrm{Spa}(R, R^+), I^\sim), (X, \mathcal{I})) .$$

Claim 3.2.6. *The functors*

$$\mathcal{C} \to Funct(\mathcal{C}_0, \mathbf{Sets}),\ X \mapsto F'_X$$

and

$$\mathcal{D} \to Funct(\mathcal{D}_0, \mathbf{Sets}),\ (X, \mathcal{I}) \mapsto G'_{(X,\mathcal{I})}$$

are fully faithful.

Proof of Claim 3.2.6. We give a proof for the second functor (the assertion for the first functor can be proved similarly). For $(X, \mathcal{I}) \in \mathrm{ob}(\mathcal{D})$, consider the functor

$$G_X : \mathcal{D} \to \mathbf{Sets},\ (Y, \mathcal{J}) \mapsto \mathrm{Hom}_{\mathcal{D}}((Y, \mathcal{J}) \to (X, \mathcal{I}))\ .$$

Then by the Yoneda lemma, the functor

$$\mathcal{D} \to Funct(\mathcal{D}, \mathbf{Sets}),\ (X, \mathcal{I}) \to G_{(X,\mathcal{I})}$$

is fully faithful. Now by the definition of distinguished ideal sheaves, every $(Y, \mathcal{J}) \in \mathrm{ob}(\mathcal{D})$ admits a family of open immersions $\{(U_i, \mathcal{I}_i) \to (Y, \mathcal{J})\}_{i \in I}$ with $U_i \in \mathrm{ob}(\mathcal{C}_0)$ $(i \in I)$. Hence $\iota : \mathcal{D}_0 \hookrightarrow \mathcal{D}$ induces an equivalence of categories of sheaves $\mathcal{D}^\sim \to \mathcal{D}_0^\sim$. In particular, the composite of functors

$$\mathcal{D} \to \mathcal{D}^\sim \to \mathcal{D}_0^\sim,\ (X, \mathcal{I}) \mapsto G_{(X,\mathcal{I})} \mapsto G'_{(X,\mathcal{I})}$$

is fully faithful. □

Thus we obtain functorial bijections

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathrm{Hom}_{Funct}(F'_X, F'_Y) \cong \mathrm{Hom}_{Funct}(G'_{(X^\flat, \mathcal{I}_X)}, G'_{(Y^\flat, \mathcal{I}_Y)}) \cong \mathrm{Hom}_{\mathcal{D}}((X^\flat, \mathcal{I}_X), (Y^\flat, \mathcal{I}_Y))\ .$$

for $(X, Y) \in \mathrm{ob}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C})$. Therefore the resulting bijection $\cdot^\flat : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}((X^\flat, \mathcal{I}_X), (Y^\flat, \mathcal{I}_Y))$ gives the fully faithful functor

$$\cdot^\flat : \mathcal{C} \to \mathcal{D},\ X \mapsto (X^\flat, \mathcal{I}_X)\ .$$

Let $\mathcal{C}'$ and $\mathcal{D}'$ be the full subcategories of $\mathcal{C}$ and $\mathcal{D}$ with

$$\mathrm{ob}(\mathcal{C}') = \{X \in \mathrm{ob}(\mathcal{C}) \mid \text{there exists an open immersion } X \hookrightarrow X' \text{ for some } X' \in \mathrm{ob}(\mathcal{C}_0)\}$$

and

$$\mathrm{ob}(\mathcal{D}') = \{(Y, \mathcal{I}) \in \mathrm{ob}(\mathcal{D}) \mid \text{there exists an open immersion } (Y, \mathcal{I}) \hookrightarrow (Y', \mathcal{I}') \text{ for some } (Y', \mathcal{I}') \in \mathrm{ob}(\mathcal{D}_0)\}\ ,$$

respectively. To prove the theorem, the following claim is essential.

Claim 3.2.7. *For any $X \in \mathrm{ob}(\mathcal{C}')$, the pair $(X^\flat, \mathcal{I}_X) \in \mathrm{ob}(\mathcal{D})$ exists and belongs to $\mathrm{ob}(\mathcal{D}')$. Moreover, the induced fully faithful functor $\mathcal{C}' \to \mathcal{D}'$ has the properties stated in Theorem 3.2.5 2.*

Proof of Claim 3.2.7. We may assume that X is an open subspace U of $\mathrm{Spa}(B, B^+)$ for some $(B, B^+) \in \mathrm{ob}(\mathscr{A})$. Let us show that $(U^\flat, I^\sim_{B^+}|_{U^\flat}) = (X^\flat, \mathcal{I}_X)$ (where $U^\flat$ is the image of U under the homeomorphism $\cdot^\flat : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(B^\flat, (B^+)^\flat))$. Put $I_{A^+} := \mathrm{Ker}(\theta_{A^+})$ and $I_{B^+} := \mathrm{Ker}(\theta_{B^+})$. Then we have the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(\mathrm{Spa}(A, A^+), U) & \overset{\iota_1}{\hookrightarrow} & \mathrm{Hom}_{\mathcal{C}}(\mathrm{Spa}(A, A^+), \mathrm{Spa}(B, B^+)) \\
 & & \downarrow \gamma \\
\mathrm{Hom}_{\mathcal{D}}((\mathrm{Spa}(A^\flat, (A^+)^\flat), I_{\widetilde{A^+}}), (U^\flat, I_{\widetilde{B^+}}|_{U^\flat})) & \overset{\iota_2}{\hookrightarrow} & \mathrm{Hom}_{\mathcal{D}}((\mathrm{Spa}(A^\flat, (A^+)^\flat), I_{\widetilde{A}}), (\mathrm{Spa}(B^\flat, (B^+)^\flat), I_{\widetilde{B^+}}))
\end{array} .$$

Here γ is bijective by Theorem 2.3.1. Thus it suffices to show that $\gamma(\mathrm{Im}(\iota_1)) = \mathrm{Im}(\iota_2)$.

First we show that $\gamma(\mathrm{Im}(\iota_1)) \subset \mathrm{Im}(\iota_2)$. Pick $\varphi \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{Spa}(A, A^+), U)$. Let $f : \mathrm{Spa}(A, A^+) \to \mathrm{Spa}(B, B^+)$ be the composite

$$\mathrm{Spa}(A, A^+) \xrightarrow{\varphi} U \overset{\iota}{\hookrightarrow} \mathrm{Spa}(B, B^+) .$$

Take a family of rational subsets $\{W_i\}_{i \in I}$ of $\mathrm{Spa}(B, B^+)$ such that $U = \bigcup_{i \in I} W_i$. Then $V_i := f^{-1}(W_i)$ is isomorphic to $\mathrm{Spa}(A_i, A_i^+)$ for some perfectoid affinoid ring (A_i, A_i^+). Thus we obtain the commutative diagram

$$\begin{array}{ccc}
\mathrm{Spa}(A, A^+) & \xrightarrow{f} & \mathrm{Spa}(B, B^+) \\
I_i \uparrow & & \uparrow J_i \\
\mathrm{Spa}(A_i, A_i^+) & \xrightarrow[f_i]{} & \mathrm{Spa}(B_{W_i}, B_{W_i}^+)
\end{array} ,$$

which induces the commutative diagram

$$\begin{array}{ccc}
(\mathrm{Spa}(A^\flat, (A^+)^\flat), I_{\widetilde{A^+}}) & \xrightarrow{f^\flat} & (\mathrm{Spa}(B^\flat, (B^+)^\flat), I_{\widetilde{B^+}}) \\
I_i^\flat \uparrow & & \uparrow J_i^\flat \\
(\mathrm{Spa}(A_i^\flat, (A_i^+)^\flat), I_{\widetilde{A_i^+}}) & \xrightarrow[f_i^\flat]{} & (\mathrm{Spa}(B_{W_i}^\flat, (B_{W_i}^+)^\flat), I_{\widetilde{B_{W_i}^+}})
\end{array} .$$

Then, $I_i^\flat$ and $J_i^\flat$ are open immersions, $I_i^\flat$ factors through $(V_i^\flat, I_{\widetilde{A^+}}|_{V_i^\flat})$, and $J_i^\flat$ factors through $(W_i^\flat, I_{\widetilde{B^+}}|_{W_i^\flat})$. Hence the composite

$$(V_i^\flat, I_{\widetilde{A}}|_{V_i^\flat}) \hookrightarrow (\mathrm{Spa}(A^\flat, (A^+)^\flat), I_{\widetilde{A^+}}) \xrightarrow{f^\flat} (\mathrm{Spa}(B^\flat, (B^+)^\flat), I_{\widetilde{B^+}})$$

factors through $(W_i^\flat, I_{\widetilde{B}}|_{W_i^\flat})$. Thus, since $\mathrm{Spa}(A^\flat, (A^+)^\flat) = \bigcup_i (f^{-1}(W_i))^\flat$, $f^\flat$ factors through $(U^\flat, I_{\widetilde{B^+}}|_{U^\flat})$. Hence we have $\gamma(\mathrm{Im}(\iota_1)) \subset \mathrm{Im}(\iota_2)$.

Similarly, one can show that $\mathrm{Im}(\iota_2) \subset \gamma(\mathrm{Im}(\iota_1))$. Indeed, pick $\psi \in \mathrm{Hom}(\mathrm{Spa}(R, R^+), U^\flat)$ and let $g : \mathrm{Spa}(R, R^+) \to X^\flat$ be the composite

$$\mathrm{Spa}(R, R^+) \xrightarrow{\psi} U^\flat \hookrightarrow X^\flat .$$

Then we can find that $g^\sharp$ factors through $(U^\flat)^\sharp$ $(= U)$ in a similar way. Hence the assertion follows.

Next we show that the induced fully faithful functor $\mathcal{C}' \to \mathcal{D}'$ is essential surjective. Pick $(Y, \mathcal{I}) \in \mathrm{ob}(\mathcal{D}')$. We may assume the following: there exists some $((S, S^+), J) \in \mathrm{ob}(\mathscr{R})$ such that Y is an open subspace V of $\mathrm{Spa}(S, S^+)$ and $\mathcal{I} = J^{\sim}|_V$. Set $(C, C^+) = ((S, S^+), J)^{\sharp}$. Let X' be the preimage of V under the homeomorphism $\cdot^{\flat} : \mathrm{Spa}(C, C^+) \to \mathrm{Spa}(C^{\flat}, (C^+)^{\flat})$ $(= \mathrm{Spa}(S, S^+))$ equipped with the structure as a perfectoid space. Then by the above argument, we find that $(V, J^{\sim}|_V) = ((X')^{\flat}, \mathcal{I}_{X'})$. Hence the assertion follows.

The remaining properties we have to check follow immediately from the construction. □

Now let us complete the proof of Theorem 3.2.5. Consider an affinoid perfectoid open covering $X = \bigcup_{i \in I} U_i$. Then by Clam 3.2.7, the tilts $U_i^{\flat}$ $(i \in I)$ exist, and we can glue them to obtain the pair $(Y, \mathcal{I}) \in \mathrm{ob}(\mathcal{D})$ together with the open covering $Y = \bigcup_{i \in I} V_i$ such that $(V_i, \mathcal{I}|_{V_i}) \cong (U^{\flat}, \mathcal{I}_{U_i})$ for every $i \in I$. In this situation, $(Y, \mathcal{I}) = (X^{\flat}, \mathcal{I}_X)$. Indeed, pick $f \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{Spa}(A, A^+), X)$ and set $f_i = f|_{f^{-1}(U_i)}$ $(i \in I)$. Then by the construction of $(Y, \mathcal{I})$, we can glue the morphisms $f_i^{\flat}$ to obtain the morphism $f' : (\mathrm{Spa}(A^{\flat}, (A^+)^{\flat}), I^{\sim}_{A^+}) \to (Y, \mathcal{I})$. By Claim 3.2.7, we find that this assigning gives a functorial bijection that we need. Now the assertion 2 is clear. □

3.3 Finite étale cover

In this section, we give the proof of the almost purity theorem. First we recall Scholze's definition of étale morphism of adic spaces.

Definition 3.3.1 (Étale morphisms, [49, Definition 7.1])**.**

1. We say that a continuous affinoid ring homomorphism $(R, R^+) \to (S, S^+)$ is *finite étale*, if $R \to S$ is finite étale and S^+ is integral over R^+.

2. We say that a morphism of adic spaces $X \to Y$ is *finite étale*, if there exists an affinoid open covering $Y = \bigcup_{i \in I} V_i$ such that each preimage $U_i := f^{-1}(V_i)$ is affinoid and the associated morphism of affinoid algebras $(\mathcal{O}_Y(V_i), \mathcal{O}_Y^+(V_i)) \to (\mathcal{O}_X(U_i)), \mathcal{O}_X^+(U_i))$ is finite étale.

3. We say that a morphism of adic spaces $f : X \to Y$ is *étale*, if for any point $x \in X$ there are open neighbourhoods U of x and V of $f(x)$, and an adic space W with a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{j} & W \\ & \searrow^{f|_U} & \downarrow p \\ & & V \end{array}$$

where j is an open immersion and p is finite étale.

Definition 3.3.2 (Strongly étale morphisms, cf. [49, Definition 7.2])**.**

1. We say that a morphism of perfectoid affinoid rings $(R, R^+) \to (S, S^+)$ is *strongly finite étale*, if it is finite étale and $S^{\circ a}$ is finite étale over $R^{\circ a}$.

2. We say that a morphism of perfectoid spaces $X \to Y$ is *strongly finite étale*, if there exists an affinoid perfectoid open covering $Y = \bigcup_{i \in I} V_i$ such that each preimage $U_i := f^{-1}(V_i)$ is affinoid perfectoid and the associated morphism of perfectoid affinoid rings $(\mathcal{O}_Y(V_i), \mathcal{O}_Y^+(V_i)) \to (\mathcal{O}_X(U_i)), \mathcal{O}_X^+(U_i))$ is strongly finite étale.

3. We say that a morphism of perfectoid spaces $f : X \to Y$ is *strongly étale*, if for any point $x \in X$ there are open neighbourhoods U of x and V of $f(x)$, and a perfectoid space W with a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\ j\ } & W \\ & {\scriptstyle f|_U}\searrow & \downarrow {\scriptstyle p} \\ & & V \end{array}$$

where j is an open immerstion and p is strongly finite étale.

