Geometric Relations among the Bott-Virasoro Group, Equicentroaffine Curves and the KdV Equation

（Bott-Virasoro 群と Equicentroaffine 曲線および KdV 方程式の間の幾何的な関係）

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Contents

Introduction

As the title says, in this thesis, we would like to build a bridge between two different mathematical objects: the Bott-Virasoro group and the space of equicentroaffine curves. The key will be the Korteweg-de Vries equation:

$$\frac{\partial u}{\partial t} = -a\frac{\partial^3 u}{\partial t^3} - 3u\frac{\partial u}{\partial x}, \tag{1}$$

where $u : S^1 \times \mathbb{R} \to \mathbb{R}$ is a function of $(x,t) \in S^1 \times \mathbb{R}$ and a a positive constant.

In [14], Khesin and Wendt provided a procedure to derive the KdV equation with the help of the Euler equation. In Chapter 1, we will review their method. Let G be a Lie group which can be infinite-dimensional and $\mathfrak{g}$ its Lie algebra. Let r_g denote the right group multiplication by $g \in G$ and r_{g*} its differential that acts on the tangent bundle. Let $\langle -,- \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a fixed inner product and $A : \mathfrak{g} \to \mathfrak{g}^*$ the corresponding isomorphism, called the inertia operator. Given a curve $c : \mathbb{R} \to G$, we define $m : \mathbb{R} \to \mathfrak{g}^*$ to be a curve in the dual Lie algebra by

$$m(t) := A({r_{c(t)^{-1}}}_*\dot{c}(t)), \tag{2}$$

where $\dot{c}$ denotes the derivative with respect to t. Then, the curve c is a geodesic in G with respect to the right-invariant metric induced from $\langle -,- \rangle$ if and only if m satisfies the following:

$$\dot{m}(t) = \mathrm{ad}^*_{A^{-1}m(t)}m(t), \tag{3}$$

which is called the Euler equation (see Section 1.1.2). Let $\mathrm{Diff}(S^1)$ denote the orientation-preserving diffeomorphism group of S^1. In Section 1.3.2, we will take G to be the Bott-Virasoro group, which is denoted by $\mathrm{Diff}(S^1) \times_B \mathbb{R}$. This group is defined as a set $\{(\varphi, a) \mid \varphi \in \mathrm{Diff}(S^1), a \in \mathbb{R}\}$ with multiplication

$$(\varphi, a)(\psi, b) := (\varphi \circ \psi, a + b + B(\varphi, \psi)).$$

Here, $B : \mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1) \to \mathbb{R}$ is the Bott cocycle given by

$$B(\varphi, \psi) := \frac{1}{2}\int_{S^1} \log(\varphi' \circ \psi) d\log\psi'.$$

Let $\mathfrak{X}(S^1) \times_\omega \mathbb{R}$ denote the Lie algebra of the Bott-Virasoro group, called the Virasoro algebra (see Section 1.2.2). As a linear space, it is isomorphic to $\mathfrak{X}(S^1) \times \mathbb{R}$. Imposing

$\langle -, - \rangle$ as the L^2-inner product over the Virasoro algebra, we will see that the Euler equation (3) yields the KdV equation (1) as the $\mathfrak{X}(S^1)^*$-part of the corresponding curve m in the dual Virasoro algebra.

By introducing the $H^1_{\alpha,\beta}$-inner product instead of the L^2-inner product, Khesin and Wendt also obtained a generalization of the KdV equation:

$$\alpha(\dot{u} + 3uu') - \beta(\dot{u}'' + 2u'u'' + uu''') + au''' = 0. \tag{4}$$

When $\alpha = 1, \beta = 0$, it reduces to the KdV equation. When $\alpha = \beta = 1$, it reduces to the Camassa-Holm equation and when $\alpha = 0, \beta = 1$, the Hunter-Saxton equation.

On the other hand, we would like to generalize the KdV equation by changing the Bott-Virasoro group while keeping the L^2-inner product. It is known that there are two independent group 2-cocycles on $\mathrm{Diff}(S^1)$: the Euler cocycle and the Bott cocycle B. By using these two cocycles, we can construct central extensions which are distinct from the Bott-Virasoro group. Let $\Lambda = \chi^\alpha_\mathbb{R} + \beta B$ be a group 2-cocycle and $\mathrm{Diff}(S^1) \times_\Lambda \mathbb{R}$ the corresponding central extension (see Section 1.2.3 for the precise definition). Then, by taking G as $\mathrm{Diff}(S^1) \times_\Lambda \mathbb{R}$, we can prove the first main result:

Theorem 1 (Theorem 1.46)**.** *Let $\varphi : \mathbb{R} \to \mathrm{Diff}(S^1)$ and $d, a : \mathbb{R} \to \mathbb{R}$ be smooth curves and let $u : S^1 \times \mathbb{R} \to \mathbb{R}$ be a smooth function of $(x,t) \in S^1 \times \mathbb{R}$. Set $c := (\varphi, d) : \mathbb{R} \to G$ and $m := (udx \otimes dx, a) : \mathbb{R} \to \mathfrak{X}(S^1)^* \times \mathbb{R}$, where $\mathfrak{X}(S^1)^* \times \mathbb{R}$ is naturally identified with* $\mathrm{Lie}(G)^*$. *Suppose that m is the curve corresponding to c in equation (2). Then, c is a geodesic with respect to the right-invariant metric induced from the L^2-inner product if and only if*

$$(\dot{u}dx \otimes dx, \dot{a}) = ((-3uu' - \beta au''' + \alpha au')dx \otimes dx, 0), \tag{5}$$

where $\dot{u}$ and $\dot{a}$ denotes the derivatives with respect to t, and u' with respect to x. ■

We rewrite (5) as

$$\dot{u} = -3uu' - \beta au''' + \alpha au'. \tag{6}$$

By setting $u = v + \frac{1}{3}a\alpha$, and $\beta = 1$, the equation (6) amounts to

$$\dot{v} = -3vv' - av''',$$

which is nothing but the KdV equation (1).

In Chapter 2, we will turn our attention to $\mathcal{M}$, the connected component of the space of equicentroaffine curves containing the unit circle c_0. An element in $\mathcal{M}$ is a plain curve $\gamma : S^1 \to \mathbb{R}^2$ satisfying

$$\det(\gamma, \gamma') = 1$$

(see Section 2.1.1 for the details). Given $\gamma \in \mathcal{M}$, there exists an equicentroaffine curvature $\kappa : S^1 \to \mathbb{R}$ which is determined by

$$\gamma'' + \kappa\gamma = 0.$$

A tangent vector $X \in T_\gamma \mathcal{M}$ over $\mathcal{M}$ is identified with a vector field along γ, which has the form

$$X = -\frac{1}{2}\lambda'\gamma + \lambda\gamma',$$

where $\lambda : S^1 \to \mathbb{R}$ is a certain function on S^1. In [7], Fujioka and Kurose discussed two presymplectic forms $\hat{\omega}_0$ and $\hat{\omega}_1$ on $\mathcal{M}$, which are given by

$$\hat{\omega}_0(X, Y) := \int_{S^1} \lambda\mu' dx,$$

called the Pinkall 2-form, and

$$\hat{\omega}_1(X, Y) := \int_{S^1} \lambda(\frac{1}{2}\mu''' + 2\kappa\mu' + \kappa'\mu)dx,$$

which we will call the Fujioka-Kurose 2-form, where $X = -\frac{1}{2}\lambda'\gamma + \lambda\gamma', Y = -\frac{1}{2}\mu'\gamma + \mu\gamma' \in T_\gamma\mathcal{M}$. They also introduced a Hamiltonian function H on $\mathcal{M}$, whose presymplectic gradient X_H with respect to $\hat{\omega}_1$ satisfies

$$X_H(\gamma) = \frac{1}{2}\kappa'\gamma - \kappa\gamma',$$

where κ is the equicentroaffine curvature of γ. Due to Pinkall as one of the main results in [20], given an integral curve $\tilde{\gamma} : \mathbb{R} \to \mathcal{M}$ of X_H, let $\tilde{\kappa}(-, t) : S^1 \to \mathbb{R}$ be the equicentroaffine curvature of $\tilde{\gamma}(t)$, where we consider $\tilde{\kappa}$ as a function of $(x, t) \in S^1 \times \mathbb{R}$. Then, we can verify that $\tilde{\kappa}$ satisfies the KdV equation (1):

$$\dot{\tilde{\kappa}} = -\frac{1}{2}\tilde{\kappa}''' - 3\tilde{\kappa}'\tilde{\kappa}. \tag{7}$$

It is very interesting that the KdV equation appears in such two completely different objects, the Bott-Virasoro group and the space of equicentroaffine curves. We think that there may exist some mathematical connection behind them, and this becomes an important motivation for us to obtain the main results in Chapter 3.

In Section 2.2, we will introduce a right S^1-action and a left $\mathrm{SL}(2, \mathbb{R})$-action on $\mathcal{M}$. Setting $\mathcal{M}_0 := \mathcal{M}/S^1$ and $\mathcal{M}_1 := \mathrm{SL}(2, \mathbb{R}) \setminus \mathcal{M}$ to be the quotient spaces, we will see that the presymplectic forms $\hat{\omega}_i (i = 0, 1)$ descend to symplectic forms ω_i on $\mathcal{M}_i$, respectively. Moreover, due to Fujioka, Kurose and Moriyoshi in [8], we will see that there are momentum maps $\mu_i (i = 0, 1)$ on $\mathcal{M}_i$ corresponding to certain actions as well as the symplectic forms ω_i on $\mathcal{M}_i$, respectively (see Section 2.4).

In Chapter 3, we will finally focus on the relations between the Bott-Virasoro group and the space of equicentroaffine curves. A powerful tool will be the well-known canonical symplectic form $d\Theta$ (see Definition 3.5) related to the cotangent bundle T^*G. In Section 3.1, we will introduce the canonical symplectic form $d\Theta$ and discuss its relation to the Kirillov-Kostant-Souriau form ω_{KKS}. Given a Lie group G, which can be infinite-dimensional, we will introduce a right action of G on the semidirect product $G \ltimes \mathfrak{g}^*$ given by

$$(g, \alpha) \cdot h = (gh, \mathrm{Ad}^*_{h^{-1}}\alpha), \tag{8}$$

where $(g,\alpha) \in G \ltimes \mathfrak{g}^*$ and $h \in G$. For $U \in \mathfrak{g}$, let $\underline{U}$ be the fundamental vector field corresponding to the action (8) (see Section 2.1.2). Let $\mathrm{pr}_2 : G \ltimes \mathfrak{g}^* \to \mathfrak{g}^*$ denote the projection onto $\mathfrak{g}^*$. We will prove that:

Theorem 2 (Theorem 3.11)**.** *For the Kirillov-Kostant-Souriau form ω_{KKS} on $\mathfrak{g}^*$ and the canonical symplectic form $d\Theta$ on $G \ltimes \mathfrak{g}^*$, we have*

$$\mathrm{pr}_2^*\omega_{KKS}(\underline{U}_{(g,\sigma)}, \underline{V}_{(g,\sigma)}) = d\Theta(\underline{U}_{(g,\sigma)}, \underline{V}_{(g,\sigma)}),$$

where $U, V \in \mathfrak{g}$ and $(g,\sigma) \in G \ltimes \mathfrak{g}^*$. ■

In Section 3.1.2, we will introduce a Hamiltonian function E on $G \ltimes \mathfrak{g}^*$. Let X_E be its symplectic gradient with respect to $d\Theta$ and (φ, ξ) an integral curve of X_E. By taking G as the Bott-Virasoro group, we can verify that the curve ξ satisfies:

$$\dot{\xi} = \frac{1}{2}\xi''' + 3\xi'\xi, \tag{9}$$

which is only different from the KdV equation (1) by a minus sign. Since the KdV equation (7) is related to the presymplectic form $\hat{\omega}_1$ and the vector field X_H, the discussion above shows that we may answer why the KdV equation (7) appears by revealing the relationship between the forms $d\Theta$ and $\hat{\omega}_1$ as well as the relationship between the vector fields X_E and X_H. It is known that there is a right $\mathrm{Diff}(S^1)$-action on $\mathcal{M}$, which is called Pinkall's right action. In Section 3.2, setting G to be the Bott-Virasoro group and $\mathfrak{g}^*$ the dual Virasoro algebra, we will prove that:

Theorem 3 (Theorem 3.12)**.** *Given $\gamma \in \mathcal{M}$, we take $\psi \in \mathrm{Diff}(S^1)$ such that $\gamma = c_0 \cdot \psi$, where $\cdot$ denotes Pinkall's right action. Let $\sigma_0 : \mathcal{M} \to G \ltimes \mathfrak{g}^*$ be a map on $\mathcal{M}$ given by*

$$\sigma_0(\gamma) := ((\psi^{-1}, 0), (-\frac{1}{2}(\psi^{-1})'^2 dx \otimes dx, 0)),$$

Then, we have

$$\sigma_0^* d\Theta = \hat{\omega}_0,$$

where $d\Theta$ is the canonical symplectic form on $G \ltimes \mathfrak{g}^*$ and $\hat{\omega}_0$ the Pinkall 2-form on $\mathcal{M}$. ■

Theorem 4 (Theorem 3.14)**.** *Given $\gamma \in \mathcal{M}$, we take $\psi \in \mathrm{Diff}(S^1)$ such that $\gamma = c_0 \cdot \psi$. Let $\sigma_1 : \mathcal{M} \to G \ltimes \mathfrak{g}^*$ be a map on $\mathcal{M}$ given by*

$$\sigma_1(\gamma) := ((\psi, 0), (-\kappa dx \otimes dx, -\frac{1}{2})),$$

where κ is the equicentroaffine curvature of γ. Then, we have

$$\sigma_1^* d\Theta = \hat{\omega}_1$$

where $d\Theta$ is the canonical symplectic form on $G \ltimes \mathfrak{g}^*$ and $\hat{\omega}_1$ the Fujioka-Kurose 2-form on $\mathcal{M}$. ■

As corollaries of the theorems above and Theorem 2, we can show the relationship between the Kirillov-Kostant-Souriau form ω_{KKS} and $\omega_i (i = 0, 1)$:

Corollary 5 (Corollary 3.13)**.** Let ω_0 be the Pinkall symplectic form and ω_{KKS} the Kirillov-Kostant-Souriau form on $\mathfrak{g}^*$. Then, we have

$$\mu_0^* \omega_{KKS} = -\omega_0,$$

where $\mu_0 : \mathcal{M}_0 \to \mathfrak{g}^*$ is the momentum map mentioned in Section 2.4.1. ■

Corollary 6 (Corollary 3.15)**.** Let ω_1 be the Fujioka-Kurose symplectic form and ω_{KKS} the Kirillov-Kostant-Souriau form on $\mathfrak{g}^*$. Then, we have

$$\mu_1^* \omega_{KKS} = \omega_1,$$

where $\mu_1 : \mathcal{M}_1 \to \mathfrak{g}^*$ is the momentum map mentioned in Section 2.4.2. ■

With the help of these theorems and corollaries, we will finally build a bridge between the Bott-Virasoro group and the space of equicentroaffine curves. To sum up, we will construct the following commutative diagram:

Equicentroaffine Curve　　　　Bott-Virasoro Group

$$\begin{tikzcd}
 & (\mathcal{M}_1, \omega_1) \arrow[drr, "\mu_1"] & & \\
(\mathcal{M}_0, \omega_0) \arrow[rrr, "\mu_0"] & & & ((\mathfrak{X}(S^1) \times_\omega \mathbb{R})^*, \omega_{KKS}) \\
 & & & ((\mathrm{Diff}(S^1) \times_B \mathbb{R}) \ltimes (\mathfrak{X}(S^1) \times_\omega \mathbb{R})^*, d\Theta) \arrow[u, "\mathrm{pr}_2"] \\
 & (\mathcal{M}, \hat{\omega}_0, \hat{\omega}_1) \arrow[uuu, "\pi_1"] \arrow[uul, "\pi_0"] \arrow[urr, "\sigma_0, \sigma_1"] & &
\end{tikzcd}$$

where

$$\mu_0 \circ \pi_0 = -\mathrm{pr}_2 \circ \sigma_0 \quad \text{and} \quad \mu_1 \circ \pi_1 = \mathrm{pr}_2 \circ \sigma_1.$$

In Section 3.2.3, by making use of the map σ_1, we will obtain a relation between X_H and X_E:

$$\sigma_1(X_H(\gamma)) = X_E(\sigma_1(\gamma)) + X \tag{10}$$

where $X \in T_{\sigma_1(\gamma)}((\mathrm{Diff}(S^1) \times_B \mathbb{R}) \ltimes (\mathfrak{X}(S^1) \times_\omega \mathbb{R})^*)$ is a tangent vector such that

$$d\Theta(X_{\sigma_1(\gamma)}, \sigma_{1*} Z) = 0 \tag{11}$$

for all $Z \in T_\gamma \mathcal{M}$. The existence of X in (10) may keep us from getting a strong relation between X_H and X_E. But what we will prove in Theorem 3.17 shows that the $(\mathfrak{X}(S^1) \times_\omega \mathbb{R})^*$-part of X vanishes. This will help us finally answer the question why $\tilde{\kappa}$ leads to the KdV equation (7) as the Euler equation applied to the Bott-Virasoro group does (see Section 3.2.3).

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Chapter 1

The Euler Equation for Generalized Bott-Virasoro Groups

In this chapter, we will provide some basic knowledge and derive a generalized version of the KdV equation. In Section 1.1, we will introduce the Euler equation. In Section 1.3, we will apply the Euler equation to the Bott-Virasoro group to obtain the KdV equation.

1.1 The Euler Equation

In this section, we will introduce the Euler equation, which appears as a characterization of geodesics in a Lie group G. Then, we will employ the Euler equation and the group $\mathrm{Diff}(S^1)$ to obtain Burgers' equation. The content in this section is mainly adapted from Khesin and Wendt [14], Michor and Ratio [17], Tu [22] and Vizman [23].

1.1.1 Adjoint and Coadjoint Actions

For a smooth map $f : M \to N$ between two manifolds, we denote by $f_{*,x} : T_xM \to T_{f(x)}N$ the differential of f at x.

Definition 1.1 ([14, Definition 2.1])**.** Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. For any $g \in G$, let $c_g : G \to G$ be a map given by

$$c_g(h) := ghg^{-1}.$$

Then, the ***adjoint action of*** G is defined to be a map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ such that for any $g \in G$ we set

$$\mathrm{Ad}_g := c_{g_{*,1_G}},$$

and the ***adjoint action of*** $\mathfrak{g}$ is defined to be the differential of the adjoint action of G at 1_G, denoted by $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$

Remark 1.2 ([22, p.119])**.** For any $X \in \mathfrak{g}$, there is a corresponding left-invariant vector field $\tilde{X} \in \mathfrak{X}(G)$ such that for any $g \in G$ we have

$$\tilde{X}_g := l_{g*,1_G} X,$$

where $l_g : G \to G$ is the left multiplication by g.

The following lemma and its proof essentially appear in Tu [22].

Lemma 1.3 ([22, Proposition 15.15])**.** For any $X, Y \in \mathfrak{g}$, we have

$$\mathrm{ad}_X Y = [X, Y],$$

where the right-hand side is the usual Lie bracket induced by the left-invariant vector fields. ∎

Definition 1.4 ([14, Definition 2.4])**.** Let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. Then, the ***coadjoint action of*** G is a map $\mathrm{Ad}^* : G \to \mathrm{Aut}(\mathfrak{g}^*)$ such that for any $\xi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$, we have

$$(\mathrm{Ad}^*_g \xi)(X) = \xi(\mathrm{Ad}_{g^{-1}} X),$$

and the ***coadjoint action of*** $\mathfrak{g}$ is the differential of the coadjoint action of G at 1_G, denoted by $\mathrm{ad}^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$.

Remark 1.5 ([14, p.20])**.** Note that for any $X, Z \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, we have

$$(\mathrm{ad}^*_Z \xi)(X) = -\xi(\mathrm{ad}_Z X),$$

by the definition of the coadjoint action of G and the coadjoint action of $\mathfrak{g}$.

1.1.2 The Euler Equation on Lie Groups

Definition 1.6 ([22, p.198])**.** Let G be a Lie group. Then, the ***left Maurer-Cartan form*** is defined to be a $\mathfrak{g}$-valued 1-form $\theta : TG \to \mathfrak{g}$ given by

$$\theta(X) := {l_{g^{-1}}}_* X,$$

where $X \in T_g G$.

Definition 1.7 ([23, p.3])**.** For any inner product $\langle -, - \rangle$ over $\mathfrak{g}$, there is an induced isomorphism $A : \mathfrak{g} \to \mathfrak{g}^*$ given by

$$A(X) := \langle X, - \rangle,$$

which is called the ***inertia operator***.

Definition 1.8 ([23, p.2])**.** For any inner product $\langle -, - \rangle$ over $\mathfrak{g}$, there is an induced left-invariant metric $(-, -)$ over G given by

$$(X, Y) := \langle \theta(X), \theta(Y) \rangle, \tag{1.1}$$

where $X, Y \in T_g G$. Let $\mathcal{C}$ denote the space of smooth curves over G. Then, the ***energy function*** $\mathcal{E} : \mathcal{C} \to \mathbb{R}$ is defined by

$$\mathcal{E}(c) = \frac{1}{2} \int_a^b (\dot{c}(t), \dot{c}(t)) dt, \tag{1.2}$$

where $c : [a, b] \to G$ is a smooth curve in G.

Definition 1.9 ([23, pp.2-3])**.** For any smooth curve $c : [a, b] \to G$, a ***variation*** of c is a smooth map $\tilde{c} : [a, b] \times [-\varepsilon, \varepsilon] \to \mathcal{C}$ such that $\tilde{c}(-, 0) = c$. The curve c is called a ***geodesic*** if any variation $\tilde{c}$ of c with fixed endpoints satisfies

$$\left. \frac{d}{ds} \right|_{s=0} \mathcal{E}(\tilde{c}(-, s)) = 0, \tag{1.3}$$

where s denotes the parameter along the variation, and t along the curve.

Given a curve $c : [a, b] \to G$, define $m : [a, b] \to \mathfrak{g}^*$ to be a curve in the dual Lie algebra $\mathfrak{g}^*$ by

$$m(t) := A(\theta(\dot{c}(t))). \tag{1.4}$$

In the following theorem, we will present the Euler equation, which appears as a characterization of geodesics in a Lie group. The proof is essentially given by Vizman [23].

Theorem 1.10 ([23, Theorem 1])**.** *A smooth curve $c : [a, b] \to G$ is a geodesic with respect to the left-invariant metric mentioned in (1.1) if and only if*

$$\dot{m}(t) = -\mathrm{ad}^*_{A^{-1}(m(t))} m(t), \tag{1.5}$$

where m is the curve given in (1.4) and $\dot{m}$ denotes the derivative with respect to t.

Proof. For any variation $\tilde{c} : [-\varepsilon, \varepsilon] \to \mathcal{C}$ of c with fixed endpoints, we define the ***velocity field*** $u : [a, b] \times [-\varepsilon, \varepsilon] \to \mathfrak{g}$ by

$$u(t, s) := \theta(\partial_1|_{(t,s)} \tilde{c})$$

and the ***Jacobian field*** $v : [a, b] \times [-\varepsilon, \varepsilon] \to \mathfrak{g}$ by

$$v(t, s) := \theta(\partial_2|_{(t,s)} \tilde{c}),$$

where ∂_1 and ∂_2 denote the directional derivatives with respect to the parameters t and s, respectively. We first show that

$$-\partial_1|_{(t,s)} v + \partial_2|_{(t,s)} u = [u(t, s), v(t, s)].$$

By the definition of the left Maurer-Cartan form, for any $X, Y \in T_gG$ we have

$$d\theta(X,Y) = d\theta(\widetilde{\theta(X)}, \widetilde{\theta(Y)})(g) = -\theta([\widetilde{\theta(X)}, \widetilde{\theta(Y)}])(g) = -[\theta(X), \theta(Y)].$$

It follows that

$$\begin{aligned}
[u(t,s), v(t,s)] &= [\theta(\partial_1|_{(t,s)}\tilde{c}), \theta(\partial_2|_{(t,s)}\tilde{c}))] \\
&= -d\theta(\partial_1|_{(t,s)}\tilde{c}, \partial_2|_{(t,s)}\tilde{c})) \\
&= -\tilde{c}^* d\theta(\partial_1|_{(t,s)}, \partial_2|_{(t,s)}) \\
&= -\partial_1|_{(t,s)}\tilde{c}^*\theta(\partial_2) + \partial_2|_{(t,s)}\tilde{c}^*\theta(\partial_1) \\
&= -\partial_1|_{(t,s)}v + \partial_2|_{(t,s)}u,
\end{aligned}$$

Therefore,

$$\left.\frac{d}{ds}\right|_{s=0} \mathcal{E}(\tilde{c}(-,s)) = \int_a^b \langle \partial_2|_{(t,0)}u, u(t,0)\rangle dt = \int_a^b \langle \partial_1|_{(t,0)}v + [u(t,0), v(t,0)], u(t,0)\rangle dt.$$

Since $v(a,s) = v(b,s) = 0$ for all $s \in [-\varepsilon, \varepsilon]$, we see that

$$\begin{aligned}
\int_a^b \langle \partial_1|_{(t,0)}v, u(t,0)\rangle dt &= \int_a^b \partial_1|_{(t,0)}\langle v, u\rangle dt - \int_a^b \langle v(t,0), \partial_1|_{(t,0)}u\rangle dt \\
&= -\int_a^b \langle v(t,0), \partial_1|_{(t,0)}u\rangle dt.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left.\frac{d}{ds}\right|_{s=0} \mathcal{E}(\tilde{c}(-,s)) &= \int_a^b (\langle [u(t,0), v(t,0)], u(t,0)\rangle - \langle v(t,0), \partial_1|_{(t,0)}u\rangle) dt \\
&= \int_a^b (A(u(t,0))(\mathrm{ad}_{u(t,0)}v(t,0)) - A(\partial_1|_{(t,0)}u)(v(t,0))) dt \\
&= \int_a^b (-\mathrm{ad}^*_{u(t,0)}A(u(t,0)) - A(\partial_1|_{(t,0)}u)(v(t,0)) dt.
\end{aligned}$$

By (1.3), the curve c is a geodesic if and only if

$$A(\partial_1|_{(t,0)}u) = -\mathrm{ad}^*_{u(t,0)}A(u(t,0)). \tag{1.6}$$

By the definition of the velocity field, we have

$$u(t,0) = \theta(\dot{c}(t)). \tag{1.7}$$

Substituting (1.7) into (1.6), we finally obtain the Euler equation. ■

Remark 1.11 ([14, Remark 4.16])**.** Instead of using the left-invariant metric on G, we can also use the right-invariant one. In this case, the corresponding curve $m : [a,b] \to \mathfrak{g}^*$ in the dual Lie algebra is given by

$$m(t) := A({r_{c(t)^{-1}}}_*\dot{c}(t)), \tag{1.8}$$

where r_g denotes the right group multiplication by $g \in G$, and the Euler equation becomes

$$\dot{m}(t) = \mathrm{ad}^*_{A^{-1}(m(t))}m(t), \tag{1.9}$$

which is only different from the Euler equation in Theorem 1.10 by a minus sign.

