On certain central extensions associated to covering spaces
(被覆空間に付随するある中心拡大について)

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April 15, 2020

Abstract

For a short exact sequence for groups, we have the five-term exact sequence for group cohomology. Applying this five-term exact sequence to the homeomorphism and diffeomorphism group, we can understand characteristic classes from central extensions. The author first introduces a new central extension, called the central extension associated to a given regular $\mathbb{Z}$-covering space over X. Second, the author give the geometrical meaning to the central extension associated to the regular $\mathbb{Z}$-covering space. The author also introduces the Euler class for an (X, α)-bundle and proves that the extension class corresponding to the central extension associated to the $\mathbb{Z}$-covering space agree with the Euler class for the universal flat (X, α)-bundle. Finally, the author construct some formula of 2-cocycles representing that extension class. Especially if X bounds on some suitable manifold, the author shows the relation between the volume flux homomorphism and that extension class.

Contents

1 Introduction

The group cohomology theory is closely related to the theory of characteristic classes on flat bundles. In the thesis, we consider group cohomology classes in low dimensions (1 and 2 in most cases) and investigate the relation to group homomorphisms and central extensions. In particular, we clarify the relations among the five-term exact sequences of group cohomology and the transgression formulas in central extensions from the viewpoint of Geometry and Topology. We will be concerned with homeomorphism and diffeomorphism groups as pivotal examples.

In the present thesis, there contain three main results. Given a regular $\mathbb{Z}$-covering space over X, there exists a cohomology class $\alpha \in H^1(X;\mathbb{Z})$ corresponding to the covering space. Let $\mathrm{Homeo}_\alpha(X)$ denote the group consisting of homeomorphisms on X that preserve α. We first construct a new central extension $\widehat{G}$ of $\mathrm{Homeo}_\alpha(X)$. It is called the *central extension associated to the regular $\mathbb{Z}$-covering space*. For instance, the central extension associated to the universal covering space $\mathbb{R} \to S^1$ is the universal covering group of the orientation-preserving circle homeomorphism group. In this case, it is well known that the resulting extension class of $\widehat{G}$ is the Euler class for the universal oriented flat circle bundle (see Ghys [7] for instance).

Second, we shall associate a new characteristic class to the central extension constructed above. Given a regular $\mathbb{Z}$-covering space over X with $\alpha \in H^1(X;\mathbb{Z})$ the corresponding class, we take an (X,α)-*bundle* $E \to B$. It is, by definition, a fiber bundle with X the fiber and $\mathrm{Homeo}_\alpha(X)$ the structure group. We then define the Euler class $e_\alpha(E) \in H^2(B;\mathbb{Z})$ in a similar manner to that of the Euler class for (not necessarily flat) circle bundle. Observe that the new class $e_\alpha(E)$ is a generalization of the Euler class for circle bundles since $\mathrm{Homeo}_\alpha(S^1)$ is exactly the group of orientation-preserving homeomorphisms on the circle S^1 with α the orientation class. Then we prove that the Euler class $e_\alpha(E)$ coincides with the pullback of the extension class $e(\widehat{G})$ given by the central extension $\widehat{G}$ of $\mathrm{Homeo}_\alpha(X)$ when the (X,α)-bundle $E \to B$ is flat. This part is based on the joint work with Moriyoshi [6].

Finally, we describe representative 2-cocycles for the extension class $e(\widehat{G})$ (Proposition 5.25). The most known one is the Euler 2-cocycle as is explained in Ghys [7]. In the case of diffeomorphisms groups, we exhibit another representative 2-cocycle for the smooth central extension associated to a regular $\mathbb{Z}$-covering space by using the double complex consisting of group cochains with values in differential forms (Proposition 5.27). Furthermore, we consider a flux homomorphisms on the volume-preserving relative diffeomorphism group and prove a transgression formula for the universal Euler class (Theorem 5.34, Corollary 5.35).

Now we describe the three results in more detail. As is mentioned, we shall construct a central extension and a characteristic class, called the *central extension associated to* α and the *Euler class for* (X, α)*-bundles*, and investigate the relation between them.

Definition 1.1 (see Definition 5.9)**.** Let $\pi : \widehat{X} \to X$ be a regular $\mathbb{Z}$-covering space and let $\alpha \in H^1(X; \mathbb{Z})$ denote the cohomology class corresponding to $\pi : \widehat{X} \to X$. Denote by $\mathrm{Homeo}_\alpha(X)$ the group consisting of homeomorphisms on X that preserve α. Denote also by $\mathrm{Homeo}(\widehat{X})^{\mathbb{Z}}$ the group of homeomorphisms on $\widehat{X}$ that commute with the $\mathbb{Z}$-action on $\widehat{X}$. The *central extension associated to the* $\mathbb{Z}$*-covering space* is defined to be:

$$0 \to \mathbb{Z} \to \mathrm{Homeo}(\widehat{X})^{\mathbb{Z}} \to \mathrm{Homeo}_\alpha(X) \to 1.$$

We also refer to it as the *central extension associated to* α.

Next, we explain a significance of the central extension above from a viewpoint of Geometry. Let E be an (X, α)*-bundle* on B, that is a fiber bundle $E \to B$ with X the typical fiber and $\mathrm{Homeo}_\alpha(X)$ the structure group. By using the five-term exact sequence of the Leray-Serre spectral sequence for $E \to B$, we can define a characteristic class, called the *Euler class*, which is denoted by $e_\alpha(E) \in H^2(B; \mathbb{Z})$ for a given $E \to B$ (Definition 5.20). Observe that the Euler class is well defined even for non-flat (X, α)-bundle since there exists the five-term exact sequence for any fiber bundle. We then prove a relation between the Euler class and the central extension associated to α when the (X, α)-bundle is flat.

