Studies on multiple zeta values, Arakawa-Kaneko zeta functions and iterated log-sine integrals
（多重ゼータ値と荒川・金子ゼータ関数, 反復log-sine積分に関する研究）

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Abstract

In this thesis, we discuss relations between multiple zeta values, Arakawa-Kaneko zeta functions and iterated log-sine integrals. In Section 1, we describe the background and known results of multiple zeta values, Arakawa-Kaneko zeta functions and log-sine integrals. In Section 2, we discuss analogues of the Arakawa-Kaneko zeta function based on [40]. In particular, we discuss an analogue of the Arakawa-Kaneko zeta function of Miyagawa-type and obtain relations among Miyagawa multiple zeta values. We also discuss a class of multiple zeta functions to which Ito's zeta functions of the case of general index are related, and we obtain relations among special values of multiple zeta functions in this class. In Section 3, we define iterated log-sine integrals and prove its fundamental properties and discuss relations between iterated log-sine integrals, multiple zeta values and multiple polylogarithms. In particular, we give a method to obtain relations among multiple zeta values, which uses iterated log-sine integrals. We based on [41] for the method and the definition and fundamental properties of iterated log-sine integrals, and based on [42] for the rest of Section 3. We also prove theorems that iterated log-sine integral can be written in terms of multiple zeta values and multiple polylogarithms. These theorems make numerical evaluations of iterated log-sine integrals possible. Finally, we state some conjectures on multiple zeta values, multiple Clausen values, multiple Glaisher values and iterated log-sine integrals suggested by numerical evaluations.

Contents

1 Introduction

This thesis is a summary of the author's papers [40], [41] and [42] on multiple zeta values, Arakawa-Kaneko zeta functions and iterated log-sine integrals. In this section, we describe the background and basics of multiple zeta values, Arakawa-Kaneko zeta functions and log-sine integrals, respectively.

1.1 Multiple zeta values and related functions

1.1.1 multiple zeta values

Multiple zeta values (MZVs) are defined by

$$\zeta(k_1, \dots, k_n) = \sum_{0<m_1<\cdots<m_n} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}},$$

where $k_1, \dots, k_n$ are positive integers with $k_n \geq 2$. This series are multiplex of the Riemann zeta values $\zeta(k) = \sum_{m=1}^{\infty} m^{-k}$. For an index $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$, we define the weight of $\mathbf{k}$ as $|\mathbf{k}| = k_1 + \cdots + k_n$, and the depth of $\mathbf{k}$ as $\mathrm{d}(\mathbf{k}) = n$. We call an index $\mathbf{k} = (k_1, \dots, k_n)$ *admissible index* if $\mathbf{k} \in \mathbb{N}^n$ and $k_n \geq 2$. We also write $\zeta(k_1, \dots, k_n)$ as $\zeta(\mathbf{k})$. Let $\{k\}^n$ denotes n repetitions of k. For example, $(1, 1, 2) = (\{1\}^2, 2)$.

Euler is believed as the first parson who studied MZVs. He studied MZVs with $n = 2$. After that, studies on MZVs began in earnest with Hoffman [21] and Zagier [45], and have been continued actively even today. It is known that MZVs are related to other branches of mathematics and physics, for example, quantum groups, knot theory, etc.

The functions defined by changing positive integers $k_1, \dots, k_n$ in the definition of multiple zeta values by complex variables $s_1, \dots, s_n$ are called multiple zeta functions (MZFs). The multiple zeta function $\zeta(s_1, \dots, s_n)$ converges absolutely when

$$\sum_{i=0}^{k} \Re(s_{n-i}) > k + 1$$

for any k with $0 \leq k \leq n - 1$ (see [34]) and can be continued meromorphically to the whole $\mathbb{C}^n$ space (see [1]).

On MZVs theory, the most famous conjecture is Zagier's dimension conjecture. Let $\mathcal{Z}_k$ be the $\mathbb{Q}$-vector space spanned by all MZVs of weight k, and $\mathcal{Z}$ the $\mathbb{Q}$-vector space spanned by all MZVs namely,

$$\mathcal{Z}_0 = \mathbb{Q}, \quad \mathcal{Z}_1 = \{0\},$$
$$\mathcal{Z}_k = \sum_{\substack{k_1+\cdots+k_n=k \\ k_1,\dots,k_{n-1}\geq 1, k_n \geq 2}} \mathbb{Q} \cdot \zeta(k_1, \dots, k_n),$$

and

$$\mathcal{Z} = \sum_{k=0}^{\infty} \mathcal{Z}_k.$$

Then, Zagier's dimension conjecture is as follows.

Conjecture 1 (Zagier [45])**.** We have

$$\dim \mathcal{Z}_k = d_k,$$

where d_k is the sequence defined by $d_0 = 1$, $d_1 = 0$, $d_2 = 1$ and $d_k = d_{k-2} + d_{k-3}$ $(k \geq 3)$.

It is known that Conjecture 1 is true when $k \leq 4$, but not proved for $k \geq 5$. On the upper bound of dimension of $\mathcal{Z}_k$, there is the following result.

Theorem 1 (Terasoma [39], Goncharov [18], Deligne-Goncharov[16])**.**

$$\dim \mathcal{Z}_k \leq d_k.$$

The following is a table of d_k.

k	2	3	4	5	6	7	8	9	10	$\cdots$
d_k	1	1	1	2	2	3	4	5	7	$\cdots$
2^{k-2}	1	2	4	8	16	32	64	128	256	$\cdots$

Here, 2^{k-2} is the number of MZVs of weight k. By Theorem 1, we can see that there are at least $2^{k-2} - d_k$ independent relations in MZVs of weight k. Hoffman conjectured on a basis of $\mathcal{Z}_k$. We define a set of multiple zeta values

$$H_{k,d} = \left\{ \zeta(k_1, \dots, k_n) \,\middle|\, \begin{array}{c} k_1 + \cdots + k_n = k,\ n \le k, \\ k_i \in \{2,3\},\ |\{i \mid k_i = 3\}| \le d \end{array} \right\}.$$

Conjecture 2 (Hoffman [22])**.** The set of multiple zeta values $H_{k,k}$ is a basis of $\mathcal{Z}_k$.

Of course, the number of elements of $H_{k,k}$ is equal to d_k. It is proved by Brown [10] that $\mathcal{Z}_k$ is generated by multiple zeta values $H_{k,k}$. It follows by Brown [10] and Goncharov [18, Theorem 1.2] that every multiple zeta value with weight k and depth d can be written as a $\mathbb{Q}$-linear combination of elements of $H_{k,d}$. In Section 3.4.2, we will give conjectures that are analogues of this assertion and Conjecture 2.

Various families of relations of MZVs have been known so far, and there are families that are conjectured to include all relations, for example, associator relations (see [17]), regularized double shuffle relations (see [24]), confluence relations (see [20]), integral-series identity (see [32]). However there are no families which are proved to include all relations. In Section 3.2, we will give a new method to obtain a family of relations among MZVs, which unfortunately probably does not obtain all relations.

Some multiple zeta values have explicit expressions. The formula

$$\zeta(\{1,3\}^n) = \frac{2\pi^{4n}}{(4n+2)!}$$

was conjectured by Zagier [45] and proved by Borwein, Bradley, Broadhurst and Lisoněk [6]. The following formula is known:

$$\zeta(\{2k\}^n) = C_n^{(k)} \frac{(2\pi i)^{2nk}}{(2nk)!},$$

where B_n is the n-th Bernoulli number and $C_n^{(k)}$ is the rational number defined by

$$C_0^{(k)} = 1, \quad C_n^{(k)} = \frac{1}{2\pi} \sum_{m=1}^{n} (-1)^m \binom{2nk}{2mk} B_{2mk} C_{n-m}^{(k)}.$$

The formulas

$$\zeta(\{3,1\}^n, 3) = 4^{-n} \sum_{k=0}^{n} (-1)^k \zeta(4k+3) \zeta(\{4\}^{n-k})$$

and

$$\zeta(\{3,1\}^n, 2) = 4^{-n} \sum_{k=0}^{n} (-1)^k \zeta(\{4\}^{n-k}) \left\{ (4k+1)\zeta(4k+2) - 4\sum_{j=1}^{k} \zeta(4j-1)\zeta(4k-4j+3) \right\}$$

are proved by Bowman and Bradley [5]. For $s + t = 2n+1$, $n \ge 1$, the formulas

$$\begin{aligned} \zeta(s,t) = {} & \frac{1}{2}\left\{ \binom{s+t}{s} - 1 \right\} \zeta(s+t) + \zeta(s)\zeta(t) \\ & - \sum_{j=1}^{n} \left\{ \binom{2j-2}{s-1} + \binom{2j-1}{t-1} \right\} \zeta(2j-1)\zeta(s+t-2j+1) \quad \text{if } s : \text{odd} \end{aligned}$$

and

$$\zeta(s,t) = -\frac{1}{2}\left\{\binom{s+t}{s}+1\right\}\zeta(s+t)$$
$$+\sum_{j=1}^{n}\left\{\binom{2j-2}{s-1}+\binom{2j-1}{t-1}\right\}\zeta(2j-1)\zeta(s+t-2j+1) \quad \text{if } s:\text{even}$$

are proved by Borwein, Borwein and Girgensohn [3]. Huard, Williams and Zhang [23] also proved an equation equivalent to these.

For non-negative integers a and b, the formula

$$\zeta(\{2\}^a,3,\{2\}^b) = 2\sum_{r=1}^{a+b+1}(-1)^r\left\{\binom{2r}{2a+2}-\left(1-\frac{1}{2^{2r}}\right)\binom{2r}{2b+1}\right\}\zeta(\{2\}^{a+b-r+1})\zeta(2r+1)$$

is proved by Zagier [46]. The equation

$$1-\sum_{m,n=1}^{\infty}\zeta(\{1\}^{n-1},m+1)X^mY^n = \exp\left(\sum_{n=2}^{\infty}\zeta(n)\frac{X^n+Y^n-(X+Y)^n}{n}\right)$$

is also known. In particular, the multiple zeta value $\zeta(\{1\}^{n-1},m+1)$ can be written as a $\mathbb{Q}$-linear combination of products of the Riemann zeta values with with rational coefficients.

We introduce some of the famous explicit relations among MZVs. The first is the duality that was conjectured by Hoffman [21] and proved by Zagier [45]. For $\mathbf{k}\in\mathbb{N}^n$ with $k_n>1$, we write $\mathbf{k}$ in the form

$$\mathbf{k} = (\{1\}^{a_1-1}, b_1+1, \{1\}^{a_2-1}, b_2+1,\dots,\{1\}^{a_h-1}, b_h+1),$$

then its dual index $\mathbf{k}^*$ is defined by

$$\mathbf{k}^* = (\{1\}^{b_h-1}, a_h+1,\dots,\{1\}^{b_2-1}, a_2+1, \{1\}^{b_1-1}, a_1+1).$$

The duality of multiple zeta values is as follows:

$$\zeta(\mathbf{k}) = \zeta(\mathbf{k}^*).$$

The second is the sum formula:

$$\sum_{\substack{k_1+\cdots+k_n=k\\ k_1,\dots,k_{n-1}\geq 1, k_n\geq 2}}\zeta(k_1,\dots,k_n) = \zeta(k)$$

proved by [19]

Ohno [37] gave the following generalization of the duality and the sum formula, which is called Ohno's relations. For $m\geq 0$,

$$\sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\geq 0})^{\mathrm{d}(\mathbf{k})}\\ |\mathbf{l}|=m}}\zeta(\mathbf{k}+\mathbf{l}) = \sum_{\substack{\mathbf{l}'\in(\mathbb{Z}_{\geq 0})^{\mathrm{d}(\mathbf{k}^*)}\\ |\mathbf{l}'|=m}}\zeta(\mathbf{k}^*+\mathbf{l}').$$

Many other explicit relations are known.

Zagier also conjectured as follows.

Conjecture 3 (Zagier [45]). We have

$$\mathcal{Z} = \bigoplus_{k\geq 0}\mathcal{Z}_k.$$

Conjecture 3 states that there are no relations including MZVs of different weight. Actually, no relations have been found so far.

The vector space $\mathcal{Z}$ is closed by product, namely $\mathcal{Z}$ is a $\mathbb{Q}$-algebra. In particular $\mathcal{Z}_{k_1} \cdot \mathcal{Z}_{k_2} \subset \mathcal{Z}_{k_1+k_2}$ holds. There are two products on $\mathcal{Z}$. The first product is obtained by decomposing the sum of definition, which is called the harmonic product. For example, we have

$$\begin{aligned}
\zeta(2)\zeta(2) &= \left(\sum_{0<m} \frac{1}{m^2}\right)\left(\sum_{0<n} \frac{1}{n^2}\right) \\
&= \sum_{0<m}\sum_{0<n} \frac{1}{m^2}\frac{1}{n^2} \\
&= \left(\sum_{0<m<n} + \sum_{0<n<m} + \sum_{0<m=n}\right) \frac{1}{m^2}\frac{1}{n^2} \\
&= \zeta(2,2) + \zeta(2,2) + \zeta(4) \\
&= 2\zeta(2,2) + \zeta(4). \qquad (1.1)
\end{aligned}$$

The second product is defined via the following iterated integral representation:

$$\begin{aligned}
\zeta(k_1, \dots, k_n) &= \int_0^1 \omega_k(t_k) \cdots \int_0^{t_3} \omega_2(t_2) \int_0^{t_2} \omega_1(t_1) \\
&= \int_{0<t_1<\cdots<t_k<1} \omega_1(t_1) \cdots \omega_k(t_k),
\end{aligned}$$

where

$$\omega_i(t) = \begin{cases} \frac{dt}{1-t} & \text{if } i \in \{1, k_1+1, k_1+k_2+1, \dots, k_1+\cdots+k_{n-1}+1\}, \\ \frac{dt}{t} & \text{otherwise.} \end{cases}$$

We note that the duality is proved by changing of variables $t_1 = 1 - t_1$ in iterated integral representation of multiple zeta values. By decomposing the integral domain, the second product, which is called the shuffle product, is obtained. For example,

$$\begin{aligned}
\zeta(2)\zeta(2) &= \left(\int_{0<t_1<t_2} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\right)\left(\int_{0<u_1<u_2} \frac{du_1}{1-u_1}\frac{du_2}{u_2}\right) \\
&= \int_{\substack{0<t_1<t_2 \\ 0<u_1<u_2}} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\frac{du_1}{1-u_1}\frac{du_2}{u_2} \\
&= \left(\int_{0<t_1<t_2<u_1<u_2} + \int_{0<t_1<u_1<t_2<u_2} + \int_{0<t_1<u_1<u_2<t_2}\right. \\
&\quad \left. + \int_{0<u_1<u_2<t_1<t_2} + \int_{0<u_1<t_1<u_2<t_2} + \int_{0<u_1<t_1<t_2<u_2}\right) \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\frac{du_1}{1-u_1}\frac{du_2}{u_2} \\
&= \zeta(2,2) + \zeta(1,3) + \zeta(1,3) + \zeta(2,2) + \zeta(1,3) + \zeta(1,3) \\
&= 2\zeta(2,2) + 4\zeta(1,3). \qquad (1.2)
\end{aligned}$$

By comparing (1.1) and (1.2(, we have

$$\zeta(4) = 4\zeta(1,3).$$

As above, the method to get relations among MZVs by comparing two products of is called the (finite) double shuffle relation.

1.1.2 Other types of multiple zeta values and functions

Various types of multiple zeta values and functions have been studied so far. We cannot introduce all of them, so we only introduce the ones used later. When there is a risk of confusion, we call multiple zeta values as Euler-Zagier multiple zeta values (EZ-type MZVs) and multiple zeta functions as Euler-Zagier multiple zeta functions (EZ-type MZFs).

For an admissible index $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$, multiple zeta star values are defined by

$$\zeta^{\star}(\mathbf{k}) = \sum_{0 \leq m_1 \leq \cdots \leq m_n} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}},$$

This series are multiplex of the Riemann zeta values that is different form multiple zeta values. It is known that multiple zeta star values can be written as $\mathbb{Q}$-linear combination of multiple zeta values.

We also introduce Mordell-Tornheim multiple zeta values which are the special values of the following functions. For $s_1, \ldots, s_{r+1} \in \mathbb{C}$, Mordell-Tornheim multiple zeta functions (MT-type MZFs) is defined by

$$\zeta_{MT}(s_1, \ldots, s_r; s_{r+1}) = \sum_{m_1=1,\ldots,m_r=1}^{\infty} \frac{1}{m_1^{s_1} \cdots m_r^{s_r} (m_1 + \cdots + m_r)^{s_{r+1}}}.$$

This series converges absolutely when

$$\sum_{l=1}^{j} \Re(s_{k_l}) + \Re(s_{r+1}) > j$$

with $1 \leq k_1 < k_2 < \cdots < k_j \leq r$ for any $j = 1, 2, \ldots, r$ (see [38]) and can be continued meromorphically to the whole $\mathbb{C}^n$ space (see [35]). The values of MT-type MZFs at non-negative integer points in the domain of convergence are called Mordell-Tornheim multiple zeta values (MT-type MZVs).

Miyagawa [36] introduced the following generalization of multiple zeta functions, which include both the EZ-type and the MT-type as special cases. For $s_1, \ldots, s_{r+1} \in \mathbb{C}$, we defined

$$\hat{\zeta}_{MT,j,r}(s_1, \ldots, s_j; s_{j+1}, \ldots, s_{r+1}) = \sum_{m_1=1,\ldots,m_r=1}^{\infty} \frac{1}{m_1^{s_1} \cdots m_j^{s_j} \prod_{u=j+1}^{r+1} (\sum_{v=1}^{u-1} m_v)^{s_u}}. \tag{1.3}$$

Miyagawa introduced these functions and showed that the function (1.3) can be continued meromorphically to the whole $\mathbb{C}^{r+1}$ space. We call the function (1.3) the Miyagawa multiple zeta function (Miyagawa-type MZF). Moreover, we call the values of Miyagawa-type MZFs at non-negative integer points in the domain of convergence Miyagawa multiple zeta values (Miyagawa-type MZVs).

For a matrix $A = (a_{i,j})_{1 \leq i \leq n, 1 \leq j \leq r},\ a_{i,j} \in \mathbb{R}_{\geq 0}$ and $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n$, Matsumoto [35] defined

$$\zeta(\mathbf{s}; A) = \sum_{m_1=1,\ldots,m_r=1}^{\infty} (a_{1,1}m_1 + \cdots + a_{1,r}m_r)^{-s_1} \cdots (a_{n,1}m_1 + \cdots + a_{n,r}m_r)^{-s_n} \tag{1.4}$$

and showed that the function $\zeta(\mathbf{s}; A)$ can be continued meromorphically to the whole $\mathbb{C}^n$ space.

1.1.3 Multiple polylogarithms

Here, we introduce multiple polylogarithms and their basic properties. Multiple polylogarithms are used for all studies on multiple zeta values, Arakawa-Kaneko zeta functions and iterated log-sine integrals. Multiple polylogarithms are defined by

$$\mathrm{Li}_{\mathbf{k}}(z) = \sum_{0 < m_1 < \cdots < m_n} \frac{z^{m_n}}{m_1^{k_1} \cdots m_n^{k_n}},$$

where $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ and $z \in \mathbb{C}$. (The condition $\mathbf{k} \in \mathbb{N}^n$ is not essential for the definition, but we mainly deal with $\mathbf{k} \in \mathbb{N}^n$.) These series are generalization of logarithm $-\log(1-z) = \sum_{m=1}^{\infty} z^m/m$. Multiple polylogarithms converge absolutely when $|z| < 1$, and can be continued holomorphically to $\mathbb{C} \setminus [1, \infty)$. If $\mathbf{k}$ is admissible, it converges absolutely on the disc $|z| \leq 1$ and $\mathrm{Li}_{\mathbf{k}}(1) = \zeta(\mathbf{k})$ holds. Multiple polylogarithms have the following iterated integral representation:

$$\mathrm{Li}_{\mathbf{k}}(z) = \int_0^z \omega_k(t_k) \cdots \int_0^{t_3} \omega_2(t_2) \int_0^{t_2} \omega_1(t_1),$$

where

$$\omega_i(t) = \begin{cases} \frac{dt}{1-t} & \text{if } i \in \{1, k_1+1, k_1+k_2+1, \dots, k_1+\cdots+k_{n-1}+1\}, \\ \frac{dt}{t} & \text{otherwise.} \end{cases}$$

By this iterated representation, we can prove that multiple polylogarithms can be continued holomorphically to $\mathbb{C} \setminus [1,\infty)$, and satisfy

$$\frac{d}{dz}\mathrm{Li}_{k_1,\dots,k_n}(z) = \begin{cases} \frac{1}{z}\mathrm{Li}_{k_1,\dots,k_n-1}(z) & (\text{if } k_n > 1), \\ \frac{1}{1-z}\mathrm{Li}_{k_1,\dots,k_{n-1}}(z) & (\text{if } k_n = 1). \end{cases} \tag{1.5}$$

We can see that multiple polylogarithms also satisfy the shuffle product formula as in the case of multiple zeta values. For example, in the same way as (1.2),

$$\mathrm{Li}_2(z)\,\mathrm{Li}_2(z) = 2\,\mathrm{Li}_{2,2}(z) + 4\,\mathrm{Li}_{1,3}(z). \tag{1.6}$$

The real and imaginary parts of multiple polylogarithms at $e^{i\sigma}$ are called as follows.

Definition 1 (Multiple Clausen function, multiple Glaisher function)**.** The multiple Clausen function $\mathrm{Cl}_{\mathbf{k}}(\sigma)$ and the multiple Glaisher function $\mathrm{Gl}_{\mathbf{k}}(\sigma)$ are defined by

$$\mathrm{Cl}_{\mathbf{k}}(\sigma) = \begin{cases} \Im(\mathrm{Li}_{\mathbf{k}}(e^{i\sigma})) & \text{if } |\mathbf{k}| : \text{even}, \\ \Re(\mathrm{Li}_{\mathbf{k}}(e^{i\sigma})) & \text{if } |\mathbf{k}| : \text{odd}, \end{cases}$$

$$\mathrm{Gl}_{\mathbf{k}}(\sigma) = \begin{cases} \Re(\mathrm{Li}_{\mathbf{k}}(e^{i\sigma})) & \text{if } |\mathbf{k}| : \text{even}, \\ \Im(\mathrm{Li}_{\mathbf{k}}(e^{i\sigma})) & \text{if } |\mathbf{k}| : \text{odd}. \end{cases}$$

In particular, we call the values $\mathrm{Cl}_{\mathbf{k}}(\pi/3)$ and $\mathrm{Gl}_{\mathbf{k}}(\pi/3)$ multiple Clausen values and multiple Glaisher values respectively, and define $\mathcal{C}_k$ as the vector space spanned by all multiple Clausen values of weight k, and $\mathcal{G}_k$ as the vector space spanned by all multiple Glaisher values of weight k. Namely,

$$\mathcal{C}_k = \sum_{\substack{0 \le n \le k \\ k_1+\cdots+k_n = k \\ k_1,\dots,k_n \ge 1}} \mathbb{Q} \cdot \mathrm{Cl}_{k_1,\dots,k_n}(\pi/3)\,,$$

$$\mathcal{G}_k = \sum_{\substack{0 \le n \le k \\ k_1+\cdots+k_n = k \\ k_1,\dots,k_n \ge 1}} \mathbb{Q} \cdot \mathrm{Gl}_{k_1,\dots,k_n}(\pi/3)\,.$$

The dimension of $\mathcal{C}_k$ and $\mathcal{G}_k$ are conjectured by Borwein, Broadhurst and Kamnitzer as follows:

Conjecture 4 (Borwein-Broadhurst-Kamnitzer [9])**.**

$$\dim \mathcal{C}_k = I(k), \quad \dim \mathcal{G}_k = R(k),$$

where $R(k)$ and $I(k)$ are sequences defined by

$$I(0) = I(1) = 0, \quad R(0) = R(1) = 1,$$
$$I(k) = I(k-1) + R(k-2) \quad (k \ge 2),$$
$$R(k) = R(k-1) + I(k-2) \quad (k \ge 2).$$

Note that $W(k) := I(k) + R(k)$ is equal to F_{k+1}, the $(k+1)$-th Fibonacci number. It is shown by Deligne [15] that the dimension of the vector space spanned by all multiple polylogarithms $\mathrm{Li}_{\mathbf{k}}(e^{\pi i/3})$ is F_{k+1} or less. In Section 3.4.3, we will give a conjecture on a basis of $\mathcal{C}_k$ and $\mathcal{G}_k$.

1.2 Arakawa-Kaneko zeta function

1.2.1 Poly-Bernoulli numbers and the Arakawa-Kaneko zeta function

The Arakawa-Kaneko zeta function was initially motivated in the study of poly-Bernoulli numbers, but it has many relations with multiple zeta values. For $k \in \mathbb{Z}$, poly-Bernoulli numbers $B_n^{(k)}$ and $C_n^{(k)}$ introduced by Kaneko [30] are defined by the following generating function:

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}, \qquad \frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1} = \sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!}.$$

When $k=1$, since $\mathrm{Li}_k(1-e^{-t}) = t$, poly-Bernoulli numbers coincide with the classical Bernoulli numbers B_n and C_n defined by the following generating functions:

$$\frac{te^t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \qquad \frac{t}{e^t-1} = \sum_{n=0}^{\infty} C_n \frac{t^n}{n!}.$$

We note that the Bernoulli numbers B_n and C_n are essentially the same because $B_n = C_n$ except $n=1$ and further $B_n = (-1)^n C_n$ holds. However, poly-Bernoulli numbers $B_n^{(k)}$ and $C_n^{(k)}$ do not have such a property. Therefore, we need to define $B_n^{(k)}$ and $C_n^{(k)}$ separately. One of important properties of the Bernoulli numbers is that they appear at non-positive integer points of the Riemann zeta function as follows:

$$\zeta(-m) = -\frac{B_{m+1}}{m+1} = \frac{(-1)^m C_{m+1}}{m+1} \quad (m \in \mathbb{Z}_{\geq 0}).$$

The Arakawa-Kaneko zeta function was introduced by Arakawa and Kaneko [2] as a function which generalizes the Riemann zeta function so that poly-Bernoulli numbers appear at non-positive integer points.

Definition 2 (Arakawa-Kaneko zeta function)**.** For $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ and $s \in \mathbb{C}$ with $\Re(s) > 1-n$, the Arakawa-Kaneko zeta function is defined by

$$\xi(\mathbf{k}; s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{\mathrm{Li}_{\mathbf{k}}(1-e^{-t})}{e^t-1}\, dt. \tag{1.7}$$

This function is a generalization of the Riemann zeta function since $\xi(1; s) = s\zeta(s+1)$. The integral of the right hand side of (1.7) converges only when $\Re(s) > 1-n$, but the Arakawa-Kaneko zeta function can be analytically continued to an entire function. The values of the Arakawa-kaneko zeta function with $n=1$ at non-positive integer points is as follows:

$$\xi(k; -m) = (-1)^m C_m^{(k)} \qquad (m \in \mathbb{Z}_{\geq 0}),$$

(see [2, Theorem 6 (i)]). For any n, multi-poly-Bernoulli numbers defined by Imatomi, Kaneko and Takeda [25] appear at non-positive integer points of the Arakawa-Kaneko zeta function. Multi-poly-Bernoulli numbers $B_m^{(k_1,\ldots,k_n)}$ and $C_m^{(k_1,\ldots,k_n)}$ are defined as a generalization of poly-Bernoulli numbers as follows:

$$\frac{\mathrm{Li}_{k_1,\ldots,k_n}(1-e^{-t})}{1-e^{-t}} = \sum_{m=0}^{\infty} B_m^{(k_1,\ldots,k_n)} \frac{t^m}{m!},$$
$$\frac{\mathrm{Li}_{k_1,\ldots,k_n}(1-e^{-t})}{e^t-1} = \sum_{m=0}^{\infty} C_m^{(k_1,\ldots,k_n)} \frac{t^m}{m!}.$$

Then, the values of the Arakawa-Kaneko zeta function at non-positive integer points are as follows:

$$\xi(k_1, \ldots, k_n; -m) = (-1)^m C_m^{(k_1,\ldots,k_n)} \qquad (m \in \mathbb{Z}_{\geq 0})$$

(see [31, Remark 2.4]).

On the other hand, there is a function such that multi-poly-Bernoulli numbers $B_m^{(k_1,\ldots,k_n)}$ appear at its non-positive integer points. The function was introduced Kaneko and Tsumura [31] and is called the Kaneko-Tsumura zeta function.

Definition 3 (Kaneko-Tsumura zeta function)**.** For $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ or $\mathbf{k} = (k_1, \ldots, k_n) \in (\mathbb{Z}_{\leq 0})^n$, and $s \in \mathbb{C}$ with $\Re(s) > 1 - n$, the Kaneko-Tsumura zeta function is defined by

$$\eta(\mathbf{k}; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\mathrm{Li}_{\mathbf{k}}(1 - e^t)}{1 - e^t}\, dt. \tag{1.8}$$

The integral of the right hand side of (1.8) converges only when $\Re(s) > 1-n$, but the Kaneko-Tsumura zeta function can be analytically continued to an entire function. The values of the Kaneko-Tsumura zeta function at non-positive integer points is as follows:

$$\eta(k_1, \ldots, k_n; -m) = B_m^{(k_1,\ldots,k_n)} \qquad (m \in \mathbb{Z}_{\geq 0})$$

(see [31, Theorem 2.3, Theorem 4.4]). The Kaneko-Tsumura zeta function has some properties which can be contrasted with these of the Arakawa-Kaneko zeta function. One of them is the following formulas.

Theorem 2 ([31, Theorem 3.2])**.** *For* $\mathbf{k} \in \mathbb{N}^n$*, we have*

$$\eta(\mathbf{k}; s) = (-1)^{n-1} \sum_{\mathbf{k} \preceq \mathbf{k}'} \xi(\mathbf{k}'; s)$$

and

$$\xi(\mathbf{k}; s) = (-1)^{n-1} \sum_{\mathbf{k} \preceq \mathbf{k}'} \eta(\mathbf{k}'; s).$$

Here, we denote $\mathbf{k} \preceq \mathbf{k}'$ when $\mathbf{k}$ is obtained from $\mathbf{k}'$ by replacing some commas by $+$'s. For example, $(3) = (1+2) \preceq (1, 2)$.

1.2.2 The Arakawa-Kaneko zeta function and multiple zeta values

The Arakawa-Kaneko zeta function and the Kaneko-Tsumura zeta function are also related to multiple zeta values and multiple zeta star values. It was proved in [2] the following relations between the Arakawa-Kaneko zeta function and multiple zeta functions.

Theorem 3 ([2, Theorem 8])**.** *For* $n, k \in \mathbb{N}$, $s \in \mathbb{C}$,

$$\begin{aligned} &\xi(\{1\}^{n-1}, k; s) \\ &= (-1)^{k-1} \sum_{\substack{a_1 + \cdots + a_k = n \\ a_i \geq 0}} \binom{a_k + s - 1}{a_k} \zeta(a_1 + 1, \ldots, a_{k-1} + 1, a_k + s) \\ &\quad + \sum_{j=0}^{k-2} (-1)^j \zeta(\{1\}^{n-1}, k - j) \zeta(\{1\}^j, s). \end{aligned}$$

The following relations between the Arakawa-Kaneko zeta function and multiple zeta values were also proved.