1.1.3 Burgers' Equation

Let $\mathrm{Diff}(S^1)$ be the orientation-preserving diffeomorphism group of S^1. By setting

$$\widetilde{\mathrm{Diff}(S^1)} := \{f \in C^\infty(\mathbb{R}) \mid f'(x) > 0 \text{ and } f(x+2\pi) = f(x) + 2\pi \text{ for all } x \in \mathbb{R}\},$$

we can identify $\mathrm{Diff}(S^1)$ with a quotient space:

$$\mathrm{Diff}(S^1) = \widetilde{\mathrm{Diff}(S^1)}/\sim,$$

where $f \sim g$ if and only if $f + 2n\pi = g$ for some $n \in \mathbb{Z}$. The Lie algebra of $\mathrm{Diff}(S^1)$ is $\mathfrak{X}(S^1)$, the space of vector fields over S^1. An element in $\mathfrak{X}(S^1)$ is denoted by $u\partial_x$, where $u \in C^\infty(S^1)$ can be regarded as a smooth function of $x \in \mathbb{R}$ with period 2π. The Lie bracket $[-,-]$ over $\mathfrak{X}(S^1)$ is given by

$$[u\partial_x, v\partial_x] = (u'v - uv')\partial_x, \tag{1.10}$$

where u' denotes the derivative with respect to x. The dual space of $\mathfrak{X}(S^1)$ is identified with the space of quadratic forms $\{udx \otimes dx \mid u \in C^\infty(S^1)\}$, denoted by $\mathfrak{X}(S^1)^*$. For any $udx \otimes dx \in \mathfrak{X}(S^1)^*$ and $v\partial_x \in \mathfrak{X}(S^1)$, we have

$$(udx \otimes dx)(v\partial_x) = \int_{S^1} uvdx.$$

Define $\langle -,- \rangle$ to be the inner product over $\mathfrak{X}(S^1)$ by

$$\langle u\partial_x, v\partial_x \rangle = \int_{S^1} uvdx, \tag{1.11}$$

and call it the ***L^2-inner product***. Its induced inertia operator $A : \mathfrak{X}(S^1) \to \mathfrak{X}(S^1)^*$ is given by

$$A(u\partial_x) = udx \otimes dx.$$

The following lemma and its proof essentially appear in Michor and Ratio [17].

Lemma 1.12 ([17, p.9])**.** For the orientation-preserving diffeomorphism group $\mathrm{Diff}(S^1)$, the coadjoint action of $\mathfrak{X}(S^1)$ is given by

$$\mathrm{ad}^*_{u\partial_x}(vdx \otimes dx) = (-2u'v - uv')dx \otimes dx.$$

for any $u\partial_x \in \mathfrak{X}(S^1)$ and $vdx \otimes dx \in \mathfrak{X}(S^1)^*$.

Proof. Let $u\partial_x, w\partial_x \in \mathfrak{X}(S^1)$ and $vdx \otimes dx \in \mathfrak{X}(S^1)^*$. By (1.10), we see that

$$\begin{aligned}
(\mathrm{ad}^*_{u\partial_x} vdx \otimes dx)(w\partial_x) &= -(vdx \otimes dx)(\mathrm{ad}_{u\partial_x} w\partial_x) \\
&= -(vdx \otimes dx)((u'w - uw')\partial_x) \\
&= -\int_{S^1} v(u'w - uw')dx \\
&= \int_{S^1} (-2u'v - uv')wdx \\
&= ((-2u'v - uv')dx \otimes dx)(w\partial_x),
\end{aligned}$$

which implies the claim. ∎

Now, by the Euler equation, Lemma 1.12 and a direct calculation, we can show that:

Theorem 1.13 ([17, p.10])**.** *Given $c : \mathbb{R} \to \mathrm{Diff}(S^1)$ and $u : S^1 \times \mathbb{R} \to \mathbb{R}$ a function of $(x, t) \in S^1 \times \mathbb{R}$, suppose that $m := u dx \otimes dx : \mathbb{R} \to \mathfrak{X}(S^1)^*$ is the curve corresponding to c in (1.4). Then, c is a geodesic in $\mathrm{Diff}(S^1)$ with respect to the right-invariant metric induced from the L^2-inner product if and only if*

$$\dot{u} dx \otimes dx = -3u'u dx \otimes dx, \tag{1.12}$$

where $\dot{u}$ denotes the derivative with respect to t and u' with respect to x. ■

By rewriting (1.12), we finally obtain

$$\dot{u} = -3u'u, \tag{1.13}$$

which is called ***Burgers' Equation***. As a corollary, we will apply Burgers' equation to obtain the geodesic equation for $\mathrm{Diff}(S^1)$ in the following. The proof essentially appears in Michor and Ratio [17].

Corollary 1.14 ([17, p.3])**.** The geodesic equation for $\mathrm{Diff}(S^1)$ is given by

$$2\dot{c}'\dot{c} + \ddot{c}c' = 0, \tag{1.14}$$

where $c : [a, b] \to \mathrm{Diff}(S^1)$ is a smooth curve in $\mathrm{Diff}(S^1)$.

Proof. The velocity field $u : [a, b] \to C^\infty(S^1)$ of c is computed as

$$u(-, t) = r_{c(-,t)^{-1}*}(\dot{c}(-, t)) = \frac{d}{ds}\bigg|_{s=t} c(-, s) \circ c(-, t)^{-1} = \dot{c}(-, t) \circ c(-, t)^{-1},$$

where r_φ denotes the right group multiplication by $\varphi \in \mathrm{Diff}(S^1)$. It follows that

$$\dot{u}(-, t) = (\dot{c}'(-, t) \circ c(-, t)^{-1})(c(-, t)^{-1})_t + \ddot{c}(-, t) \circ c(-, t)^{-1}$$

Since

$$c(-, t) \circ c(-, t)^{-1} = \mathrm{id}_{S^1},$$

we see that

$$(c'(-, t) \circ c(-, t)^{-1})(c(-, t)^{-1})_t + \dot{c}(-, t) \circ c(-, t)^{-1} = 0,$$

which implies that

$$(c(-, t)^{-1})_t = -\frac{\dot{c}(-, t) \circ c(-, t)^{-1}}{c'(-, t) \circ c(-, t)^{-1}}.$$

Therefore, we have

$$\dot{u}(-, t) = -\frac{(\dot{c}(-, t) \circ c(-, t)^{-1})(\dot{c}'(-, t) \circ c(-, t)^{-1})}{c'(-, t) \circ c(-, t)^{-1}} + \ddot{c}(-, t) \circ c(-, t)^{-1} \tag{1.15}$$

Similarly, we have

$$u'(-,t) = \frac{\dot{c}'(-,t) \circ c(-,t)^{-1}}{c'(-,t) \circ c(-,t)^{-1}}. \tag{1.16}$$

Substituting (1.15) and (1.16) into Burgers' equation, we obtain

$$2\dot{c}'\dot{c} + \ddot{c}c' = 0,$$

which implies the claim. ■

Definition 1.15 ([22, Definition 15.8])**.** The ***exponential map*** for a Lie group G with Lie algebra $\mathfrak{g}$ is the map $\exp : \mathfrak{g} \to G$ given by

$$\exp(X) = \varphi_{\tilde{X}}(1),$$

where $\varphi_{\tilde{X}}$ represents the integral curve of the vector field $\tilde{X}$ with initial vector X, called the ***exponential curve***. Since

$$\exp(tX) = \varphi_{\widetilde{tX}}(1) = \varphi_{\tilde{X}}(t),$$

we see that $\exp((-)X)$ can also be used to denote the exponential curve.

As a corollary, we will apply the geodesic equation (1.14) to provide a condition to determine whether an exponential curve in $\mathrm{Diff}(S^1)$ is a geodesic or not in the following.

Corollary 1.16. For any $u\partial_x \in \mathfrak{X}(S^1)$, the exponential curve $\exp(tu\partial_x)$ in $\mathrm{Diff}(S^1)$ is a geodesic if and only if $u = \mathrm{const} \neq 0$.

Proof. Denote by $\varphi_{\widetilde{u\partial_x}}$ the exponential curve. It's easy to see that

$$\dot{\varphi}_{\widetilde{u\partial_x}}(t) = \widetilde{u\partial_x}_{\varphi_{\widetilde{u\partial_x}}(t)} = l_{\varphi_{\widetilde{u\partial_x}}(t)}{}_*(u\partial_x) = \frac{d}{ds}\bigg|_{s=0} \varphi_{\widetilde{u\partial_x}}(t) \circ c_{u\partial_x}(s) = \varphi_{\widetilde{u\partial_x}}(t)'u, \tag{1.17}$$

where $c_{u\partial_x}$ is a curve in $\mathrm{Diff}(S^1)$ with initial vector $u\partial_x \in \mathfrak{X}(S^1)$. It follows that

$$\ddot{\varphi}_{\widetilde{u\partial_x}}(t) = \dot{\varphi}_{\widetilde{u\partial_x}}(t)'u = (\varphi_{\widetilde{u\partial_x}}(t)'u)'u = \varphi_{\widetilde{u\partial_x}}(t)''u^2 + \varphi_{\widetilde{u\partial_x}}(t)'u'u. \tag{1.18}$$

Moreover, we have

$$\dot{\varphi}_{\widetilde{u\partial_x}}(t)' = (\varphi_{\widetilde{u\partial_x}}(t)'u)' = \varphi_{\widetilde{u\partial_x}}(t)''u + \varphi_{\widetilde{u\partial_x}}(t)'u'. \tag{1.19}$$

Substituting (1.17), (1.18) and (1.19) into (1.14), we have

$$(\varphi''_{\widetilde{u\partial_x}}u^2 + \varphi'_{\widetilde{u\partial_x}}u'u)\varphi'_{\widetilde{u\partial_x}} + 2(\varphi''_{\widetilde{u\partial_x}}u + \varphi'_{\widetilde{u\partial_x}}u')(\varphi'_{\widetilde{u\partial_x}}u) = 0.$$

If $\varphi'_{\widetilde{u\partial_x}} = 0$, then $\varphi_{\widetilde{u\partial_x}}(0) = \mathrm{const} \neq \mathrm{id}_{S^1}$, which contradicts to the definition of the integral curve $\varphi_{\widetilde{u\partial_x}}$. It follows that

$$(\varphi'_{\widetilde{u\partial_x}}u)' = 0,$$

which implies that

$$\varphi_{\widetilde{u\partial_x}}(t)'u = f(t), \tag{1.20}$$

where f is a smooth function with the parameter t. Substituting $t = 0$ into (1.20) , we have

$$u = f(0)$$

which implies the claim. ■

1.2 Generalized Bott-Virasoro Groups

In this section, we will use the connection cochain which is introduced by Moriyoshi [19] to obtain a smooth Euler cocycle (see Ghys [9] for instance) on $\mathrm{Diff}(S^1)$. In Section 1.2.1, we will discuss the Bott-Virasoro group and its relationship to Hill's operator (see [14] for instance for the details of Hill's operator). In Section 1.2.3, with the help of the smooth Euler cocycle, we will construct the generalized Bott-Virasoro group. The content in this section is mainly adapted from Khesin and Wendt [14] and Moriyoshi [19].

1.2.1 Connection Cochain

Definition 1.17 ([19, Definition 1])**.** A ***central extension*** of a group G is an exact sequence of groups

$$1 \longrightarrow A \longrightarrow \tilde{G} \xrightarrow{\pi} G \longrightarrow 1 \tag{1.21}$$

such that the image of A lies in the center of $\tilde{G}$. A map $\tau : \tilde{G} \to A$ such that

$$\tau(\tilde{g}a) = \tau(\tilde{g}) + a,$$

for all $\tilde{g} \in \tilde{G}$ and $a \in A$, is called a ***connenction cochain***.

Remark 1.18. If all the groups in (1.21) are Lie groups and all maps are smooth, then (1.21) is called a ***smooth central extension***. In our thesis, when focusing on a smooth central extension, it is enough to consider A as a vector space.

The following lemma essentially appears in Moriyoshi [19] and we refer the readers to it for the proof.

Lemma 1.19 ([19, Proposition 1])**.** Let $\tau : \tilde{G} \to A$ be a connection cochain of the central extension (1.21). Then, there exists a well-defined 2-cocycle $\sigma : G \times G \to A$ given by

$$\sigma(g_1, g_2) := \tau(\tilde{g}_1) - \tau(\tilde{g}_1\tilde{g}_2) + \tau(\tilde{g}_2),$$

where $\tilde{g}_1, \tilde{g}_2 \in \tilde{G}$ lift $g_1, g_2 \in G$, respectively, called the ***curvature*** of τ. ■

Definition 1.20. For the central extension (1.21), let $s : G \to \tilde{G}$ be a section of π. Then, the ***Euler cocycle*** $\chi : G \times G \to A$ of (1.21) induced from s is defined by

$$\chi(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1},$$

for any $g_1, g_2 \in G$.

We will discuss the relationship between the connection cochain and the Euler cocycle in the following remark.

Remark 1.21 ([19, Proposition 2])**.** Let $\tau : \tilde{G} \to A$ be a connection cochain of the central extension (1.21) and σ its curvature. Choose a section $s : G \to \tilde{G}$ given by

$$s(g) := \tilde{g}\tau(\tilde{g})^{-1},$$

where $\tilde{g} \in \tilde{G}$ is a lift of $g \in G$. By the definition of connection cochain, the section s is well-defined. Let χ be the Euler cocycle induced from s. Then, for any $g_1, g_2 \in G$, we have

$$\begin{aligned}
\chi(g_1, g_2) &= s(g_1)s(g_2)s(g_1g_2)^{-1} \\
&= \tilde{g}_1\tau(\tilde{g}_1)^{-1}\tilde{g}_2\tau(\tilde{g}_2)^{-1}\tau(\tilde{g}_1\tilde{g}_2)\tilde{g}_2^{-1}\tilde{g}_1^{-1} \\
&= -\tau(\tilde{g}_1) + \tau(\tilde{g}_1\tilde{g}_2) - \tau(\tilde{g}_2) \\
&= -\sigma(g_1, g_2),
\end{aligned}$$

where $\tilde{g}_1, \tilde{g}_2 \in \tilde{G}$ lift $g_1, g_2 \in G$, respectively.

Definition 1.22 ([14, Definition 2.11])**.** Two central extensions are said to be ***equivalent*** if there exists a commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}"'] & \tilde{G} \arrow[r] \arrow[d, "\Phi"] & G \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
1 \arrow[r] & A \arrow[r] & \tilde{G}' \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

where $\Phi : \tilde{G} \to \tilde{G}'$ is an isomorphism.

Example 1.23 ([14, p.25])**.** Let χ be the Euler cocycle for (1.21) induced from s, and $G \times_\chi A$ a group with multiplication

$$(g, a)(h, b) = (gh, a + b + \chi(g, h)). \tag{1.22}$$

It's easy to see that the Euler cocycle satisfies the cocycle condition

$$\chi(g_1g_2, g_3)\chi(g_1, g_2) = \chi(g_1, g_2g_3)\chi(g_2, g_3),$$

which ensures that $G \times_\chi A$ becomes a group. By (1.22) and the definition of the Euler cocycle, the unit element of this group is $(1_G, -s(1_G))$. Now, consider the following central extension

$$1 \longrightarrow A \longrightarrow G \times_\chi A \xrightarrow{\mathrm{pr}_1} G \longrightarrow 1\ . \tag{1.23}$$

We can see that it is equivalent to the central extension (1.21) with the isomorphism $\Phi : \tilde{G} \to G \times_\chi A$ given by

$$\Phi(\tilde{g}) := (g, \tilde{g}s(g)^{-1})$$

where $\tilde{g} \in \tilde{G}$ lifts $g \in G$.

Remark 1.24. When the central extension (1.21) is smooth, it is still possible to construct the Euler cocycle χ and the group $G \times_\chi A$, with the procedures used in Definition 1.20 and Example 1.23. However, only when the section s is smooth, can the group $G \times_\chi A$ be a Lie group and (1.23) be equivalent to (1.21) as smooth central extensions.

By Lemma 1.3, and a direct calculation, we can prove that:

Lemma 1.25 ([14, Proposition 3.14])**.** Suppose that (1.21) is a smooth central extension with A being a vector space. Let χ be the Euler cocycle induced from a smooth section s and $e : \mathfrak{g} \times \mathfrak{g} \to A$ a map given by

$$e(X,Y) = \frac{d^2}{dsdt}\bigg|_{t=s=0} \chi(c_X(t), c_Y(s)) - \frac{d^2}{dsdt}\bigg|_{t=s=0} \chi(c_Y(s), c_X(t)),$$

where c_X and c_Y are curves in G with initial vectors X and Y, respectively. Then, the Lie algebra of $G \times_\chi A$ assumes the form $\mathfrak{g} \times_e A$ with Lie bracket $[-,-]$ given by

$$[(X,a),(Y,b)] = ([X,Y], e(X,Y)),$$

where $[-,-]$ on the right-hand side is the Lie bracket over $\mathfrak{g}$. ■

Remark 1.26 ([19, Remark 1])**.** Suppose that we have the central extension (1.21) and a homomorphism $\iota : A \to B$, where B is an abelian group. Then, a map $\tau : \tilde{G} \to B$ such that

$$\tau(\tilde{g}a) = \tau(\tilde{g}) + \iota(a)$$

for all $\tilde{g} \in G$ and $a \in A$, is called a ***connection cochain with values in*** B. Setting the quotient space $\tilde{G} \times_{A,\iota} B := (\tilde{G} \times B)/\sim$ where $(\tilde{g}_1, b_1) \sim (\tilde{g}_2, b_2)$ if and only if $(\tilde{g}_1 a, b_1 - \iota(a)) = (\tilde{g}_2, b_2)$ for some $a \in A$, and defining its multiplication by

$$[\tilde{g}_1, b_1][\tilde{g}_2, b_2] = [\tilde{g}_1\tilde{g}_2, b_1 + b_2],$$

we construct the following central extension

$$1 \longrightarrow B \longrightarrow \tilde{G} \times_{A,\iota} B \longrightarrow G \longrightarrow 1\ . \tag{1.24}$$

Let $\tau_B : \tilde{G} \times_{A,\iota} B \to B$ be a map given by

$$\tau_B([\tilde{g}, b]) := \tau(\tilde{g}) + b, \tag{1.25}$$

which is obviously well-defined. Since

$$\tau_B([\tilde{g}, b_1]b_2) = \tau_B([\tilde{g}, b_1 + b_2]) = \tau(\tilde{g}) + b_1 + b_2 = \tau_B([\tilde{g}, b_1]) + b_2,$$

by Definition 1.17, we see that the map τ_B is a connection cochain of (1.24).

1.2.2 The Bott-Virasoro Group

The ***Bott-Virasoro group***, denoted by $\mathrm{Diff}(S^1) \times_B \mathbb{R}$, is defined to be a set $\{(\varphi, a) \mid \varphi \in \mathrm{Diff}(S^1), a \in \mathbb{R}\}$ with multiplication

$$(\varphi, a)(\psi, b) = (\varphi \circ \psi, a + b + B(\varphi, \psi)),$$

where $B : \mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1) \to \mathbb{R}$ is the ***Bott cocycle*** given by

$$B(\varphi, \psi) = \frac{1}{2} \int_{S^1} \log(\varphi' \circ \psi) d \log \psi'.$$

The Lie algebra of the Bott-Virasoro group has the form $\mathfrak{X}(S^1) \times_\omega \mathbb{R}$, where $\omega : \mathfrak{X}(S^1) \times \mathfrak{X}(S^1) \to \mathbb{R}$ is the ***Gelfand-Fuchs cocycle*** given by

$$\omega(u\partial_x, v\partial_x) = \int_{S^1} u'v''dx,$$

called the ***Virasoro algebra***. The dual space of the Virasoro algebra is identified with $\{(udx \otimes dx, a) | u \in C^\infty(S^1), a \in \mathbb{R}\}$, denoted by $(\mathfrak{X}(S^1) \times_\omega \mathbb{R})^*$. For any $(udx \otimes dx, a) \in (\mathfrak{X}(S^1) \times_\omega \mathbb{R})^*$ and $(v\partial_x, b) \in \mathfrak{X}(S^1) \times_\omega \mathbb{R}$, we have

$$(udx \otimes dx, a)(v\partial_x, b) = \int_{S^1} uvdx + ab.$$

Given $\varphi \in \mathrm{Diff}(S^1)$, the ***Schwarzian derivative*** is defined to be

$$S(\varphi) := \frac{\varphi'\varphi''' - \frac{3}{2}(\varphi'')^2}{(\varphi')^2}.$$

In the following, we will present an expression of the coadjoint action of the Bott-Virasoro group, and provide an alternative proof.

Lemma 1.27 ([14, Proposition 2.7])**.** For $(\varphi, a) \in \mathrm{Diff}(S^1) \times_B \mathbb{R}$ and $(udx \otimes dx, b) \in (\mathfrak{X}(S^1) \times_\omega \mathbb{R})^*$, the coadjoint action of the Bott-Virasoro group is given by

$$\mathrm{Ad}^*_{(\varphi,a)}(udx \otimes dx, b) = (((u \circ \varphi^{-1})((\varphi^{-1})')^2 + bS(\varphi^{-1}))dx \otimes dx, b). \tag{1.26}$$

Proof. First, we compute the adjoint action of the Bott-Virasoro group. Given $(\varphi, a) \in \mathrm{Diff}(S^1) \times_B \mathbb{R}$ and $(v\partial_x, c) \in \mathfrak{X}(S^1) \times_\omega \mathbb{R}$, let $\varphi_{v\partial_x}$ be the flow of $v\partial_x \in \mathfrak{X}(S^1)$. It is known that $\varphi_{v\partial_x}$ can be regarded as a curve in $\mathrm{Diff}(S^1)$ with initial vector $v\partial_x$. By the definition of the adjoint action, we have

$$\begin{aligned}
\mathrm{Ad}_{(\varphi^{-1},-a)}(v\partial_x, c) &= \frac{d}{dt}\bigg|_{t=0} (\varphi^{-1}, -a)(\varphi_{v\partial_x}(t), ct)(\varphi, a) \\
&= \frac{d}{dt}\bigg|_{t=0} (\varphi^{-1} \circ \varphi_{v\partial_x}(t) \circ \varphi, ct + B(\varphi_{v\partial_x}(t), \varphi) + B(\varphi^{-1}, \varphi_{v\partial_x}(t) \circ \varphi)) \\
&= (((\varphi^{-1})' \circ \varphi)(v \circ \varphi)\partial_x, c + \frac{d}{dt}\bigg|_{t=0} (B(\varphi_{v\partial_x}(t), \varphi) + B(\varphi^{-1}, \varphi_{v\partial_x}(t) \circ \varphi))).
\end{aligned}$$

Note that

$$
\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} B(\varphi_{v\partial_x}(t),\varphi) &= \frac{1}{2}\left.\frac{d}{dt}\right|_{t=0}\int_{S^1}\log(\varphi_{v\partial_x}(t)'\circ\varphi)d\log\varphi' \\
&= \frac{1}{2}\int_{S^1}(v'\circ\varphi)d\log\varphi' \\
&= \frac{1}{2}\int_{S^1}\frac{(v'\circ\varphi)\varphi''}{\varphi'}dx \\
&= \frac{1}{2}\int_{S^1}\frac{v'(\varphi''\circ\varphi^{-1})}{\varphi'\circ\varphi^{-1}}(\varphi^{-1})'dx \\
&= \frac{1}{2}\int_{S^1}v'(\varphi''\circ\varphi^{-1})((\varphi^{-1})')^2dx \\
&= -\frac{1}{2}\int_{S^1}((\varphi''\circ\varphi^{-1})'((\varphi^{-1})')^2+2(\varphi''\circ\varphi^{-1})(\varphi^{-1})'(\varphi^{-1})'')vdx.
\end{aligned}
$$

Since

$$
0=(\varphi\circ\varphi^{-1})''=((\varphi'\circ\varphi^{-1})(\varphi^{-1})')'=(\varphi''\circ\varphi^{-1})((\varphi^{-1})')^2+(\varphi'\circ\varphi^{-1})(\varphi^{-1})'',
$$

we see that

$$
\varphi''\circ\varphi^{-1}=-\frac{(\varphi'\circ\varphi^{-1})(\varphi^{-1})''}{((\varphi^{-1})')^2}=-\frac{(\varphi^{-1})''}{((\varphi^{-1})')^3}. \tag{1.27}
$$

It follows that

$$
\begin{aligned}
(\varphi''\circ\varphi^{-1})' &= -\frac{(\varphi^{-1})'''((\varphi^{-1})')^3-3(\varphi^{-1})''((\varphi^{-1})')^2(\varphi^{-1})''}{((\varphi^{-1})')^6} \\
&= \frac{-(\varphi^{-1})'''(\varphi^{-1})'+3((\varphi^{-1})'')^2}{((\varphi^{-1})')^4}.
\end{aligned} \tag{1.28}
$$

Substituting (1.27) and (1.28), we have

$$
\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} B(\varphi_{v\partial_x}(t),\varphi) &= -\frac{1}{2}\int_{S^1}(\frac{-(\varphi^{-1})'''(\varphi^{-1})'+3((\varphi^{-1})'')^2}{((\varphi^{-1})')^2}-2\frac{((\varphi^{-1})'')^2}{((\varphi^{-1})')^2})vdx \\
&= \frac{1}{2}\int_{S^1}\frac{(\varphi^{-1})'''(\varphi^{-1})'-((\varphi^{-1})'')^2}{((\varphi^{-1})')^2}vdx.
\end{aligned}
$$

Similarly, we can obtain

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} B(\varphi^{-1}, \varphi_{v\partial_x}(t)\circ\varphi)) &= \frac{1}{2}\left.\frac{d}{dt}\right|_{t=0} \int_{S^1} \log(\varphi^{-1}\circ\varphi_{v\partial_x}(t)\circ\varphi)' d\log(\varphi_{v\partial_x}(t)\circ\varphi)' \\
&= \frac{1}{2}\int_{S^1} (((\varphi^{-1})'\circ\varphi)(v\circ\varphi))' d\log\varphi' \\
&= \frac{1}{2}\int_{S^1} \frac{(((\varphi^{-1})''\circ\varphi)\varphi'(v\circ\varphi) + ((\varphi^{-1})'\circ\varphi)(v'\circ\varphi)\varphi')\varphi''}{\varphi'} dx \\
&= \frac{1}{2}\int_{S^1} (((\varphi^{-1})''\circ\varphi)(v\circ\varphi)\varphi'' + ((\varphi^{-1})'\circ\varphi)(v'\circ\varphi)\varphi'')dx \\
&= \frac{1}{2}\int_{S^1} ((\varphi^{-1})''(\varphi''\circ\varphi^{-1})(\varphi^{-1})'v + ((\varphi^{-1})')^2(\varphi''\circ\varphi^{-1})v')dx \\
&= \frac{1}{2}\int_{S^1} (-\frac{((\varphi^{-1})'')^2}{((\varphi^{-1})')^2} + \frac{(\varphi^{-1})'''(\varphi^{-1})' - ((\varphi^{-1})'')^2}{((\varphi^{-1})')^2})vdx.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathrm{Ad}_{(\varphi^{-1},-a)}(v\partial_x, c) &= (((\varphi^{-1})'\circ\varphi)(v\circ\varphi)\partial_x, c + \int_{S^1} \frac{(\varphi^{-1})'''(\varphi^{-1})' - \frac{3}{2}((\varphi^{-1})'')^2}{((\varphi^{-1})')^2} vdx) \\
&= (((\varphi^{-1})'\circ\varphi)(v\circ\varphi)\partial_x, c + \int_{S^1} S(\varphi^{-1})vdx)
\end{aligned}$$

Now, for $(udx\otimes dx, b) \in (\mathfrak{X}(S^1)\times_\omega \mathbb{R})^*$, since

$$\begin{aligned}
(\mathrm{Ad}^*_{(\varphi,a)}(udx\otimes dx, b))(v\partial_x, c) &= (udx\otimes dx, b)(\mathrm{Ad}_{(\varphi^{-1},-a)}(v\partial_x, c)) \\
&= (udx\otimes dx, b)(((\varphi^{-1})'\circ\varphi)(v\circ\varphi)\partial_x, c + \int_{S^1} S(\varphi^{-1})vdx) \\
&= \int_{S^1} ((u\circ\varphi^{-1})((\varphi^{-1})')^2 + bS(\varphi^{-1}))vdx + bc \\
&= (((u\circ\varphi^{-1})((\varphi^{-1})')^2 + bS(\varphi^{-1}))dx\otimes dx, b)(v\partial_x, c),
\end{aligned}$$

we finally get the claim. ■

By a direct computation, we can show that:

Lemma 1.28 ([14, p.73])**.** The Schwarzian derivative satisfies

$$S(\varphi\circ\psi) = (S(\varphi)\circ\psi)(\psi')^2 + S(\psi), \tag{1.29}$$

for any $\varphi, \psi \in \mathrm{Diff}(S^1)$. ■

Now, we will show the significance of the expression (1.26) of the coadjoint action of the Bott-Virasoro group in the following remark.

