APPROXIMATING A FIXED POINT OF A FINITE FAMILY OF MAPPINGS ON A CAT(1) SPACE
(CAT(1) 空間における有限個の写像の不動点近似)

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Contents

1. Introduction 2
2. Preliminaries 7
3. Halpern iteration 12
4. Applications of Halpern iteration 23
5. CQ projection method and Shrinking projection method 26
References 34

1. Introduction

Various problems reduce to those of finding a fixed point of a mapping. For example, there are problems of approximation of distance projection and minimization of functions. Such studies are common in Hilbert space and Banach spaces. Recently, a similar problem has been considered in the CAT(1) spaces. In this paper, we concentrate in the problem of establishing approximation schemes assuming the existence of at least one fixed point.

Let us begin with a historical explanation on Halpern schemes. In 1967, Halpern [7] considered an iterative method to find a fixed point of a nonexpansive mapping from the unit ball of a real Hilbert space into itself. In 1992, Wittmann [28] considered the following Halpern type iteration scheme in a real Hilbert space H: Let $C \subset H$ be a closed convex subset, and $u, x_1 \in C$ be given. The iteration scheme is

$$x_{n+1} := \alpha_n u + (1 - \alpha_n) T x_n$$

for all $n \in \mathbb{N}$, where T is a nonexpansive mapping from C into itself such that the set $F(T)$ of its fixed points is nonempty, and where the real sequence $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. He showed that $\{x_n\}$ converges strongly to a fixed point which is nearest to u in $F(T)$.

In 1997, Shioji-Takahashi [23] extended Wittmann's result to the case where the Hilbert space H is replaced by a Banach space. In 1998, motivated by results of Ishikawa [8] and Das-Debata [3], Atsushiba-Takahashi [1] considered a variation of Halpern iteration using W-mappings $\{W_n\}$ defined by

$$\begin{aligned} U_{n,1} &:= \alpha_{n,1}T_1 \oplus (1-\alpha_{n,1})I, \\ U_{n,2} &:= \alpha_{n,2}T_2U_{n,1} \oplus (1-\alpha_{n,2})I, \\ &\cdots \\ U_{n,r} &:= \alpha_{n,r}T_rU_{n,r-1} \oplus (1-\alpha_{n,r})I, \\ W_n &:= U_{n,r}. \end{aligned}$$

in a Banach space, where $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ are given (see Definition 2.2). Let u, x_1 are given and

$$x_{n+1} := \beta_n u + (1-\beta_n)W_n x_n$$

for all $n \in \mathbb{N}$.

A CAT(0) space is a generalization of Hilbert space in a derecton different from that of a Banach space. In 2011, Saejung [21] considered the Halpern iteration using single nonexpansive mapping in a CAT(0) space. In 2011, Phuengrattana-Suantai [20] considered the same iteration scheme using W-mapping in a convex metric space. Remark that a CAT(0) space is a convex metric space, so that their result covers the case of CAT(0) space. In 2013, Kimura-Satô [16] considered the Halpern iteration using single strongly quasinonexpansive mapping in a CAT(1) space. Remark that a CAT(1) space is not necessarily a convex metric space.

In this paper, we consider the Halpern iteration with W-mapping generated by a finite family of quasinonexpansive mappings in a CAT(1) space, that is, we showed the following theorem under the similar condition in the result of Kimura-Satô:

Theorem 1.1 (Theorem 3.1). *Let X be a complete* CAT(1) *space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let $T_1, T_2, \dots, T_r$ be a finite number of quasinonexpansive and Δ-demiclosed mappings of X into itself such that $F := \bigcap_{i=1}^r F(T_i) \neq \emptyset$, and let $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ be real numbers for $n \in \mathbb{N}$ such that $\alpha_{n,i} \in [a, 1-a]$ for every $i = 1, 2, \dots, r$, where $0 < a < 1/2$. Let W_n be the W-mappings of X into itself generated by $T_1, T_2, \dots, T_r$ and $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ for $n \in \mathbb{N}$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 < \beta_n < 1$ for every $n \in \mathbb{N}, \lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^\infty \beta_n = \infty$. For given points $u, x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by*

$$x_{n+1} = \beta_n u \oplus (1 - \beta_n) W_n x_n$$

for $n \in \mathbb{N}$. Suppose that one of the following conditions holds:

(a) $\sup_{v,v' \in X} d(v, v') < \pi/2$;
(b) $d(u, P_F u) < \pi/4$ *and* $d(u, P_F u) + d(x_1, P_F u) < \pi/2$;
(c) $\sum_{n=1}^\infty \beta_n^2 = \infty$.

Then $\{x_n\}$ converges to $P_F u$, where P_F is the metric projection onto F from X.

Next, let us begin with a historical explanation on CQ-projection method and Shrinking projection method.

Let C be a nonempty closed convex subset of a real Hilbert space and T a nonexpansive mapping from C onto itself such that fixed points set $F(T)$ is nonempty. Let P_C denote the metric projection to a nonempty closed convex subset. In 2000, Solodov-Svaiter [24] introduced CQ projection method and in 2003, Nakajo-Takahashi [18] considered the following iteration:

$$
\begin{aligned}
x_1 &:= x \in C, \\
y_n &:= \alpha_n x_n + (1-\alpha_n) T x_n, \\
C_n &:= \{ z \in C \mid \|y_n - z\| \leq \|x_n - z\| \}, \\
Q_n &:= \{ z \in C \mid \langle x_n - z, x_1 - x_n \rangle \geq 0 \}, \\
x_{n+1} &:= P_{C_n \cap Q_n} x_1,
\end{aligned}
$$

where $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $P_{C_n \cap Q_n}$ is the metric projection from C onto $C_n \cap Q_n$. Then they showed $\{x_n\}$ is converget to $P_{F(T)} x_1$. In 2006, Nakajo-Shimoji-Takahashi [19] used W-mapping generated to consider following iteration:

$$
\begin{aligned}
x_1 &:= x \in C, \\
y_n &:= W_n x_n, \\
C_n &:= \{ z \in C \mid \|y_n - z\| \leq \|x_n - z\| \}, \\
Q_n &:= \{ z \in C \mid \langle x_n - z, x_1 - x_n \rangle \geq 0 \}, \\
x_{n+1} &:= P_{C_n \cap Q_n} x_1,
\end{aligned}
$$

and showed $\{x_n\}$ is convergent to $P_F x_1$, where F is a common fixed point set of generaters of W-mapping. In 2012, Kimura-Satô [13] considered similar iteration in real Hilbert sphere as following:

$$
\begin{aligned}
x_1 &:= x \in C, \\
y_n &:= T x_n, \\
C_n &:= \{ z \in C \mid d(y_n, z) \leq d(x_n, z) \}, \\
Q_n &:= \{ z \in C \mid \cos d(x_1, x_n) \cos d(x_0, x_n) \geq \cos d(x_1, z) \}, \\
x_{n+1} &:= P_{C_n \cap Q_n} x_1
\end{aligned}
$$

On the other hand, Takahashi-Takeuchi-Kubota [27] introduced shrinking projection method in real Hilbert space as the following iteration:

$$\begin{aligned} x_1 &:= x \in C, \\ y_n &:= \alpha_n x_n + (1-\alpha_n) T x_n, \\ C_n &:= \{z \in C \mid \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} &:= P_{C_n} x_1, \end{aligned}$$

where $0 \leq \alpha_n < a < 1$ for all $n \in \mathbb{N}$. Then they showed $\{x_n\}$ converges to $P_{F(T)}$. In 2009, Kimura-Takahashi [17] considered shrinking projection method in Banach space and Kimura [12] considered shrinking projection method in a real Hilbert ball which is a example of a Hadamard space, that is, a complete CAT(0) space.

In this paper, we consider CQ projection method and shrinking projection method for a finite family of nonexpansive mappings in a real Hilbert sphere which is a example of a complete CAT(1) space, that is, we show the following results:

Theorem 1.2 (Theorem 5.1)**.** *Let C be a closed convex subset in real Hilbert sphere S_H such that $d(v, v') \leq \pi/2$ for every $v, v' \in C$. Let $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ be real numbers for $n \in \mathbb{N}$ such that $\alpha_{n,i} \in [a, 1-a]$ for every $i = 1, 2, \dots, r$ where $0 < a < 1/2$, and let $T_1, T_2, \dots, T_r$ be a finite number of nonexpansive mappings of C into itself such that $F := \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let W_n be the W-mappings of X into itself generated by $T_1, T_2, \dots, T_r$ and $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ for $n \in \mathbb{N}$. For a given point $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$\begin{aligned} y_n &:= W_n x_n, \\ C_n &:= \{z \in C \mid d(y_n, z) \leq d(x_n, z)\}, \\ Q_n &:= \{z \in C \mid \cos d(x_1, x_n) \cos d(x_n, z) \geq \cos d(x_1, z)\}, \\ x_{n+1} &:= P_{C_n \cap Q_n} x_1 \end{aligned}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ is well-defined and convergent to $P_F x_1$.

Theorem 1.3 (Theorem 5.2)**.** *Let C be a closed convex subset in real Hilbert sphere S_H such that $d(v, v') < \pi/2$ for every $v, v' \in C$. Let $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ be real numbers for $n \in \mathbb{N}$ such that $\alpha_{n,i} \in [a, 1-a]$ for every $i = 1, 2, \dots, r$ where $0 < a < 1/2$, and let $T_1, T_2, \dots, T_r$ be a finite number of nonexpansive mappings of C into itself such that $F := \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let W_n be the W-mappings of X into itself generated by $T_1, T_2, \dots, T_r$ and $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ for $n \in \mathbb{N}$. For a given point $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$\begin{aligned} x_1 &:= x \in C, \\ y_n &:= W_n x_n, \\ C_n &:= \{z \in C \mid d(y_n, z) \leq d(x_n, z)\} \cap C_{n-1}, \\ x_{n+1} &:= P_{C_n} x_1, \end{aligned}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ is well defined and convergent to $P_F x_1$.