Lemma 3.3.3. *Let $(R, R^+) \to (S, S^+)$ be a finite étale morphism of perfectoid affinoid rings of characterisic p.*

1. *$(R, R^+) \to (S, S^+)$ is strongly finite étale.*
2. *The morphism $(R^\sharp, (R^+)^\sharp) \to (S^\sharp, (S^+)^\sharp)$ is also strongly finite étale.*

Proof. The assertion 1 follows from Proposition 2.4.3. Let us show the assertion 2. Put $(A, A^+) := (R^\sharp, (R^+)^\sharp)$ and $(B, B^+) := (S^\sharp, (S^+)^\sharp)$. Then $B^{\circ a}$ is finite étale over $A^{\circ a}$ by Corollary 2.4.5. Thus it suffices to show that $A^+ \to B^+$ is integral. Pick $b \in B^+$. Since $A^+/(\varpi^p) \cong R^+/(t^p)$, $B^+/(\varpi^p) \cong S^+/(t^p)$, and $R^+ \to S^+$ is integral, the map $A^+/(\varpi^p) \to B^+/(\varpi^p)$ is also integral. Hence there exists a monic polynomial $f \in A^+[X]$ such that $f(b) \in \varpi^p B^+$. Thus, since $\varpi^p B^+ \subset \varpi^p B^\circ \subset (A^+)^+_B$, b is integral over A^+. Hence the assertion follows. □

Let us recall a basic technique on Henselian approximation.

Proposition 3.3.4 ([21, Proposition 5.4.53])**.** *Let A_0 be a ring, $t \in A_0$ a non-zerodivisor, and $\widehat{A_0}$ be the t-adic completion. Suppose that A_0 is t-adically Henselian. Then the functor*

$$(A_0[\tfrac{1}{t}])_{\text{fét}} \to (\widehat{A_0}[\tfrac{1}{t}])_{\text{fét}},\ B \mapsto B \otimes_{A_0[\frac{1}{t}]} \widehat{A_0}[\tfrac{1}{t}]$$

is an equivalence of categories.

Corollary 3.3.5 ([49, Lemma 7.5])**.** *Let A_0 be a ring and $t \in A_0$ a non-zerodivisor. Let $\{B_{0,\lambda}\}_{\lambda \in \Lambda}$ be a filtered direct system of t-torsion free A_0-algebras, B_0 the direct limit, and $\widehat{B_0}$ the t-adic completion. Suppose that $B_{0,i}$ is t-adically Hausdorff complete for each $i \in I$. Then we have an equivalence of categories*

$$(\widehat{B_0}[\tfrac{1}{t}])_{\text{fét}} \cong 2 - \varinjlim_i (B_{0,i}[\tfrac{1}{t}])_{\text{fét}}\ .$$

Lemma 3.3.6 (cf. [49, Lemma 7.3])**.** *Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of perfectoid spaces. Assume that f is finite étale. Then the morphism $X \times_Y Z \to Z$ is also finite étale.*

Proof. We may assume that $Y = \mathrm{Spa}(A, A^+)$, $X = \mathrm{Spa}(B, B^+)$, and $Z = \mathrm{Spa}(C, C^+)$ where (A, A^+), (B, B^+), and (C, C^+) are perfectoid. Put $D := B \otimes_A C$, let D^+ be the integral closure of $B^+ \otimes_{A^+} C^+$ in D, and equip D with the canonical topology. As then D is finite étale over C, it is Hausdorff complete (and so perfectoid). Hence the affinoid ring (D, D^+) is perfectoid (notice that the image of $B^+ \otimes_{A^+} C^+$ in D forms a ring of definition). Thus, $X \otimes_Y Z$ is given by $\mathrm{Spa}(D, D^+)$. Moreover, D^+ agrees with the integral closure of C^+ in $B \otimes_A C$ because B^+ is integral over A^+. Hence $X \times_Y Z \to Z$ is finite étale. □

Proposition 3.3.7 (cf. [49, Proposition 7.6])**.** *Let $f : X \to Y$ be a strongly finitely étale morphism of perfectoid spaces. Then for every affinoid perfectoid open set $V \subset Y$, $U := f^{-1}(V)$ is also affinoid perfectoid and*

$$(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$$

is strongly finite étale.

Proof. By Corollary 2.4.6 and Theorem 3.2.5, we may assume that X is of characteristic $p > 0$. Notice that in this case any finite étale morphism is strongly finite étale by Proposition 2.4.3. By Lemma 3.3.6, we may assume further that $Y = V = \mathrm{Spa}(R, R^+)$ where (R, R^+) is a perfectoid affinoid ring. Then by assumption, we have an affinoid perfectoid open covering $Y = \bigcup_{i \in I} V_i$ with I a finite set, such that each preimage $U_i := f^{-1}(V_i)$ is affinoid perfectoid and the associated morphism of perfectoid affinoid rings $(\mathcal{O}_Y(V_i), \mathcal{O}_Y^+(V_i)) \to (\mathcal{O}_X(U_i)), \mathcal{O}_X^+(U_i))$ is strongly finite étale. We may assume that each V_i is a rational subset by Lemma 3.3.6 again. Put $V_{ijk} := V_i \cap V_j \cap V_k$ and $U_{ijk} := U_i \cap U_j \cap U_k$ for every $i, j, k \in I$.

By Lemma 3.1.3, (R, R^+) is written as the Hausdorff completion of the direct limit of p-finite perfectoid affinoid (F, F°)-algebras (R_λ, R_λ^+). Thus we have the homeomorphism $\mathrm{Spa}(R, R^+) \xrightarrow{\cong} \varprojlim_\lambda \mathrm{Spa}(R_\lambda, R_\lambda^+)$. For each $\lambda \in \Lambda$, let $V_\lambda = \mathrm{Spa}(R_\lambda, R_\lambda^+)$ and let $V_{\lambda,ijk}$ be the rational subset of V_λ corresponding to V_{ijk}. Then by Corollary 3.3.5 and Lemma 3.3.6, there exists some $\lambda \in \Lambda$ and morphisms of affinoid adic spaces $U_{\lambda,ijk} \to V_{\lambda,ijk}$ $(i, j, k \in I)$ such that the morphism

$$(\mathcal{O}_{V_\lambda}(V_{\lambda,ijk}), \mathcal{O}_{V_\lambda}^+(V_{\lambda,ijk})) \to (\mathcal{O}_{U_{\lambda,ijk}}(U_{\lambda,ijk}), \mathcal{O}_{U_{\lambda,ijk}}^+(U_{\lambda,ijk}))$$

is finite étale, $U_{ijk} = U_{\lambda,ijk} \times_{V_{\lambda,ijk}} V_{ijk}$, and $U_{\lambda,ijk} = U_{\lambda,i} \times_{V_{\lambda,i}} V_{\lambda,ijk}$ (where $U_{\lambda,i} = U_{\lambda,iii}$ and $V_{\lambda,i} = V_{\lambda,iii}$) for every $i, j, k \in I$. Hence one can glue $\{U_{\lambda,i}\}_{i \in I}$ and then obtain a finite étale morphism $X_\lambda \to V_\lambda$ such that $X = X_\lambda \times_{V_\lambda} V$. Thus by Lemma 3.3.6, the proof is reduced to the case when (R, R^+) is p-finite, i.e., the Hausdorff completion of the direct perfection of a reduced affinoid F-algebra (S, S^+) of topologically of finite type.

Hence by the argument similar to the above, we are reduced to showing the assertion for a finite étale morphism $X^0 \to \mathrm{Spa}(S, S^+)$. Then, since S is strongly Noetherian, one can apply the conclusion of [32, Example 1.6.6(ii)] and complete the proof. □

Lemma 3.3.8 (cf. [49, Corollary 7.8])**.** *Let $f : X \to Y$ and $g : Y \to Z$ be étale morphisms of perfectoid spaces. Then the composite $g \circ f : X \to Z$ is also étale.*

Proof. In this situation, one can shrink U, V, and W appeared in Definition 3.3.1 3 to affinoid pefectoid spaces. Moreover, by Corollary 2.4.5 and Theorem 3.2.5, we may assume that X, Y, and Z are of characteristic p. Thus, by the argument in the proof of Proposition 3.3.7, we are reduced to the assertion for étale morphisms of locally Noetherian adic spaces. Hence one can apply [32, Proposition 1.6.7(ii)] and complete the proof. □

3.3.1 Proof of the almost purity theorem

Let us give a proof for Theorem 2.4.1. By the argument in §2.4, it suffices to show the following.

Theorem 3.3.9. *Let A be a perfectoid ring, and $A \to B$ a finite étale ring homomorphism. Equip B with the canonical topology as in Lemma 1.4.7. Then B is also a perfectoid ring. Moreover, $A^\circ \to B^\circ$ is almost finite étale.*

Proof. Put $X := \mathrm{Spa}(A, A^+)$ and $Y := \mathrm{Spa}(B, B^+)$. Pick $x \in X$, and let $x^\flat \in X^\flat$ be the corresponding point. Then we have

$$2\text{-}\varinjlim_{x \in U} \mathcal{O}_X(U)_{\text{fét}} \cong ((\varinjlim_{x \in U} \mathcal{O}_X^+(U))^\wedge_\varpi[\tfrac{1}{\varpi}])_{\text{fét}} \cong (\widehat{k(x)})_{\text{fét}} \cong (\widehat{k(x^\flat)})_{\text{fét}} \cong ((\varinjlim_{x^\flat \in V} \mathcal{O}_{X^\flat}(V))^\wedge_{\varpi^\flat}[\tfrac{1}{\varpi^\flat}])_{\text{fét}}$$
$$\cong 2\text{-}\varinjlim_{x^\flat \in V} \mathcal{O}_{X^\flat}(V)_{\text{fét}}$$

by Proposition 2.4.7 and Corollary 3.3.5. Hence there exist a rational open neighbourhoods V of $x^\flat$ and a finite étale $\mathcal{O}_{X^\flat}(V)$-algebra S such that

$$\widehat{k(x)} \otimes_{\mathcal{O}_X(V^\sharp)} (\mathcal{O}_X(V^\sharp) \otimes_A B) \cong \widehat{k(x)} \otimes_A B \cong (\widehat{k(x^\flat)} \otimes_{\mathcal{O}_{X^\flat}(V)} S)^\sharp \cong \widehat{k(x)} \otimes_{\mathcal{O}_X(V^\sharp)} S^\sharp$$

where the finite étale $\mathcal{O}_{X^\flat}(V)$-algebra (resp. $\widehat{k(x^\flat)}$-algebra) S (resp. $\widehat{k(x^\flat)} \otimes_{\mathcal{O}_{X^\flat}(V)} S$) is equipped with the canonical topology. Hence there exists some rational open neighbourhood U of x and a ring isomorphism

$$\mathcal{O}_X(U) \otimes_A B \cong \mathcal{O}_X(U) \otimes_{\mathcal{O}_X(V^\sharp)} S^\sharp\ . \tag{3.2}$$

Equip the module-finite $\mathcal{O}_X(U)$-algebra $\mathcal{O}_X(U) \otimes_{\mathcal{O}_X(V^\sharp)} S^\sharp$ with the canonical topology. Then (3.2) gives an isomorphism of topological rings, $\mathcal{O}_X(U) \otimes_{\mathcal{O}_X(V^\sharp)} S^\sharp$ is Hausdorff complete by Corollary 1.8.10(5), and the image of $\mathcal{O}_X^+(U) \otimes_{\mathcal{O}_X^+(V^\sharp)} (S^+)^\sharp$ in $\mathcal{O}_X(U) \otimes_{\mathcal{O}_X(V^\sharp)} S^\sharp$ is a ring of definition by Lemma 1.4.9. Hence we have isomorphisms of topological rings

$$\mathcal{O}_X(U) \otimes_A B \cong \mathcal{O}_X(U) \otimes_{\mathcal{O}_X(V^\sharp)} S^\sharp \cong (\mathcal{O}_{X^\flat}(U^\flat) \otimes_{\mathcal{O}_X(V)} S)^\sharp$$

by Proposition 2.2.2 (in particular, $\mathcal{O}_X(U) \otimes_A B$ is perfectoid), and so

$$(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to (\mathcal{O}_X(U) \otimes_A B, (\mathcal{O}_X^+(U) \otimes_{A^+} B^+)^+_{\mathcal{O}_X(U) \otimes_A B})$$

is strongly finite étale by Corollary 2.4.5. Thus, since $\mathcal{O}_X(U) \otimes_A B$ is Hausdorff complete by Corollary 1.8.10(5), the morphism $Y \to X$ is strongly finite étale. Hence by Proposition 3.3.7, the assertion follows. $\square$

By Lemma 3.1.5, 3.3.6, and 3.3.8, we can define the étale site of a perfectoid space.

Definition 3.3.10. Let X be a perfectoid space. The *étale site of* X is the category $X_{\text{ét}}$ of perfectoid spaces that are étale over X equipped with the Grothendieck topology such that a family $\{f_i : U_i \to U\}_{i \in I}$ of morphisms in $X_{\text{ét}}$ is a covering if $U = \bigcup_{i \in I} f_i(U_i)$.

From the above discussions, we obtain the following.

Theorem 3.3.11. *Let $X \in \mathrm{ob}(\mathcal{C})$. Then the functor $\flat : \mathcal{C}/X \to \mathcal{D}/X^\flat$ gives an equivalence of sites $X_{\text{ét}} \cong X^\flat_{\text{ét}}$.*

Proof. It follows immediately from Theorem 2.4.1 and Theorem 3.2.5. $\square$

3.4 Examples

Let the notations be as in Example 2.1.4. In this section, as an example that the theory developed above can be applied to, we construct a perfectoid space from $\widehat{R_\infty}$ functorially. Let $X_n := \operatorname{Spf} R_n$ and $\mathcal{X}_n := X_n^{\mathrm{rig}}$ for every integer $n \geq 0$. Let $X_\infty := \operatorname{Spf}(\widehat{R_\infty})$ and $\mathcal{X}_\infty := X_\infty^{\mathrm{rig}}$. We prove the following.

Theorem 3.4.1. *The valued triple* $\mathbf{ZR}(\mathcal{X}_\infty)$ *is an adic space. Furthermore, it is a perfectoid space.*

Consider an admissible blow-up $\pi : X'_\infty \to X_\infty$ along $\mathcal{I} := (\mathfrak{m}\widehat{R_\infty})^{\Delta}$. For each $i = 1, \ldots, d$, let U_i be the affine part of X'_∞ where the ideal $\mathcal{I}\mathcal{O}_{U_i}$ is generated by x_i. Put $\mathcal{U}_{\infty,i} := U_i^{\mathrm{rig}}$. Then we have an open covering $\langle \mathcal{X}_\infty \rangle = \bigcup_{i=1}^{d} \langle \mathcal{U}_{\infty,i} \rangle$. Set $Z_{\infty,i} := \operatorname{Spa}(S_{\infty,i}, S^\circ_{\infty,i})$. Since $Z_{\infty,i}$ is a perfectoid space in view of Example 2.1.4, it suffices to construct an isomorphism of valued triples $Z_{\infty,i} \xrightarrow{\cong} \mathbf{ZR}(\mathcal{U}_i)$.