Remark 1.29. Given $a, b \in \mathbb{R}$, define $\Pi_a : \mathrm{Diff}(S^1) \to (\mathfrak{X}(S^1) \times_\omega \mathbb{R})^*$ to be a map on $\mathrm{Diff}(S^1)$ by

$$\Pi_{a,b}(\varphi) := ((aS(\varphi) + b(\varphi')^2)dx \otimes dx, a).$$

The Bott-Virasoro group acts on $\mathrm{Diff}(S^1)$ naturally such that for any $(\psi, d) \in \mathrm{Diff}(S^1) \times_B \mathbb{R}$, we have

$$\varphi \cdot (\psi, d) := \varphi \circ \psi \tag{1.30}$$

By the expression of the coadjoint action, since

$$\begin{aligned} \Pi_{a,b}(\varphi \cdot (\psi, d)) &= \Pi_{a,b}(\varphi \circ \psi) \\ &= ((aS(\varphi \circ \psi) + b((\varphi \circ \psi)')^2)dx \otimes dx, a) \\ &= (((aS(\varphi) + b(\varphi')^2) \circ \psi)(\psi')^2 + aS(\psi))dx \otimes dx, a) \\ &= \mathrm{Ad}^*_{(\psi^{-1}, -d)}((aS(\varphi) + b(\varphi')^2)dx \otimes dx, a) \\ &= \mathrm{Ad}^*_{(\psi^{-1}, -d)}\Pi_{a,b}(\varphi), \end{aligned} \tag{1.31}$$

we see that $\Pi_{a,b}$ is equivariant with respect to the natural action (1.30) and the coadjoint action of the Bott-Virasoro group. In this computation, the expression of the coadjoint action (1.26) is essential. Moreover, it is the $\mathbb{R}$-part of the map $\Pi_{a,b}$ that ensures it to be equivariant. This shows the significance of the $\mathbb{R}$-part.

Remark 1.30 ([14, Corollary 2.10])**.** For $a, b, c, d \in \mathbb{R}$ such that $ad - bc = 1$, the diffeomorphism $\varphi : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$ given by

$$\varphi(x) := \frac{ax + b}{cx + d},$$

is called a Mobius transformation. It's easy to verify that

$$S(\varphi) = 0.$$

Since $\mathbb{R} \cup \{\infty\}$ is diffeomorphic to S^1, such Mobius transformation φ can be regarded as an element in $\mathrm{Diff}(S^1)$.

By the similar method we have used in the proof of Lemma 1.12, we can show that:

Lemma 1.31 ([14, p.73])**.** The coadjoint action of the Virasoro algebra is given by

$$\mathrm{ad}^*_{(u\partial_x, a)}(vdx \otimes dx, b) = ((-bu''' - 2u'v - uv')dx \otimes dx, 0), \tag{1.32}$$

where $(u\partial_x, a) \in \mathfrak{X}(S^1) \times_\omega \mathbb{R}$ and $(vdx \otimes dx, b) \in (\mathfrak{X}(S^1) \times_\omega \mathbb{R})^*$. ■

Remark 1.32 ([14, p.74])**.** It is convenient to consider the dual Virasoro algebra as the space of ***Hill's operator*** $\{a\partial_x^2 + u \mid u \in C^\infty(S^1), a \in \mathbb{R}\}$. Let $a = 1$ and f, g be two independent solutions of the corresponding Hill's differential equation

$$(\partial_x^2 + u)y = 0, \tag{1.33}$$

for an unknown function y. In this case, although the equation (1.33) has periodic coefficients, the solutions need not be periodic, but instead are functions over $\mathbb{R}$.

The following proposition and its proof essentially appear in Khesin and Wendt [14].

Proposition 1.33 ([14, Proposition 2.9])**.** Define $\eta : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ to be the ***ratio map*** on $\mathbb{R}$ by

$$\eta(x) := \frac{f(x)}{g(x)}.$$

Then, for the differential equation (1.33), we have $u = S(\eta)/2$.

Proof. First, we note tht the ***Wronskian*** given by

$$W(f,g) := \det \begin{bmatrix} f & f' \\ g & g' \end{bmatrix} = fg' - f'g$$

is constant, since $W' = 0$. It follows that

$$\eta' = \frac{f'g - fg'}{g^2} = -\frac{W}{g^2},$$

and

$$\eta'' = \frac{2Wg'}{g^3}.$$

Moreover, we have

$$\eta''' = \frac{2W(g''g - 3(g')^2)}{g^4}.$$

Therefore, we see that

$$S(\eta) = \frac{\eta'\eta''' - \frac{3}{2}(\eta'')^2}{(\eta')^2} = \frac{-\frac{W}{g^2}\frac{2W(g''g-3(g')^2)}{g^4} - \frac{3}{2}(\frac{2Wg'}{g^3})^2}{(-\frac{W}{g^2})^2} = \frac{-2g''g + 6(g')^2 - 6(g')^2}{g^2} = 2u,$$

which implies the claim. ∎

The following lemma and its proof essentially appear in Khesin and Wendt [14].

Lemma 1.34 ([14, Corollary 2.10])**.** For $a, b, c, d \in \mathbb{R}$ such that $ad - bc = 1$, let η be the ratio map of two independent solutions of (1.33). Then, we have

$$S(\eta) = S(\frac{a\eta + b}{c\eta + d}).$$

This means that S is invariant under the Mobius transformation. ∎

1.2.3 Generalized Bott-Virasoro Groups

Consider the central extension of the diffeomorphism group $\mathrm{Diff}(S^1)$:

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Diff}(S^1)} \longrightarrow \mathrm{Diff}(S^1) \longrightarrow 1,$$

where $\widetilde{\mathrm{Diff}(S^1)}$ is defined in Section 1.1.3. An integer $k \in \mathbb{Z}$ is regarded as a element in $\widetilde{\mathrm{Diff}(S^1)}$ such that

$$k(x) = x + 2k\pi.$$

For any $\alpha \in \mathbb{R}$, we regard it as a function over $\mathbb{Z}$ such that for any $k \in \mathbb{Z}$ we have

$$\alpha(k) = -\alpha 2k\pi^2.$$

Let $\tau^\alpha : \widetilde{\mathrm{Diff}(S^1)} \to \mathbb{R}$ be a map given by

$$\tau^\alpha(\tilde{f}) := -\frac{\alpha}{2}\int_0^{2\pi} \tilde{f}(x)dx + \alpha\pi^2,$$

where $\tilde{f} \in \widetilde{\mathrm{Diff}(S^1)}$. Since

$$\begin{aligned}\tau^\alpha(\tilde{f} \circ k) &= -\frac{\alpha}{2}\int_0^{2\pi} \tilde{f}(x + 2k\pi)dx + \alpha\pi^2 \\ &= -\frac{\alpha}{2}\int_0^{2\pi} \tilde{f}(x)dx - \alpha 2k\pi^2 + \alpha\pi^2 \\ &= \tau^\alpha(\tilde{f}) + \alpha(k),\end{aligned}$$

we see that τ^α is a connection cochain with values in $\mathbb{R}$, where the homomorphism ι mentioned in Remark 1.26 is replaced by the function α. As (1.23), we can construct the following central extension

$$1 \longrightarrow \mathbb{R} \longrightarrow \widetilde{\mathrm{Diff}(S^1)} \times_{\mathbb{Z},\alpha} \mathbb{R} \longrightarrow \mathrm{Diff}(S^1) \longrightarrow 1\ , \tag{1.34}$$

By the discussion in Remark 1.26, the map $\tau^\alpha_{\mathbb{R}} : \widetilde{\mathrm{Diff}(S^1)} \times_{\mathbb{Z},\alpha} \mathbb{R} \to \mathbb{R}$ defined by

$$\tau^\alpha_{\mathbb{R}}([\tilde{f}, a]) = \tau^\alpha(\tilde{f}) + a$$

is a connection cochain of (1.34). Let $s^\alpha_{\mathbb{R}} : \mathrm{Diff}(S^1) \to \widetilde{\mathrm{Diff}(S^1)} \times_{\mathbb{Z},\alpha} \mathbb{R}$ be a section defined by

$$s^\alpha_{\mathbb{R}}(f) := [\tilde{f}, -\tau^\alpha_{\mathbb{R}}([\tilde{f}, 0])].$$

Since for any $k \in \mathbb{Z}$,

$$[\tilde{f} \circ k, -\tau^\alpha_{\mathbb{R}}([\tilde{f} \circ k, 0])] = [\tilde{f}, -\tau^\alpha_{\mathbb{R}}([\tilde{f}, 0])], \tag{1.35}$$

we see that the section $s^\alpha_{\mathbb{R}}$ is well-defined. Note that $s^\alpha_{\mathbb{R}}$ is smooth since we can always choose local smooth sections $s : \mathrm{Diff}(S^1) \to \widetilde{\mathrm{Diff}(S^1)}$ such that

$$s^\alpha_{\mathbb{R}}(f) = [s(f), -\tau^\alpha_{\mathbb{R}}([s(f), 0])].$$

and by (1.35), $s^\alpha_{\mathbb{R}}$ is independent of the choices of s. By Definition 1.20, the Euler cocycle $\chi^\alpha_{\mathbb{R}} : \mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1) \to \mathbb{R}$ of the central extension (1.34) induced from $s^\alpha_{\mathbb{R}}$ is given by

$$\begin{aligned}
\chi^\alpha_{\mathbb{R}}(f_1, f_2) &= s^\alpha_{\mathbb{R}}(f_1)s^\alpha_{\mathbb{R}}(f_2)s^\alpha_{\mathbb{R}}(f_1 f_2)^{-1} \\
&= [\tilde{f}_1, -\tau^\alpha_{\mathbb{R}}([\tilde{f}_1, 0])][\tilde{f}_2, -\tau^\alpha_{\mathbb{R}}([\tilde{f}_2, 0])][\tilde{f}_2^{-1} \circ \tilde{f}_1^{-1}, \tau^\alpha_{\mathbb{R}}([\tilde{f}_1 \circ \tilde{f}_2, 0])] \\
&= -\frac{\alpha}{2}\int_0^{2\pi} (\tilde{f}_1 \circ \tilde{f}_2(x) - \tilde{f}_1(x) - \tilde{f}_2(x))dx - \alpha\pi^2.
\end{aligned}$$

Then, we can define the generalized Bott-Virasoro group in the following.

Definition 1.35. Given $\alpha, \beta \in \mathbb{R}$, the ***generalized Bott-Virasoro group*** is defined to be $\mathrm{Diff}(S^1) \times_{\chi^\alpha_{\mathbb{R}} + \beta B} \mathbb{R}$.

Denote by $e : \mathfrak{X}(S^1) \times \mathfrak{X}(S^1) \to \mathbb{R}$ the cocycle on $\mathfrak{X}(S^1)$ given by

$$e(u\partial_x, v\partial_x) = \int_{S^1} uv' dx.$$

By Lemma 1.25, since for any $u\partial_x, v\partial_x \in \mathfrak{X}(S^1)$ we have

$$\begin{aligned}
\left.\frac{d^2}{dsdt}\right|_{t=s=0} \chi^\alpha_{\mathbb{R}}(c_{u\partial_x}(t), c_{v\partial_x}(s)) &= -\frac{\alpha}{2}\left.\frac{d^2}{dsdt}\right|_{t=s=0} \int_0^{2\pi} (\tilde{c}_{u\partial x}(t) \circ \tilde{c}_{v\partial x}(s)(x))dx \\
&= \frac{\alpha}{2}\int_{S^1} uv' dx,
\end{aligned}$$

where $c_{u\partial_x}$ and $c_{v\partial_x}$ are two curves in $\mathrm{Diff}(S^1)$ with initial vectors $u\partial_x$ and $v\partial_x$ respectively, we see that the Lie algebra of the generalized Bott-Virasoro group assumes the form $\mathfrak{X}(S^1) \times_{\alpha e + \beta\omega} \mathbb{R}$, which is called the ***generalized Virasoro algebra***. With the similar method we have used in the proof of Lemma 1.12, we can prove that:

Lemma 1.36. The coadjoint action of the generalized Virasoro algebra on $(\mathfrak{X}(S^1) \times_{\alpha e + \beta\omega} \mathbb{R})^*$ is computed as

$$\mathrm{ad}^*_{(u\partial_x, a)}(vdx \otimes dx, b) = ((-2u'v - uv' - \beta bu''' + \alpha bu')dx \otimes dx, 0).$$

where $(u\partial_x, a) \in \mathfrak{X}(S^1) \times_{\alpha e + \beta\omega} \mathbb{R}$ and $(vdx \otimes dx, b) \in (\mathfrak{X}(S^1) \times_{\alpha e + \beta\omega} \mathbb{R})^*$. ∎

1.3 The Generalized Korteweg–de Vries Equation

In this section, we will employ the generalized Bott-Virasoro group and the Euler equation to obtain a generalized version of the KdV equation. This will be our first main result. Moreover, we will provide some basic knowledge on symplectic forms and Poisson brackets. The content in Section 1.3.1 and Section 1.3.2 is mainly adapted from Khesin and Wendt [14] and Kolev [15].

1.3.1 Symplectic Forms and Poisson Brackets

Definition 1.37 ([15, Definition 2.1])**.** For a manifold M, a ***symplectic form*** is a closed nondegenerate 2-form ω over M. Since ω is nondegenerate, for any smooth function $f \in C^\infty(M)$, there exists a vector field $X_f \in \mathfrak{X}(M)$ such that

$$i_{X_f}\omega = -df,$$

called the ***symplectic gradient***.

Definition 1.38 ([15, Definition 2.2])**.** Let M be a manifold. Then, a ***Poisson bracket*** over M is a skew-symmetric bilinear map $\{-,-\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying

1. (Jacobi identity) for any $f, g, h \in C^\infty(M)$ we have

$$\sum_{\text{cycl}} \{\{f, g\}, h\} = 0;$$

2. (Leibniz rule) for any $f, g, h \in C^\infty(M)$ we have

$$\{f, gh\} = \{f, g\}h + \{f, h\}g.$$

Note that by the Leibniz rule, for any smooth function $H \in C^\infty(M)$, there exists a vector field X_H such that

$$X_H g = \{H, g\}$$

for any $g \in C^\infty(M)$, called the ***Hamiltonian vector field***.

The following lemma and its proof essentially appear in Kolev [15].

Lemma 1.39 ([15, pp.3-5])**.** Let ω be a symplectic form over M. Then, there exists a Poisson bracket such that the Hamiltonian vector field of a function $f \in C^\infty(M)$ coincides with its symplectic gradient.

Proof. For any $f, g \in C^\infty(M)$, let $X_f, X_g \in \mathfrak{X}(M)$ be the symplectic gradients of f, g, respectively. Then, define the Poisson bracket $\{-,-\}_\omega$ over $C^\infty(M)$ by

$$\{f, g\}_\omega := \omega(X_f, X_g).$$

It's easy to see that $\{-,-\}_\omega$ is indeed a Poisson bracket. For any $f \in C^\infty(M)$, denote by $\bar{X}_f$ its Hamiltonian vector field. Since

$$\bar{X}_f g = \{f, g\} = \omega(X_f, X_g) = -i_{X_g}\omega(X_f) = dg(X_f),$$

where $g \in C^\infty(M)$, we see that the symplectic gradient of f coincides with its the Hamiltonian vector field. ∎

Definition 1.40 ([14, Definition 4.8]). Let G be a Lie group with Lie algebra $\mathfrak{g}$. The ***Lie-Poisson*** bracket $\{-,-\}_{LP} : C^\infty(\mathfrak{g}^*) \times C^\infty(\mathfrak{g}^*) \to C^\infty(\mathfrak{g}^*)$ is defined by

$$\{f,g\}_{LP}(x) := x([df(x), dg(x)]),$$

where $df(x)$ is regarded as an element in $\mathfrak{g}$ such that

$$m(df(x)) = \frac{d}{dt}\bigg|_{t=0} f(x+tm),$$

for any $m \in \mathfrak{g}^*$.

The following proposition and its proof essentially appear in Khesin and Wendt [14].

Proposition 1.41 ([14, Proposition 4.9]). The Hamiltonian equation of a function H with respect to the Lie-Poisson bracket over $C^\infty(\mathfrak{g}^*)$ is given by

$$\dot{m}(t) = -\mathrm{ad}^*_{dH(m(t))} m(t),$$

where m is the integral curve of the Hamiltonian vector field X_H.

Proof. For any $x \in \mathfrak{g}^*$, the tangent vector $X_H(x)$ is identified with an element in $\mathfrak{g}^*$. It follows that

$$\begin{aligned}(X_H(x))(df(x)) &= \frac{d}{dt}\bigg|_{t=0} f(x+tX_H(x)) \\ &= (X_H f)(x) \\ &= \{H,f\}(x) \\ &= x([dH(x), df(x)]) \\ &= (-\mathrm{ad}^*_{dH(x)} x)(df(x)),\end{aligned}$$

where $f \in C^\infty(M)$, which implies the claim. ∎

Definition 1.42 ([14, Definition 4.20]). The ***constant Poisson bracket*** associated to a point $x_0 \in \mathfrak{g}^*$ is defined to be a bracket $\{-,-\}_{x_0} : C^\infty(\mathfrak{g}^*) \times C^\infty(\mathfrak{g}^*)$ such that

$$\{f,g\}_{x_0}(x) := x_0([df(x), dg(x)]),$$

for any $f, g \in C^\infty(\mathfrak{g}^*)$.

We refers the readers to Khesin and Wendt [14] for the proof of the following lemma.

Lemma 1.43 ([14, Lemma 4.21]). The Hamiltonian equation corresponding to a function H and the constant Lie-Poisson bracket over $C^\infty(\mathfrak{g}^*)$ with freezing point m_0 is given by

$$\dot{m}(t) = -\mathrm{ad}^*_{dH(m(t))} m_0(t),$$

where m is the integral curve of the Hamiltonian vector field X_H. ∎

Remark 1.44. Let M be a manifold and $C^\infty(M)_0$ a subset of $C^\infty(M)$. Then, a Poisson bracket $\{-,-\}_{C^\infty(M)_0}$ over $C^\infty(M)_0$ can also be defined similarly as Definition 1.38, as long as for any $f, g \in C^\infty(M)_0$ we have

$$\{f,g\}_{C^\infty(M)_0} \in C^\infty(M)_0,$$

and $\{-,-\}_{C^\infty(M)_0}$ satisfies the conditions for the Poisson brackets.

1.3.2 The Korteweg-de Vries Equation

Consider the Bott-Virasoro group and the Virasoro algebra. Define the L^2-inner product $\langle -, - \rangle$ over the Virasoro algebra by

$$\langle (u\partial_x, a), (v\partial_x, b) \rangle := \int_{S^1} uv dx + ab, \tag{1.36}$$

where $(u\partial_x, a), (v\partial_x, b) \in \mathfrak{X}(S^1) \times_\omega \mathbb{R}$. By Definition 1.7, the induced inertia operator A on the Virasoro algebra is given by

$$A(u\partial_x, a) = (udx \otimes dx, a).$$

Then, by Remark 1.11 and (1.32), the Euler equation applied to the Bott-Virasoro group corresponding to the right-invariant metric induced from the L^2-inner product is expressed as

$$\begin{aligned}(\dot{u}dx \otimes dx, \dot{a}) &= \mathrm{ad}^*_{A^{-1}(udx\otimes dx,a)}(udx \otimes dx, a)\\ &= \mathrm{ad}^*_{(u\partial_x,a)}(udx \otimes dx, a)\\ &= ((-3u'u - au''')dx \otimes dx, 0),\end{aligned} \tag{1.37}$$

where $(udx \otimes dx, a)$ is a curve in the dual Virasoro algebra. We rewrite (1.37) as

$$\dot{u} = -3u'u - au''', \tag{1.38}$$

which is called the ***Korteweg–de Vries equation***.

Remark 1.45 ([14, Theorem 2.20])**.** The KdV equation can also obtained from the Hamiltonian equation over the dual Virasoro algebra. Let $\{-, -\}_{(-\frac{1}{2}dx\otimes dx,0)}$ be the constant Lie-Poisson bracket on $C^\infty((\mathfrak{X}(S^1)\times_\omega\mathbb{R})^*)$ with freezing point $(-\frac{1}{2}dx\otimes dx, 0) \in (\mathfrak{X}(S^1)\times_\omega\mathbb{R})^*$. Define $H : (\mathfrak{X}(S^1) \times_\omega \mathbb{R})^* \to \mathbb{R}$ by

$$H(udx \otimes dx, a) := \int_{S^1} (\frac{1}{2}u^3 - \frac{a}{2}(u')^2)dx.$$

Then, it's easy to see that for any $(udx \otimes dx, a), (vdx \otimes dx, b) \in (\mathfrak{X}(S^1) \times_\omega \mathbb{R})^*$, we have

$$\begin{aligned}(vdx \otimes dx, b)(dH(udx \otimes dx, a)) &= \frac{d}{dt}\bigg|_{t=0} H((udx \otimes dx, a) + t(vdx \otimes dx, b))\\ &= \frac{d}{dt}\bigg|_{t=0} \int_{S^1} (\frac{1}{2}(u + tv)^3 - \frac{a + tb}{2}((u + tv)')^2)dx\\ &= \int_{S^1} (\frac{3}{2}u^2 v - \frac{b}{2}(u')^2 - au'v')dx\\ &= (vdx \otimes dx, b)((\frac{3}{2}u^2 + au'')\partial_x, -\frac{1}{2}\int_{S^1}(u')^2 dx),\end{aligned}$$

which implies that

$$dH(udx \otimes dx, a) = ((\frac{3}{2}u^2 + au'')\partial_x, -\frac{1}{2}\int_{S^1}(u')^2 dx).$$

Then, by Lemma 1.43, we see that the Hamiltonian equation of H is expressed as

$$\begin{aligned}(\dot{u}dx \otimes dx, \dot{a}) &= -\mathrm{ad}^*_{((\frac{3}{2}u^2+au'')\partial_x, -\frac{1}{2}\int_{S^1}(u')^2dx)}(-\frac{1}{2}dx \otimes dx, 0)\\ &= (-(\frac{3}{2}u^2 + au'')'dx \otimes dx, 0)\\ &= ((-3u'u - au''')dx \otimes dx, 0),\end{aligned}$$

where $(udx \otimes dx, a)$ is a curve in the dual Virasoro algebra, which is nothing but the KdV equation.