2. Preliminaries

Let (X, d) be a metric space. For $x, y \in X$, a mapping $c : [0, l] \to X$ is a geodesic of $x, y \in X$ if $c(0) = x, c(l) = y$ and $d(c(s), c(t)) = |s - t|$ for all $s, t \in [0, l]$. For $r > 0$, if a geodesic exists for every $x, y \in X$ with $d(x, y) < r$, then X is called an r-geodesic metric space. If a geodesic is unique for every $x, y \in X$, we define $[x, y] := c([0, l])$ and it is called a geodesic segment of $x, y \in X$. In what follows, a metric space X is always assumed to be π-geodesic and every geodesic is unique. For $x, y \in X$, let $c : [0, l] \to X$ be a geodesic of joinning $x, y \in X$. For $t \in [0, l]$, we denote

$$tx \oplus (1-t)y := c((1-t)l).$$

In other words, $z := tx \oplus (1-t)y$ satisfies $d(x, z) = (1-t)d(x, y)$. Let X be a geodesic metric space. A geodesic triangle is defined by the union of segment $\triangle(x,y,z) := [x,y] \cup [y,z] \cup [z,x]$. Let $\mathbb{S}^2$ be the unit sphere of the Euclidean space $\mathbb{R}^3$ and $d_{\mathbb{S}^2}$ is the spherical metric on $\mathbb{S}^2$. Then, for $x,y,z \in X$ satisfying $d(x,y)+d(y,z)+d(z,x) < 2\pi$, there exist $\overline{x},\overline{y},\overline{z} \in \mathbb{S}^2$ such that $d(x,y) = d_{\mathbb{S}^2}(\overline{x},\overline{y}), d(y,z) = d_{\mathbb{S}^2}(\overline{y},\overline{z})$ and $d(z,x) = d_{\mathbb{S}^2}(\overline{z},\overline{x})$. A point $\overline{p} \in [\overline{x},\overline{y}]$ is called a comparison point for $p \in [x,y]$ if $d_{\mathbb{S}^2}(\overline{x},\overline{p}) = d(x,p)$. If every p,q on the triangle $\triangle(x,y,z)$ with $d(x,y)+d(y,z)+d(z,x) < 2\pi$ and their comparison points $\overline{p},\overline{q} \in \triangle(\overline{x},\overline{y},\overline{z})$ satisfy that

$$d(p,q) \leq d_{\mathbb{S}^2}(\overline{p},\overline{q}),$$

X is called a CAT(1) space. In addition, if we take a compation triangle $\triangle(\overline{x},\overline{y},\overline{z})$ in $\mathbb{R}^2$ and use Euclid metric $d_{\mathbb{R}^2}$, we can also define a CAT(0) space.

Definition 2.1. Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and $\|\cdot\|$ be its norm. A real Hilbert sphere S_H is defined by $S_H := \{x \in H \mid \|x\| = 1\}$ and we can define metric function of S_H as $d(x,y) := \arccos\langle x,y\rangle$.

Then S_H is an example of complete CAT(1) space, and thus a nonempty closed convex subset of S_H is a complete CAT(1) space (see [2]). It is known that

$$\{z \in S_H \mid d(x,z) \leq d(y,z)\}$$

and

$$\{z \in S_H \mid \cos d(x,y) \cos d(y,z) \geq \cos d(x,z)\}$$

are closed convex subsets of S_H. If $x \neq y$, these two subsets are hemispheres of S_H . We refer more details and examples of CAT(1) space to [2].

Theorem 2.1. [15, p.5081, Lemma 2.1] *Let x,y,z be points in* CAT(1) *space such that $d(x,y)+d(y,z)+d(z,x) < 2\pi$. Let $v := tx \oplus (1-t)y$*

for some $t \in [0,1]$. *Then*

$$\begin{aligned} &\cos d(v,z)\sin d(x,y) \\ &\geq \cos d(x,z)\sin(td(x,y)) + \cos d(y,z)\sin((1-t)d(x,y)). \end{aligned}$$

Corollary 2.1. [16, p.4, Remark] *Let* x, y, z *be points in* CAT(1) *space such that* $d(x,y) + d(y,z) + d(z,x) < 2\pi$. *Let* $v := tx \oplus (1-t)y$ *for some* $t \in [0,1]$. *Then*

$$\cos d(v,z) \geq t\cos d(x,z) + (1-t)\cos d(y,z).$$

Let X be a complete CAT(1) space such that $d(v,v') < \pi/2$ for all $v, v' \in X$, and let C be a nonempty closed convex subset of X. Then for any $x \in X$, there exists a unique point $P_C x \in C$ such that

$$d(x, P_C x) = \inf_{y \in C} d(x,y).$$

That is, using similar techniques to the cace of Hilbert space, we can define metric projection P_C from X onto C such that $P_C x$ is the nearest point of C to x. Let X be a metric space and $\{x_n\}$ a bounded sequence of X. The asymptotic center $AC(\{x_n\})$ of $\{x_n\}$ is defined by

$$AC(\{x_n\}) := \left\{ z \mid \limsup_{n\to\infty} d(z,x_n) = \inf_{x\in X} \limsup_{n\to\infty} d(x,x_n) \right\}.$$

We say that $\{x_n\}$ is Δ-convergent to a point z if for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$, its asympotic center consists only of z, that is, $AC(\{x_{n_i}\}) = \{z\}$. Let X be a metric space. Let T be a mapping of X into itself. Then, T is said to be nonexpansive if $d(Tx,Ty) \leq d(x,y)$ for all $x, y \in X$. Hereafter we denote by

$$F(T) := \{z \mid Tz = z\}$$

the set of fixed points. Then T is said to be quasinonexpansive if $d(Tx,p) \leq d(x,p)$ for all $x \in X$ and $p \in F(T)$. Using similar techniques to the case of Hilbert space, we can prove that $F(T)$ is a closed convex subset of X. T is said to be strongly quasinonexpansive if it is quasinonexpansive, and for every $p \in F(T)$ and every sequence in X satisfying that $\sup_{n\in\mathbb{N}} d(x_n, p) < \pi/2$ and $\lim_{n\to\infty}(\cos d(x_n, p)/\cos d(Tx_n, p)) = 0$, it follows that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. T is said to be Δ-demiclosed if for any Δ-convergent sequence $\{x_n\}$ in X, its Δ-limit belongs to $F(T)$ whenever $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

The notation of W-mapping is originally proposed by Takahashi. We use the same notation in the setting of geodesic space as following:

Definition 2.2 (Takahashi [25]). Let X be a geodesic metric space. Let $T_1, T_2, \dots, T_r$ be a finite number of mappings of X into itself and $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 \le \alpha_i \le 1$ for every $i = 1, 2, \dots, r$. Then, we define a mapping W of X into itself as follows:

$$\begin{aligned}
U_1 &:= \alpha_1 T_1 \oplus (1-\alpha_1)I, \\
U_2 &:= \alpha_2 T_2 U_1 \oplus (1-\alpha_2)I, \\
&\cdots \\
U_r &:= \alpha_r T_r U_{r-1} \oplus (1-\alpha_r)I, \\
W &:= U_r.
\end{aligned}$$

Such a mapping W is called a W-mapping generated by $T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$.

The following lemmas are important for our main result.

Lemma 2.1. [16, p.8, Lemma 4.4] *Let T be a quasinonexpansive mapping defined on a* CAT(1) *space. For any real number $\alpha \in [0, 1]$, the mapping $\alpha T \oplus (1-\alpha)I$ is quasinonexpansive.*

The proof of Lemma 2.1 is essentially obtained in [16], so we omit the proof.

Lemma 2.2. [16, p.8, Lemma 4.5] *Let T be a nonexpansive mapping on a* CAT(1) *space. For a any real number $\alpha \in (0, 1]$, the mapping $\alpha T \oplus (1-\alpha)I$ is Δ-demiclosed.*

The following Lemma 2.3 is applied to show Theorem 3.1.

Lemma 2.3. [22, p.745, Lemma 2.6] *Let* $\{s_n\}, \{t_n\}$ *be sequences of real numbers such that* $s_n \geq 0$ *for every* $n \in \mathbb{N}$*. Let* $\{\gamma_n\}$ *be a sequence in* $(0, 1)$ *such that* $\sum_{n=0}^{\infty} \gamma_n = \infty$*. Suppose that* $s_{n+1} \leq (1 - \gamma_n)s_n + \gamma_n t_n$ *for every* $n \in \mathbb{N}$*. If* $\limsup_{j\to\infty} t_{n_j} \leq 0$ *for every subsequence* $\{n_j\}$ *of* $\mathbb{N}$ *satisfying* $\liminf_{j\to\infty}(s_{n_j+1} - s_{n_j}) \geq 0$*, then* $\lim_{n\to\infty} s_n = 0$*.*

Lemma 2.4. [4, p.417, Proposition 4.4] *Let* X *be a complete* CAT(1) *space, and* $\{x_n\}$ *be a sequence in* X*. If there exists* $x \in X$ *such that* $\limsup_{n\to\infty} d(x_n, x) < \pi/2$*, then* $\{x_n\}$ *has a* Δ*-convergent subsequence.*

Lemma 2.5. [9, p.447, Proposition 2.3] *Let* X *be a complete* CAT(1) *space and* $p \in X$*. If a sequence* $\{x_n\}$ *in* X *satisfies that* $\limsup_{n\to\infty} d(x_n, p) < \pi/2$ *and that* $\{x_n\}$ *is* Δ*-convergent to* $x \in X$*, then* $d(x, p) \leq \liminf_{n\to\infty} d(x_n, p)$*.*

Lemma 2.6. [16, p.3, Lemma 3.1] *Let* X *be a* CAT(1) *space such that* $d(v, v') < \pi/2$ *for every* $v, v' \in X$*. Let* $\alpha \in [0, 1]$ *and* $u, y, z \in X$*. Then*

$$\begin{aligned} &1 - \cos d(\beta u \oplus (1 - \beta)y, z) \\ &\leq (1 - \gamma)(1 - \cos d(y, z)) \\ &\qquad + \gamma \left(1 - \frac{\cos d(u, z)}{\sin d(u, y) \tan(2^{-1}\beta d(u, y)) + \cos d(u, y)}\right), \end{aligned}$$

where

$$\gamma := \begin{cases} 1 - \dfrac{\sin((1 - \beta)d(u, y))}{\sin(\beta d(u, y))} & (u \neq y), \\ \beta & (u = y). \end{cases}$$

Lemma 2.7. [15, p.5082, Theorem 3.1] *Let* X *be a complete* CAT(1) *space such that* $d(v, v') < \pi/2$ *for every* $v, v' \in X$ *and* $p \in X$*. If a sequence* $\{x_n\}$ *in* X *is* Δ*-convergent to* $x \in X$ *and* $\lim_{n\to\infty} d(x_n, p) = d(x, p)$*. Then* $\{x_n\}$ *is convergent to* x*.*

Lemma 2.8. [13, p.951, Remark 2.2] *If x_n is included in some closed convex subset of Hilbert sphere S_H and is Δ-convergent to $x \in S_H$, then $x \in C$.*

3. Halpern iteration

We begin this section with the following useful lemma.