3.4.1 Structure of underlying topological spaces

In this subsection, we construct a homeomorphism $|Z_{\infty,i}| \xrightarrow{\cong} |\langle \mathcal{U}_{\infty,i} \rangle|$ between the underlying topological spaces. Let $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Pick $z_m \in Z_{m,i}$. Then we have a semivalutaion $v_{z_m} : S_{m,i} \to \Gamma_{z_m} \cup \{\infty\}$. Put $\mathfrak{p}_{z_m} := v_{z_m}^{-1}(\{\infty\})$. Then v_{z_m} factors through the valuation $\widetilde{v}_{z_m} : \operatorname{Frac}(S_{m,i}/\mathfrak{p}_{z_m}) \to \Gamma_{z_m} \cup \{\infty\}$. Let V_{z_m} be the valuation ring of $\operatorname{Frac}(S_{m,i}/\mathfrak{p}_{z_m})$ corresponding to $\widetilde{v}_{z_m}$, and $\widehat{V_{z_m}}$ the Hausdorff completion. Then we have a continuous ring homomorphism $R_{m,i} \to \widehat{V_{z_m}}$. Hence we obtain the rigid point $\alpha_{z_m} : \operatorname{Spf} \widehat{V_{z_m}} \to \operatorname{Spf} R_{m,i}$ and hence the rigid point $\langle \alpha_{z_m} \rangle : \operatorname{Spf} \widehat{V_{z_m}} \to \langle \mathcal{U}_{m,i} \rangle$ by Proposition 1.7.16. Let $\mathfrak{m}_{z_m}$ be the maximal ideal of $\widehat{V_{z_m}}$. Then we obtain a map

$$\varphi_{m,i} : |Z_{m,i}| \to |\langle \mathcal{U}_{m,i} \rangle|, \ z_m \mapsto \alpha_{z_m}(\mathfrak{m}_{z_m}) \ .$$

By [20, Theorem A4.7], if $m \in \mathbb{Z}_{\geq 0}$, then $\varphi_{m,i}$ is a homeomorphism. Let us show that $\varphi_{\infty,i}$ is also a homeomorphism.

Take $m, m' \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that $m \leq m'$. Then the map $\widehat{R_{m,i}} \to \widehat{R_{m',i}}$ induces the continuous ring homomorphism $S_{m,i} \to S_{m',i}$. It induces a morphism $\lambda_{m,m'} : Z_{m',i} \to Z_{m,i}$. Pick $z_{m'} \in Z_{m',i}$ and put $z_m := \lambda_{m,m'}(z_{m'})$. Then we obtain $\operatorname{Frac}(S_{m,i}/\mathfrak{p}_{z_m}) \hookrightarrow \operatorname{Frac}(S_{m',i}/\mathfrak{p}_{z_{m'}})$, which induces the local homomorphism $\widehat{V_{z_m}} \to \widehat{V_{z_{m'}}}$. Thus we obtain the commutative diagram of continuous ring homomorphisms

$$\begin{array}{ccc} \widehat{R_{m,i}} & \longrightarrow & \widehat{V_{z_m}} \\ \downarrow & & \downarrow \\ \widehat{R_{m',i}} & \longrightarrow & \widehat{V_{z_{m'}}} \end{array} ,$$

which induces the commutative diagram of morphisms of locally ringed spaces

$$\begin{array}{ccc} \operatorname{Spf} \widehat{V_{z_{m'}}} & \xrightarrow{\langle \alpha_{z_{m'}} \rangle} & \langle \mathcal{U}_{m',i} \rangle \\ \downarrow & & \downarrow {\scriptstyle \mu_{m,m'}} \\ \operatorname{Spf} \widehat{V_{z_m}} & \xrightarrow[\langle \alpha_{z_m} \rangle]{} & \langle \mathcal{U}_{m,i} \rangle \end{array} .$$

Thus, we obtain the commutative diagram of maps

$$\begin{array}{ccc} |Z_{\infty,i}| & \xrightarrow{\varphi_{\infty,i}} & |\langle \mathcal{U}_{\infty,i}\rangle| \\ \lambda \downarrow & & \downarrow \mu \\ \varprojlim_n |Z_{n,i}| & \underset{\varprojlim_n \varphi_{n,i}}{\xrightarrow{\cong}} & \varprojlim_n |\langle \mathcal{U}_{n,i}\rangle| \end{array} .$$

λ is a homeomorphism. Hence the following lemma implies that $\varphi_{\infty,i}$ is a homeomorphism, and induces a correspondence between rational subsets of $Z_{\infty,i}$ and rational subdomains of $\mathcal{U}_{\infty,i}$.

Lemma 3.4.2. *The following assertions hold.*

1. *μ is injective.*

2. *$\varphi_{\infty,i}(Z_{\infty,i}(\frac{f_1,\ldots,f_r}{f_0})) = \langle \mathcal{U}_{\infty,i}(\frac{f_1}{f_0},\ldots,\frac{f_r}{f_0})\rangle$.*

Proof. 1: Let us construction the inverse map μ^{-1}. Pick an element $(x_n)_{n\geq 0} \in \varprojlim_n |\langle \mathcal{U}_{n,i}\rangle|$. Let $\alpha_{x_n} : \mathrm{Spf}(\widehat{V_{x_n}}) \to X_n$ be the rigid point corresponding to x_n for each $n \geq 0$ (cf. Proposition 1.7.15 1 and 1.7.16). Put $V_n := \widehat{V_{x_n}}$. Then there exist morphisms $\mathrm{Spf}(V_{n+1}) \to \mathrm{Spf}(V_n)$ ($n \geq 0$) such that the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathrm{Spf}(V_{n+1}) & \longrightarrow & \mathrm{Spf}(V_n) & \longrightarrow & \cdots \\ & & \downarrow \alpha_{x_{n+1,i}} & & \downarrow \alpha_{x_{n,i}} & & \\ \cdots & \longrightarrow & X_{n+1} & \longrightarrow & X_n & \longrightarrow & \cdots \end{array} .$$

Hence we obtain

$$\widehat{\varinjlim_n R_{n,i}} \to \widehat{\varinjlim_n V_n} \ . \tag{3.3}$$

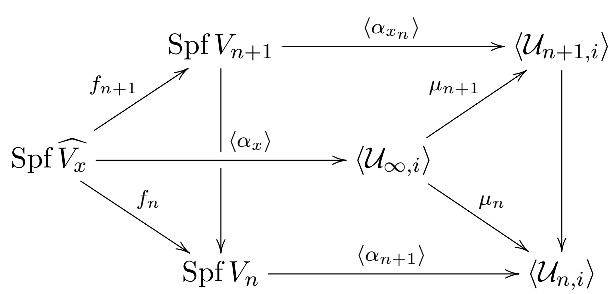
Set $V_\infty := \varinjlim_n V_n$. Then V_∞ is a valuation ring with a maximal ideal

$$\mathfrak{m}_{V_\infty} = \{r \in V_\infty \mid r = [r_n] \text{ for some } r_n \in \mathfrak{m}_{V_n}\} \ .$$

Now (3.3) induces rigid points $\alpha_{(x_n)} : \mathrm{Spf}\,\widehat{V_\infty} \to \mathrm{Spf}\,\widehat{R_{\infty,i}}$ and $\langle \alpha_{(x_n)}\rangle : \mathrm{Spf}\,\widehat{V_\infty} \to \langle \mathcal{U}_{\infty,i}\rangle$ (cf. Proposition 1.7.16). Thus we obtain a map

$$\mu' : \varprojlim_n \langle \mathcal{U}_{n,i}\rangle \to \langle \mathcal{U}_{\infty,i}\rangle,\ (x_n) \mapsto \alpha_{(x_n)}(\mathfrak{m}_{V_\infty}) \ . \tag{3.4}$$

Let us show that $\mu' \circ \mu = \mathbf{Id}_{\langle \mathcal{U}_{\infty,i}\rangle}$. Pick $x \in \langle \mathcal{U}_{\infty,i}\rangle$. Put $x_n := \mu_{\infty,n}(x)$ for every $n \geq 1$. It suffices to show that $\mu'((x_n)_{n\geq 1}) = x$. Let $\langle \alpha_x\rangle : \mathrm{Spf}\,\widehat{V_x} \to \langle \mathcal{U}_{\infty,i}\rangle$ be the rigid point corresponding to x (cf. Proposition 1.7.15 1). Then by Proposition 1.7.15 2, we obtain the commutative diagrams



such that $f_n(\mathfrak{m}_{\widehat{V_x}}) = \mathfrak{m}_{V_n}$ $(n \geq 0)$. Hence we have $\mathrm{sp}_{X_{\infty,i}} \circ \langle \alpha_x \rangle = \alpha_{(x_n)}$, which yields the assertion. Next let us show the assertion 2. It suffices to check the following.

Claim 3.4.3. *There exist $f_{n,0}, \dots f_{n,r} \in R_{n,i}$ such that*

$$Z_{\infty,i}(\tfrac{f_1,\dots,f_r}{f_0}) = p_{n,i}^{-1}(Z_{n,i}(\tfrac{f_{n,1},\dots,f_{n,r}}{f_{n,0}})) \tag{3.5}$$

and

$$\langle \mathcal{U}_{\infty,i}(\tfrac{f_1}{f_0}, \dots, \tfrac{f_r}{f_0}) \rangle \cong q_{n,i}^{-1}(\langle \mathcal{U}_{n,i}(\tfrac{f_{n,1}}{f_{n,0}}, \dots, \tfrac{f_{n,r}}{f_{n,0}}) \rangle). \tag{3.6}$$

Proof of Claim 3.4.3. There exists some $c > 0$ such that $v(f_j) < v(x_i^c)$ for every $v \in Z_{\infty,i}$ and each $j = 1, \dots, r$. Take $f_{n,0}, \dots, f_{n,r} \in S_{n,i}$ such that $f_j \equiv f_{n,j} \mod x_i^c \widehat{R_{\infty,i}}$ for each $j = 1, \dots, r$. Then $Z_{\infty,i}(\frac{f_1,\dots,f_r}{f_0}) = Z_{\infty,i}(\frac{f_{n,1},\dots,f_{n,r}}{f_{n,0}})$. Moreover, for every $v \in Z_{\infty,i}$, we have $v(f_{n,j}) = p_{n,i}(v)(f_{n,j})$ for each $j = 1, \dots, r$. Therefore $Z_{\infty,i}(\frac{f_{n,1},\dots,f_{n,r}}{f_{n,0}}) = p_{n,i}^{-1}(Z_{n,i}(\frac{f_{n,1},\dots,f_{n,r}}{f_{n,0}}))$. Thus (3.5) follows. Since $\varphi_{n,i}(Z_{n,i}(\frac{f_{n,1},\dots,f_{n,r}}{f_{n,0}})) = \langle \mathcal{U}_{n,i}(\frac{f_{n,1}}{f_{n,0}}, \dots, \frac{f_{n,r}}{f_{n,0}}) \rangle$ and λ and μ are bijective, (3.6) follows from (3.5). □

□

3.4.2 Comparison between integral structure sheaves

Notice that the map $\varphi_{\infty,i} : Z_{\infty,i} \to \langle \mathcal{U}_i \rangle$ treated above and the map φ in Proposition 1.7.19 are constructed in the same way. Hence by Proposition 1.7.19, we obtain an isomorphism of sheaves $h_\infty : \mathcal{O}_{\mathcal{U}_{\infty,i}} \xrightarrow{\cong} \varphi_{\infty,i*}\mathcal{O}_{Z_{\infty,i}}$. First we show the following.

Lemma 3.4.4. *Let $z \in Z_{\infty,i}$, and let $h_z : \mathcal{O}_{\mathcal{U}_{\infty,i},\varphi_{\infty,i}(z)} \to \mathcal{O}_{Z_{\infty,i},z}$ be the homomorphism induced by h_∞. Then the semivaluation $v_z \circ h_z$ is equivalent to $v_{\varphi_{\infty,i}(z)}$.*

Proof. For every integer $n \geq 0$, let $h_n : \mathcal{O}_{\mathcal{U}_{n,i}} \xrightarrow{\cong} \varphi_{n,i*}\mathcal{O}_{Z_{n,i}}$ be the isomorphism constructed in Proposition 1.7.19. Then we have the commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{U}_{n,i}} & \xrightarrow{h_n} & \varphi_{n,i*}\mathcal{O}_{Z_{n,i}} \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{U}_{\infty,i}} & \xrightarrow[h_\infty]{} & \varphi_{\infty,i*}\mathcal{O}_{Z_{\infty,i}} \end{array} .$$

commutes. Set $x := \varphi_{\infty,i}(z)$, and set $z_n := \lambda_{\infty,n}(z)$ and $x_n := \mu_{\infty,n}(x)$ for every $n \geq 0$. Then by [20, **II**, Theorem A4.7], the above diagram induces

$$\begin{array}{ccc} \varinjlim_n k(z_n) & \longrightarrow & \varinjlim_n \mathrm{Frac}(V_{x_n}) \\ \downarrow & & \downarrow \\ k(z) & & \mathrm{Frac}(V_x) \end{array}$$

where $k(z)$ (resp. $k(z_n)$) is the residue field of $\mathcal{O}_{Z_{\infty,i},z}$ (resp. $\mathcal{O}_{Z_{n,i},z_n}$) and V_x (resp. V_{x_n}) is the valuation ring $\mathcal{O}^{\mathrm{int}}_{\mathcal{U}_{\infty,i},x_i}/\cap_{k\geq 1}(x_i)^k$ (resp. $\mathcal{O}^{\mathrm{int}}_{\mathcal{U}_{n,i},x_i}/\cap_{k\geq 1}(x_i)^k$). Moreover, by [20, **II**, Theorem A4.7], the restriction of $\varinjlim_n k(z_n) \to \varinjlim_n \mathrm{Frac}(V_{x_n})$ induces an isomorphism $\varinjlim_n \widehat{k(z_n)^+} \to \varinjlim_n \widehat{V_{x_n}}$. On the other hand, the restriction of the vertical maps $\varinjlim_n k(z_n) \to k(z)$ and $\varinjlim_n \mathrm{Frac}(V_{x_n}) \to \mathrm{Frac}(V_x)$ induces isomorphisms $\widehat{\varinjlim_n k(z_n)^+} \xrightarrow{\cong} \widehat{k(z)^+}$ and $\widehat{\varinjlim_n V_{x_n}} \xrightarrow{\cong} \widehat{V_x}$. Thus h_∞ induces an isomorphism $\widehat{k(z)^+} \xrightarrow{\cong} \widehat{V_x}$. Hence the assertion follows. □

Thus we find the following.