1.3.3 The Generalized Korteweg-de Vries Equation

In this section, we would like to apply the Euler equation to the generalized Bott-Virasoro group $G := \mathrm{Diff}(S^1) \times_{\chi^\alpha_{\mathbb{R}}+\beta B} \mathbb{R}$. Let $\langle -, - \rangle$ be the L^2-inner product over the generalized Virasoro algebra $\mathrm{Lie}(G) = \mathfrak{X}(S^1) \times_{\alpha e+\beta\omega} \mathbb{R}$. Then, as one of our main results, we will prove that:

Theorem 1.46. *Let $\varphi : \mathbb{R} \to \mathrm{Diff}(S^1)$ and $d, a : \mathbb{R} \to \mathbb{R}$ be smooth curves and let $u : S^1 \times \mathbb{R} \to \mathbb{R}$ be a smooth function of $(x, t) \in S^1 \times \mathbb{R}$. Set $c := (\varphi, d) : \mathbb{R} \to G$ and $m := (udx \otimes dx, a) : \mathbb{R} \to \mathfrak{X}(S^1)^* \times \mathbb{R}$, where $\mathfrak{X}(S^1)^* \times \mathbb{R}$ is naturally identified with* $\mathrm{Lie}(G)^*$. *Suppose that m and c satisfy the condition (1.4). Then, c is a geodesic with respect to the right-invariant metric induced from the L^2-inner product if and only if*

$$(\dot{u}dx \otimes dx, \dot{a}) = ((-3uu' - \beta au''' + \alpha au')dx \otimes dx, 0), \tag{1.39}$$

where $\dot{u}$ and $\dot{a}$ denotes the derivatives with respect to t, and u' with respect to x.

Proof. It's easy to see that the induced inertia operator $A : \mathrm{Lie}(G) \to \mathrm{Lie}(G)^*$ is given by

$$(v\partial_x, b) \mapsto (vdx \otimes dx, b).$$

Then, with the help of the computation of the coadjoint action of the generalized Virasoro algeara in Lemma 1.36, and applying the Euler equation, we see that

$$\begin{aligned}(\dot{u}dx \otimes dx, \dot{a}) &= \mathrm{ad}^*_{A^{-1}(udx\otimes dx, a)}(udx \otimes dx, a)\\ &= \mathrm{ad}^*_{(u\partial_x, a)}(udx \otimes dx, a)\\ &= ((-3uu' - \beta au''' + \alpha au')dx \otimes dx, 0),\end{aligned}$$

which implies the claim. ■

We rewrite (1.39) as

$$\dot{u} = -3uu' - \beta au''' + \alpha au'. \tag{1.40}$$

By setting $u = v + \frac{1}{3}a\alpha$, and $\beta = 1$, the equation (1.40) amounts to

$$\dot{v} = -3vv' - av''',$$

which is nothing but the KdV equation.

Chapter 2

Hamiltonian Equations for the Space of Equicentroaffine Curves

In this chapter, we will focus on the space of equicentroaffine curves $\mathcal{M}$, which is introduced by Ulrich Pinkall in [20]. In Section 2.2, we will provide two presymplectic forms on $\mathcal{M}$, one of which, denoted by $\hat{\omega}_1$, is constructed in [7], by Fujioka and Kurose. We will see that a presymplectic gradient with respect to $\hat{\omega}_1$ also leads to the KdV equation as the Euler equation does in (1.38). However, in this case, there is no metric involved. We will compare these two cases in Chapter 3.

2.1 The Space of Equicentroaffine Curves

In this section, we will introduce the space of equicentroaffine curves $\mathcal{M}$. In Section 2.1.1, we will provide some basic knowledge of $\mathcal{M}$, making use of Hill's operator. In Remark 2.6, we will see that the KdV equation appears, which involves a presymplectic form on $\mathcal{M}$. In Section 2.1.2, we will present a right action introduced by Pinkall, and derive a formula for the Lie bracket of the fundamental vector fields corresponding to this action. The content in this section is mainly adapted from Fujioka-Kurose [7], Pinkall [20] and Tu [22].

2.1.1 Basic Concepts

Definition 2.1 ([20, p.328]). An ***equicentroaffine curve*** is a smooth curve $\gamma : S^1 \to \mathbb{R}^2$ satisfying

$$\det(\gamma, \gamma') = 1. \tag{2.1}$$

We will denote by $\hat{\mathcal{M}}$ the space of equicentroaffine curves and $\mathcal{M}$ the connected component of $\hat{\mathcal{M}}$ contaning the unit circle c.

Remark 2.2 ([20, p.328]). Given a equicentroaffine curve $\gamma = \begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix}$, there is a unique function $\kappa : S^1 \to \mathbb{R}$ determined by

$$\gamma'' + \kappa\gamma = 0,$$

called the ***equicentroaffine curvature of*** γ. By Remark 1.32, $\begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix}$ is nothing but a pair of independent solutions of Hill's equation

$$(\partial_x^2 + \kappa)y = 0.$$

Here, both the potential κ and and the solutions $\begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix}$ have period 2π. By Proposision 1.33, setting $\eta = \gamma_1/\gamma_2$ to be the ratio map, we have $\kappa = S(\eta)/2$.

Example 2.3. Let $\gamma = \begin{bmatrix}a\cos x\\ b\sin x\end{bmatrix}$ be an ellipse, where $a, b \in \mathbb{R}$ not equal to zero. Since

$$\det(\gamma, \gamma') = \det\begin{bmatrix}a\cos x & -a\sin x\\ b\sin x & b\cos x\end{bmatrix} = ab,$$

we see that when $ab = 1$, the ellipse γ will be an equicentroaffine curve.

The following lemma and its proof essentially appear in Pinkall [20].

Lemma 2.4 ([20, p.330]). Let $X \in T_\gamma\mathcal{M}$ be a tangent vector over $\mathcal{M}$. Then X has the form

$$X = -\frac{1}{2}\lambda'\gamma + \lambda\gamma',$$

where $\lambda \in C^\infty(S^1)$.

Proof. Note that a tangent vector $X \in T_\gamma\mathcal{M}$ over $\mathcal{M}$ could be expressed as

$$X = \mu\gamma + \lambda\gamma',$$

where $\mu, \lambda \in C^\infty(S^1)$. Suppose that $\tilde{\gamma}$ is a variation of γ with initial vector X. By (2.1), we have

$$\det(\dot{\tilde{\gamma}}, \tilde{\gamma}') + \det(\tilde{\gamma}, \dot{\tilde{\gamma}}') = 0,$$

where $\tilde{\gamma}'$ stands for the derivative with respect to the parameter x along the curve and $\dot{\tilde{\gamma}}$ to the parameter t along the variation. Setting $t = 0$, we have

$$\det(\mu\gamma + \lambda\gamma', \gamma') + \det(\gamma, (\mu\gamma + \lambda\gamma')') = 0.$$

It follows that

$$\mu = -\frac{1}{2}\lambda',$$

which implies the claim. ∎

Remark 2.5 ([7, p.2])**.** By (2.1), we have

$$\kappa = \det(\kappa\gamma, \gamma') = \det(\gamma', \gamma''). \tag{2.2}$$

Given $\gamma \in \mathcal{M}$, let $\tilde{\gamma}$ be a variation of γ with respect to $(x,t) \in S^1 \times [-\varepsilon, \varepsilon]$, whose initial vector is $\dot{\tilde{\gamma}}|_{t=0} = -\frac{1}{2}\lambda'\gamma + \lambda\gamma'$ for some $\lambda \in C^\infty(S^1)$. Let $\tilde{\kappa}(-,t)$ be the equicentroaffine curvature of $\tilde{\gamma}(-,t)$ and call $\tilde{\kappa}$ the ***equicentroaffine curvature flow*** of $\tilde{\gamma}$. By (2.2), we see that

$$\dot{\tilde{\kappa}} = \det(\dot{\tilde{\gamma}}', \tilde{\gamma}'') + \det(\tilde{\gamma}', \dot{\tilde{\gamma}}'').$$

Setting $t = 0$, we have

$$\dot{\tilde{\kappa}}|_{t=0} = \det((-\frac{1}{2}\lambda'\gamma + \lambda\gamma')', \gamma'') + \det(\gamma', (-\frac{1}{2}\lambda'\gamma + \lambda\gamma')'') = \frac{1}{2}\lambda''' + 2\kappa\lambda' + \kappa'\lambda. \tag{2.3}$$

If we make use the operator $\Omega := \frac{1}{2}\partial_x^2 + 2\kappa + \kappa'\partial_x^{-1}$ introduced by Fujioka and Kurose [7], the equation (2.3) can be rewritten as

$$\dot{\tilde{\kappa}}|_{t=0} = \Omega\lambda'. \tag{2.4}$$

By the expression of the coadjoint action of the Virasoro algebra mentioned in Lemma 1.31, for $(\kappa dx \otimes dx, \frac{1}{2}) \in (\mathfrak{X}(S^1) \times_\omega \mathbb{R})^*$ and $(\lambda\partial_x, 0) \in \mathfrak{X}(S^1) \times_\omega \mathbb{R}$, we have

$$((\Omega\lambda')dx \otimes dx, 0) = -\mathrm{ad}^*_{(\lambda\partial_x,0)}(\kappa dx \otimes dx, \frac{1}{2}). \tag{2.5}$$

This implies that Ω may not be considered as a new operator.

Remark 2.6 ([20, p.331])**.** Let $X_H^1 \in \mathfrak{X}(M)$ be a vector filed over $\mathcal{M}$ defined by

$$X_H^1(\gamma) = \frac{1}{2}\kappa'\gamma - \kappa\gamma',$$

where κ is the equicentroaffine curvature of γ. Suppose that $\tilde{\gamma}$ is an integral curve of X_H^1 in $\mathcal{M}$ and $\tilde{\kappa}$ its equicentroaffine curvature flow. By the discussion in Remark 2.5, we have

$$\dot{\tilde{\kappa}} = \Omega(-\tilde{\kappa})' = -\frac{1}{2}\tilde{\kappa}''' - 3\tilde{\kappa}'\tilde{\kappa}, \tag{2.6}$$

which means that the equicentroaffine curvature flow $\tilde{\kappa}$ also leads to the KdV equation as the Euler equation applied to the Bott-Virasoro group does. In order to find an explanation for (2.6), we will study the vector field X_H^1 in Section 2.3.2

2.1.2 Fundamental Vector Fields over $\mathcal{M}$

Definition 2.7 ([20, p.332])**.** There is a well-defined right action of $\mathrm{Diff}(S^1)$ on $\mathcal{M}$ such that for any $\varphi \in \mathrm{Diff}(S^1)$ and any $\gamma \in \mathcal{M}$, we have

$$\gamma \cdot \varphi := (\sqrt{(\varphi^{-1})'}\gamma) \circ \varphi = \frac{\gamma \circ \varphi}{\sqrt{\varphi'}},$$

which is called ***Pinkall's right action of*** $\mathbf{Diff}(S^1)$ ***on*** $\mathcal{M}$.

Remark 2.8. It is known that there is a bijection $I : \mathrm{Diff}(S^1) \to \mathcal{M}$ given by

$$\psi \mapsto c \cdot \psi.$$

For any $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R})$, let $f_A : [-\frac{\pi}{2}, \frac{3}{2}\pi] \to [-\frac{\pi}{2}, \frac{3}{2}\pi]$ be a function defined by

$$f_A(x) = \begin{cases} \arctan \frac{c\cos x + d \sin x}{a \cos x + b \sin x} + \pi & a\cos x + b \sin x < 0 \\ \arctan \frac{c\cos x + d \sin x}{a \cos x + b \sin x} & a\cos x + b \sin x \geq 0 \end{cases}.$$

We can consider f_A as an element in $\mathrm{Diff}(S^1)$. Thus, there is a left action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathrm{Diff}(S^1)$ given by

$$A \cdot \psi = f_A \circ \psi.$$

On the other hand, there is a left action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathcal{M}$ given by

$$A \cdot \gamma = A\gamma.$$

We claim that the bijection I is an equivariant map with respect to these two actions. Indeed, it's easy to see that

$$I(f_A \circ \mathrm{id}_A) = c \cdot f_A = \frac{c \circ f_A}{\sqrt{f_A'}} = \frac{\begin{bmatrix} (\cos(\arctan \frac{c\cos x + d\sin x}{a\cos x + b \sin x})) \\ (\sin(\arctan \frac{c\cos x + d\sin x}{a\cos x + b \sin x})) \end{bmatrix}}{\frac{1}{\sqrt{(a\cos x + b\sin x)^2 + (c\cos x + d \sin x)^2}}} = Ac,$$

which implies the claim. As you can see, the action of $\mathrm{SL}(2, \mathbb{R})$ is very complicated on $\mathrm{Diff}(S^1)$. However, by using the bijection I and identifying an element in $\mathrm{Diff}(S^1)$ with a geometric element in $\mathcal{M}$, such an action becomes very simple. This is one of the geometric natures of the space of equicentroaffine curves $\mathcal{M}$.

Definition 2.9 ([22, p.247])**.** Let G be a Lie group acting on a manifold M. Then, for any element $V \in \mathfrak{g}$, the corresponding ***fundamental vector field*** over M is given by

$$\underline{V}_x = \left.\frac{d}{dt}\right|_{t=0} c_V(t) \cdot x, \tag{2.7}$$

where $c_V(t)$ is a curve in G with initial vector V.

The following lemma provides an expression for the Lie bracket of the fundamental vector fields corresponding to Pinkall's right action of $\mathrm{Diff}(S^1)$ on $\mathcal{M}$, which essentially appears in Pinkall [20]. We will give an alternative proof here.

Lemma 2.10 ([20, p.332])**.** Consider Pinkall's right action of $\mathrm{Diff}(S^1)$ on $\mathcal{M}$. For any $\lambda\partial_x, \mu\partial_x \in \mathfrak{X}(S^1)$ in the Lie algebra of $\mathrm{Diff}(S^1)$, let $\underline{\lambda\partial_x}, \underline{\mu\partial_x} \in \mathfrak{X}(\mathcal{M})$ be the corresponding fundamental vector fields respectively. Then, we have

$$[\underline{\lambda\partial_x}, \underline{\mu\partial_x}] = \underline{(\lambda'\mu - \lambda\mu')\partial_x},$$

for the Lie bracket of vector fields.

Proof. For any $\lambda\partial_x \in \mathfrak{X}(S^1)$, let $\varphi_{\lambda\partial_x}$ be the flow of $\lambda\partial_x$ and regard it as a curve in $\mathrm{Diff}(S^1)$ with initial vector $\lambda\partial_x$. By (2.7), the fundamental vector field is computed as

$$\underline{\lambda\partial_x}_\gamma = \frac{d}{dt}\Big|_{t=0} \gamma \cdot \varphi_{\lambda\partial_x}(t) = \frac{d}{dt}\Big|_{t=0} (\sqrt{(\varphi_{\lambda\partial_x}(t)^{-1})'}\gamma) \circ \varphi_{\lambda\partial_x}(t) = -\frac{1}{2}\lambda'\gamma + \lambda\gamma'. \tag{2.8}$$

It follows that for any $\gamma \in \mathcal{M}$, we have

$$\frac{d}{ds}\Big|_{s=t} \gamma \cdot \varphi_{\lambda\partial_x}(s) = \frac{d}{ds}\Big|_{s=0} \gamma \cdot \varphi_{\lambda\partial_x}(s+t) = \frac{d}{ds}\Big|_{s=0} (\gamma \cdot \varphi_{\lambda\partial_x}(t)) \cdot \varphi_{\lambda\partial_x}(s) = \underline{\lambda\partial_x}_{\gamma\cdot\varphi_{\lambda\partial_x}(t)}.$$

Therefore, we see that

$$\varphi_{\underline{\lambda\partial_x}}(-,t) = (-) \cdot \varphi_{\lambda\partial_x}(t), \tag{2.9}$$

where $\varphi_{\underline{\lambda\partial_x}}(-,t)$ stands for the flow of the fundamental vector field $\underline{\lambda\partial_x} \in \mathfrak{X}(\mathcal{M})$ over $\mathcal{M}$. It follows that for any $\lambda\partial_x, \mu\partial_x \in \mathfrak{X}(S^1)$ and $f \in C^\infty(\mathcal{M})$ we have

$$\begin{aligned}
(\underline{\lambda\partial_x}\,\underline{\mu\partial_x} f)(\gamma) &= \frac{d}{dt}\Big|_{t=0} (\underline{\mu\partial_x}_{\gamma\cdot\varphi_{\lambda\partial_x}(t)} f) \\
&= \frac{d}{dt}\Big|_{t=0} (f_*(\frac{d}{ds}\Big|_{s=0} \gamma \cdot \varphi_{\lambda\partial_x}(t) \cdot \varphi_{\mu\partial_x}(s))) \\
&= f_*(\frac{d^2}{dtds}\Big|_{t=s=0} \gamma \cdot \varphi_{\lambda\partial_x}(t) \cdot \varphi_{\mu\partial_x}(s)) \\
&= f_*(-\frac{1}{2}\mu'(-\frac{1}{2}\lambda'\gamma + \lambda\gamma') + \mu(-\frac{1}{2}\lambda'\gamma + \lambda\gamma')'),
\end{aligned}$$

which implies that

$$(\underline{\lambda\partial_x}\,\underline{\mu\partial_x})(\gamma) = -\frac{1}{2}\mu'(-\frac{1}{2}\lambda'\gamma + \lambda\gamma') + \mu(-\frac{1}{2}\lambda'\gamma + \lambda\mu')'.$$

Similarly, we can compute $(\underline{\mu\partial_x}\,\underline{\lambda\partial_x})(\gamma)$. It follows that

$$\begin{aligned}
[\underline{\lambda\partial_x}, \underline{\mu\partial_x}]_\gamma &= (\underline{\lambda\partial_x}\,\underline{\mu\partial_x} - \underline{\mu\partial_x}\,\underline{\lambda\partial_x})(\gamma) \\
&= -\frac{1}{2}(\lambda'\mu - \lambda\mu')'\gamma + (\lambda'\mu - \lambda\mu')\gamma' \\
&= \underline{(\lambda'\mu - \lambda\mu')\partial_x}_\gamma,
\end{aligned}$$

which implies the claim. ∎

2.2 Symplectic Forms for the Space of Equicentroaffine Curves

In this section, we will present two presymplectic forms $\hat{\omega}_i (i = 0, 1)$ on $\mathcal{M}$, introduced by Pinkall, Fujioka and Kurose. In Theorem 2.14 and Theorem 2.20, we will show that $\hat{\omega}_i$ descends to a symplectic form on quotient spaces $\mathcal{M}_i$. The content in this section is mainly adapted from Fujioka and Kurose [7] and Pinkall [20].

2.2.1 The Pinkall 2-Form

Definition 2.11 ([20, p.330])**.** Given tangent vectors $X = -\frac{1}{2}\lambda'\gamma+\lambda\gamma', Y = -\frac{1}{2}\mu'\gamma+\mu\gamma' \in T_\gamma\mathcal{M}$, define $\hat{\omega}_0$ to be a 2-form on $\mathcal{M}$ by

$$\hat{\omega}_0(X, Y) := \int_{S^1} \lambda\mu' dx,$$

called the ***Pinkall* 2*-form***.

The following lemma essentially appears in Pinkall [20]. We will provide a proof here.

Lemma 2.12 ([20, p.330])**.** The Pinkall 2-form $\hat{\omega}_0$ is closed.

Proof. For any tangent vectors $X = -\frac{1}{2}\lambda'\gamma + \lambda\gamma', Y = -\frac{1}{2}\mu'\gamma + \mu\gamma', Z = -\frac{1}{2}\nu'\gamma + \nu\gamma' \in T_\gamma\mathcal{M}$, by the discussion in Lemma 2.10, the fundamental vector fields $\underline{\lambda\partial_x}, \underline{\mu\partial_x}, \underline{\nu\partial_x} \in \mathfrak{X}(\mathcal{M})$ of $\lambda\partial_x, \mu\partial_x, \nu\partial_x \in \mathfrak{X}(S^1)$ extend X, Y, Z, respectively. It follows that

$$\begin{aligned} d\hat{\omega}_0(X, Y, Z) &= d\hat{\omega}_0(\underline{\lambda\partial_x}, \underline{\mu\partial_x}, \underline{\nu\partial_x})(\gamma) \\ &= \sum_{\text{cycl}} \underline{\lambda\partial_x}_\gamma \hat{\omega}_0(\underline{\mu\partial_x}, \underline{\nu\partial_x}) - \sum_{\text{cycl}} \hat{\omega}_0([\underline{\lambda\partial_x}, \underline{\mu\partial_x}], \underline{\nu\partial_x})(\gamma). \end{aligned}$$

By Definition 2.11, the value of $\hat{\omega}_0(\underline{\mu\partial_x}, \underline{\nu\partial_x})$ is independent of $\gamma \in \mathcal{M}$. Thus, we have

$$\sum_{\text{cycl}} \underline{\lambda\partial_x}_\gamma \hat{\omega}_0(\underline{\mu\partial_x}, \underline{\nu\partial_x}) = 0.$$

On the other hand, by Lemma 2.10, we have

$$\begin{aligned} d\hat{\omega}_0(X, Y, Z) &= -\sum_{\text{cycl}} \hat{\omega}_0([\underline{\lambda\partial_x}, \underline{\mu\partial_x}], \underline{\nu\partial_x})(\gamma) \\ &= -\sum_{\text{cycl}} \hat{\omega}_0(\underline{(\lambda'\mu - \lambda\mu')\partial_x}, \underline{\nu\partial_x})(\gamma) \\ &= -\sum_{\text{cycl}} \int_{S^1} (\lambda'\mu - \lambda\mu')\nu' dx \\ &= 0, \end{aligned}$$

which implies the claim. ∎

Remark 2.13 ([20, p.330])**.** Let $\in S^1$ act on $\mathcal{M}$ such that for any $z \in S^1$ and $\gamma \in \mathcal{M}$ we have

$$\alpha \cdot z := \alpha \circ z,$$

where we regard $z \in S^1$ as an element in $\text{Diff}(S^1)$. Set $\mathcal{M}_0 := \mathcal{M}/S^1$. For any tangent vector $V \in T_{[\gamma]}\mathcal{M}_0$ over $\mathcal{M}_0$, there exists a lift $\hat{V} \in T_\gamma\mathcal{M}$ over $\mathcal{M}$ such that $\pi_{0*}\hat{V} = V$, where $\pi_0 : \mathcal{M} \to \mathcal{M}_0$ represents the projection. If $X \in T_\gamma\mathcal{M}$ satisfies $\pi_{0*}X = 0$, then we have

$$X = \frac{d}{dt}\bigg|_{t=0} \gamma \cdot (t\lambda) = \frac{d}{dt}\bigg|_{t=0} \gamma \circ (t\lambda) = \lambda\gamma', \tag{2.10}$$

for some $\lambda \in \mathbb{R}$.

The following theorem and its proof essentially appear in Pinkall [20].

Theorem 2.14 ([20, p.330])**.** *The Pinkall* 2*-form* $\hat{\omega}_0$ *descends to a symplectic form* ω_0 *on* $\mathcal{M}_0$*, called the* ***Pinkall symplectic form****.*

Proof. Given tangent vectors $X, Y \in T_{[\gamma]}\mathcal{M}_0$, define ω_0 to be a 2-form on $\mathcal{M}_0$ by

$$\omega_0(X, Y) := \hat{\omega}_0(\hat{X}, \hat{Y}), \tag{2.11}$$

where $\hat{X} = -\frac{1}{2}\lambda'\gamma + \lambda\gamma', \hat{Y} = -\frac{1}{2}\mu'\gamma + \mu\gamma' \in T_\gamma\mathcal{M}$ lift X, Y, respectively. First, we show that (2.11) is well-defined. Suppose that $\hat{X}' \in T_\gamma\mathcal{M}$ is another lift of X. Since

$$\pi_{0*}(\hat{X} - \hat{X}') = X - X = 0,$$

by (2.10), we see that there exists $\lambda \in \mathbb{R}$ such that

$$\hat{X} - \hat{X}' = \lambda\gamma'.$$

It follows that

$$\hat{\omega}_0(\hat{X}, \hat{Y}) - \hat{\omega}_0(\hat{X}', \hat{Y}) = \int_{S^1} \lambda\mu' dx = 0. \tag{2.12}$$

Suppose that $\hat{Y}' \in T_\gamma\mathcal{M}$ is another lift of Y. By (2.12), we have

$$\hat{\omega}_0(\hat{X}', \hat{Y}') = \hat{\omega}_0(\hat{X}', \hat{Y}) = \hat{\omega}_0(\hat{X}, \hat{Y}),$$

which implies that (2.11) is independent of the choices of the lifts of tangent vectors at a fixed representative γ of $[\gamma]$. For another representative $\gamma \circ z$ of $[\gamma]$, where $z \in S^1$, let $\hat{X}'', \hat{Y}'' \in T_{\gamma\circ z}\mathcal{M}$ lift X, Y, respectively. Suppose that $\tilde{\gamma}$ is a smooth curve in $\mathcal{M}$ with initial vector $\hat{X}$. Then, we have

$$r_{z*}\hat{X} = \frac{d}{dt}\bigg|_{t=0} \tilde{\gamma}(t) \circ z = (-\frac{1}{2}\lambda'\gamma + \lambda\gamma') \circ z,$$

where $r_z : \mathcal{M} \to \mathcal{M}$ denote the action of z. Similarly, we can compute $r_{z*}\hat{Y}$. It follows that

$$\hat{\omega}_0(r_{z*}\hat{X}, r_{z*}\hat{Y}) = \int_{S^1} (\lambda \circ z)(\mu \circ z)' dx = \int_{S^1} \lambda\mu' dx = \hat{\omega}_0(\hat{X}, \hat{Y}).$$

Since $r_{z*}\hat{X}, r_{z*}\hat{Y}$ and $\hat{X}'', \hat{Y}''$ are both lifts over $\mathcal{M}$ at $\gamma \circ z$ of X, Y respectively, by the first part of the proof, we have

$$\hat{\omega}_0(\hat{X}, \hat{Y}) = \hat{\omega}_0(r_{z*}\hat{X}, r_{z*}\hat{Y}) = \hat{\omega}_0(\hat{X}'', \hat{Y}'').$$

This implies that ω_0 is well-defined, and

$$\pi_0^* \omega_0 = \hat{\omega}_0.$$

By Lemma 2.12, ω_1 is closed. Therefore, it leaves us to prove that ω_0 is nondegenerate. Suppose that $X \in T_{[\gamma]}\mathcal{M}_0$ belongs to the kernel of ω_0. Then, a lift $\hat{X} = -\frac{1}{2}\lambda'\gamma + \lambda\gamma' \in T_\gamma\mathcal{M}$ of X belongs to the kernel of $\hat{\omega}_0$. By the definition of $\hat{\omega}_1$, we have $\lambda' = 0$. Hence,

$$X = \pi_{0*}\hat{X} = \left.\frac{d}{dt}\right|_{t=0} \pi_0(\gamma \circ (t\lambda)) = \left.\frac{d}{dt}\right|_{t=0} [\gamma] = 0,$$

which implies that $\hat{\omega}_0$ is nondegenerate. ■

2.2.2 The Fujioka-Kurose 2-Form

Definition 2.15 ([7, p.3])**.** Given tangent vectors $X = -\frac{1}{2}\lambda'\gamma + \lambda\gamma', Y = -\frac{1}{2}\mu'\gamma + \mu\gamma' \in T_\gamma\mathcal{M}$, with κ the equicentroaffine curvature of γ, define $\hat{\omega}_1$ to be a 2-form on $\mathcal{M}$ by

$$\hat{\omega}_1(X, Y) := \int_{S^1} \lambda(\frac{1}{2}\mu''' + 2\kappa\mu' + \kappa'\mu)dx = \int_{S^1} \lambda\Omega\mu' dx,$$

which is called the ***Fujioka-Kurose* 2*-form***.