Lemma 3.1. *If $\delta \in [0, \pi/2]$ satisfies*

$$\sin \delta \geq \sin(\alpha\delta) + \sin((1-\alpha)\delta)$$

for some $\alpha \in (0,1)$, then $\delta = 0$.

Proof. It is obtained by an elementary calculation. □

Next we study the set of fixed points of a W-mapping.

Proposition 3.1. *Let X be a* CAT(1) *space. Let $T_1, T_2, \ldots, T_r$ be quasinonexpansive mappings of X into itself such that $\bigcap_{i=1}^r F(T_i) \neq \emptyset$ and let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for every $i = 1, 2, \ldots, r$. Let W be the W-mappig of X into itself generated by $T_1, T_2, \ldots, T_r$ and $\alpha_1, \alpha_2, \ldots, \alpha_r$. Then, $F(W) = \bigcap_{i=1}^r F(T_i)$.*

Proof. It is obvious that $\bigcap_{i=1}^r F(T_i) \subset F(W)$. So, we shall prove $F(W) \subset \bigcap_{i=1}^r F(T_i)$. Let $z \in F(W)$ and $w \in \bigcap_{i=1}^r F(T_i)$. Then it follows that

$$0 = d(z,z) = d(Wz,z) = d(\alpha_r T_r U_{r-1} z \oplus (1-\alpha_r) z, z) = \alpha_r d(z, T_r U_{r-1} z).$$

Since $0 < \alpha_r \leq 1$, we obtain $z = T_r U_{r-1} z$ and hence

$$\begin{aligned}
\cos d(z,w) &= \cos d(T_r U_{r-1} z, w) \\
&\geq \cos d(U_{r-1} z, w) \\
&= \cos d(\alpha_{r-1} T_{r-1} U_{r-2} z \oplus (1-\alpha_{r-1}) z, w)
\end{aligned}$$

By Corollary 2.1,

$$\geq \alpha_{r-1} \cos d(T_{r-1}U_{r-2}z, w) + (1 - \alpha_{r-1}) \cos d(z, w)$$

and similary,

$$\begin{aligned}
&\geq \alpha_{r-1} \cos d(U_{r-2}z, w) + (1 - \alpha_{r-1}) \cos d(z, w) \\
&\geq \alpha_{r-1} \cos d(\alpha_{r-2}T_{r-2}U_{r-3}z \oplus (1 - \alpha_{r-2})z, w) + (1 - \alpha_{r-1}) \cos d(z, w) \\
&\geq \alpha_{r-1}\alpha_{r-2} \cos d(T_{r-2}U_{r-3}z, w) + (1 - \alpha_{r-1}\alpha_{r-2}) \cos d(z, w) \\
&\geq \cdots \\
&\geq \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2 \cos d(T_2U_1z, w) + (1 - \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2) \cos d(z, w) \\
&\geq \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2 \cos d(U_1z, w) + (1 - \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2) \cos d(z, w) \\
&\geq \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2 \cos d(\alpha_1T_1z \oplus (1 - \alpha_1)z, w) \\
&\quad + (1 - \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2) \cos d(z, w) \\
&\geq \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2\alpha_1 \cos d(T_1z, w) + (1 - \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2\alpha_1) \cos d(z, w) \\
&\geq \cos d(z, w).
\end{aligned}$$

Then it follows that

$$d(z, w) = \cos d(T_1z, w) = d(U_1z, w) = d(\alpha_1T_1z \oplus (1 - \alpha_1)z, w).$$

By Theorem 2.1,

$$\begin{aligned}
&\cos d(\alpha_1T_1z \oplus (1 - \alpha_1)z, w) \sin d(T_1z, z) \\
&\geq \cos d(T_1z, w) \sin(\alpha_1 d(T_1z, z)) + \cos d(z, w) \sin((1 - \alpha_1)d(T_1z, z)).
\end{aligned}$$

By, Lemma 3.1 $d(T_1z, z) = 0$, Thus we obtain $T_1z = z$. Similarly, we have

$$d(z, w) = d(T_2U_1z, w) = d(U_2z, w) = d(\alpha_2T_2U_1z \oplus (1 - \alpha_2)z, w).$$

By Theorem 2.1,

$$\begin{aligned}&\cos d(\alpha_2 T_2 U_1 z \oplus (1-\alpha_2)z, w)\sin d(T_2 U_1 z, z)\\&\geq \cos d(T_2 U_1 z, w)\sin(\alpha_2 d(T_2 U_1 z, z))\\&\quad + \cos d(z, w)\sin((1-\alpha_2)d(T_2 U_1 z, z)).\end{aligned}$$

By Lemma 3.1, we obtain $T_2U_1z = z$. Since $U_1z = z$, we obtain $T_2z = z$. Using such techniques, we obtain $T_iz = z$ and $U_iz = z$ for all $i = 1, 2, \dots, r$, and hence $z \in \bigcap_{i=1}^{r} F(T_i)$. This implies $F(W) \subset \bigcap_{i=1}^{r} F(T_i)$. Therefore we have $F(W) = \bigcap_{i=1}^{r} F(T_i)$. □

Remark 3.1. Let W_n be the W-mappings of X into itself generated by $T_1, T_2, \dots, T_r$ and $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ for $n \in \mathbb{N}$. By Proposition 3.1, all the sets of fixed points $\{F(W_n)\}$ is identical.

The following Lemma 3.2 is essentially given by Kasahara [11]. For the sake of completeness, we give the proof.

Lemma 3.2. [11, p.8, Lemma 3.6] *Let $\{S_n\}$ be a sequence of quasinonexpansive mappings of a* CAT(1) *space X into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. Then for given real numbers $\alpha_n \in [a, 1-a] \subset (0,1)$ and $p \in \bigcap_{n=1}^{\infty} F(S_n)$, if $\{x_n\}$ satisfies that $\sup_{n\in\mathbb{N}} d(x_n, p) < \pi/2$ and*

$$\lim_{n\to\infty} \frac{\cos d(x_n, p)}{\cos d(\alpha_n S_n x_n \oplus (1-\alpha_n)x_n, p)} = 1,$$

then $\lim_{n\to\infty} d(S_n x_n, x_n) = 0$.

Proof. Let $\delta_n := d(S_n x_n, x_n)$. Assume that $\{x_n\} \subset X$ and $p \in \bigcap_{n=1}^{\infty} F(S_n)$ such that $\sup_{n\in\mathbb{N}} d(x_n, p) < \pi/2$ and $\lim_{n\to\infty}(\cos d(x_n, p)/\cos d(\alpha_n S_n x_n \oplus$

$(1-\alpha_n)x_n, p)) = 1$. Then by Theorem 2.1, we have

$$\begin{aligned}
&\cos d(\alpha_n S_n x_n \oplus (1-\alpha_n)x_n, p) \sin d(S_n x_n, x_n) \\
&\geq \cos d(S_n x_n, p) \sin(\alpha d(S_n x_n, x_n)) + \cos d(x_n, p) \sin((1-\alpha_n) d(S_n x_n, x_n)) \\
&\geq \min\{\cos d(S_n x_n, p), \cos d(x_n, p)\}(\sin(\alpha_n d(S_n x_n, x_n)) \\
&\quad + \sin((1-\alpha_n) d(S_n x_n, x_n))) \\
&= 2 \cos d(x_n, p) \sin \frac{d(S_n x_n, x_n)}{2} \cos \frac{(2\alpha_n - 1) d(S_n x_n, x_n)}{2}.
\end{aligned}$$

Hence

$$\cos d(\alpha_n S_n x_n \oplus (1-\alpha_n)x_n, p) \sin \delta_n \geq 2 \cos d(x_n, p) \sin \frac{\delta_n}{2} \cos \frac{(2\alpha_n - 1)\delta_n}{2}.$$

We assume that $\delta_n \neq 0$. Dividing above by $2\sin(\delta_n/2)$, we have

$$\begin{aligned}
\cos d(\alpha_n S_n x_n \oplus (1-\alpha_n)x_n, p) \cos \frac{\delta_n}{2} &\geq \cos d(x_n, p) \cos \frac{(2\alpha_n - 1)\delta_n}{2} \\
&\geq \cos d(x_n, p) \cos \frac{(1-2a)\delta_n}{2}.
\end{aligned}$$

Moreover, dividing above by $\cos((1-2a)\delta_n/2)$, we have

$$\cos d(x_n, p) \leq \cos d(\alpha S_n x_n \oplus (1-\alpha_n)x_n, p) \frac{\cos \frac{\delta_n}{2}}{\cos \frac{(1-2a)\delta_n}{2}}.$$

Then

$$\begin{aligned}
&\cos d(x_n, p) \\
&\leq \cos d(\alpha_n S_n x_n \oplus (1-\alpha_n)x_n, p) \frac{\cos \frac{(1-2a)\delta_n}{2} \cos(a\delta_n) - \sin \frac{(1-2a)\delta_n}{2} \sin(a\delta_n)}{\cos \frac{(1-2a)\delta_n}{2}} \\
&\leq \cos d(\alpha_n S_n x_n \oplus (1-\alpha_n)x_n, p) \cos(a\delta_n).
\end{aligned}$$

Thus we have that

$$\cos d(a\delta_n) \geq \frac{\cos d(x_n, p)}{\cos d(\alpha_n S_n x_n \oplus (1-\alpha_n)x_n, p)} \to 1 \ (n \to \infty),$$

which implies $\lim_{n\to\infty} \delta_n = 0$, that is, $\lim_{n\to\infty} d(S_n x_n, x_n) = 0$. □

The following is a first result. It is essential to use a Lemma 2.3. In order to show that assumptions of Lemma 2.3 are satisfied, we find good approximate sequence $\{y_j^{(k)}\}$.