Lemma 3.4.5. *The isomorphism $h_\infty : \mathcal{O}_{\mathcal{U}_{\infty,i}} \xrightarrow{\cong} \varphi_{\infty,i*}\mathcal{O}_{Z_{\infty,i}}$ induces an isomorphism $\mathcal{O}^{\mathrm{int}}_{\mathcal{U}_{\infty,i}} \xrightarrow{\cong} \varphi_{\infty,i*}\mathcal{O}^+_{Z_{\infty,i}}$.*

Proof. Let $\mathcal{W}$ be a rational subdomain of $\mathcal{U}_{\infty,i}$. Pick $f \in \Gamma(\langle\mathcal{W}\rangle, \mathcal{O}_{\mathcal{W}})$. Then by §3.4.1 and Lemma 3.4.4, we have

$$\begin{aligned} f \in \Gamma(\langle\mathcal{W}\rangle, \mathcal{O}^{\mathrm{int}}_{\mathcal{W}}) \quad &\Longleftrightarrow \quad f_x \in \mathcal{O}^{\mathrm{int}}_{\mathcal{W},x} \qquad (\forall x \in \langle\mathcal{W}\rangle) \\ &\Longleftrightarrow \quad v_x(f_x) \geq 0 \qquad (\forall x \in \langle\mathcal{W}\rangle) \\ &\Longleftrightarrow \quad f \in \Gamma(\varphi^{-1}_{\infty,i}(\langle\mathcal{W}\rangle), \mathcal{O}^+_{Z_{\infty,i}})\ . \end{aligned}$$

Hence the assertion follows. □

By the above arguments, we find that the morphism $(\varphi_{\infty,i}, h_\infty) : (|Z_{\infty,i}|, \mathcal{O}_{Z_{\infty,i}}) \to (\langle\mathcal{U}_{\infty,i}\rangle, \mathcal{O}_{\mathcal{X}})$ gives an isomorphism of valued triples $Z_{\infty,i} \xrightarrow{\cong} \mathbf{ZR}(\mathcal{U}_{\infty,i})$, which completes the proof of Theorem 3.4.1.

Chapter 4

Witt-perfect Abhyankar's lemma

Recall that all perfectoid rings are Hausdorff complete by definition. In this chapter, we introduce *preperfectoid rings* as a class of decompleted perfectoid rings, and compare them with Davis-Kedlaya's Witt-perfect rings. We then prove a variant of Perfectoid Abhyankar's lemma for Witt-perfect rings. This result is applied to construction of almost Cohen-Macaulay's algebras to answer Question 0.2.1 in Appendix A.

4.1 Witt-perfect rings and preperfectoid rings

Definition 4.1.1 (Preperfectoid ring)**.** Fix a prime number $p > 0$.

1. A *preperfectoid ring* (resp.*preperfectoid field*) is a Tate ring of which the Hausdorff completion is a perfectoid ring (resp. perfectoid field).

2. A *preperfectoid pair* is a pair $(A_0, (t))$ such that $t \in A_0$ is a non-zerodivisor and the associated Tate ring $A_0[\frac{1}{t}]$ is preperfectoid.

In the case when $t = p$, this notion can be characterized by combining preuniformity and *Witt-perfectness* introduced by Davis and Kedlaya (cf. [11] and [12]).

Definition 4.1.2 (Witt-perfect ring)**.** We say that a p-torsion free ring A_0 is *Witt-perfect*, if the Witt-Frobenius map $F : W_{p^n}(A_0) \to W_{p^{n-1}}(A_0)$ is surjective for all $n > 2$.

We describe the relationship between Witt-perfect and perfectoid algebras in Proposition 4.1.4 and 4.1.7. These results are based on the following criterion for being Witt-perfect.

Lemma 4.1.3 ([11, Theorem 3.2])**.** *For a prime number $p > 0$, assume that A_0 is a p-torsion free ring. Then the following statements are equivalent.*

1. *A_0 is a Witt-perfect ring.*

2. *The Frobenius endomorphism on $A_0/(p)$ is surjective and for every $a \in A_0$, one can find $b \in A_0$ such that $b^p \equiv pa \pmod{p^2}$.*

Lemma 4.1.4.

1. *Let A_0 be a p-torsion free ring. Then the following conditions are equivalent.*

(a) $(A_0,(p))$ *is preperfectoid.*

(b) $(A_0,(p))$ *is preuniform, and* $(A_0)^+_{A_0[\frac{1}{p}]}$ *is Witt-perfect.*

2. *Let V be a p-torsion free valuation ring that is p-adically Hausdorff. Then the following conditions are equivalent.*

   (a) *The Tate ring associated to $(V,(p))$ is a preperfectoid field.*

   (b) *V is Witt-perfect.*

Proof. 1: Let A be the Tate ring associated to $(A_0,(p))$, and set $A^+ := (A_0)^+_A$. First we assume that $\widehat{A}$ is perfectoid. Then A^+ is integrally closed in A by Corollary 1.2.8. Moreover, $(A^+)^\wedge_p/(p)$ ($\simeq A^+/(p)$) is semiperfect and there is some $t \in (A^+)^\wedge_p$ for which $t^p \equiv p \mod p^2(A^+)^\wedge_p$ by Lemma 2.1.9 2. Hence A^+ is Witt-perfect. Conversely, we then assume that A preuniform and A^+ is Witt-perfect. Then $\widehat{A}$ is uniform by Corollary 1.2.8, $(A^+)^\wedge_p/(p)$ is semiperfect, and there is some $\varpi \in (A^+)^\wedge_p$ such that $\varpi^p \equiv p \mod p^2(A^+)^\wedge_p$: In particular, $p = \varpi^p u$ holds for some unit $u \in (A^+)^\wedge_p$, because $(A^+)^\wedge_p$ is p-adically Zariskian. Thus we also have some $t \in (A^+)^\wedge_p$ for which $t^p \equiv \varpi \mod p(A^+)^\wedge_p$ (and so $\varpi = t^p u'$ for some unit $u' \in (A^+)^\wedge_p$). Hence by Lemma 1.3.5, $(\widehat{A})^\circ/(p)$ is semiperfect. Therefore, $\widehat{A}$ is perfectoid, as desired.

2: Since $(V)^\wedge_p$ is a valuation ring with the field of fractions $(V)^\wedge_p[\frac{1}{p}]$, the assertion follows from Lemma 1.5.7 and the assertion 1. □

Corollary 4.1.5. *Let A_0 be a p-torsion free Witt-perfect ring that is integrally closed in $A_0[\frac{1}{p}]$. Denote by $\widehat{A_0}$ the p-adic completion of A_0. Denote by I_0 and I'_0 the ideals $\sqrt{(p)} \subset A_0$ and $\sqrt{(p)} \subset \widehat{A_0}$, respectively. Then one has $I'_0 = I_0\widehat{A_0}$ and $I_0 = I_0^2$.*

Proof. By Lemma 4.1.3, there is a sequence $\{\varpi_n\}_{n\geq 0}$ in A_0 such that $\varpi_0 = p$, $\varpi_1^p \equiv \varpi_0 \mod (p^2)$, and $\varpi_{n+1}^p \equiv \varpi_n \mod (p)$ for every $n \geq 0$. By induction on n, we have $\varpi_{n+1}^p = \varpi_n u_n$ for some unit $u_n \in \widehat{A_0}$, because I'_0 is contained in the Jacobson radical of $\widehat{A_0}$. Let R be a Banach ring associated to $(\widehat{A_0},(p))$. Then $R^{\circ\circ} = I'_0$. Moreover, by Lemma 1.5.6 2 and Lemma 4.1.4, we may assume that the norm on A is powermultiplicative. Hence I'_0 is generated by $\{\varpi_n\}_{n\geq 0}$. Therefore, one has $I'_0 = I_0\widehat{A_0}$ and $I'_0 = {I'_0}^2$. In particular, $I_0\widehat{A_0} = I_0^2\widehat{A_0}$. Next pick an element $x \in I_0$. Then by the equality stated just now, we have $x = \sum_{i=1}^r y_i\alpha_i$ for some $y_i \in I_0^2$ and $\alpha_i \in \widehat{A_0}$ $(i = 1,\ldots,r)$. Take $a_i \in A_0$ for which $a_i \equiv \alpha_i \mod (p^2)$ $(i = 1,\ldots,r)$. Then we have $x - \sum_{i=1}^r y_i a_i \in p^2\widehat{A_0} \cap A_0 = p^2A_0$. Therefore, $x \in I_0^2$, as wanted. □

4.1.1 Almost Witt-perfect and almost perfectoid algebras

We then introduce the following class of rings to establish a variant of Lemma 4.1.4 fitting for almost mathematics.

Definition 4.1.6 (Almost Witt-perfect ring)**.** Let A_0 be a p-torsion free ring with an element $g \in A_0$ admitting a compatible system of p-power roots $g^{\frac{1}{p^n}} \in A_0$. Then we say that A_0 is $(g)^{\frac{1}{p^\infty}}$-*almost Witt-perfect*, if the following conditions are satisfied.

1. The Frobenius endomorphism on $A_0/(p)$ is $(g)^{\frac{1}{p^\infty}}$-almost surjective.

2. For every $a \in A_0$ and every $n > 0$, there is an element $b \in A_0$ such that $b^p \equiv pg^{\frac{1}{p^n}} a \pmod{p^2}$.

Lemma 4.1.7. *Let V be a p-torsion free valuation ring tat is p-adically Hausdorff, and let A_0 be a p-torsion free $V[T^{1/p^\infty}]$-algebra. Put $g^{1/p^n} := T^{1/p^n} \cdot 1 \in A_0$ for every $n \geq 0$ and denote by $\widehat{V}$ and $\widehat{A_0}$ the p-adic completions of V and A_0, respectively. Then the following conditions are equivalent.*

- (a) *V is a Witt-perfect valuation ring of rank 1 and A_0 is $(g)^{\frac{1}{p^\infty}}$-almost Witt-perfect and integrally closed (resp. completely integrally closed) in $A_0[\frac{1}{p}]$.*
- (b) *Let K be the Tate ring associated to $(\widehat{V}, (p))$, and R the Tate ring associated to $(\widehat{A_0}, (p))$. Then K is a perfectoid field with $K^\circ = \widehat{V}$, and the map $V[T^{1/p^\infty}] \to A_0$ induces a structure as an almost perfectoid $K\langle T^{1/p^\infty}\rangle$-algebra on R.*

Proof. We first prove $(a) \Rightarrow (b)$. Clearly, the ring map $V[T^{1/p^\infty}] \to A_0$ induces a continuous ring homomorphism $K\langle T^{1/p^\infty}\rangle \to R$. Moreover, K is a perfectoid field by Lemma 4.1.4, and $K^\circ = \widehat{V}$ and $\widehat{A_0}$ forms an open and integrally closed subring of R° by Lemma 1.2.5. Let $\varpi \in K^\circ$ be a perfectoid pseudo-uniformizer that satisfies $p = \varpi^p u$ for some unit $u \in K^\circ$ and admits a compatible system of p-power roots (it exists by Lemma 2.1.9 and [39, Lemma 1.1(iii)]). Then, since the Frobenius endomorphism on $\widehat{A_0}/(p)$ is $(g)^{1/p^\infty}$-almost surjective, the Frobenius endomorphism on $R^\circ/(p)$ is $(\varpi g)^{1/p^\infty}$-almost surjective by Corollary 1.2.8, Corollary 1.3.4 and Lemma 1.3.5. So we find that R is an almost perfectoid $K\langle T^{1/p^\infty}\rangle$-algebra.

Next we prove $(b) \Rightarrow (a)$. Assume (b). In view of Corollary 1.2.8 and Lemma 4.1.4, it suffices to show that A_0 is $(g)^{1/p^\infty}$-almost Witt-perfect. Let $\varpi \in K^\circ$ be a perfectoid pseudo-uniformizer with the property mentioned above. Since $K^\circ = \widehat{V}$ and the map $K \to R$ carries $\widehat{V}$ into $\widehat{A_0}$, there is some unit $u \in \widehat{A_0}$ such that $p = \varpi^p u$. Moreover, the Frobenius endomorphism on $R^\circ/(p)$ is $(\varpi g)^{1/p^\infty}$-almost surjective by assumption. Hence by Corollary 1.3.4 and Lemma 1.3.5, the Frobenius endomorphism on $\widehat{A_0}/(p)$ $(\cong A_0/(p))$ is $(g)^{1/p^\infty}$-almost surjective. Thus it is enough to check the condition 2 in Definition 4.1.6 for $a = 1$. Fix an integer $n > 0$. Then we have $g^{1/p^n}p = \varpi^p(g^{1/p^n}u)$, and there exist some $b, c \in \widehat{A_0}$ such that $g^{1/p^n}u = b^p + pc$. Thus we have $g^{1/p^n}p = (\varpi b)^p + \varpi^p pc = (\varpi b)^p + p^2u^{-1}c$, which yields $(\varpi b)^p \equiv pg^{1/p^n} \mod p^2\widehat{A_0}$. Since $\widehat{A_0}/(p^2) \cong A_0/(p^2)$, the assertion follows. □

Lemma 4.1.8. *Let R be a preperfectoid ring, $t \in R$ be a pseudo-uniformizer, and A an almost preperfectoid $R[T^{1/p^\infty}]$-algebra. Let $g \in A$ be the image of T for every $n \geq 0$. Suppose that $\widehat{A}$ is g-torsion free[1]. Then $A(\frac{t}{g})$ is preperfectoid.*

Proof. Let $\mathcal{A}$ be the Hausdorff completion of A. Then by Lemma 2.3.7, there exist a perfectoid $\mathcal{R}\langle T^{1/p^\infty}\rangle$-algebra $\mathcal{A}^\natural$ and an $\mathcal{R}^\circ\langle T^{1/p^\infty}\rangle$-algebra homomorphism $\mathcal{A}^{\natural\circ} \hookrightarrow \mathcal{A}^\circ$ that is $(T)^{1/p^\infty}$-almost bijective. Hence φ induces the isomorphism of topological rings $\mathcal{A}^\natural(\frac{t}{g}) \xrightarrow{\cong} \mathcal{A}(\frac{t}{g})$. Taking Hausdorff completion, we get $\mathcal{A}^\natural\langle\frac{t}{g}\rangle \xrightarrow{\cong} \mathcal{A}\langle\frac{t}{g}\rangle$. Thus it suffices to show that we have a ring isomorphism $(A[\frac{t}{g}])^\wedge_t \cong \mathcal{A}\langle\frac{t}{g}\rangle$, and it follows form the assumption that A is g-torsion free. □

In this subsection, we fix a prime number $p > 0$, and for any ring R, we denote by $\widehat{R}$ the p-adic completion of R. Moreover for a ring map $f : R \to S$, we denote by $\widehat{f}$ the ring map $\widehat{R} \to \widehat{S}$ induced by f.