Remark 2.16. Since for any $X = -\frac{1}{2}\lambda'\gamma + \lambda\gamma', Y = -\frac{1}{2}\mu'\gamma + \mu\gamma' \in T_\gamma\mathcal{M}$ we have

$$\begin{aligned}\hat{\omega}_1(Y, X) &= \int_{S^1} \mu\Omega\lambda' dx \\ &= \int_{S^1} \mu(\frac{1}{2}\lambda''' + 2\kappa\lambda' + \kappa'\lambda)dx \\ &= -\int_{S^1} \lambda(\frac{1}{2}\mu''' + 2\kappa\mu' + \kappa'\mu)dx \\ &= -\hat{\omega}_1(X, Y),\end{aligned}$$

we see that $\hat{\omega}_1$ does be a 2-form.

The following lemma essentially appears in Fujioka and Kurose [7]. We will provide an alternative proof here.

Lemma 2.17 ([7, Theorem 2])**.** The Fujioka-Kurose 2-form $\hat{\omega}_1$ is closed.

Proof. By the discussion in the proof of Lemma 2.10, for any tangent vectors $X = -\frac{1}{2}\lambda'\gamma + \lambda\gamma', Y = -\frac{1}{2}\mu'\gamma + \mu\gamma', Z = -\frac{1}{2}\nu'\gamma + \nu\gamma' \in T_\gamma\mathcal{M}$, we have

$$d\hat{\omega}_1(X, Y, Z) = \sum_{\text{cycl}} \underline{\lambda\partial_x}_\gamma \hat{\omega}_1(\underline{\nu\partial_x}, \underline{\nu\partial_x}) - \sum_{\text{cycl}} \hat{\omega}_1([\underline{\lambda\partial_x}, \underline{\mu\partial_x}], \underline{\nu\partial_x})(\gamma). \tag{2.13}$$

Let $\varphi_{\lambda\partial_x}$ be the flow of $\lambda\partial_x \in \mathfrak{X}(S^1)$. By (2.9), we have

$$\underline{\lambda\partial_x}_\gamma \hat{\omega}_1(\underline{\mu\partial_x}, \underline{\nu\partial_x}) = \frac{d}{dt}\bigg|_{t=0} \hat{\omega}_1(\underline{\mu\partial_x}_{\gamma\cdot\varphi_{\lambda\partial_x}(t)}, \underline{\nu\partial_x}_{\gamma\cdot\varphi_{\lambda\partial_x}(t)})$$

Let $\tilde{\kappa}$ be the equicentroaffine curvature flow of $\gamma \cdot \varphi_{\lambda\partial_x}$. By (2.3), we have

$$\begin{aligned}
\underline{\lambda\partial_x}_\gamma \hat{\omega}_1(\underline{\mu\partial_x}, \underline{\nu\partial_x}) &= \frac{d}{dt}\bigg|_{t=0} \int_{S^1} \mu(\frac{1}{2}\nu''' + 2\tilde{\kappa}(t)\nu' + \tilde{\kappa}(t)'\nu)dx \\
&= \frac{d}{dt}\bigg|_{t=0} \int_{S^1} \mu(2\tilde{\kappa}(t)\nu' + \tilde{\kappa}(t)'\nu)dx \\
&= \int_{S^1} \mu(2(\Omega\lambda')\nu' + (\Omega\lambda')'\nu)dx \\
&= \int_{S^1} (\mu\nu' - \mu'\nu)\Omega\lambda' dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{\text{cycl}} \underline{\lambda\partial_x}_\gamma \hat{\omega}_1(\underline{\mu\partial_x}, \underline{\nu\partial_x}) &= \sum_{\text{cycl}} \int_{S^1} (\mu\nu' - \mu'\nu)\Omega\lambda' dx \\
&= \sum_{\text{cycl}} \int_{S^1} (\mu\nu' - \mu'\nu)(\frac{1}{2}\lambda''' + 2\kappa\lambda' + \kappa'\lambda)dx \\
&= 0.
\end{aligned}$$

For the second term in (2.13), we have

$$\begin{aligned}
\sum_{\text{cycl}} \hat{\omega}_1([\underline{\lambda\partial_x}, \underline{\mu\partial_x}], \underline{\nu\partial_x})(\gamma) &= \sum_{\text{cycl}} \hat{\omega}_1(\underline{(\lambda'\mu - \lambda\mu')\partial_x}, \underline{\nu\partial_x})(\gamma) \\
&= \sum_{\text{cycl}} \int_{S^1} (\lambda'\mu - \lambda\mu')\Omega\nu' dx \\
&= -\sum_{\text{cycl}} \underline{\lambda\partial_x}_\gamma \hat{\omega}_1(\underline{\mu\partial_x}, \underline{\nu\partial_x}) \\
&= 0.
\end{aligned}$$

It follows that the Fujioka-Kurose 2-form $\hat{\omega}_1$ is closed. ■

Remark 2.18 ([7, p.4])**.** We have already defined a left action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathcal{M}$, such that for any $A \in \mathrm{SL}(2, \mathbb{R})$ and $\gamma \in \mathcal{M}$ we have

$$A \cdot \gamma = A\gamma.$$

Let $\mathcal{M}_1 := \mathrm{SL}(2,\mathbb{R}) \setminus \mathcal{M}$. For any $X \in T_\gamma\mathcal{M}$ such that $\pi_{1*}X = 0$, where $\pi_1 : \mathcal{M} \to \mathcal{M}_0$ represents the projection, there exists $V \in \mathfrak{sl}(2,\mathbb{R})$ such that

$$X = \frac{d}{dt}\bigg|_{t=0} \exp(tV) \cdot \gamma.$$

By Lemma 1.34, the equicentroaffine curvature flow of the curve $\exp((-)X)\cdot\gamma$ in $\mathcal{M}$ is independent of t. On the other hand, if $\tilde{\gamma}$ is a variation of γ with equicentroaffine curvature flow $\tilde{\kappa}$ such that $\dot{\tilde{\kappa}}|_{t=0} = 0$, then there exists a curve A in $\mathrm{SL}(2,\mathbb{R})$ such that

$$\dot{\tilde{\gamma}}|_{t=0} = \frac{d}{dt}\bigg|_{t=0} A(t)\cdot\gamma,$$

for the initial vector of $\tilde{\gamma}$.

Remark 2.19 ([7, p.3])**.** Given tangent vectors $X = -\frac{1}{2}\lambda\gamma + \lambda\gamma', Y = -\frac{1}{2}\mu'\gamma + \mu\gamma' \in T_\gamma\mathcal{M}$, it's easy to verify that

$$\hat{\omega}_1(X,Y) = \int_{S^1} \det(X, (\partial_x^2 + \kappa)Y)dx, \tag{2.14}$$

where κ is the equicentroaffine curvature of γ. On the other hand, by (2.4), setting $\tilde{\gamma}$ to be a variation of γ with initial vector Y, we have

$$\hat{\omega}_1(X,Y) = \int_{S^1} \lambda\Omega\mu' dx = \frac{d}{dt}\bigg|_{t=0} \int_{S^1} \lambda\tilde{\kappa}(t)dx. \tag{2.15}$$

where $\tilde{\kappa}(t)$ is the equicentroaffine curvature of $\tilde{\gamma}(t)$.

The following theorem and its proof essentially appear in Fujioka and Kurose [7].

Theorem 2.20 ([7, p.4])**.** *The Fujioka-Kurose 2-form $\hat{\omega}_1$ descends to a symplectic form ω_1 on $\mathcal{M}_1$, which is called the **Fujioka-Kurose symplectic form**.*

Proof. Given tangent vectors $X, Y \in T_{[\gamma]}\mathcal{M}_1$, define ω_1 to be a 2-form on $\mathcal{M}_1$ by

$$\omega_1(X,Y) := \hat{\omega}_1(\hat{X},\hat{Y}), \tag{2.16}$$

where $\hat{X}, \hat{Y} \in T_\gamma\mathcal{M}$ lift X, Y, respectively. First, we prove that (2.16) is well-deifned. Let $\hat{X}' \in T_\gamma\mathcal{M}$ be another lift of X over $\mathcal{M}$ at γ . By the discussion in Remark 2.18, there exists $V \in \mathfrak{sl}(2,\mathbb{R})$ such that

$$\hat{X} - \hat{X}' = \frac{d}{dt}\bigg|_{t=0} \exp(tV)\cdot\gamma.$$

Let $\tilde{\kappa}$ be the equicentroaffine curvature flow of $\exp((-)V)\cdot\gamma$. By Remark 2.18 and Remark 2.19, we have

$$\hat{\omega}_1(\hat{X},\hat{Y}) - \omega(\hat{X}',\hat{Y}) = -\hat{\omega}_1(\hat{Y},\hat{X}-\hat{X}') = -\frac{d}{dt}\bigg|_{t=0} \int_{S^1} \mu\tilde{\kappa}(t)dx = 0. \tag{2.17}$$

Suppose that $\hat{Y}' \in T_\gamma\mathcal{M}$ is another lift at γ of Y. By (2.17), we have

$$\hat{\omega}_1(\hat{X}', \hat{Y}') = \hat{\omega}_1(\hat{X}, \hat{Y}') = \hat{\omega}_1(\hat{X}, \hat{Y}).$$

It follows that (2.16) is independent of the choices of the lifts of tangent vectors at a fixed representative γ of $[\gamma]$. For another representative $A\gamma$ of $[\gamma]$, where $A \in \mathrm{SL}(2, \mathbb{R})$, let $\hat{X}'', \hat{Y}'' \in T_{A\gamma}\mathcal{M}$ be tangent vectors over $\mathcal{M}$ at $A\gamma$ lifting X, Y respectively. Since the equicentroaffine curvature of $A\gamma$ is same to the one of γ, by (2.14), we have

$$\begin{aligned}
\omega(l_{A*}\hat{X}, l_{A*}\hat{Y}) &= \hat{\omega}_1(A\hat{X}, A\hat{Y}) \\
&= \int_{S^1} \det(A\hat{X}, (\partial_x^2 + \kappa)A\hat{Y})dx \\
&= \int_{S^1} \det(A)\det(\hat{X}, (\partial_x^2 + \kappa)\hat{Y})dx \\
&= \hat{\omega}_1(\hat{X}, \hat{Y}),
\end{aligned}$$

where $l_A : \mathcal{M} \to \mathcal{M}$ denotes the left action of $A \in \mathrm{SL}(2, \mathbb{R})$. Since both $l_{A*}\hat{X}, l_{A*}\hat{Y} \in T_{A\gamma}\mathcal{M}$ and $\hat{X}'', \hat{Y}''$ are lifts of X, Y over $\mathcal{M}$ at $A\gamma$ respectively, by the first part of the proof, we see that

$$\hat{\omega}_1(\hat{X}, \hat{Y}) = \hat{\omega}_1(l_{A*}\hat{X}, l_{A*}\hat{Y}) = \hat{\omega}_1(\hat{X}'', \hat{Y}'').$$

It follows that ω_1 is well-defined, and

$$\pi_1^*\omega_1 = \hat{\omega}_1.$$

By Lemma 2.17, ω_1 is closed. Therefore, it leaves us to prove that ω_1 is nondegenerate. Suppose that $X \in T_{[\gamma]}\mathcal{M}_1$ belongs to the kernel of ω_1. Then, a lift $\hat{X} = -\frac{1}{2}\lambda'\gamma + \lambda\gamma' \in T_\gamma\mathcal{M}$ of X over $\mathcal{M}$ belongs to the kernel of $\hat{\omega}_1$, which means that $\Omega\lambda' = 0$. Let $\tilde{\gamma}$ be a variation of γ with inital vector $\hat{X}$ and $\tilde{\kappa}$ its equicentroaffine curvature flow. Then,

$$\left.\frac{d}{dt}\right|_{t=0} \tilde{\kappa}(t) = \Omega\lambda' = 0.$$

By the discussion in Remark 2.18, the variation $\tilde{\gamma}$ can be expressed as $A\gamma$, where A is a curve in $\mathrm{SL}(2, \mathbb{R})$. It follows that

$$X = \pi_{1*}\hat{X} = \left.\frac{d}{dt}\right|_{t=0} \pi_1(A(t)\gamma) = \left.\frac{d}{dt}\right|_{t=0} [\gamma] = 0,$$

which implies that ω_1 is nondegenerate. ■

2.3 Poisson Brackets for the Space of Equicentroaffine Curves

In this section, we will present two Poisson brackets $\{-,-\}_i (i = 0, 1)$ for $\mathcal{M}$ (see Fujioka and Kurose [7]). We will show the relationship between $\{-,-\}_i$ and $\{-,-\}_{\omega_i}$. In Example 2.25, we will see that the vector field X_H^1 mentioned in Remark 2.6 is nothing but the presymplectic gradient of a Hamiltonian function on $\mathcal{M}$. The content in this section is mainly adapted from Fujioka and Kurose [7].

2.3.1 A Poisson Bracket for $\mathcal{M}_0$

Let $C^\infty(\mathcal{M})_\kappa$ be the space consisting of the functions over $\mathcal{M}$ in the following form

$$F(\gamma) = \int_{S^1} f(\kappa, \kappa', \cdots, \kappa^{(n)}) dx,$$

where κ is the equicentroaffine curvature of γ and f is a polynomial of $\kappa, \kappa', \cdots, \kappa^{(n)}$ for some $n \in \mathbb{N}_+$. The following lemma shows that such F is a Hamiltonian function with respect to $\hat{\omega}_0$, which essentially appears in Fujioka and Kurose [7]. We will provide a proof here.

Lemma 2.21 ([7, Proposition 1])**.** For any $F \in C^\infty(\mathcal{M})_\kappa$, there is a well-defined vector field $X_F^0 \in \mathfrak{X}(\mathcal{M})$ given by

$$X_F^0(\gamma) = \frac{1}{2}(\Omega(\delta F)')\gamma - (\frac{1}{2}(\delta F)'' + 2(\kappa(\delta F))' - f(\kappa, \kappa', \cdots, \kappa^{(n)}))\gamma'$$

where

$$\delta F := \frac{\partial f}{\partial \kappa}(\kappa, \kappa', \cdots, \kappa^{(n)}) - \frac{\partial f}{\partial \kappa'}(\kappa, \kappa', \cdots, \kappa^{(n)})' + \cdots$$

such that

$$i_{X_F^0}\hat{\omega}_0 = -dF. \tag{2.18}$$

This means that X_F^0 is the ***presymplectic gradient*** of F with respect to $\hat{\omega}_0$.

Proof. Since

$$\begin{aligned} f(\kappa, \kappa', \cdots, \kappa^{(n)})' &= (\int_0^{(-)} f(\kappa, \kappa', \cdots, \kappa^{(n)})' ds)' \\ &= (\int_0^{(-)} (\frac{\partial f}{\partial \kappa}(\kappa, \kappa', \cdots, \kappa^{(n)})\kappa' + \frac{\partial f}{\partial \kappa'}(\kappa, \kappa', \cdots, \kappa^{(n)})\kappa'' + \cdots) ds)' \\ &= (\int_0^{(-)} \kappa'(\delta F) ds)' \\ &= \kappa'(\delta F), \end{aligned}$$

we see that

$$(\frac{1}{2}(\delta F)'' + 2(\kappa(\delta F))' - f(\kappa, \kappa', \cdots, \kappa^{(n)}))' = \frac{1}{2}(\delta F)''' + 2\kappa(\delta F)' + \kappa'(\delta F) = \Omega(\delta F)',$$

which implies that $X_F^0(\gamma)$ does be a well-defined tangent vector over $\mathcal{M}$ by Lemma 2.4. For any $X = -\frac{1}{2}\lambda'\gamma + \lambda\gamma' \in T_\gamma\mathcal{M}$, let $\tilde{\gamma}$ be a variation of γ with initial vector X and $\tilde{\kappa}$

its equicentroaffine curvature flow. Since

$$
\begin{aligned}
dF(\gamma)(X) &= \frac{d}{dt}\Big|_{t=0} F(\tilde{\gamma}(t)) \\
&= \frac{d}{dt}\Big|_{t=0} \int_{S^1} f(\tilde{\kappa}(t), \tilde{\kappa}(t)', \cdots, \tilde{\kappa}(t)^{(n)})dx \\
&= \int_{S^1} (\frac{\partial f}{\partial \kappa}(\tilde{\kappa}, \tilde{\kappa}', \cdots, \tilde{\kappa}^{(n)})\dot{\tilde{\kappa}} + \frac{\partial f}{\partial \kappa'}(\tilde{\kappa}, \tilde{\kappa}', \cdots, \tilde{\kappa}^{(n)})\dot{\tilde{\kappa}}' + \cdots)dx\Big|_{t=0} \\
&= \int_{S^1} (\frac{\partial f}{\partial \kappa}(\kappa, \kappa', \cdots \kappa^{(n)})\Omega\lambda' + \frac{\partial f}{\partial \kappa'}(\kappa, \kappa', \cdots \kappa^{(n)})(\Omega\lambda')' + \cdots)dx \\
&= \int_{S^1} (\frac{\partial f}{\partial \kappa}(\kappa, \kappa', \cdots \kappa^{(n)}) - \frac{\partial f}{\partial \kappa'}(\kappa, \kappa', \cdots \kappa^{(n)})' + \cdots)(\Omega\lambda')dx \\
&= \int_{S^1} (\delta F)(\Omega\lambda')dx
\end{aligned}
$$

we see that

$$
-i_{X_F^0}\hat{\omega}_0(X) = \hat{\omega}_0(X, X_F^0(\gamma)) = -\int_{S^1} \lambda(\Omega(\delta F)')dx = \int_{S^1} (\delta F)(\Omega\lambda')dx = dF(\gamma)(X),
$$

which implies the claim. ■

Theorem 2.22. *Given $F \in C^\infty(\mathcal{M})_\kappa$ and $[\gamma] \in \mathcal{M}$, define $\iota_0 : C^\infty(\mathcal{M})_\kappa \to C^\infty(\mathcal{M}_0)$ to be a map by*

$$
\iota_0(F)([\gamma]) := F(\gamma)
$$

where $\gamma \in \mathcal{M}$ is a representative of $[\gamma]$. Let $\{-,-\}_0 : C^\infty(\mathcal{M})_\kappa \times C^\infty(\mathcal{M})_\kappa \to C^\infty(\mathcal{M})_\kappa$ be a bracket over $C^\infty(\mathcal{M})_\kappa$ given by

$$
\{F, G\}_0(\gamma) := \hat{\omega}_0(X_F^0(\gamma), X_G^0(\gamma)).
$$

Then, for any $F, G \in C^\infty(\mathcal{M})_\kappa$ and $\gamma \in \mathcal{M}_0$ we have

$$
\{F, G\}_0(\gamma) = \{\iota_0(F), \iota_0(F)\}_{\omega_0}([\gamma]),
$$

where $\{-,-\}_{\omega_0}$ is the Poisson bracket induced by the Pinkall symplectic form ω_0.

Proof. First, we prove that ι_0 is well-defined. For any $[\gamma] \in \mathcal{M}$, let γ and $\gamma \circ z$ be two representatives for $[\gamma]$, where $z \in S^1$. Suppose that η is the ratio of γ. Then, $\eta \circ z$ is the ratio of $\gamma \circ z$. By Remark 1.28 and Lemma 1.33, we have

$$
\kappa_{\gamma \circ z} = \frac{1}{2}S(\eta \circ z) = \frac{1}{2}((S(\eta) \circ z)(z')^2 + S(z)) = \frac{1}{2}S(\eta) \circ z = \kappa \circ z,
$$

where κ is the equicentroaffine curvature of γ and $\kappa_{\gamma \circ z}$ the one of $\gamma \circ z$. It follows that

$$
F(\gamma \circ z) = \int_{S^1} f(\kappa \circ z, \kappa' \circ z, \cdots, \kappa^{(n)} \circ z)dx = \int_{S^1} f(\kappa, \kappa', \cdots, \kappa^{(n)})dx = F(\gamma),
$$

which implies ι_0 is well-defined. Since for any $\gamma \in \mathcal{M}$, we have

$$\pi_0^*(\iota_0(F))(\gamma) = \iota_0(F)([\gamma]) = F(\gamma),$$

we see that

$$\pi_0^*(d\iota_0(F)) = d(\pi_0^*(\iota_0(F))) = dF.$$

It follows that for any tangent vector $X \in T_{[\gamma]}\mathcal{M}_0$ over $\mathcal{M}_0$, we have

$$\begin{aligned}
\omega_0(\pi_{0*}(X_F^0(\gamma)), X) &= \omega_0(\pi_{0*}(X_F^0(\gamma)), \pi_{0*}\hat{X}) \\
&= \pi_0^*\omega_0(X_F^0(\gamma), \hat{X}) \\
&= \hat{\omega}_0(X_F^0(\gamma), \hat{X}) \\
&= -dF(\hat{X}) \\
&= -\pi_0^*(d\iota_0(F))(\hat{X}) \\
&= -d(\iota_0(F))(X) \\
&= \omega_0(X_{\iota_0(F)}([\gamma]), X),
\end{aligned}$$

where $\hat{X} \in T_\gamma\mathcal{M}$ is a lift of X over $\mathcal{M}$ and $X_{\iota_0(F)}$ is the symplectic gradient of $\iota_0(F)$ with respect to ω_0. Since the Pinkall symplectic form ω_0 is nondegenerate, we see that

$$\pi_{0*}(X_F^0(\gamma)) = X_{\iota_0(F)}([\gamma]).$$

Therefore, by the definition of ω_0, we have

$$\{F, G\}_0(\gamma) = \hat{\omega}_0(X_F^0(\gamma), X_G^0(\gamma)) = \omega_0(X_{\iota_0(F)}([\gamma]), X_{\iota_0(G)}[\gamma]) = \{\iota_0(F), \iota_0(G)\}_{\omega_0}([\gamma]),$$

which implies the claim. ■

The following theorem essentially appears in Fujioka and Kurose [7]. We will provide a proof here.

Theorem 2.23 ([7, p.6])**.** *The bracket $\{-,-\}_0$ on $C^\infty(\mathcal{M})_\kappa$ is a Poisson bracket.*

Proof. For any $F, G \in C^\infty(\mathcal{M})_\kappa$, since

$$\{F, G\}_0 = \hat{\omega}_0(X_F^0, X_G^0) = -\hat{\omega}_0(X_G^0, X_F^0) = \{G, F\}_0,$$

we see that $\{-,-\}_0$ is anti-symmetric. For any $F, G \in C^\infty(\mathcal{M})_0$ and $[\gamma] \in \mathcal{M}_0$, by the discussion in Theorem 2.22, we have

$$\iota_0(\{F, G\}_0)([\gamma]) = \{F, G\}_0(\gamma) = \{\iota_0(F), \iota_0(G)\}_{\omega_0}([\gamma]).$$

It follows that for any $F, G, H \in C^\infty(\mathcal{M})_\kappa$ and $\gamma \in \mathcal{M}$, we have

$$\begin{aligned}
\sum_{\text{cycl}} \{\{F, G\}_0, H\}_0(\gamma) &= \sum_{\text{cycl}} \{\iota_0(\{F, G\}_0), \iota_0(H)\}_{\omega_0}([\gamma]) \\
&= \sum_{\text{cycl}} \{\{\iota_0(F), \iota_0(G)\}_{\omega_0}, \iota_0(H)\}_{\omega_0} \\
&= 0.
\end{aligned}$$

Therefore, $\{-,-\}_0$ satisfies the Jacobi identity. Moreover, since

$$\begin{aligned}\{F, GH\}_0(\gamma) &= \{\iota_0(F), \iota_0(G)\iota_0(H)\}_{\omega_0}([\gamma]) \\ &= \iota_0(G)([\gamma])\{\iota_0(F), \iota_0(H)\}_{\omega_0}([\gamma]) + \iota_0(H)([\gamma])\{\iota_0(F), \iota_0(G)\}_{\omega_0}([\gamma]) \\ &= G(\gamma)\{F, H\}_0(\gamma) + H(\gamma)\{F, G\}_0(\gamma),\end{aligned}$$

we see that $\{-,-\}_0$ satisfies the Leibniz rule. It follows that $\{-,-\}_0$ is a Poisson bracket on $C^\infty(\mathcal{M})_\kappa$. ■

2.3.2 A Poisson Bracket for $\mathcal{M}_1$

Definition 2.24 ([7, Theorem 2])**.** For any $F \in C^\infty(\mathcal{M})_\kappa$, define $X_F^1 \in \mathfrak{X}(\mathcal{M})$ to be a vector field over $\mathcal{M}$ such that for any $\gamma \in \mathcal{M}$ we have

$$X_F^1(\gamma) := \frac{1}{2}(\delta F)'\gamma - (\delta F)\gamma'. \tag{2.19}$$

Note that by the computation in Lemma 2.21, for any $X = -\frac{1}{2}\lambda'\gamma + \lambda\gamma' \in T_\gamma\mathcal{M}$ we have

$$dF(\gamma)(X) = \int_{S^1} (\delta F)(\Omega\lambda')dx = -\hat{\omega}_1(X_F^1(\gamma), X) = -i_{X_F^1}\hat{\omega}_1(X).$$

This means that X_F^1 is the presymplectic gradient of F with respect to $\hat{\omega}_1$.