Theorem 1.2 (see Theorem 5.23 [1])**.** *Let* $e(\widehat{G})$ *denote the extension class of the central extension associated to* α *which belongs to* $H^2_{\mathrm{grp}}(\mathrm{Homeo}_\alpha(X); \mathbb{Z})$.

[1] Theorem 5.23 grew out of a discussion with my supervisor, Professor Moriyoshi, trying to construct a geometric central extension for the identity component of the homeomorphism group. Finally it culminated in the central extension associated to α.

Let E_α be the universal flat (X,α)-bundle over $B\mathrm{Homeo}_\alpha(X)^\delta$ and $e_\alpha(E_\alpha)$ denote the Euler class for E_α. One has $e(\widehat{G}) = e_\alpha(E_\alpha)$ under the natural identification $H^2_{\mathrm{grp}}(\mathrm{Homeo}_\alpha(X);\mathbb{Z}) = H^2(B\mathrm{Homeo}_\alpha(X)^\delta;\mathbb{Z})$. Here $\mathrm{Homeo}_\alpha(X)^\delta$ is the group $\mathrm{Homeo}_\alpha(X)$ with the discrete topology.

We also exhibit a formula of representative group 2-cocycles corresponding to the smooth central extension associated to α in Proposition 5.27. Furthermore, we give another formula of representative cocycles for the restricted smooth central extension associated to α concerning the flux homomorphism. Let $\pi : \widehat{M} \to M$ be a smooth regular $\mathbb{Z}$-covering space with the classifying class $\alpha \in H^1(M;\mathbb{Z})$. We have the central extension, called the *restricted smooth central extension associated to* α:

$$0 \to \mathbb{Z} \to p^{-1}(G_\partial) \to G_\partial = \mathrm{Diff}_+(M)_0 \to 1,$$

where $p : \mathrm{Diff}(\widehat{M})^{\mathbb{Z}} \to \mathrm{Diff}_\alpha(M)$ is the projection. Let W be an $(n+1)$-dimensional compact, connected and oriented manifold such that M is diffeomorphic to the boundary of W. Assume that there exists a regular $\mathbb{Z}$-covering space $\widehat{W} \to W$ over W such that the covering on the boundary coincides with $\widehat{M} \to M$. We take an n-form $\eta \in \Omega^n(W)$ such that $\Omega = -d\eta$ is the volume form on W whose volume is 1. By the assumption, we have the corresponding class $\bar{\alpha} \in H^1(W;\mathbb{R})$ to $\widehat{W} \to W$ that satisfies $\bar{\alpha}|_{\partial W} = \alpha \in H^1(M;\mathbb{Z})$. Denote by $G = \mathrm{Diff}_\Omega(W)_0$ the identity component of the group volume-preserving diffeomorphisms on W. Also denote by G_{rel} the kernel of the restriction map $G \to G_\partial$. We then have a short exact sequence:

$$1 \to G_{\mathrm{rel}} \to G \to G_\partial \to 1.$$

Let $\mathrm{Flux} : G_{\mathrm{rel}} \to H^n(W, \partial W;\mathbb{R})$ the flux homomorphism defined in Definition 5.32.

Theorem 1.3 (see Theorem 5.34, Corollary 5.35)**.** *There exists a transgression formula (see Theorem 5.34 and Corollary 5.35 for the precise statement) that connects the pairing homomorphism $\langle \mathrm{Flux}, i_*\bar{\alpha}\rangle : G_{\mathrm{rel}} \to \mathbb{R}$, $\phi \mapsto \langle \mathrm{Flux}(\phi) \cup i_*\bar{\alpha}, [W,\partial W]\rangle$ with the extension class in $H^2_{\mathrm{grp}}(G_\partial;\mathbb{R})$ of the restricted smooth central extension associated to α. Here $i : \mathbb{Z} \hookrightarrow \mathbb{R}$ is the natural inclusion map.*

This thesis is organized as follows. Sections 2, 3 and 4 are preliminary parts. The main part of the thesis is Section 5. In Section 2, we begin with motivating the readers to consider the group cohomology theory for discrete groups and overview its geometric background. In fact, the theory plays a significant role even in Geometry and Topology since it is closely related to the classification of flat bundles and foliation structures. In Section 3, we recall the group cohomology theory and prepare mandatory notations. In particular, we explain the relation among the lower degree cohomology groups, central extensions and so on. We also provide certain examples that will be relevant in the main part, Section 5. In Section 4, we introduce the Lyndon–Hochschild–Serre spectral sequence, Lerya–Serre spectral sequence and the five-term exact sequence for these spectral sequences. Those will play crucial roles in Section 5 in order to establish the main theorems in the thesis. In Section 5, we construct the central extension associated to α and a characteristic class for (X, α)-bundle. We also clarify the relation between them from a viewpoint of Geometry. In Subsection 5.1, we define the central extension associated to α or a given $\mathbb{Z}$-covering space (see Definition 5.9). In Subsection 5.2, we recall the definition of the Euler class for the oriented S^1-bundle. For the universal oriented flat S^1-bundle, it is known that the Euler class corresponds to the central extension given by the universal covering group of the circle homeomorphism group. In Subsection 5.3, we introduce the Euler class for the (X, α)-bundle (see Definition 5.19 and 5.20) and prove that the extension class of the central extension associated to α coincides with the Euler class for the universal flat (X, α)-bundle (see Theorem 5.23). In Subsection 5.4, we exhibit explicit formulas of group 2-cocycles that correspond to the (smooth) central extension associated to α. In Subsection 5.5, we prove a relation between the restricted central extension associated to α and the flux homomorphism for the volume-preserving diffeomorphism group on certain compact manifolds (see Theorem 5.34). In Subsection 5.6, we discuss some problems for the further study and developments.