Theorem 4 ([2, Theorem 9 (i)])**.** *For* $n \geq 1$, $k \geq 1$ *and* $m \geq 0$,

$$\xi(\{1\}^{n-1}, k; m+1) = \sum_{a_1 + \cdots + a_k = m} \binom{a_k + n}{n} \zeta(a_1 + 1, \ldots, a_{k-1} + 1, a_k + n + 1).$$

Theorem 5 ([2, Theorem 9 (ii)])**.** *For* $m \geq 1$, $n \geq 1$ *and* $k \geq 2$,

$$\begin{aligned} &\xi(\{1\}^{n-1}, k; m+1) + (-1)^k \xi(\{1\}^{m-1}, k; n+1) \\ &= \sum_{j=0}^{k-2} (-1)^j \zeta(\{1\}^{n-1}, k - j) \zeta(\{1\}^{m-1}, 2 + j). \end{aligned}$$

Moreover, by combining these theorems, the following relations among MZVs were proved in [2].

Theorem 6 ([2, Corollary 11]). *For $m \geq 1$, $n \geq 1$ and $k \geq 2$,*

$$\sum_{a_1+\cdots+a_k=m} \binom{a_k+n}{n} \zeta(a_1+1,\ldots,a_{k-1}+1,a_k+n+1)$$
$$+ (-1)^k \sum_{a_1+\cdots+a_k=n} \binom{a_k+m}{m} \zeta(a_1+1,\ldots,a_{k-1}+1,a_k+m+1)$$
$$= \sum_{j=0}^{k-2} (-1)^j \zeta(\{1\}^{n-1}, k-j)\zeta(\{1\}^{m-1}, 2+j).$$

Therefore, the Arakawa-Kaneko zeta function is not only related to MZVs but also applicable to the theory of MZVs.

In [31], on the values of the Arakawa-Kaneko zeta function and the Kaneko-Tsumura zeta function at positive integers, the following formulas were proved.

Theorem 7 ([31, Theorem 2.5]). *We write*

$$b(\mathbf{k};\mathbf{j}) = \prod_{i=1}^{n} \binom{k_i+j_i-1}{j_i}.$$

For $m \in \mathbb{N}$ and $\mathbf{k} = (k_1,\ldots,k_n) \in \mathbb{N}^n$, we have

$$\xi(\mathbf{k};m) = \sum_{|\mathbf{j}|=m-1,\, \mathrm{d}(\mathbf{j})=d((\mathbf{k}_+)^*)} b((\mathbf{k}_+)^*;\mathbf{j})\zeta((\mathbf{k}_+)^* + \mathbf{j})$$

and

$$\eta(\mathbf{k};m) = (-1)^{n-1} \sum_{|\mathbf{j}|=m-1,\, \mathrm{d}(\mathbf{j})=d((\mathbf{k}_+)^*)} b((\mathbf{k}_+)^*;\mathbf{j})\zeta^\star((\mathbf{k}_+)^* + \mathbf{j}),$$

where the sum is over all $\mathbf{j} = (j_1,\ldots,j_n) \in \mathbb{Z}_{\geq 0}^n$ satisfying $|\mathbf{j}| = m-1$, $\mathrm{d}(\mathbf{j}) = d((\mathbf{k}_+)^*)$.

There are the following functional relations between the Arakawa-Kaneko zeta function and multiple zeta functions.

Theorem 8 ([31, Theorem 3.6]). *The Arakawa-Kankeo zeta function $\xi(\mathbf{k};s)$ can be written in terms of EZ-type MZFs as*

$$\xi(\mathbf{k};s) = \sum_{\mathbf{k}',\, j\geq 0} c_{\mathbf{k}}(\mathbf{k}';j) \binom{s+j-1}{j} \zeta(\mathbf{k}', j+s).$$

Here, the sum is over indices $\mathbf{k}'$ and integers $j \geq 0$ satisfying $|\mathbf{k}'| + j \leq |\mathbf{k}|$, and $c_{\mathbf{k}}(\mathbf{k}';j)$ is a $\mathbb{Q}$-linear combination of EZ-type MZVs of weight $|\mathbf{k}| - |\mathbf{k}'| - j$.

1.2.3 The Mordell-Tornheim type analogue of the Arakawa-Kaneko zeta function

Ito [26] focused on the fact that the properties of the Arakawa-Kaneko zeta function lead to relations among MZVs, and considered a Mordell-Tornheim analogue of the Arakawa-Kaneko zeta function. Ito introduced the following function as an analogue of the Arakawa-Kaneko zeta function of Mordell-Tornheim type.

Definition 4 (Ito zeta function). For $k_1,\ldots,k_r \in \mathbb{N}$ and $s \in \mathbb{C}$ with $\Re(s) > 1-r$, we define

$$\xi_{MT}(k_1,\ldots,k_r;s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\prod_{i=1}^r \mathrm{Li}_{k_i}(1-e^{-t})}{e^t-1}\, dt. \tag{1.9}$$

We note that this notation is due to the arXiv version of [26]. Ito proved the following two theorems on the Ito zeta function. The first is an evaluation of the special values of the Ito zeta function in terms of MT-type MZVs.

Theorem 9 ([26, Proposition 5])**.** *For* $m \in \mathbb{Z}_{\geq 0}$*,*

$$\xi_{MT}(k_1, \dots, k_r; m+1) = \frac{1}{m!}\zeta_{MT}(k_1, \dots, k_r, \{1\}^m; 1).$$

The second is a functional relation between Ito zeta functions and MT-type MZFs.

Theorem 10 ([26, Theorem 8])**.** *For* $r \in \mathbb{N}$ *and* $s \in \mathbb{C}$*,*

$$\sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j \zeta(2)^{r-1-j} \Gamma(s) \xi_{MT}(\{2\}^j; s)$$
$$= \sum_{j=0}^{r-1} \binom{r-1}{j} \Gamma(s+j) \zeta_{MT}(0, \{2\}^{r-1-j}, \{1\}^j; j+s).$$

Then, Ito obtained relations among MT-type MZVs by putting $s = m+1$ in Theorem 10 and using Theorem 9 for $\xi_{MT}(\{2\}^j; m+1)$, for example

$$\zeta(2)^2\zeta(m+1) - 2\zeta(2)\frac{1}{m!}\zeta_{MT}(2, \{1\}^m; 1) + \frac{1}{m!}\zeta(2, 2, \{1\}^m; 1)$$
$$= \zeta_{MT}(2, 2, 0; m+1) + 2(m+1)\zeta_{MT}(2, 1, 0; m+2) + (m+1)(m+2)\zeta_{MT}(1, 1, 0; m+3).$$

There is a generalization of the Ito zeta function, which was given by Ito himself as follows.

Definition 5 (Generalized Ito zeta function with $r = 1$)**.** For $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$, $k_{n+1} \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{C}$ with $\Re(s) > 1 - n$, we define

$$\xi_{MT}((\mathbf{k}; k_{n+1}); s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} \mathrm{Li}_{\mathbf{k}; k_{n+1}}(1 - e^{-t})\, dt, \tag{1.10}$$

where

$$\mathrm{Li}_{\mathbf{k}; k_{n+1}}(z) = \sum_{m_1=1, \dots, m_n=1}^{\infty} \frac{z^{\sum_{j=1}^n m_j}}{m_1^{k_1} \cdots m_n^{k_n} (\sum_{j=1}^n m_j)^{k_{n+1}}} \quad (|z| < 1).$$

Ito considered a version of the function (1.10), in which $\mathrm{Li}_{\mathbf{k}; k_{n+1}}(1 - e^{-t})$ is replaced by a product of r quantities of the form $\mathrm{Li}_{\mathbf{k}; k_{n+1}}(1 - e^{-t})$ ([26, Definition 13]). Therefore, we call the function (1.10) the generalized Ito zeta function with $r = 1$. In Section 2.1, we will follow Ito's work and use generalized Ito zeta function with $r = 1$ to obtain relations among Miyagawa-type MZVs.

1.3 Log-sine integrals

1.3.1 Log-sine integrals

Log-sine integrals are defined by the following integrals:

$$\mathrm{Ls}_k^{(l)}(\sigma) = -\int_0^\sigma \theta^l \log^{k-1-l} \left| 2 \sin \frac{\theta}{2} \right| d\theta,$$

where $k \in \mathbb{N}$ and $l \in \mathbb{Z}_{\geq 0}$. When $l = 0$, $\mathrm{Ls}_k(\sigma) := \mathrm{Ls}_k^{(0)}(\sigma)$ is often called the (basic) log-sine integrals and $\mathrm{Ls}_k^{(l)}(\sigma)$ is often called the generalized log-sine integrals. Various applications of log-sine integrals are known, for example Mahler measures (see [4, 8]) and Feynman diagrams (see [12, 13, 14, 27, 29]). The generalized log-sine integrals are also related to multiple (or Riemann) zeta values, for example [7, 11, 14, 33, 47].

It is believed that Euler was the first who studied integrals involving logarithm of sine, and his results:

$$\int_0^{\frac{\pi}{2}} \log(\sin x)\, dx = -\frac{\pi}{2} \log 2$$

and

$$\int_0^{\frac{\pi}{2}} x \log(\sin x)\, dx = \frac{7}{16}\zeta(3) - \frac{\pi^2}{8} \log 2$$

are famous. Lewin's book [33], published in 1981, contains many results on log-sine integrals, Riemann zeta values and polylogarithms. For example, it includes the description of a method of expressing $\mathrm{Ls}_k(\pi)$ and $\mathrm{Ls}_k^{(l)}(2\pi)$ in terms of the Riemann zeta values. We will discuss relations between values of iterated log-sine integrals at $\pi/3$, which are a generalization of the generalized log-sine integrals, and multiple zeta values in Section 3.2 and Section 3.4.2. Therefore, we focus on values of log-sine integrals at $\pi/3$. In 1979, van der Poorten [43] proved remarkable results:

$$\mathrm{Ls}_3^{(0)}\left(\frac{\pi}{3}\right) = -\frac{7}{108}\pi^3 \tag{1.11}$$

and

$$\mathrm{Ls}_4^{(1)}\left(\frac{\pi}{3}\right) = -\frac{17}{6480}\pi^4, \tag{1.12}$$

then Zucker [48] proved

$$\frac{3}{2}\mathrm{Ls}_5^{(2)}\left(\frac{\pi}{3}\right) - \mathrm{Ls}_5^{(0)}\left(\frac{\pi}{3}\right) = \frac{253}{3240}\pi^5 \tag{1.13}$$

and

$$\frac{1}{2}\mathrm{Ls}_6^{(3)}\left(\frac{\pi}{3}\right) - \mathrm{Ls}_6^{(1)}\left(\frac{\pi}{3}\right) = \frac{313}{408240}\pi^6. \tag{1.14}$$

After that, Zhang and Williams [47] proved formulas that include (1.11), (1.12), (1.13) and (1.14).

Log-sine integrals at $\pi/3$ are also related to multiple zeta values. For $k \in \mathbb{Z}_{\geq 0}$, the formula

$$\begin{aligned}
&(-1)^k \int_0^{\frac{\pi}{3}} \left(\theta - \frac{\pi}{3}\right)^{2k+1} \log\left(2\sin\frac{\theta}{2}\right) d\theta \\
&= -\frac{1}{2}(2k+1)!(1-2^{-2k-2})(1-3^{-2k-2})\zeta(2k+3) \\
&\quad + (2k+1)! \sum_{m=0}^{k} (-1)^m \left(\frac{\pi}{3}\right)^{2m} \frac{\zeta(2k+3-2m)}{(2m)!}.
\end{aligned} \tag{1.15}$$

is obtained by Choi, Cho and Srivistava [11, (4.14)] from the result of Lewin [33, (7.160)]. In particular,

$$\int_0^{\frac{\pi}{3}} \left(\theta - \frac{\pi}{3}\right) \log\left(2\sin\frac{\theta}{2}\right) d\theta = \frac{2}{3}\zeta(3) \tag{1.16}$$

is obtained in Lewin's book. The formula

$$\begin{aligned}
&\zeta(\{1\}^{a-1}, b+1) \\
&= -\frac{1}{a!b!}\left(\frac{\pi}{3}i\right)^a \left(-\frac{\pi}{3}i\right)^b + \frac{i^a}{(a-1)!b!}\int_0^{\frac{\pi}{3}} \theta^{a-1}\left(-\log\left(2\sin\frac{\theta}{2}\right) + i\frac{\theta-\pi}{2}\right)^b d\theta \\
&\quad + \frac{(-i)^b}{a!(b-1)!}\int_0^{\frac{\pi}{3}} \theta^{b-1}\left(-\log\left(2\sin\frac{\theta}{2}\right) - i\frac{\theta-\pi}{2}\right)^a d\theta.
\end{aligned} \tag{1.17}$$

is obtained by Borwein, Broadhurst and Kamnitzer [9].

1.4 Organization of this thesis

1.4.1 Organization of this thesis

In Section 2, we discuss the Ito zeta function and the generalized Ito zeta function with $r = 1$. Specifically, we obtain relations among Miyagawa multiple zeta values using the generalized Ito zeta function with $r = 1$ in Section 2.1. In Section 2.2, we introduce generalized Mordell-Tornheim multiple zeta functions first. Then we generalize Theorem 10 and Theorem 12 proved in Section 2.1 to the general index using generalized Mordell-Tornheim multiple zeta functions. As a consequence, we can obtain relations among special values of generalized Mordell-Tornheim multiple zeta functions.

In Section 3, we discuss iterated log-sine integrals. Specifically, we introduce iterated log-sine integrals and prove its fundamental properties in Section 3.1. In Section 3.2, we give a new method of obtaining relations among multiple zeta values, which uses iterated log-sine integrals. In Section 3.3, we evaluate iterated log-sine integrals in terms of multiple zeta values and multiple polylogarithms. This evaluation makes numerical evaluation of iterated log-sine integrals possible. In Section 3.4, we give some conjectures on iterated log-sine integrals, multiple zeta values, multiple Clausen values and multiple Glaisher values. These conjectures have been checked by numerical experiments.

2 Analogues of the Arakawa-Kaneko zeta function

In this section, we discuss the Ito zeta function and the generalized Ito zeta function with $r = 1$ based on author's paper [40] entitled "On an analog of the Arakawa-Kaneko zeta function and relations of some multiple zeta values" published Tsukuba Journal of Mathematics. As explained in Section 1.2.3, Ito obtained relations among Mordell-Tornheim multiple zeta values using the Ito zeta function. In Section 2.1, we obtain relations among Miyagawa multiple zeta values using the generalized Ito zeta function with $r = 1$. Namely, this can be regarded as Miyagawa type analogue of Ito's work. In Section 2.2, we generalize Theorem 10 and Theorem 12 to the general index. For this purpose, we need to introduce generalized Mordell-Tornheim multiple zeta functions defined in Definition 7. We describe the definition and basic properties of generalized Mordell-Tornheim multiple zeta functions first, then we generalize Theorem 10 and Theorem 12. As a consequence, we can obtain relations among special values of generalized Mordell-Tornheim multiple zeta functions.

2.1 Miyagawa-type analogue of Ito's work

2.1.1 Miyagawa multiple zeta function

Here, we describe properties on the Miyagawa-type MZF proved in [40].

First, we rewrite the Miyagawa-type MZF (1.3) as follows.

Definition 6. For $\mathbf{s}_{r+1} = (s_{r+1,1}, \dots, s_{r+1,n_{r+1}}) \in \mathbb{C}^{n_{r+1}}$, we write

$$\zeta_{MT}(s_1, \dots, s_r; \mathbf{s}_{r+1}) = \sum_{0<m_1,\dots,0<m_r} \sum_{m_{r+1,1}=1,\dots,m_{r+1,n_{r+1}-1}=1}^{\infty} \frac{1}{m_1^{s_1} \cdots m_r^{s_r} \prod_{u=1}^{n_{r+1}} (\sum_{v=1}^{r} m_v + \sum_{w=1}^{u-1} m_{r+1,w})^{s_{r+1,u}}}.$$

The following propositions on the Miyagawa-type MZF hold.

Proposition 1 ([40, Proposition 1])**.** *The Miyagawa multiple zeta function $\zeta_{MT}(s_1, s_2, \dots, s_r; \mathbf{s}_{r+1})$ converges absolutely when*

$$\sum_{i=0}^{k} \Re(s_{r+1,n_{r+1}-i}) > k+1$$

for any $k = 1, \dots, n_{r+1} - 2$ and

$$\sum_{l=1}^{j} \Re(s_{k_l}) + \sum_{i=0}^{n_{r+1}-1} \Re(s_{r+1,n_{r+1}-i}) - n_{r+1} + 1 > j$$

with $1 \le k_1 < k_2 < \cdots < k_j \le r$ for any $j = 1, 2, \dots, r$ are all satisfied.

Proposition 2 ([40, Remark 5])**.** *If $\Re(s_i) > 0$ $(2 \le i \le r)$, $\Re(s_{r+1,i}) > 0$ $(1 \le i \le n_{r+1})$ and the Miyagawa multiple zeta function $\zeta_{MT}(0, s_2, \dots, s_r; \mathbf{s}_{r+1})$ converges absolutely, we have*

$$\begin{aligned} &\zeta_{MT}(0, s_2, \dots, s_r; \mathbf{s}_{r+1}) \\ &= \frac{1}{\prod_{i=2}^{r} \Gamma(s_i) \prod_{j=1}^{n_{r+1}} \Gamma(s_{r+1,j})} \int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=2}^{r} \frac{t_i^{s_i-1}\, dt_i}{e^{t_i + t_{r+1,1} + \cdots + t_{r+1,n_{r+1}}} - 1} \\ &\quad \times \prod_{u=1}^{n_{r+1}} \frac{t_{r+1,u}^{s_{r+1,u}-1}\, dt_{r+1,u}}{e^{t_{r+1,u} + \cdots + t_{r+1,n_{r+1}}} - 1}. \end{aligned} \tag{2.1}$$

2.1.2 Special values of the generalized Ito zeta function

Ito obtained relations among MT-type MZVs by showing an evaluation of special values of the Ito zeta function by MT-type MZVs (Theorem 9) and functional relations between the Ito zeta function and MT-type MZFs (Theorem 10). We obtain relations among Miyagawa-type MZVs by showing evaluation of special values of the generalized Ito zeta function with $r = 1$ by Miyagawa-type MZVs and functional relations between the generalized Ito zeta function with $r = 1$ and Miyagawa-type MZFs in accordance with Ito's work. First, we prove that the generalized Ito zeta function with $r = 1$ can be written in terms of Miyagawa-type MZVs.

Theorem 11. *For* $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$, $k_{n+1} \in \mathbb{Z}_{\geq 0}$ *and* $m \in \mathbb{Z}_{\geq 0}$, *we have*

$$\begin{aligned}\xi_{MT}((\mathbf{k}; k_{n+1}); m+1) = &\sum_{a_1+\cdots+a_{k_{n+1}+1}=m} \frac{1}{a_{k_{n+1}+1}!}\\ &\times \zeta_{MT}(\{1\}^{a_{k_{n+1}+1}}, k_1, \ldots, k_n; {}_-((a_1+1, \ldots, a_{k_{n+1}}+1, 2)^*)),\end{aligned}$$

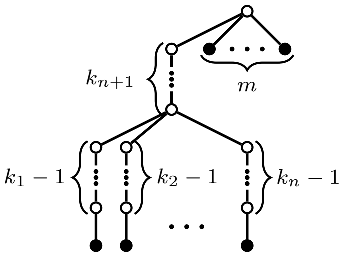
where the sum is over all $a_1, \ldots, a_{k_{n+1}+1} \in \mathbb{Z}_{\geq 0}$ *satisfying* $a_1 + \cdots + a_{k_{n+1}+1} = m$.

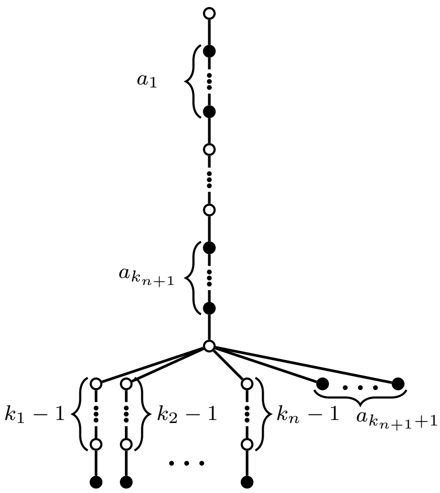
Proof. In [40], two proofs are given. We prove this theorem in the second way that is based on the Yamamoto-integral defined by Yamamoto [44]. Since

$$\begin{aligned}m!\xi_{MT}((\mathbf{k}; k_{n+1}); m+1) &= \int_0^\infty \frac{\left(\mathrm{Li}_1(1-e^{-t})\right)^m}{e^t - 1} \mathrm{Li}_{\mathbf{k};k_{n+1}}(1-e^{-t})\, dt\\ &= \int_0^1 \frac{\left(\mathrm{Li}_1(t)\right)^m}{t} \mathrm{Li}_{\mathbf{k};k_{n+1}}(t)\, dt,\end{aligned}$$

by using the Yamamoto-integral, we have

$m!\xi_{MT}((\mathbf{k}; k_{n+1}); m+1)$

$= I \Big($  $\Big)$

$= \displaystyle\sum_{a_1+\cdots+a_{k_{n+1}+1}=m} \frac{m!}{a_{k_{n+1}+1}!} I \Big($  $\Big)$.

The second equality is obtained by ordering m black vertices and $1 + k_{n+1}$ white vertices. Here, the special values of the Miyagawa-type MZF is written as

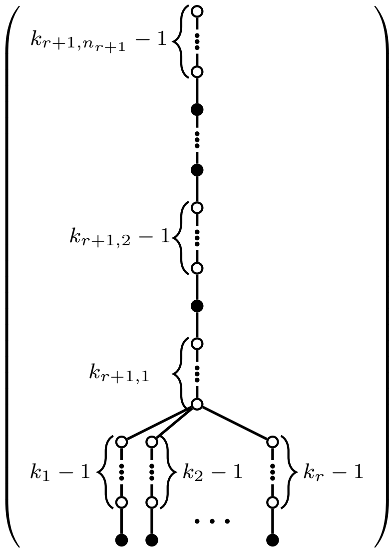
$$\begin{aligned}&\zeta_{MT}(k_1, \ldots, k_r; \mathbf{k}_{r+1})\\ &= \int_{0<t_1<\cdots<t_{k_{r+1,1}+\cdots+k_{r+1,n_{r+1}}}<1} \left(\prod_{i=2}^{k_{r+1,n_{r+1}}} \frac{1}{t_{k_{r+1,1}+\cdots+k_{r+1,n_{r+1}-1}+i}} \right)\end{aligned}$$

$$\times \frac{1}{1-t_{k_{r+1,1}+\cdots+k_{r+1,n_{r+1}-1}+1}}\left(\prod_{i=2}^{k_{r+1,n_{r+1}-1}}\frac{1}{t_{k_{r+1,1}+\cdots+k_{r+1,n_{r+1}-2}+i}}\right)\cdots$$

$$\cdots\times \frac{1}{1-t_{k_{r+1,1}+k_{r+1,2}+1}}\left(\prod_{i=2}^{k_{r+1,2}}\frac{1}{t_{k_{r+1,1}+i}}\right)$$

$$\times \frac{1}{1-t_{k_{r+1,1}+1}}\left(\prod_{i=1}^{k_{r+1,1}}\frac{1}{t_i}\right)\prod_{i=1}^{r}\mathrm{Li}_{k_i}(t_1)\,dt_1\cdots dt_{k_{r+1,1}+\cdots+k_{r+1,n_{r+1}}}$$

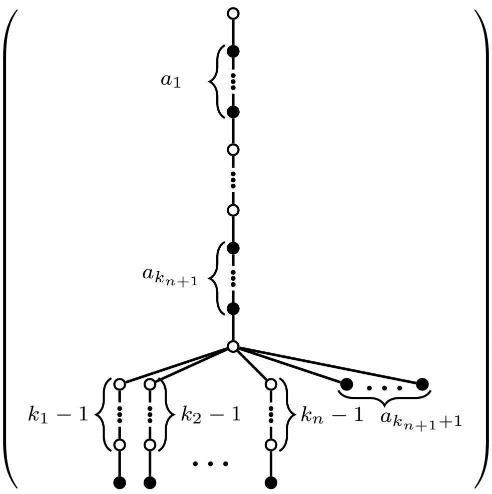
$$= I$$



Therefore, we obtain

$$\zeta_{MT}(\{1\}^{a_{k_{n+1}+1}},k_1,\ldots,k_n;{}_-((a_1+1,\ldots,a_{k_{n+1}}+1,2)^*))$$

$$= I$$



Therefore, we obtain Theorem 11. □

2.1.3 Functional relations

The generalized Ito zeta function with $r=1$ has the following functional relations with Miyagawa-type MZFs.

Theorem 12. *For $l,\, k \in \mathbb{N},\ s \in \mathbb{C}$, we have*

$$\begin{aligned}
&\zeta(2)^l\zeta(\{1\}^k,s)+\sum_{j=1}^{l}\binom{l}{j}\zeta(2)^{l-j}(-1)^j\\
&\quad\times\left(\sum_{i=1}^{k}(-1)^{i-1}\zeta_{MT}(\{2\}^j;i)\zeta(\{1\}^{k-i},s)+(-1)^k\xi_{MT}((\{2\}^j;k);s)\right)\\
&=\sum_{a+b_1+\cdots+b_{k+1}=l}\frac{l!}{a!}\binom{s+b_{k+1}-1}{b_{k+1}}\zeta_{MT}(0,\{2\}^a,\{1\}^{l-a};b_1+1,\ldots,b_k+1,b_{k+1}+s),
\end{aligned}$$

where the sum on the right hand side is over all $a \in \mathbb{Z}_{\geq 0}$ and $b_i \in \mathbb{Z}_{\geq 0}$ satisfying $a+b_1+\cdots+b_{k+1}=l$.

Remark 1. If we understand the sum with respect to i as 0 when $k=0$, then Theorem 12 holds also when $k=0$ and coincides with Theorem 10.

Proof of Theorem 12. For $s \in \mathbb{C}$ with $\Re(s)>1$, let

$$\begin{aligned}
J=&\int_0^\infty\cdots\int_0^\infty t_{k+1}^{s-1}\left(\prod_{i=1}^{l}\frac{u_i+t_1+\cdots+t_{k+1}}{e^{u_i+t_1+\cdots+t_{k+1}}-1}\right)\\
&\times\frac{1}{e^{t_1+\cdots+t_{k+1}}-1}\frac{1}{e^{t_2+\cdots+t_{k+1}}-1}\cdots\frac{1}{e^{t_{k+1}}-1}\,du_1\cdots du_l dt_1\cdots dt_{k+1}.
\end{aligned}$$

We calculate J in two different ways.

The first calculation is to integrate directly by using the equation (1.5) and [26, Lemma 4]. By integrating with respect to $u_1,\ldots,u_l$, we have

$$\begin{aligned}
J=&\int_0^\infty\cdots\int_0^\infty t_{k+1}^{s-1}\Big(\zeta(2)-\mathrm{Li}_2(1-e^{-(t_1+\cdots+t_{k+1})})\Big)^l\\
&\times\frac{1}{e^{t_1+\cdots+t_{k+1}}-1}\frac{1}{e^{t_2+\cdots+t_{k+1}}-1}\cdots\frac{1}{e^{t_{k+1}}-1}\,dt_1\cdots dt_{k+1}\\
=&\,\Gamma(s)\zeta(2)^l\zeta(\{1\}^k,s)\\
&+\sum_{j=1}^{l}\binom{l}{j}\zeta(2)^{l-j}(-1)^j\int_0^\infty\cdots\int_0^\infty t_{k+1}^{s-1}\Big(\mathrm{Li}_2(1-e^{-(t_1+\cdots+t_{k+1})})\Big)^j\\
&\times\frac{1}{e^{t_1+\cdots+t_{k+1}}-1}\frac{1}{e^{t_2+\cdots+t_{k+1}}-1}\cdots\frac{1}{e^{t_{k+1}}-1}\,dt_1\cdots dt_{k+1}.
\end{aligned}$$

We compute the above integral by parts in order of $t_1,\ldots,t_k$ to obtain

$$\begin{aligned}
&\int_0^\infty\cdots\int_0^\infty t_{k+1}^{s-1}\Big(\mathrm{Li}_2(1-e^{-(t_1+\cdots+t_{k+1})})\Big)^j\\
&\quad\times\frac{1}{e^{t_1+\cdots+t_{k+1}}-1}\frac{1}{e^{t_2+\cdots+t_{k+1}}-1}\cdots\frac{1}{e^{t_{k+1}}-1}\,dt_1\cdots dt_{k+1}\\
&=\int_0^\infty\cdots\int_0^\infty t_{k+1}^{s-1}\Big(\zeta_{MT}(\{2\}^j;1)-\mathrm{Li}_{\{2\}^j;1}(1-e^{-(t_2+\cdots+t_{k+1})})\Big)\\
&\quad\times\frac{1}{e^{t_2+\cdots+t_{k+1}}-1}\frac{1}{e^{t_3+\cdots+t_{k+1}}-1}\cdots\frac{1}{e^{t_{k+1}}-1}\,dt_2\cdots dt_{k+1}\\
&=\Gamma(s)\zeta_{MT}(\{2\}^j;1)\zeta(\{1\}^{k-1},s)\\
&\quad-\int_0^\infty\cdots\int_0^\infty t_{k+1}^{s-1}\mathrm{Li}_{\{2\}^j;1}(1-e^{-(t_2+\cdots+t_{k+1})})\\
&\quad\times\frac{1}{e^{t_2+\cdots+t_{k+1}}-1}\frac{1}{e^{t_3+\cdots+t_{k+1}}-1}\cdots\frac{1}{e^{t_{k+1}}-1}\,dt_2\cdots dt_{k+1}\\
&=\Gamma(s)\zeta_{MT}(\{2\}^j;1)\zeta(\{1\}^{k-1},s)-\Gamma(s)\zeta_{MT}(\{2\}^j;2)\zeta(\{1\}^{k-2},s)\\
&\quad+\int_0^\infty\cdots\int_0^\infty t_{k+1}^{s-1}\mathrm{Li}_{\{2\}^j;2}(1-e^{-(t_3+\cdots+t_{k+1})})
\end{aligned}$$

$$\times \frac{1}{e^{t_3+\cdots+t_{k+1}}-1}\frac{1}{e^{t_4+\cdots+t_{k+1}}-1}\cdots\frac{1}{e^{t_{k+1}}-1}\,dt_3\cdots dt_{k+1}$$
$$=\cdots$$
$$=\Gamma(s)\sum_{i=1}^{k}(-1)^{i-1}\zeta_{MT}(\{2\}^j;i)\zeta(\{1\}^{k-i},s)$$
$$+(-1)^k\int_0^\infty t_{k+1}^{s-1}\mathrm{Li}_{\{2\}^j;k}(1-e^{-t_{k+1}})\frac{1}{e^{t_{k+1}}-1}\,dt_{k+1}.$$

Therefore, we have

$$J=\Gamma(s)\zeta(2)^l\zeta(\{1\}^k,s)$$
$$+\Gamma(s)\sum_{j=1}^{l}\binom{l}{j}\zeta(2)^{l-j}(-1)^j\left(\sum_{i=1}^{k}(-1)^{i-1}\zeta_{MT}(\{2\}^j;i)\zeta(\{1\}^{k-i},s)+(-1)^k\xi_{MT}((\{2\}^j;k);s)\right).$$

The second calculation is to use the polynomial expansion and the equation (2.1). By symmetry of $u_1,\ldots,u_l$, we have

$$J=\sum_{a+b_1+\cdots+b_{k+1}=l}\frac{l!}{a!b_1!\cdots b_{k+1}!}\int_0^\infty\cdots\int_0^\infty t_{k+1}^{s-1}u_1\cdots u_a t_1^{b_1}\cdots t_{k+1}^{b_{k+1}}$$
$$\times\left(\prod_{i=1}^{l}\frac{1}{e^{u_i+t_1+\cdots+t_{k+1}}-1}\right)\left(\prod_{i=1}^{k+1}\frac{1}{e^{t_i+\cdots+t_{k+1}}-1}\right)du_1\cdots du_l\,dt_1\cdots dt_{k+1}$$
$$=\sum_{a+b_1+\cdots+b_{k+1}=l}\frac{l!\Gamma(b_{k+1}+s)}{a!b_{k+1}!}\zeta(0,\{2\}^a,\{1\}^{l-a};1+b_1,\ldots,1+b_k,b_{k+1}+s).$$

By the first and second calculations, we obtain the desired identity for $\Re(s)>1$. By the analytic continuation of the EZ-type MZF, the generalized Ito zeta function ([26, Theorem 14]) and the Miyagawa-type MZF, we obtain the stated theorem. ☐

Note that J was calculated by Arakawa and Kaneko when $l=1$, and Ito when $k=0$ (see the proof of [2, Theorem 6.(ii)] and [26, Theorem 8]). Therefore, we may regard this proof as a fusion of the method of Arakawa and Kaneko, and the method of Ito.