[1] This assumption is realized if (p, g) is a regular sequence in A.

Theorem 4.1.9 (Almost purity for Witt-perfect rings, [11], [12])**.** *Let A be a preperfectoid ring, and $f : A \to B$ be a finite étale ring homomorphism. Let $(A_0, (t))$ be a pair of definition and $I = \sqrt{tA_0}$. Assume that (A_0, I) is a basic setup*[2]*. Then the following assertions hold.*

1. *B is also preperfectoid.*
2. *The induced homomorphism $A^\circ \to B^\circ$ is I-almost finite étale.*

4.2 Weak extension theorem

Here we establish a weak form of Riemann's extension theorem. This result is crucial for the proof of Witt-perfect Abhyankar's lemma. Notice that this theorem itself is independent from the theory of perfectoid rings and spaces.

Theorem 4.2.1. *Let A_0 be a ring with a non-zerodivisor ϖ that is ϖ-adically Zariskian and integral over a Noetherian ring. Let $g \in A_0$ be a non-zerodivisor. Let A be the Tate ring associated to $(A_0, (\varpi))$, and A^j the Tate ring associated to $(A_0[\frac{\varpi^j}{g}], (\varpi))$ for every integer $j > 0$. Then we have an isomorphism of rings*

$$(A_0)^+_{A_0[\frac{1}{\varpi g}]} \xrightarrow{\cong} \varprojlim_j A^{j\circ}$$

where the transition map $A^{j+1\circ} \to A^{j\circ}$ is the natural one.

Proof. By assumption, we have a canonical ring isomorphism $\varphi_j : A_0[\frac{1}{\varpi g}] \xrightarrow{\cong} A^j$ for each $j > 0$. By restricting φ_j to $(A_0)^+_{A_0[\frac{1}{\varpi g}]}$, we obtain the ring map $\varphi_j^+ : (A_0)^+_{A_0[\frac{1}{\varpi g}]} \to A^{j\circ}$. Then $\{\varphi_j\}_{j>0}$ and $\{\varphi_j^+\}_{j>0}$ induce the commutative diagram of ring maps

$$\begin{array}{ccc} (A_0)^+_{A_0[\frac{1}{\varpi g}]} & \xrightarrow{\varphi^+} & \varprojlim_j A^{j\circ} \\ \downarrow & & \downarrow \\ A_0[\frac{1}{\varpi g}] & \xrightarrow[\varphi]{\cong} & \varprojlim_j A^j \end{array} \tag{4.1}$$

where φ is an isomorphism and the vertical maps are injective. Thus it suffices to prove that (4.1) is cartesian. Pick $c \in A_0[\frac{1}{\varpi g}]$ such that $\varphi_j(c) \in A^{j\circ}$ for every $j > 0$. Let us show that c lies in $(A_0)^+_{A_0[\frac{1}{\varpi g}]}$ by applying Proposition 1.6.13. For this, we consider the (ϖg)-adic topology: let $A_{(\varpi g)}$ be the Tate ring associated to $(A_0, (\varpi g))$ (notice that each A^j is also the Tate ring associated to $(A_0[\frac{\varpi^j}{g}], (\varpi g))$). Let $X_{(\varpi g)} = \mathrm{Spa}(A_{(\varpi g)}, (A_0)^+_{A_{(\varpi g)}})$, $X_j = \mathrm{Spa}(A^j, A^{j\circ})$ for each $j > 0$, and U be the subspace

$$U = \{x \in X_{(\varpi g)} \mid v_{\widetilde{x}}(\varpi) > 0 \ \text{ for the maximal generalization } \widetilde{x} \text{ of } x\}$$

of $X_{(\varpi g)}$. Then the underlying ring of $A_{(\varpi g)}$ is equal to $A_0[\frac{1}{\varpi g}]$, and we have

$$(A_0)^+_{A_{(\varpi g)}} = \{a \in A_{(\varpi g)} \mid v_x(a) > 0 \text{ for all } x \in [U]\}$$

[2]For example, this assumption is realized if A_0 admits $\{t^{\frac{1}{p^n}}\}_{n \geq 0}$, A_0 is t-adically Zariskian, or A_0 is an algebra over a t-adically Hausdorff Witt-perfect valuation ring of rank 1 (cf. Example 1.1.1 and Lemma 1.3.2).

by Proposition 1.6.13. On the other hand, since we also have

$$A^{j\circ} = \{a \in A^j \mid v_{x_j}(a) > 0 \text{ for all } x_j \in [X_j]\}$$

by Corollary 1.6.14, $v_{x_j}(\varphi_j(c)) > 0$ for all $j > 0$ and all $x_j \in [X_j]$. Now since φ_j gives a continuous map $A_{(\varpi g)} \to A^j$, (4.1) induces the continuous map $\varinjlim_j [X_j] \to [X_{(\varpi g)}]$, which factors through $[U]$ because $\varpi \in A^j$ is topologically nilpotent. Thus we are reduced to showing that the resulting map $f : \varinjlim_j [X_j] \to [U]$ is surjective.

Pick $x \in [U]$ and let $v_x : A_{(\varpi g)} \to \mathbb{R}_{\geq 0}$ be a corresponding semivaluation. Let us find some $j_0 > 0$ such that the composite

$$v_{x,j_0} : A^{j_0} \to A_{(\varpi g)} \xrightarrow{v_x} \mathbb{R}_{\geq 0}$$

gives a point $x_{j_0} \in [X_{j_0}]$ for which $f([x_{j_0}]) = x$. Since $v_x(\varpi) > 0$ and v_x is of rank 1, there exists some $j_0 > 0$ such that $v_x(\frac{\varpi^{j_0}}{g}) > 0$. Then we have $v_x(A_0[\frac{\varpi^{j_0}}{g}]) \geq 0$ because $v_x(A_0) \geq 0$ and v_x is of rank 1. Thus, since any $a \in A^{j_0 \circ}$ is almost integral over $A_0[\frac{\varpi^{j_0}}{g}]$ and v_x is of rank 1, we have $v_{x,j_0}(A^{j_0\circ}) \geq 0$. Hence v_{x,j_0} gives the desired point $x_{j_0} \in [X_{j_0}]$. □

Corollary 4.2.2. *Let A be a preunifrom Tate ring that admits a pair of definition $(A_0, (\varpi))$ such that A_0 is ϖ-adically Zariskian and integral over a Noetherian ring. Let $g \in A_0$ be a non-zerodivisor such that A is completely integrally closed in $A[\frac{1}{g}]$. Let A^j be the Tate ring associated to $(A_0[\frac{\varpi^j}{g}], (\varpi))$ for every integer $j > 0$. Then we have an isomorphism of rings*

$$A^\circ \xrightarrow{\cong} \varprojlim_j A^{j\circ} .$$

This corollary is regarded as an algebraic variant of Bartenwerfer's extension theorem.

Theorem 4.2.3 (Bartenwerfer)**.** *Let K be a non-archimedean field with $0 < |\varpi| < 1$, let A be an affinoid algebra over K, and suppose that A is normal as a ring. Let $g \in A$ be a non-zerodivisor. Then we have an isomorphism*

$$A^\circ \xrightarrow{\cong} \varprojlim_j A\langle \frac{\varpi^j}{g} \rangle^\circ .$$

4.3 Witt-perfect Abhyankar's lemma

In this section we prove the main theorem of this chapter, which is a Witt-perfect version of André's Perfectoid Abhyankar's Lemma. Let us start setting up some notation.

Lemma 4.3.1. *Let V be a p-torsion free valuation ring of rank 1. Let A be a p-torsion free V-algebra together with a non-zerodivisor $g \in A$ admitting a compatible system of p-power roots $\{g^{1/p^n}\}_{n \geq 1}$ in A. Suppose that*

1. *V is Witt perfect;*

2. *A is a $(g)^{\frac{1}{p^\infty}}$-almost Witt-perfect ring that is completely integrally closed in $A[\frac{1}{p}]$;*

3. *(p, g) is a regular sequence on A (or more generally, it suffices to assume that p, g are non-zerodivisors on the p-adic completion $\widehat{A}$).*

Let K be the Hausdorff completion of the Tate ring associated to $(V, (p))$. For each $j > 0$, let A^j be the Tate ring associated to $(A[\frac{p^j}{g}], (p))$ and $\mathcal{A}^j$ the Hausdorff completion of A^j. Then the following assertions hold.

(a) K is a perfectoid field, and $\mathcal{A}^j$ is a perfectoid K-algebra for each $j > 0$.

(b) The subring

$$A_0^j := \left\{ a \in A[\tfrac{1}{pg}] \;\middle|\; \psi(a) \in \mathcal{A}^{j\circ} \right\} \tag{4.2}$$

of A^j is a ring of definition.

Proof. The assertion (a) follows from Lemma 4.1.7 and 4.1.8. To prove (b), let us show that $A_0^j (\subset A^j)$ is bounded. By Lemma 1.2.7, the diagram of ring homomorphisms:

$$\begin{array}{ccc} A[\frac{p^j}{g}] & \longrightarrow & (A[\frac{p^j}{g}])_p^\wedge \\ \big\downarrow & & \big\downarrow \\ A^j & \xrightarrow[\psi]{} & \mathcal{A}^j \end{array}$$

is cartesian. Moreover, by the assertion (a), $t^l \mathcal{A}^{j\circ} \subset (A[\frac{p^j}{g}])_p^\wedge$ for some $l > 0$. Thus we have $t^l A_0^j \subset A[\frac{p^j}{g}]$, which yields the assertion (b). □

The following proposition is crucial in the proof of Witt-perfect Abhyankar's lemma.

Proposition 4.3.2 ([42, Proposition 4.16])**.** *Let the notation and hypotheses be as in Lemma 4.3.1. Suppose further that A is p-adically Zariskian and integral over a Noetherian ring. Then we have an isomorphism of rings*

$$\widehat{\varprojlim_{j>0} A^{j\circ}} \cong \varprojlim_{j>0} \widehat{A^{j\circ}},$$

where $\widehat{(\ \ )}$ is p-adic completion.

Proof. By Riemann's extension theorem for (almost) perfectoid K-algebras [1, Théorème 4.2.2], we have

$$g^{-1/p^\infty} \mathcal{A}^\circ \cong \varprojlim_{j>0} \mathcal{A}^{j\circ}. \tag{4.3}$$

Since the left-hand side of (4.3) is p-adically complete,

$$\varprojlim_{j>0} \widehat{A^{j\circ}} \cong \varprojlim_{j>0} \mathcal{A}^{j\circ} \tag{4.4}$$

is also p-adically complete. On the other hand, we have a natural map:

$$\varprojlim_{j>0} A^{j\circ} \to \varprojlim_{j>0} \widehat{A^{j\circ}}. \tag{4.5}$$

Notice that $p^n \widehat{A^{j\circ}} \cap A^{j\circ} = p^n A^{j\circ}$ for $n > 0$. Since p is a non-zerodivisor on both $A^{j\circ}$ and $\widehat{A^{j\circ}}$ and intersection commutes with inverse limit, it follows that

$$\left(p^n \varprojlim_{j>0} \widehat{A^{j\circ}}\right) \cap \varprojlim_{j>0} A^{j\circ} = p^n \varprojlim_{j>0} A^{j\circ}.$$

This says that the topology on $\varprojlim_{j>0} A^{j\circ}$ induced from the inverse image of the filtration $\{p^n \varprojlim_{j>0} \widehat{A^{j\circ}}\}_{n>0}$ via (4.5) coincides with the p-adic topology. So $\widehat{\varprojlim_{j>0} A^{j\circ}}$ is the topological closure of the image of (4.5), which implies that the natural map $\widehat{\varprojlim_{j>0} A^{j\circ}} \to \varprojlim_{j>0} \widehat{A^{j\circ}}$ is injective. It remains to prove that this is surjective. Since $A \hookrightarrow A^{j\circ}$ and $A^{j\circ}$ is completely integrally closed in $A^{j\circ}[\frac{1}{pg}]$, Lemma 1.2.3 shows that $g^{-1/p^\infty} A \hookrightarrow \varprojlim_{j>0} A^{j\circ}$, which induces a map

$$\varprojlim_{j>0} \widehat{A^{j\circ}} \cong g^{-1/p^\infty} \mathcal{A}^\circ \cong g^{-1/p^\infty} \widehat{A} \cong \widehat{g^{-1/p^\infty} A} \to \widehat{\varprojlim_{j>0} A^{j\circ}}, \tag{4.6}$$

where the first isomorphism follows from (4.3) and (4.4). Therefore, we find that $\widehat{\varprojlim_{j>0} A^{j\circ}} \to \varprojlim_{j>0} \widehat{A^{j\circ}}$ splits the map (4.6), and $\widehat{\varprojlim_{j>0} A^{j\circ}} \to \varprojlim_{j>0} \widehat{A^{j\circ}}$ is surjective, as desired. □

Now we assert the main theorem of this chapter.

Theorem 4.3.3 (Witt-perfect Abhyankar's lemma, [42, Theorem 4.19])**.** *Let the notation and hypotheses be as in Lemma 4.3.1. Suppose further that*

(a) A is p-adically Zariskian and integral over a Noetherian ring;

(b) A is completely integrally closed in $A[\frac{1}{pg}]$;

(c) V admits a compatible system of p-power roots $\{p^{1/p^n}\}_{n\geq 0}$ of p.

Let $A[\frac{1}{pg}] \hookrightarrow B'$ be a finite étale extension and put $B := (g^{-1/p^\infty} A)^+_{B'}$. Then the following statements hold:

1. *The Frobenius endomorphism $Frob : B/(p) \to B/(p)$ is $(pg)^{1/p^\infty}$-almost surjective and it induces an injection $B/(p^{1/p}) \hookrightarrow B/(p)$.*
2. *Assume that A is a normal ring that is torsion free and integral over a Noetherian normal domain. Then the induced map $A/(p^m) \to B/(p^m)$ is $(pg)^{1/p^\infty}$-almost finite étale for all $m > 0$.*

To prove this, we shall check the following statements.