Acknowledgments

I am indebted to my supervisor, Professor Hitoshi Moriyoshi for an enlightening discussion and invaluable suggestion. I would also thank the professors, staffs, graduate students I have met at Graduate School of Nagoya University and who have made my time here so enjoyable.

2 Flat Bundles and Discrete Groups

In this section, we motivate us to consider group cohomology theory in differential topology.

Geometrical (or topological) objects have been studied by using some cohomology classes called characteristic classes, such as the Euler class, the Chern class, the Pontrjagin class and so on. For characteristic classes of smooth principal bundles, the Chern-Weil theory is well known.

Proposition 2.1 (the Chern–Weil theory)**.** *Let $P \to M$ be a smooth principal G-bundle with connection 1-form $\theta \in \Omega^1(P;\mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra associated with the Lie group G. Set $\Omega = d\theta + \frac{1}{2}[\theta,\theta] \in \Omega^2(P;\mathfrak{g})$ the curvature 2-form.*

Denote by $I^k(G)$ the symmetric k-linear functions from the k-tuple $\mathfrak{g} \times \cdots \times \mathfrak{g}$ to $\mathbb{R}$ which are invariant by G-adjoint action, that is,

$$f(\mathrm{Ad}_g(A_1),\ldots,\mathrm{Ad}_g(A_k)) = f(A_1,\ldots,A_k)$$

for $g \in G$ and $A_1,\ldots,A_k \in \mathfrak{g}$.

Then a homomorphism $I^k(G) \to H^{2k}(M;\mathbb{R})$ is induced by the correspondence from $f \in I^k(G)$ to $f(\Omega^k) \in \Omega^{2k}(P)$. Here, the $2k$-form $f(\Omega^{2k})$ is defined as

$$f(\Omega^{2k})(X_1,\ldots,X_{2k}) = \sum_{\sigma\in\mathfrak{S}_{2k}} \mathrm{sgn}(\sigma) f(\Omega(X_{\sigma(1)},X_{\sigma(2)}),\ldots,\Omega(X_{\sigma(2k-1)},X_{\sigma(2k)}))$$

for $2k$ tangent vector fields $X_1,\ldots,X_{2k} \in \mathfrak{X}(P)$.

This homomorphism is called the *Chern–Weil homomorphism.* Remark that the Chern–Weil homomorphism is independent on the choice of connection 1-forms on P.

For a flat principal bundle which has some connection forms with the zero curvature, however, any characteristic classes from the Chern–Weil theory vanish since $\Omega = 0$, so that, $0 = f(\Omega^k) \in \Omega^{2k}(P)$. We need non-zero characteristic classes for flat principal bundles. For flat principal bundles, the following is well known:

Proposition 2.2. *Isomorphism classes of flat principal G-bundles over M are one-to-one corresponding to conjugate classes of homomorphisms from $\pi_1(M)$ to G.*

$$\left\{\begin{matrix}\textit{flat principal } G\textit{-bundle} \\ \textit{over } M\end{matrix}\right\}/\mathrm{isom.} \overset{1:1}{\longleftrightarrow} \mathrm{Hom}(\pi_1(M),G)/\mathrm{conj.}$$

A flat principal G-bundle over M corresponds to its *holonomy representation* $\mathrm{Hol} : \pi_1(M) \to G$, and a homomorphism $\psi : \pi_1(M) \to G$ corresponds to a flat principal G-bundle $E \to M$ defined by

$$E = (\widetilde{M} \times G)/\pi_1(M).$$

Here, $\widetilde{M} \to M$ is the universal covering space of M and the equivalence relation is defined by a diagonal action of $\pi_1(M)$ along the homomorphism ψ. This proposition implies that the structure group G of a flat principal G-bundle can reduce the group G with the discrete topology. We write G^δ as a topological group G with the discrete topology. Hence flat principal G-bundles are classified by the universal bundle $EG^\delta \to BG^\delta$.

On the other hand, the following is well-known about the singular cohomology group of the Eilenberg–MacLane space $BG^\delta = K(G^\delta, 1)$ for a discrete group G^δ.

Proposition 2.3 (see e.g. Brown [3, Chapter II, (4.1) Proposition]). *Let G be a discrete group and assume that A is a G-module. Then the singular cohomology $H^*(BG; \mathcal{A})$ and the group cohomology $H^*_{\mathrm{grp}}(G^\delta; A)$ are isomorphic, where $\mathcal{A}$ is the local system of coefficients on BG^δ of A.*

Therefore we can regard group cohomology classes in $H^*_{\mathrm{grp}}(G^\delta; \mathbb{R})$ as characteristic classes for flat principal G-bundle.

3 The Group Cohomology Theory

In this section we recall the group cohomology theory based on Brown's book [3].

3.1 Definition of the group cohomology theory

Let G be a group and A be an Abelian group. Assume that A has a left G-action by $g \cdot a$ for $a \in A$ and $g \in G$.

Definition 3.1 (Brown [3, Chapter III, 1]). The *group cochain complex* $(C^*(G; A), \delta)$ with coefficients in A is given by the pair

$$C^p(G; A) = \{c : G^p \to A\}, \quad \delta : C^p(G; A) \to C^{p+1}(G; A).$$

Here a group p-cochain $c : G^p \to A$ is an arbitrary function on the p-tuple product of G and where the coboundary δ is defined to be

$$\begin{aligned}\delta c(g_1, \ldots, g_{p+1}) = g_1 \cdot c(g_2, \ldots, g_{p+1}) + \sum_{i=1}^{p} (-1)^i c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{p+1}) \\ + (-1)^{p+1} c(g_1, \ldots, g_p).\end{aligned}$$

We denote by $H^*_{\mathrm{grp}}(G; A)$ the cohomology group of $C^*(G; A)$, called the *group cohomology* of G with coefficients in A.