Example 2.25 ([7], Theorem 1)**.** Take $H : \mathcal{M} \to \mathbb{R}$ as the function F in Definition 2.3.2 defined by

$$H(\gamma) := \frac{1}{2}\int_{S^1} \kappa^2 dx,$$

where κ is the equicentroaffine curvature of γ. By (2.19) its presymplectic gradient X_H^1 with respect to $\hat{\omega}_1$ assumes the form

$$X_H^1(\gamma) = \frac{1}{2}\kappa'\gamma - \kappa\gamma'. \tag{2.20}$$

Observing the expression (2.20), we see that X_H^1 is precisely the vector field mentioned in Remark 2.6 which yields the KdV equation.

By the similar method we have used in the proofs of Theorem 2.22 and Theorem 2.23, we can show the following theorem, which essentially appears in Fujioka and Kurose [7].

Theorem 2.26 ([7, p.6])**.** *The bracket $\{-,-\}_1 : C^\infty(\mathcal{M})_\kappa \times C^\infty(\mathcal{M})_\kappa \to C^\infty(\mathcal{M})_\kappa$ defined by*

$$\{F, G\}_1(\gamma) := \hat{\omega}_1(X_F^1(\gamma), X_G^1(\gamma))$$

is a Poisson bracket over $C^\infty(\mathcal{M})_\kappa$. ■

By the similar method we have used in the proof of Theorem 2.22, we can show the following theorem.

Theorem 2.27. *Let $\iota_1 : C^\infty(\mathcal{M})_\kappa \to C^\infty(\mathcal{M}_1)$ be the inclusion given by*

$$\iota_1(F)([\gamma]) := F(\gamma).$$

Then, we have

$$\{F, G\}_1(\gamma) = \{\iota_1(F), \iota_1(G)\}_{\omega_1}([\gamma])$$

for all $\gamma \in \mathcal{M}$. ■

2.4 Momentum Maps for the Space of Equicentroaffine Curves

In this section, we will present a left action of the Bott-Virasoro group on $\mathcal{M}$, called the spatial left action, and two momentum maps $\mu_i (i = 0, 1)$ on $\mathcal{M}_i$, which are introduced by Fujioka, Kurose and Moriyoshi, such that the spatial left action and Pinkall's right action are Hamiltonian. The content in this section is mainly adapted from Fujioka, Kurose and Moriyoshi [8] and Heckman [10].

2.4.1 A Momentum Map on $\mathcal{M}_0$

Definition 2.28 ([10, Definition 1.2])**.** Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Suppose that G acts on a manifold M with a symplectic form ω. Then, a equivariant map $\mu : M \to \mathfrak{g}^*$ such that

$$d(\mu(-)V) = -i_{\underline{V}}\omega$$

for all $V \in \mathfrak{g}$ is called a ***momentum map*** of the action, where the momentum map μ is equivariant in the sense that

$$\mu(g \cdot x) = \mathrm{Ad}_g^*(\mu(x)),$$

for all $g \in G$ and $x \in M$. The action is called ***Hamiltonian*** if there exists a momentum map and the action leaves ω invariant.

Definition 2.29. The Bott-Virasoro group acts on $\mathcal{M}$ from the left such that for any $(\varphi, a) \in \mathrm{Diff}(S^1) \times_B \mathbb{R}$ and $\gamma \in \mathcal{M}$, we have

$$(\varphi, a) \cdot \gamma := c \cdot (\varphi \circ \psi) \tag{2.21}$$

where $\gamma = c \cdot \psi$, for some $\psi \in \mathrm{Diff}(S^1)$, with the help of Pinkall's right action discussed in Definition 2.7, called the ***spatial left action of the Bott-Virasoro group on $\mathcal{M}$***.

Definition 2.30. The spatial left action of the Bott-Virasoro group can also be defined on $\mathcal{M}_0$. For any $(\varphi, a) \in \mathrm{Diff}(S^1) \times_B \mathbb{R}$ and $[\gamma] \in \mathcal{M}_0$, we have

$$(\varphi, a) \cdot [\gamma] = [c \cdot (\varphi \circ \psi)], \tag{2.22}$$

where γ is a representative of $[\gamma]$ such that $\gamma = c \cdot \psi$ for some $\psi \in \mathrm{Diff}(S^1)$, called the ***spatial left action of the Bott-Virasoro group on*** $\mathcal{M}_0$. Note that for any other representative $\gamma \circ z$ of $[\gamma]$, where $z \in S^1$, since

$$\gamma \circ z = (c \cdot \psi) \circ z = (\sqrt{(\psi^{-1})'}\gamma) \circ \psi \circ z = (\sqrt{((\psi \circ z)^{-1})'}) \circ (\psi \circ z) = c \cdot (\psi \circ z),$$

and

$$c \cdot (\varphi \circ \psi \circ z) = (c \cdot (\varphi \circ \psi)) \circ z,$$

we see that the spatial left action (2.22) is well-defined.

Definition 2.31. Given $[\gamma] \in \mathcal{M}_0$, define $\mu_0 : \mathcal{M}_0 \to (\mathfrak{X}(S^1) \times_\omega \mathbb{R})^*$ to be a map by

$$\mu_0([\gamma]) := (\frac{1}{2}((\psi^{-1})')^2 dx \otimes dx, 0), \tag{2.23}$$

where $\psi \in \mathrm{Diff}(S^1)$ such that $\gamma = c \cdot \psi$ for some representative γ of $[\gamma]$. Let $\gamma \circ z$ be another representative of $[\gamma]$, where $z \in S^1$. Since

$$((\psi \circ z)^{-1})' = ((z^{-1})' \circ \psi^{-1})(\psi^{-1})' = (\psi^{-1})',$$

we see that μ_0 is well-defined.

Lemma 2.32. The map μ_0 is equivariant.

Proof. For any $[\gamma] \in \mathcal{M}_0$, let $\gamma = c \cdot \psi$ be a representative of $[\gamma]$, where $\psi \in \mathrm{Diff}(S^1)$. By (1.26), for any $(\varphi, a) \in \mathrm{Diff}(S^1) \times_B \mathbb{R}$ we have

$$\begin{aligned}
\mu_0((\varphi, a) \cdot [\gamma]) &= \mu_0([c \cdot (\varphi \circ \psi)]) \\
&= (\frac{1}{2}((\psi^{-1} \circ \varphi^{-1})')^2 dx \otimes dx, 0) \\
&= (\frac{1}{2}(((\psi^{-1})')^2 \circ \varphi^{-1})((\varphi^{-1})')^2 dx \otimes dx, 0) \\
&= \mathrm{Ad}^*_{(\varphi,a)}(\frac{1}{2}((\psi^{-1})')^2 dx \otimes dx, 0) \\
&= \mathrm{Ad}^*_{(\varphi,a)}(\mu_0([\gamma])),
\end{aligned}$$

which implies the claim. ∎

Lemma 2.33. For any $(\nu\partial_x, a) \in \mathfrak{X}(S^1) \times_\omega \mathbb{R}$, the fundamental vector field $\underline{(\nu\partial_x, a)} \in \mathfrak{X}(\mathcal{M})$ with respect to the spatial left action of the Bott-Virasoro group on $\mathcal{M}$ has the form

$$\underline{(\nu\partial_x, a)}_\gamma = -\frac{1}{2}(\frac{\nu \circ \psi}{\psi'})'\gamma + \frac{\nu \circ \psi}{\psi'}\gamma'$$

where $\gamma = c \cdot \psi$ with some $\psi \in \mathrm{Diff}(S^1)$.

Proof. Let $\varphi_{c\partial_x}$ be the flow of $\nu\partial_x \in \mathfrak{X}(S^1)$. Then, we have

$$\begin{aligned}
\underline{(\nu\partial_x, a)}_\gamma &= \frac{d}{dt}\Big|_{t=0} (\varphi_{\nu\partial_x}(t), at)\cdot\gamma \\
&= \frac{d}{dt}\Big|_{t=0} c\cdot(\varphi_{\nu\partial_x}(t)\circ\psi) \\
&= \frac{d}{dt}\Big|_{t=0} (c\cdot\varphi_{\nu\partial_x}(t))\cdot\psi \\
&= (\sqrt{(\psi^{-1})'}\circ\psi)((\frac{d}{dt}\Big|_{t=0} c\cdot\varphi_{\nu\partial_x}(t))\circ\psi) \\
&= ((-\frac{1}{2}\nu' c + \nu c')\circ\psi)(\sqrt{(\psi^{-1})'}\circ\psi) \\
&= -\frac{1}{2}(\nu'\circ\psi)(c\cdot\psi) + (\nu\circ\psi)(c'\circ\psi)(\sqrt{(\psi^{-1})'}\circ\psi) \\
&= -\frac{1}{2}(\nu'\circ\psi)(c\cdot\psi) + (\nu\circ\psi)\frac{c'\circ\psi}{\sqrt{\psi'}}
\end{aligned}$$

where we have used the computation in Lemma 2.10 for the fundamental vector fields of Pinkall's right action of Diff(S^1) on $\mathcal{M}$ and

$$(\psi^{-1})' = \frac{1}{\psi'\circ\psi^{-1}}.$$

Since

$$(c\cdot\psi)' = (\frac{c\circ\psi}{\sqrt{\psi'}})' = \frac{(c'\circ\psi)\psi'\sqrt{\psi'} - (c\circ\psi)\frac{1}{2\sqrt{\psi'}}\psi''}{\psi'} = \frac{c'\circ\psi}{\sqrt{\psi'}}\psi' - \frac{1}{2}(c\cdot\psi)\frac{\psi''}{\psi'},$$

we see that

$$\begin{aligned}
\underline{(\nu\frac{d}{dx}, a)}_\gamma &= -\frac{1}{2}(\nu'\circ\psi)(c\cdot\psi) + (\nu\circ\psi)\frac{(c\cdot\psi)' + \frac{1}{2}(c\cdot\psi)\frac{\psi''}{\psi'}}{\psi'} \\
&= -\frac{1}{2}(\nu'\circ\psi - \frac{(\nu\circ\psi)\psi''}{(\psi')^2})(c\cdot\psi) + \frac{\nu\circ\psi}{\psi'}(c\cdot\psi)' \\
&= -\frac{1}{2}(\frac{\nu\circ\psi}{\psi'})'\gamma + \frac{\nu\circ\psi}{\psi'}\gamma',
\end{aligned}$$

which implies the claim. ∎

Lemma 2.34. The map μ_0 is the momentum map with respect to the spatial left action of the Bott-Virasoro group on $\mathcal{M}_0$.

Proof. By Definition 2.28 and Lemma 2.32, it leaves us to to prove that

$$d(\mu_0(-)(\nu\partial_x, a)) = -i_{\underline{(\nu\partial_x,a)}}\omega_0.$$

For any tangent vector $X \in T_{[\gamma]}\mathcal{M}_0$ over $\mathcal{M}_0$, suppose that $\hat{X} = -\frac{1}{2}\lambda'\gamma + \lambda\gamma' \in T_\gamma\mathcal{M}$ is a lift of X over $\mathcal{M}$. Let $\varphi_{\lambda\partial_x}$ be the flow of $\lambda\partial_x \in \mathfrak{X}(S^1)$. It's easy to see that

$$\begin{aligned}
d(\mu_0(-)(\nu\partial_x, a))(X) &= \left.\frac{d}{dt}\right|_{t=0} \mu_0([\gamma \cdot \varphi_{\lambda\partial_x}(t)])(\nu\partial_x, a) \\
&= \left.\frac{d}{dt}\right|_{t=0} \mu_0([c \cdot (\psi \circ \varphi_{\lambda\partial_x}(t))])(\nu\partial_x, a),
\end{aligned}$$

where $\psi \in \mathrm{Diff}(S^1)$ such that $\gamma = c \cdot \psi$ for some representative γ of $[\gamma]$. By the definition of μ_0 in (2.23), we have

$$\begin{aligned}
d(\mu_0(-)(\nu\partial_x, a))(X) &= \left.\frac{d}{dt}\right|_{t=0} (\frac{1}{2}((\varphi_{\lambda\partial_x}(t)^{-1} \circ \psi^{-1})')^2 dx \otimes dx, 0)(\nu\partial_x, a) \\
&= \left.\frac{d}{dt}\right|_{t=0} \int_{S^1} \frac{1}{2}((\varphi_{\lambda\partial_x}(t)^{-1} \circ \psi^{-1})')^2 \nu dx \\
&= -\int_{S^1} (\lambda' \circ \psi^{-1})((\psi^{-1})')^2 \nu dx \\
&= -\int_{S^1} \frac{\nu \circ \psi}{\psi'} \lambda' dx \\
&= -\hat{\omega}_0(\underline{(\nu\partial_x, a)}_\gamma, \hat{X}),
\end{aligned}$$

where $\underline{(\nu\partial_x, a)}_\gamma$ denotes the fundamental vector field of $(v\partial_x, a)$ valued at γ with respect to the spatial left action of the Bott-Virasoro group on $\mathcal{M}$, computed in Lemma 2.33. By (2.22), we have

$$\begin{aligned}
\pi_{0*}\underline{(\nu\partial_x, a)}_\gamma &= \left.\frac{d}{dt}\right|_{t=0} [(\varphi_{\nu\partial_x}(t), at) \cdot \gamma] \\
&= \left.\frac{d}{dt}\right|_{t=0} [c \cdot (\varphi_{\nu\partial_x}(t) \circ \psi)] \\
&= \left.\frac{d}{dt}\right|_{t=0} (\varphi_{\nu\partial_x}(t), at) \cdot [\gamma] \\
&= \underline{(\nu\partial_x, a)}_{[\gamma]},
\end{aligned}$$

where $\underline{(\nu\partial_x, a)}_{[\gamma]}$ denotes the fundamental vector field of $(v\partial_x, a)$ valued at $[\gamma]$ with respect to the spatial left action of the Bott-Virasoro group on $\mathcal{M}_0$. By the definition of ω_0, we have

$$d(\mu_0(-)(\nu\partial_x, a))(X) = -\hat{\omega}_0(\underline{(\nu\partial_x, a)}_\gamma, \hat{X}) = -\omega_0(\underline{(\nu\partial_x, a)}_{[\gamma]}, X) = -i_{\underline{(\nu\partial_x, a)}}\omega_0(X),$$

which implies the claim. ∎

Lemma 2.35. The spatial left action of the Bott-Virasoro group on $\mathcal{M}_0$ leaves ω_0 invariant.

Proof. Suppose that $\hat{X} = -\frac{1}{2}\lambda'\gamma + \lambda\gamma' \in T_\gamma\mathcal{M}$ is a tangent vector over $\mathcal{M}$, where $\gamma = c\cdot\psi$ for some $\psi \in \mathrm{Diff}(S^1)$. Let $\varphi_{\lambda\partial_x}$ be the flow of $\lambda\partial_x \in \mathfrak{X}(S^1)$. Then, it's easy to see that for any $(\varphi, a) \in \mathrm{Diff}(S^1) \times_B \mathbb{R}$, we have

$$\begin{aligned} l_{(\varphi,a)_*}\hat{X} &= \frac{d}{dt}\bigg|_{t=0} (\varphi, a)\cdot(\gamma\cdot\varphi_{\lambda\partial_x}(t)) \\ &= \frac{d}{dt}\bigg|_{t=0} c\cdot\varphi\cdot\psi\cdot\varphi_{\lambda\partial_x}(t) \\ &= \frac{d}{dt}\bigg|_{t=0} ((\varphi, a)\cdot\gamma)\cdot\varphi_{\lambda\partial_x}(t) \\ &= -\frac{1}{2}\lambda'((\varphi, a)\cdot\gamma) + \lambda((\varphi, a)\cdot\gamma)', \end{aligned} \tag{2.24}$$

where $l_{(\varphi,a)}$ denotes the spatial left action of the Bott-Virasoro group on $\mathcal{M}$. Let $l_{(\varphi,a)}$ also denote the spatial left action of the Bott-Virasoro group on $\mathcal{M}_0$. Since

$$l_{(\varphi,a)} \circ \pi_0(\gamma) = l_{(\varphi,a)}([\gamma]) = [c\cdot(\varphi\circ\psi)] = [l_{(\varphi,a)}(\gamma)] = \pi_0 \circ l_{(\varphi,a)}(\gamma), \tag{2.25}$$

we see that for any $X, Y \in T_{[\gamma]}\mathcal{M}_0$, with lifts $\hat{X} = -\frac{1}{2}\lambda'\gamma + \lambda\gamma'$, $\hat{Y} = -\frac{1}{2}\mu'\gamma + \mu\gamma' \in T_\gamma\mathcal{M}$ respectively,

$$\begin{aligned} l^*_{(\varphi,a)}\omega_0(X, Y) &= \pi_0^* l^*_{(\varphi,a)}\omega_0(\hat{X}, \hat{Y}) \\ &= l^*_{(\varphi,a)}\pi_0^*\omega_0(\hat{X}, \hat{Y}) \\ &= l^*_{(\varphi,a)}\hat{\omega}_0(\hat{X}, \hat{Y}) \\ &= \hat{\omega}_0(l_{(\varphi,a)_*}\hat{X}, l_{(\varphi,a)_*}\hat{Y}). \end{aligned}$$

Thus, by the computation in (2.24), we have

$$l^*_{(\varphi,a)}\omega_0(X, Y) = \hat{\omega}_0(l_{(\varphi,a)_*}\hat{X}, l_{(\varphi,a)_*}\hat{Y}) = \int_{S^1} \lambda\mu' dx = \hat{\omega}_0(\hat{X}, \hat{Y}) = \omega_0(X, Y),$$

which implies the claim. ∎

By showing Lemma 2.32, Lemma 2.34 and Lemma 2.35, we have indeed proved the following theorem, which is one of the main results in Fujioka, Kurose and Moriyoshi [8].

Theorem 2.36 ([8])**.** *The spatial left action of the Bott-Virasoro group on $\mathcal{M}_0$ is Hamiltonian with momentum map μ_0.* ∎

2.4.2 A Momentum Map on $\mathcal{M}_1$

Definition 2.37. Recalling Definition 2.7, given $(\varphi, a) \in \mathrm{Diff}(S^1) \times_B \mathbb{R}$ and $\gamma \in \mathcal{M}$, we define a similar action on $\mathcal{M}$ by

$$\gamma\cdot(\varphi, a) := \gamma\cdot\varphi = (\sqrt{((\varphi^{-1})')}\gamma)\circ\varphi = \frac{\gamma\circ\varphi}{\sqrt{\varphi'}}, \tag{2.26}$$

and call it ***Pinkall's right action of the Bott-Virasoro group on*** $\mathcal{M}$.

Remark 2.38. The action (2.26) can also be defined on $\mathcal{M}_1$. Given $(\varphi, a) \in \mathrm{Diff}(S^1)\times_B\mathbb{R}$ and $[\gamma] \in \mathcal{M}_1$, we set

$$[\gamma]\cdot(\varphi, a) := [\gamma\cdot\varphi], \tag{2.27}$$

and call it ***Pinkall's right action of the Bott-Virasoro group on*** $\mathcal{M}_1$. Let $A \in \mathrm{SL}(2,\mathbb{R})$ and $A\gamma$ be another representative of γ. Since

$$(A\gamma)\cdot\varphi = \frac{(A\gamma)\circ\varphi}{\sqrt{\varphi'}} = \frac{A(\gamma\circ\varphi)}{\sqrt{\varphi'}} = A(\gamma\cdot\varphi),$$

we see that (2.27) is well-defined.

Lemma 2.39. Pinkall's right action leaves ω_1 invariant.

Proof. For any tangent vector $\hat{X} = \frac{1}{2}\lambda'\gamma + \lambda\gamma' \in T_\gamma\mathcal{M}$ over $\mathcal{M}$, let $\varphi_{\lambda\partial_x}$ be the flow of $\lambda\partial_x$. By the computation in the proof of Lemma 2.33, given $(\varphi, a) \in \mathrm{Diff}(S^1)\times_B\mathbb{R}$, we have

$$\begin{aligned} r_{(\varphi,a)_*}\hat{X} &= \frac{d}{dt}\Big|_{t=0}(\gamma\cdot\varphi_{\lambda\partial_x}(t))\cdot(\varphi, a) \\ &= \frac{d}{dt}\Big|_{t=0}\gamma\cdot\varphi_{\lambda\partial_x}(t)\cdot\varphi \\ &= -\frac{1}{2}(\frac{\lambda\circ\varphi}{\varphi'})'(\gamma\cdot(\varphi,a)) + \frac{\lambda\circ\varphi}{\varphi'}(\gamma\cdot(\varphi,a))', \end{aligned} \tag{2.28}$$

where $r_{(\varphi,a)}$ denotes Pinkall's right action of the Bott-Virasoro group on $\mathcal{M}$. Let $r_{(\varphi,a)}$ also denote Pinkall's right action of the Bott-Virasoro group on $\mathcal{M}_1$. It's easy to see that

$$r_{(\varphi,a)}\circ\pi_1 = \pi_1\circ r_{(\varphi,a)}.$$

Given tangent vectors $X, Y \in T_{[\gamma]}\mathcal{M}_1$, suppose that $\hat{X} = -\frac{1}{2}\lambda'\gamma + \lambda\gamma'$, $\hat{Y} = -\frac{1}{2}\mu'\gamma + \mu\gamma' \in T_\gamma\mathcal{M}$ lift X, Y, respectively. Then, we have

$$r^*_{(\varphi,a)}\omega_1(X.Y) = \pi_1^* r^*_{(\varphi,a)}\omega_1(\hat{X},\hat{Y}) = r^*_{(\varphi,a)}\pi_1^*\omega_1(\hat{X},\hat{Y}) = \hat{\omega}_1(r_{(\varphi,a)_*}\hat{X}, r_{(\varphi,a)_*}\hat{Y}).$$

Let $\varphi_{\mu\partial_x}$ be the flow of $\mu\partial_x$ and $\tilde{\kappa}$ the equicentroaffine curvature flow of the variation $\gamma\cdot\varphi_{\mu\partial_x}\cdot\varphi$ whose initial vector is $r_{(\varphi,a)_*}\hat{Y}$. By the computation in (2.28) and the discussion in Remark 2.19, we have

$$r^*_{(\varphi,a)}\omega_1(X,Y) = \hat{\omega}_1(r_{(\varphi,a)_*}\hat{X}, r_{(\varphi,a)_*}\hat{Y}) = \int_{S^1}\frac{\lambda\circ\varphi}{\varphi'}\Omega(\frac{\mu\circ\varphi}{\varphi'})'dx = \frac{d}{dt}\Big|_{t=0}\int_{S^1}\frac{\lambda\circ\varphi}{\varphi'}\tilde{\kappa}(t)dx.$$

Let η be the ratio of γ, which is mentioned in Proposition 1.33. By the definition of Pinkall's right action of $\mathrm{Diff}(S^1)$ on $\mathcal{M}$, the ratio of $\gamma\cdot\varphi_{\mu\partial_x}\cdot\varphi$ is $\eta\circ\varphi_{\mu\partial_x}\circ\varphi$. Then, by the discussion in Remark 2.2, we have

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\tilde{\kappa}(t) &= \frac{d}{dt}\Big|_{t=0}\frac{S(\eta\circ\varphi_{\mu\partial_x}(t)\circ\varphi)}{2} \\
&= \frac{d}{dt}\Big|_{t=0}\frac{(S(\eta\circ\varphi_{\mu\partial_x}(t))\circ\varphi)(\varphi')^2+S(\varphi)}{2} \\
&= \frac{d}{dt}\Big|_{t=0}((\frac{S(\eta)}{2}\circ\varphi_{\mu\partial_x}(t)\circ\varphi)((\varphi_{\mu\partial_x}(t)')^2\circ\varphi)+\frac{S(\varphi_{\mu\partial_x}(t))}{2}\circ\varphi)(\varphi')^2 \\
&= \frac{d}{dt}\Big|_{t=0}((\kappa\circ\varphi_{\mu\partial_x}(t)\circ\varphi)((\varphi_{\mu\partial_x}(t)')^2\circ\varphi)+\frac{S(\varphi_{\mu\partial_x}(t))}{2}\circ\varphi)(\varphi')^2 \\
&= ((\kappa'\circ\varphi)(\mu\circ\varphi)+2(\kappa\circ\varphi)(\mu'\circ\varphi)+\frac{1}{2}(\mu'''\circ\varphi))(\varphi')^2,
\end{aligned}$$

where κ is the equicentroaffine curvature of γ and we have used the property of the Schwarzian derivative mentioned in Lemma 1.28. It follows that

$$\begin{aligned}
r^*_{(\varphi,a)}\omega_1(X,Y) &= \int_{S^1}\frac{\lambda\circ\varphi}{\varphi'}((\kappa'\circ\varphi)(\mu\circ\varphi)+2(\kappa\circ\varphi)(\mu'\circ\varphi)+\frac{1}{2}(\mu'''\circ\varphi))(\varphi')^2dx. \\
&= \int_{S^1}\lambda\Omega\mu'dx \\
&= \hat{\omega}_1(\hat{X},\hat{Y}) \\
&= \omega_1(X,Y),
\end{aligned}$$

which implies the claim. ■

Definition 2.40 ([8])**.** Given $[\gamma]\in\mathcal{M}_1$, define $\mu_1:\mathcal{M}_1\to(\mathfrak{X}(S^1)\times_\omega\mathbb{R})^*$ to be a map on $\mathcal{M}_1$ by

$$\mu_1([\gamma]):=(-\kappa dx\otimes dx,-\frac{1}{2}),$$

where κ is the equicentroaffine curvature of a representative γ of $[\gamma]$. By Lemma 1.34, the map μ_1 is well-defined.

Lemma 2.41. The map μ_1 is equivariant. ■

In the following remark, we will discuss the role of the $\mathbb{R}$-part in the definition of the map μ_1, which will also contain the proof of Lemma 2.41.