Theorem 3.1. *Let X be a complete $\mathrm{CAT}(1)$ space such that $d(v,v') < \pi/2$ for every $v, v' \in X$. Let $T_1, T_2, \dots, T_r$ be a finite number of quasi-nonexpansive and Δ-demiclosed mappings of X into itself such that $F := \bigcap_{i=1}^r F(T_i) \neq \emptyset$, and let $\alpha_{n,1}, a_{n,2}, \dots, \alpha_{n,r}$ be real numbers for $n \in \mathbb{N}$ such that $\alpha_{n,i} \in [a, 1-a]$ for every $i = 1, 2, \dots, r$, where $0 < a < 1/2$. Let W_n be the W-mappings of X into itself generated by $T_1, T_2, \dots, T_r$ and $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ for $n \in \mathbb{N}$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 < \beta_n < 1$ for every $n \in \mathbb{N}, \lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^\infty \beta_n = \infty$. For a given points $u, x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by*

$$x_{n+1} = \beta_n u \oplus (1-\beta_n) W_n x_n$$

for $n \in \mathbb{N}$. Suppose that one of the following conditions holds:

(a) $\sup_{v,v' \in X} d(v,v') < \pi/2$;

(b) $d(u, P_F u) < \pi/4$ *and* $d(u, P_F u) + d(x_1, P_F u) < \pi/2$;

(c) $\sum_{n=1}^\infty \beta_n^2 = \infty$.

Then $\{x_n\}$ converges to $P_F u$.

Proof. Let $p := P_F u$ and let

$$s_n := 1 - \cos d(x_n, p),$$

$$t_n := 1 - \frac{\cos d(u,p)}{\sin d(u, W_n x_n) \tan(2^{-1}\beta_n d(u, W_n x_n)) + \cos d(u, W_n x_n)},$$

$$\gamma_n := \begin{cases} 1 - \dfrac{\sin((1-\beta_n) d(u, W_n x_n))}{\sin(\beta_n d(u, W_n x_n))} & (u \neq W_n x_n), \\ \beta_n & (u = W_n x_n) \end{cases}$$

for $n \in \mathbb{N}$. If $\{s_n\}, \{t_n\}$ and $\{\gamma_n\}$ satisfy the conditions of Lemma 2.3, then we will have $\lim_{n\to\infty} s_n = 0$, that is, $\{x_n\}$ converges to $p = P_F u$.

Thus the proof of Theorem 3.1 will be completed. First, it is obvious that $s_n \geq 0$. By Lemma 2.1, W_n is quasinonexpansive. Then, it follows from Lemma 2.6 that

$$s_{n+1} \leq (1-\gamma_n)(1-\cos d(W_n x_n, p)) + \gamma_n t_n \leq (1-\gamma_n)s_n + \gamma_n t_n$$

for every $n \in \mathbb{N}$. Now, it is also obvious that $\{\gamma_n\}$ is a sequence in $(0,1)$. We show that $\sum_{n=1}^{\infty} \gamma_n = \infty$ holds under each condition (a),(b) and (c). We have

$$\begin{aligned}\cos d(x_{n+1}, p) &= \cos d(\beta_n u \oplus (1-\beta_n)W_n x_n, p)\\ &\geq \beta_n \cos d(u,p) + (1-\beta_n)\cos d(W_n x_n, p)\\ &\geq \beta_n \cos d(u,p) + (1-\beta_n)\cos d(x_n, p)\\ &\geq \min\{\cos d(u,p), \cos d(x_n,p)\}\end{aligned}$$

for all $n \in \mathbb{N}$. Thus we have

$$\begin{aligned}\cos d(x_n, p) &\geq \min\{\cos d(u,p), \cos d(x_1,p)\}\\ &= \cos \max\{d(u,p), d(x_1,p)\}\\ &> 0\end{aligned}$$

for all $n \in \mathbb{N}$ and hence $\sup_{n\in\mathbb{N}} d(x_n,p) \leq \max\{d(u,p), d(x_1,p)\} < \pi/2$. For the case of (a) and (b), let $M = \sup_{n\in\mathbb{N}} d(u, W_n x_n)$. Then we show that $M < \pi/2$. For (a), it is trivial. For (b), since $\sup_{n\in\mathbb{N}} d(x_n,p) \leq \max\{d(u,p), d(x_1,p)\}$, we have

$$\begin{aligned}M &= \sup_{n\in\mathbb{N}} d(u, W_n x_n)\\ &\leq \sup_{n\in\mathbb{N}} (d(u,p) + d(p, W_n x_n))\\ &\leq \sup_{n\in\mathbb{N}} (d(u,p) + d(p, x_n))\\ &\leq \max\{2d(u,p), d(u,p) + d(x_1,p)\}\\ &< \frac{\pi}{2}.\end{aligned}$$

Thus, in each case of (a) and (b), we have

$$\begin{aligned}\gamma_n &\geq 1 - \frac{\sin((1-\beta_n)M)}{\sin M} \\ &= \frac{2}{\sin M} \sin\left(\frac{\beta_n}{2} M\right) \cos\left(\left(1 - \frac{\beta_n}{2}\right) M\right) \\ &\geq \beta_n \cos M.\end{aligned}$$

Since $\sum_{n=1}^{\infty} \beta_n = \infty$, it follows that $\sum_{n=1}^{\infty} \gamma_n = \infty$. For the case of (c), we have

$$\gamma_n \geq 1 - \sin\frac{(1-\beta_n)\pi}{2} = 1 - \cos\frac{\beta_n \pi}{2} \geq \frac{\beta_n^2 \pi^2}{16}$$

for every $n \in \mathbb{N}$. Therefore, in the case of (c) we also have $\sum_{n=1}^{\infty} \gamma_n = \infty$. Finally, we show that $\limsup_{j\to\infty} t_{n_j} \leq 0$ for any subsequence $\{n_j\}$ of $\mathbb{N}$ with $\liminf_{j\to\infty}(s_{n_j+1} - s_{n_j}) \geq 0$. Let $\{s_{n_j}\}$ be a subsequence of $\{s_n\}$ satisfying that $\liminf_{j\to\infty}(s_{n_j+1} - s_{n_j}) \geq 0$, and put

$$\alpha := \min_{k=1,\dots,r} \left(\inf_{n\in\mathbb{N}} \alpha_{n,k} \right).$$

Then we have

$$\begin{aligned}0 &\leq \liminf_{j\to\infty}(s_{n_j+1} - s_{n_j}) \\ &= \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(x_{n_j+1}, p)) \\ &= \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(\beta_{n_j} u \oplus (1-\beta_{n_j}) W_{n_j} x_{n_j}, p))\end{aligned}$$

By Corollary 2.1,

$$\leq \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - (\beta_{n_j} \cos d(u, p) + (1-\beta_{n_j}) \cos d(W_{n_j} x_{n_j}, p))$$

Since $\lim_{n\to\infty} \beta_n = 0$,

$$= \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(W_{n_j} x_{n_j}, p))$$

By Definition 2.2,

$$= \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,r} T_r U_{n_j,r-1} x_{n_j} \oplus (1-\alpha_{n_j,r}) x_{n_j}, p))$$

By Corollary 2.1,

$$\begin{aligned}
&\le \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - (\alpha_{n_j,r} \cos d(T_r U_{n_j,r-1} x_{n_j}, p) + (1-\alpha_{n_j,r}) \cos d(x_{n_j}, p)))\\
&= \liminf_{j\to\infty}(\alpha_{n_j,r} \cos d(x_{n_j}, p) - \alpha_{n_j,r} \cos d(T_r U_{n_j,r-1} x_{n_j}, p))\\
&\le \alpha \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(T_r U_{n_j,r-1} x_{n_j}, p))
\end{aligned}$$

Since T_r is quasinonexpansive,

$$\begin{aligned}
&\le \alpha \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(U_{n_j,r-1} x_{n_j}, p))\\
&= \alpha \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,r-1} T_{r-1} U_{n_j,r-2} x_{n_j} \oplus (1-\alpha_{n_j,r-1}) x_{n_j}, p))\\
&\le \alpha \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - (\alpha_{n_j,r-1} \cos d(T_{r-1} U_{n_j,r-2} x_{n_j}, p) + (1-\alpha_{n_j,r-1}) \cos d(x_{n_j}, p)))\\
&= \alpha \liminf_{j\to\infty}(\alpha_{n_j,r-1} \cos d(x_{n_j}, p) - \alpha_{n_j,r-1} \cos d(T_{r-1} U_{n_j,r-2} x_{n_j}, p))\\
&\le \alpha^2 \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(T_{r-1} U_{n_j,r-2} x_{n_j}, p))\\
&\le \cdots\\
&\le \alpha^{r-1} \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(T_2 U_{n_j,1} x_{n_j}, p))\\
&\le \alpha^{r-1} \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(U_{n_j,1} x_{n_j}, p))\\
&= \alpha^{r-1} \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,1} T_1 x_{n_j} \oplus (1-\alpha_{n_j,1}) x_{n_j}, p))\\
&\le \alpha^{r-1} \limsup_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,1} T_1 x_{n_j} \oplus (1-\alpha_{n_j,1}) x_{n_j}, p))\\
&\le 0.
\end{aligned}$$