2.1.4 Relations among Miyagawa-type MZVs

By putting $s=m+1$ with $m\in\mathbb{N}$ in Theorem 12 and using Theorem 11 for $\xi_{MT}((\{2\}^j;k);s)$, we can obtain the following relations among Miyagawa-type MZVs.

Theorem 13. *For $l,k,m\in\mathbb{N}$, we have*

$$\zeta(2)^l\zeta(\{1\}^k,m+1)$$
$$+\sum_{j=1}^{l}\binom{l}{j}\zeta(2)^{l-j}(-1)^j\left(\sum_{i=1}^{k}(-1)^{i-1}\zeta_{MT}(\{2\}^j;i)\zeta(\{1\}^{k-i},m+1)\right.$$
$$\left.+(-1)^k\sum_{a_1+\cdots+a_{k+1}=m}\frac{1}{a_{k+1}!}\zeta_{MT}(\{1\}^{a_{k+1}},\{2\}^j;{}_{-}((a_1+1,\ldots,a_k+1,2)^*))\right)$$
$$=\sum_{a+b_1+\cdots+b_{k+1}=l}\frac{l!}{a!}\binom{m+b_{k+1}}{b_{k+1}}$$
$$\times\zeta_{MT}(0,\{2\}^a,\{1\}^{l-a};b_1+1,\ldots,b_k+1,b_{k+1}+m+1).$$

Example 1. Let $l=1,k=1,m=1$. Then we obtain

$$\zeta(2)\zeta(1,2)-\zeta_{MT}(2;1)\zeta(2)+\zeta_{MT}(2;1,2)+\zeta_{MT}(1,2;2)$$
$$=\zeta_{MT}(0,2;1,2)+\zeta_{MT}(0,1;2,2)+2\zeta_{MT}(0,1;1,3).$$

When $l=1,k=1,m=2$ and $l=1,k=2,m=1$. Then we obtain

$$\begin{aligned}&\zeta(2)\zeta(1,3)-\zeta_{MT}(2;1)\zeta(3)+\zeta_{MT}(2;1,1,2)+\zeta_{MT}(1,2;1,2)+\frac{1}{2}\zeta_{MT}(1,1,2;2)\\&=\zeta_{MT}(0,2;1,3)+\zeta_{MT}(0,1;2,3)+3\zeta_{MT}(0,1;1,4)\end{aligned}$$

and

$$\begin{aligned}&\zeta(2)\zeta(1,1,2)-\zeta_{MT}(2;1)\zeta(1,2)+\zeta_{MT}(2;2)\zeta(2)-\zeta_{MT}(2;2,2)-\zeta_{MT}(2;1,3)-\zeta_{MT}(1,2;3)\\&=\zeta_{MT}(0,2;1,1,2)+\zeta_{MT}(0,1;2,1,2)+\zeta_{MT}(0,1;1,2,2)+2\zeta_{MT}(0,1;1,1,3),\end{aligned}$$

respectively.

2.2 Generalization of Theorem 10 and Theorem 12

2.2.1 Generalized Mordell-Tornheim multiple zeta function

We introduce a class of multiple zeta functions (generalized multiple zeta functions), which is necessary to generalize Theorem 10 and Theorem 12.

Definition 7 (Generalized Mordell-Tornheim multiple zeta function (GMT-type MZF))**.** For $\mathbf{s}_i=(s_{i,1},\ldots,s_{i,n_i})\in\mathbb{C}^{n_i}$ $(1\le i\le r+1,\ n_i\in\mathbb{N})$, we define

$$\begin{aligned}&\zeta_{MT}(\mathbf{s}_1,\mathbf{s}_2,\ldots,\mathbf{s}_r;\mathbf{s}_{r+1})\\&=\sum_{\substack{0<m_{1,1}<m_{1,2}<\cdots<m_{1,n_1}\\ \vdots\\ 0<m_{r,1}<m_{r,2}<\cdots<m_{r,n_r}}}\ \sum_{m_{r+1,1}=1,\ldots,m_{r+1,n_{r+1}-1}=1}^{\infty}\\&\frac{1}{\prod_{i=1}^{r}\prod_{j=1}^{n_i}m_{i,j}^{s_{i,j}}\prod_{u=1}^{n_{r+1}}(\sum_{v=1}^{r}m_{v,n_v}+\sum_{w=1}^{u-1}m_{r+1,w})^{s_{r+1,u}}}\\&=\sum_{\substack{0<m_{1,1},0<m_{1,2},\ldots,0<m_{1,n_1}\\ \vdots\\ 0<m_{r,1},0<m_{r,2},\ldots,0<m_{r,n_r}}}\ \sum_{m_{r+1,1}=1,\ldots,m_{r+1,n_{r+1}-1}=1}^{\infty}\\&\frac{1}{\prod_{i=1}^{r}\prod_{k=1}^{n_i}(\sum_{j=1}^{k}m_{i,j})^{s_{i,k}}\prod_{u=1}^{n_{r+1}}(\sum_{i=1}^{r}\sum_{j=1}^{n_i}m_{i,j}+\sum_{w=1}^{u-1}m_{r+1,w})^{s_{r+1,u}}}.\end{aligned}\tag{2.2}$$

We call this function the generalized Mordell-Tornheim multiple zeta function (GMT-type MZF) and call the values of GMT-type MZFs at non-negative integer points in the domain of convergence generalized Mordell-Tornheim multiple zeta values (GMT-type MZVs).

Remark 2. A class of functions (2.2) contains EZ-type MZFs, MT-type MZFs and Miyagawa-type MZFs as special cases. For example,

$$\zeta_{MT}((s_1,\ldots,s_j);(s_{j+1},\ldots,s_n))=\zeta(s_1,\ldots,s_{j-1},s_j+s_{j+1},s_{j+2},\ldots,s_n),$$

$$\zeta_{MT}((s_1),\ldots,(s_r);(s_{r+1}))=\zeta_{MT}(s_1,\ldots,s_r;s_{r+1}),$$

and

$$\zeta_{MT}((s_1),\ldots,(s_j);(s_{j+1},\ldots,s_{r+1}))=\hat{\zeta}_{MT,j,r}(s_1,\ldots,s_j;s_{j+1},\ldots,s_{r+1}).$$

Therefore, for the case $n_i=1$ $(1\le i\le r)$ and $\mathbf{s}_{r+1}=(s_{r+1,1},\ldots,s_{r+1,n_{r+1}})$, we may omit the parentheses. For example, $\zeta_{MT}((s_{1,1}),(s_{2,1},s_{2,2});(s_{3,1},s_{3,2}))$ is written as $\zeta_{MT}(s_{1,1},(s_{2,1},s_{2,2});s_{3,1},s_{3,2})$. Under this notation, the Miyagawa-type MZF is written as

$$\hat{\zeta}_{MT,r,r+n_{r+1}-1}(s_1,\ldots,s_r;s_{r+1,1},\ldots,s_{r+1,n_{r+1}})=\zeta_{MT}(s_1,\ldots,s_r;\mathbf{s}_{r+1}).$$

The function (2.2) is included as a special case of the function (1.4). Therefore, the function (2.2) can be continued meromorphically to the whole space $\mathbb{C}^{n_1+\cdots+n_{r+1}}$.

On the GMT-type MZFs, the following propositions hold. See [40] for details of their proof.

Proposition 3. *If one of the following conditions is satisfied, then the function* (2.2) *converges absolutely.*

(i)

$$\begin{aligned} &\Re(s_{i,j}) \geq 1 && (1 \leq i \leq r, 1 \leq j \leq n_r), \\ &\sum_{i=0}^{k} \Re(s_{r+1,n_{r+1}-i}) > k+1 && (0 \leq k \leq n_{r+1}-2), \\ &\sum_{i=0}^{n_{r+1}-1} \Re(s_{r+1,n_{r+1}-i}) > n_{r+1}-1, \end{aligned}$$

(ii)

$$\begin{aligned} &\mathbf{s_1} = (0), \\ &\Re(s_{i,j}) \geq 1 && (2 \leq i \leq r, 1 \leq j \leq n_r), \\ &\sum_{i=0}^{k} \Re(s_{r+1,n_{r+1}-i}) > k+1 && (0 \leq k \leq n_{r+1}-1). \end{aligned}$$

Proposition 4. *If* $\zeta_{MT}(\mathbf{s}_1, \ldots, \mathbf{s}_r; \mathbf{s}_{r+1})$ *satisfying* $\Re(s_{i,j}) > 0$ *for* $1 \leq i \leq r+1$ *and* $1 \leq j \leq n_i$ *converges absolutely, especially when the condition (i) of Proposition 3 is satisfied, we have*

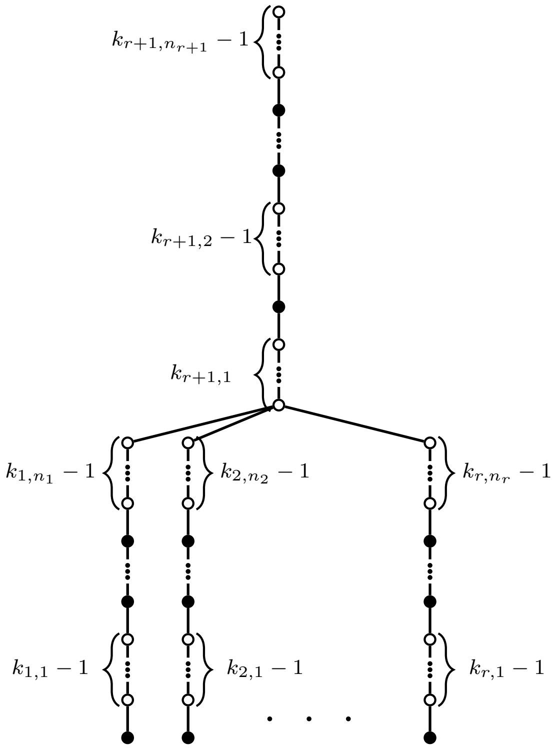
$$\begin{aligned} &\zeta_{MT}(\mathbf{s}_1, \ldots, \mathbf{s}_r; \mathbf{s}_{r+1}) \\ &= \frac{1}{\prod_{i=1}^{r+1} \prod_{j=1}^{n_i} \Gamma(s_{i,j})} \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} \prod_{j=1}^{n_i} \frac{t_{i,j}^{s_{i,j}-1}\, dt_{i,j}}{e^{t_{i,j}+\cdots+t_{i,n_i}+t_{r+1,1}+\cdots+t_{r+1,n_{r+1}}} - 1} \\ &\quad \times t_{r+1,1}^{s_{r+1,1}-1}\, dt_{r+1,1} \prod_{u=2}^{n_{r+1}} \frac{t_{r+1,u}^{s_{r+1,u}-1}\, dt_{r+1,u}}{e^{t_{r+1,u}+\cdots+t_{r+1,n_{r+1}}} - 1}. \end{aligned}$$

If $\zeta_{MT}(0, \mathbf{s}_2, \ldots, \mathbf{s}_r; \mathbf{s}_{r+1})$ *satisfying* $\Re(s_{i,j}) > 0$ *for* $2 \leq i \leq r+1$ *and* $1 \leq j \leq n_i$ *converges absolutely, especially when the condition (ii) of Proposition 3 is satisfied, we have*

$$\begin{aligned} &\zeta_{MT}(0, \mathbf{s}_2, \ldots, \mathbf{s}_r; \mathbf{s}_{r+1}) \\ &= \frac{1}{\prod_{i=2}^{r+1} \prod_{j=1}^{n_i} \Gamma(s_{i,j})} \int_0^\infty \cdots \int_0^\infty \prod_{i=2}^{r} \prod_{j=1}^{n_i} \frac{t_{i,j}^{s_{i,j}-1}\, dt_{i,j}}{e^{t_{i,j}+\cdots+t_{i,n_i}+t_{r+1,1}+\cdots+t_{r+1,n_{r+1}}} - 1} \\ &\quad \times \prod_{u=1}^{n_{r+1}} \frac{t_{r+1,u}^{s_{r+1,u}-1}\, dt_{r+1,u}}{e^{t_{r+1,u}+\cdots+t_{r+1,n_{r+1}}} - 1}. \end{aligned} \tag{2.3}$$

Proposition 5. *For* $\mathbf{k}_i = (k_{i,1}, \ldots, k_{i,n_i}) \in \mathbb{N}^{n_i}$ $(1 \leq i \leq r+1,\ n_i \in \mathbb{N})$*, we have*

$$\zeta_{MT}(\mathbf{k}_1, \ldots, \mathbf{k}_r; \mathbf{k}_{r+1})$$

$= I \Bigg($ 

$\Bigg).$

2.2.2 Generalization of Theorem 10

We generalize Theorem 10 and give relations among GMT-type MZVs with $n_{r+1} = 1$ as its consequence. It is difficult to generalize Theorem 10 and Theorem 12 in their original forms. First, in order to make the idea easier to understand, we rewrite Theorem 10.

Proposition 6. *For $r \in \mathbb{N}, s \in \mathbb{C}$, we have*

$$\begin{aligned}
&\xi_{MT}(\{2\}^r; s) \\
&= \sum_{a_1+a_2+a_3=r} \frac{r!}{a_1!a_3!}\zeta(2)^{a_1}(-1)^{a_2+a_3}\binom{s+a_2-1}{a_2}\zeta_{MT}(0, \{1\}^{a_2}, \{2\}^{a_3}; a_2+s),
\end{aligned}$$

where the sum is over all $a_1, a_2, a_3 \in \mathbb{Z}_{\geq 0}$ satisfying $a_1 + a_2 + a_3 = r$.

Proof. By using the formula (1.5), we have

$$\mathrm{Li}_2(1-e^{-t}) = \zeta(2) - \int_0^\infty \frac{t}{e^{t+u}-1}\,du - \int_0^\infty \frac{u}{e^{t+u}-1}\,du.$$

Therefore, for $\Re(s) > 1$, we have

$$\begin{aligned}
&\Gamma(s)\xi_{MT}(\{2\}^r; s) \\
&= \int_0^\infty \frac{t^{s-1}}{e^t-1}\Big(\mathrm{Li}_2(1-e^{-t})\Big)^r dt \\
&= \int_0^\infty \frac{t^{s-1}}{e^t-1}\Big(\zeta(2) - \int_0^\infty \frac{t}{e^{t+u}-1}\,du - \int_0^\infty \frac{u}{e^{t+u}-1}\,du\Big)^r dt \\
&= \sum_{a_1+a_2+a_3=r} \frac{r!}{a_1!a_2!a_3!} \\
&\quad \times \int_0^\infty \frac{t^{s-1}}{e^t-1}\zeta(2)^{a_1}\Big(-\int_0^\infty \frac{t}{e^{t+u}-1}\,du\Big)^{a_2}\Big(-\int_0^\infty \frac{u}{e^{t+u}-1}\,du\Big)^{a_3} dt \\
&= \sum_{a_1+a_2+a_3=r} \frac{r!}{a_1!a_2!a_3!}\zeta(2)^{a_1}(-1)^{a_2+a_3}
\end{aligned}$$

$$\times \int_0^\infty \frac{t^{s+a_2-1}}{e^t-1}\Big(\int_0^\infty \frac{1}{e^{t+u}-1}\,du\Big)^{a_2}\Big(\int_0^\infty \frac{u}{e^{t+u}-1}\,du\Big)^{a_3}\,dt$$
$$= \sum_{a_1+a_2+a_3=r} \frac{r!}{a_1!a_2!a_3!}\zeta(2)^{a_1}(-1)^{a_2+a_3}\Gamma(a_2+s)\zeta_{MT}(0,\{1\}^{a_2},\{2\}^{a_3};a_2+s).$$

By the analytic continuation of the Ito zeta function ([26, Theorem 2]) and the MT-type MZF, we obtain the stated theorem. □

The key of this proof is to use the formula

$$\mathrm{Li}_2(1-e^{-t}) = \zeta(2) - \int_0^\infty \frac{t}{e^{t+u}-1}\,du - \int_0^\infty \frac{u}{e^{t+u}-1}\,du$$

directly. We generalize this formula to any k.

Lemma 1. *For $k \in \mathbb{Z}_{\geq 2}$ and $t > 0$, we have*

$$\mathrm{Li}_k(1-e^{-t}) = \sum_{j=0}^{2k-2} f(t;j,k),$$

where

$$f(t;j,k) = \begin{cases} (-1)^j\zeta(k-j)\int_0^\infty\cdots\int_0^\infty \prod_{i=1}^{j}\frac{du_i}{e^{t+u_1+\cdots+u_i}-1} & (\text{if } j<k-1),\\ (-1)^{k-1}\int_0^\infty\cdots\int_0^\infty t\prod_{i=1}^{k-1}\frac{du_i}{e^{t+u_1+\cdots+u_i}-1} & (\text{if } j=k-1),\\ (-1)^{k-1}\int_0^\infty\cdots\int_0^\infty u_{j-k+1}\prod_{i=1}^{k-1}\frac{du_i}{e^{t+u_1+\cdots+u_i}-1} & (\text{if } j>k-1),\end{cases}$$

and we understand if $j=0$ then

$$\int_0^\infty\cdots\int_0^\infty \frac{1}{e^{t+u_1}-1}\cdots\frac{1}{e^{t+u_1+\cdots+u_j}-1}\,du_1\cdots du_j = 1.$$

Proof. We use induction. If $k=2$, it is true since

$$\mathrm{Li}_2(1-e^{-t}) = \zeta(2) - \int_0^\infty \frac{t}{e^{t+u}-1}\,du - \int_0^\infty \frac{u}{e^{t+u}-1}\,du.$$

Assume that the formula holds for k, and prove it for $k+1$. We have

$$\begin{aligned}\mathrm{Li}_{k+1}(1-e^{-t}) &= \zeta(k+1) - \int_0^\infty \frac{\mathrm{Li}_k(1-e^{-(t+u_1)})}{e^{t+u_1}-1}\,du_1\\ &= \zeta(k+1) - \int_0^\infty \frac{1}{e^{t+u_1}-1}\Big(\sum_{j=0}^{2k-2} f(t+u_1;j,k)\Big)du_1\\ &= \zeta(k+1) + \sum_{j=0}^{k-2} f(t;j+1,k+1) + f(t;k,k+1) + f(t;k+1,k+1)\\ &\quad + \sum_{j=k}^{2k-2} f(t;j+2,k+1)\\ &= \sum_{j=0}^{2k} f(t;j,k+1).\end{aligned}$$

This completes the proof. □

Lemma 2. *With the assumption of Lemma 1, for $k_i \geq 2\,(1 \leq i \leq r)$ and $t > 0$, we have*

$$\prod_{i=1}^{r} \mathrm{Li}_{k_i}(1-e^{-t}) = \sum_{\substack{0\leq j_1\leq 2k_1-2\\ \vdots \\ 0\leq j_r\leq 2k_r-2}} \prod_{i=1}^{r} f(t;j_i,k_i).$$

Proof. By Lemma 1, we have

$$\prod_{i=1}^{r} \mathrm{Li}_{k_i}(1-e^{-t}) = \prod_{i=1}^{r} \sum_{j=0}^{2k_i-2} f(t;j,k_i) = \sum_{\substack{0\leq j_1\leq 2k_1-2\\ \vdots \\ 0\leq j_r\leq 2k_r-2}} \prod_{i=1}^{r} f(t;j_i,k_i).$$

□

Then Proposition 6 is generalized as follows.

Theorem 14. *For $l \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{C}$ and $k_i \geq 2$ $(1 \leq i \leq r)$, we have*

$$\xi_{MT}(\{1\}^l, k_1, \ldots, k_r; s) = \sum_{\substack{0\leq j_1\leq 2k_1-2\\ \vdots \\ 0\leq j_r\leq 2k_r-2}} a_r(\mathbf{j},\mathbf{k}) \frac{\Gamma(l+b_r(\mathbf{j},\mathbf{k})+s)}{\Gamma(s)} \times \zeta_{MT}(0, \mathbf{k}(j_1,k_1), \ldots, \mathbf{k}(j_r,k_r); l+b_r(\mathbf{j},\mathbf{k})+s),$$

where

$$a_r(\mathbf{j},\mathbf{k}) = \prod_{i=1}^{r} a(j_i,k_i), \qquad a(j_i,k_i) = \begin{cases} (-1)^{j_i}\zeta(k_i-j_i) & (j_i < k_i-1), \\ (-1)^{k_i-1} & (j_i \geq k_i-1), \end{cases}$$

$$b_r(\mathbf{j},\mathbf{k}) = |\{i \in \{1,\ldots,r\} \mid j_i = k_i-1\}|,$$

and

$$\mathbf{k}(j_i,k_i) = \begin{cases} (\{1\}^{j_i}) & (j_i \leq k_i-1), \\ (\underbrace{\{1\}^{j_i-k_i}, 2, \{1\}^{2k_i-2-j_i}}_{k_i-1}) & (j_i > k_i-1). \end{cases}$$

Proof. By Lemma 2 and the formula (2.3), for $\Re(s) > 1$, we have

$$\begin{aligned}
\Gamma(s)\xi_{MT}(\{1\}^l, k_1, \ldots, k_r; s) &= \int_0^\infty \frac{t^{s+l-1}}{e^t-1} \prod_{i=1}^{r} \mathrm{Li}_{k_i}(1-e^{-t})\,dt \\
&= \sum_{\substack{0\leq j_1\leq 2k_1-2\\ \vdots \\ 0\leq j_r\leq 2k_r-2}} \int_0^\infty \frac{t^{s+l-1}}{e^t-1} \prod_{i=1}^{r} f(t;j_i,k_i)\,dt \\
&= \sum_{\substack{0\leq j_1\leq 2k_1-2\\ \vdots \\ 0\leq j_r\leq 2k_r-2}} a_r(\mathbf{j},\mathbf{k})\Gamma(l+b_r(\mathbf{j},\mathbf{k})+s) \\
&\quad \times \zeta_{MT}(0, \mathbf{k}(j_1,k_1), \ldots, \mathbf{k}(j_r,k_r); l+b_r(\mathbf{j},\mathbf{k})+s).
\end{aligned}$$

By the analytic continuation, we obtain the stated theorem. □

By putting $s = m+1$ in Theorem 14 and using Theorem 9, we can obtain the following relations among GMT-type MZVs with $n_{r+1} = 1$.

Theorem 15. *For $m \in \mathbb{N}$ and $k_i \geq 2$ $(1 \leq i \leq r)$, we have*

$$\begin{aligned}&\zeta_{MT}(k_1, \ldots, k_r, \{1\}^m; 1)\\ &= \sum_{\substack{0 \leq j_1 \leq 2k_1-2 \\ \vdots \\ 0 \leq j_r \leq 2k_r-2}} (b_r(\mathbf{j},\mathbf{k}) + m)! a_r(\mathbf{j},\mathbf{k}) \zeta_{MT}(0, \mathbf{k}(j_1,k_1), \ldots, \mathbf{k}(j_r,k_r); b_r(\mathbf{j},\mathbf{k}) + m + 1).\end{aligned}$$

Example 2. In the case $k_1 = 2, \ldots, k_r = 2$, Theorem 15 gives relations among MT-type MZVs. For example, if $k_1 = 2, r = 1$ then we obtain

$$\begin{aligned}&\zeta_{MT}(2, \{1\}^m; 1)\\ &= m!\zeta(2)\zeta_{MT}(0; m+1) - (m+1)!\zeta_{MT}(0,1; m+2) - m!\zeta_{MT}(0,2; m+1).\end{aligned}$$

If there exists i satisfying $k_i \geq 3$ then Theorem 15 gives relations among GMT-type MZVs with $n_{r+1} = 1$, not MT-type MZVs. For example, if $k_1 = 3, r = 1$ then we obtain

$$\begin{aligned}\zeta_{MT}(3, \{1\}^m; 1) &= m!\zeta(3)\zeta_{MT}(0; m+1) - m!\zeta(2)\zeta_{MT}(0,1; m+1) + (m+1)!\zeta_{MT}(0,(1,1); m+2)\\ &\quad + m!\zeta_{MT}(0,(2,1); m+1) + m!\zeta_{MT}(0,(1,2); m+1).\end{aligned}$$

2.2.3 Generalization of Theorem 12

We generalize Theorem 12 and give relations among GMT-type MZVs as its consequence.

Theorem 16. *With the assumption of Theorem 14, for $l \in \mathbb{Z}_{\geq 0}$, $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}_{\geq 2}^n$, $k_{n+1} \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{C}$, we have*

$$\begin{aligned}&(-1)^{k_{n+1}} \xi_{MT}((\{1\}^l, \mathbf{k}; k_{n+1}); s)\\ &= \sum_{\substack{0 \leq j_1 \leq 2k_1-2 \\ \vdots \\ 0 \leq j_n \leq 2k_n-2}} \sum_{c_1 + \cdots + c_{k_{n+1}+1} = l + b_n(\mathbf{j},\mathbf{k})} \big(l + b_n(\mathbf{j},\mathbf{k})\big)! a_n(\mathbf{j},\mathbf{k}) \binom{s + c_{k_{n+1}+1} - 1}{c_{k_{n+1}+1}}\\ &\quad \times \zeta_{MT}(0, \mathbf{k}(j_1,k_1), \ldots, \mathbf{k}(j_n,k_n); c_1 + 1, \ldots, c_{k_{n+1}} + 1, c_{k_{n+1}+1} + s)\\ &\quad - \sum_{i=1}^{k_{n+1}} (-1)^{i-1} \zeta_{MT}(\{1\}^l, k_1, \ldots, k_n; i) \zeta(\{1\}^{k_{n+1}-i}, s).\end{aligned}$$

Proof. Let $q = k_{n+1} + 1$. For $\Re(s) > 1$, let

$$J = \int_0^\infty \cdots \int_0^\infty t_q^{s-1} (t_1 + \cdots + t_q)^l \left(\prod_{i=1}^n \mathrm{Li}_{k_i}(1 - e^{-(t_1 + \cdots + t_q)}) \right) \prod_{i=1}^q \frac{dt_i}{e^{t_i + \cdots + t_q} - 1}.$$

We calculate J in two different ways. By using [26, Lemma 4], we have

$$\begin{aligned}J &= \int_0^\infty \cdots \int_0^\infty t_q^{s-1} \Big(\zeta_{MT}(\{1\}^l, k_1, \ldots, k_n; 1) - \mathrm{Li}_{\{1\}^l, k_1, \ldots, k_n; 1}(1 - e^{-(t_2 + \cdots + t_q)}) \Big)\\ &\quad \times \frac{1}{e^{t_2 + \cdots + t_q} - 1} \cdots \frac{1}{e^{t_q} - 1} \, dt_2 \cdots dt_q\\ &= \Gamma(s) \zeta_{MT}(\{1\}^l, k_1, \ldots, k_n; 1) \zeta(\{1\}^{q-2}, s)\\ &\quad - \int_0^\infty \cdots \int_0^\infty t_q^{s-1} \mathrm{Li}_{\{1\}^l, k_1, \ldots, k_n; 1}(1 - e^{-(t_2 + \cdots + t_q)})\\ &\quad \times \frac{1}{e^{t_2 + \cdots + t_q} - 1} \cdots \frac{1}{e^{t_q} - 1} \, dt_2 \cdots dt_q\\ &= \cdots\\ &= \Gamma(s) \sum_{i=1}^{q-1} (-1)^{i-1} \zeta_{MT}(\{1\}^l, k_1, \ldots, k_n; i) \zeta(\{1\}^{q-1-i}, s)\\ &\quad + (-1)^{q-1} \Gamma(s) \xi_{MT}((\{1\}^l, \mathbf{k}; q-1); s).\end{aligned}$$

On the other hand, by using Lemma 2, we have

$$J = \sum_{\substack{0\le j_1\le 2k_1-2\\ \vdots \\ 0\le j_n\le 2k_n-2}} \int_0^\infty \cdots \int_0^\infty t_q^{s-1}(t_1+\cdots+t_q)^l \times \left(\prod_{i=1}^n f(t_1+\cdots+t_q;j_i,k_i)\right)\prod_{i=1}^q \frac{dt_i}{e^{t_i+\cdots+t_q}-1}$$

$$= \sum_{\substack{0\le j_1\le 2k_1-2\\ \vdots \\ 0\le j_n\le 2k_n-2}} \int_0^\infty \cdots \int_0^\infty t_q^{s-1}(t_1+\cdots+t_q)^{l+b_n(\mathbf{j},\mathbf{k})} \times \left(\prod_{i=1}^n \frac{f(t_1+\cdots+t_q;j_i,k_i)}{(t_1+\cdots+t_q)^{b_n(\mathbf{j},\mathbf{k})}}\right)\prod_{i=1}^q \frac{dt_i}{e^{t_i+\cdots+t_q}-1}$$

$$= \sum_{\substack{0\le j_1\le 2k_1-2\\ \vdots \\ 0\le j_n\le 2k_n-2}} \sum_{c_1+\cdots+c_q=l+b_n(\mathbf{j},\mathbf{k})} \frac{(l+b_n(\mathbf{j},\mathbf{k}))!}{c_1!\cdots c_q!}\int_0^\infty \cdots \int_0^\infty t_q^{s-1}t_1^{c_1}\cdots t_q^{c_q} \times \left(\prod_{i=1}^n \frac{f(t_1+\cdots+t_q;j_i,k_i)}{(t_1+\cdots+t_q)^{b_n(\mathbf{j},\mathbf{k})}}\right)\prod_{i=1}^q \frac{dt_i}{e^{t_i+\cdots+t_q}-1}$$

$$= \sum_{\substack{0\le j_1\le 2k_1-2\\ \vdots \\ 0\le j_n\le 2k_n-2}} \sum_{c_1+\cdots+c_q=l+b_n(\mathbf{j},\mathbf{k})} \frac{(l+b_n(\mathbf{j},\mathbf{k}))!}{c_1!\cdots c_q!}a_n(\mathbf{j},\mathbf{k})\left(\prod_{i=1}^{q-1}\Gamma(c_i+1)\right) \times \Gamma(c_q+s)\zeta_{MT}(0,\mathbf{k}(j_1,k_1),\ldots,\mathbf{k}(j_n,k_n);c_1+1,\ldots,c_{q-1}+1,c_q+s)$$

$$= \sum_{\substack{0\le j_1\le 2k_1-2\\ \vdots \\ 0\le j_n\le 2k_n-2}} \sum_{c_1+\cdots+c_q=l+b_n(\mathbf{j},\mathbf{k})} \frac{(l+b_n(\mathbf{j},\mathbf{k}))!}{c_q!}a_n(\mathbf{j},\mathbf{k})\Gamma(c_q+s) \times \zeta_{MT}(0,\mathbf{k}(j_1,k_1),\ldots,\mathbf{k}(j_n,k_n);c_1+1,\ldots,c_{q-1}+1,c_q+s).$$

By the analytic continuation, we obtain the stated theorem. □

By putting $s=m+1$ in Theorem 16 and using Theorem 11, we can obtain the following relations among GMT-type MZVs.