Lemma 4.3.4 ([42, Lemma 4.18])**.** *Let the notation and hypotheses be as in Theorem 4.3.3. Let $B[\frac{p^j}{g}] \subset B'$ be the B-subalgebra that is generated by $\frac{p^j}{g}$. Then the following statements hold:*

1. *$A = g^{-\frac{1}{p^\infty}} A$ holds. In particular, B is equal to the integral closure of A in B'.*
2. *Let B^j be the module-finite A^j-algebra B' equipped with the canonical structure as a Tate ring as in Lemma 1.4.6. Then $\left(B[\frac{p^j}{g}], (p)\right)$ is a pair of definition of B^j.*

3. *Let $\mathcal{B}$ and $\mathcal{B}\{\frac{p^j}{g}\}$ be the Hausdorff completions of Tate rings associated to $\big(B,(p)\big)$ and $\big(B[\frac{p^j}{g}],(p)\big)$, respectively and let $\mathcal{B}^j$ be the module-finite $\mathcal{A}^j$-algebra $B' \otimes_{A[\frac{1}{pg}]} \mathcal{A}^j$ equipped with the canonical topology as in Lemma 1.4.7. Then we have a canonical isomorphism of topological rings:*
$$\mathcal{B}^j \xrightarrow{\cong} \mathcal{B}\{\tfrac{p^j}{g}\}.$$
4. *$p \in B$ is in the Jacobson radical and B is completely integrally closed in $B[\frac{1}{pg}]$.*
5. *Under the hypotheses as in Theorem 4.3.3(c), (p,g) is a regular sequence on B.*

Proof of Lemma 4.3.4. 1: This just follows from Lemma 1.2.4 and 1.2.3 1 by the assumption that A is completely integrally closed in $A[\frac{1}{pg}]$.

2: By the assertion 1 and our assumption, the map $A[\frac{p^j}{g}] \to B[\frac{p^j}{g}]$ is integral and becomes finite étale after inverting p. We have $(A[\frac{p^j}{g}])$ is preuniform in view of Lemma 1.2.5. Now we can apply Proposition 1.8.7 1 and deduce that $\big(B[\frac{p^j}{g}],(p)\big)$ is preuniform. Since $\big(B[\frac{p^j}{g}]\big)^+_{\mathcal{B}^j} = \big(A[\frac{p^j}{g}]\big)^+_{\mathcal{B}^j}$ and $\big((A[\frac{p^j}{g}])^+_{\mathcal{B}^j},(p)\big)$ is a pair of definition of $\mathcal{B}^j$ by Corollary 1.8.10 1, the assertion follows.

3: We denote by $\widehat{B}$, $\widehat{A[\frac{p^j}{g}]}$ and $\widehat{B[\frac{p^j}{g}]}$ the p-adic completions, respectively. Let $\mathcal{B}^j_0$ be the image of the natural map $B[\frac{p^j}{g}] \otimes_{A[\frac{p^j}{g}]} \widehat{A[\frac{p^j}{g}]} \to \mathcal{B}^j$. Then by the assertion 1 and Lemma 1.4.9, we see that $(\mathcal{B}^j_0,(p))$ is a pair of definition of $\mathcal{B}^j$. Moreover by Proposition 1.8.11 1, the natural map $B[\frac{p^j}{g}] \otimes_{A[\frac{p^j}{g}]} \widehat{A[\frac{p^j}{g}]} \to \widehat{B[\frac{p^j}{g}]}$ induces an isomorphism

$$\mathcal{B}^j_0 \xrightarrow{\cong} \widehat{B[\tfrac{p^j}{g}]}. \tag{4.7}$$

Inverting p in (4.7), we obtain the desired isomorphism of topological rings $\mathcal{B}^j \xrightarrow{\cong} \mathcal{B}\{\frac{p^j}{g}\}$.

4: Since B is integral over A and $p \in A$ is contained in the Jacobson radical, $p \in B$ is also contained in the Jacobson radical of B by [38, §9, Lemma 2]. Since B is integral over a Noetherian ring and integrally closed in $B' = B[\frac{1}{pg}]$, it is also completely integrally closed in B' by Corollary 1.6.14.

5: By assumption, A is integral over the Noetherian normal domain R, the field of fractions of R has characteristic zero, A is normal, and $A[\frac{1}{pg}] \to B'$ is finite étale. These facts combine together to show that B' is a normal ring and B is the filtered colimit of normal rings that are torsion free and module-finite over R. Thus, (p,g) forms a regular sequence on B in view of Serre's normality criterion. □

Now we are prepared to prove Theorem 4.3.3.

Proof of Theorem 4.3.3. To prove the statements, we use Galois theory of commutative rings. By decomposing A into the direct product of rings, we may assume and do that $A[\frac{1}{pg}] \to B'$ is finite étale of constant rank (indeed, one can check the conditions $1 \sim 4$ remain true for each direct factor of the ring A). By [1, Lemme 1.9.2] applied to the finite étale extension $A[\frac{1}{pg}] \hookrightarrow B' = B[\frac{1}{pg}]$, there is the decomposition
$$A[\tfrac{1}{pg}] \hookrightarrow B' = B[\tfrac{1}{pg}] \hookrightarrow C', \tag{4.8}$$

where $A[\frac{1}{pg}] \to C'$ and $B' = B[\frac{1}{pg}] \to C'$ are Galois coverings and let G be the Galois group for $A[\frac{1}{pg}] \to C'$. Let B^j (resp. C^j) be the resulting Tate ring according to Lemma 4.3.4.

We shall fix algebras $\mathcal{A}$, K and $\mathcal{A}^j$ as defined in Lemma 4.3.1. Recall that K is a perfectoid field, $\mathcal{A}$ is a $(pg)^{\frac{1}{p^\infty}}$-almost perfectoid and $\mathcal{A}^j$ are perfectoid K-algebras. Consider the Tate ring: $\mathcal{B}^j := B' \otimes_{A[\frac{1}{pg}]} \mathcal{A}^j$ (resp. $\mathcal{C}^j := C' \otimes_{A[\frac{1}{pg}]} \mathcal{A}^j$) as in Lemma 4.3.4. Since $A[\frac{1}{pg}] \to B'$ (resp. $A[\frac{1}{pg}] \to C'$) is finite étale, $\mathcal{A}^j \to \mathcal{B}^j$ (resp. $\mathcal{A}^j \to \mathcal{C}^j$) is also finite étale. By Theorem 3.3.9, $\mathcal{B}^j$ (resp. $\mathcal{C}^j$) is a perfectoid K-algebra. Moreover, we have a natural commutative diagram

$$\begin{array}{ccc} B' & \xrightarrow{\psi_{j+1}} & \mathcal{B}^{j+1} \\ \| & & \downarrow \\ B' & \xrightarrow{\psi_j} & \mathcal{B}^j \end{array}$$

and the set of $\mathcal{A}$-algebras $\{\mathcal{B}^j\}_{j>0}$ forms an inverse system, where $\mathcal{B}^{j+1} \to \mathcal{B}^j$ is the natural inclusion. Since $B^{j\circ} = \psi_j^{-1}(\mathcal{B}^{j\circ})$ by Corollary 1.8.12, we obtain the following commutative diagram:

$$\begin{array}{ccccccc} A & \longrightarrow & A^{j+1\circ} & \longrightarrow & B^{j+1\circ} & \longrightarrow & B' \\ \| & & \downarrow & & \downarrow & & \| \\ A & \longrightarrow & A^{j\circ} & \longrightarrow & B^{j\circ} & \longrightarrow & B' \end{array}$$

Taking inverse limits, we have compositions of ring maps:

$$A \cong \widetilde{A} := \varprojlim_j A^{j\circ} \to \widetilde{B} := \varprojlim_j B^{j\circ} \to B', \tag{4.9}$$

where the first isomorphism is due to Theorem 4.2.1. Similarly, after setting $\widetilde{C} := \varprojlim_j C^{j\circ}$, we obtain the compositions of ring maps $A \cong \widetilde{A} \to \widetilde{C} \to C'$.

Let us prove the assertion (b). By Lemma 1.8.13, the action of G preserves $C^{j\circ}$ and we have $(C^{j\circ})^G = A^{j\circ}$. Hence $A^{j\circ} \to C^{j\circ}$ is integral. Since $B^j \to C^j$ carries $B^{j\circ}$ into $C^{j\circ}$ by Lemma 1.4.8, $A^{j\circ} \to B^{j\circ}$ is also integral. Taking G-invariants of rings appearing in (4.9), we get

$$\widetilde{C}^G \cong (\varprojlim_j C^{j\circ})^G \cong \varprojlim_j (C^{j\circ})^G \cong \varprojlim_j A^{j\circ} \cong \widetilde{A},$$

which implies that $\widetilde{A} \to \widetilde{C}$ is integral. Hence $\widetilde{A} \to \widetilde{B}$ is integral. Since $A^{j\circ}[\frac{1}{p}] \to B^{j\circ}[\frac{1}{p}]$ is identified with the finite étale extension $A[\frac{1}{pg}] \to B' = B[\frac{1}{pg}]$ and $B^{j\circ}$ is the integral closure of $A^{j\circ}$ in B', it follows from the almost purity theorem for Witt-perfect rings [11, Theorem 5.2] or [12, Theorem 2.9], that $A^{j\circ} \to B^{j\circ}$ is $(p)^{\frac{1}{p^\infty}}$-almost finite étale. In particular, $B^{j\circ}$ is a Witt-perfect V-algebra. Retain the notation as in (4.9). We already know that $A \cong g^{-\frac{1}{p^\infty}} A \cong \widetilde{A}$. Since A is $(pg)^{\frac{1}{p^\infty}}$-almost Witt-perfect by assumption, the p-adic completion $\widehat{\widetilde{A}}$ is an integral $(pg)^{\frac{1}{p^\infty}}$-almost perfectoid ring.[3] Then we claim that

$$\widehat{\widetilde{B}} \text{ is integral } (pg)^{\frac{1}{p^\infty}}\text{-almost perfectoid.} \tag{4.10}$$

[3] In [1, Question 3.5.1], a question is raised as to whether $g^{-\frac{1}{p^\infty}} \widehat{A}[\frac{1}{p}]^\circ$ is integral perfectoid.

By applying [1, Proposition 4.4.1], for any fixed $r = \frac{n}{p}$ with $n \in \mathbb{N}$, we get a $(pg)^{\frac{1}{p^\infty}}$-almost isomorphism:

$$\varprojlim_j{}^1 \big(B^{j\circ}/(p^r)\big) \approx 0. \tag{4.11}$$

After applying $\varprojlim$ to the short exact sequence $0 \to B^{j\circ}/(p^{\frac{p-1}{p}}) \to B^{j\circ}/(p) \to B^{j\circ}/(p^{\frac{1}{p}}) \to 0$, the following $(pg)^{\frac{1}{p^\infty}}$-almost surjection follows from (4.11):

$$\varprojlim_j B^{j\circ}/(p) \to \varprojlim_j B^{j\circ}/(p^{\frac{1}{p}}). \tag{4.12}$$

By Witt-perfectness of $B^{j\circ}$, the Frobenius isomorphism $B^{j\circ}/(p^{\frac{1}{p}}) \cong B^{j\circ}/(p)$ yields that

$$\varprojlim_j \big(B^{j\circ}/(p)\big) \xrightarrow{\text{Frob}} \varprojlim_j \big(B^{j\circ}/(p)\big) \text{ is } (pg)^{\frac{1}{p^\infty}}\text{-almost surjective.} \tag{4.13}$$

Consider the commutative diagram

$$\begin{array}{ccc} \varprojlim_j \big(B^{j\circ}/(p)\big) & \xrightarrow{\text{Frob}} & \varprojlim_j \big(B^{j\circ}/(p)\big) \\ \uparrow & & \uparrow \\ \big(\varprojlim_j B^{j\circ}\big)/(p) & \xrightarrow{\text{Frob}} & \big(\varprojlim_j B^{j\circ}\big)/(p) \end{array}$$

In order to prove (4.10), it suffices to show that $\big(\varprojlim_j B^{j\circ}\big)/(p) \to \varprojlim_j \big(B^{j\circ}/(p)\big)$ is a $(pg)^{\frac{1}{p^\infty}}$-almost isomorphism in view of (4.13). By [1, Lemme 2.8.1], this map is injective. On the other hand, the above map is $(pg)^{\frac{1}{p^\infty}}$-almost surjective by applying the almost surjectivity of (4.12) to [1, Proposition 4.3.1 and Remarque 4.3.1] and thus, the Frobenius endomorphism on $\widetilde{B}/(p)$ is $(pg)^{\frac{1}{p^\infty}}$-almost surjective. Notice that $B = \widetilde{B}$, which follows from the fact that $A \cong \widetilde{A} \to \widetilde{B}$ is integral. So the assertion (b) is proved.

Finally we prove (c). Let the notation be as previously. Then $A[\frac{1}{pg}] \to C'$ is a G-Galois covering, $\mathcal{A}^j \to \mathcal{C}^j$ is also a G-Galois covering by [17, Lemma 12.2.7]. Let $\widehat{C^{j\circ}}$ be the p-adic completion of $C^{j\circ}$. Since $C^{j\circ}[\frac{1}{p}] = C'$, there is a natural $\mathcal{A}^j$-algebra homomorphism

$$\mathcal{C}^j = C' \otimes_{A[\frac{1}{pg}]} \mathcal{A}^j \to \widehat{C^{j\circ}}[\tfrac{1}{p}]. \tag{4.14}$$

Since $\widehat{A^{j\circ}} \cong \mathcal{A}^{j\circ}$, the map (4.14) is an isomorphism, which induces $\mathcal{C}^{j\circ} \cong \widehat{C^{j\circ}}$ in view of Corollary 1.8.12. Thus, G acts on $\widehat{C^{j\circ}}$ and

$$(\widehat{C^{j\circ}})^G \cong (\mathcal{C}^{j\circ})^G \cong \mathcal{A}^{j\circ} \tag{4.15}$$

by applying Lemma 1.8.13. In particular, $\mathcal{A}^{j\circ} \to \widehat{C^{j\circ}}$ is an integral extension. In summary,

$$\mathcal{A}^{j\circ} \to \mathcal{C}^{j\circ} \cong \widehat{C^{j\circ}} \text{ is } (p)^{\frac{1}{p^\infty}}\text{-almost étale and } \mathcal{A}^j \to \mathcal{C}^j \cong \widehat{C^{j\circ}}[\tfrac{1}{p}] \text{ is Galois with Galois group } G. \tag{4.16}$$

To finish the proof, let us apply the proof of [1, Proposition 5.2.3] via Galois theory of commutative rings to (4.16). Refer the reader to [1, (5.6), (5.7), (5.8), (5.9) and (5.10) of Proposition 5.2.3] for the following discussions. Let

$$\mathcal{C} := (\varprojlim_j \widehat{C^{j\circ}})[\tfrac{1}{p}]$$

and in particular, $\mathcal{C}^\circ \cong \varprojlim_j \mathcal{C}^{j\circ}$. Now we have the following crucial result after applying Lemma 4.3.4 to Proposition 4.3.2:

$$\mathcal{C}^\circ \cong \widehat{\widetilde{C}} \text{ or equivalently, } \widehat{\varprojlim_j C^{j\circ}} \cong \varprojlim_j \widehat{C^{j\circ}}. \tag{4.17}$$

Hence

$$(\widehat{\widetilde{C}})^G \cong \big(\widehat{\varprojlim_j C^{j\circ}}\big)^G \cong \big(\varprojlim_j \widehat{C^{j\circ}}\big)^G \cong \varprojlim_j (\mathcal{C}^{j\circ})^G \cong \varprojlim_j \mathcal{A}^{j\circ} \cong \varprojlim_j \widehat{A^{j\circ}} \cong \widehat{\widetilde{A}}, \tag{4.18}$$

where the third isomorphism follows from the commutativity of inverse limits with taking G-invariants and (4.16), and the fourth one from (4.15). The last one follows from Proposition 4.3.2.