Remark 2.42. Given $\gamma\in\mathcal{M}$, define $\Pi:\mathcal{M}\to(\mathfrak{X}(S^1)\times_\omega\mathbb{R})^*$ to be a map by

$$\Pi(\gamma)=(-\frac{1}{2}S(\eta)dx\otimes dx,-\frac{1}{2}),$$

where $\eta=\gamma_1/\gamma_2$ is the ratio map and S the Schwarzian derivative. It's easy to see that we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[dr, "\Pi"] & \\
\mathrm{Diff}(S^1) \arrow[u, "I"] \arrow[r, "\Pi_{-\frac{1}{2},-1}"'] & (\mathfrak{X}(S^1)\times_\omega\mathbb{R})^*
\end{tikzcd}$$

where I is the bijection mentioned in Remark 2.8, and $\Pi_{-1/2.-1}$ in Remark 1.29. Indeed, for any $\psi \in \mathrm{Diff}(S^1)$, we have

$$\begin{aligned}\Pi \circ I(\psi) &= \Pi(c \cdot \psi)\\ &= (-\frac{1}{2}S(\cot \circ \psi)dx \otimes dx, -\frac{1}{2})\\ &= ((-\frac{1}{2}(S(\cot) \circ \psi)(\psi')^2 - \frac{1}{2}S(\psi))dx \otimes dx, -\frac{1}{2})\\ &= ((-\frac{1}{2}S(\psi) - (\psi')^2)dx \otimes dx, -\frac{1}{2})\\ &= \Pi_{-\frac{1}{2},-1}(\psi),\end{aligned}$$

since $S(\cot) = 2$. Now, by the discussion in Remark 1.29, we see that Π is equivariant with respect to Pinkall's right action on $\mathcal{M}$ and the coadjoint action on the Virasoro algebra. Given $[\gamma] \in \mathcal{M}$, choose a representative $\gamma \in \mathcal{M}$ and denote it by $s([\gamma])$. By the discussion above, we see that $\mu_1 = \Pi \circ s$ is equivariant. Note that without the occurrence of $-\frac{1}{2}$ in the expression of μ_1, the map $\Pi_{-1/2.-1}$ will not be involved, and μ_1 can not be equivariant. This shows the role of the $\mathbb{R}$-part in the definition of μ_1.

Lemma 2.43. The map μ_1 is a momentum map of Pinkall's right action of the Bott-Virasoro group on $\mathcal{M}_1$.

Proof. By Definition 2.28 and Lemma 2.39, it is sufficient for us to prove that for any $(\nu\partial_x, a) \in \mathfrak{X}(S^1) \times_\omega \mathbb{R}$, we have

$$d(\mu_1(-)(\nu\partial_x, a)) = -i_{\underline{(\nu\partial_x,a)}}\omega_1.$$

Let $X \in T_{[\gamma]}\mathcal{M}_1$ be a tangent vector over $\mathcal{M}_1$ and $\hat{X} = -\frac{1}{2}\lambda'\gamma + \lambda\gamma' \in T_\gamma\mathcal{M}$ a lift of X over $\mathcal{M}$. Suppose that $\tilde{\gamma}$ is a variation in $\mathcal{M}$ with initial vector $\hat{X}$, and $\tilde{\kappa}$ its equicentroaffine curvature flow. It's easy to see that

$$\begin{aligned}d(\mu_1(-)(\nu\partial_x, a))(X) &= \frac{d}{dt}\Big|_{t=0}\mu_1([\tilde{\gamma}(t)])(\nu\partial_x, a)\\ &= -\frac{d}{dt}\Big|_{t=0}(\tilde{\kappa}(t)dx \otimes dx, \frac{1}{2})(\nu\partial_x, a)\\ &= -\frac{d}{dt}\Big|_{t=0}\int_{S^1}\nu\tilde{\kappa}(t)dx\\ &= -\int_{S^1}\nu\Omega\lambda' dx\\ &= -\hat{\omega}_1(\underline{(\nu\partial_x, a)}_\gamma, \hat{X}),\end{aligned}$$

where $\underline{(\nu\partial_x, a)}_\gamma$ is the fundamental vector field of $(\nu\partial_x, a)$ with respect to Pinkall's right action of the Bott-Virasoro group on $\mathcal{M}$ valued at γ, given by

$$\underline{(\nu\partial_x, a)}_\gamma = -\frac{1}{2}\nu'\gamma + \nu\gamma'.$$

Let $\varphi_{\nu\partial_x}$ be the flow of $\nu\partial_x \in \mathfrak{X}(S^1)$. By (2.27), we have

$$\pi_{1*}\underline{(\nu\partial_x, a)}_{\gamma} = \left.\frac{d}{dt}\right|_{t=0} [\gamma \cdot \varphi_{\nu\partial_x}(t)] = \left.\frac{d}{dt}\right|_{t=0} [\gamma] \cdot (\varphi_{\nu\partial_x}(t), at) = \underline{(\nu\partial_x, a)}_{[\gamma]},$$

where $\underline{(\nu\partial_x, a)}_{[\gamma]}$ is the fundamental vector field of $(\nu\partial_x, a)$ with respect to Pinkall's right action of the Bott-Virasoro group on $\mathcal{M}_1$ valued at $[\gamma]$. By the definition of ω_1, we see that

$$d(\mu_1(-)(\nu\partial_x, a))(X) = -\hat{\omega}_1(\underline{(\nu\partial_x, a)}_{\gamma}, \hat{X}) = -\omega_1(\underline{(\nu\partial_x, a)}_{[\gamma]}, X) = -i_{\underline{(\nu\frac{d}{dx}, a)}}\omega_1(X),$$

which implies the claim. ■

By showing Lemma 2.41 and Lemma 2.43, we have indeed proved the following theorem, which is one of the main results in Fujioka, Kurose and Moriyoshi [8].

Theorem 2.44 ([8])**.** *Pinkall's right action of the Bott-Virasoro group on $\mathcal{M}_1$ is Hamiltonian with momentum map μ_1.* ■

Chapter 3

Relations between the Bott-Virasoro Group and the Space of Equicentroaffine Curves

In this chapter, we will discover the relationship between the Kirillov-Kostant-Souriau form, the Pinkall 2-form, the Fujioka-Kurose form, and the canonical symplectic form. In particular, by clarifying the connection between the Fujioka-Kurose 2-form and the canonical symplectic form for the Bott-Virasoro group, we can show why the equicentroaffine curvature flow of the vector field X^1_H mentioned in Example 2.25 must satisfy the KdV equation.

3.1 The Kirillov-Kostant-Souriau Form and Canonical Symplectic Form

In this section, we will show the relationship between the canonical symplectic form on $G \ltimes \mathfrak{g}^*$ and the Kirillov-Kostant-Souriau form on $\mathfrak{g}^*$. We will derive an explicit formula for the canonical symplectic form and show how it yields the Euler equation. In Theorem 3.11, we will construct an action on $G \ltimes \mathfrak{g}^*$, and show that the pullback of the KKS form by a projection map is precisely the canonical symplectic form restricted to the oribts. This will be one of our main results. The content in Section 3.1.1 and Section 3.1.2 is mainly adapted from Baues and Cortes [2], Dwivedi, Herman, Jeffrey and Hurk [11] and Kumar [16].

3.1.1 The Kirillov-Kostant Souriau Form

The following theorem essentially appears in Dwivedi, Herman, Jeffrey, and Hurk T. [11]. We refer the readers to it for the proof.

Theorem 3.1 ([11], Theorem 5.1)**.** *Suppose that G is a Lie group and $\mathfrak{g}$ its Lie algebra. For any $\lambda \in \mathfrak{g}^*$, let $\mathcal{O}_\lambda$ be the coajoint orbit of G at λ. Then, $\mathcal{O}_\lambda$ carries a symplectic* *form $\omega_{\mathcal{O}_\lambda}$ such that for any tangent vectors $X, Y \in T_\xi\mathcal{O}_\lambda$ along $\mathcal{O}_\lambda$ with $\xi \in \mathcal{O}_\lambda$, we have*

$$\omega_{\mathcal{O}_\lambda}(X, Y) = \xi([U, V]),$$

where $U, V \in \mathfrak{g}$ such that $\mathrm{ad}_U^*(\xi) = X, \mathrm{ad}_V^*(\xi) = Y$. ■

Remark 3.2. In Theorem 3.1, we have used the fact that any tangent vector $X \in T_\xi\mathcal{O}_\lambda$ along the coadjoint orbit of G has the form

$$X = \mathrm{ad}_U^*(\xi),$$

for some $U \in \mathfrak{g}$.

Definition 3.3. The family of 2-forms $\{\omega_{\mathcal{O}_\lambda} \in \Omega^2(\mathcal{O}_\lambda) \mid \lambda \in \mathfrak{g}\}$ is called the ***Kirillov-Kostant-Souriau form***, denoted by ω_{KKS}.

Remark 3.4. For any $U \in \mathfrak{g}$, let $F : \mathfrak{g}^* \to \mathbb{R}$ be a function on $\mathfrak{g}^*$ defined by

$$F(m) := m(U). \tag{3.1}$$

In Definition 1.40, we have shown that $dF(\xi)$ can be regarded as an element in $\mathfrak{g}$. For any $\eta \in \mathfrak{g}^*$, since

$$\eta(dF(\xi)) = \left.\frac{d}{dt}\right|_{t=0} F(\xi + t\eta) = \left.\frac{d}{dt}\right|_{t=0} (\xi + t\eta)(U) = \eta(U),$$

we see that

$$dF(\xi) = U.$$

Similarly, for any $V \in \mathfrak{g}$, let $H : \mathfrak{g}^* \to \mathbb{R}$ be the function over $\mathfrak{g}^*$ defined by

$$H(m) := m(V).$$

It follows that given tangent vectors $\mathrm{ad}_U^*(\xi), \mathrm{ad}_V^*(\xi) \in T_\xi\mathcal{O}_\lambda$ along the coadjoint orbit $\mathcal{O}_\lambda$ of G, we have

$$\begin{aligned}\omega_{KKS}(\mathrm{ad}_U^*(\xi), \mathrm{ad}_V^*(\xi)) &= \omega_{KKS}(\mathrm{ad}_{dF(\xi)}^*(\xi), \mathrm{ad}_{dH(\xi)}^*(\xi)) \\ &= \xi([dF(\xi), dH(\xi)]) \\ &= \{F, H\}_{LP}(\xi).\end{aligned}$$

This shows the relationship between the KKS Form and the Lie-Poisson bracket.

3.1.2 Canonical Symplectic Form

Definition 3.5 ([2, pp.49])**.** Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Set $G \ltimes \mathfrak{g}^*$ to be the semidirect product with multiplication

$$(g,\alpha)(h,\beta) = (gh, \mathrm{Ad}^*_{h^{-1}}\alpha + \beta).$$

Given $\alpha \in \mathfrak{g}^*$, a tangent vector in $T_\alpha\mathfrak{g}^*$ is expressed as β_α, where $\beta \in \mathfrak{g}^*$. Then, the 1-form $\Theta : T(G \ltimes \mathfrak{g}^*) \to \mathbb{R}$ on $G \ltimes \mathfrak{g}^*$ defined by

$$\Theta(X_g, \beta_\alpha) := \alpha({l_{g^{-1}}}_* X_g),$$

where $(X_g, \beta_\alpha) \in T_gG \times T_\alpha\mathfrak{g}^* = T_{(g,\alpha)}(G \ltimes \mathfrak{g}^*)$, is called the ***tautological*** **1*-form***, the exterior differential $d\Theta$ of which is called the ***canonical symplectic form***.

Remark 3.6 ([2, pp.48-49])**.** Usually, the tautological 1-form refers to the 1-form $\tilde{\Theta} : T(T^*G) \to \mathbb{R}$ defined by

$$\tilde{\Theta}(X_{\alpha_g}) = \alpha_g(\pi_* X_{\alpha_g}),$$

where $X_{\alpha_g} \in T_{\alpha_g}(T^*G)$ is a tangent vector over T^*G, and the canonical symplectic form refers to the exterior differential $d\tilde{\Theta}$. However, there is a group isomorphism $\Phi : T^*G \to G \ltimes \mathfrak{g}^*$ given by

$$\Phi(\alpha_g) := (g, \alpha_g \circ l_{g*}),$$

where the cotangent bundle T^*G holds the multiplication

$$\alpha_g\beta_h := \alpha_g \circ {r_{h^{-1}}}_* + \beta_h \circ {l_{g^{-1}}}_*,$$

such that

$$\Phi^*\Theta = \tilde{\Theta},$$

since for any $X_{\alpha_g} \in T_{\alpha_g}(T^*G)$, we have

$$\begin{aligned}
\Phi^*\Theta(X_{\alpha_g}) &= \Theta(\frac{d}{dt}\bigg|_{t=0} \Phi(c_{X_{\alpha_g}}(t))) \\
&= \Theta(\frac{d}{dt}\bigg|_{t=0} \pi(c_{X_{\alpha_g}}(t)), \frac{d}{dt}\bigg|_{t=0} c_{X_{\alpha_g}}(t) \circ {l_{\pi(c_{X_{\alpha_g}}(t))}}_*) \\
&= \alpha_g \circ l_{g*} \circ {l_{g^{-1}}}_*(\frac{d}{dt}\bigg|_{t=0} \pi(c_{X_{\alpha_g}}(t))) \\
&= \alpha_g(\pi_* X_{\alpha_g}) \\
&= \tilde{\Theta}(X_{\alpha_g}),
\end{aligned}$$

where $c_{X_{\alpha_g}}$ is a curve in T^*G with initial vector X_{α_g} and $\pi : T^*G \to G$ the projection on T^*G. This is the reason why we use Θ as the tautological 1-form and it turns out that it is enough for our later discussion.

The following theorem essentially appear in Baues O., Cortes V. [2]. We provide a proof here.

Theorem 3.7 ([2, Proposition 4.3]). *The canonical symplectic form $d\Theta$ can be computed as*

$$d\Theta((X_g, \alpha_\sigma), (Y_g, \beta_\sigma)) = \alpha(l_{g^{-1}}{}_* Y_g) - \beta(l_{g^{-1}}{}_* X_g) - \sigma([l_{g^{-1}}{}_* X_g, l_{g^{-1}}{}_* Y_g]),$$

where $(X_g, \alpha_\sigma), (Y_g, \beta_\sigma) \in T_{(g,\sigma)}(G \ltimes \mathfrak{g}^*)$ are tangent vectors.

Proof. For any element $(V, \beta) \in \mathfrak{g} \times \mathfrak{g}^*$ in the Lie algebra of $G \ltimes \mathfrak{g}^*$, we have

$$\begin{aligned}
\Theta(\widetilde{(V,\beta)}_{(g,\sigma)}) &= \Theta(\frac{d}{dt}\Big|_{t=0} (g,\sigma)(c_V(t), \beta t)) \\
&= \Theta(\frac{d}{dt}\Big|_{t=0} (gc_V(t), \mathrm{Ad}^*_{c_V(t)^{-1}}\sigma + \beta t)) \\
&= \Theta(\tilde{V}_g, -\mathrm{ad}^*_V \sigma + \beta) \\
&= \sigma(V),
\end{aligned}$$

where c_V is a curve in G with initial vector V. It follows that for any $(U, \alpha), (V, \beta) \in \mathfrak{g} \times \mathfrak{g}^*$ we have

$$\begin{aligned}
\widetilde{(U,\alpha)}_{(g,\sigma)}(\Theta(\widetilde{(V,\beta)})) &= (\Theta(\widetilde{(V,\beta)}))_* \widetilde{(U,\alpha)}_{(g,\sigma)} \\
&= (\Theta(\widetilde{(V,\beta)}))_* l_{(g,\sigma)_*}(U,\alpha) \\
&= \frac{d}{dt}\Big|_{t=0} \Theta(\widetilde{(V,\beta)}_{(g,\sigma)(c_U(t),\alpha t)}) \\
&= \frac{d}{dt}\Big|_{t=0} \Theta(\widetilde{(V,\beta)}_{(gc_U(t), \mathrm{Ad}^*_{c_U(t)^{-1}}\sigma + \alpha t)}) \\
&= \frac{d}{dt}\Big|_{t=0} (\mathrm{Ad}^*_{c_U(t)^{-1}}\sigma + \alpha t)(V) \\
&= \sigma([U,V]) + \alpha(V).
\end{aligned}$$

where $(g, \sigma) \in G \ltimes \mathfrak{g}^*$ and c_U is a curve in G with initial vector U. Since

$$\begin{aligned}
[(U,\alpha),(V,\beta)] &= \mathrm{ad}_{(U,\alpha)}(V,\beta) \\
&= \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{(c_U(t),\alpha t)}(V,\beta) \\
&= \frac{d^2}{dsdt}\Big|_{t=s=0} (c_U(t),\alpha t)(c_V(s),\beta s)(c_U(t)^{-1}, -\mathrm{Ad}^*_{c_U(t)}\alpha t) \\
&= \frac{d^2}{dsdt}\Big|_{t=s=0} (c_U(t)c_V(s), \mathrm{Ad}^*_{c_V(s)^{-1}}\alpha t + \beta s)(c_U(t)^{-1}, -\mathrm{Ad}^*_{c_U(t)}\alpha t) \\
&= \frac{d^2}{dsdt}\Big|_{t=s=0} (c_U(t)c_V(s)c_U(t)^{-1}, \mathrm{Ad}^*_{c_U(t)}(\mathrm{Ad}^*_{c_V(s)^{-1}}\alpha t + \beta s) - \mathrm{Ad}^*_{c_U(t)}\alpha t) \\
&= ([U,V], \mathrm{ad}^*_U\beta - \mathrm{ad}^*_V\alpha).
\end{aligned}$$

we see that

$$\begin{aligned}\Theta([\widetilde{(U,\alpha)},\widetilde{(V,\beta)}]_{(g,\sigma)}) &= \Theta(\widetilde{[(U,\alpha),(V,\beta)]}_{(g,\sigma)}) \\ &= \Theta((\widetilde{[U,V],\mathrm{ad}^*_U\beta} - \mathrm{ad}^*_V\alpha)_{(g,\sigma)}) \\ &= \sigma([U,V]).\end{aligned}$$

It follows that

$$\begin{aligned}d\Theta(\widetilde{(U,\alpha)}_{(g,\sigma)},\widetilde{(V,\beta)}_{(g,\sigma)}) &= \widetilde{(U,\alpha)}_{(g,\sigma)}(\Theta(\widetilde{(V,\beta)})) - \widetilde{(V,\beta)}_{(g,\sigma)}(\Theta(\widetilde{(U,\alpha)})) - \Theta([\widetilde{(U,\alpha)},\widetilde{(V,\beta)}]_{(g,\sigma)}) \\ &= \sigma([U,V]) + \alpha(V) - \sigma([V,U]) - \beta(U) - \sigma([U,V]) \\ &= \alpha(V) - \beta(U) + \sigma([U,V]).\end{aligned}$$

Note that for any tangent vector $(X_g,\alpha_\sigma) \in T_{(g,\sigma)}(G \ltimes \mathfrak{g}^*)$, we have

$$\begin{aligned}l_{(g^{-1},-\mathrm{Ad}^*_g\sigma)}(X_g,\alpha_\sigma) &= \left.\frac{d}{dt}\right|_{t=0}(g^{-1},-\mathrm{Ad}^*_g\sigma)(c_{X_g}(t),\sigma+\alpha t) \\ &= \left.\frac{d}{dt}\right|_{t=0}(g^{-1}c_{X_g}(t),-\mathrm{Ad}^*_{c_{X_g}(t)^{-1}g}\sigma+\sigma+\alpha t) \\ &= (l_{g^{-1}*}X_g,\mathrm{ad}^*_{l_{g^{-1}*}X_g}\sigma+\alpha),\end{aligned}$$

where c_{X_g} is a curve in G with initial vector X_g. It follows that

$$\begin{aligned}d\Theta((X_g,\alpha_\sigma),(Y_g,\beta_\sigma)) &= d\Theta((l_{g^{-1}*}X_g,\widetilde{\mathrm{ad}^*_{l_{g^{-1}*}X_g}\sigma+\alpha)}_{(g,\sigma)},(l_{g^{-1}*}Y_g,\widetilde{\mathrm{ad}^*_{l_{g^{-1}*}Y_g}\sigma+\beta)}_{(g,\sigma)}) \\ &= (\mathrm{ad}^*_{l_{g^{-1}*}X_g}\sigma+\alpha)(l_{g^{-1}*}Y_g) - (\mathrm{ad}^*_{l_{g^{-1}*}Y_g}\sigma+\beta)(l_{g^{-1}*}X_g) + \sigma([l_{g^{-1}*}X_g,l_{g^{-1}*}Y_g]) \\ &= \alpha(l_{g^{-1}*}Y_g) - \beta(l_{g^{-1}*}X_g) - \sigma([l_{g^{-1}*}X_g,l_{g^{-1}*}Y_g]),\end{aligned}$$

which gives the claim. ■

The following lemma essentially appears in Baues O., Cortes V. [2]. We will provide a proof here.

Lemma 3.8 ([2, p.49])**.** The canonical symplectic form $d\Theta$ does be symplectic over $G \ltimes \mathfrak{g}^*$.

Proof. It's sufficient to show that $d\Theta$ is nondegenerate. Suppose that $(X_g,\alpha_\sigma) \in T_{(g,\sigma)}(G \ltimes \mathfrak{g}^*)$ belongs to the kernel of $d\Theta$. Choose an inner product $\langle -,- \rangle$ over $\mathfrak{g}$ and let A be the induced inertia operator. By Theorem 3.7, we have

$$d\Theta((X_g,\alpha_\sigma),(0_g,(A(l_{g^{-1}*}X_g))_\sigma)) = -A(l_{g^{-1}*}X_g)(l_{g^{-1}*}X_g) = 0.$$

It follows that $X_g = 0$. Since

$$d\Theta((0_g,\alpha_\sigma),(l_{g*}(A^{-1}(\alpha)),0_\sigma)) = \alpha(A^{-1}(\alpha)) = 0,$$

we see that $\alpha = 0$, which implies the claim. ■

The following theorem essentially appears in Kumar [16]. We will provide a proof here.

Theorem 3.9 ([16, Proposition 13])**.** *Let $\langle -,- \rangle$ be an inner product and let $A : \mathfrak{g} \to \mathfrak{g}^*$ be the induced inertia operator. Define $E : G \ltimes \mathfrak{g}^* \to \mathbb{R}$ by*

$$E(g, \alpha) := \frac{1}{2}\alpha(A^{-1}(\alpha)).$$

Suppose that $X_E \in \mathfrak{X}(G \ltimes \mathfrak{g}^*)$ is the symplectic gradient of E and (φ, ξ) its integral curve. Then, φ is a geodesic in G.

Proof. For any tangent vector $(X_g, \alpha_\sigma) \in T_{(g,\sigma)}(G \ltimes \mathfrak{g}^*)$, let c_{X_g} be a curve in G with initial vector X_g. Then, we have

$$dE(X_g, \alpha_\sigma) = \left.\frac{d}{dt}\right|_{t=0} E(c_{X_g}(t), \sigma + \alpha t) = \left.\frac{d}{dt}\right|_{t=0} \frac{1}{2}(\sigma + \alpha t)(A^{-1}(\sigma + \alpha t)) = \alpha(A^{-1}(\sigma)).$$

By Theorem 3.7, since

$$d\Theta((l_{g*}(A^{-1}(\sigma)), -\mathrm{ad}^*_{A^{-1}(\sigma)}\sigma), (X_g, \alpha_\sigma)) = -\alpha(A^{-1}(\sigma)) = -dE(X_g, \alpha_\sigma), \tag{3.2}$$

we see that

$$X_E(g, \sigma) = (l_{g*}A^{-1}(\sigma), -\mathrm{ad}^*_{A^{-1}(\sigma)}\sigma). \tag{3.3}$$

Suppose that m is the curve in $\mathfrak{g}^*$ corresponding to the curve φ in G satisfying (1.4), i.e.,

$$m(t) = A(\theta(\dot{\varphi}(t))).$$

Then, by (3.3), we have

$$m(t) = A(l_{\varphi(t)^{-1}}{}_*\dot{\varphi}(t)) = A(l_{\varphi(t)^{-1}}{}_* l_{\varphi(t)*}A^{-1}(\xi(t))) = \xi(t).$$

It follows that

$$\dot{m}(t) = \dot{\xi}(t) = -\mathrm{ad}^*_{A^{-1}(\xi(t))}\xi(t) = -\mathrm{ad}^*_{A^{-1}(m(t))}m(t),$$

which means that m satisfies the Euler equation in Theorem 1.10. This implies that φ is a geodesic in G with respect to the left-invariant metric induced from the inner product $\langle -,- \rangle$. ■

Remark 3.10. From the proof of Theorem 3.9, in terms of the integral curve (φ, ξ) of X_E, we see that φ and ξ satisfies the condition (1.4) in Theorem 1.10, i.e.,

$$\dot{\xi}(t) = A(\theta(\dot{\varphi}(t))).$$

From (3.3), we see that ξ satisfies the Euler equation.

3.1.3 The Kirillov-Kostant-Sauriau Form and Canonical Symplectic Form

Let G act on $G \ltimes \mathfrak{g}^*$ such that for any $h \in G$ and $(g, \alpha) \in G \ltimes \mathfrak{g}^*$ we have

$$(g, \alpha) \cdot h = (gh, \mathrm{Ad}^*_{h^{-1}}\alpha), \tag{3.4}$$

Given $U \in \mathfrak{g}$, let $\underline{U} \in \mathfrak{X}(G \ltimes \mathfrak{g}^*)$ denote the fundamental vector field with respect to the action (3.4). Moreover, denoting by $\mathrm{pr}_2 : G \ltimes \mathfrak{g}^* \to \mathfrak{g}^*$ the projection onto $\mathfrak{g}^*$, we can prove that:

Theorem 3.11. *For the Kirillov-Kostant-Souriau form ω_{KKS} on $\mathfrak{g}^*$ and the canonical symplectic form $d\Theta$ on $G \ltimes \mathfrak{g}^*$, we have*

$$\mathrm{pr}_2^*\omega_{KKS}(\underline{U}_{(g,\sigma)}, \underline{V}_{(g,\sigma)}) = d\Theta(\underline{U}_{(g,\sigma)}, \underline{V}_{(g,\sigma)}),$$

where $U, V \in \mathfrak{g}$ and $(g, \sigma) \in G \ltimes \mathfrak{g}^*$.