Remark 3.2. In the definition, A is a "left" G-module. On the other hand, we can define group cohomology with coefficients in a "right" G-module A. In that case the coboundary δ is defined to be

$$\begin{aligned}\delta c(g_1, \ldots, g_{p+1}) = c(g_2, \ldots, g_{p+1}) + \sum_{i=1}^{p} (-1)^i c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{p+1}) \\ + (-1)^{p+1} c(g_1, \ldots, g_p) \cdot g_{p+1}.\end{aligned}$$

Example 3.3. Let M be a smooth manifold and G be a subgroup of diffeomorphisms on M. Since G acts on M from the left, it acts on the de Rham complex $\Omega^*(M)$ from the right by pull-back. Then a cochain complex $C^p(G; \Omega^q(M))$ has two coboundary maps: the coboundary map δ for group cochains and the differential d for differential forms. This complex is called the *group-de Rham double-complex*.

In general, cohomology theory can be equipped with a product called cup product whose cohomology group called *cohomology ring*. In the case of group cohomology theory, the cohomology group has the cup product when the coefficient group has suitable product.

Definition 3.4 (Brown [3, Chapter V, 3])**.** Let G be a group and A be a left G-module. Assume that A has a $\mathbb{Z}$-bilinear map $[\ ,\ ] : A \times A \to A$ satisfying $[g \cdot a, g \cdot b] = g \cdot [a, b]$ for any $a, b \in A$ and $g \in G$. For a group p-cochain $c \in C^p(G; A)$ and a group q-cochain $c' \in C^q(G; A)$, the *cup product* $c \cup c' \in C^{p+q}(G; A)$ is defined as follows:

$$c \cup c'(g_1, \ldots, g_{p+q}) = [c(g_1, \ldots, g_p), (g_1 \cdots g_p).c'(g_{p+1}, \ldots, g_{p+q})]$$

for $g_1, \ldots, g_{p+q} \in G$.

Remark 3.5. The cup product $\cup : C^p(G;A) \times C^q(G;A) \to C^{p+q}(G;A)$ and the coboundary map $\delta : C^k(G;A) \to C^{k+1}(G;A)$ satisfy the *Leibniz's rule*:

$$\delta(c \cup c') = (\delta c) \cup c' + (-1)^p c \cup (\delta c')$$

for $c \in C^p(G;A)$ and $c' \in C^q(G;A)$. Thus, this cup product for group cocycles induces the cup product for cohomology classes by $[c] \cup [c'] = [c \cup c']$.

Example 3.6. Let M be a smooth manifold and let G be a subgroup of diffeomorphisms on M. Since the differential forms $\Omega^*(M)$ on M has the exterior product,it induces a cup product on the double complex $C^*(G; \Omega^*(M))$:

$$(c \cup c')(g_1, \ldots, g_p) = (g_{p+1} \cdots g_{p+q})^* c(g_1, \ldots, g_p) \wedge c'(g_{p+1}, \ldots, g_{p+q}) \in \Omega^{k+l}(M)$$

for $c \in C^p(G; \Omega^k(M))$ and $c' \in C^q(G; \Omega^l(M))$.

The following remark is easy to check.

Remark 3.7. Let G be an arbitrary group and A be a left G-module. The first cohomology group $H^1_{\mathrm{grp}}(G;A)$ of G with coefficients in A is the group of crossed-homomorphisms from G to A as G-modules. Especially when G acts on A trivially, $H^1(G;A)$ is just the homomorphism group $\mathrm{Hom}(G, A)$.

The group cohomology theory for discrete groups is related to the singular cohomology theory for certain topological spaces:

Proposition 3.8 (Brown [3, Chapter II, (4.1) Proposition])**.** *Let G be a discrete group and assume that A is a G-module. Then the singular cohomology $H^*(BG; \mathcal{A})$ and the group cohomology $H^*_{\mathrm{grp}}(G^\delta; A)$ are isomorphic to each other, where $\mathcal{A}$ is the local system of coefficients on BG^δ of A.*

3.2 Central extensions

It is well-known that there is an alternative definition for the second cohomology group. First we recall the notions of group extensions and central extensions.

Definition 3.9 (Brown [3, Chapter IV, 1])**.** Let A, $\widehat{G}$ and G be arbitrary groups. A *group extension* $\widehat{G}$ of G by A is a short exact sequence of groups:

$$1 \to A \to \widehat{G} \to G \to 1.$$

Furthermore, $\widehat{G}$ is called a *central extension* if the inclusion $A \to \widehat{G}$ factors through the center of $\widehat{G}$.

Remark 3.10. When $\widehat{G}$ is a central extension of G by A, A is in the center of $\widehat{G}$, so that A is Abelian.

Consider a group extension $\widehat{G}$ of G by an Abelian group A:

$$0 \to A \hookrightarrow \widehat{G} \xrightarrow{p} G \to 1$$

Set $s : G \to \widehat{G}$ an arbitrary set-theorical section of p. There is a left group action of G on A defined by

$$g \cdot a := s(g)as(g)^{-1} \in A$$

for $g \in G$ and $a \in A$ since $p(g \cdot a) = gp(a)g^{-1} = 1$. This group action is independent of the choice of sections s. For a left G-module A, we call a *group extension* $\widehat{G}$ of G by the left G-module A if the above left G-action on A coincides with the one on A originally. For $g, h \in G$, we have $s(g)s(h)s(gh)^{-1} \in A$ by $p(s(g)s(h)s(gh)^{-1}) = 1$. Then we can define a group 2-cochain $\chi \in C^2(G; A)$ in the following way:

$$\chi(g, h) = s(g)s(h)s(gh)^{-1} \in A. \tag{3.1}$$

It is easy to check that $\chi \in C^2(G; A)$ is a group 2-cocycle and the cohomology class $[\chi] \in H^2_{\mathrm{grp}}(G; A)$ is independent of the choice of sections $s : G \to \widehat{G}$, so that the class $[\chi]$ depends only on the group extension.