Theorem 17. *For $m\in\mathbb{N}, k_{n+1}, l\in\mathbb{Z}_{\ge 0}$ and $k_i\ge 2$ $(1\le i\le n)$, we have*

$$(-1)^{k_{n+1}} \sum_{a_1+\cdots+a_{k_{n+1}+1}=m} \frac{1}{a_{k_{n+1}+1}!} \times \zeta_{MT}(\{1\}^{a_{k_{n+1}+1}+l},k_1,\ldots,k_n;{}_-((a_1+1,\ldots,a_{k_{n+1}}+1,2)^*))$$

$$= \sum_{\substack{0\le j_1\le 2k_1-2\\ \vdots \\ 0\le j_n\le 2k_n-2}} \sum_{c_1+\cdots+c_{k_{n+1}+1}=l+b_n(\mathbf{j},\mathbf{k})} (l+b_n(\mathbf{j},\mathbf{k}))!a_n(\mathbf{j},\mathbf{k})\binom{m+c_{k_{n+1}+1}}{c_{k_{n+1}+1}} \times \zeta_{MT}(0,\mathbf{k}(j_1,k_1),\ldots,\mathbf{k}(j_n,k_n);c_1+1,\ldots,c_{k_{n+1}}+1,c_{k_{n+1}+1}+m+1)$$

$$-\sum_{i=1}^{k_{n+1}}(-1)^{i-1}\zeta_{MT}(\{1\}^l,k_1,\ldots,k_n;i)\zeta(\{1\}^{k_{n+1}-i},m+1).$$

Example 3. In the case $k_{n+1} = 0$, Theorem 17 coincides with Theorem 15. If $k_1 = 2, \ldots, k_n = 2$ then Theorem 17 gives relations among Miyagawa-type MZVs. For example, if $m = 1, l = 0, k_1 = 2, k_2 = 1, n = 1$ then we obtain

$$\begin{aligned}
&-\zeta_{MT}(2;1,2) - \zeta_{MT}(1,2;2) \\
&= \zeta(2)\zeta_{MT}(0;1,2) - \zeta_{MT}(0,1;2,2) - 2\zeta_{MT}(0,1;1,3) \\
&\quad - \zeta_{MT}(0,2;1,2) - \zeta_{MT}(2;1)\zeta(2).
\end{aligned}$$

If there exists i satisfying $k_i \geq 3$ then Theorem 17 gives relations among GMT-type MZVs, not Miyagawa-type MZVs or GMT-type MZVs with $n_{r+1} = 1$. For example, if $m = 1, l = 0, k_1 = 3, k_2 = 1, n = 1$ then we obtain

$$\begin{aligned}
&-\zeta_{MT}(3;1,2) - \zeta_{MT}(1,3;2) \\
&= \zeta(3)\zeta_{MT}(0;1,2) - \zeta(2)\zeta_{MT}(0,1;1,2) + \zeta_{MT}(0,(1,1);2,2) \\
&+ 2\zeta_{MT}(0,(1,1);1,3) + \zeta_{MT}(0,(2,1);1,2) + \zeta_{MT}(0,(1,2);1,2) \\
&\quad - \zeta_{MT}(3;1)\zeta(2).
\end{aligned}$$

3 Iterated log-sine integrals

In this section, we introduce iterated log-sine integrals and discuss relations between iterated log-sine integrals, multiple zeta values and multiple polylogarithms. Section 3.1.1 and Section 3.2 are based on the author's paper [41] entitled "Multiple zeta values and iterated log-sine integrals" accepted by Kyushu Journal of Mathematics. Section 3.1.2, Section 3.3 and Section 3.4 are based on the author's preprint [42] entitled "Evaluation of iterated log-sine integrals in terms of multiple polylogarithms". In Section 3.1, we introduce iterated log-sine integrals and prove its fundamental properties. We also discuss the vector space spanned by iterated log-sine integrals. In Section 3.2, we give a new method of obtaining relations among multiple zeta values, which uses iterated log-sine integrals. In Section 3.3, we evaluate iterated log-sine integrals in terms of multiple zeta values and multiple polylogarithms. This evaluation makes numerical evaluation of iterated log-sine integrals possible. In Section 3.4, we give some conjectures on iterated log-sine integrals, multiple zeta values, multiple Clausen values and multiple Glaisher values. These conjectures have been checked by numerical experiments.

3.1 Iterated log-sine integrals

3.1.1 The definition and fundamental properties

The author defined the iterated log-sine integrals to study on MZVs in [41]. Iterated log-sine integrals are defined by iterated integration of the generalized log-sine integrals as follows.

Definition 8 (Iterated log-sine integrals)**.** For $\sigma \in \mathbb{R}$, $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ and $\mathbf{l} = (l_1, \ldots, l_n) \in \mathbb{Z}_{\geq 0}^n$, we define

$$\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma) = (-1)^n \int_0^{\sigma} \int_0^{\theta_n} \cdots \int_0^{\theta_2} \prod_{u=1}^{n} \theta_u^{l_u} A^{k_u-1-l_u}(\theta_u)\, d\theta_1 \cdots d\theta_n,$$

where $A(\theta) = \log|2\sin(\theta/2)|$.

If $k_u - 1 - l_u \geq 0$ for all $u \in \{1, \ldots, n\}$, then this integral converges absolutely for any $\sigma \in \mathbb{R}$. For an iterated log-sine integral $\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma)$, we call $|\mathbf{k}|$ the weight of $\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma)$ and n the depth of $\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma)$. We regard $\emptyset$ as an index of weight 0 and depth 0, and $\mathrm{Ls}_{\emptyset}^{\emptyset}(\sigma)$ as 1.

There are two fundamental properties on iterated log-sine integrals. The first proposition is as follows. We call this proposition the shuffle product formula.

Proposition 7 (Shuffle product formula)**.** *Let $\mathfrak{S}_n$ be the symmetric group of degree n. For $\mathbf{k} = (k_1, \ldots, k_n)$, $\mathbf{l} = (l_1, \ldots, l_n)$, $\mathbf{k}' = (k_{n+1}, \ldots, k_{n+n'})$, $\mathbf{l}' = (l_{n+1}, \ldots, l_{n+n'})$ and $\tau \in \mathfrak{S}_{n+n'}$, we define*

$$\tau(\mathbf{k}, \mathbf{k}') = \tau(k_1, \ldots, k_{n+n'}) = (k_{\tau(1)}, \ldots, k_{\tau(n+n')}),$$
$$\tau(\mathbf{l}, \mathbf{l}') = \tau(l_1, \ldots, l_{n+n'}) = (l_{\tau(1)}, \ldots, l_{\tau(n+n')}),$$

and

$$\mathfrak{S}_{n,n'} = \left\{ \tau \in \mathfrak{S}_{n+n'} \;\middle|\; \begin{array}{c} \tau^{-1}(1) < \cdots < \tau^{-1}(n) \\ \tau^{-1}(n+1) < \cdots < \tau^{-1}(n+n') \end{array} \right\}.$$

If $\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma)$ and $\mathrm{Ls}_{\mathbf{k}'}^{\mathbf{l}'}(\sigma)$ converges absolutely then we have

$$\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma) \cdot \mathrm{Ls}_{\mathbf{k}'}^{\mathbf{l}'}(\sigma) = \sum_{\tau \in \mathfrak{S}_{n,n'}} \mathrm{Ls}_{\tau(\mathbf{k},\mathbf{k}')}^{\tau(\mathbf{l},\mathbf{l}')}(\sigma). \tag{3.1}$$

In particular, the product of two iterated log-sine integrals at σ can be written as a $\mathbb{Q}$-linear combination of iterated log-sine integrals at σ. Moreover, this decomposition satisfies the commutativity and the associative law.

Proof. We have

$$\begin{aligned} &\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma) \cdot \mathrm{Ls}_{\mathbf{k}'}^{\mathbf{l}'}(\sigma) \\ &= (-1)^{n+n'} \int_{\substack{0<\theta_1<\cdots<\theta_n<\sigma \\ 0<\theta_{n+1}<\cdots<\theta_{n+n'}<\sigma}} \prod_{u=1}^{n+n'} \theta_u^{l_u} A^{k_u-1-l_u}(\theta_u)\, d\theta_u. \end{aligned}$$

By decomposing the domain of integration, we obtain the desired formula. The commutativity is clear by the above equation.

Let $\mathbf{k}'' = (k_{n+n'+1}, \dots, k_{n+n'+n''})$ and $\mathbf{l}'' = (l_{n+n'+1}, \dots, l_{n+n'+n''})$. Then we obtain that

$$\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma)\left(\mathrm{Ls}_{\mathbf{k}'}^{\mathbf{l}'}(\sigma)\cdot \mathrm{Ls}_{\mathbf{k}''}^{\mathbf{l}''}(\sigma)\right)$$

and

$$\left(\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma)\cdot \mathrm{Ls}_{\mathbf{k}'}^{\mathbf{l}'}(\sigma)\right)\mathrm{Ls}_{\mathbf{k}''}^{\mathbf{l}''}(\sigma)$$

are equal to

$$\begin{aligned}&(-1)^{n+n'+n''}\int_{\substack{0<\theta_1<\cdots<\theta_n<\sigma\\ 0<\theta_{n+1}<\cdots<\theta_{n+n'}<\sigma\\ 0<\theta_{n+n'+1}<\cdots<\theta_{n+n'+n''}<\sigma}}\prod_{u=1}^{n+n'+n''}\theta_u^{l_u}A^{k_u-1-l_u}(\theta_u)\,d\theta_u\\ &=\sum_{\tau\in\mathfrak{S}_{n,n',n''}}\mathrm{Ls}_{\tau(\mathbf{k},\mathbf{k}',\mathbf{k}'')}^{\tau(\mathbf{l},\mathbf{l}',\mathbf{l}'')}(\sigma),\end{aligned}$$

where

$$\begin{aligned}\tau(\mathbf{k},\mathbf{k}',\mathbf{k}'') &= (k_{\tau(1)},\dots,k_{\tau(n+n'+n'')}),\\ \tau(\mathbf{l},\mathbf{l}',\mathbf{l}'') &= (l_{\tau(1)},\dots,l_{\tau(n+n'+n'')}),\end{aligned}$$

and

$$\mathfrak{S}_{n,n',n''}=\left\{\tau\in\mathfrak{S}_{n+n'+n''}\;\middle|\;\begin{array}{l}\tau^{-1}(1)<\cdots<\tau^{-1}(n),\\ \tau^{-1}(n+1)<\cdots<\tau^{-1}(n+n'),\\ \tau^{-1}(n+n'+1)<\cdots<\tau^{-1}(n+n'+n'')\end{array}\right\}.$$

Therefore, we obtain the desired proposition. □

Example 4. We obtain

$$\begin{aligned}&\mathrm{Ls}_{1,3}^{(0,1)}(\sigma)\cdot\mathrm{Ls}_{2}^{(1)}(\sigma)\\ &=-\int_{\substack{0<\theta_1<\theta_2<\sigma\\ 0<\theta_3<\sigma}}\theta_2A(\theta_2)\theta_3\,d\theta_1d\theta_2d\theta_3\\ &=-\left(\int_{0<\theta_3<\theta_1<\theta_2<\sigma}+\int_{0<\theta_1<\theta_3<\theta_2<\sigma}+\int_{0<\theta_1<\theta_2<\theta_3<\sigma}\right)\\ &\qquad\theta_2A(\theta_2)\theta_3\,d\theta_1d\theta_2d\theta_3\\ &=\mathrm{Ls}_{2,1,3}^{(1,0,1)}(\sigma)+\mathrm{Ls}_{1,2,3}^{(0,1,1)}(\sigma)+\mathrm{Ls}_{1,3,2}^{(0,1,1)}(\sigma).\end{aligned}$$

The second proposition is as follows. We call this proposition the reduction formula.

Proposition 8 (Reduction formula)**.** *For a pair of indices $\mathbf{k}=(k_1,\dots,k_n)$ and $\mathbf{l}=(l_1,\dots,l_n)$ with $n\geq 2$, we define*

$$\begin{aligned}\mathbf{k}_{j+}&=(k_1,\dots,k_{j-1},k_j+k_{j+1},k_{j+2},\dots,k_n) && (j\in\{1,\dots,n-1\}),\\ \mathbf{k}_{j-}&=(k_1,\dots,k_{j-2},k_{j-1}+k_j,k_{j+1},\dots,k_n) && (j\in\{2,\dots,n\}),\\ \mathbf{k}_{-}&=(k_1,\dots,k_{n-1}),\end{aligned}$$

and

$$\begin{aligned}\mathbf{l}_{j+}&=(l_1,\dots,l_{j-1},k_j+l_{j+1},l_{j+2},\dots,l_n) && (j\in\{1,\dots,n-1\}),\\ \mathbf{l}_{j-}&=(l_1,\dots,l_{j-2},l_{j-1}+k_j,l_{j+1},\dots,l_n) && (j\in\{2,\dots,n\}),\\ \mathbf{l}_{-}&=(l_1,\dots,l_{n-1}).\end{aligned}$$

For $j \in \{1,\dots,n\}$, if $\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma)$ converges absolutely and $k_j - 1 - l_j = 0$, then the following formula holds:

$$\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma) = \begin{cases} -\frac{1}{k_1}\sigma^{k_1} & \text{if } n = 1, j = 1, \\ -\frac{1}{k_1}\mathrm{Ls}_{\mathbf{k}_{1+}}^{\mathbf{l}_{1+}}(\sigma) & \text{if } n \geq 2, j = 1, \\ \frac{1}{k_j}\left(\mathrm{Ls}_{\mathbf{k}_{j-}}^{\mathbf{l}_{j-}}(\sigma) - \mathrm{Ls}_{\mathbf{k}_{j+}}^{\mathbf{l}_{j+}}(\sigma)\right) & \text{if } n \geq 2, 1 < j < n, \\ \frac{1}{k_n}\left(\mathrm{Ls}_{\mathbf{k}_{n-}}^{\mathbf{l}_{n-}}(\sigma) - \sigma^{k_n}\mathrm{Ls}_{\mathbf{k}_{-}}^{\mathbf{l}_{-}}(\sigma)\right) & \text{if } n \geq 2, j = n. \end{cases} \tag{3.2}$$

Remark 3. When $n \geq 2$, if there exist two or more distinct j's in the set $\{j \mid k_j - 1 - l_j = 0, 1 \leq j \leq n\}$, then we can use the formula (3.2) with any of these j. Let $j_1, \dots, j_m$ $(j_1 < \cdots < j_m)$ be all elements of $\{j \mid k_j - 1 - l_j = 0, 1 \leq j \leq n\}$. If we use the formula (3.2) with j_a $(1 \leq a \leq m)$, then on the right hand side, the indices of the log-sine integrals have depth $n-1$ and $j_1, \dots, j_{a-1}, j_{a+1} - 1, \dots, j_m - 1$ are all elements of $\{j \mid k_j - 1 - l_j = 0, 1 \leq j \leq n-1\}$. If we choose any j_1, j_2 $(j_1 < j_2)$ in $j_1, \dots, j_m$, then the expressions given by applying the formula (3.2) first with j_1 and then with $j_2 - 1$, and those by applying the formula (3.2) first with j_2 and then with j_1 are the same. Moreover, by applying the formula (3.2) repeatedly, we can see that any $\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma)$ satisfying $k_j - 1 - l_j \geq 0$ for all $j \in \{1,\dots,n\}$ can be written as a $\mathbb{Q}$-linear combination of the products of σ^m $(m \geq 0)$ and an iterated log-sine integral at σ satisfying $k_j - 1 - l_j > 0$ for all $j \in \{1,\dots,n\}$.

Proof. We have

$$\begin{aligned} \mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma) &= (-1)^n \int_{0<\theta_1<\cdots<\theta_n<\sigma} \theta_j^{l_j}\, d\theta_j \prod_{\substack{1\leq u\leq n \\ u\neq j}} \theta_u^{l_u} A^{k_u-1-l_u}(\theta_u)\, d\theta_u \\ &= (-1)^n \int_{D_\sigma^{(j)}} \left(\frac{\theta_{j+1}^{l_j+1} - \theta_{j-1}^{l_j+1}}{l_j+1}\right) \prod_{\substack{1\leq u\leq n \\ u\neq j}} \theta_u^{l_u} A^{k_u-1-l_u}(\theta_u)\, d\theta_u \\ &= \begin{cases} -\frac{1}{k_1}\sigma^{k_1} & \text{if } n = 1, j = 1, \\ -\frac{1}{k_1}\mathrm{Ls}_{\mathbf{k}_{1+}}^{\mathbf{l}_{1+}}(\sigma) & \text{if } n \geq 2, j = 1, \\ \frac{1}{k_j}\left(\mathrm{Ls}_{\mathbf{k}_{j-}}^{\mathbf{l}_{j-}}(\sigma) - \mathrm{Ls}_{\mathbf{k}_{j+}}^{\mathbf{l}_{j+}}(\sigma)\right) & \text{if } n \geq 2, 1 < j < n, \\ \frac{1}{k_n}\left(\mathrm{Ls}_{\mathbf{k}_{n-}}^{\mathbf{l}_{n-}}(\sigma) - \sigma^{k_n}\mathrm{Ls}_{\mathbf{k}_{-}}^{\mathbf{l}_{-}}(\sigma)\right) & \text{if } n \geq 2, j = n, \end{cases} \end{aligned}$$

where the domain of integration $D_\sigma^{(j)}$ is defined by

$$D_\sigma^{(j)} = \{0 = \theta_0 < \cdots < \theta_{j-1} < \theta_{j+1} < \cdots < \theta_{n+1} = \sigma\}.$$

Therefore, we obtain the desired proposition. □

Example 5. We obtain

$$\mathrm{Ls}_{1,3}^{(0,1)}(\sigma) = -\mathrm{Ls}_4^{(2)}(\sigma), \quad \mathrm{Ls}_2^{(1)}(\sigma) = -\frac{\sigma^2}{2}.$$

By applying the formula (3.2) to $\mathrm{Ls}_{2,1,3}^{(1,0,1)}(\sigma)$ with $j = 1$ and $j = 2$, we obtain

$$\mathrm{Ls}_{2,1,3}^{(1,0,1)}(\sigma) = -\frac{1}{2}\mathrm{Ls}_{3,3}^{(2,1)}(\sigma), \quad \mathrm{Ls}_{2,1,3}^{(1,0,1)}(\sigma) = \mathrm{Ls}_{3,3}^{(2,1)}(\sigma) - \mathrm{Ls}_{2,4}^{(1,2)}(\sigma),$$

respectively. Since we obtain

$$\mathrm{Ls}_{3,3}^{(2,1)}(\sigma) = -\frac{1}{3}\mathrm{Ls}_6^{(4)}(\sigma), \quad \mathrm{Ls}_{2,4}^{(1,2)}(\sigma) = -\frac{1}{2}\mathrm{Ls}_6^{(4)}(\sigma),$$

we also obtain

$$\mathrm{Ls}_{2,1,3}^{(1,0,1)}(\sigma) = \frac{1}{6}\mathrm{Ls}_6^{(4)}(\sigma),$$

independently of the choice of order.

In the same way, we can obtain

$$\mathrm{Ls}_{1,2,3}^{(0,1,1)}(\sigma)=\frac{1}{3}\mathrm{Ls}_6^{(4)}(\sigma),\quad \mathrm{Ls}_{1,3,2}^{(0,1,1)}(\sigma)=-\frac{1}{2}\mathrm{Ls}_6^{(4)}(\sigma)+\frac{\sigma^2}{2}\mathrm{Ls}_4^{(2)}(\sigma).$$

By the shuffle product formula and the reduction formula, we obtain the following theorem.

Theorem 18. *The product of iterated log-sine integrals at σ satisfying $k_j-1-l_j\geq 0$ for all $j\in\{1,\dots,n\}$ can be written as a $\mathbb{Q}$-linear combination of the products of σ^m $(m\geq 0)$ and an iterated log-sine integral at σ satisfying $k_j-1-l_j>0$ for all $j\in\{1,\dots,n\}$. Moreover, this expression does not depend on the order of applying the formula (3.1) and the formula (3.2).*

Proof. We prove only that the computations by using the formula (3.1) and by using the formula (3.2) is commutative. It is enough to show that when $k_j-1-l_j=0$, the expressions given by applying first the formula (3.1) and then the formula (3.2), and those by applying first the formula (3.2) and then the formula (3.1) are the same. We assume $1<j<n$. Also in the case $j=1$ and $j=n$, we can prove it in the same way. By applying the formula (3.1) to $\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma)\cdot\mathrm{Ls}_{\mathbf{k}'}^{\mathbf{l}'}(\sigma)$, we have

$$\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma)\cdot\mathrm{Ls}_{\mathbf{k}'}^{\mathbf{l}'}(\sigma)=\sum_{\tau\in\mathfrak{S}_{n,n'}}\mathrm{Ls}_{\tau(\mathbf{k},\mathbf{k}')}^{\tau(\mathbf{l},\mathbf{l}')}(\sigma).$$

Here, we define

$$H_j=\left\{(\mathbf{h}_1,\mathbf{h}_2,\mathbf{h}_3,\mathbf{h}_4)\;\middle|\;\substack{1\leq a<b<c\leq n+n'\\ \tau\in S_{n,n'}\\ \mathbf{h}_1=(\tau(1),\dots,\tau(a-1))\\ \tau(a)=j-1\\ \mathbf{h}_2=(\tau(a+1),\dots,\tau(b-1))\\ \tau(b)=j\\ \mathbf{h}_3=(\tau(b+1),\dots,\tau(c-1)\\ \tau(c)=j+1\\ \mathbf{h}_4=(\tau(c+1),\dots,\tau(n+n'))}\right\},$$

$$H_j'=\{(\mathbf{h}_1,\mathbf{h}_4)\mid(\mathbf{h}_1,\mathbf{h}_2,\mathbf{h}_3,\mathbf{h}_4)\in H_j\},$$

and for fixed $(\mathbf{h}_1,\mathbf{h}_4)\in H_j'$,

$$H_j^{(\mathbf{h}_1,\mathbf{h}_4)}=\{(\mathbf{h}_2',\mathbf{h}_3')\mid(\mathbf{h}_1,\mathbf{h}_2',\mathbf{h}_3',\mathbf{h}_4)\in H_j\}.$$

Moreover, for $\mathbf{h}=(h_1,\dots,h_m)$, we define

$$\mathbf{l}(\mathbf{h})=(l_{h_1},\dots,l_{h_m}),\quad \mathbf{k}(\mathbf{h})=(k_{h_1},\dots,k_{h_m}).$$

Then we have

$$\sum_{\tau\in\mathfrak{S}_{n,n'}}\mathrm{Ls}_{\tau(\mathbf{k},\mathbf{k}')}^{\tau(\mathbf{l},\mathbf{l}')}(\sigma)=\sum_{(\mathbf{h}_1,\mathbf{h}_4)\in H_j'}\sum_{(\mathbf{h}_2,\mathbf{h}_3)\in H_j^{(\mathbf{h}_1,\mathbf{h}_4)}}\mathrm{Ls}_{\mathbf{k}(\mathbf{h}_1,j-1,\mathbf{h}_2,j,\mathbf{h}_3,j+1,\mathbf{h}_4)}^{\mathbf{l}(\mathbf{h}_1,j-1,\mathbf{h}_2,j,\mathbf{h}_3,j+1,\mathbf{h}_4)}(\sigma).$$

Here, if we regard $(\mathbf{h}_2,\mathbf{h}_3)\in\mathbb{N}^{b-a-1}\times\mathbb{N}^{c-b-1}$ as $(\mathbf{h}_2,\mathbf{h}_3)\in\mathbb{N}^{c-a-2}$, then all $(\mathbf{h}_2,\mathbf{h}_3)\in H_j^{(\mathbf{h}_1,\mathbf{h}_4)}$ can be regarded as the same $(\mathbf{h}_2,\mathbf{h}_3)\in\mathbb{N}^{c-a-2}$. Therefore we denote such $(\mathbf{h}_2,\mathbf{h}_3)\in\mathbb{N}^{c-a-2}$ by $\mathbf{h}_{2,3}\in\mathbb{N}^{c-a-2}$. We note that $\mathbf{h}_{2,3}$ is uniquely determined for $(\mathbf{h}_1,\mathbf{h}_4)\in H_j'$. By applying the formula (3.2) with j, the above equation is equal to

$$\begin{aligned}&\sum_{(\mathbf{h}_1,\mathbf{h}_4)\in H_j'}\frac{1}{k_j}\Big(\mathrm{Ls}_{\mathbf{k}(\mathbf{h}_1),k_{j-1}+k_j,\mathbf{k}(\mathbf{h}_{2,3},j+1,\mathbf{h}_4)}^{\mathbf{l}(\mathbf{h}_1),l_{j-1}+k_j,\mathbf{l}(\mathbf{h}_{2,3},j+1,\mathbf{h}_4)}(\sigma)\\&\qquad\qquad-\mathrm{Ls}_{\mathbf{k}(\mathbf{h}_1,j-1,\mathbf{h}_{2,3}),k_j+k_{j+1},\mathbf{k}(\mathbf{h}_4)}^{\mathbf{l}(\mathbf{h}_1,j-1,\mathbf{h}_{2,3}),k_j+l_{j+1},\mathbf{l}(\mathbf{h}_4)}(\sigma)\Big)\\&=\frac{1}{k_j}\left(\sum_{\tau\in\mathfrak{S}_{n-1,n'}}\mathrm{Ls}_{\tau(\mathbf{k}_{j-},\mathbf{k}')}^{\tau(\mathbf{l}_{j-},\mathbf{l}')}(\sigma)-\sum_{\tau\in\mathfrak{S}_{n-1,n'}}\mathrm{Ls}_{\tau(\mathbf{k}_{j+},\mathbf{k}')}^{\tau(\mathbf{l}_{j+},\mathbf{l}')}(\sigma)\right).\end{aligned}$$

On the other hand, by applying the formula (3.2) to $\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma)\cdot\mathrm{Ls}_{\mathbf{k}'}^{\mathbf{l}'}(\sigma)$, we have

$$\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma)\cdot\mathrm{Ls}_{\mathbf{k}'}^{\mathbf{l}'}(\sigma)=\frac{1}{k_j}\left(\mathrm{Ls}_{\mathbf{k}_{j-}}^{\mathbf{l}_{j-}}(\sigma)-\mathrm{Ls}_{\mathbf{k}_{j+}}^{\mathbf{l}_{j+}}(\sigma)\right)\cdot\mathrm{Ls}_{\mathbf{k}'}^{\mathbf{l}'}(\sigma).$$

By the formula (3.1), we have

$$\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma)\cdot\mathrm{Ls}_{\mathbf{k}'}^{\mathbf{l}'}(\sigma)=\frac{1}{k_j}\left(\sum_{\tau\in\mathfrak{S}_{n-1,n'}}\mathrm{Ls}_{\tau(\mathbf{k}_{j-},\mathbf{k}')}^{\tau(\mathbf{l}_{j-},\mathbf{l}')}(\sigma)-\sum_{\tau\in\mathfrak{S}_{n-1,n'}}\mathrm{Ls}_{\tau(\mathbf{k}_{j+},\mathbf{k}')}^{\tau(\mathbf{l}_{j+},\mathbf{l}')}(\sigma)\right).$$

Therefore the proof is complete. □

Example 6. For example, by applying the formula (3.1) to $\mathrm{Ls}_{1,3}^{(0,1)}(\sigma)\cdot\mathrm{Ls}_{2}^{(1)}(\sigma)$, we have

$$\mathrm{Ls}_{1,3}^{(0,1)}(\sigma)\cdot\mathrm{Ls}_{2}^{(1)}(\sigma)=\mathrm{Ls}_{2,1,3}^{(1,0,1)}(\sigma)+\mathrm{Ls}_{1,2,3}^{(0,1,1)}(\sigma)+\mathrm{Ls}_{1,3,2}^{(0,1,1)}(\sigma)$$

(see Example 4). By applying the formula (3.2) to each term on the right hand side, we obtain

$$\mathrm{Ls}_{1,3}^{(0,1)}(\sigma)\cdot\mathrm{Ls}_{2}^{(1)}(\sigma)=\frac{\sigma^2}{2}\mathrm{Ls}_{4}^{(2)}(\sigma)$$

(see Example 5). On the other hand, by applying the formula (3.2) to $\mathrm{Ls}_{1,3}^{(0,1)}(\sigma)$ and $\mathrm{Ls}_{2}^{(1)}(\sigma)$, we obtain the same result:

$$\mathrm{Ls}_{1,3}^{(0,1)}(\sigma)\cdot\mathrm{Ls}_{2}^{(1)}(\sigma)=-\mathrm{Ls}_{4}^{(2)}(\sigma)\cdot-\frac{\sigma^2}{2}=\frac{\sigma^2}{2}\mathrm{Ls}_{4}^{(2)}(\sigma)$$

(see Example 5).

3.1.2 On the vector space spanned by iterated log-sine integrals

We define $\mathcal{L}_k^o$ as the vector space spanned by all iterated log-sine integrals at $\pi/3$ whose weight is k and $|\mathbf{k}-\mathbf{1}_n-\mathbf{l}|$ is an odd number, where we note that $|\mathbf{k}-\mathbf{1}_n-\mathbf{l}|$ is the sum of exponents of $A(\theta):=\log|2\sin(\theta/2)|$. Similarly, we define $\mathcal{L}_k^e$ as the vector space spanned by all iterated log-sine integrals at $\pi/3$ whose weight is k and $|\mathbf{k}-\mathbf{1}_n-\mathbf{l}|$ is an even number. Namely,

$$\mathcal{L}_k^o=\sum_{\substack{k_1+\cdots+k_n=k,0\le n\le k\\ k_1,\dots,k_n\ge1,l_1,\dots,l_n\ge0\\ k_1-1-l_1,\dots,k_n-1-l_n\ge0\\ \sum_{u=1}^n k_u-1-l_u:\mathrm{odd}}}\mathbb{Q}\cdot\mathrm{Ls}_{k_1,\dots,k_n}^{(l_1,\dots,l_n)}(\pi/3),$$

$$\mathcal{L}_k^e=\sum_{\substack{k_1+\cdots+k_n=k,0\le n\le k\\ k_1,\dots,k_n\ge1,l_1,\dots,l_n\ge0\\ k_1-1-l_1,\dots,k_n-1-l_n\ge0\\ \sum_{u=1}^n k_u-1-l_u:\mathrm{even}}}\mathbb{Q}\cdot\mathrm{Ls}_{k_1,\dots,k_n}^{(l_1,\dots,l_n)}(\pi/3),$$

where we understand $\mathrm{Ls}_{\emptyset}^{\emptyset}(\sigma)$ satisfies $\sum_{u=1}^n k_u-1-l_u=0$.