Proof. It's easy to see that

$$\begin{aligned}
\mathrm{pr}_2^*\omega_{KKS}(\underline{U}_{(g,\sigma)}, \underline{V}_{(g,\sigma)}) &= \omega_{KKS}(\mathrm{pr}_{2*}(\tilde{U}_g, -\mathrm{ad}^*_U\sigma), \mathrm{pr}_{2*}(\tilde{V}_g, -\mathrm{ad}^*_V\sigma)) \\
&= \omega_{KKS}(-\mathrm{ad}^*_U\sigma, -\mathrm{ad}^*_V\sigma) \\
&= \sigma([U, V]).
\end{aligned}$$

On the other hand, by the computation in Theorem 3.7, we have

$$d\Theta(\underline{U}_{(g,\sigma)}, \underline{V}_{(g,\sigma)}) = d\Theta((\tilde{U}_g, -\mathrm{ad}^*_U\sigma), (\tilde{V}_g, -\mathrm{ad}^*_V\sigma)) = \sigma([U, V])$$

which implies the claim. ∎

3.2 Canonical Symplectic Form, the Pinkall 2-Form and the Fujioka-Kurose 2-Form

In Theorem 3.12 and Theorem 3.14 of this section, we will construct two maps σ_0 and σ_1 on $\mathcal{M}$ and show the relationships between the canonical symplectic form, the Pinkall 2-form and the Fujioka-Kurose 2-form. In Corollary 3.13 and Corollary 3.15, with the help of these results and Theorem 3.11, we will show the relationship between the KKS form, the Pinkall symplectic form and the Fujioka-Kurose symplectic form. In Section 3.2.3, we will reveal the reason why the presymplectic gradient X_H also leads to the KdV equation as the Euler equation applied to the Bott-Virasoro group does, by using the map σ_1 and the vector field X_E mentioned in Theorem 3.9. Even if it seems to be difficult to obtain a strong relationship between X_E and X_H, but Theorem 3.17 we will prove will clear the obstruction.

3.2.1 Canonical Symplectic Form and the Pinkall 2-Form

Let c be the unit circle. Denoting by G the Bott-Virasoro group and $\mathfrak{g}^*$ its Lie algebra, we can prove that:

Theorem 3.12. *Given $\gamma \in \mathcal{M}$, we take $\psi \in \mathrm{Diff}(S^1)$ such that $\gamma = c \cdot \psi$, where $\cdot$ denotes Pinkall's right action. Let $\sigma_0 : \mathcal{M} \to G \ltimes \mathfrak{g}^*$ be a map on $\mathcal{M}$ given by*

$$\sigma_0(\gamma) := ((\psi^{-1}, 0), (-\frac{1}{2}(\psi^{-1})'^2 dx \otimes dx, 0)),$$

Then, we have

$$\sigma_0^* d\Theta = \hat{\omega}_0,$$

where $d\Theta$ is the canonical symplectic form on $G \ltimes \mathfrak{g}^*$ and $\hat{\omega}_0$ the Pinkall 2-form on $\mathcal{M}$.

Proof. Let $X = -\frac{1}{2}\lambda'\gamma + \lambda\gamma' \in T_\gamma\mathcal{M}$ be a tangent vector over $\mathcal{M}$ with $\gamma = c \cdot \psi$ for some $\psi \in \mathrm{Diff}(S^1)$. Suppose that $\varphi_{\lambda\partial_x}$ is the flow of $\lambda\partial_x \in \mathfrak{X}(S^1)$. Then, by the discussion in the proof of Lemma 2.10, $c \cdot \psi \cdot \varphi_{\lambda\partial_x}$ is a curve in $\mathcal{M}$ with initial vector X. It follows that

$$\begin{aligned} \sigma_{0*}X &= \frac{d}{dt}\bigg|_{t=0} ((\varphi_{\lambda\partial_x}(t)^{-1} \circ \psi^{-1}, 0), (-\frac{1}{2}(\varphi_{\lambda\partial_x}(t)^{-1} \circ \psi^{-1})'^2 dx \otimes dx, 0)) \\ &= ((-\lambda \circ \psi^{-1}, 0), ((\lambda' \circ \psi^{-1})(\psi^{-1})'^2 dx \otimes dx, 0)). \end{aligned} \tag{3.5}$$

Therefore, we have

$$\sigma_0^* d\Theta(X, Y) = d\Theta((-\lambda \circ \psi^{-1}, 0), ((\lambda' \circ \psi^{-1})(\psi^{-1})'^2, 0)), ((-\mu \circ \psi^{-1}, 0), ((\mu' \circ \psi^{-1})(\psi^{-1})'^2, 0))),$$

where the symbol $dx \otimes dx$ is omitted. Note that

$$\begin{aligned} l_{(\psi,0)_*}(-\lambda \circ \psi^{-1}, 0) &= \frac{d}{dt}\bigg|_{t=0} (\psi, 0)(\varphi_{\lambda\partial_x}(t)^{-1} \circ \psi^{-1}, 0) \\ &= \frac{d}{dt}\bigg|_{t=0} (\psi \circ \varphi_{\lambda\partial_x}(t)^{-1} \circ \psi^{-1}, B(\psi, \varphi_{\lambda\partial_x}(t)^{-1} \circ \psi^{-1})) \\ &= (-(\psi' \circ \psi^{-1})(\lambda \circ \psi^{-1})\partial_x, B_\lambda) \\ &= (-\frac{\lambda \circ \psi^{-1}}{(\psi^{-1})'}\partial_x, B_\lambda) \end{aligned} \tag{3.6}$$

where B_λ denotes the value of the derivative of the corresponding Bott cocycle. Moreover, it's easy to see that

$$[(\frac{\lambda \circ \psi^{-1}}{(\psi^{-1})'}\partial_x, B_\lambda), (\frac{\mu \circ \psi^{-1}}{(\psi^{-1})'}\partial_x, B_\mu)] = (\frac{(\lambda' \circ \psi^{-1})(\mu \circ \psi^{-1}) - (\lambda \circ \psi^{-1})(\mu' \circ \psi^{-1})}{(\psi^{-1})'}\partial_x, \omega_{\lambda\mu}),$$

where $\omega_{\lambda\mu}$ denotes value of the corresponding Gelfand-Fuchs cocycle. By the formula for the canonical symplectic form mentioned in Theorem 3.7 and subsituting these results, we have

$$\sigma_0^* d\Theta(X, Y) = \int_{S^1} (-\lambda'\mu + \lambda\mu' + \frac{1}{2}(\lambda'\mu - \lambda\mu'))dx = \int_{S^1} \lambda\mu' dx,$$

which implies the claim. ∎

The following corollary is one of the main results in Fujioka, Kurose and Moriyoshi [8]. By using Theorem 3.11 and Theorem 3.12, we can provide an alternative proof here.

Corollary 3.13 ([8]). Let ω_0 be the Pinkall symplectic form and ω_{KKS} the Kirillov-Kostant-Souriau form on $\mathfrak{g}^*$. Then, we have

$$\mu_0^*\omega_{KKS} = -\omega_0,$$

where $\mu_0 : \mathcal{M}_0 \to \mathfrak{g}^*$ is the momentum map mentioned in Section 2.4.1.

Proof. For any tangent vector $X = \frac{1}{2}\lambda'\gamma + \lambda\gamma' \in T_\gamma\mathcal{M}$ over $\mathcal{M}$, where $\gamma = c \cdot \psi$ with $\psi \in \mathrm{Diff}(S^1)$, by the computation in (3.5) and (3.6), we have

$$\begin{aligned}
\sigma_{0*}X &= (-\lambda \circ \psi^{-1}, 0), ((\lambda' \circ \psi^{-1})((\psi^{-1})')^2 dx \otimes dx, 0) \\
&= (\widetilde{(-\frac{\lambda \circ \psi^{-1}}{\psi^{-1}}\partial_x, B_\lambda)}_{(\psi^{-1},0)}, -\mathrm{ad}^*_{(-\frac{\lambda\circ\psi^{-1}}{\psi^{-1}}\partial_x, B_\lambda)}(-\frac{1}{2}((\psi^{-1})')^2 dx \otimes dx, 0)) \\
&= \underline{(-\frac{\lambda \circ \psi^{-1}}{\psi^{-1}}\partial_x, B_\lambda)}_{((\psi^{-1},0),(-\frac{1}{2}((\psi^{-1})')^2)dx\otimes dx,0))}
\end{aligned}$$

where B_λ is the value of the derivative of the corresponding Bott cocycle. This implies that $\sigma_{0*}X$ does be the value of a fundamental vector field with respect to the action (3.4). Note that

$$\mathrm{pr}_2 \circ \sigma_0 = -\mu_0 \circ \pi_0.$$

By Theorem 3.12 and Theorem 3.11, we have

$$\pi_0^*\omega_0 = \hat{\omega}_0 = \sigma_0^* d\Theta = \sigma_0^*\mathrm{pr}_2^*\omega_{KKS} = -\pi_0^*\mu_0^*\omega_{KKS},$$

which implies the claim. ∎

3.2.2 Canonical Symplectic Form and the Fujioka-Kurose 2-Form

Let c be the unit circle. Denoting by G the Bott-Virasoro group and $\mathfrak{g}^*$ its Lie algebra, we can prove that:

Theorem 3.14. *Given $\gamma \in \mathcal{M}$, we take $\psi \in \mathrm{Diff}(S^1)$ such that $\gamma = c \cdot \psi$. Let $\sigma_1 : \mathcal{M} \to G \ltimes \mathfrak{g}^*$ be a map on $\mathcal{M}$ given by*

$$\sigma_1(\gamma) := ((\psi, 0), (-\kappa dx \otimes dx, -\frac{1}{2})),$$

where κ is the equicentroaffine curvature of γ. Then, we have

$$\sigma_1^* d\Theta = \hat{\omega}_1$$

where $d\Theta$ is the canonical symplectic form on $G \ltimes \mathfrak{g}^*$ and $\hat{\omega}_1$ the Fujioka-Kurose 2-form on $\mathcal{M}$.

Proof. Let $X = -\frac{1}{2}\lambda'\gamma + \lambda\gamma' \in T_\gamma\mathcal{M}$ be a tangent vector with $\gamma = c \cdot \psi$ for some $\psi \in \mathrm{Diff}(S^1)$. Suppose that $\varphi_{\lambda\partial_x}$ is the flow of $\lambda\partial_x \in \mathfrak{X}(S^1)$. Then, we have

$$\begin{aligned}\sigma_{1*}X &= \frac{d}{dt}\bigg|_{t=0} \sigma_1(c \cdot \psi \cdot \varphi_{\lambda\partial_x}(t)) \\ &= \frac{d}{dt}\bigg|_{t=0} ((\psi \circ \varphi_{\lambda\partial_x}(t), 0), (-\tilde{\kappa}(t)dx \otimes dx, -\frac{1}{2}))\end{aligned} \tag{3.7}$$

where $\tilde{\kappa}$ is the equicentroaffine curvature flow of $c \cdot \psi \cdot \varphi_{\lambda\partial_x}$. Now, by (2.4), we see that

$$\sigma_{1*}X = ((\psi'\lambda, 0), (-\Omega\lambda' dx \otimes dx, 0)). \tag{3.8}$$

It follows that for tangent vectors $X = -\frac{1}{2}\lambda'\gamma + \lambda\gamma', Y = -\frac{1}{2}\mu'\gamma + \mu\gamma' \in T_\gamma\mathcal{M}$, we have

$$\sigma_1^* d\Theta(X, Y) = d\Theta(((\psi'\lambda, 0), (-\Omega\lambda' dx \otimes dx, 0)), ((\psi'\mu, 0), (-\Omega\mu' dx \otimes dx, 0))).$$

Note that

$$l_{(\psi^{-1},0)_*}(\psi'\mu, 0) = \frac{d}{dt}\bigg|_{t=0} (\psi^{-1}, 0)(\psi \circ \varphi_{\mu\partial_x}(t), 0) = (\mu\partial_x, B_\mu),$$

where B_μ denotes the value of the derivatie of the corresponding Bott-cocycle. Moreover, it's easy to see that

$$[(\lambda\partial_x, B_\lambda), (\mu\partial_x, B_\mu)] = ((\lambda'\mu - \lambda\mu)\partial_x, \omega(\lambda\partial_x, \mu\partial_x)).$$

Thus, by Theorem 3.7, we have

$$\begin{aligned}\sigma_1^* d\Theta(X, Y) =& (-\Omega\lambda', 0)(\mu, B_\mu) - (-\Omega\mu', 0)(\lambda, B_\lambda) - (-\kappa, -\frac{1}{2})(\lambda'\mu - \lambda\mu, \omega(\lambda, \mu)) \\ =& -\int_{S^1} \mu\Omega\lambda' dx + \int_{S^1} \lambda\Omega\mu' dx + \int_{S^1} \kappa(\lambda'\mu - \lambda\mu')dx + \frac{1}{2}\int_{S^1} \lambda'\mu'' dx \\ =& \int_{S^1} \lambda\Omega\mu' dx,\end{aligned}$$

where the symbols $dx \otimes dx$ and ∂_x are omitted, which implies the claim. ■

By the similar method we have used in the proof of Lemma 3.13, we can prove that the following corollary, which is indeed one of the main results in Fujioka, Kurose and Moriyoshi [8].

Corollary 3.15 ([8]). Let ω_1 be the Fujioka-Kurose symplectic form and ω_{KKS} the Kirillov-Kostant-Souriau form on $\mathfrak{g}^*$. Then, we have

$$\mu_1^* \omega_{KKS} = \omega_1,$$

where $\mu_1 : \mathcal{M}_1 \to \mathfrak{g}^*$ is the momentum map mentioned in Section 2.4.2. ■

3.2.3 Conclusions

In Section 2.3.2, we have introduced a function $H : \mathcal{M} \to \mathbb{R}$, whose presymplectic gradient with respect to $\hat{\omega}_1$ is given by

$$X_H(\gamma) = \frac{1}{2}\kappa'\gamma - \kappa\gamma',$$

where κ is the equicentroaffine curvature of γ. Let $\tilde{\gamma}$ be the integral curve of X_H^1 in $\mathcal{M}$ and $\tilde{\kappa}$ its equicentroaffine curvature flow. Then, by (2.4), we have

$$\dot{\tilde{\kappa}} = \Omega(-\tilde{\kappa})' = -\frac{1}{2}\tilde{\kappa}''' - 3\tilde{\kappa}'\tilde{\kappa}, \tag{3.9}$$

which is precisely the KdV equation.

Remark 3.16. We are going to give another explanation that why the equicentroaffine curvature flow $\tilde{\kappa}$ also leads to the KdV equation in (3.9) as the Euler equation applied to the Bott-Virasoro group does. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Let $\langle -, - \rangle$ be an inner product on $\mathfrak{g}$ and A its induced inertia operator. In Theorem 3.9, we have introduced a symplectic gradient $X_E \in \mathfrak{X}(G \ltimes \mathfrak{g}^*)$ of a function E with respect to the canonical symplectic form $d\Theta$. In particular, the $\mathfrak{g}^*$-part of its integral curve (φ, ξ) satisfies the Euler equation, i.e.,

$$\dot{\xi}(t) = -\mathrm{ad}^*_{A^{-1}(\xi(t))}\xi(t). \tag{3.10}$$

When G is setted to be the Bott-Virasoro group and $\langle -, - \rangle$ the L^2-inner product, the equation (3.10) becomes

$$\dot{\xi} = \frac{1}{2}\xi''' + 3\xi'\xi.$$

By observing this, the main method we would like to try is to find the relationship between the vector fields X_E and X_H^1, as well as the relationship between the forms $d\Theta$ and $\hat{\omega}_1$. Since the relationship between the forms has already been showed in Theorem 3.14, i.e.,

$$\sigma_1^* d\Theta = \hat{\omega}_1,$$

where $\sigma_1 : \mathcal{M} \to G \ltimes \mathfrak{g}^*$ is a map on $\mathcal{M}$ given by

$$\sigma_1(\gamma) := ((\psi, 0), (-\kappa dx \otimes dx, -\frac{1}{2})),$$

what we need to to show now is the relationship between the vector fields. First, we study the relationship between the Hamiltonian functions E and H. Here, the function $E : G \ltimes \mathfrak{g}^* \to \mathbb{R}$ becomes

$$E((\varphi, a), (u dx \otimes dx, b)) = \frac{1}{2}\int_{S^1} u^2 dx + \frac{b^2}{2}.$$

By compositing E with the map σ_1, we see that the map $E \circ \sigma_1 : \mathcal{M} \to \mathbb{R}$ is given by

$$E \circ \sigma_1(\gamma) = E((\psi, 0), (-\kappa dx \otimes dx, -\frac{1}{2})) = \frac{1}{2} \int_{S^1} \kappa^2 dx + \frac{1}{8} = H(\gamma) + \frac{1}{8}.$$

Since X_H^1 is the presymplectic gradient of H with respect to $\hat{\omega}_1$, we have

$$d(E \circ \sigma_1) = d(H + \frac{1}{8}) = -i_{X_H^1} \hat{\omega}_1 = -\hat{\omega}_1(X_H^1, -) = -\sigma_1^* d\Theta(X_H^1, -). \tag{3.11}$$

On the other hand, since X_E is the symplectic gradient of E with respect to $d\Theta$, we see that

$$d(E \circ \sigma_1) = dE \circ \sigma_{1*} = -i_{X_E}(d\Theta) \circ \sigma_{1*} = -d\Theta(X_E, \sigma_{1*}(-)). \tag{3.12}$$

It follows that as a 1-form on $\mathcal{M}$, we have

$$d\Theta(\sigma_{1*} X_H^1, \sigma_{1*}(-)) = d\Theta(X_E, \sigma_{1*}(-)).$$

Since $d\Theta$ is a symplectic form by Lemma 3.8, we see that for any $\gamma \in \mathcal{M}$,

$$\sigma_{1*}(X_H^1(\gamma)) = X_E(\sigma_1(\gamma)) + X, \tag{3.13}$$

where $X \in T_{\sigma_1(\gamma)}(G \ltimes \mathfrak{g}^*)$ is a tangent vector over $G \ltimes \mathfrak{g}^*$ such that

$$d\Theta(X, \sigma_{1*}(Z)) = 0, \tag{3.14}$$

for all tangent vectors $Z \in T_\gamma \mathcal{M}$.

In Remark 3.16, the existence of the tangent vector X in the equation (3.13) is an obstruction that keeps us from obtaining a strong relationship between X_E and X_H^1. However, by proving the following theorem, such obstruction will be cleared.

Theorem 3.17. *Let $\gamma \in \mathcal{M}$ be an element in $\mathcal{M}$. Suppose that $X \in T_{\sigma_1(\gamma)}(G \ltimes \mathfrak{g}^*)$ is the tangent vector in (3.13) which satisfies (3.14). Then X has the form*

$$X = (K, (0dx \otimes dx, 0))$$

where K represents a tangent vector over G.

Proof. Let $((u\partial_x, a), (vdx \otimes dx, b)) \in \mathfrak{g} \times \mathfrak{g}^*$ be an element in the Lie algebra of $G \ltimes \mathfrak{g}^*$ such that

$$l_{\sigma_1(\gamma)^{-1}}{}_* X = ((u\partial_x, a), (vdx \otimes dx, b)).$$

Suppose that $\gamma = c \cdot \psi$ for some $\psi \in \mathrm{Diff}(S^1)$. Let $\varphi_{u\partial_x}$ be the flow of $u\partial_x \in \mathfrak{X}(S^1)$. Then, we have

$$\begin{aligned}
X &= l_{\sigma_1(\gamma)}{}_*((u\partial_x, a), (vdx \otimes dx, b)) \\
&= \frac{d}{dt}\bigg|_{t=0} ((\psi, 0), (-\kappa dx \otimes dx, -\frac{1}{2}))((\varphi_{u\partial_x}(t), at), (vtdx \otimes dx, bt)) \\
&= \frac{d}{dt}\bigg|_{t=0} ((\psi \circ \varphi_{u\partial_x}(t), at + B(\psi, \varphi_{u\partial_x}(t)), \mathrm{Ad}^*_{(\varphi_{u\partial_x}(t)^{-1}, -at)}(-\kappa dx \otimes dx, -\frac{1}{2}) + (vdx \otimes dx, b))).
\end{aligned}$$

where κ is the equicentroaffine curvature of γ. Denoting by B_u the value of the derivative of the corresponding Bott cocycle, we have

$$X = ((\psi' u, B_u + a), \mathrm{ad}^*_{(-u\partial_x, -a)}(-\kappa dx \otimes dx, -\frac{1}{2}) + (v dx \otimes dx, b)).$$

By the expression of the coadjoint action of $\mathfrak{g}$ in (1.32), we see that

$$X = ((\psi' u, B_u + a), ((-\frac{1}{2}u''' - 2\kappa u' - \kappa' u + v)dx \otimes dx, b)). \tag{3.15}$$

By using the operator $\Omega := \frac{1}{2}\partial_x^2 + 2\kappa + \kappa'\partial_x^{-1}$ mentioned in (2.4), the equation (3.15) can be rewritten as

$$X = ((\psi' u, B_u + a), ((-\Omega u' + v)dx \otimes dx, b)). \tag{3.16}$$

By (3.8), all the tangent vectors $\sigma_{1*}Z \in T_{\sigma_1(\gamma)}(G \ltimes \mathfrak{g}^*)$ has the following form

$$\sigma_{1*}Z = ((\psi'\lambda, 0), (-\Omega\lambda' dx \otimes dx, 0)), \tag{3.17}$$

for some $\lambda \in C^\infty(S^1)$. Thus, by condition (3.14), we have

$$d\Theta(X, ((\psi'\lambda, 0), (-\Omega\lambda' dx \otimes dx, 0))) = 0, \tag{3.18}$$

for all $\lambda \in C^\infty(S^1)$. Substituting (3.16) into (3.18) and using the formula for the canonical symplectic form provided in Theorem 3.7, we have

$$\begin{aligned}
d\Theta(X, ((\psi'\lambda, 0), (-\Omega\lambda', 0))) &= (-\Omega u' + v, b)(\lambda, B_\lambda) - (-\Omega\lambda', 0)(u, a) - (-\kappa, -\frac{1}{2})(u'\lambda - u\lambda', \omega(u, \lambda)) \\
&= \int_{S^1} (-\lambda\Omega u' + \lambda v + u\Omega\lambda' - u\Omega\lambda')dx + bB_\lambda \\
&= \int_{S^1} \lambda(v - \Omega u')dx + bB_\lambda.
\end{aligned}$$

where B_λ is the value of the derivative of the corresponding Bott cocycle and the symbols $dx \otimes dx$ and ∂_x are omitted. It follows that

$$\int_{S^1} \lambda(v - \Omega u')dx + bB_\lambda = 0, \tag{3.19}$$

for all $\lambda \in C^\infty(S^1)$. By the expression for the vector field X_E in (3.3), we have

$$\begin{aligned}
X_E(\sigma_1(\gamma)) &= (K_0, -\mathrm{ad}^*_{(-\kappa\partial_x, -\frac{1}{2})}(-\kappa dx \otimes dx, -\frac{1}{2})) \\
&= (K_0, ((\frac{1}{2}\kappa''' + 3\kappa'\kappa)dx \otimes dx, 0)),
\end{aligned} \tag{3.20}$$

where K_0 is a tangent vector over G. On the other hand, by (3.17) we have

$$\sigma_{1*}(X_H(\gamma)) = (K_1, (\alpha dx \otimes dx, 0)) \tag{3.21}$$

where K_1 is a tangent vector over G and $\alpha \in C^\infty(S^1)$. Substituting (3.20) and (3.21) into (3.13) and comparing the coefficients, we see that

$$b = 0.$$

Thus, by (3.19), we have

$$v = \Omega u'.$$

It follows that

$$X = ((\psi' u, B_u + a), (0dx \otimes dx, 0))$$

which implies the claim. ■

Now, with all these theorems in hands, we can give an alternative explanation that why the equicentroaffine curvature flow $\tilde{\kappa}$ must satisfy the KdV equation (3.9), i.e.,

Corollary 3.18. Let $\tilde{\gamma}$ be an integral curve of X_H^1. Then, we have

$$\dot{\tilde{\kappa}} = -\frac{1}{2}\tilde{\kappa}''' - 3\tilde{\kappa}'\tilde{\kappa},$$

where $\tilde{\kappa}$ is the equicentroaffine curvature of $\tilde{\gamma}$.

Proof. Let $\tilde{\psi}$ be a curve in $\mathrm{Diff}(S^1)$ such that $\tilde{\gamma}(t) = c \cdot \tilde{\psi}(t)$. By the definition of σ_1, we have

$$\sigma_{1*}(\dot{\tilde{\gamma}}) = (((\dot{\tilde{\psi}}, 0), (-\dot{\tilde{\kappa}} dx \otimes dx, 0))).$$

By the expression for X_E in (3.3), the condition (3.13) and Theorem 3.17, we have

$$(((\dot{\tilde{\psi}}, 0), (-\dot{\tilde{\kappa}} dx \otimes dx, 0))) = (K_4, -\mathrm{ad}^*_{(-\tilde{\kappa} dx \otimes dx, -\frac{1}{2})}(-\tilde{\kappa} dx \otimes dx, -\frac{1}{2}) + (0dx \otimes dx, 0)),$$

where K_4 is a tangent vector over G. It follows that

$$-(\dot{\tilde{\kappa}} dx \otimes dx, 0) = -\mathrm{ad}^*_{(-\tilde{\kappa}\partial_x, -\frac{1}{2})}(-\tilde{\kappa} dx \otimes dx, -\frac{1}{2})$$

which implies that

$$\dot{\tilde{\kappa}} = -\frac{1}{2}\tilde{\kappa}''' - 3\tilde{\kappa}'\tilde{\kappa},$$

which is nothing but the KdV equation. ■

Thus, by revealing the relationship between the forms σ_1 and $d\Theta$ in Theorem 3.14, as well as the relationship between the vector fields X_E and X_H^1 in this section, we finally understand why the equicentraoffine curvature flow $\tilde{\kappa}$ must satisfy the KdV equation.

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