Definition 3.11 (Brown [3, Chapter IV, 3])**.** The cohomology class $[\chi] \in H^2_{\mathrm{grp}}(G; A)$ defined by Equation (3.1) is called the *extension class* or the *Euler class* of the group extension $\widehat{G}$, denote by $e(\widehat{G})$.

If a section $s : G \to \widehat{G}$ is a homomorphism then the corresponding 2-cocycle $\chi \in C^2(G; A)$ is obviously trivial. This implies that the splitting extension has a trivial extension class. The following is well known.

Proposition 3.12 (Brown [3, Chapter IV, (3.12) Theorem])**.** *Let G be a group and A be a left G-module. The second cohomology group $H^2_{\mathrm{grp}}(G; A)$ is isomorphic to the equivalence classes of extensions of G by the left G-module A:*

$$H^2_{\mathrm{grp}}(G; A) \cong \{\textit{extensions of } G \textit{ by the } G\textit{-module } A\}/\{\textit{splitting extensions}\}.$$

Remark 3.13. Let G be a group, N be a normal subgroup of G. We have a group extension $1 \to N \to G \to G/N \to 1$. Let A be a left G-module and $f : N \to A$ be a G-*equivariant homomorphism*, that is, $f(gng^{-1}) = g.f(n)$ for $g \in G$ and $n \in N$.

Then we obtain a central extension of G/N by A:

$$0 \to A \to G_A \to G/N \to 1,$$

where $G_A = (A \rtimes G)/\sim$ with the equivalence relation $(a, ng) \sim (f(n)+a, g)$ for $(g, a) \in G \ltimes A$ and $n \in N$. Given $a \in A$ and $g \in G$, we denote the corresponding element in G_A by $[a, g] \in G_A$.

3.3 Connection cochain

We introduce the notion of connection cochain. It is a useful tool to construct representative 2-cocycle of the extension class of a given central extension.

Definition 3.14 (Moriyoshi [15, Definition 1])**.** Assume that $1 \to N \to G \to G/N \to 1$ is a short exact sequence for groups. Let A be a left G-module on which N acts trivially. A map $\tau : G \to A$ is a *connection cochain* if it satisfies the following conditions:

$$\tau(gn) = \tau(g) + g \cdot \tau(n), \quad \tau(gng^{-1}) = g \cdot \tau(n)$$

for $n \in N$ and $g \in G$.

Remark 3.15. The restriction $f = \tau|_N : N \to A$ is a G-equivariant homomorphism:

$$f(nn') = f(n) + f(n'), \quad f(gng^{-1}) = f(n)$$

for $n \in N$ and $g \in G$. Hence we call $\tau : G \to A$ a *connection cochain over* f.

Proposition 3.16 (Moriyoshi [15, Proposition 1])**.** *Let* $1 \to N \to G \xrightarrow{p} G/N \to 1$ *be a short exact sequence for groups and let* A *be a left* G*-module on which* N *acts trivially. Take a* G*-equivariant homomorphism* $f : N \to A$*. We have a central extension*

$$0 \to A \to G_A \xrightarrow{p_A} G/N \to 1$$

defined in Remark 3.13.

Then the following holds:

1. *For a connection cochain $\tau : G \to A$ over f, there exists a group 2-cocycle $\sigma_\tau \in C^2(G/N; A)$ such that*

$$\sigma_\tau([g],[h]) = \delta\tau(g,h) \in A$$

for each $g, h \in G$ and their equivalence class $[g],[h] \in G/N$.

2. *Assume that $\tau, \tau' : G \to A$ are different connection cochains over f. Then $\sigma_{\tau'} \in C^2(G/N; A)$ is cohomologous to $\sigma_\tau \in C^2(G/N; A)$.*

3. *For a connection cochain $\tau : G \to A$ over f, the cohomology class $[-\sigma_\tau] \in H^2_{\mathrm{grp}}(G/N; A)$ coincides with the extension class $e(G_A) \in H^2_{\mathrm{grp}}(G/N; A)$.*

Proof. 1. For $n \in N$ and $g, h \in G$, we have

$$\begin{aligned}\delta\tau(g,hn) =& g\cdot\tau(hn) - \tau(ghn) + \tau(g)\\ =& g\cdot(\tau(h) + h\cdot\tau(n)) - (\tau(gh) + (gh)\cdot\tau(n)) + \tau(g) = \delta\tau(g,h).\end{aligned}$$

since $\tau(gn) = \tau(g) + g\cdot\tau(n)$. Similarly, we obtain

$$\begin{aligned}\delta\tau(gn,h) =& (gn)\cdot\tau(h) - \tau(gnh) + \tau(gn)\\ =& g\cdot\tau(h) - \tau(ghh^{-1}nh) + \tau(gn)\\ =& g\cdot\tau(h) - (\tau(gh) + (gh)\cdot\tau(h^{-1}nh)) + (\tau(g) + g\cdot\tau(n)) = \delta\tau(g,h).\end{aligned}$$

since $\tau(gn) = \tau(g) + g\cdot\tau(n)$ and $\tau(h^{-1}nh) = h^{-1}\cdot\tau(n)$. These imply that $\delta\tau(g,h)$ depends only on the equivalence class $[g],[h] \in G/N$, so that there exists a group 2-cocycle $\sigma \in C^2(G; A)$ such as

$$\sigma([g],[h]) = \delta\tau(g,h)$$

for $g, h \in G$ and their equivalence class $[g],[h] \in G/N$.