We count the number of all iterated log-sine integrals whose weight is k and $|\mathbf{k}-\mathbf{1}_n-\mathbf{l}|$ is an odd or even number less than or equal to d. For non-negative integers k, d and a fixed real number σ, we define

$$L_{k,d}^o(\sigma)=\left\{\mathrm{Ls}_{k_1,\dots,k_n}^{(l_1,\dots,l_n)}(\sigma)\;\middle|\;\begin{array}{c}k_1+\cdots+k_n=k,0\le n\le k\\ k_1,\dots,k_n\ge1,l_1,\dots,l_n\ge0\\ k_1-1-l_1,\dots,k_n-1-l_n\ge0\\ \displaystyle\sum_{u=1}^n k_u-1-l_u\le d:odd.\end{array}\right\}$$

and define $L_{k,d}^e(\sigma)$ analogously for the even case. Then, the number of elements of those sets is evaluated as follows.

Theorem 19. *We have*

$$\left|L^o_{0,d}(\sigma)\right| = \left|L^o_{1,d}(\sigma)\right| = 0, \qquad \left|L^e_{0,d}(\sigma)\right| = \left|L^e_{1,d}(\sigma)\right| = 1, \tag{3.3}$$

$$\left|L^o_{2,d}(\sigma)\right| = \begin{cases} 0 & (d=0), \\ 1 & (d \geq 1), \end{cases} \qquad \left|L^e_{2,d}(\sigma)\right| = 2, \tag{3.4}$$

$$\left|L^o_{k,0}(\sigma)\right| = 0, \qquad \left|L^e_{k,0}(\sigma)\right| = \begin{cases} 1 & (k=0), \\ 2^{k-1} & (k \geq 1), \end{cases} \tag{3.5}$$

and

$$\begin{aligned} \left|L^o_{k,d}(\sigma)\right| &= 2\left|L^o_{k-1,d}(\sigma)\right| + \left|L^e_{k-1,d-1}(\sigma)\right| - \left|L^e_{k-2,d-1}(\sigma)\right| \quad (k \geq 3, d \geq 1), \\ \left|L^e_{k,d}(\sigma)\right| &= 2\left|L^e_{k-1,d}(\sigma)\right| + \left|L^o_{k-1,d-1}(\sigma)\right| - \left|L^o_{k-2,d-1}(\sigma)\right| \quad (k \geq 3, d \geq 1). \end{aligned} \tag{3.6}$$

In particular, we have

$$\begin{aligned} &\left|L^o_{0,0}(\sigma)\right| = 0, \quad \left|L^e_{0,0}(\sigma)\right| = 1, \\ &\left|L^o_{k,k}(\sigma)\right| = (F_{2k} - F_k)/2 \quad (k \geq 1), \\ &\left|L^e_{k,k}(\sigma)\right| = (F_{2k} + F_k)/2 \quad (k \geq 1), \end{aligned} \tag{3.7}$$

where F_k represents the k-th Fibonacci number.

Proof. We have (3.3) since iterated log-sine integrals of weight 0 is only $\mathrm{Ls}^{\emptyset}_{\emptyset}(\sigma)$ and iterated log-sine integrals of weight 1 is only $\mathrm{Ls}^{(0)}_1(\sigma)$. We also have (3.4) since iterated log-sine integrals of weight 2 are $\mathrm{Ls}^{(0)}_2(\sigma)$, $\mathrm{Ls}^{(1)}_2(\sigma)$ and $\mathrm{Ls}^{(0,0)}_{1,1}(\sigma)$. The number of elements of $L^e_{k,0}(\sigma)$ is equal to the number of indices of weight k because iterated log-sine integrals satisfying $|\mathbf{k} - \mathbf{1}_n - \mathbf{l}| = 0$ are of the form $\mathrm{Ls}^{(k_1-1,\ldots,k_n-1)}_{k_1,\ldots,k_n}(\sigma)$. Therefore, we obtain (3.5).

For $k \geq 3$ and $d \geq 1$, we have

$$\begin{aligned} \left|L^o_{k,d}(\sigma)\right| = &\left|\left\{\mathrm{Ls}^{(l_1,\ldots,l_n,0)}_{k_1,\ldots,k_n,1}(\sigma) \,\middle|\, \mathrm{Ls}^{(l_1,\ldots,l_n)}_{k_1,\ldots,k_n}(\sigma) \in L^o_{k-1,d}(\sigma)\right\}\right| \\ &+ \left|\left\{\mathrm{Ls}^{(l_1,\ldots,l_n+1)}_{k_1,\ldots,k_n+1}(\sigma) \,\middle|\, \mathrm{Ls}^{(l_1,\ldots,l_n)}_{k_1,\ldots,k_n}(\sigma) \in L^o_{k-1,d}(\sigma)\right\}\right| \\ &+ \left|\left\{\mathrm{Ls}^{(l_1,\ldots,l_n)}_{k_1,\ldots,k_n+1}(\sigma) \,\middle|\, \mathrm{Ls}^{(l_1,\ldots,l_n)}_{k_1,\ldots,k_n}(\sigma) \in L^e_{k-1,d-1}(\sigma)\right\}\right| \\ &- \left|\left\{\mathrm{Ls}^{(l_1,\ldots,l_n+1)}_{k_1,\ldots,k_n+2}(\sigma) \,\middle|\, \mathrm{Ls}^{(l_1,\ldots,l_n)}_{k_1,\ldots,k_n}(\sigma) \in L^e_{k-2,d-1}(\sigma)\right\}\right|, \end{aligned}$$

and the corresponding formula holds for $|L^e_{k,d}(\sigma)|$. Therefore, we obtain (3.6).

By (3.6), for $k \geq 3$, we obtain

$$\begin{aligned} &\left|L^e_{k,k}(\sigma)\right| + \left|L^o_{k,k}(\sigma)\right| \\ &= 3\left(\left|L^e_{k-1,k-1}(\sigma)\right| + \left|L^o_{k-1,k-1}(\sigma)\right|\right) - \left(\left|L^e_{k-2,k-2}(\sigma)\right| + \left|L^o_{k-2,k-2}(\sigma)\right|\right). \end{aligned}$$

By adding $\left|L^e_{1,1}(\sigma)\right| + \left|L^o_{1,1}(\sigma)\right| = 1$ and $\left|L^e_{2,2}(\sigma)\right| + \left|L^o_{2,2}(\sigma)\right| = 3$ to this, we have

$$\left|L^e_{k,k}(\sigma)\right| + \left|L^o_{k,k}(\sigma)\right| = F_{2k} \quad (k \geq 1). \tag{3.8}$$

On the other hand, by (3.6), for $k \geq 3$ we have

$$\begin{aligned} &\left|L^e_{k,k}(\sigma)\right| - \left|L^o_{k,k}(\sigma)\right| \\ &= \left(\left|L^e_{k-1,k-1}(\sigma)\right| - \left|L^o_{k-1,k-1}(\sigma)\right|\right) + \left(\left|L^e_{k-2,k-2}(\sigma)\right| - \left|L^o_{k-2,k-2}(\sigma)\right|\right). \end{aligned}$$

By adding $\left|L^e_{1,1}(\sigma)\right| - \left|L^o_{1,1}(\sigma)\right| = 1$ and $\left|L^e_{2,2}(\sigma)\right| - \left|L^o_{2,2}(\sigma)\right| = 1$ to this, we have

$$\left|L^e_{k,k}(\sigma)\right| - \left|L^o_{k,k}(\sigma)\right| = F_k, \quad (k \geq 1). \tag{3.9}$$

From (3.8) and (3.9), (3.7) follows. □

By Theorem 19, we have $\dim \mathcal{L}_k^o \le (F_{2k} - F_k)/2$ and $\dim \mathcal{L}_k^e \le (F_{2k} + F_k)/2$ for $k \ge 1$. However, these evaluations can be improved by the reduction formula. For non-negative integers k, d and a fixed real number σ, we define

$$M_{k,d}^o(\sigma) = \left\{ \sigma^m \operatorname{Ls}_{k_1,\dots,k_n}^{(l_1,\dots,l_n)}(\sigma) \;\middle|\; \begin{array}{c} m + k_1 + \cdots + k_n = k, 0 \le n \le k, \\ m \ge 0, k_1, \dots, k_n \ge 2, l_1, \dots, l_n \ge 0, \\ k_1 - 1 - l_1, \dots, k_n - 1 - l_n \ge 1, \\ \displaystyle\sum_{u=1}^{n} k_u - 1 - l_u \le d : \text{odd}. \end{array} \right\},$$

and define $M_{k,d}^e(\sigma)$ analogously for the even case. Then, the following theorem holds.

Theorem 20. *We have*

$$\begin{aligned} \operatorname{span}_{\mathbb{Q}}(L_{k,d}^o(\sigma)) &= \operatorname{span}_{\mathbb{Q}}(M_{k,d}^o(\sigma)), \\ \operatorname{span}_{\mathbb{Q}}(L_{k,d}^e(\sigma)) &= \operatorname{span}_{\mathbb{Q}}(M_{k,d}^e(\sigma)). \end{aligned}$$

Proof. By applying the reduction formula, repeatedly, we can see that elements of $L_{k,d}^o(\sigma)$ and $L_{k,d}^e(\sigma)$ belong to $\operatorname{span}_{\mathbb{Q}}(M_{k,d}^o(\sigma))$ and $\operatorname{span}_{\mathbb{Q}}(M_{k,d}^e(\sigma))$, respectively. Conversely, by $\sigma^m = -m \operatorname{Ls}_m^{(m-1)}(\sigma)$ and the shuffle product formula, elements of $M_{k,d}^o(\sigma)$ and $M_{k,d}^e(\sigma)$ belong to $\operatorname{span}_{\mathbb{Q}}(L_{k,d}^o(\sigma))$ and $\operatorname{span}_{\mathbb{Q}}(L_{k,d}^e(\sigma))$. □

The number of elements of $M_{k,d}^o(\sigma)$ and $M_{k,d}^e(\sigma)$ are evaluated as follows.

Theorem 21. *We have*

$$\left|M_{0,d}^o(\sigma)\right| = \left|M_{1,d}^o(\sigma)\right| = 0, \qquad \left|M_{0,d}^e(\sigma)\right| = \left|M_{1,d}^e(\sigma)\right| = 1, \tag{3.10}$$

and

$$\begin{aligned} \left|M_{k,d}^o(\sigma)\right| &= \left|M_{k-1,d}^o(\sigma)\right| + \left|M_{k-1,d-1}^e(\sigma)\right| \quad (k \ge 2, d \ge 1), \\ \left|M_{k,d}^e(\sigma)\right| &= \left|M_{k-1,d}^e(\sigma)\right| + \left|M_{k-1,d-1}^o(\sigma)\right| \quad (k \ge 2, d \ge 1). \end{aligned} \tag{3.11}$$

In particular, we have

$$\begin{aligned} &\left|M_{0,0}^o(\sigma)\right| = \left|M_{1,1}^o(\sigma)\right| = 0, \quad \left|M_{0,0}^e(\sigma)\right| = \left|M_{1,1}^e(\sigma)\right| = 1, \\ &\left|M_{k,k}^o(\sigma)\right| = \left|M_{k,k}^e(\sigma)\right| = 2^{k-2} \quad (k \ge 2). \end{aligned} \tag{3.12}$$

Proof. The equation (3.10) is clear by the definition. In order to prove (3.11), we define

$$N_{k,d}^o(\sigma) = \left\{ \operatorname{Ls}_{k_1,\dots,k_n}^{(l_1,\dots,l_n)}(\sigma) \;\middle|\; \begin{array}{c} k_1 + \cdots + k_n = k, 0 \le n \le k, \\ k_1, \dots, k_n \ge 2, l_1, \dots, l_n \ge 0, \\ k_1 - 1 - l_1, \dots, k_n - 1 - l_n \ge 1, \\ \displaystyle\sum_{u=1}^{n} k_u - 1 - l_u \le d : \text{odd}. \end{array} \right\}$$

and define $N_{k,d}^e(\sigma)$ analogously for the even case. Then, we have

$$\begin{aligned} \left|M_{k,d}^o(\sigma)\right| &= \sum_{i=0}^{k} \left|N_{i,d}^o(\sigma)\right|, \\ \left|M_{k,d}^e(\sigma)\right| &= \sum_{i=0}^{k} \left|N_{i,d}^e(\sigma)\right|. \end{aligned}$$

Here, for $k \geq 3$ and $d \geq 1$, we have

$$\begin{aligned}
\left|N^o_{k,d}(\sigma)\right| &= \left|\left\{\mathrm{Ls}^{(l_1,\ldots,l_n+1)}_{k_1,\ldots,k_n+1}(\sigma) \;\middle|\; \mathrm{Ls}^{(l_1,\ldots,l_n)}_{k_1,\ldots,k_n}(\sigma) \in N^o_{k-1,d}(\sigma)\right\}\right| \\
&\quad + \left|\left\{\mathrm{Ls}^{(l_1,\ldots,l_n)}_{k_1,\ldots,k_n+1}(\sigma) \;\middle|\; \mathrm{Ls}^{(l_1,\ldots,l_n)}_{k_1,\ldots,k_n}(\sigma) \in N^e_{k-1,d-1}(\sigma)\right\}\right| \\
&\quad + \left|\left\{\mathrm{Ls}^{(l_1,\ldots,l_n,0)}_{k_1,\ldots,k_n,2}(\sigma) \;\middle|\; \mathrm{Ls}^{(l_1,\ldots,l_n)}_{k_1,\ldots,k_n}(\sigma) \in N^e_{k-2,d-1}(\sigma)\right\}\right| \\
&\quad - \left|\left\{\mathrm{Ls}^{(l_1,\ldots,l_n+1)}_{k_1,\ldots,k_n+2}(\sigma) \;\middle|\; \mathrm{Ls}^{(l_1,\ldots,l_n)}_{k_1,\ldots,k_n}(\sigma) \in N^e_{k-2,d-1}(\sigma)\right\}\right| \\
&= \left|N^o_{k-1,d}(\sigma)\right| + \left|N^e_{k-1,d-1}(\sigma)\right|.
\end{aligned}$$

Similarly, for $k \geq 3$ and $d \geq 1$, we obtain

$$\left|N^e_{k,d}(\sigma)\right| = \left|N^e_{k-1,d}(\sigma)\right| + \left|N^o_{k-1,d-1}(\sigma)\right|.$$

Therefore, for $k \geq 2$ and $d \geq 1$, we have

$$\begin{aligned}
\left|M^o_{k,d}(\sigma)\right| &= \sum_{i=2}^{k-1}\left(\left|N^o_{i,d}(\sigma)\right| + \left|N^e_{i,d-1}(\sigma)\right|\right) + \left|N^o_{2,d}(\sigma)\right| + \left|N^o_{1,d}(\sigma)\right| + \left|N^o_{0,d}(\sigma)\right| \\
&= \left|M^o_{k-1,d}(\sigma)\right| + \left|M^e_{k-1,d-1}(\sigma)\right| - \left|N^e_{1,d-1}(\sigma)\right| - \left|N^e_{0,d-1}(\sigma)\right| + \left|N^o_{2,d}(\sigma)\right| \\
&= \left|M^o_{k-1,d}(\sigma)\right| + \left|M^e_{k-1,d-1}(\sigma)\right|,
\end{aligned}$$

and

$$\begin{aligned}
\left|M^e_{k,d}(\sigma)\right| &= \sum_{i=2}^{k-1}\left(\left|N^e_{i,d}(\sigma)\right| + \left|N^o_{i,d-1}(\sigma)\right|\right) + \left|N^e_{2,d}(\sigma)\right| + \left|N^e_{1,d}(\sigma)\right| + \left|N^e_{0,d}(\sigma)\right| \\
&= \left|M^e_{k-1,d}(\sigma)\right| + \left|M^o_{k-1,d-1}(\sigma)\right| - \left|N^o_{1,d-1}(\sigma)\right| - \left|N^o_{0,d-1}(\sigma)\right| + \left|N^e_{2,d}(\sigma)\right| \\
&= \left|M^e_{k-1,d}(\sigma)\right| + \left|M^o_{k-1,d-1}(\sigma)\right|
\end{aligned}$$

which prove (3.11). By (3.11), for $k \geq 2$, we obtain

$$\left|M^e_{k,k}(\sigma)\right| + \left|M^o_{k,k}(\sigma)\right| = 2\left(\left|M^e_{k-1,k-1}(\sigma)\right| + \left|M^o_{k-1,k-1}(\sigma)\right|\right).$$

By adding $\left|M^e_{1,1}(\sigma)\right| + \left|M^o_{1,1}(\sigma)\right| = 1$ to this, for $k \geq 1$, we obtain

$$\left|M^e_{k,k}(\sigma)\right| + \left|M^o_{k,k}(\sigma)\right| = 2^{k-1}.$$

On the other hand, by (3.11), for $k \geq 2$, we obtain

$$\left|M^e_{k,k}(\sigma)\right| - \left|M^o_{k,k}(\sigma)\right| = 0.$$

Therefore, we can obtain (3.12). □

By Theorem 21, we can obtain $\dim \mathcal{L}^o_k \leq 2^{k-2}$ and $\dim \mathcal{L}^e_k \leq 2^{k-2}$ for $k \geq 2$.

3.2 A method to obtain relations among multiple zeta values

3.2.1 A method to obtain relations among multiple zeta values

As one of applications of iterated log-sine integrals to multiple zeta values, we can obtain relations among multiple zeta values by using iterated log-sine integrals. Iterated log-sine integrals were first introduced for this study. We introduce the method here. We define $\mathbf{k}^{(0)} = \mathbf{k}$,

$$\mathbf{k}^{(1)} = \begin{cases} (k_1, \ldots, k_{n-1}, k_n - 1) & \text{if } k_n > 1, \\ (k_1, \ldots, k_{n-1}) & \text{if } k_n = 1, \\ \emptyset & \text{if } n = 1, k_n = 1, \end{cases}$$

and $\mathbf{k}^{(m)} = (\mathbf{k}^{(m-1)})^{(1)}$ $(m > 1)$. The following theorem on relations between multiple zeta values and multiple polylogarithms is known.

Theorem 22 (Borwein-Broadhurst-Kamnitzer [9, theorem 4.4])**.** *For* $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ *with* $k_n > 1$*, we have*

$$\zeta(\mathbf{k}) = \sum_{m=0}^{|\mathbf{k}|} \mathrm{Li}_{\mathbf{k}^{(m)}}(e^{\frac{\pi}{3}i})\overline{\mathrm{Li}_{(\mathbf{k}^*)^{(|\mathbf{k}|-m)}}(e^{\frac{\pi}{3}i})}. \tag{3.13}$$

Next, we prove the following theorem which connects multiple polylogarithms and iterated log-sine integrals.

Theorem 23. *For* $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ *and* $0 \le \sigma < \pi$*, we have*

$$\mathrm{Li}_{\mathbf{k}}(1 - e^{\pm i\sigma}) = (\mp i)^n \int_{0<\theta_1<\cdots<\theta_n<\theta_{n+1}=\sigma} \prod_{u=1}^{n} \frac{\left(A(\theta_{u+1}) - A(\theta_u) \pm \frac{i\theta_{u+1}}{2} \mp \frac{i\theta_u}{2}\right)^{k_u-1}}{(k_u-1)!} d\theta_u.$$

In particular,

$$\mathrm{Li}_{\mathbf{k}}(e^{\frac{\pi}{3}i}) = i^n \int_{0<\theta_1<\cdots<\theta_n<\theta_{n+1}=\frac{\pi}{3}} \prod_{u=1}^{n} \frac{\left(A(\theta_{u+1}) - A(\theta_u) - \frac{i\theta_{u+1}}{2} + \frac{i\theta_u}{2}\right)^{k_u-1}}{(k_u-1)!} d\theta_u,$$

since $e^{\pi i/3} = 1 - e^{-\pi i/3}$.

We can see that any $\mathrm{Li}_{\mathbf{k}}(1 - e^{\pm i\sigma})$ can be written as a $\mathbb{Q}(i)$-linear combination of iterated log-sine integrals at σ satisfying $k_u - 1 - l_u \ge 0$ for all $u \in \{1, \ldots, n\}$. In particular, $\mathrm{Li}_{\mathbf{k}}(e^{\pi i/3})$ belongs to $\mathcal{L}^o_{|\mathbf{k}|} + i\mathcal{L}^e_{|\mathbf{k}|}$ when $|\mathbf{k}|$ is odd, and belongs to $\mathcal{L}^e_{|\mathbf{k}|} + i\mathcal{L}^o_{|\mathbf{k}|}$ when $|\mathbf{k}|$ is even.

Proof. For $z \in \mathbb{C} \setminus [1, \infty)$, the multiple polylogarithm is written as

$$\mathrm{Li}_{\mathbf{k}}(z) = \int_0^z \frac{\log^{k_n-1}(z/z_n)}{(k_n-1)!} \frac{dz_n}{1-z_n} \int_0^{z_n} \frac{\log^{k_{n-1}-1}(z_n/z_{n-1})}{(k_{n-1}-1)!} \frac{dz_{n-1}}{1-z_{n-1}} \cdots \int_0^{z_2} \frac{\log^{k_1-1}(z_2/z_1)}{(k_1-1)!} \frac{dz_1}{1-z_1}.$$

We put $z = 1 - e^{\pm i\sigma}$ and $z_u = 1 - e^{\pm i\theta_u}$ $(u \in \{1, \ldots, n\})$. Then we have

$$\mathrm{Li}_{\mathbf{k}}(1 - e^{\pm i\sigma}) = (\mp i)^n \int_{0<\theta_1<\cdots<\theta_n<\theta_{n+1}=\sigma} \prod_{u=1}^{n} \frac{\left(\log(1 - e^{\pm i\theta_{u+1}}) - \log(1 - e^{\pm i\theta_u})\right)^{k_u-1}}{(k_u-1)!} d\theta_u.$$

Finally, we use the formula

$$\log(1 - e^{\pm i\theta}) = A(\theta) \pm i\frac{\theta - \pi}{2} \qquad (0 < \theta < 2\pi),$$

and the proof is complete. ☐

By applying Theorem 23 and Theorem 18 to the right hand side of the formula in Theorem 22, we see that any multiple zeta value can be written as a $\mathbb{Q}(i)$-linear combination of products of a power of π and an iterated log-sine integral at $\pi/3$ satisfying $k_u - 1 - l_u > 0$ for all $u \in \{1, \ldots, n\}$. Note that in the case $\mathbf{k} = (\{1\}^{a-1}, b+1)$, the expression obtained by this calculation is the same as (1.17). After writing the right hand side of (3.13) in terms of iterated log-sine integrals, by taking the real part of it, we can obtain the iterated log-sine integral expression of multiple zeta values. In addition, by taking the imaginary part, we can obtain relations among iterated log-sine integrals. Therefore, by applying these relations to the iterated log-sine integral expression of multiple zeta values, we obtain relations among multiple zeta values.

3.2.2 Examples

We give some examples. First, we calculate $\zeta(2)$. By using Theorem 22, Theorem 23 and Theorem 18, we obtain

$$\begin{aligned}\zeta(2) &= \mathrm{Li}_2(e^{\frac{\pi}{3}i}) + \mathrm{Li}_1(e^{\frac{\pi}{3}i})\overline{\mathrm{Li}_1(e^{\frac{\pi}{3}i})} + \overline{\mathrm{Li}_2(e^{\frac{\pi}{3}i})}\\ &= i\int_{0<\theta_1<\frac{\pi}{3}}\left(-A(\theta_1) - \frac{i\pi}{6} + \frac{i\theta_1}{2}\right)d\theta_1 + \int_{0<\theta_1<\frac{\pi}{3}} d\theta_1 \cdot \int_{0<\theta_1<\frac{\pi}{3}} d\theta_1\\ &\quad - i\int_{0<\theta_1<\frac{\pi}{3}}\left(-A(\theta_1) + \frac{i\pi}{6} - \frac{i\theta_1}{2}\right)d\theta_1\\ &= \frac{\pi^2}{6}.\end{aligned}$$

Similarly, by calculating $\zeta(3)$, we obtain

$$\zeta(3) = \frac{1}{2}\pi\mathrm{Ls}_2^{(0)}\left(\frac{\pi}{3}\right) - \frac{3}{2}\mathrm{Ls}_3^{(1)}\left(\frac{\pi}{3}\right) - \frac{i}{2}\mathrm{Ls}_3^{(0)}\left(\frac{\pi}{3}\right) - \frac{7i}{216}\pi^3. \tag{3.14}$$

By taking the real part of (3.14), we have

$$\zeta(3) = \frac{1}{2}\pi\mathrm{Ls}_2^{(0)}\left(\frac{\pi}{3}\right) - \frac{3}{2}\mathrm{Ls}_3^{(1)}\left(\frac{\pi}{3}\right)$$

which is equal to (1.16).

Next, we calculate MZVs of weight 4. The real parts of all MZVs of weight 4 are as follows:

$$\begin{aligned}\zeta(4) &= -\frac{23}{5184}\pi^4 - \frac{1}{4}\pi\mathrm{Ls}_3^{(0)}\left(\frac{\pi}{3}\right) + \frac{1}{4}\mathrm{Ls}_4^{(1)}\left(\frac{\pi}{3}\right),\\ \zeta(1,1,2) &= -\frac{23}{5184}\pi^4 - \frac{1}{4}\pi\mathrm{Ls}_3^{(0)}\left(\frac{\pi}{3}\right) + \frac{1}{4}\mathrm{Ls}_4^{(1)}\left(\frac{\pi}{3}\right),\\ \zeta(1,3) &= \frac{7}{1296}\pi^4 + \mathrm{Ls}_4^{(1)}\left(\frac{\pi}{3}\right),\\ \zeta(2,2) &= \frac{1}{324}\pi^4 - 2\mathrm{Ls}_4^{(1)}\left(\frac{\pi}{3}\right).\end{aligned} \tag{3.15}$$

Here, by taking the imaginary part of (3.14), we obtain

$$\mathrm{Ls}_3^{(0)}\left(\frac{\pi}{3}\right) = -\frac{7}{108}\pi^3$$

which is equal to (1.11). Moreover, by taking the imaginary part of the following expression of $\zeta(1,4)$:

$$\begin{aligned}\zeta(1,4) = &-\frac{17}{25920}i\,\pi^5 + \frac{1}{8}\,\pi^2\mathrm{Ls}_3^{(1)}\left(\frac{\pi}{3}\right) - \frac{1}{2}\,\pi\mathrm{Ls}_4^{(2)}\left(\frac{\pi}{3}\right)\\ &-\frac{1}{4}i\,\pi\mathrm{Ls}_4^{(1)}\left(\frac{\pi}{3}\right) + \frac{3}{8}\,\mathrm{Ls}_5^{(3)}\left(\frac{\pi}{3}\right) - \frac{1}{6}\,\mathrm{Ls}_5^{(1)}\left(\frac{\pi}{3}\right),\end{aligned}$$

we obtain

$$\mathrm{Ls}_4^{(1)}\left(\frac{\pi}{3}\right) = -\frac{17}{6480}\pi^4$$

which is equal to (1.12). By applying (1.11) and (1.12) to (3.15), we obtain

$$\zeta(4) = \frac{\pi^4}{90}, \quad \zeta(1,1,2) = \frac{\pi^4}{90}, \quad \zeta(1,3) = \frac{\pi^4}{360}, \quad \zeta(2,2) = \frac{\pi^4}{120}.$$

Therefore, we obtain a relation among MZVs:

$$3\zeta(4) = 3\zeta(1,1,2) = 12\zeta(1,3) = 4\zeta(2,2).$$

In this way, by applying relations among iterated log-sine integrals to the iterated log-sine integral expressions of multiple zeta values, we can obtain relations among multiple zeta values. We can prove that $\zeta(\mathbf{k})$ and $\zeta(\mathbf{k}^*)$ have the same iterated log-sine integral expression and $\zeta(2k)$ is equal to $(-1)^{k+1}(2\pi)^{2k}B_{2k}/2(2k)!$ only by our method (see [41] for details).

3.2.3 A result obtained by using computers

We compute the number of relations among MZVs obtained by our method using computers. Let l_k be the upper bound of $\dim \mathcal{Z}_k$ obtained by our method, which is the number of all MZVs of weight k minus the number of independent relations among MZVs obtained by our method. The result of the calculation of l_k by using computers is as follows. Here, d_k is the sequence defined in Conjecture 1, and 2^{k-2} is the number of all MZVs of weight k.

k	2	3	4	5	6	7	8	9	10	$\cdots$
d_k	1	1	1	2	2	3	4	5	7	$\cdots$
l_k	1	1	1	2	2	4	4	9	9	$\cdots$
2^{k-2}	1	2	4	8	16	32	64	128	256	$\cdots$

We can see d_k equals l_k up to weight 6. However, d_k does not equal to l_k when the weight is 7. Therefore, we can not obtain all relations among MZVs by using our method. However, by comparing d_k, l_k and 2^{k-2}, we can see that many relations are obtained by our method. Note that there is a possibility that more relations can be obtained, because the author's program is only to use relations among iterated log-sine integrals obtained by calculations of weight up to $k+1$ for obtaining relations among MZVs of weight k. However, the last one relation of MZVs of weight 7 can not obtained from relations among iterated log-sine integrals obtained by calculations of weight 8, 9, 10 and 11 with the author's program.

Here, we describe the algorithm of calculating l_k. The author's program is implemented in the computer algebra system SageMath. We use the example of the case $k = 5$ for explanation. First, we make a vertical vector consisting of all MZVs of weight k. Note that $\zeta(\mathbf{k})$ and $\zeta(\mathbf{k}^*)$ have the same log-sine integral expression. For that reason we omit $\zeta(\mathbf{k}^*)$ when $\zeta(\mathbf{k})$ is included as an element of the vector. Next, we apply Theorem 22, Theorem 23 and Theorem 18 to each entries, and write each entries as a $\mathbb{Q}$-linear combination of products of a power of π and an iterated log-sine integral at $\pi/3$ satisfying $k_u - 1 - l_u > 0$ for all $u \in \{1, \ldots, n\}$, and take their real parts. For the case $k = 5$, we have

$$\begin{pmatrix} \zeta(5) \\ \zeta(1,4) \\ \zeta(2,3) \\ \zeta(3,2) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{1}{12} & \frac{3}{16} & \frac{1}{12} & -\frac{1}{16} & \frac{1}{16} & -\frac{1}{48} \\ 0 & 0 & -\frac{1}{6} & \frac{3}{8} & 0 & -\frac{1}{2} & \frac{1}{8} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -\frac{7}{8} & 0 & \frac{5}{4} & -\frac{5}{12} & \frac{7}{216} \\ 0 & -\frac{3}{2} & -\frac{1}{2} & \frac{3}{8} & 0 & -\frac{3}{4} & \frac{1}{4} & -\frac{1}{72} \end{pmatrix} \begin{pmatrix} \mathrm{Ls}_{3,2}^{(0,0)}\left(\frac{\pi}{3}\right) \\ \mathrm{Ls}_{2,3}^{(0,0)}\left(\frac{\pi}{3}\right) \\ \mathrm{Ls}_{5}^{(1)}\left(\frac{\pi}{3}\right) \\ \mathrm{Ls}_{5}^{(3)}\left(\frac{\pi}{3}\right) \\ \pi\mathrm{Ls}_{4}^{(0)}\left(\frac{\pi}{3}\right) \\ \pi\mathrm{Ls}_{4}^{(2)}\left(\frac{\pi}{3}\right) \\ \pi^2\mathrm{Ls}_{3}^{(1)}\left(\frac{\pi}{3}\right) \\ \pi^3\mathrm{Ls}_{2}^{(0)}\left(\frac{\pi}{3}\right) \end{pmatrix}. \tag{3.16}$$

Here, the vertical vector of the right hand side is composed of all $\pi^m \mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\pi/3)$ satisfying $m + |\mathbf{k}| = k$ and $\sum_{u=1}^{n} k_u - 1 - l_u \equiv 1 \bmod 2$.