Thus we have

$$\lim_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,1} T_1 x_{n_j} \oplus (1-\alpha_{n_j,1}) x_{n_j}, p)) = 0.$$

Using the inequality $\sup_{j\in\mathbb{N}} d(x_{n_j}, p) < \pi/2$, we also have

$$\lim_{j\to\infty} \frac{\cos d(x_{n_j}, p)}{\cos d(\alpha_{n_j,1} T_1 x_{n_j} \oplus (1-\alpha_{n_j,1}) x_{n_j}, p)} = 1.$$

By Lemma 3.2, it follows that

$$\lim_{j\to\infty} d(T_1 x_{n_j}, x_{n_j}) = 0.$$

Put

$$y_j^{(k)} := U_{n_j,k} x_{n_j}$$

for $k = 1, 2, \dots, r-1$. We show that

$$\lim_{j\to\infty} d(x_{n_j}, y_j^{(k)}) = 0, \quad \lim_{j\to\infty} d(T_{k+1} y_j^{(k)}, y_j^{(k)}) = 0$$

by induction on $k = 1, 2, \dots, r-1$. First, we consider the case $k = 1$. We have

$$\begin{aligned}
\lim_{j\to\infty} d(x_{n_j}, y_j^{(1)}) &= \lim_{j\to\infty} d(x_{n_j}, U_{n_j,1} x_{n_j}) \\
&= \lim_{j\to\infty} d(x_{n_j}, \alpha_{n_j,1} T_1 x_{n_j} \oplus (1-\alpha_{n_j,1}) x_{n_j}) \\
&= \lim_{j\to\infty} \alpha_{n_j} d(T_1 x_{n_j}, x_{n_j}) \\
&= 0.
\end{aligned}$$

On the other hand, by the calculation above we have

$$\begin{aligned}
0 &\le \liminf_{j\to\infty} (\cos d(x_{n_j}, p) - \cos d(U_{n_j,2} x_{n_j}, p)) \\
&= \liminf_{j\to\infty} (\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,2} T_2 U_{n_j,1} x_{n_j} \oplus (1-\alpha_{n_j,2}) x_{n_j}, p)) \\
&\le \limsup_{j\to\infty} (\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,2} T_2 U_{n_j,1} x_{n_j} \oplus (1-\alpha_{n_j,2}) x_{n_j}, p)) \\
&\le 0.
\end{aligned}$$

Therefore

$$\lim_{j\to\infty} (\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,2} T_2 U_{n_j,1} x_{n_j} \oplus (1-\alpha_{n_j,2}) x_{n_j}, p)) = 0.$$

Using the inequality $\sup_{j\in\mathbb{N}} d(x_{n_j}, p) < \pi/2$, we also have

$$\lim_{j\to\infty} \frac{\cos d(x_{n_j}, p)}{\cos d(\alpha_{n_j,2} T_2 U_{n_j,1} x_{n_j} \oplus (1-\alpha_{n_j,2}) x_{n_j}, p)} = 1.$$

By Lemma 3.2, and since $\lim_{j\to\infty} d(x_{n_j}, y_j^{(1)}) = 0$,

$$\lim_{j\to\infty} d(T_2 y_j^{(1)}, y_j^{(1)}) \leq \lim_{j\to\infty} (d(T_2 y_j^{(1)}, x_{n_j}) + d(x_{n_j}, y_j^{(1)})) = 0.$$

Hence we have that case $k = 1$, that is,

$$\lim_{j\to\infty} d(x_{n_j}, y_j^{(1)}) = 0, \quad \lim_{j\to\infty} d(T_2 y_j^{(1)}, y_j^{(1)}) = 0.$$

holds. Next, assume the hypothesis with $k = l$, that is,

$$\lim_{j\to\infty} d(x_{n_j}, y_j^{(l)}) = 0, \quad \lim_{j\to\infty} d(T_{l+1} y_j^{(l)}, y_j^{(l)}) = 0$$

holds. Then by assumption, we have

$$\begin{aligned}
\lim_{j\to\infty} d(x_{n_j}, y_j^{(l+1)}) &= \lim_{j\to\infty} d(x_{n_j}, U_{n_j,l+1} x_{n_j}) \\
&= \lim_{j\to\infty} d(x_{n_j}, \alpha_{n_j,l+1} T_{l+1} U_{n_j,l} x_{n_j} \oplus (1 - \alpha_{n_j,l+1}) x_{n_j}) \\
&= \lim_{j\to\infty} d(x_{n_j}, \alpha_{n_j,l+1} T_{l+1} y_j^{(l)} \oplus (1 - \alpha_{n_j,l+1}) x_{n_j}) \\
&= \lim_{j\to\infty} d(x_{n_j}, \alpha_{n_j,l+1} y_j^{(l)} \oplus (1 - \alpha_{n_j,l+1}) x_{n_j}) \\
&= \lim_{j\to\infty} \alpha_{n_j,l+1} d(x_{n_j}, y_j^{(l)}) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq \liminf_{j\to\infty} (\cos d(x_{n_j}, p) - \cos d(U_{n_j,l+2} x_{n_j}, p)) \\
&= \liminf_{j\to\infty} (\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,l+2} T_{l+2} U_{n_j,l+1} x_{n_j} \oplus (1 - \alpha_{n_j,l+1}) x_{n_j}, p)) \\
&= \limsup_{j\to\infty} (\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,l+2} T_{l+2} U_{n_j,l+1} x_{n_j} \oplus (1 - \alpha_{n_j,l+1}) x_{n_j}, p)) \\
&\leq 0.
\end{aligned}$$

Therefore

$$\lim_{j\to\infty} (\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,l+2} T_{l+2} U_{n_j,l+1} x_{n_j} \oplus (1 - \alpha_{n_j,l+1}) x_{n_j}, p)) = 0.$$

Using inequality $\sup_{j\in\mathbb{N}} d(x_{n_j} p) < \pi/2$, we have

$$\lim_{j\to\infty} \frac{\cos d(x_{n_j}, p)}{\cos d(\alpha_{n_j,l+2} T_{l+2} U_{n_j,l+1} x_{n_j} \oplus (1 - \alpha_{n_j,l+1}) x_{n_j}, p)} = 1.$$

Since $\lim_{j\to\infty} d(x_{n_j}, y_j^{(l+1)}) = 0$ and by Lemma 3.2, we have

$$\lim_{j\to\infty} d(T_{l+2}y_j^{(l+1)}, y_j^{(l+1)}) = \lim_{j\to\infty} d(T_{l+2}y_j^{(l+1)}, x_{n_j})$$
$$= \lim_{j\to} d(T_{l+2}U_{n_j,l+1}x_{n_j}, x_{n_j}) = 0.$$

So, we have the hypothesis $k = l+1$, that is,

$$\lim_{j\to\infty} d(x_{n_j}, y_j^{(l+1)}) = 0, \quad \lim_{j\to\infty} d(T_{l+2}y_j^{(l+1)}, y_j^{(l+1)}) = 0$$

for $k = 1, 2, \ldots, r-1$. By induction, we obtain

$$\lim_{j\to\infty} d(x_{n_j}, y_j^{(k)}) = 0, \quad \lim_{j\to\infty} d(T_{k+1}y_j^{(k)}, y_j^{(k)}) = 0$$

for all $k = 1, 2, \ldots, r-1$. By Lemma 2.4, let $\{x_{n_{j_k}}\}$ be a Δ-convergent subsequence of $\{x_{n_j}\}$ with the Δ-limit z such that $\lim_{k\to\infty} d(u, x_{n_{j_k}}) = \liminf_{j\to\infty} d(u, x_{n_j})$. Then, since T_1 is Δ-demiclosed and $\lim_{j\to\infty} d(x_{n_j}, T_1x_{n_j}) = 0$, the Δ-limit z of $\{x_{n_{j_k}}\}$ belongs to $F(T_1)$. Similarly, since T_2 is Δ-demiclosed and $\lim_{j\to\infty} d(x_{n_j}, y_j^{(1)}) = \lim_{j\to\infty} d(y_j^{(1)}, T_2y_j^{(1)}) = 0$, $\{y_{j_k}^{(1)}\}$ is Δ-convergent to z and the Δ-limit z is belongs to $F(T_2)$. Using such techniques, we obtain $z \in F(T_i)$ for all $i = 1, 2, \ldots r$, and hence $z \in \bigcap_{i=1}^r F(T_i) = F$. Using Lemma 2.5 and the definition of the metric projection, we have

$$\begin{aligned}
\liminf_{j\to\infty} d(u, W_{n_j}x_{n_j}) &= \liminf_{j\to\infty} d(u, \alpha_{n_j,r}T_rU_{n_j,r-1}x_{n_j} \oplus (1-\alpha_{n_j})x_{n_j}) \\
&= \liminf_{j\to\infty} d(u, \alpha_{n_j,r}T_ry_j^{(r-1)} \oplus (1-\alpha_{n_j})x_{n_j}) \\
&= \liminf_{j\to\infty} d(u, \alpha_{n_j,r}y_j^{(r-1)} \oplus (1-\alpha_{n_j})x_{n_j}) \\
&= \liminf_{j\to\infty} d(u, \alpha_{n_j,r}x_{n_j} \oplus (1-\alpha_{n_j})x_{n_j}) \\
&= \liminf_{j\to\infty} d(u, x_{n_j}) \\
&= \lim_{k\to\infty} d(u, x_{n_{j_k}}) \\
&\geq d(u, z) \\
&\geq d(u, P_Fu).
\end{aligned}$$

Therefore, we obtain

$$
\begin{aligned}
&\limsup_{j\to\infty} t_{n_j} \\
&= \limsup_{j\to\infty} \left(1 - \frac{\cos d(u,p)}{\sin d(u, W_{n_j} x_{n_j}) \tan(2^{-1}\beta_{n_j} d(u, W_{n_j} x_{n_j})) + \cos d(u, W_{n_j} x_{n_j})}\right) \\
&= \limsup_{j\to\infty} \left(1 - \frac{\cos d(u,p)}{0 + \cos d(u, W_{n_j} x_{n_j})}\right) \\
&= 1 - \frac{\cos d(u,p)}{\cos(\liminf_{j\to\infty} d(u, W_{n_j} x_{n_j}))} \\
&\leq 1 - \frac{\cos d(u,p)}{\cos d(u,z)} \\
&\leq 0.
\end{aligned}
$$

By Lemma 2.3, we have that $\lim_{n\to\infty} s_n = 0$, that is, $\{x_n\}$ converges to $p = P_F u$, and we finish the proof. ☐

Remark 3.2. By Lemma 2.2, a nonexpansive mapping defined on a CAT(1) space having a fixed point is quasinonexpansive and Δ-demiclosed.