2. Assume that τ and τ' are different connection cochains over f. Then for $g \in G$ and $n \in N$, we have

$$(\tau' - \tau)(gn) = (\tau'(g) + g\cdot\tau'(n)) - (\tau(g) + g\cdot\tau(n)) = (\tau' - \tau)(g).$$

Therefore there exists a group 1-cochain $\lambda \in C^1(G/N; A)$ such that

$$\lambda([g]) = \tau'(g) - \tau(g)$$

for $g \in G$ and its equivalence class $[g] \in G/N$. Thus we obtain $\sigma_{\tau'} = \sigma_\tau + \delta\lambda \in C^2(G/N; A)$.

3. Let $s : G/N \to G$ be a set-theorical section of the quotient map $p : G \to G/N$. We define a group 1-cochain $\tau_s : G \to A$ as

$$\tau_s(g) = f(g\{s \circ p(g)\}^{-1}) \in A$$

for $g \in G$ since $p(g\{s \circ p(g)\}^{-1}) = 1 \in G/N$, so that $g\{s \circ p(g)\}^{-1} \in N$. By definition, the 1-cochain τ_s is a connection cochain over f:

$$\tau_s(gn) = f(gn\{s \circ p(gn)\}^{-1}) = f((gng^{-1}) \cdot g\{s \circ p(g)\}^{-1}) = g \cdot f(n) + \tau_s(g)$$

for $g \in G$ and $n \in N$. Then we obtain

$$\begin{aligned}
\delta\tau_s(g,h) =& g \cdot \tau_s(h) - \tau_s(gh) + \tau_s(g) \\
=& g.f(h\{s \circ p(h)\}^{-1}) - f(gh\{s \circ p(gh)\}^{-1}) + f(g\{s \circ p(g)\}^{-1}) \\
=& f(g\{s \circ p(g)\}^{-1}) + f(gh\{s \circ p(h)\}^{-1}g^{-1}) - f(gh\{s \circ p(gh)\}^{-1}) \\
=& - f(\{s \circ p(g)\}g^{-1}g\{s \circ p(h)\}h^{-1}g^{-1}gh\{s \circ p(gh)\}^{-1}) \\
=& - f(s \circ p(g) s \circ p(h)\{s \circ p(gh)\}^{-1})
\end{aligned}$$

for $g, h \in G$. Thus $\sigma_{\tau_s}(x,y) = -f(s(x)s(y)s(xy)^{-1}) \in A$ for $x, y \in G/N$.

Recall $G_A = (A \rtimes G)/\sim$ discussed in Remark 3.13, where the equivalence relation is defined by $(a, ng) \sim (f(n) + a, g)$ for $(a, g) \in A \rtimes G$ and $n \in N$. For the section $s : G/N \to G$, we have a set-theorical section $s_A : G/N \to G_A$ of $p_A : G_A \to G/N$ defined as $s_A(x) = [0, s(x)]$ for $x \in G/N$. Then we obtain

$$\begin{aligned}
s_A(x)s_A(y)s_A(xy)^{-1} =& [0, s(x)][0, s(y)][0, s(xy)]^{-1} \\
=& [0, s(x)s(y)s(xy)^{-1}] \\
=& [f(s(x)s(y)s(xy)^{-1}), 1] = [-\sigma_{\tau_s}(x,y), 1]
\end{aligned}$$

for $x, y \in G/N$. The left hand side is a group 2-cocycle representing the extension class $e(G_A)$ by Definition 3.11. Thus, σ_{τ_s} also represents the extension class $e(G_A) \in H^2_{\mathrm{grp}}(G/N; A)$. Since σ_τ is cohomologous to σ_{τ_s} for each connection cochain τ over f, the cohomology class $[-\sigma_{\tau_s}] \in H^2_{\mathrm{grp}}(G/N; A)$ coincides with the extension class $e(G_A) \in H^2_{\mathrm{grp}}(G/N; A)$.

$\square$

Remark 3.17. A connection cochain $\tau : G \to A$ for the short exact sequence $1 \to N \to G \to G/N \to 1$ is an analogy to a connection 1-form of a principal bundle.

Let $P \to M$ be a principal $U(1)$-bundle over M with a connection 1-form $\theta \in \Omega^1(P)$. The the exact 2-form $d\theta \in \Omega^2(P)$, called the curvature 2-form, induces a closed 2-form on the base space M and represents $2\pi c_1(P) \in H^2(M;\mathbb{R})$, where $c_1(P)$ is the first Chern class of the principal bundle $P \to M$.

One the other hand, for a connection cochain $\tau \in C^1(G;A)$, the coboundary $\delta\tau \in C^2(G;A)$ induces a group 2-cocycle $\sigma \in C^2(G/N;A)$ and represents the extension class $e(G_A) \in H^2_{\mathrm{grp}}(G/N;A)$ corresponding to the central extension

$$0 \to A \to G_A \to G/N \to 1.$$

3.4 Examples

Example 3.18 (see e.g. Ghys [7, 6.2], Moriyoshi [15, Example 1]). Denote by $G_\partial = \mathrm{Homeo}_+(S^1)$ the group of orientation-preserving homeomorphisms on a unit circle S^1. We consider the group G_∂ as a topological group by Whitney topology. It is well-known that the universal covering group $\widetilde{G}_\partial$ of G_∂ coincides with the group of homeomorphisms on the real line $\mathbb{R}$ which commutes with a homeomorphism $T : \mathbb{R} \to \mathbb{R}$, $T(x) = x + 1$.

Then we obtain a central extension

$$0 \to \mathbb{Z} \to \widetilde{G}_\partial \to G_\partial \to 1$$

and the extension class $e(\widetilde{G}_\partial) \in H^2_{\mathrm{grp}}(G_\partial;\mathbb{Z})$ corresponding to this central extension.