On the other hand, we make a vertical vector consisting of all MZVs of weight $k+1$ except when it is self-dual (but when $\zeta(\mathbf{k})$ is included as an element of the vector, then we omit $\zeta(\mathbf{k}^*)$). The reason for omitting them is that no relations among iterated log-sine integrals obtained from multiple zeta values for self-dual indices and relations among iterated log-sine integrals obtained from $\zeta(\mathbf{k})$ and $\zeta(\mathbf{k}^*)$ differ only in sign. We write each entries of the vector as a $\mathbb{Q}$-linear combination of products of a power of π and an iterated log-sine integral at $\pi/3$ satisfying $k_u - 1 - l_u > 0$ for all $u \in \{1, \ldots, n\}$, and take their imaginary parts. Namely,

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \Im(\zeta(6)) \\ \Im(\zeta(1,5)) \\ \Im(\zeta(2,4)) \\ \Im(\zeta(3,3)) \\ \Im(\zeta(4,2)) \\ \Im(\zeta(1,3,2)) \end{pmatrix} = A \begin{pmatrix} \mathrm{Ls}_{2,2,2}^{(0,0,0)}\left(\frac{\pi}{3}\right) \\ \mathrm{Ls}_{4,2}^{(1,0)}\left(\frac{\pi}{3}\right) \\ \mathrm{Ls}_{3,3}^{(0,1)}\left(\frac{\pi}{3}\right) \\ \mathrm{Ls}_{3,3}^{(1,0)}\left(\frac{\pi}{3}\right) \\ \mathrm{Ls}_{2,4}^{(0,1)}\left(\frac{\pi}{3}\right) \\ \mathrm{Ls}_{6}^{(0)}\left(\frac{\pi}{3}\right) \\ \mathrm{Ls}_{6}^{(2)}\left(\frac{\pi}{3}\right) \\ \mathrm{Ls}_{6}^{(4)}\left(\frac{\pi}{3}\right) \\ \pi\mathrm{Ls}_{3,2}^{(0,0)}\left(\frac{\pi}{3}\right) \\ \pi\mathrm{Ls}_{2,3}^{(0,0)}\left(\frac{\pi}{3}\right) \\ \pi\mathrm{Ls}_{5}^{(1)}\left(\frac{\pi}{3}\right) \\ \pi\mathrm{Ls}_{5}^{(3)}\left(\frac{\pi}{3}\right) \\ \pi^2\mathrm{Ls}_{4}^{(0)}\left(\frac{\pi}{3}\right) \\ \pi^2\mathrm{Ls}_{4}^{(2)}\left(\frac{\pi}{3}\right) \\ \pi^3\mathrm{Ls}_{3}^{(1)}\left(\frac{\pi}{3}\right) \\ \pi^4\mathrm{Ls}_{2}^{(0)}\left(\frac{\pi}{3}\right) \end{pmatrix}, \tag{3.17}$$

where A represents a matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{120} & -\frac{1}{48} & -\frac{5}{128} & 0 & 0 & \frac{1}{24} & -\frac{1}{96} & -\frac{1}{48} & \frac{1}{64} & -\frac{1}{96} & \frac{1}{384} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{12} & -\frac{1}{16} & 0 & 0 & \frac{1}{12} & \frac{1}{48} & 0 & \frac{1}{16} & -\frac{1}{48} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{7}{24} & \frac{11}{64} & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{16} & 0 & -\frac{1}{24} & \frac{1}{24} & -\frac{23}{5184} \\ 0 & 0 & \frac{3}{4} & \frac{3}{4} & -\frac{3}{4} & 0 & -\frac{3}{8} & -\frac{9}{64} & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{7}{16} & 0 & -\frac{1}{4} & \frac{7}{144} & -\frac{5}{1728} \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & \frac{1}{4} & \frac{3}{32} & \frac{1}{2} & 0 & -\frac{1}{6} & -\frac{7}{24} & \frac{1}{36} & \frac{11}{48} & -\frac{1}{18} & \frac{7}{2592} \\ 0 & 0 & 0 & 0 & -1 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & \frac{5}{24} & 0 & -\frac{13}{48} & \frac{1}{12} & -\frac{7}{1296} \end{pmatrix}.$$

The vertical vector of the right hand side of (3.17) is composed of all $\pi^m \mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\pi/3)$ satisfying $m+|\mathbf{k}| = k+1$ and $\sum_{u=1}^{n} k_u - 1 - l_u \equiv 1 \bmod 2$. The row echelon form of the matrix A is

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & \frac{1}{36} & \frac{5}{48} & 0 & -\frac{1}{12} & \frac{25}{432} & -\frac{1}{72} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\frac{3}{16} & 0 & 0 & \frac{1}{4} & -\frac{7}{48} & 0 & \frac{11}{24} & -\frac{7}{48} & \frac{7}{1296} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{45}{16} & 0 & 0 & \frac{5}{2} & -\frac{15}{8} & 0 & \frac{45}{8} & \frac{25}{8} & -\frac{25}{16} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{3}{4} & 0 & 0 & -1 & -\frac{1}{4} & 0 & -\frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{12} & \frac{5}{16} & 0 & -\frac{1}{4} & -\frac{1}{48} & \frac{5}{216} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{9}{4} & \frac{3}{2} & -\frac{3}{4} \end{pmatrix}.$$

Because the eight entries from the bottom of the vertical vector of the right hand side of (3.17) are multiples of π, we focus on the rows where the leading coefficient of the row echelon form of the matrix A is within 8 from the right. Then we have

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & \frac{1}{12} & \frac{5}{16} & 0 & -\frac{1}{4} & -\frac{1}{48} & \frac{5}{216} \\ 0 & 0 & 0 & 0 & 1 & \frac{9}{4} & \frac{3}{2} & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} \mathrm{Ls}_{3,2}^{(0,0)}\left(\frac{\pi}{3}\right) \\ \mathrm{Ls}_{2,3}^{(0,0)}\left(\frac{\pi}{3}\right) \\ \mathrm{Ls}_{5}^{(1)}\left(\frac{\pi}{3}\right) \\ \mathrm{Ls}_{5}^{(3)}\left(\frac{\pi}{3}\right) \\ \pi\mathrm{Ls}_{4}^{(0)}\left(\frac{\pi}{3}\right) \\ \pi\mathrm{Ls}_{4}^{(2)}\left(\frac{\pi}{3}\right) \\ \pi^2\mathrm{Ls}_{3}^{(1)}\left(\frac{\pi}{3}\right) \\ \pi^3\mathrm{Ls}_{2}^{(0)}\left(\frac{\pi}{3}\right) \end{pmatrix}. \tag{3.18}$$

We also calculate the imaginary parts of MZVs of weight $k-1-2m$ $(0 \le m \le (k-4)/2)$. On our example, we only need to calculate when the weight is 4. By calculating the imaginary parts of all MZVs of weight 4 except when it is self-dual (but when $\zeta(\mathbf{k})$ is included as an element of the vector, then we omit $\zeta(\mathbf{k}^*)$), we have

$$\begin{pmatrix} 0 \end{pmatrix} = \begin{pmatrix} \Im(\zeta(4)) \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} \end{pmatrix} \begin{pmatrix} \mathrm{Ls}_4^{(0)}\left(\frac{\pi}{3}\right) \\ \mathrm{Ls}_4^{(2)}\left(\frac{\pi}{3}\right) \\ \pi\mathrm{Ls}_3^{(1)}\left(\frac{\pi}{3}\right) \\ \pi^2\mathrm{Ls}_2^{(0)}\left(\frac{\pi}{3}\right) \end{pmatrix}.$$

Next, we multiply relations obtained from the imaginary parts of MZVs of weight $k-1-2m$ by π^{1+2m}. On our example, we have

$$\begin{pmatrix} 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} \end{pmatrix} \begin{pmatrix} \pi\mathrm{Ls}_4^{(0)}\left(\frac{\pi}{3}\right) \\ \pi\mathrm{Ls}_4^{(2)}\left(\frac{\pi}{3}\right) \\ \pi^2\mathrm{Ls}_3^{(1)}\left(\frac{\pi}{3}\right) \\ \pi^3\mathrm{Ls}_2^{(0)}\left(\frac{\pi}{3}\right) \end{pmatrix}. \tag{3.19}$$

By applying (3.18) and (3.19) to (3.16), we obtain

$$\begin{pmatrix} \zeta(5) \\ \zeta(1,4) \\ \zeta(2,3) \\ \zeta(3,2) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{1}{12} & \frac{3}{16} & 0 & -\frac{1}{4} & -\frac{1}{16} & \frac{1}{24} \\ 0 & 0 & -\frac{1}{6} & \frac{3}{8} & 0 & -\frac{1}{2} & \frac{1}{8} & 0 \\ 0 & 0 & \frac{11}{24} & -\frac{33}{32} & 0 & \frac{11}{8} & -\frac{13}{32} & \frac{1}{48} \\ 0 & 0 & -\frac{3}{8} & \frac{27}{32} & 0 & -\frac{9}{8} & \frac{7}{32} & \frac{1}{48} \end{pmatrix} \begin{pmatrix} \mathrm{Ls}_{3,2}^{(0,0)}\left(\frac{\pi}{3}\right) \\ \mathrm{Ls}_{2,3}^{(0,0)}\left(\frac{\pi}{3}\right) \\ \mathrm{Ls}_5^{(1)}\left(\frac{\pi}{3}\right) \\ \mathrm{Ls}_5^{(3)}\left(\frac{\pi}{3}\right) \\ \pi\mathrm{Ls}_4^{(0)}\left(\frac{\pi}{3}\right) \\ \pi\mathrm{Ls}_4^{(2)}\left(\frac{\pi}{3}\right) \\ \pi^2\mathrm{Ls}_3^{(1)}\left(\frac{\pi}{3}\right) \\ \pi^3\mathrm{Ls}_2^{(0)}\left(\frac{\pi}{3}\right) \end{pmatrix}. \tag{3.20}$$

Here, l_k is given as the rank of the first matrix of the right hand side of (3.20). Moreover, the row echelon form of

$$\left(\begin{array}{cccccccc|c} 0 & 0 & -\frac{1}{12} & \frac{3}{16} & 0 & -\frac{1}{4} & -\frac{1}{16} & \frac{1}{24} & \zeta(5) \\ 0 & 0 & -\frac{1}{6} & \frac{3}{8} & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \zeta(1,4) \\ 0 & 0 & \frac{11}{24} & -\frac{33}{32} & 0 & \frac{11}{8} & -\frac{13}{32} & \frac{1}{48} & \zeta(2,3) \\ 0 & 0 & -\frac{3}{8} & \frac{27}{32} & 0 & -\frac{9}{8} & \frac{7}{32} & \frac{1}{48} & \zeta(3,2) \end{array}\right)$$

is

$$\left(\begin{array}{cccccccc|c} 0 & 0 & 1 & -\frac{9}{4} & 0 & 3 & 0 & -\frac{1}{4} & -6\zeta(5) - 3\zeta(1,4) \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{3} & -8\zeta(5) + 4\zeta(1,4) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\zeta(5) + \zeta(2,3) + 3\zeta(1,4) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\zeta(5) + \zeta(3,2) - 2\zeta(1,4) \end{array}\right).$$

Therefore, we can obtain independent relations $0 = -\zeta(5)/2 + \zeta(2,3) + 3\zeta(1,4)$ and $0 = -\zeta(5)/2 + \zeta(3,2) - 2\zeta(1,4)$.

3.3 Evaluation of iterated log-sine integrals in terms of multiple polylogarithms

3.3.1 Statement of the theorems

We evaluate iterated log-sine integrals by using multiple polylogarithms and multiple zeta values. Such a subject for generalized log-sine integrals are discussed in [7, Section 5], [13], [14] and [28, Section 2.3]. When $\sigma \in [0, 2\pi]$, iterated log-sine integrals $\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma)$ are evaluated as follows.

Theorem 24. *Let*

$$\mathbf{1}_n = (\{1\}^n).$$

For $\sigma \in [0, 2\pi]$, $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ *and* $\mathbf{l} = (l_1, \ldots, l_n) \in (\mathbb{Z}_{\geq 0})^n$ *satisfying* $k_u - 1 - l_u \geq 0$ *for all* $u \in \{1, \ldots, n\}$, *we have*

$$\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma) = i^{|\mathbf{l}|+n}(-1)^{|\mathbf{k}|+n} \sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{k}-\mathbf{1}_n-\mathbf{l}} \frac{(-i\pi)^{|\mathbf{p}|}}{2^{|\mathbf{p}|+|\mathbf{q}|}} \binom{\mathbf{k}-\mathbf{1}_n-\mathbf{l}}{\mathbf{p}, \mathbf{q}, \mathbf{r}} F_{\mathbf{q}+\mathbf{1}}^{\mathbf{r}}(\sigma), \tag{3.21}$$

where the sum is over all $\mathbf{p} = (p_1, \ldots, p_n) \in (\mathbb{Z}_{\geq 0})^n$, $\mathbf{q} = (q_1, \ldots, q_n) \in (\mathbb{Z}_{\geq 0})^n$ *and* $\mathbf{r} = (r_1, \ldots, r_n) \in (\mathbb{Z}_{\geq 0})^n$ *satisfying* $\mathbf{p} + \mathbf{q} + \mathbf{r} = \mathbf{k} - \mathbf{1}_n - \mathbf{l}$, *and*

$$\binom{\mathbf{k}-\mathbf{1}_n-\mathbf{l}}{\mathbf{p}, \mathbf{q}, \mathbf{r}} = \prod_{u=1}^{n} \frac{(k_u - 1 - l_u)!}{p_u! q_u! r_u!}.$$

The function $F_{\mathbf{q}+\mathbf{1}}^{\mathbf{r}}(\sigma)$ will be defined in Section 3.3.3. In particular, the function $F_{\mathbf{q}+\mathbf{1}}^{\mathbf{r}}(\sigma)$ is written as a $\mathbb{Q}(i)$-linear combination of products of a power of σ, multiple zeta values and a multiple polylogarithm with admissible index at $e^{i\sigma}$. This theorem is proved in Section 3.3.2 and 3.3.3.

In Section 3.3.5, we discuss the iterated log-sine integrals at a general argument not only in the case $\sigma \in [0, 2\pi]$ and give the following theorem.

Theorem 25. *For* $\sigma \in [0, 2\pi]$ *and* $m \geq 0$, *the iterated log-sine integral* $\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\pm(2m\pi + \sigma))$ *satisfying* $k_u - 1 - l_u \geq 0$ *for all* $u \in \{1, \ldots, n\}$ *can be written as a* $\mathbb{Q}(i)$*-linear combination of products of a power of* σ, *a power of* π, *multiple zeta values and multiple polylogarithms with admissible index at* $e^{i\sigma}$.

The equation (3.21) makes numerical evaluation of iterated log-sine integrals possible. Another algorithm for numerical evaluation of the generalized log-sine integral was given by Kalmykov and Sheplyakov [28]. Conjectures suggested by numerical experiments are stated in Section 3.4.

3.3.2 Evaluation of iterated log-sine integrals in terms of $\mathrm{ILi}_{\mathbf{q}}^{\mathbf{r}}(\sigma)$

In order to prove Theorem 24, we first introduce $\mathrm{ILi}_{\mathbf{q}}^{\mathbf{r}}(\sigma)$ defined as follows.

Definition 9. For $\mathbf{q} = (q_1, \ldots, q_n) \in (\mathbb{Z}_{\geq 0})^n$, $\mathbf{r} = (r_1, \ldots, r_n) \in (\mathbb{Z}_{\geq 0})^n$ and $\sigma \in [0, 2\pi]$, we define

$$\begin{aligned}\mathrm{ILi}_{\mathbf{q}}^{\mathbf{r}}(\sigma) &= \int_0^{\sigma} \int_0^{\theta_n} \cdots \int_0^{\theta_2} \prod_{u=1}^{n} i(i\theta_u)^{q_u} \left(\mathrm{Li}_1(e^{i\theta_u})\right)^{r_u} \, d\theta_1 \cdots d\theta_n \\ &= \int_0^{\sigma} i(i\theta_n)^{q_n} \left(\mathrm{Li}_1(e^{i\theta_n})\right)^{r_n} \mathrm{ILi}_{\mathbf{q}_-}^{\mathbf{r}_-}(\theta_n)\, d\theta_n,\end{aligned}$$

where $\mathrm{ILi}_{\emptyset}^{\emptyset}(\sigma)$ is regarded as 1.

Then, we can evaluate iterated log-sine integrals in terms of $\mathrm{ILi}_{\mathbf{q}}^{\mathbf{r}}(\sigma)$ as follows.

Lemma 3. *For* $\sigma \in [0, 2\pi]$, $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ *and* $\mathbf{l} = (l_1, \ldots, l_n) \in (\mathbb{Z}_{\geq 0})^n$ *satisfying* $k_u - 1 - l_u \geq 0$ *for all* $u \in \{1, \ldots, n\}$, *we have*

$$\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma) = i^{|\mathbf{l}|+n}(-1)^{|\mathbf{k}|+n} \sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{k}-\mathbf{1}_n-\mathbf{l}} \frac{(-i\pi)^{|\mathbf{p}|}}{2^{|\mathbf{p}|+|\mathbf{q}|}} \binom{\mathbf{k}-\mathbf{1}_n-\mathbf{l}}{\mathbf{p}, \mathbf{q}, \mathbf{r}} \mathrm{ILi}_{\mathbf{q}+\mathbf{1}}^{\mathbf{r}}(\sigma).$$

Proof. Noting $\mathrm{Li}_1(e^{i\theta}) = -\log(1-e^{i\theta}) = -A(\theta) - i\frac{\theta-\pi}{2}$ $(0<\theta<2\pi)$, we have

$$\begin{aligned}
(-i)^{|\mathbf{l}|+n}(-1)^{|\mathbf{k}|+n}\,\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma) &= (-i)^{|\mathbf{l}|+n}(-1)^{|\mathbf{k}|}\int_{0<\theta_1<\cdots<\theta_n<\sigma}\prod_{u=1}^{n}\theta_u^{l_u}A^{k_u-1-l_u}(\theta_u)\,d\theta_u \\
&= \int_{0<\theta_1<\cdots<\theta_n<\sigma}\prod_{u=1}^{n} i\,(i\theta_u)^{l_u}\,(-A(\theta_u))^{k_u-1-l_u}\;d\theta_u \\
&= \int_{0<\theta_1<\cdots<\theta_n<\sigma}\prod_{u=1}^{n} i\,(i\theta_u)^{l_u}\left(\mathrm{Li}_1(e^{i\theta_u})+\frac{i\theta_u}{2}-\frac{i\pi}{2}\right)^{k_u-1-l_u}\;d\theta_u \\
&= \int_{0<\theta_1<\cdots<\theta_n<\sigma}\prod_{u=1}^{n} i\,(i\theta_u)^{l_u}\sum_{p_u+q_u+r_u=k_u-1+l_u}\frac{(k_u-1-l_u)!}{p_u!q_u!r_u!} \\
&\quad\times\left(-\frac{i\pi}{2}\right)^{p_u}\left(\frac{i\theta_u}{2}\right)^{q_u}\left(\mathrm{Li}_1(e^{i\theta_u})\right)^{r_u}\;d\theta_u \\
&= \sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{k}-\mathbf{1}_n-\mathbf{l}}\frac{(-i\pi)^{|\mathbf{p}|}}{2^{|\mathbf{p}|+|\mathbf{q}|}}\left(\prod_{u=1}^{n}\frac{(k_u-1-l_u)!}{p_u!q_u!r_u!}\right) \\
&\quad\times\int_{0<\theta_1<\cdots<\theta_n<\sigma}\prod_{u=1}^{n} i\,(i\theta_u)^{q_u+l_u}\left(\mathrm{Li}_1(e^{i\theta_u})\right)^{r_u}\;d\theta_u.
\end{aligned}$$

□

In order to prove Theorem 24, we only need to prove $\mathrm{ILi}_{\mathbf{q}}^{\mathbf{r}}(\sigma) = F_{\mathbf{q}}^{\mathbf{r}}(\sigma)$.

3.3.3 Evaluation of $\mathrm{ILi}_{\mathbf{r}}^{\mathbf{q}}(\sigma)$

Let $\mathbf{q}$, $\mathbf{r}$, $\mathbf{j}\in(\mathbb{Z}_{\geq 0})^n$. We define the rational number $B_{\mathbf{q}}$ by

$$B_{\mathbf{q}} = \frac{1}{|\mathbf{q}|+n}B_{\mathbf{q}_-} \quad\text{and}\quad B_\emptyset = 1,$$

and the rational number $C_{\mathbf{q}}^{\mathbf{j}}$ by

$$C_{\mathbf{q}}^{\mathbf{j}} = (-1)^{j_n}\frac{(|\mathbf{q}|-|\mathbf{j}_-|)!}{(|\mathbf{q}|-|\mathbf{j}|)!}C_{\mathbf{q}_-}^{\mathbf{j}_-} \quad\text{and}\quad C_\emptyset^\emptyset = 1.$$

We write $\mathbf{j}\preceq\mathbf{q}$ when $\mathbf{j}$ and $\mathbf{q}$ satisfy

$$\begin{gathered}
j_1 \leq q_1 \\
j_1+j_2 \leq q_1+q_2 \\
\vdots \\
j_1+j_2+\cdots+j_n \leq q_1+q_2+\cdots+q_n,
\end{gathered}$$

and understand $\emptyset\preceq\emptyset$. Note that this symbol "$\preceq$" is not the same as used in Theorem 2. We define the noncommutative polynomial ring $\mathfrak{H}=\mathbb{Q}\langle e_0,e_1\rangle$ and its subspace $\mathfrak{H}^0=\mathbb{Q}+e_1\mathfrak{H}e_0$ and $\mathfrak{H}^1=\mathbb{Q}+e_1\mathfrak{H}$, and the element $w_{\mathbf{j}}^{\mathbf{r}}$ in $\mathfrak{H}$ by

$$w_{\mathbf{j}}^{\mathbf{r}} = (w_{\mathbf{j}_-}^{\mathbf{r}_-}\sqcup\!\sqcup\, e_1^{\sqcup\!\sqcup r_n})e_0^{1+j_n} \quad\text{and}\quad w_\emptyset^\emptyset = 1,$$

where the shuffle product $\sqcup\!\sqcup$ is recursively defined by

$$\begin{aligned}
w\sqcup\!\sqcup 1 = w,\quad 1\sqcup\!\sqcup w = w &\qquad (w:\text{word}), \\
u_1w_1\sqcup\!\sqcup u_2w_2 = u_1(w_1\sqcup\!\sqcup u_2w_2)+u_2(u_1w_1\sqcup\!\sqcup w_2) &\qquad (u_1,u_2\in\{e_0,e_1\}, w_1,w_2:\text{words})
\end{aligned}$$

with $\mathbb{Q}$-bilinearity, and $e_1^{\sqcup\!\sqcup r_n}$ represents

$$\underbrace{e_1\sqcup\!\sqcup\cdots\sqcup\!\sqcup e_1}_{r_n}.$$

Note that $w_{\mathbf{j}}^{\mathbf{r}} \in \mathfrak{H}^0$ when $r_1 \geq 1$. We define $\mathbb{Q}$-linear maps $Z\colon \mathfrak{H}^0 \to \mathbb{R}$ and $L(\,\cdot\,; z)\colon \mathfrak{H}^1 \to \mathbb{C}$ by

$$Z(e_1 e_0^{k_1-1} \cdots e_1 e_0^{k_n-1}) = \zeta(k_1, \dots, k_n),$$
$$L(e_1 e_0^{k_1-1} \cdots e_1 e_0^{k_n-1}; z) = \mathrm{Li}_{k_1,\dots,k_n}(z),$$

and

$$Z(1) = \zeta(\emptyset) = 1,$$
$$L(1; z) = \mathrm{Li}_{\emptyset}(z) = 1,$$

respectively. Note that $L(w_1; z)L(w_2; z) = L(w_1 ш w_2; z)$ for any $w_1, w_2 \in \mathfrak{H}^1$.

For a pair of indices $\mathbf{q}, \mathbf{r} \in (\mathbb{Z}_{\geq 0})^n$, we also write $\mathbf{r} = (\{0\}^{n'}, r''_1, \dots, r''_{n''}) = (\mathbf{r}', \mathbf{r}'') \in (\mathbb{Z}_{\geq 0})^{n'+n''}$ with $r''_1 \geq 1$, and $\mathbf{q} = (q'_1, \dots, q'_{n'}, q''_1, \dots, q''_{n''}) = (\mathbf{q}', \mathbf{q}'') \in (\mathbb{Z}_{\geq 0})^{n'+n''}$. Note that $q''_{n''} = q_n$, $r''_{n''} = r_n$ and $n = n' + n''$. We define $\overline{\mathbf{q}} = (|\mathbf{q}'| + n' + q''_1, q''_2, \dots, q''_{n''})$ when $\mathbf{q}'' \neq \emptyset$, and $\overline{\mathbf{q}} = \emptyset$ when $\mathbf{q}'' = \emptyset$. For example, if $\mathbf{q} = (1, 2, 3, 4)$ and $\mathbf{r} = (0, 0, 1, 2)$ then $\mathbf{q}' = (1, 2)$, $\mathbf{q}'' = (3, 4)$, $\mathbf{r}' = (0, 0)$, $\mathbf{r}'' = (1, 2)$ and $\overline{\mathbf{q}} = (8, 4)$.

Definition 10. For $\sigma \in [0, 2\pi]$, $\mathbf{q} \in (\mathbb{Z}_{\geq 0})^n$ and $\mathbf{r} \in (\mathbb{Z}_{\geq 0})^n$, we define

$$f_{\mathbf{q}}^{\mathbf{r}}(\sigma) = B_{\mathbf{q}'} \sum_{\mathbf{j} \preceq \overline{\mathbf{q}}} C_{\overline{\mathbf{q}}}^{\mathbf{j}} (i\sigma)^{|\mathbf{q}|+n'-|\mathbf{j}|} L(w_{\mathbf{j}}^{\mathbf{r}''}; e^{i\sigma}),$$

where the sum is over $\mathbf{j} = (j_1, \dots, j_{n''}) \in (\mathbb{Z}_{\geq 0})^{n''}$ satisfying $\mathbf{j} \preceq \overline{\mathbf{q}}$.

The function $f_{\mathbf{q}}^{\mathbf{r}}(\sigma)$ can also be written as follows:

$$f_{\mathbf{q}}^{\mathbf{r}}(\sigma) = \begin{cases} 1 & \text{if } \mathbf{r} = \emptyset, \\ B_{\mathbf{q}} (i\sigma)^{|\mathbf{q}|+n} & \text{if } \mathbf{r} = (\{0\}^n), n \geq 1, \\ B_{\mathbf{q}'} \sum_{\mathbf{j} \preceq \overline{\mathbf{q}}} C_{\overline{\mathbf{q}}}^{\mathbf{j}} (i\sigma)^{|\overline{\mathbf{q}}|-|\mathbf{j}|} L(w_{\mathbf{j}}^{\mathbf{r}''}; e^{i\sigma}) & \text{otherwise.} \end{cases} \tag{3.22}$$

Remark 4. We have

$$f_{\mathbf{q}}^{\mathbf{r}}(0) = \begin{cases} 1 & \text{if } \mathbf{r} = \emptyset, \\ 0 & \text{if } \mathbf{r} = (\{0\}^n), n \geq 1, \\ B_{\mathbf{q}'} \displaystyle\sum_{\mathbf{j}_- \preceq \overline{\mathbf{q}}_-} C_{\overline{\mathbf{q}}_-}^{\mathbf{j}_-} (-1)^{|\overline{\mathbf{q}}|-|\mathbf{j}_-|} (|\overline{\mathbf{q}}| - |\mathbf{j}_-|)! & \\ \qquad \times Z\left((w_{\mathbf{j}_-}^{\mathbf{r}''_-} ш e_1^{ш r_n}) e_0^{1+|\overline{\mathbf{q}}|-|\mathbf{j}_-|} \right) & \text{otherwise.} \end{cases} \tag{3.23}$$

Proof. We prove only the case "otherwise". In this case, we obtain

$$\begin{aligned} f_{\mathbf{q}}^{\mathbf{r}}(\sigma) &= B_{\mathbf{q}'} \sum_{\mathbf{j} \preceq \overline{\mathbf{q}}} C_{\overline{\mathbf{q}}}^{\mathbf{j}} (i\sigma)^{|\overline{\mathbf{q}}|-|\mathbf{j}|} L\left(w_{\mathbf{j}}^{\mathbf{r}''}; e^{i\sigma} \right) \\ &= B_{\mathbf{q}'} \sum_{\mathbf{j}_- \preceq \overline{\mathbf{q}}_-} C_{\overline{\mathbf{q}}_-}^{\mathbf{j}_-} \sum_{j_{n''}=0}^{|\overline{\mathbf{q}}|-|\mathbf{j}_-|} (-1)^{j_{n''}} \frac{(|\overline{\mathbf{q}}| - |\mathbf{j}_-|)!}{(|\overline{\mathbf{q}}| - |\mathbf{j}|)!} \\ &\quad \times (i\sigma)^{|\overline{\mathbf{q}}|-|\mathbf{j}|} L\left((w_{\mathbf{j}_-}^{\mathbf{r}''_-} ш e_1^{ш r_n}) e_0^{1+j_{n''}}; e^{i\sigma} \right). \end{aligned} \tag{3.24}$$

By putting $\sigma = 0$, we find that the factor $(i\sigma)^{|\overline{\mathbf{q}}|-|\mathbf{j}|}$ is equal to 0 when $j_{n''} \neq |\overline{\mathbf{q}}| - |\mathbf{j}_-|$, and 1 when $j_{n''} = |\overline{\mathbf{q}}| - |\mathbf{j}_-|$. Therefore, we obtain the desired formula. □

Proposition 9. *For $\mathbf{q}, \mathbf{r} \in (\mathbb{Z}_{\geq 0})^n$ with $n \geq 1$, we have*

$$\frac{d}{d\sigma} f_{\mathbf{q}}^{\mathbf{r}}(\sigma) = i (i\sigma)^{q_n} \left(\mathrm{Li}_1(e^{i\sigma}) \right)^{r_n} f_{\mathbf{q}_-}^{\mathbf{r}_-}(\sigma).$$

Proof. In the case $\mathbf{r} = (\{0\}^n)$, $n \geq 1$, we have

$$\begin{aligned}\frac{d}{d\sigma} f_{\mathbf{q}}^{\mathbf{r}}(\sigma) &= \frac{d}{d\sigma} B_{\mathbf{q}} (i\sigma)^{|\mathbf{q}|+n} \\ &= B_{\mathbf{q}_-} i(i\sigma)^{|\mathbf{q}|+n-1} \\ &= i(i\sigma)^{q_n} B_{\mathbf{q}_-} (i\sigma)^{|\mathbf{q}_-|+n-1} \\ &= i(i\sigma)^{q_n} f_{\mathbf{q}_-}^{\mathbf{r}_-}(\sigma).\end{aligned}$$