In view of (4.16) and applying [1, Proposition 3.3.4], the map

$$\mathcal{C}^{j\circ}\widehat{\otimes}_{\mathcal{A}^{j\circ}}\mathcal{C}^{j\circ} \to \prod_G \mathcal{C}^{j\circ} \text{ defined by } b\otimes b' \mapsto \big(\gamma(b)b'\big)_{\gamma\in G} \tag{4.19}$$

is a $(p)^{\frac{1}{p^\infty}}$-almost isomorphism, where the completed tensor product is p-adic. By [1, Proposition 4.4.4], we have $\mathcal{C}\{\frac{p^j}{g}\} \cong \mathcal{C}^j$ and $\mathcal{C}$ is an $\mathcal{A}$-algebra. Using this, we obtain

$$\big(\mathcal{C}\widehat{\otimes}_{\mathcal{A}}\mathcal{C}\big)\{\tfrac{p^j}{g}\} \cong \mathcal{C}\widehat{\otimes}_{\mathcal{A}}\mathcal{C}\widehat{\otimes}_{\mathcal{A}}\mathcal{A}^j \cong \big(\mathcal{C}\widehat{\otimes}_{\mathcal{A}}\mathcal{A}^j\big)\otimes_{\mathcal{A}^j}\big(\mathcal{C}\widehat{\otimes}_{\mathcal{A}}\mathcal{A}^j\big) \cong \mathcal{C}\{\tfrac{p^j}{g}\}\otimes_{\mathcal{A}^j}\mathcal{C}\{\tfrac{p^j}{g}\} \cong \mathcal{C}^j\otimes_{\mathcal{A}^j}\mathcal{C}^j.$$

By Riemann's extension theorem [1, Théorème 4.2.2] and by [1, Proposition 3.3.4], we have $(pg)^{\frac{1}{p^\infty}}$-almost isomorphisms:

$$\varprojlim_j \big(\mathcal{C}^{j\circ}\widehat{\otimes}_{\mathcal{A}^{j\circ}}\mathcal{C}^{j\circ}\big) \xrightarrow{\approx} \varprojlim_j \big(\mathcal{C}^j\otimes_{\mathcal{A}^j}\mathcal{C}^j\big)^\circ \xrightarrow{\cong} \varprojlim_j \big(\mathcal{C}\widehat{\otimes}_{\mathcal{A}}\mathcal{C}\big)\{\tfrac{p^j}{g}\}^\circ \xrightarrow{\approx} \big(\mathcal{C}\widehat{\otimes}_{\mathcal{A}}\mathcal{C}\big)^\circ \xrightarrow{\approx} \mathcal{C}^\circ\widehat{\otimes}_{\mathcal{A}^\circ}\mathcal{C}^\circ. \tag{4.20}$$

Putting (4.19) and (4.20) together, we get the following $(pg)^{\frac{1}{p^\infty}}$-almost isomorphism:

$$\mathcal{C}^\circ\widehat{\otimes}_{\mathcal{A}^\circ}\mathcal{C}^\circ \xrightarrow{\approx} \prod_G \mathcal{C}^\circ. \tag{4.21}$$

After reducing (4.18) and (4.21) modulo p^m for $m > 0$, we find that

$$\mathcal{A}^\circ/(p^m) \xrightarrow{\approx} \big(\mathcal{C}^\circ/(p^m)\big)^G$$

is a $(pg)^{\frac{1}{p^\infty}}$-almost isomorphism via Discussion 4.3.5 2. So the induced map:

$$\widetilde{A}/(p^m) \to \widetilde{C}/(p^m)$$

is a $(pg)^{\frac{1}{p^\infty}}$-almost G-Galois covering in view of (4.17). This map factorizes as

$$\widetilde{A}/(p^m) \to \widetilde{B}/(p^m) \to \widetilde{C}/(p^m).$$

It then follows from [1, Proposition 1.9.1(3)] that $\widetilde{A}/(p^m) \to \widetilde{B}/(p^m)$ is $(pg)^{\frac{1}{p^\infty}}$-almost finite étale, as desired. □

Discussion 4.3.5 ([42, Discussion 4.20]). Consider the short exact sequence $0 \to C^{j\circ} \xrightarrow{p^m} C^{j\circ} \to C^{j\circ}/(p^m) \to 0$. Applying the Galois cohomology $H^i(G,\ )$ to this exact sequence, we get an injection $(C^{j\circ})^G/(p^m) \hookrightarrow \big(C^{j\circ}/(p^m)\big)^G$ whose cokernel embeds into $H^1(G, C^{j\circ})$. By applying [14, Theorem 2.4] or [43, Proposition 3.4], $H^1(G, C^{j\circ})$ is $(p^{\frac{1}{p^\infty}})$-almost zero.

Let us prove that

$$\mathcal{A}^\circ/(p^m) \xrightarrow{\approx} \big(\mathcal{C}^\circ/(p^m)\big)^G$$

is a $(pg)^{\frac{1}{p^\infty}}$-almost isomorphism for any integer $m > 0$. We have already seen the $(pg)^{\frac{1}{p^\infty}}$-almost isomorphisms: $\mathcal{A}^{j\circ}/(p^m) \approx (\mathcal{C}^{j\circ})^G/(p^m) \approx \big(\mathcal{C}^{j\circ}/(p^m)\big)^G$. Taking the inverse limits $j \to \infty$ and using [1, Proposition 4.2.1], we get $(pg)^{\frac{1}{p^\infty}}$-almost isomorphisms:

$$\mathcal{A}^\circ/(p^m) \xrightarrow{\approx} \varprojlim_{j>0} \big(\mathcal{C}^{j\circ}/(p^m)\big)^G \xrightarrow{\approx} \big(\varprojlim_{j>0}(\mathcal{C}^{j\circ}/(p^m))\big)^G \xrightarrow{\approx} \big(\mathcal{C}^\circ/(p^m)\big)^G,$$

as wanted.

Appendix A

Applications of Witt-perfect Abhyankar's lemma (due to Shimomoto)

In this appendix, we explain Shimomoto's construction of almost Cohen-Macaulay algebras that is an application of Witt-perfect Abhyankar's lemma and gives a positive answer to Roberts's question (Question 0.2.1).

First we recall the definition of big Cohen-Macaulay algebras, due to Hochster.

Definition A.0.1 (Big Cohen-Macaulay algebra)**.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and let T be an R-algebra. Then T is a *big Cohen-Macaulay R-algebra*, if there is a system of parameters $x_1, \ldots, x_d$ such that $x_1, \ldots, x_d$ is a regular sequence on T and $(x_1, \ldots, x_d)T \neq T$. Moreover, we say that a big Cohen-Macaulay algebra is *balanced*, if every system of parameters satisfies the above conditions.

We also recall the definition of almost Cohen-Macaulay algebras from [2, Definition 4.1.1]. Refer the reader to [3, Proposition 2.5.1] for a subtle point on this definition.

Definition A.0.2 (Almost Cohen-Macaulay algebra)**.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$, and let (T, I) be a basic setup equipped with an R-algebra structure. Fix a system of parameters $x_1, \ldots, x_d$. We say that T is *I-almost Cohen-Macaulay with respect to $x_1, \ldots, x_d$*, if $T/\mathfrak{m}T$ is not I-almost zero and

$$c \cdot \big((x_1, \ldots, x_i) :_T x_{i+1}\big) \subset (x_1, \ldots, x_i)T$$

for any $c \in I$ and $i = 0, \ldots, d-1$.

It is important to keep in mind that the permutation of the sequence $x_1, \ldots, x_d$ in the above definition may fail to form an almost regular sequence. We consider the sequence $p, x_2, \ldots, x_d$ for the main theorem below.

André's construction: For the applications given below, we take I to be the ideal $\bigcup_{n>0} \pi^{\frac{1}{p^n}} T$ as the basic setup (T, I) for some non-zerodivisor $\pi \in R$. Following [2], we introduce some auxiliary algebras. Let $W(k)$ be the ring of Witt vectors for a perfect field k of characteristic $p > 0$ and let

$$A := W(k)[[x_2, \dots, x_d]]$$

be an unramified complete regular local ring and $V_j := W(k)[p^{\frac{1}{p^j}}]$. Then V_j is a complete discrete valuation ring and set $V_\infty := \varinjlim_j V_j$. Then this is a Witt-perfect valuation domain. For a fixed element $0 \neq g \in A$, we set

$$B_{jk} := V_j[[x_2^{\frac{1}{p^j}}, \dots, x_d^{\frac{1}{p^j}}]][g^{\frac{1}{p^k}}][\tfrac{1}{p}] := \Big(V_j[[x_2^{\frac{1}{p^j}}, \dots, x_d^{\frac{1}{p^j}}]][T]/(T^{p^k} - g)\Big)[\tfrac{1}{p}]$$

for any pair of non-negative integers (j,k). For any pairs (j,k) and (j',k') with $j \leq j'$ and $k \leq k'$, we can define the natural map $B_{jk} \to B_{j'k'}$. Let us define the A-algebra A_{jk} to be the integral closure of A in B_{jk}. Let us also define

$$A_{\infty\infty} := \varinjlim_{j,k} A_{jk} \text{ and } A_{\infty g} := \text{the integral closure of } A_{\infty\infty} \text{ in } A_{\infty\infty}[\tfrac{1}{pg}]. \tag{A.1}$$

For brevity, let us write

$$A_\infty := A_{\infty 0} := \varinjlim_j V_j[[x_2^{\frac{1}{p^j}}, \dots, x_d^{\frac{1}{p^j}}]]. \tag{A.2}$$

Then we have towers of integral ring maps:

$$A \to A_\infty \to A_{\infty\infty} \to A_{\infty g}.$$

Lemma A.0.3 ([42, Lemma 5.3]). *Let R be a Noetherian domain with a proper ideal I and let T be a normal ring that is a torsion free integral extension of R. Assume that $\pi \in I$ is a nonzero element such that T admits a compatible system of p-power roots $\pi^{\frac{1}{p^n}}$. Then T/IT is not $(\pi^{\frac{1}{p^\infty}})$-almost zero.*

Proof. In order to prove that T/IT is not $(\pi)^{\frac{1}{p^\infty}}$-almost zero, it suffices to prove that $T_\mathfrak{m}/IT_\mathfrak{m}$ is not $(\pi)^{\frac{1}{p^\infty}}$-almost zero, where $\mathfrak{m}$ is any maximal ideal of T containing IT, since $T_\mathfrak{m}/IT_\mathfrak{m}$ is the localization of T/IT. Then $T_\mathfrak{m}$ is a normal domain that is an integral extension over the Noetherian domain $R_{\mathfrak{m}\cap R}$, in which I is a proper ideal. To derive a contradiction, we suppose that $T_\mathfrak{m}/IT_\mathfrak{m}$ is $(\pi^{\frac{1}{p^\infty}})$-almost zero. Notice that $T_\mathfrak{m}$ is contained in the absolute integral closure $(R_{\mathfrak{m}\cap R})^+$. In particular, it implies that

$$(\pi)^{\frac{1}{p^n}} \in IT_\mathfrak{m} \text{ for all } n > 0.$$

Raising p^n-th power on both sides, we get by [51, Lemma 4.2];

$$\pi \in \bigcap_{n>0} I^{p^n} T_\mathfrak{m} = 0,$$

which is a contradiction. □

Proposition A.0.4 ([42, Proposition 5.4]). *Let the notation be as in (A.1) and (A.2). Then the following assertions hold:*

1. *A_∞ is completely integrally closed in its field of fractions that is an integral and faithfully flat extension over A. Moreover, the localization map $A_\infty[\frac{1}{pg}] \to A_{\infty\infty}[\frac{1}{pg}]$ is ind-étale.*

2. *$A_{\infty g}$ is a $(g)^{\frac{1}{p^\infty}}$-almost Witt-perfect algebra over the Witt-perfect valuation domain V_∞ such that $p^{\frac{1}{p^n}} \in V_\infty$, $g^{\frac{1}{p^n}} \in A_{\infty g}$. Moreover, $A_{\infty g}$ is a $(pg)^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay normal ring that is completely integrally closed in $A_{\infty g}[\frac{1}{pg}]$. In particular, the localization of $A_{\infty g}$ at its any maximal ideal is a $(pg)^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay normal domain.*

Proof. 1: It is clear that $A \to A_\infty$ is integral by construction. Since A_∞ is a filtered colimit of regular local subrings with module-finite transition maps, one readily checks that $A \to A_\infty$ is faithfully flat. By Lemma 1.2.4 , A_∞ is a completely integrally closed domain in its field of fractions. By looking at the discriminant, it is easy to check that $A_\infty[\frac{1}{pg}] \to A_{\infty\infty}[\frac{1}{pg}]$ is ind-étale.