This central extension has a connection cochain τ over the identity $\mathrm{id} : \mathbb{Z} \to \mathbb{Z}$ defined as

$$\tau(\tilde{\phi}) = \lfloor \tilde{\phi}(0) \rfloor$$

for $\tilde{\phi} \in \widetilde{G}_\partial \subset \mathrm{Diff}(\mathbb{R})$. Here, $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ is the floor function.

Example 3.19. Denote by $G_\partial = \mathrm{Diff}_+(S^1)$ the group of orientation-preserving diffeomorphisms on a unit circle S^1 as the case of the homeomorphism group. We consider the group $G_\partial = \mathrm{Diff}_+(S^1)$ as a topological group by C^∞-Whitney topology. It is well-known that the universal covering group $\widetilde{G}_\partial$ of $G_\partial = \mathrm{Diff}_+(S^1)$ coincides with the group of diffeomorphisms on the real line $\mathbb{R}$ which commutes with a diffeomorphism $T : \mathbb{R} \to \mathbb{R}$, $T(x) = x + 1$.

Then we obtain a central extension

$$0 \to \mathbb{Z} \to \widetilde{G}_\partial \to G_\partial \to 1$$

and the extension class $e(\widetilde{G}_\partial) \in H^2_{\mathrm{grp}}(G_\partial;\mathbb{Z})$ corresponding to this central extension. This class $e(\widetilde{G}_\partial) \in H^2_{\mathrm{grp}}(G_\partial;\mathbb{Z})$ corresponds to the Euler class for the universal oriented flat S^1-bundle in $H^2(BG^\delta_\partial;\mathbb{Z})$, where the group G^δ_∂ is a group G_∂ with discrete topology.

Put $\omega = \frac{d\theta}{2\pi} \in \Omega^1(S^1)$, where θ is an angular coordinate on S^1. We know that the de Rham cohomology class in $H^1(S^1;\mathbb{R})$ containing ω is the generator of the cohomology ring $H^*(S^1;\mathbb{R})$. We identify the 1-form $\omega \in \Omega^1(S^1)$ as a group 0-cochain in $C^0(G_\partial;\Omega^1(S^1))$. Since $\phi \in G_\partial$ is an orientation-preserving diffeomorphism, the group coboundary $\delta\omega(\phi) = \omega - \phi^*\omega$ is exact 1-form, so that there exists a group 1-cochain $\sigma \in C^1(G_\partial;\Omega^0(S^1))$ such that

$$\delta\omega(\phi) = \omega - \phi^*\omega = d(\sigma(\phi))$$

for any $\phi \in G_\partial$. Furthermore, the coboundary $\delta\sigma \in C^2(G_\partial;\Omega^0(S^1))$ defines a group 2-cocycle in $C^2(G_\partial;\mathbb{R})$ since $d(\delta\sigma(\phi_1,\phi_2)) = \delta(\delta\omega)(\phi_1,\phi_2) = 0$. It is easy to check that the group cohomology class $-\delta\sigma \in C^2(G_\partial;\mathbb{R})$ equals to the above Euler class $e(\widetilde{G}_\partial)_\mathbb{R} \in H^2_{\mathrm{grp}}(G_\partial;\mathbb{R})$.

4 Spectral Sequence

In this section, we recall the notion of a spectral sequence. Especially, we introduce the Leray-Serre spectral sequence for a fibration and the Lyndon-Hochschild-Serre spectral sequence for a short exact sequence for groups. Their notations rely on McCleary [12] and Hochschild-Serre [9].

4.1 Spectral sequence of a filtered complex

In this subsection, we recall the theory of spectral sequence. The spectral sequence is known as the most important tool in algebraic topology. First, we will define a spectral sequence formally.

Definition 4.1 (see e.g. McCleary [12, 2.1, Definition 2.1])**.** Fix a non-negative integer $r \in \mathbb{Z}$. For each integers $p, q \in \mathbb{Z}$, let $E^{p,q}$ be an Abelian group and put

$$E^{*,*} = \bigoplus_{p,q\in\mathbb{Z}} E^{p,q}.$$

Let $d : E^{*,*} \to E^{*,*}$ be a homomorphism such that

$$d(E^{p,q}) \subset E^{p+r,q-r+1}, \quad d \circ d = 0$$

for each integers $p, q \in \mathbb{Z}$.

Then the pair $(E^{*,*}, d)$ is called a *bigraded cochain complex* and d is called the *differential* of bidigree $(r, 1-r)$.

For a bigraded cochain complex $(E^{*,*}, d)$ with the differential d of bidigree $(r, 1-r)$, we define a cohomology group $H^{*,*}(E^{*,*})$ as

$$H^{p,q}(E^{*,*}, d) := (\mathrm{Ker} d \cap E^{p,q})/(\mathrm{Im} d \cap E^{p,q}).$$

Definition 4.2 (see e.g. McCleary [12, 2.1, Definition 2.2]). A *spectral sequence* is a collection $\{(E_r^{*,*}, d_r)\}$ satisfying the following:

- For each $r \in \mathbb{Z}$, $(E_r^{*,*}, d_r)$ is a bigraded cochain complex with the differential d_r of bidigree $(r, 1-r)$.
- For each $r \in \mathbb{Z}$, the cohomology group $H^{p,q}(E_r^{*,*}, d_r)$ is isomorphic to $E_{r+1}^{p,q}$.

Furthermore, the spectral sequence $\{(E_r^{*,*}, d_r)\}$ is *first quadrant* if $E_2^{p,q} = 0$ for $p < 0$ or $q < 0$.