In the case "otherwise", that is $n'' \geq 1$, by equation (3.24) we obtain

$$\begin{aligned}\frac{d}{d\sigma} f_{\mathbf{q}}^{\mathbf{r}}(\sigma) = B_{\mathbf{q}'} \sum_{\mathbf{j}_- \preceq \overline{\mathbf{q}}_-} C_{\overline{\mathbf{q}}_-}^{\mathbf{j}_-} \sum_{j_{n''}=0}^{|\overline{\mathbf{q}}|-|\mathbf{j}_-|} (-1)^{j_{n''}} \frac{(|\overline{\mathbf{q}}| - |\mathbf{j}_-|)!}{(|\overline{\mathbf{q}}| - |\mathbf{j}|)!} \\ \times \frac{d}{d\sigma} (i\sigma)^{|\overline{\mathbf{q}}|-|\mathbf{j}|} L\left((w_{\mathbf{j}_-}^{\mathbf{r}''_-} \sqcup\!\sqcup\, e_1^{\sqcup\!\sqcup r_n}) e_0^{1+j_{n''}}; e^{i\sigma}\right).\end{aligned}$$

Here,

$$\begin{aligned}&\sum_{j_{n''}=0}^{|\overline{\mathbf{q}}|-|\mathbf{j}_-|} (-1)^{j_{n''}} \frac{(|\overline{\mathbf{q}}| - |\mathbf{j}_-|)!}{(|\overline{\mathbf{q}}| - |\mathbf{j}|)!} \frac{d}{d\sigma} (i\sigma)^{|\overline{\mathbf{q}}|-|\mathbf{j}|} L\left((w_{\mathbf{j}_-}^{\mathbf{r}''_-} \sqcup\!\sqcup\, e_1^{\sqcup\!\sqcup r_n}) e_0^{1+j_{n''}}; e^{i\sigma}\right) \\ &= \sum_{j_{n''}=0}^{|\overline{\mathbf{q}}|-|\mathbf{j}_-|-1} (-1)^{j_{n''}} \frac{(|\overline{\mathbf{q}}| - |\mathbf{j}_-|)!}{(|\overline{\mathbf{q}}| - |\mathbf{j}|)!} \frac{d}{d\sigma} (i\sigma)^{|\overline{\mathbf{q}}|-|\mathbf{j}|} L\left((w_{\mathbf{j}_-}^{\mathbf{r}''_-} \sqcup\!\sqcup\, e_1^{\sqcup\!\sqcup r_n}) e_0^{1+j_{n''}}; e^{i\sigma}\right) \\ &\quad + (-1)^{|\overline{\mathbf{q}}|-|\mathbf{j}_-|} (|\overline{\mathbf{q}}| - |\mathbf{j}_-|)! \frac{d}{d\sigma} L\left((w_{\mathbf{j}_-}^{\mathbf{r}''_-} \sqcup\!\sqcup\, e_1^{\sqcup\!\sqcup r_n}) e_0^{1+|\overline{\mathbf{q}}|-|\mathbf{j}_-|}; e^{i\sigma}\right) \\ &= \sum_{j_{n''}=0}^{|\mathbf{q}|-|\mathbf{j}_-|-1} \left((-1)^{j_{n''}} \frac{(|\overline{\mathbf{q}}| - |\mathbf{j}_-|)!}{(|\overline{\mathbf{q}}| - |\mathbf{j}| - 1)!} i(i\sigma)^{|\overline{\mathbf{q}}|-|\mathbf{j}|-1} L\left((w_{\mathbf{j}_-}^{\mathbf{r}''_-} \sqcup\!\sqcup\, e_1^{\sqcup\!\sqcup r_n}) e_0^{1+j_{n''}}; e^{i\sigma}\right)\right. \\ &\quad \left. + (-1)^{j_{n''}} \frac{(|\overline{\mathbf{q}}| - |\mathbf{j}_-|)!}{(|\overline{\mathbf{q}}| - |\mathbf{j}|)!} i(i\sigma)^{|\overline{\mathbf{q}}|-|\mathbf{j}|} L\left((w_{\mathbf{j}_-}^{\mathbf{r}''_-} \sqcup\!\sqcup\, e_1^{\sqcup\!\sqcup r_n}) e_0^{j_{n''}}; e^{i\sigma}\right)\right) \\ &\quad + (-1)^{|\overline{\mathbf{q}}|-|\mathbf{j}_-|} (|\overline{\mathbf{q}}| - |\mathbf{j}_-|)! i L\left((w_{\mathbf{j}_-}^{\mathbf{r}''_-} \sqcup\!\sqcup\, e_1^{\sqcup\!\sqcup r_n}) e_0^{|\overline{\mathbf{q}}|-|\mathbf{j}_-|}; e^{i\sigma}\right) \\ &= i(i\sigma)^{|\overline{\mathbf{q}}|-|\mathbf{j}_-|} L\left(w_{\mathbf{j}_-}^{\mathbf{r}''_-} \sqcup\!\sqcup\, e_1^{\sqcup\!\sqcup r_n}; e^{i\sigma}\right).\end{aligned}$$

Therefore, we obtain

$$\frac{d}{d\sigma} f_{\mathbf{q}}^{\mathbf{r}}(\sigma) = B_{\mathbf{q}'} \sum_{\mathbf{j}_- \preceq \overline{\mathbf{q}}_-} C_{\overline{\mathbf{q}}_-}^{\mathbf{j}_-} i(i\sigma)^{|\overline{\mathbf{q}}|-|\mathbf{j}_-|} L\left(w_{\mathbf{j}_-}^{\mathbf{r}''_-} \sqcup\!\sqcup\, e_1^{\sqcup\!\sqcup r_n}; e^{i\sigma}\right).$$

If $n'' \geq 2$, then we have

$$\begin{aligned}\frac{d}{d\sigma} f_{\mathbf{q}}^{\mathbf{r}}(\sigma) &= i(i\sigma)^{q_n} \left(\mathrm{Li}_1(e^{i\sigma})\right)^{r_n} B_{\mathbf{q}'} \sum_{\mathbf{j}_- \preceq \overline{\mathbf{q}}_-} C_{\overline{\mathbf{q}}_-}^{\mathbf{j}_-} (i\sigma)^{|\overline{\mathbf{q}}_-|-|\mathbf{j}_-|} L\left(w_{\mathbf{j}_-}^{\mathbf{r}''_-}; e^{i\sigma}\right) \\ &= i(i\sigma)^{q_n} \left(\mathrm{Li}_1(e^{i\sigma})\right)^{r_n} f_{\mathbf{q}_-}^{\mathbf{r}_-}(\sigma).\end{aligned}$$

In the case $n'' = 1$, since $\overline{\mathbf{q}} = (|\mathbf{q}'| + n' + q_n)$, $\mathbf{q}' = \mathbf{q}_-$ and $\mathbf{r}' = \mathbf{r}_-$, we have

$$\begin{aligned}\frac{d}{d\sigma} f_{\mathbf{q}}^{\mathbf{r}}(\sigma) &= B_{\mathbf{q}'} i(i\sigma)^{|\mathbf{q}'|+n'+q_n} \left(\mathrm{Li}_1(e^{i\sigma})\right)^{r_n} \\ &= i(i\sigma)^{q_n} \left(\mathrm{Li}_1(e^{i\sigma})\right)^{r_n} B_{\mathbf{q}'} (i\sigma)^{|\mathbf{q}'|+n'} \\ &= i(i\sigma)^{q_n} \left(\mathrm{Li}_1(e^{i\sigma})\right)^{r_n} f_{\mathbf{q}_-}^{\mathbf{r}_-}(\sigma).\end{aligned}$$

Therefore, in all cases, this proposition holds. □

Definition 11. For $\sigma \in [0, 2\pi]$, $\mathbf{q} \in (\mathbb{Z}_{\geq 0})^n$ and $\mathbf{r} \in (\mathbb{Z}_{\geq 0})^n$, we define

$$F_{\mathbf{q}}^{\mathbf{r}}(\sigma) = \sum_{\substack{(\mathbf{q}^{(1)},\dots,\mathbf{q}^{(h)})=\mathbf{q} \\ (\mathbf{r}^{(1)},\dots,\mathbf{r}^{(h)})=\mathbf{r} \\ \mathrm{d}(\mathbf{q}^{(j)})=\mathrm{d}(\mathbf{r}^{(j)})\geq 1 \\ (1\leq j\leq h)}} (-1)^{h-1} \left(\prod_{j=1}^{h-1} f_{\mathbf{q}^{(j)}}^{\mathbf{r}^{(j)}}(0)\right) \left(f_{\mathbf{q}^{(h)}}^{\mathbf{r}^{(h)}}(\sigma) - f_{\mathbf{q}^{(h)}}^{\mathbf{r}^{(h)}}(0)\right),$$

where the sum is over all partitions of $\mathbf{q}$ and $\mathbf{r}$, for example

$$\begin{aligned} F_{(q_1,q_2,q_3)}^{(r_1,r_2,r_3)}(\sigma) &= f_{(q_1)}^{(r_1)}(0) f_{(q_2)}^{(r_2)}(0) \left(f_{(q_3)}^{(r_3)}(\sigma) - f_{(q_3)}^{(r_3)}(0)\right) \\ &\quad - f_{(q_1,q_2)}^{(r_1,r_2)}(0) \left(f_{(q_3)}^{(r_3)}(\sigma) - f_{(q_3)}^{(r_3)}(0)\right) - f_{(q_1)}^{(r_1)}(0) \left(f_{(q_2,q_3)}^{(r_2,r_3)}(\sigma) - f_{(q_2,q_3)}^{(r_2,r_3)}(0)\right) \\ &\quad + \left(f_{(q_1,q_2,q_3)}^{(r_1,r_2,r_3)}(\sigma) - f_{(q_1,q_2,q_3)}^{(r_1,r_2,r_3)}(0)\right). \end{aligned}$$

We understand $F_{\emptyset}^{\emptyset}(\sigma) = 1$.

Remark 5. By (3.22) and (3.23), we can see that the function $F_{\mathbf{q+1}}^{\mathbf{r}}(\sigma)$ is written as a $\mathbb{Q}(i)$-linear combination of products of a power of σ, multiple zeta values and a multiple polylogarithm with admissible index at $e^{i\sigma}$.

Proposition 10. *For $\mathbf{q}, \mathbf{r} \in (\mathbb{Z}_{\geq 0})^n$ with $n \geq 1$, we have*

$$\frac{d}{d\sigma} F_{\mathbf{q}}^{\mathbf{r}}(\sigma) = i(i\sigma)^{q_n} \left(\mathrm{Li}_1(e^{i\sigma})\right)^{r_n} F_{\mathbf{q}_-}^{\mathbf{r}_-}(\sigma).$$

Proof. In the case $n = 1$, by Proposition 9, we have

$$\frac{d}{d\sigma} F_{(q_1)}^{(r_1)}(\sigma) = \frac{d}{d\sigma} \left(f_{(q_1)}^{(r_1)}(\sigma) - f_{(q_1)}^{(r_1)}(0)\right) = i(i\sigma)^{q_1} \left(\mathrm{Li}_1(e^{i\sigma})\right)^{r_1}.$$

In the case $n \geq 2$, we have

$$\begin{aligned} \frac{d}{d\sigma} F_{\mathbf{q}}^{\mathbf{r}}(\sigma) &= i(i\sigma)^{q_n} \left(\mathrm{Li}_1(e^{i\sigma})\right)^{r_n} \sum_{\substack{(\mathbf{q}^{(1)},\dots,\mathbf{q}^{(h)})=\mathbf{q} \\ (\mathbf{r}^{(1)},\dots,\mathbf{r}^{(h)})=\mathbf{r} \\ \mathrm{d}(\mathbf{q}^{(j)})=\mathrm{d}(\mathbf{r}^{(j)})\geq 1 \\ (1\leq j\leq h)}} (-1)^{h-1} \left(\prod_{j=1}^{h-1} f_{\mathbf{q}^{(j)}}^{\mathbf{r}^{(j)}}(0)\right) f_{(\mathbf{q}^{(h)})_-}^{(\mathbf{r}^{(h)})_-}(\sigma) \\ &= i(i\sigma)^{q_n} \left(\mathrm{Li}_1(e^{i\sigma})\right)^{r_n} \left(\sum_{\substack{(\mathbf{q}^{(1)},\dots,\mathbf{q}^{(h)})=\mathbf{q} \\ (\mathbf{r}^{(1)},\dots,\mathbf{r}^{(h)})=\mathbf{r} \\ \mathrm{d}(\mathbf{q}^{(j)})=\mathrm{d}(\mathbf{r}^{(j)})\geq 1 \\ (1\leq j\leq h) \\ \mathrm{d}(\mathbf{q}^{(h)})=\mathrm{d}(\mathbf{r}^{(h)})\geq 2}} + \sum_{\substack{(\mathbf{q}^{(1)},\dots,\mathbf{q}^{(h)})=\mathbf{q} \\ (\mathbf{r}^{(1)},\dots,\mathbf{r}^{(h)})=\mathbf{r} \\ \mathrm{d}(\mathbf{q}^{(j)})=\mathrm{d}(\mathbf{r}^{(j)})\geq 1 \\ (1\leq j\leq h) \\ \mathrm{d}(\mathbf{q}^{(h)})=\mathrm{d}(\mathbf{r}^{(h)})=1}} \right) \\ &\qquad (-1)^{h-1} \left(\prod_{j=1}^{h-1} f_{\mathbf{q}^{(j)}}^{\mathbf{r}^{(j)}}(0)\right) f_{(\mathbf{q}^{(h)})_-}^{(\mathbf{r}^{(h)})_-}(\sigma) \\ &= i(i\sigma)^{q_n} \left(\mathrm{Li}_1(e^{i\sigma})\right)^{r_n} \sum_{\substack{(\mathbf{q}^{(1)},\dots,\mathbf{q}^{(h)})=\mathbf{q}_- \\ (\mathbf{r}^{(1)},\dots,\mathbf{r}^{(h)})=\mathbf{r}_- \\ \mathrm{d}(\mathbf{q}^{(j)})=\mathrm{d}(\mathbf{r}^{(j)})\geq 1 \\ (1\leq j\leq h)}} (-1)^{h-1} \left(\prod_{j=1}^{h-1} f_{\mathbf{q}^{(j)}}^{\mathbf{r}^{(j)}}(0)\right) f_{\mathbf{q}^{(h)}}^{\mathbf{r}^{(h)}}(\sigma) \\ &\quad + i(i\sigma)^{q_n} \left(\mathrm{Li}_1(e^{i\sigma})\right)^{r_n} \sum_{\substack{(\mathbf{q}^{(1)},\dots,\mathbf{q}^{(h-1)})=\mathbf{q}_- \\ (\mathbf{r}^{(1)},\dots,\mathbf{r}^{(h-1)})=\mathbf{r}_- \\ \mathrm{d}(\mathbf{q}^{(j)})=\mathrm{d}(\mathbf{r}^{(j)})\geq 1 \\ (1\leq j\leq h-1)}} (-1)^{h-1} \left(\prod_{j=1}^{h-1} f_{\mathbf{q}^{(j)}}^{\mathbf{r}^{(j)}}(0)\right). \end{aligned}$$

Because

$$\sum_{\substack{(\mathbf{q}^{(1)},\dots,\mathbf{q}^{(h-1)})=\mathbf{q}_- \\ (\mathbf{r}^{(1)},\dots,\mathbf{r}^{(h-1)})=\mathbf{r}_- \\ \mathrm{d}(\mathbf{q}^{(j)})=\mathrm{d}(\mathbf{r}^{(j)})\geq 1 \\ (1\leq j\leq h-1)}} (-1)^{h-1} \left(\prod_{j=1}^{h-1} f_{\mathbf{q}^{(j)}}^{\mathbf{r}^{(j)}}(0) \right)$$

$$= - \sum_{\substack{(\mathbf{q}^{(1)},\dots,\mathbf{q}^{(h)})=\mathbf{q}_- \\ (\mathbf{r}^{(1)},\dots,\mathbf{r}^{(h)})=\mathbf{r}_- \\ \mathrm{d}(\mathbf{q}^{(j)})=\mathrm{d}(\mathbf{r}^{(j)})\geq 1 \\ (1\leq j\leq h)}} (-1)^{h-1} \left(\prod_{j=1}^{h-1} f_{\mathbf{q}^{(j)}}^{\mathbf{r}^{(j)}}(0) \right) f_{\mathbf{q}^{(h)}}^{\mathbf{r}^{(h)}}(0),$$

we obtain the desired formula. □

Proposition 11. *We have*

$$\mathrm{ILi}_{\mathbf{q}}^{\mathbf{r}}(\sigma) = F_{\mathbf{q}}^{\mathbf{r}}(\sigma).$$

Therefore, by Lemma 3 and Proposition 11, Theorem 24 holds.

Proof. We prove this proposition by induction on n. In the case $n = 1$, by Proposition 10, we have

$$\frac{d}{d\sigma} F_{(q_1)}^{(r_1)}(\sigma) = i(i\sigma)^{q_1} \left(\mathrm{Li}_1(e^{i\sigma})\right)^{r_1}.$$

Therefore, we obtain

$$\begin{aligned}\mathrm{ILi}_{(q_1)}^{(r_1)}(\sigma) &= \int_0^\sigma i(i\theta_1)^{q_1} \left(\mathrm{Li}_1(e^{i\theta_1})\right)^{r_1} \, d\theta_1 \\ &= F_{(q_1)}^{(r_1)}(\sigma) - F_{(q_1)}^{(r_1)}(0) \\ &= F_{(q_1)}^{(r_1)}(\sigma).\end{aligned}$$

We prove $\mathrm{ILi}_{\mathbf{q}}^{\mathbf{r}}(\sigma) = F_{\mathbf{q}}^{\mathbf{r}}(\sigma)$ with the assumption $\mathrm{ILi}_{\mathbf{q}_-}^{\mathbf{r}_-}(\sigma) = F_{\mathbf{q}_-}^{\mathbf{r}_-}(\sigma)$. By Proposition 10

$$\frac{d}{d\sigma} F_{\mathbf{q}}^{\mathbf{r}}(\sigma) = i(i\sigma)^{q_n} \left(\mathrm{Li}_1(e^{i\sigma})\right)^{r_n} \mathrm{ILi}_{\mathbf{q}_-}^{\mathbf{r}_-}(\sigma),$$

and hence, we obtain

$$\begin{aligned}\mathrm{ILi}_{\mathbf{q}}^{\mathbf{r}}(\sigma) &= \int_0^\sigma i(i\theta_n)^{q_n} \left(\mathrm{Li}_1(e^{i\theta_n})\right)^{r_n} \mathrm{ILi}_{\mathbf{q}_-}^{\mathbf{r}_-}(\theta_n) \, d\theta_n \\ &= F_{\mathbf{q}}^{\mathbf{r}}(\sigma) - F_{\mathbf{q}}^{\mathbf{r}}(0) \\ &= F_{\mathbf{q}}^{\mathbf{r}}(\sigma).\end{aligned}$$

□

3.3.4 Example

We give some examples of Theorem 24. For $\sigma \in [0, 2\pi]$, we obtain the following evaluations:

$$\begin{aligned}
\mathrm{Ls}_1^{(0)}(\sigma) &= -\sigma, \\
\mathrm{Ls}_2^{(0)}(\sigma) &= -\frac{1}{2} i\pi\sigma + \frac{1}{4} i\sigma^2 + i\zeta(2) - i\mathrm{Li}_2(e^{i\sigma}), \\
\mathrm{Ls}_2^{(1)}(\sigma) &= -\frac{1}{2}\sigma^2, \\
\mathrm{Ls}_{1,1}^{(0,0)}(\sigma) &= \frac{1}{2}\sigma^2, \\
\mathrm{Ls}_3^{(0)}(\sigma) &= \frac{1}{4}\pi^2\sigma - \frac{1}{4}\pi\sigma^2 + \frac{1}{12}\sigma^3 - \pi\zeta(2) + \pi\mathrm{Li}_2(e^{i\sigma}) - \sigma\mathrm{Li}_2(e^{i\sigma}) \\
&\quad + i\zeta(3) - i\mathrm{Li}_3(e^{i\sigma}) - 2i\zeta(1,2) + 2i\mathrm{Li}_{1,2}(e^{i\sigma}), \\
\mathrm{Ls}_3^{(1)}(\sigma) &= -\frac{1}{4} i\pi\sigma^2 + \frac{1}{6} i\sigma^3 - i\sigma\mathrm{Li}_2(e^{i\sigma}) - \zeta(3) + \mathrm{Li}_3(e^{i\sigma}), \\
\mathrm{Ls}_3^{(2)}(\sigma) &= -\frac{1}{3}\sigma^3, \\
\mathrm{Ls}_{1,2}^{(0,0)}(\sigma) &= \frac{1}{4} i\pi\sigma^2 - \frac{1}{6} i\sigma^3 + i\sigma\mathrm{Li}_2(e^{i\sigma}) + \zeta(3) - \mathrm{Li}_3(e^{i\sigma}), \\
\mathrm{Ls}_{1,2}^{(0,1)}(\sigma) &= \frac{1}{3}\sigma^3, \\
\mathrm{Ls}_{2,1}^{(0,0)}(\sigma) &= \frac{1}{4} i\pi\sigma^2 - \frac{1}{12} i\sigma^3 - i\sigma\zeta(2) - \zeta(3) + \mathrm{Li}_3(e^{i\sigma}), \\
\mathrm{Ls}_{2,1}^{(1,0)}(\sigma) &= \frac{1}{6}\sigma^3, \\
\mathrm{Ls}_{1,1,1}^{(0,0,0)}(\sigma) &= -\frac{1}{6}\sigma^3.
\end{aligned}$$

We note that iterated log-sine integrals satisfying $k_u - 1 - l_u = 0$ for all $u \in \{1, \ldots, n\}$ such as $\mathrm{Ls}_1^{(0)}(\sigma)$, $\mathrm{Ls}_2^{(1)}(\sigma)$ and $\mathrm{Ls}_{1,1}^{(0,0)}(\sigma)$ can be evaluated by the definition trivially. In the case $\mathbf{k} = (k)$ and $\mathbf{l} = (k-2)$, the log-sine integral $\mathrm{Ls}_k^{(k-2)}(\sigma)$ has the following easy expression:

$$\begin{aligned}
\mathrm{Ls}_k^{(k-2)}(\sigma) = &\frac{i}{2k}\sigma^k - \frac{i\pi}{2(k-1)}\sigma^{k-1} + i^{k-1}(k-2)!\zeta(k) \\
&- \sum_{j=0}^{k-2} i^{j+1} \frac{(k-2)!}{(k-2-j)!}\sigma^{k-2-j}\,\mathrm{Li}_{j+2}(e^{i\sigma})
\end{aligned}$$

which is equivalent to [33, (7.51)].

3.3.5 Evaluation of iterated log-sine integrals at general argument

We discuss iterated log-sine integrals at general argument and give a proof of Theorem 25. First, we note the following reflection formula.

Lemma 4. *We have*

$$\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma) = (-1)^{|\mathbf{l}|+n}\,\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(-\sigma).$$

This lemma is given by changing of variables $\theta_u = -\theta_u$. By Lemma 4, we only need to prove Theorem 25 for $2m\pi + \sigma \geq 0$.

We introduce the following integrals for convenience.

Definition 12. For $\rho, \sigma \in \mathbb{R}$, $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ and $\mathbf{l} = (l_1, \ldots, l_n) \in (\mathbb{Z}_{\geq 0})^n$, we define

$$\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\rho; \sigma) = (-1)^n \int_\rho^\sigma \int_\rho^{\theta_n} \cdots \int_\rho^{\theta_2} \prod_{u=1}^{n} \theta_u^{l_u} A^{k_u-1-l_u}(\theta_u)\, d\theta_1 \cdots d\theta_n.$$

Then, the following lemmas hold.

Lemma 5. *Let* $0 < \rho < \sigma$, $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$, $\mathbf{l} = (l_1, \dots, l_n) \in (\mathbb{Z}_{\geq 0})^n$, *and*

$$\mathbf{k}_h = (k_1, \dots, k_h), \qquad \mathbf{k}^h = (k_{h+1}, \dots, k_n),$$
$$\mathbf{l}_h = (l_1, \dots, l_h), \qquad \mathbf{l}^h = (l_{h+1}, \dots, l_n).$$

If $\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma)$ *converges absolutely, then*

$$\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(\sigma) = \sum_{h=0}^{n} \mathrm{Ls}_{\mathbf{k}_h}^{\mathbf{l}_h}(\rho) \, \mathrm{Ls}_{\mathbf{k}^h}^{\mathbf{l}^h}(\rho; \sigma).$$

This decomposition formula is proved by decomposing the domain of the integral

$$0 < \theta_1 < \theta_2 < \cdots < \theta_n < \sigma$$

into

$$\begin{gathered} 0 < \rho < \theta_1 < \theta_2 < \cdots < \theta_n < \sigma, \\ 0 < \theta_1 < \rho < \theta_2 < \cdots < \theta_n < \sigma, \\ \vdots \\ 0 < \theta_1 < \theta_2 < \cdots < \theta_n < \rho < \sigma. \end{gathered}$$

We write $\mathbf{j} \leq \mathbf{k}$ when $\mathbf{j} = (j_1, \dots, j_n)$ and $\mathbf{k} = (k_1, \dots, k_n)$ satisfy $j_1 \leq k_1, j_2 \leq k_2, \dots, j_n \leq k_n$, and define

$$\binom{\mathbf{l}}{\mathbf{j}} = \prod_{u=1}^{n} \binom{l_u}{j_u}.$$

Lemma 6. *For* $m \in \mathbb{Z}$, *we have*

$$\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(2m\pi, 2m\pi + \sigma) = \sum_{\mathbf{j} \leq \mathbf{l}} (2m\pi)^{|\mathbf{j}|} \binom{\mathbf{l}}{\mathbf{j}} \mathrm{Ls}_{\mathbf{k}-\mathbf{j}}^{\mathbf{l}-\mathbf{j}}(\sigma),$$

where the sum is over all $\mathbf{j} = (j_1, \dots, j_n) \in (\mathbb{Z}_{\geq 0})^n$ *satisfying* $\mathbf{j} \leq \mathbf{l}$.

Proof. By changing of variables $\theta_u = 2m\pi - \theta_u$, we have

$$\begin{aligned} &\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(2m\pi, 2m\pi + \sigma) \\ &= (-1)^n \int_{2m\pi}^{2m\pi+\sigma} \int_{2m\pi}^{\theta_n} \cdots \int_{2m\pi}^{\theta_2} \prod_{u=1}^{n} \theta_u^{l_u} A^{k_u - 1 - l_u}(\theta_u) \, d\theta_1 \cdots d\theta_n \\ &= (-1)^n \int_0^{-\sigma} \int_0^{\theta_n} \cdots \int_0^{\theta_2} \prod_{u=1}^{n} - (2m\pi - \theta_u)^{l_u} A^{k_u - 1 - l_u}(\theta_u) \, d\theta_1 \cdots d\theta_n \\ &= \int_0^{-\sigma} \int_0^{\theta_n} \cdots \int_0^{\theta_2} \prod_{u=1}^{n} \sum_{j_u=0}^{l_u} (2m\pi)^{j_u} \binom{l_u}{j_u} (-\theta_u)^{l_u - j_u} A^{k_u - 1 - l_u}(\theta_u) \, d\theta_1 \cdots d\theta_n \\ &= \sum_{\mathbf{j} \leq \mathbf{l}} (2m\pi)^{|\mathbf{j}|} \binom{\mathbf{l}}{\mathbf{j}} (-1)^{|\mathbf{l}-\mathbf{j}|+n} \mathrm{Ls}_{\mathbf{k}-\mathbf{j}}^{\mathbf{l}-\mathbf{j}}(-\sigma). \end{aligned}$$

Here, we use Lemma 4 and obtain the desired formula. □

Lemma 7. *Iterated log-sine integrals* $\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(2m\pi)$ *satisfying* $k_u - 1 - l_u \geq 0$ *for all* $u \in \{1, \dots, n\}$ *can be written as a* $\mathbb{Q}$*-linear combination of products of a power of* π *and multiple zeta values.*

We note that the assertion for the generalized log-sine integrals $\mathrm{Ls}_k^{(l)}(2m\pi)$ was proved by Borwein and Straub [7, Section 4.1].

Proof. We assume $m \geq 1$ and prove this lemma by induction on m. In the case $m = 1$, we obtain the assertion by putting $\sigma = 2\pi$ in (3.21), since $\mathrm{Li}_{\mathbf{k}}(e^{2\pi i}) = \zeta(\mathbf{k})$ when $k_n \geq 2$. We assume the assertion for $2m\pi$ and prove for $2(m+1)\pi$. By putting $\rho = 2m\pi$ and $\sigma = 2(m+1)\pi$ in Lemma 5, we obtain

$$\begin{aligned}\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(2(m+1)\pi) &= \sum_{j=0}^{n} \mathrm{Ls}_{\mathbf{k}_h}^{\mathbf{l}_h}(2m\pi)\,\mathrm{Ls}_{\mathbf{k}^h}^{\mathbf{l}^h}(2m\pi; 2(m+1)\pi) \\ &= \sum_{j=0}^{n} \mathrm{Ls}_{\mathbf{k}_h}^{\mathbf{l}_h}(2m\pi) \sum_{\mathbf{j} \leq \mathbf{l}^h} \binom{\mathbf{l}^h}{\mathbf{j}} (2m\pi)^{|\mathbf{j}|}\,\mathrm{Ls}_{\mathbf{k}^h - \mathbf{j}}^{\mathbf{l}^h - \mathbf{j}}(2\pi).\end{aligned}$$

Here, the second equality is obtained by putting $\sigma = 2\pi$ in Lemma 6. Therefore, the assertion is proved. □

Proof of Theorem 25. Let $\sigma \in [0, 2\pi]$ and $m \geq 0$. By putting $\rho = 2m\pi$ and $\sigma = 2m\pi + \sigma$ in Lemma 5, we have

$$\mathrm{Ls}_{\mathbf{k}}^{\mathbf{l}}(2m\pi + \sigma) = \sum_{j=0}^{n} \mathrm{Ls}_{\mathbf{k}_h}^{\mathbf{l}_h}(2m\pi)\,\mathrm{Ls}_{\mathbf{k}^h}^{\mathbf{l}^h}(2m\pi; 2m\pi + \sigma).$$

Here, $\mathrm{Ls}_{\mathbf{k}_h}^{\mathbf{l}_h}(2m\pi)$ can be written in terms of multiple zeta values by Lemma 7, and $\mathrm{Ls}_{\mathbf{k}^h}^{\mathbf{l}^h}(2m\pi, 2m\pi + \sigma)$ can be written in terms of multiple zeta values and multiple polylogarithms with admissible index at $e^{i\sigma}$ by Lemma 6 and Theorem 24. Therefore, Theorem 25 is proved. □

3.4 Conjectures

3.4.1 Shifted log-sine integrals

We state some conjectures on multiple zeta values, multiple Clausen values, multiple Glaisher values and iterated log-sine integrals. On the conjectures, Shifted log-sine integrals play an important role. Here, we introduce and discuss Shifted log-sine integrals.

Shifted log-sine integrals are defined as a kind of log-sine integrals as follows.