Remark 3.3. *In general, even if $T_1, T_2, \ldots, T_r$ are nonexpansive, the W-mapping generated by $T_1, T_2, \ldots, T_r$ and $\alpha_1, \alpha_2, \ldots, \alpha_r$ is not necessarily nonexpansive.*

4. Applications of Halpern iteration

Let us recall some basic notation about functions on metric space. Let X be a geodesic metric space and let f be a function from X into $(-\infty, \infty]$. We say f is lower semicontinuous if the set $\{x \in X \mid f(x) \leq a\}$ is closed for all $a \in \mathbb{R}$. The function f is said to be proper if the set $\{x \in X \mid f(x) \neq \infty\}$ is nonempty. We say f is convex if

$$f(tx \oplus (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in X$ and $t \in (0,1)$. Let X be a complete CAT(1) space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let f be a proper lower semicontinuous convex function from X into $(-\infty, \infty]$. For f, we can define a map $J_f : X \to X$ called a resolvent. One of resolvent of f is defined by

$$R_f x := \operatorname*{argmin}_{y \in X} \{ f(y) + \tan d(y,x) \sin d(y,x) \} \tag{4.1}$$

in [14]. Another type of a resolvent of f is defined by

$$R_f x := \operatorname*{argmin}_{y \in X} \{ f(y) - \log \cos d(y,x) \} \tag{4.2}$$

in [10], where $\operatorname{argmin}_X f := \{ y \in X \mid f(y) = \min_X f \}$ is a minimizer set of f. Both resolvents are well-defined, quasinonexpansive, Δ-demiclosed, and satisfy $F(R_f) = \operatorname{argmin}_X f$ ([14, 10]). So, we can approximate a common minimizer of a finite number of functions by the following theorem.

Theorem 4.1. *Let X be a complete* $\mathrm{CAT}(1)$ *space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let $f_1, f_2, \dots, f_r$ be a finite number of convex function from X into $(-\infty, \infty]$ such that $F := \bigcap_{i=1}^r \operatorname{argmin}_X f_i \neq \emptyset$, and let $\alpha_{n,1}, a_{n,2}, \dots, \alpha_{n,r}$ be real numbers for $n \in \mathbb{N}$ such that $\alpha_{n,i} \in [a, 1-a]$ for every $i = 1, 2, \dots, r$, where $0 < a < 1/2$. Let R_{f_i} be one of resolvent defined by either* (4.1) *or* (4.2) *for $i = 1, 2, \dots, r$. Let W_n be the W-mappings of X into itself generated by $R_{f_1}, R_{f_2}, \dots, R_{f_r}$ and $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ for $n \in \mathbb{N}$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 < \beta_n < 1$ for every $n \in \mathbb{N}, \lim_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^\infty \beta_n = \infty$. For given points $u, x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by*

$$x_{n+1} = \beta_n u \oplus (1 - \beta_n) W_n x_n$$

for $n \in \mathbb{N}$. Suppose that one of the following conditions holds:

(a) $\sup_{v,v' \in X} d(v, v') < \pi/2$;

(b) $d(u, P_F u) < \pi/4$ *and* $d(u, P_F u) + d(x_1, P_F u) < \pi/2$;

(c) $\sum_{n=1}^\infty \beta_n^2 = \infty$.

Then $\{x_n\}$ converges to $P_F u$.

Let us consider a more specialized situation. For a closed convex subset C of a complete CAT(1) space X, put

$$i_C(x) := \begin{cases} 0 & (x \in C) \\ \infty & (x \notin C). \end{cases}$$

This function i_C is a proper lower semicontinuous convex function. Thus the resolvent R_{i_C} of i_C is defined by either (4.1) or (4.2), and it is quasinonexpansive and Δ-demiclosed. In fact, we know $R_{i_C} = P_C$ and $F(R_{i_C}) = \operatorname{argmin} i_C = C$ for both definitions (4.1) and (4.2). Thus we can apply Theorem 3.1 and have an approximation of the nearest point in the intersection of finite family of closed convex subsets from a given point by using corresponding metric projection of each subset by the following theorem.

Theorem 4.2. *Let X be a complete* CAT(1) *space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let $C_1, C_2, \ldots, C_r$ be a finite number of closed convex subset of X such that $C := \bigcap_{i=1}^r C_i \neq \emptyset$, and let $\alpha_{n,1}, a_{n,2}, \ldots, \alpha_{n,r}$ be real numbers for $n \in \mathbb{N}$ such that $\alpha_{n,i} \in [a, 1-a]$ for every $i = 1, 2, \ldots, r$, where $0 < a < 1/2$. Let W_n be the W-mappings of X into itself generated by $P_{C_1}, P_{C_2}, \ldots, P_{C_r}$ and $\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,r}$ for $n \in \mathbb{N}$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 < \beta_n < 1$ for every $n \in \mathbb{N}, \lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. For a given points $u, x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by*

$$x_{n+1} = \beta_n u \oplus (1 - \beta_n) W_n x_n$$

for $n \in \mathbb{N}$. Suppose that one of the following conditions holds:

(a) $\sup_{v,v' \in X} d(v, v') < \pi/2$;

(b) $d(u, P_C u) < \pi/4$ *and* $d(u, P_C u) + d(x_1, P_C u) < \pi/2$;

(c) $\sum_{n=1}^{\infty} \beta_n^2 = \infty$.

Then $\{x_n\}$ converges to $P_C u$.

In the introduction we mentioned that there existed an example which is quasinonexpansive but not strongly quasinonexpansive. The following is such an example.

Example 4.1. A closed interval $[-1,1]$ is a complete CAT(1) space. Let $T:[-1,1]\to[-1,1]$ be defined by $Tx:=-x$. Then $F(T)=\{0\}$. It is easy to obtain that T is quasinonexpansive and Δ-demiclosed but it is not strongly quasinonexpansive.

5. CQ projection method and Shrinking projection method

In this section, we consider the CQ projection method and shrinking projection method for a finite family of quasinonexpansive mappings and then prove a strong convergence theorem to their common fixed point in a real Hilbert sphere. A real Hilbert sphere is an nexample of CAT(1) space. It is essential to show that $lim_{n\to\infty}d(T_ix_n,x_n)=0$ for every $i=1,2,\dots,r$ for both Theorems 5.1 and 5.2. For that we used the Theorem 2.1. This idea is an original approach in this result.

Theorem 5.1. *Let C be a closed convex subset in real Hilbert shpere S_H such that $d(v,v')<\pi/2$ for every $v,v'\in C$. Let $\alpha_{n,1},a_{n,2},\dots,\alpha_{n,r}$ be real numbers for $n\in\mathbb{N}$ such that $\alpha_{n,i}\in[a,1-a]$ for every $i=1,2,\dots,r$ where $0<a<1/2$, and let $T_1,T_2,\dots,T_r$ be a finite number of nonexpansive mappings of C into itself such that $F:=\bigcap_{i=1}^{r}F(T_i)\neq\emptyset$. Let W_n be the W-mappings of X into itself generated by $T_1,T_2,\dots,T_r$ and $\alpha_{n,1},\alpha_{n,2},\dots,\alpha_{n,r}$ for $n\in\mathbb{N}$. For a given point $x_1\in C$, let $\{x_n\}$*

be a sequence in C generated by

$$\begin{aligned} y_n &:= W_n x_n, \\ C_n &:= \{z \in C \mid d(y_n, z) \leq d(x_n, z)\}, \\ Q_n &:= \{z \in C \mid \cos d(x_1, x_n) \cos d(x_n, z) \geq \cos d(x_1, z)\}, \\ x_{n+1} &:= P_{C_n \cap Q_n} x_1 \end{aligned}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ is well defined and convergent to $P_F x_1$.

Proof. First, we show that $\{x_n\}$ is well defined, that is, we show that $C_n \cap Q_n$ is nonempty closed convex subset. By the definition, C_n and Q_n are closed subsets. Since C_n and Q_n are hemisphere, C_n and Q_n are convex subsets. Thus $C_n \cap Q_n$ is closed convex subset in C. In order to show that it is nonempty, we show $F \subset C_n \cap Q_n$ by induction for $n \in \mathbb{N}$. Since $C_1 = Q_1 = C$, we have $F \subset C_1 \cap Q_1$, and $C_1 \cap Q_1$ is nonempty closed convex subset. We assume the induction hypothesis that $F \subset C_k \cap Q_k$ and show $F \subset C_{k+1} \cap Q_{k+1}$. For all $z \in F$, by Lemma 2.1, W_n is quasinonexpansive. By Proposition 3.1, it follows $d(W_{k+1} x_{k+1}, z) \leq d(x_{k+1}, z)$ and thus $z \in C_{k+1}$. By induction hypothesis, $z \in C_k \cap Q_k$, and therefore, for any $t \in [0, 1]$, $tz \oplus (1-t) x_{k+1} = tz \oplus (1-t) P_{C_k \cap Q_k} x_1 \in C_k \cap Q_k$. Then we have

$$\begin{aligned} &2 \cos d(x_1, x_{k+1}) \cos\left(\left(1 - \frac{t}{2}\right) d(x_{k+1}, z)\right) \sin\left(\frac{t}{2} d(x_{k+1}, z)\right) \\ &= \cos d(x_1, x_{k+1})(\sin d(x_{k+1}, z) - \sin((1-t) d(x_{k+1}, z))) \\ &= \cos d(x_1, P_{C_k \cap Q_k} x_1) \sin d(x_{k+1}, z) \\ &\qquad - \cos d(x_1, x_{k+1}) \sin((1-t) d(x_{k+1}, z)) \\ &\geq \cos d(x_1, tz \oplus (1-t) x_{k+1}) \sin d(x_{k+1}, z) \\ &\qquad - \cos d(x_1, x_{k+1}) \sin((1-t) d(x_{k+1}, z)). \end{aligned}$$