2: By André's crucial result [2, Théorème 2.5.2][1] combined with Lemma A.0.3, we find that $A_{\infty\infty}$ is a $(p)^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay and Witt-perfect algebra. Next let us study $A_{\infty g}$ and consider $\widetilde{A}_{\infty\infty} := \varprojlim_j A^{j\circ}_{\infty\infty}$ attached to $A_{\infty\infty}$ as defined in Theorem 4.2.1. Then we claim that

$$A_{\infty g} \cong \widetilde{A}_{\infty\infty}. \tag{A.3}$$

Notice that since $A_{\infty g}$ is integrally closed in $A_{\infty\infty}[\frac{1}{pg}] = A_{\infty g}[\frac{1}{pg}]$, it follows from Proposition **??** that $A_{\infty g}$ is completely integrally closed in $A_{\infty\infty}[\frac{1}{pg}]$.

Now by applying Theorem 4.2.1(c) to $A_{\infty\infty}$, the equality $(A.3)$ follows, where one should notice that p is in the Jacobson radical of $A_{\infty\infty}$ and g remains a non-zerodivisor on the p-adic completion $\widehat{A}_{\infty\infty}$ in view of [2, Remarques 2.6.1]. It follows from $(A.3)$ and the Riemann's extension theorem [1, Théorème 4.2.2] that

$$g^{-\frac{1}{p^\infty}}\widehat{A}_{\infty\infty} \cong \varprojlim_j \widehat{A^{j\circ}_{\infty\infty}} \cong \widehat{\varprojlim_j A^{j\circ}_{\infty\infty}} \cong \widehat{A}_{\infty g},$$

where the middle isomorphism is due to Proposition 4.3.2. In particular, $\widehat{A}_{\infty\infty} \to \widehat{A}_{\infty g}$ is a $(g)^{\frac{1}{p^\infty}}$-almost isomorphism.

From the property of $A_{\infty\infty}$ mentioned in 1, one finds that $\widehat{A}_{\infty g}$ is an integral $(g)^{\frac{1}{p^\infty}}$-almost perfectoid and $(pg)^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay algebra. By the fact that $A_\infty[\frac{1}{pg}]$ is a normal domain and $A_\infty[\frac{1}{pg}] \to A_{\infty\infty}[\frac{1}{pg}]$ is obtained as a filtered colimit of finite étale $A_\infty[\frac{1}{pg}]$-algebras, we see that $A_{\infty\infty}[\frac{1}{pg}]$ is a normal ring. Since $A_{\infty g}$ is integrally closed in $A_{\infty\infty}[\frac{1}{pg}]$, it follows that $A_{\infty g}$ is also normal. □

As a corollary, we obtain the following theorem.

Theorem A.0.5 ([42, Theorem 5.5]). *Let $(R, \mathfrak{m})$ be a complete Noetherian local domain of mixed characteristic $p > 0$ with perfect residue field k. Let $p, x_2, \dots, x_d$ be a system of parameters and let R^+ be the absolute integral closure of R. Then there exists an R-algebra T together with a nonzero element $g \in R$ such that the following hold:*

[1] A similar construction also appears in [22, Theorem 16.9.17], where they apply p-integral closure instead of integral closure. This makes it possible to get rid of "$(p)^{\frac{1}{p^\infty}}$-almost" from the statement.

1. *T admits compatible systems of p-power roots $p^{\frac{1}{p^n}}, g^{\frac{1}{p^n}} \in T$ for all $n > 0$.*

2. *The Frobenius endomorphism $\mathrm{Frob} : T/(p) \to T/(p)$ is surjective.*

3. *T is a $(pg)^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay normal domain with respect to $p, x_2, \ldots, x_d$ and $R \subset T \subset R^+$.*

Furthermore, one can arrange T such that $R[\frac{1}{pg}] \to T[\frac{1}{pg}]$ is a filtered colimit of finite étale $R[\frac{1}{pg}]$-algebras contained in $T[\frac{1}{pg}]$.

Proof. In the following, we may assume $\dim R \geq 2$ without loss of generality. By Cohen's structure theorem, there is a module-finite extension

$$A := W(k)[[x_2, \ldots, x_d]] \hookrightarrow R.$$

As the induced field extension $\mathrm{Frac}(A) \to \mathrm{Frac}(R)$ is separable, there is an element $g \in A \setminus pA$ such that $A[\frac{1}{pg}] \to R[\frac{1}{pg}]$ is étale. As in Proposition A.0.4, we set

$$A_\infty := \bigcup_{n>0} W(k)[p^{\frac{1}{p^n}}][[x_2^{\frac{1}{p^n}}, \ldots, x_d^{\frac{1}{p^n}}]].$$

Now consider the integral extensions $A \to A_\infty \to A_{\infty\infty} \to A_{\infty g}$ as in Proposition A.0.4. Let $\mathfrak{n}$ be a maximal ideal of $A_{\infty g}$. Then the localization $(A_{\infty g})_\mathfrak{n}$ is a normal domain that is an integral extension over A and enjoys the same properties as $A_{\infty g}$. Since (p, g) forms part of a system of parameters of A and $(A_{\infty g})_\mathfrak{n}$ is a filtered colimit of module-finite normal A-algebras, it follows that (p, g) is a regular sequence on $(A_{\infty g})_\mathfrak{n}$ by Serre's normality criterion.[2] By base change, the map

$$(A_{\infty g})_\mathfrak{n}[\tfrac{1}{pg}] \to R \otimes_A (A_{\infty g})_\mathfrak{n}[\tfrac{1}{pg}] \tag{A.4}$$

is finite étale. Then $R \otimes_A (A_{\infty g})_\mathfrak{n}[\frac{1}{pg}]$ is a normal ring; the localization at any maximal ideal is an integrally closed domain by Lemma 1.2.12. Letting the notation be as in $(A.4)$, set

$$B := \text{the integral closure of } R \text{ in } R \otimes_A (A_{\infty g})_\mathfrak{n}[\tfrac{1}{pg}].$$

Then by the normality of $R \otimes_A (A_{\infty g})_\mathfrak{n}[\frac{1}{pg}]$ and Lemma 1.2.12, it follows that B is a normal ring that fits into the commutative diagram:

$$\begin{array}{ccc} (A_{\infty g})_\mathfrak{n} & \longrightarrow & B \\ \uparrow & & \uparrow \\ A & \longrightarrow & R \end{array}$$

in which every map is injective and integral. Let $\mathfrak{n}'$ be any maximal ideal of B. Since A is a local domain and $A \to B$ is a torsion free integral extension, one finds that $A \cap \mathfrak{n}'$ is the unique maximal ideal of A and the induced localization map $A \to B_{\mathfrak{n}'}$ is an injective integral extension between normal domains. By setting $A := (A_{\infty g})_\mathfrak{n}$ in the notation of Theorem 4.3.3 and applying Lemma

[2] In what follows, if necessary, we repeat the same argument for deriving the regularity of (p, g) in order to apply Theorem 4.3.3.

A.0.3, it follows that B is a $(pg)^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay normal ring with respect to $p, x_2, \dots, x_d$ and $(pg)^{\frac{1}{p^\infty}}$-almost Witt-perfect. Since these properties are preserved under localization, it follows that the normal domain $B_{\mathfrak{n}'}$ enjoys the same properties.

To finish the proof, let us put $C := B_{\mathfrak{n}'}$ for brevity of notation. Set

$$T := \text{the integral closure of } C \text{ in } C[\tfrac{1}{p}]^{\text{ét}},$$

where $C[\frac{1}{p}]^{\text{ét}}$ is the maximal étale extension of $C[\frac{1}{p}]$ contained in the absolute integral closure $C[\frac{1}{p}]^+$. Then T is a Witt-perfect normal domain in view of [52, Lemma 5.1] or [53, Lemma 10.1]. Therefore, it remains to establish that T is $(pg)^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay with respect to $p, x_2, \dots, x_d$. Let us note that the composite map

$$(A_{\infty g})_{\mathfrak{n}}[\tfrac{1}{pg}] \to C[\tfrac{1}{pg}] \to T[\tfrac{1}{pg}]$$

is an ind-étale extension. So we find that $T[\frac{1}{pg}]$ is the filtered colimit of finite étale $(A_{\infty g})_{\mathfrak{n}}[\frac{1}{pg}]$-algebras. As T is integrally closed in its field of fractions, the integral closure of $(A_{\infty g})_{\mathfrak{n}}$ in $T[\frac{1}{pg}]$ is the same as T. Summing up, we conclude from Theorem 4.3.3 applied to $A := (A_{\infty g})_{\mathfrak{n}}$, together with the fact that $A_{\infty\infty}/(p) \to A_{\infty g}/(p)$ is a $(g)^{\frac{1}{p^\infty}}$-almost isomorphism, that $T/(p)$ is the filtered colimit of $(pg)^{\frac{1}{p^\infty}}$-almost finite étale $A_{\infty\infty}/(p)$-algebras. By Lemma A.0.3, T is $(pg)^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay. $\square$

Appendix B

Remarks on Henselian approximation

For proving the almost purity theorem, we used Proposition 3.3.4. This proposition is based on the following result on Henselian approximation, called *Elkik's approximation theorem.*

Theorem B.0.1 ([13, Théorème 2 *bis* and p.587])**.** *Let (A, I) be a Henselian pair, and suppose that (A, I) satisfies* $(\mathbf{E_a})$ *or* $(\mathbf{E_b})$*. Let $\widehat{A}$ be the I-adic completion of A, B a finitely presented A-algebra, and $\overline{B} = B \otimes_A \widehat{A}$. Let $V \subset \operatorname{Spec} B$ be an open set which is smooth over $\operatorname{Spec} A$, and $\overline{V} \subset \operatorname{Spec} \overline{B}$ the preimage of V.*

Then for every integer $n \geq 0$ and every $\widehat{A}$-section $\overline{\varepsilon} : \operatorname{Spec} \widehat{A} \to \operatorname{Spec} \overline{B}$ whose restriction to $\operatorname{spec} \widehat{A} \setminus V(I\widehat{A})$ *factors through $\overline{V}$, there exists an A-section $\varepsilon : \operatorname{Spec} A \to \operatorname{Spec} B$ congruent to $\overline{\varepsilon}$ modulo I^n, whose restriction to* $\operatorname{Spec} A \setminus V(I)$ *factors through V.*

Here the conditions $(\mathbf{E_a})$ are $(\mathbf{E_b})$ are the following, which provide generalization of Noetherianness.

$(\mathbf{E_a})$ *I is principal, and A has bounded I-torsion (i.e. there exists an integer $l > 0$ such that $I^l A_{I\text{-tor}} = (0)$);*

$(\mathbf{E_b})$ *there exists a noetherian pair (A_0, I_0) such that A is flat over A_0 and $I = I_0 A$.*

On the course of generalization of the theory of perfectoid spaces, we might use this theorem in a more generalized situation. Now we remark that the assertion of this theorem would fail for some Henselian pairs.

Claim B.0.2 ([40, Main result])**.** *In Elkik's approximation theorem (Theorem B.0.1), the condition '$(\mathbf{E_a})$ or $(\mathbf{E_b})$' cannot be weakened to either one of the following conditions:*

$(\mathbf{E'_a})$ *I is principal;*

$(\mathbf{E''_a})$ *I is finitely generated, and A has bounded I-torsion;*

$(\mathbf{E'_b})$ *there exists a pair (A_0, I_0) such that A_0 is Noetherian outside I_0 (i.e. the scheme* $\operatorname{Spec} A_0 \setminus V(I_0)$ *is Noetherian), A is flat over A_0, and $I = I_0 A$.*

Here we list counter-examples (see [40] for details).

Example B.0.3 ([40, Example 3.2]). Let k be a field. We consider a polynomial ring in countably many indeterminates $k[x_0, x_1, x_2, \ldots][y]$, and its ideal

$$\mathfrak{a} = (x_0 y, x_0 - x_1 y, x_1 - x_2 y, x_2 - x_3 y, \ldots) \ .$$

Let us denote the quotient $k[y, x_0, x_1, \ldots]/\mathfrak{a}$ by R. Consider the polynomial ring $R[t]$ and its ideal

$$\mathfrak{n} = (x_0 t^2, x_1 t^3, x_2 t^4, \ldots) \ .$$

Let A_0 be the quotient $R[t]/\mathfrak{n}$, A the t-adic Henselization of A_0, and $\widehat{A}$ the t-adic completion of A. Consider the polynomial system $f = (f_1, f_2)$ where

$$\begin{cases} f_1 = (t - y)X \\ f_2 = t^2 X \end{cases}$$

in $A[X]$ and the solution $\widehat{\alpha} = \sum_{i=0}^{\infty} x_i t^i \in \widehat{A}$ of $f = 0$. Let B be the A-algebra $A[X]/(f_1, f_2)$, $\overline{B} = B \otimes_A \widehat{A}$, $V = \operatorname{Spec} B \setminus V(tB)$, and $\overline{\varepsilon} : \operatorname{Spec} \widehat{A} \to \operatorname{Spec} \overline{B}$ the $\widehat{A}$-section corresponding to $\widehat{\alpha}$. Then this example provides a proof of Claim B.0.2 on $(\mathbf{E}'_{\mathbf{a}})$ and $(\mathbf{E}'_{\mathbf{b}})$.

Example B.0.4 ([40, Example 3.4]). Let k and R be the same as in Example B.0.3. Consider the polynomial ring $R[t, u]$ and its ideals

$$\begin{cases} \mathfrak{n} = (x_0 t^2, x_1 t^3, x_2 t^4, \ldots) \\ \mathfrak{n}' = (x_0 u, x_1 t u, x_2 t^2 u, x_3 t^3 u, \ldots) \ . \end{cases}$$

Put $A'_0 = R[t, u]/(\mathfrak{n} + \mathfrak{n}')$ and $I_0 = (t, u)A'_0$. Let A' be the I_0-adic Henselization of A'_0, $I \subset A'$ the ideal $I_0 A'$, and $\widehat{A'}$ the I-adic completion of A'. Consider the polynomial system $f' = (f_1, f_2, f_3)$ where

$$\begin{cases} f_1 = (t - y)X \\ f_2 = t^2 X \\ f_3 = uX \end{cases}$$

in $A'[X]$ and the solution $\widehat{\alpha} = \sum_{i=0}^{\infty} x_i t^i \in \widehat{A'}$ of $f' = 0$. Let B' be the A'-algebra $A'[X]/(f_1, f_2, f_3)$, $\overline{B'} = B' \otimes_{A'} \widehat{A'}$, $V' = \operatorname{Spec} B' \setminus V(IB')$, and $\overline{\varepsilon'} : \operatorname{Spec} \widehat{A'} \to \operatorname{Spec} \overline{B'}$ the $\widehat{A'}$-section corresponding to $\widehat{\alpha}$. Then this example provides a proof of Claim B.0.2 on $(\mathbf{E}''_{\mathbf{a}})$.

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