Definition 13 (Shifted log-sine integrals)**.** For $\sigma \in \mathbb{R}$, $\mathbf{k} = (k_1, \ldots, k_n) \in (\mathbb{Z}_{\geq 2})^n$, we define

$$\mathrm{SLs}(\mathbf{k}; \sigma) = \int_0^{\sigma} \int_0^{\theta_n} \cdots \int_0^{\theta_2} \prod_{u=1}^{n} (\theta_u - \sigma)^{k_u - 2} A(\theta_u)\, d\theta_1 \cdots d\theta_n.$$

Shifted log-sine integrals satisfy the following shuffle product formula.

Proposition 12. *For $n, m \in \mathbb{N}$ and $k_1, \ldots, k_{n+m} \in \mathbb{Z}_{\geq 2}$, we have*

$$\begin{aligned}&\mathrm{SLs}(k_1, \ldots, k_n; \sigma) \cdot \mathrm{SLs}(k_{n+1}, \ldots, k_{n+m}; \sigma) \\ &= \sum_{\tau \in \mathfrak{S}_{n,m}} \mathrm{SLs}(k_{\tau(1)}, \ldots, k_{\tau(n+m)}; \sigma),\end{aligned}$$

where $\mathfrak{S}_{n,m}$ is the same as defined in Proposition 7.

This proposition is proved by decomposing the domain of the integration the same as Proposition 7.

Shifted log-sine integrals can be written in terms of iterated log-sine integrals as follows.

Theorem 26. *Let*

$$\mathbf{2}_n = (\{2\}^n).$$

For $\sigma \in \mathbb{R}$ and $\mathbf{k} = (k_1, \ldots, k_n) \in (\mathbb{Z}_{\geq 2})^n$, we have

$$\mathrm{SLs}(\mathbf{k}; \sigma) = (-1)^n \sum_{\mathbf{j} \leq \mathbf{k} - \mathbf{2}_n} \binom{\mathbf{k} - \mathbf{2}_n}{\mathbf{j}} (-\sigma)^{|\mathbf{k} - \mathbf{2}_n - \mathbf{j}|}\,\mathrm{Ls}_{\mathbf{j} + \mathbf{2}_n}^{\mathbf{j}}(\sigma),$$

where the sum is over all $\mathbf{j} = (j_1, \ldots, j_n) \in (\mathbb{Z}_{\geq 0})^n$ satisfying $\mathbf{j} \leq \mathbf{k} - \mathbf{2}_n$.

Proof. We calculate the integrand $\prod_{u=1}^{n}(\theta_u-\sigma)^{k_u-2}A(\theta_u)$ of the definition of the shifted log-sine integral. Then, we have

$$\begin{aligned}\prod_{u=1}^{n}(\theta_u-\sigma)^{k_u-2}A(\theta_u)&=\prod_{u=1}^{n}\sum_{j_u=0}^{k_u-2}\binom{k_u-2}{j_u}\theta_u^{j_u}(-\sigma)^{k_u-2-j_u}A(\theta_u)\\&=\sum_{\mathbf{j}\le\mathbf{k}-\mathbf{2}_n}\binom{\mathbf{k}-\mathbf{2}_n}{\mathbf{j}}(-\sigma)^{|\mathbf{k}-\mathbf{2}_n-\mathbf{j}|}\prod_{u=1}^{n}\theta_u^{j_u}A(\theta_u).\end{aligned}$$

Therefore, we obtain the desired formula. □

We write $\mathrm{SLs}(\mathbf{k}):=\mathrm{SLs}(\mathbf{k};\pi/3)$, because values of shifted log-sine integrals at $\pi/3$ are especially important on the conjectures.

Conjecture 5. Elements of the set

$$\{\pi^m\,\mathrm{SLs}(k_1,\dots,k_n)\mid m\ge 0,n\ge 0,k_i\ge 2\}$$

are $\mathbb{Q}$-linearly independent.

The author calculated numerical values of shifted log-sine integrals by using Theorem 26 and Theorem 24. Then, by using the lindep of Pari/GP, there seems to be no $\mathbb{Q}$-linear relations among these values satisfying $m+k_1+\cdots+k_n\le 8$.

3.4.2 On multiple zeta values

For $k\in\mathbb{Z}_{\ge0}$ and $d\in\mathbb{Z}_{\ge0}$, we define

$$S'_{k,d}=\left\{\pi^{2m}\,\mathrm{SLs}(k_1,\dots,k_n)\;\middle|\;\begin{array}{c}2m+k_1+\cdots+k_n=k,\\ m\ge 0,d\ge n\ge 0,\\ k_i\ge 3:\text{odd}.\end{array}\right\}.$$

Conjecture 6. Every multiple zeta value with weight k and depth d can be written as a $\mathbb{Q}$-linear combination of elements of $S'_{k,d}$.

Conjecture 7. The set of the real numbers $S'_{k,k}$ is a basis of $\mathcal{Z}_k$ defined in Section 1.1.1.

If Conjecture 7 is true, then Conjecture 1 is true, because the numbers $|S'_{k,d}|$ of elements of $S'_{k,d}$ satisfy

$$\begin{aligned}&|S'_{0,d}|=1,\ |S'_{1,d}|=0,\ |S'_{2,d}|=1\quad(d\ge 0),\\&|S'_{k,d}|=|S'_{k-2,d}|+|S'_{k-3,d-1}|\quad(k\ge 3,d\ge 1).\end{aligned}$$

It is known that every multiple zeta value with weight k and depth d can be written as a $\mathbb{Q}$-linear combination of elements of $H_{k,d}$ as mentioned in Section 1.1.1. Conjecture 6 can be regarded as an analogue of this assertion. Conjecture 7 can be regarded as an analogue of Conjecture 2. However, the real numbers in $S'_{k,d}$ have different properties from those of the multiple zeta values in $H_{k,d}$. We define

$$\mathcal{S}'_{k,d}=\mathrm{span}_{\mathbb{Q}}(S'_{k,d})\ \text{ and }\ \mathcal{H}_{k,d}=\mathrm{span}_{\mathbb{Q}}(H_{k,d}).$$

Then, we can see that $\mathcal{S}'_{k_1,d_1}\cdot\mathcal{S}'_{k_2,d_2}\subset\mathcal{S}'_{k_1+k_2,d_1+d_2}$ just by using the shuffle product formula (Proposition 12). More strictly, we can see that the product of an element of S'_{k_1,d_1} and an element of S'_{k_2,d_2} can be written as the sum of products of a positive integer and an element of $S'_{k_1+k_2,d_1+d_2}$. However, we cannot see that $\mathcal{H}_{k_1,d_1}\cdot\mathcal{H}_{k_2,d_2}\subset\mathcal{H}_{k_1+k_2,d_1+d_2}$ just by using the harmonic product formula or the shuffle product formula. For example, each products of two $\zeta(2)\in H_{2,0}$ are

$$\zeta(2)\cdot\zeta(2)=\begin{cases}2\zeta(2,2)+\zeta(4)&\text{(harmonic product)},\\2\zeta(2,2)+4\zeta(1,3)&\text{(shuffle product)}.\end{cases}$$

Here, $\zeta(2,2)\in H_{4,0}$, but $\zeta(4)\notin H_{4,0}$ and $\zeta(1,3)\notin H_{4,0}$.

The author checked Conjecture 6 and Conjecture 7 up to weight 13 by numerical experiments.

Theoretically, we can check that some multiple zeta values can be written as a $\mathbb{Q}$-linear combination of elements of $S'_{k,d}$. First, we consider the Riemann zeta values.

Theorem 27. *The Riemann zeta value $\zeta(k)$ can be written as a $\mathbb{Q}$-linear combination of elements of $S'_{k,1}$.*

Proof. When the argument is even, it is known that $\zeta(2k) \in \mathbb{Q} \cdot \pi^{2k}$. Therefore, we can see that $\zeta(2k)$ can be written as a $\mathbb{Q}$-linear combination of elements of $S'_{2k,1} = S'_{2k,0} = \{\pi^{2k}\}$.

When the argument is odd, we use the formula (1.15). We show that $\zeta(2k+3)$ can be written as a $\mathbb{Q}$-linear combination of elements of $S'_{2k+3,1}$ by induction on k. By the formula (1.16), we can see that $\zeta(3)$ can be written as a $\mathbb{Q}$-linear combination of elements of $S'_{3,1} = \{\mathrm{SLs}(3)\}$. By (1.15), we have

$$\begin{aligned} &-\frac{1}{2}(2k+1)!\left((1-2^{-2k-2})(1-3^{-2k-2})-2\right)\zeta(2k+3) \\ &= (-1)^k \,\mathrm{SLs}(2k+3) - (2k+1)! \sum_{m=1}^{k} (-1)^m \left(\frac{\pi}{3}\right)^{2m} \frac{\zeta(2k+3-2m)}{(2m)!}. \end{aligned}$$

We assume $\zeta(2k+3-2m)$ with $m \in \{1,\dots,k\}$ can be written as a $\mathbb{Q}$-linear combination of elements of $S'_{2k+3-2m,1}$. Then, we can see that $\zeta(2k+3)$ can be written as a $\mathbb{Q}$-linear combination of elements of $S'_{2k+3,1}$ because the product of π^{2m} and an element of $S'_{2k+3-2m,1}$ is included in $S'_{2k+3,1}$. ☐

By Theorem 27 and the shuffle product formula, multiple zeta values such that

$$\zeta(\mathbf{k}) \in \sum_{\substack{2m+l_1+\cdots+l_h=|\mathbf{k}| \\ m\geq 0,\ 0\leq h\leq d}} \mathbb{Q}\cdot\pi^{2m}\cdot\zeta(l_1)\cdots\zeta(l_h)$$

can be written as a $\mathbb{Q}$-linear combination of elements of $S'_{|\mathbf{k}|,d}$. In particular, multiple zeta values of the weight up to 7 can be written as a $\mathbb{Q}$-linear combination of elements of $S'_{k,d}$. In addition, by evaluations introduced in Section 1.1.1, we fined that $\zeta(\{1,3\}^{2n})$, $\zeta(\{2k\}^n)$, $\zeta(\{3,1\}^n,3)$, $\zeta(\{3,1\}^n,2)$, $\zeta(s,t)$ with $s+t$:odd, $\zeta(\{2\}^a,3,\{2\}^b)$ and $\zeta(\{1\}^{n-1},m+1)$ can also be written as a $\mathbb{Q}$-linear combination of elements of $S'_{k,d}$.

3.4.3 On multiple Clausen values and multiple Glaisher values

For $k \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{\geq 0}$, we define

$$S^o_{k,d} = \left\{ \pi^m\,\mathrm{SLs}(k_1,\dots,k_n) \;\middle|\; \begin{array}{c} m+k_1+\cdots+k_n=k, \\ d\geq n\geq 0,\ n:\text{odd}, \\ m\geq 0, k_i\geq 2. \end{array} \right\},$$

$$S^e_{k,d} = \left\{ \pi^m\,\mathrm{SLs}(k_1,\dots,k_n) \;\middle|\; \begin{array}{c} m+k_1+\cdots+k_n=k, \\ d\geq n\geq 0,\ n:\text{even}, \\ m\geq 0, k_i\geq 2. \end{array} \right\}.$$

Conjecture 8. The following assertions hold.

(i) Every multiple Clausen values with weight k and depth d can be written as a $\mathbb{Q}$-linear combination of elements of $S^o_{k,d}$.

(ii) Every multiple Glaisher values with weight k and depth d can be written as a $\mathbb{Q}$-linear combination of elements of $S^e_{k,d}$.

Conjecture 9. The following assertions hold.

(i) The set of the real numbers $S^o_{k,k}$ is a basis of $\mathcal{C}_k$.

(ii) The set of the real numbers $S^e_{k,k}$ is a basis of $\mathcal{G}_k$.

(iii) The set of the real numbers $S^o_{k,k} \cup S^e_{k,k}$ is a basis of $\mathcal{C}_k + \mathcal{G}_k$.

If Conjecture 9 is true, then conjecture 4 is true, because $|S^o_{k,d}|$ and $|S^e_{k,d}|$ satisfy

$$\begin{aligned}
&|S^o_{0,0}| = |S^o_{1,1}| = 0, \quad |S^e_{0,0}| = |S^e_{1,1}| = 1,\\
&|S^o_{k,d}| = |S^o_{k-1,d}| + |S^e_{k-2,d-1}| \quad (k \geq 2, d \geq 1),\\
&|S^e_{k,d}| = |S^e_{k-1,d}| + |S^o_{k-2,d-1}| \quad (k \geq 2, d \geq 1).
\end{aligned}$$

The author checked Conjecture 8 and Conjecture 9 up to weight 9 by numerical experiments.

Theoretically, we can check that some multiple Clausen values and multiple Glaisher values can be written as a $\mathbb{Q}$-linear combination of elements of $S^o_{k,d}$ and $S^e_{k,d}$, respectively. First, we consider the case when the depth is 1.

Theorem 28. *The following assertions hold.*

(i) The Clausen value $\mathrm{Cl}_k(\pi/3)$ *can be written as a* $\mathbb{Q}$*-linear combination of elements of* $S^o_{k,1}$.

(ii) The Glaisher value $\mathrm{Gl}_k(\pi/3)$ *can be written as a* $\mathbb{Q}$*-linear combination of elements of* $S^e_{k,1}(= S^e_{k,0})$.

Proof. Clausen values of odd index are evaluated as follows (see [33, p.198]):

$$\mathrm{Cl}_{2k+1}(\pi/3) = \frac{1}{2}(1-2^{-2k})(1-3^{-2k})\zeta(2k+1).$$

By Theorem 27 and $S'_{2k+1,1} \subset S^o_{2k+1,1}$, the assertion (i) for odd indices holds.

On even index, we use the following formula for $\mathrm{SLs}(2k+2)$:

$$\begin{aligned}
\mathrm{SLs}(2k+2) = {}& \frac{i}{2k+1}\left(-\frac{\pi}{3}\right)^{2k+2} + \frac{i}{2(2k+2)}\left(-\frac{\pi}{3}\right)^{2k+2}\\
&+ i(-1)^k(2k)!\,\mathrm{Li}_{2+2k}(e^{i\pi/3}) - i(-1)^k(2k)!\sum_{j=0}^{2k}\frac{(i\pi/3)^{2k-j}}{(2k-j)!}\zeta(2+j),
\end{aligned}$$

which is obtained by Theorem 26 and Theorem 24. Therefore, we obtain

$$\begin{aligned}
&\mathrm{Cl}_{2+2k}(\pi/3)\\
&= -\frac{(-1)^k}{(2k)!}\mathrm{SLs}(2k+2) + \sum_{j=0}^{k-1}(-1)^{k-j-1}\frac{(\pi/3)^{2k-2j-1}}{(2k-2j-1)!}\zeta(3+2j)
\end{aligned}$$

and obtain the assertion (i).

On the other hand, the following explicit evaluation of Glaisher functions is known (see [33, (7.60)], for example):

$$\mathrm{Gl}_k(2\pi x) = (-1)^{1+[k/2]}2^{k-1}\pi^k B_k(x)/k! \qquad (0 \leq x \leq 1, k > 1),$$

where $B_n(x)$ denotes the n-th Bernoulli polynomial. Therefore, we have

$$\mathrm{Gl}_k(\pi/3) = (-1)^{1+[k/2]}2^{k-1}\pi^k B_k(1/6)/k!.$$

Since $S^e_{k,1} = S^e_{k,0} = \{\pi^k\}$, we can see that Glaisher value $\mathrm{Gl}_k(\pi/3)$ can be written as a $\mathbb{Q}$-linear combination of elements of $S^e_{k,1}$. □

We consider indices the form $\mathbf{k} = (\{1\}^a, 2, \{1\}^b)$. By Theorem 23, we have

$$\begin{aligned}
&\mathrm{Li}_{\{1\}^a,2,\{1\}^b}(e^{\frac{\pi}{3}i}) \\
&= i^{a+b+1} \int_{0<\theta_1<\cdots<\theta_{a+b+1}<\frac{\pi}{3}} A(\theta_{a+2}) - A(\theta_{a+1}) - \frac{i\theta_{a+2}}{2} + \frac{i\theta_{a+1}}{2}\, d\theta_1 \cdots d\theta_{a+b+1} \\
&= i^{a+b+1} \frac{1}{(a+1)!} \frac{1}{(b-1)!} \int_0^{\frac{\pi}{3}} \theta^{a+1} A(\theta) \left(\frac{\pi}{3} - \theta\right)^{b-1} d\theta \\
&\quad - i^{a+b+1} \frac{1}{a!} \frac{1}{b!} \int_0^{\frac{\pi}{3}} \theta^{a} A(\theta) \left(\frac{\pi}{3} - \theta\right)^{b} d\theta \\
&\quad + \frac{i^{a+b}}{2} \left(\frac{a+2}{(a+b+2)!} \left(\frac{\pi}{3}\right)^{a+b+2} - \frac{a+1}{(a+b+2)!} \left(\frac{\pi}{3}\right)^{a+b+2} \right) \\
&= i^{a+b+1} \sum_{j=0}^{a+1} \frac{1}{(a+1)!} \frac{(-1)^{b-1}}{(b-1)!} \binom{a+1}{j} \left(\frac{\pi}{3}\right)^j \mathrm{SLs}(a+b+2-j) \\
&\quad - i^{a+b+1} \sum_{j=0}^{a} \frac{1}{a!} \frac{(-1)^b}{b!} \binom{a}{j} \left(\frac{\pi}{3}\right)^j \mathrm{SLs}(a+b+2-j) + i^{a+b} \frac{(\pi/3)^{a+b+2}}{2(a+b+2)!}
\end{aligned}$$

for $a \geq 0$ and $b \geq 1$, and similarly

$$\mathrm{Li}_{\{1\}^a,2}(e^{\frac{\pi}{3}i}) = -i^{a+1} \sum_{j=0}^{a} \frac{1}{a!} \binom{a}{j} \left(\frac{\pi}{3}\right)^j \mathrm{SLs}(a+2-j) + i^a \frac{(\pi/3)^{a+2}}{2(a+2)!}$$

for $a \geq 0$. Therefore, for $a \geq 0$ and $b \geq 0$, we can see that $\mathrm{Cl}_{\{1\}^a,2,\{1\}^b}(\pi/3)$ and $\mathrm{Gl}_{\{1\}^a,2,\{1\}^b}(\pi/3)$ can be written as a $\mathbb{Q}$-linear combination of elements of $S^o_{a+b+2,1}$ and $S^e_{a+b+2,1} (= S^e_{a+b+2,0})$, respectively.

3.4.4 On iterated log-sine integrals

On iterated log-sine integrals, we conjecture as follows.

Conjecture 10. The following assertions hold.

(i) Every iterated log-sine integral at $\pi/3$ of weight k and odd $d := |\mathbf{k} - \mathbf{1}_n - \mathbf{l}|$ can be written as a $\mathbb{Q}$-linear combination of elements of $S^o_{k,d}$.

(ii) Every iterated log-sine integral at $\pi/3$ of weight k and even $d := |\mathbf{k} - \mathbf{1}_n - \mathbf{l}|$ can be written as a $\mathbb{Q}$-linear combination of elements of $S^e_{k,d}$.

Conjecture 11. The following assertions hold.

(i) The set of the real numbers $S^o_{k,k}$ is a basis of $\mathcal{L}^o_k$.

(ii) The set of the real numbers $S^e_{k,k}$ is a basis of $\mathcal{L}^e_k$.

(iii) The set of the real numbers $S^o_{k,k} \cup S^e_{k,k}$ is a basis of $\mathcal{L}^o_k + \mathcal{L}^e_k$.

If Conjecture 11 is true, then $\dim \mathcal{L}^o_k = I(k)$ and $\dim \mathcal{L}^e_k = R(k)$ hold. We showed $\dim \mathcal{L}^o_k \leq 2^{k-2}$ and $\dim \mathcal{L}^e_k \leq 2^{k-2}$ for $k \geq 2$ using the reduction formulas in Section 3.1.2. Therefore, if Conjecture 11 is true, then there are $2^{k-2} - I(k)$ non-trivial relations among iterated log-sine integrals at $\pi/3$ whose weight is k and $|\mathbf{k} - \mathbf{1}_n - \mathbf{l}|$ is an odd number, and $2^{k-2} - R(k)$ non-trivial relations among iterated log-sine integrals at $\pi/3$ whose weight is k and $|\mathbf{k} - \mathbf{1}_n - \mathbf{l}|$ is an even number.

The following is a table on the dimensions of $\mathcal{L}^o_k$.

k	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\lvert L^o_{k,k}(\sigma)\rvert$	0	0	1	3	9	25	68	182	483	1275	3355	$\cdots$
$\lvert M^o_{k,k}(\sigma)\rvert$	0	0	1	2	4	8	16	32	64	128	256	$\cdots$
$I(k)$	0	0	1	2	3	4	6	10	17	28	45	$\cdots$

Here, $|L^o_{k,k}(\sigma)|$, $|M^o_{k,k}(\sigma)|$, and $I(k)$ are the number of generators of the definition of $\mathcal{L}^o_k$, the upper bound of $\dim \mathcal{L}^o_k$ given by the reduction formulas, and the conjectured dimension of $\mathcal{L}^o_k$, respectively.

The following table is the even version of the table above.

k	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\lvert L^e_{k,k}(\sigma)\rvert$	1	1	2	5	12	30	76	195	504	1309	3410	$\cdots$
$\lvert M^e_{k,k}(\sigma)\rvert$	1	1	1	2	4	8	16	32	64	128	256	$\cdots$
$R(k)$	1	1	1	1	2	4	7	11	17	27	44	$\cdots$

Let $|L_{k,k}(\sigma)| = |L^o_{k,k}(\sigma)| + |L^e_{k,k}(\sigma)|$ and $|M_{k,k}(\sigma)| = |M^o_{k,k}(\sigma)| + |M^e_{k,k}(\sigma)|$. Then, a table on the dimensions of $\mathcal{L}^o_k + \mathcal{L}^e_k$ is as follows.

k	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\lvert L_{k,k}(\sigma)\rvert$	1	1	3	8	21	55	144	377	987	2584	6765	$\cdots$
$\lvert M_{k,k}(\sigma)\rvert$	1	1	2	4	8	16	32	64	128	256	512	$\cdots$
$W(k)$	1	1	2	3	5	8	13	21	34	55	89	$\cdots$

Conjecture 10 follows from Conjecture 6 and Conjecture 8 as follows. Therefore, Conjecture 10 can be checked numerically for the weight up to 9, indirectly.

Theorem 29. *If Conjecture 6 and Conjecture 8 are true, then Conjecture 10 is true.*

Proof. Let $\mathcal{S}^o_{k,d} = \mathrm{span}_{\mathbb{Q}}(S^o_{k,d})$ and $\mathcal{S}^e_{k,d} = \mathrm{span}_{\mathbb{Q}}(S^e_{k,d})$. We show that the right hand side of (3.21) with $\sigma = \pi/3$ belongs to $\mathcal{S}^o_{|\mathbf{k}|,|\mathbf{k}-\mathbf{1}_n-\mathbf{l}|}$ if $|\mathbf{k}-\mathbf{1}_n-\mathbf{l}|$ is odd, and belongs to $\mathcal{S}^e_{|\mathbf{k}|,|\mathbf{k}-\mathbf{1}_n-\mathbf{l}|}$ if $|\mathbf{k}-\mathbf{1}_n-\mathbf{l}|$ is even assuming Conjecture 6 and Conjecture 8. By (3.23) and Conjecture 6, we obtain

$$f^{\mathbf{r}}_{\mathbf{q}}(0) \in \mathcal{S}'_{|\mathbf{q}|+|\mathbf{r}|+n,|\mathbf{r}|}.$$

Therefore, we obtain

$$\begin{aligned}\prod_{j=1}^{h-1} f^{\mathbf{r}^{(j)}}_{\mathbf{q}^{(j)}}(0) &\in \mathcal{S}'_{\sum_{j=1}^{h-1}|\mathbf{q}^{(j)}|+|\mathbf{r}^{(j)}|+\mathrm{d}(\mathbf{r}^{(j)}),\sum_{j=1}^{h-1}|\mathbf{r}^{(j)}|} \\ &\subset \begin{cases}\mathcal{S}^o_{\sum_{j=1}^{h-1}|\mathbf{q}^{(j)}|+|\mathbf{r}^{(j)}|+\mathrm{d}(\mathbf{r}^{(j)}),\sum_{j=1}^{h-1}|\mathbf{r}^{(j)}|} \\ \qquad (\sum_{j=1}^{h-1}|\mathbf{q}^{(j)}|+|\mathbf{r}^{(j)}|+\mathrm{d}(\mathbf{r}^{(j)}) : \text{odd}), \\ \mathcal{S}^e_{\sum_{j=1}^{h-1}|\mathbf{q}^{(j)}|+|\mathbf{r}^{(j)}|+\mathrm{d}(\mathbf{r}^{(j)}),\sum_{j=1}^{h-1}|\mathbf{r}^{(j)}|} \\ \qquad (\sum_{j=1}^{h-1}|\mathbf{q}^{(j)}|+|\mathbf{r}^{(j)}|+\mathrm{d}(\mathbf{r}^{(j)}) : \text{even}).\end{cases}\end{aligned} \tag{3.25}$$

On the other hand, by Conjecture 8 we have

$$L(w^{\mathbf{j}}_{\mathbf{r}''}; e^{i\pi/3}) \in \begin{cases}\mathcal{S}^o_{|\mathbf{r}|+|\mathbf{j}|+n'',|\mathbf{r}|} + i\mathcal{S}^e_{|\mathbf{r}|+|\mathbf{j}|+n'',|\mathbf{r}|} & |\mathbf{r}|+|\mathbf{j}|+n'' : \text{odd}, \\ \mathcal{S}^e_{|\mathbf{r}|+|\mathbf{j}|+n'',|\mathbf{r}|} + i\mathcal{S}^o_{|\mathbf{r}|+|\mathbf{j}|+n'',|\mathbf{r}|} & |\mathbf{r}|+|\mathbf{j}|+n'' : \text{even}.\end{cases}$$

By multiplying this expression by $(i\pi/3)^{|\overline{\mathbf{q}}|-|\mathbf{j}|}$, we obtain

$$(i\pi/3)^{|\overline{\mathbf{q}}|-|\mathbf{j}|} L(w^{\mathbf{j}}_{\mathbf{r}''}; e^{i\pi/3}) \in \begin{cases}\mathcal{S}^o_{|\mathbf{q}|+|\mathbf{r}|+n,|\mathbf{r}|} + i\mathcal{S}^e_{|\mathbf{q}|+|\mathbf{r}|+n,|\mathbf{r}|} & |\mathbf{q}|+|\mathbf{r}|+n : \text{odd}, \\ \mathcal{S}^e_{|\mathbf{q}|+|\mathbf{r}|+n,|\mathbf{r}|} + i\mathcal{S}^o_{|\mathbf{q}|+|\mathbf{r}|+n,|\mathbf{r}|} & |\mathbf{q}|+|\mathbf{r}|+n : \text{even}.\end{cases}$$

Therefore, we obtain

$$f^{\mathbf{r}^{(h)}}_{\mathbf{q}^{(h)}}(\pi/3) - f^{\mathbf{r}^{(h)}}_{\mathbf{q}^{(h)}}(0) \in \begin{cases}\mathcal{S}^o_{|\mathbf{q}^{(h)}|+|\mathbf{r}^{(h)}|+\mathrm{d}(\mathbf{q}^{(j)}),|\mathbf{r}^{(h)}|} + i\mathcal{S}^e_{|\mathbf{q}^{(h)}|+|\mathbf{r}^{(h)}|+\mathrm{d}(\mathbf{q}^{(j)}),|\mathbf{r}^{(h)}|} \\ \qquad (|\mathbf{q}^{(h)}|+|\mathbf{r}^{(h)}|+\mathrm{d}(\mathbf{q}^{(j)}) : \text{odd}), \\ \mathcal{S}^e_{|\mathbf{q}^{(h)}|+|\mathbf{r}^{(h)}|+\mathrm{d}(\mathbf{q}^{(j)}),|\mathbf{r}^{(h)}|} + i\mathcal{S}^o_{|\mathbf{q}^{(h)}|+|\mathbf{r}^{(h)}|+\mathrm{d}(\mathbf{q}^{(j)}),|\mathbf{r}^{(h)}|} \\ \qquad (|\mathbf{q}^{(h)}|+|\mathbf{r}^{(h)}|+\mathrm{d}(\mathbf{q}^{(j)}) : \text{even}).\end{cases} \tag{3.26}$$

By (3.25) and (3.26), we obtain

$$F^{\mathbf{r}}_{\mathbf{q}}(\pi/3) \in \begin{cases} \mathcal{S}^{o}_{|\mathbf{q}|+|\mathbf{r}|+n,|\mathbf{r}|} + i\mathcal{S}^{e}_{|\mathbf{q}|+|\mathbf{r}|+n,|\mathbf{r}|} & |\mathbf{q}|+|\mathbf{r}|+n : \text{odd}, \\ \mathcal{S}^{e}_{|\mathbf{q}|+|\mathbf{r}|+n,|\mathbf{r}|} + i\mathcal{S}^{o}_{|\mathbf{q}|+|\mathbf{r}|+n,|\mathbf{r}|} & |\mathbf{q}|+|\mathbf{r}|+n : \text{even}. \end{cases}$$

Noting $\mathbf{p}+\mathbf{q}+\mathbf{r} = \mathbf{k} - \mathbf{1}_n - \mathbf{l}$, we obtain

$$i^{|\mathbf{l}|+n}(-i\pi)^{|\mathbf{p}|} F^{\mathbf{r}}_{\mathbf{q}+\mathbf{l}}(\pi/3) \in \begin{cases} \mathcal{S}^{o}_{|\mathbf{k}|,|\mathbf{r}|} + i\mathcal{S}^{e}_{|\mathbf{k}|,|\mathbf{r}|} & |\mathbf{k}-\mathbf{1}_n-\mathbf{l}| : \text{odd}, \\ \mathcal{S}^{e}_{|\mathbf{k}|,|\mathbf{r}|} + i\mathcal{S}^{o}_{|\mathbf{k}|,|\mathbf{r}|} & |\mathbf{k}-\mathbf{1}_n-\mathbf{l}| : \text{even}. \end{cases}$$

Since $|\mathbf{r}|$ takes the maximum value $|\mathbf{k}-\mathbf{1}_n-\mathbf{l}|$ when $|\mathbf{p}| = |\mathbf{q}| = 0$, we obtain

$$\mathrm{Ls}^{\mathbf{l}}_{\mathbf{k}}(\sigma) \in \begin{cases} \mathcal{S}^{o}_{|\mathbf{k}|,|\mathbf{k}-\mathbf{1}_n-\mathbf{l}|} & |\mathbf{k}-\mathbf{1}_n-\mathbf{l}| : \text{odd}, \\ \mathcal{S}^{e}_{|\mathbf{k}|,|\mathbf{k}-\mathbf{1}_n-\mathbf{l}|} & |\mathbf{k}-\mathbf{1}_n-\mathbf{l}| : \text{even}. \end{cases}$$

□

Acknowledgment

The author would like to express my deepest gratitude to Prof. Kohji Matsumoto for giving me various advice, including mathematics and English. The author would also like to express my gratitude to Prof. Kentaro Ihara, Prof. Iwao Kimura, Prof. Yayoi Nakamura, Prof. Yasuo Ohno, Prof. Koji Tasaka, Dr. Minoru Hirose and Mr. Takuma Ito, for giving me valuable advice which has helped advance my studies. The author would also like to thank Prof. Shin-ya Kadota, Prof. Yuta Suzuki, Dr. Tomokazu Onozuka, Mr. Kenta Endo and Mr. Tomohiro Ikkai, members or graduates of the laboratory.

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