By Theorem 2.1, the last expression is estimated as

$$\begin{aligned}
&\geq \cos d(x_1, z)\sin(td(z, x_{k+1})) + \cos d(x_1, x_{k+1})\sin((1-t)d(z, x_{k+1})) \\
&- \cos d(x_1, x_{k+1})\sin((1-t)d(z, x_{k+1})) \\
&= \cos d(x_1, z)\sin(td(z, x_{k+1})) \\
&= 2\cos d(x_1, z)\sin\left(\frac{t}{2}d(z, x_{k+1})\right)\cos\left(\frac{t}{2}d(z, x_{k+1})\right).
\end{aligned}$$

If $z = x_{k+1}$, by the definition of Q_k, it is obvious that $z \in Q_{k+1}$. So, we assume that $z \neq x_{k+1}$. Dividing above by $2\sin(td(z, x_{k+1})/2)$ and letting $t \to 0$, we have

$$\cos d(x_1, x_{k+1})\cos d(x_k, z) \geq \cos d(x_1, z)$$

and thus $z \in Q_{k+1}$. From the above, we get $z \in C_{k+1} \cap Q_{k+1}$ and $F \subset C_{k+1} \cap Q_{k+1}$. Therefore, $C_n \cap Q_n$ is nonempty closed convex subset and $\{x_n\}$ is well defined.

Next, we show that $\lim_{n\to\infty} d(T_i x_n, x_n) = 0$ for all $i = 1, 2, \ldots, r$ to get our result. By definition of metric projection, we have

$$d(x_1, x_n) = d(x_1, P_{C_{n-1}\cap Q_{n-1}}x_1) \leq d(x_1, P_F x_1) < \frac{\pi}{2}$$

for all $n \in \mathbb{N} \setminus \{1\}$ and hence $\sup_{n\in\mathbb{N}} d(x_1, x_n) \leq d(x_1, P_F x_1) < \pi/2$. By the definition of Q_n, we have

$$d(x_1, x_n) = d(x_1, P_{Q_n}x_1) \leq d(x_1, P_{C_n\cap Q_n}x_1) = d(x_1, x_{n+1}).$$

Thus, $\{\cos d(x_1, x_n)\}$ is a monotonically non-increasing sequence of real numbers. Then we can put

$$a := \lim_{n\to\infty} \cos d(x_1, x_n) > \cos\frac{\pi}{2} = 0$$

By the definition $x_{n+1} = P_{C_n\cap Q_n}x_1 \in Q_n$,

$$\cos d(x_1, x_n)\cos d(x_n, x_{n+1}) \geq \cos d(x_1, x_{n+1})$$

for all $n \in \mathbb{N}$, and letting $n \to \infty$, we obtain

$$a \liminf_{n\to\infty} \cos d(x_n, x_{n+1}) \geq a.$$

It follow that

$$1 \geq \cos\left(\limsup_{n\to\infty} d(x_n, x_{n+1})\right) = \liminf_{n\to\infty} \cos d(x_n, x_{n+1}) \geq 1,$$

we have that $\limsup_{n\to\infty} d(x_n, x_{n+1}) = 0$. Hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. By the definition C_n and x_{n+1}, it follows that $x_{n+1} \in C_n$ and then

$$d(W_n x_n, x_{n+1}) \leq d(x_n, x_{n+1}).$$

We have that

$$\begin{aligned} 0 &\leq \alpha_{n,r} d(T_r U_{n,r-1} x_n, x_n) \\ &= d(\alpha_{n,r} T_r U_{n,r-1} x_n \oplus (1-\alpha_{n,r}) x_n, x_n) \\ &= d(W_n x_n, x_n) \\ &\leq d(W_n x_n, x_{n+1}) + d(x_{n+1}, x_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_n) \\ &= 2d(x_n, x_{n+1}) \to 0 \ (n \to \infty). \end{aligned}$$

Since $\inf_{n\in\mathbb{N}} \alpha_{n,r} > 0$, we get

$$\lim_{n\to\infty} d(T_r, U_{n,r-1} x_n, x_n) = 0.$$

Next we show

$$\lim_{n\to\infty} d(T_{r-1} U_{n,r-2} x_n, x_n) = 0.$$

Since $P_{C_n \cap Q_n}$ is quasinonexpansive and $z \in F \subset C_n \cap Q_n$, we have that

$$d(x_{n+1}, z) = d(P_{C_n\cap Q_n} x_1, z) \leq d(x_1, z) < \frac{\pi}{2}.$$

Thus we have that

$$\inf_{n\in\mathbb{N}} \cos d(x_n, z) = \cos\left(\sup_{n\in\mathbb{N}} d(x_n, z)\right) > \cos\frac{\pi}{2} = 0.$$

Put $\varepsilon_n := d(T_r U_{n,r-1} x_n, x_n)$ and $\delta_n := d(T_{r-1} U_{n,r-2} x_n, x_n)$. Then, we have

$$\begin{aligned}
&\cos d(x_n, z) \sin d(T_{r-1} U_{n,r-2} x_n, x_n) \\
&\geq \cos(d(x_n, T_r U_{n,r-1} x_n) + d(T_r U_{n,r-1} x_n, z)) \sin d(T_{r-1} U_{n,r-2} x_n, x_n) \\
&= \cos(\varepsilon_n + d(T_r U_{n,r-1} x_n, z)) \sin d(T_{r-1} U_{n,r-2} x_n, x_n) \\
&= \{\cos \varepsilon_n \cos d(T_r U_{n,r-1} x_n, z) \\
&\qquad - \sin \varepsilon_n \sin d(T_r U_{n,r-1} x_n, z)\} \sin d(T_{r-1} U_{n,r-2} x_n, x_n) \\
&= \cos \varepsilon_n \cos d(T_r U_{n,r-1} x_n, z) \sin d(T_{r-1} U_{n,r-2} x_n, x_n) \\
&\quad - \sin \varepsilon_n \sin d(T_r U_{n,r-1} x_n, z) \sin d(T_{r-1} U_{n,r-2} x_n, x_n).
\end{aligned}$$

Since T_r is nonexpansive,

$$\begin{aligned}
&\geq \cos \varepsilon_n \cos d(U_{n,r-1} x_n, z) \sin d(T_{r-1} U_{n,r-2} x_n, x_n) \\
&\quad - \sin \varepsilon_n \sin d(T_r U_{n,r-1} x_n, z) \sin \delta_n.
\end{aligned}$$

By Theorem 2.1,

$$\begin{aligned}
&\geq \cos \varepsilon_n \{\cos d(T_{r-1} U_{n,r-2} x_n, z) \sin(\alpha_{n,r-1} d(T_{r-1} U_{n,r-2} x_n, x_n)) \\
&\quad + \cos d(x_n, z) \sin((1 - \alpha_{n,r-1}) d(T_{r-1} U_{n,r-2} x_n, x_n))\} \\
&\qquad - \sin \varepsilon_n \sin d(T_r U_{n,r-1} x_n, z) \sin \delta_n \\
&\geq \cos \varepsilon_n \{\cos d(x_n, z) \sin(\alpha_{n,r-1} d(T_{r-1} U_{n,r-2} x_n, x_n)) \\
&\quad + \cos d(x_n, z) \sin((1 - \alpha_{n,r-1}) d(T_{r-1} U_{n,r-2} x_n, x_n))\} \\
&\qquad - \sin \varepsilon_n \sin d(T_r U_{n,r-1} x_n, z) \sin \delta_n \\
&= \cos \varepsilon_n \cos d(x_n, z) \{\sin(\alpha_{n,r-1} d(T_{r-1} U_{n,r-2} x_n, x_n)) \\
&\quad + \sin((1 - \alpha_{n,r-1}) d(T_{r-1} U_{n,r-2} x_n, x_n))\} \\
&\qquad - \sin \varepsilon_n \sin d(T_r U_{n,r-1} x_n, z) \sin \delta_n
\end{aligned}$$

and hence

$$\begin{aligned}&\cos d(x_n, z)\sin\delta_n\\&\geq \cos\varepsilon_n \cos d(x_n, z)\{\sin(\alpha_{n,r-1}\delta_n) + \sin((1-\alpha_{n,r-1})\delta_n)\}\\&\quad - \sin\varepsilon_n \sin d(T_r U_{n,r-1}x_n, z)\sin\delta_n.\end{aligned}$$

Then dividing above by $\cos d(x_n, z)$, we have

$$\begin{aligned}\sin\delta_n \geq \cos\varepsilon_n&\{\sin(\alpha_{n,r-1}\delta_n) + \sin((1-\alpha_{n,r-1})\delta_n)\}\\&- \frac{\sin\varepsilon_n \sin d(T_r U_{n,r-1}x_n, z)\sin\delta_n}{\cos d(x_n, z)}.\end{aligned}$$

Let $\{\delta_{n_i}\}$ be a convergent subsequence whose limit is $\delta \in [0, \pi/2]$. There exists subsequence $\{\alpha_{n_{i_j},r-1}\}$ of $\{\alpha_{n_i,r-1}\}$ and $\alpha \in (0,1)$ such that $\alpha_{n_{i_j},r-1} \to \alpha$ as $j \to \infty$. Then since $\varepsilon_{n_{i_j}} \to 0$ as $j \to \infty$, we get

$$\sin\delta \geq \sin(\alpha\delta) + \sin((1-\alpha)\delta).$$

By Lemma 3.1 $\delta = 0$. Therefore $\{\delta_n\}$ converges to 0, that is,

$$\lim_{n\to\infty} d(T_{r-1}U_{n,r-1}x_n, x_n) = 0.$$

Using a similar caluculation inductively, we have

$$\lim_{n\to\infty} d(T_i U_{n,i-1}x_n, x_n) = 0$$

for all $i = 1, 2, \ldots, r$. Since

$$\begin{aligned}d(T_i x_n, x_n) &\leq d(T_i x_n, T_i U_{n,i-1}x_n) + d(T_i U_{n,i-1}x_n, x_n)\\&\leq d(x_n, U_{n,i-1}x_n) + d(T_i U_{n,i-1}x_n, x_n)\\&= d(x_n, \alpha_{n,i-1}T_{i-1}U_{n,i-2}x_n \oplus (1-\alpha_{n,i-1})x_n)\\&\quad + d(T_i U_{n,i-1}x_n, x_n)\\&= \alpha_{n,i-1}d(T_{i-1}U_{n,i-2}x_n, x_n) + d(T_i U_{n,i-1}x_x, x_n)\\&\to 0 \ (n \to \infty),\end{aligned}$$

we obtain $\lim_{n\to\infty} d(T_i x_n, x_n) = 0$ for all $i = 1, 2, \ldots, r$. Let $\{x_{n_i}\}$ be an arbitrary subsequence of $\{x_n\}$. By inequality $\sup_{j\in\mathbb{N}} d(x_1, x_{n_j}) \leq \sup_{n\in\mathbb{N}} d(x_1, x_n) < \pi/2$ and Lemma 2.4, there exists subsequence $\{x_{n_{i_j}}\}$

of $\{x_{n_i}\}$ and w_∞ such that $\{x_{n_{i_j}}\}$ is Δ-convergent to w_∞. For the sake of simplicity, we put $w_j := x_{n_{i_j}}$. Then we can show $w_\infty \in \bigcap_{i=1}^r F(T_i)$. For all $i = 1, 2, \dots, r$,

$$\begin{aligned}\limsup_{j\to\infty} d(w_j, T_i w_\infty) &\le \limsup_{j\to\infty}(d(w_j, T_i w_j) + d(T_i w_j, T_i w_\infty))\\ &\le \limsup_{j\to\infty}(d(w_j, T_i w_j) + d(w_j, w_\infty))\\ &= \limsup_{j\to\infty} d(w_j, w_\infty).\end{aligned}$$

By the definition of Δ-convergence, we get $T_i w_\infty = w_\infty$. Hence $w_\infty \in \bigcap_{i=1}^r F(T_i)$. Since

$$w_j = x_{n_{i_j}} = P_{C_{n_{i_j}-1} \cap Q_{n_{i_j}-1}} x_1 \in C_{n_{i_j}-1} \cap Q_{n_{i_j}-1},$$

$F \subset C_{n_{i_j}-1} \cap Q_{n_{i_j}-1}$ and Lemma 2.5,

$$d(x_1, P_F x_1) \le d(x_1, w_\infty) \le \lim_{j\to\infty} d(x_1, w_j) \le d(x_1, P_F x_1).$$

Thus we get $\lim_{j\to\infty} d(x_1, w_j) = d(x_1, w_\infty)$ and by Lemma 2.7, $\{w_j\}$ convergent to w_∞. On the other hand, we get $d(x_1, P_F x_1) = d(x_1, w_\infty)$ and by definition of metric projection P_F, we get $w_\infty = P_F x_1$. From the above, any subsequence $\{x_{n_i}\}$ of $\{x_n\}$ has subsequence $\{x_{n_{i_j}}\}$ such that $\{x_{n_{i_j}}\}$ is convergent to $P_F x_1$. Threfore $\{x_n\}$ convergent to $P_F x_1$. $\square$

Next, we consider shrinking projection method using W-mapping on a complete CAT(1) space.

Theorem 5.2. *Let C be a closed convex subset in real Hilbert shpere S_H such that $d(v, v') < \pi/2$ for every $v, v' \in C$. Let $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ be real numbers for $n \in \mathbb{N}$ such that $\alpha_{n,i} \in [a, 1-a]$ for every $i = 1, 2, \dots, r$ where $0 < a < 1/2$, and let $T_1, T_2, \dots, T_r$ be a finite number of nonexpansive mappings of C into itself such that $F := \bigcap_{i=1}^r F(T_i) \neq \emptyset$. Let W_n be the W-mappings of X into itself generated by $T_1, T_2, \dots, T_r$ and $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ for $n \in \mathbb{N}$. For a given point $x_1 \in C$, let $\{x_n\}$*

be a sequence in C generated by

$$
\begin{aligned}
x_1 &:= x \in C,\\
y_n &:= W_n x_n,\\
C_n &:= \{z \in C \mid d(y_n, z) \le d(x_n, z)\} \cap C_{n-1},\\
x_{n+1} &:= P_{C_n} x_1,
\end{aligned}
$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ is well defined and convergent to $P_F x_1$.

Proof. First, we show that $\{x_n\}$ is well defined, that is, we show that C_n is nonempty closed convex subset of C. By the definition of C_n, we have that C_n is a closed convex subset of C. So, we shall show C_n is nonempty. Since $T_1, T_2, \ldots, T_r$ are nonexpansive, by Lemma 2.1, W_n is quasinonexpansive. Then by Proposition 3.1, we have that $F \subset \{z \in C \mid d(y_n, z) \le d(x_n, z)\}$. Then by induction, we have that $F \subset C_n$ for all $n \in \mathbb{N}$, that is, C_n is nonempty for all $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is well defined.

Next, we show that $\{x_n\}$ is convergent to $P_F x_1$. Let $\{x_{n_i}\}$ be an arbitrary subsequence of $\{x_n\}$. By the inequality

$$\sup_{j\in\mathbb{N}} d(x_1, x_{n_j}) \le \sup_{n\in\mathbb{N}} d(x_1, x_n) < \pi/2,$$

there exists subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ and w_∞ such that $\{x_{n_{i_j}}\}$ is Δ-convergent to w_∞. For the sake of simplicity, we put $w_j := x_{n_{i_j}}$. For all $k \in \mathbb{N}$, there exists $j_0 \in \mathbb{N}$ such that, for every $j \ge j_0, w_j \in C_k$. By Lemma 2.8, we get $w_\infty \in C_k$. Thus we have that $w_\infty \in \bigcap_{k=1}^{\infty} C_k$, and by Lemma 2.8,

$$
\begin{aligned}
d(x_1, P_{\bigcap_{k=1}^{\infty} C_k} x_1) \le d(x_1, w_\infty) &\le \lim_{j\to\infty} d(x_1, w_j)\\
&= \lim_{j\to\infty} d(x_1, P_{C_{n_{i_j}}} x_1) \le d(x_1, P_{\bigcap_{k=1}^{\infty} C_k} x_1).
\end{aligned}
$$

Thus we get $d(x_1, w_\infty) = \lim_{j\to\infty} d(x_1, w_j)$ and by Lemma 2.4, $\{w_j\}$ convergent to w_∞. On the other hand, we get $d(x_1, P_{\bigcap_{k=1}^{\infty} C_k} x_1) = d(x_1, w_\infty)$ and by the definition of metric projection $P_{\bigcap_{k=1}^{\infty} C_k}$, we get

$w_\infty = P_{\bigcap_{k=1}^\infty C_k} x_1$. From the above, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$ has subsequence $\{x_{n_{i_j}}\}$ such that $\{x_{n_{i_j}}\}$ convergent to $P_{\bigcap_{k=1}^\infty C_k} x_1$. Since $P_{\bigcap_{k=1}^\infty C_k} x_1 \in \bigcap_{k=1}^\infty C_k \subset C_n$ for all $n \in \mathbb{N}$ and the definition of C_n,

$$d(W_n x_n, P_{\bigcap_{k=1}^\infty C_k} x_1) \leq d(x_n, P_{\bigcap_{k=1}^\infty C_k} x_1).$$

Thus

$$\limsup_{n\to\infty} d(W_n x_n, P_{\bigcap_{k=1}^\infty C_k} x_1) \leq \lim_{n\to\infty} d(x_n, P_{\bigcap_{k=1}^\infty C_k} x_1) = 0$$

holds. Hence $\{W_n x_n\}$ is convergent to $P_{\bigcap_{k=1}^\infty C_k} x_1$. Therefore,

$$\lim_{n\to\infty} d(W_n x_n, x_n) = 0.$$

Then, as in the proof of Theorem 5.1, we have that $\lim_{n\to\infty} d(T_i x_n, x_n) = 0$ for all $i = 1, 2, \ldots, r$. Thus $P_{\bigcap_{k=1}^\infty C_k} x_1 \in F(T_i)$ for all $r = 1, 2, \ldots, r$. Thus $P_{\bigcap_{k=1}^\infty C_k} x_1 \in \bigcap_{i=1}^r F(T_i)$. Since $F \subset \bigcap_{k=1}^\infty C_k$ and $P_{\bigcap_{k=1}^\infty C_k} x_1 \in F$,

$$d(x_1, P_{\bigcap_{k=1}^\infty C_k} x_1) \leq d(x_1, P_F x_1) \leq d(x_1, P_{\bigcap_{k=1}^\infty C_k} x_1).$$

By the definition of metric projection, we have $P_{\bigcap_{k=1}^\infty C_k} x_1 = P_F x_1$, that is, $\{x_n\}$ is convergent to $P_F x_1$, and we finish the proof. □

Example 5.1. Let $\triangle$ be a spherical equilateral triangle in $\mathbb{S}^2$. Let T_1, T_2, T_3 be reflection of vertical bisector of each edge of $\triangle$. Then T_1, T_2, T_3 are nonexpansive and $\bigcap_{i=1}^3 F(T_i)$ is one point set which is the intersection of vertical bisector of edges. By Theorem 5.1 and 5.2, a sequence generated by CQ projection method or shrinking projection method is convergent to the point $\bigcap_{i=1}^3 F(T_i)$.

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