Let $\{(E_r^{*,*}, d_r)\}$ be a spectral sequence. We will start the $(E_1^{*,*}, d_1)$. Put $Z_1^{p,q} = \mathrm{Im}(d_1) \cap E_1^{p,q}$ and $B_1^{p,q} = \mathrm{Ker}(d_1) \cap E_1^{p,q}$. By definition, we have $E_2^{p,q} \cong Z_1^{p,q}/B_1^{p,q}$. We also have $E_3^{p,q} \cong \bar{Z}_2^{p,q}/\bar{B}_2^{p,q}$, where $\bar{Z}_2^{p,q} = \mathrm{Im}(d_2) \cap E_2^{p,q}$ and $\bar{B}_2^{p,q} = \mathrm{Ker}(d_2) \cap E_2^{p,q}$. Since both of $Z_2^{p,q}$ and $B_2^{p,q}$ are subgroups of $E_2^{p,q}$, there exists subgroups $Z_2^{p,q}$ and $B_2^{p,q}$ of $E_1^{p,q}$ such that $\bar{Z}_2^{p,q} \cong Z_2^{p,q}/B_1^{p,q}$ and $\bar{B}_2^{p,q} \cong B_2^{p,q}/B_1^{p,q}$, respectively. Thus we obtain a sequence of inclusions:

$$0 \subset B_1^{p,q} \subset B_2^{p,q} \subset Z_2^{p,q} \subset Z_1^{p,q} \subset E_1^{p,q}.$$

Iterating this process, we obtain

$$0 \subset B_1^{p,q} \subset \cdots \subset B_r^{p,q} \subset \cdots \subset Z_r^{p,q} \subset \cdots \subset Z_1^{p,q} \subset E_1^{p,q}.$$

Put

$$Z_\infty^{p,q} = \bigcap_{r \geq 1} Z_r^{p,q}, \quad B_\infty^{p,q} = \bigcup_{r \geq 1} B_r^{p,q}$$

and define

$$E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}.$$

Some construction methods for spectral sequences are known. Here, we introduce a construction method for a spectral sequence by using the filtration of a cochain complex.

Definition 4.3 (see e.g. McCleary [12, 2.2, Definition 2.4])**.** A spectral sequence $\{(E_r^{*,*}, d_r)\}$ *converges* to a graded module H^* if there is a filtration F^*H^* of H^* such that

$$E_\infty^{p,q} \cong F^pH^{p+q}/F^{p+1}H^{p+q}.$$

Definition 4.4 (see e.g. McCleary [12, 2.2, Definition 2.5])**.** Let (C^*, δ) be a cochain complex. The complex (C^*, δ) is a *filtered cochain complex* if the cochain complex C^* has an increasing sequence of Abelian groups, which called a *filtration* of the cochain complex:

$$\cdots \subset F^{p+1}C^* \subset F^pC^* \subset \cdots \subset C^*,$$

where $\delta(F^pC^*) \subset F^pC^*$ for each p.

Furthermore, the filtration is *bounded* if for each $n \in \mathbb{Z}$, there exists some integer $s, t \in \mathbb{Z}$ satisfying

$$0 \subset F^sC^n \subset F^{s-1}C^n \subset \cdots \subset F^{t+1}C^n \subset F^tC^n = C^n.$$

Let (C^*, δ) be a filtered cochain complex with a filtration F^pC^*. We set

$$Z_r^{p,q} = \{c \in F^pC^{p+q} \mid \delta c \in F^{p+r}C^{p+q+1}\}$$

Then we define a collection of cochain complexes $\{(E_r^{*,*}, d_r)\}_{r\in\mathbb{Z}}$

$$E_r^{p,q} = Z_r^{p,q}/(Z_{r-1}^{p-1,q+1} + \delta Z_{r-1}^{p+1-r,q+r-2})$$

and differential operators $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ are induced by δ.

Proposition 4.5 (see e.g. McCleary [12, 2.2, Theorem 2.6])**.** *For the given filtered cochain complex* (C^*, δ)*, the collection of cochain complex* $\{(E_r^{*,*}, d_r)\}_{r\in\mathbb{Z}}$ *is a spectral sequence with*

$$E_1^{p,q} \cong H^{p+q}(F^pC^*/F^{p+1}C^*).$$

Especially if the filtration is bounded, then the spectral sequence converges to $H^*(C^*)$, that is, there is a filtration

$$F^p H^{p+q} = \mathrm{Im}(H^{p+q}(F^p C^*) \to H^{p+q}(C^*))$$

of $H^*(C^*)$ such that $E_\infty^{p,q} \cong F^p H^{p+q}/F^{p+1}H^{p+q}$.

For a filtered cochain complex (C^*, δ) with a filtration F^*C^*, we call this spectral sequence $E_1^{p,q} \cong H^{p+q}(F^pC^*/F^{p+1}C^*)$ is called the *spectral sequence of the filtered cochain complex*.

We obtain a useful tool from a spectral sequence to compute cohomology groups.

Proposition 4.6 (see e.g. McCleary [12, 1.2, Example 1.A]). *Assume that $\{(E_r^{*,*}, d_r)\}$ is a first quadrant spectral sequence converging to a graded module H^*.*

Then we have an exact sequence

$$0 \to E_2^{1,0} \to H^1 \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to H^2,$$

which is called the five-term exact sequence.

4.2 Th Lyndon-Hochschild-Serre spectral sequence

Hereinafter, we consider group cochain complexes as normalized ones.

Let G be a group and N be a normal subgroup of G. Then there is a group extension

$$1 \to N \to G \to G/N \to 1.$$

For a left G-moddule A, we define a filtration $(F^pC^*)_{p\in\mathbb{Z}}$ of $C^*(G;A)$ as follows:

$$\begin{cases} F^pC^q = C^q(G;A), & \text{if } p \le 0, \\ F^pC^q = 0, & \text{if } q < p \end{cases}$$

and for $0 < p \le q$, we set F^pC^q the group of q-cochains $c \in C^q(G;A)$ such that $c(g_1, \dots, g_p) = 0$ whenever $(q-p+1)$ of the arguments belong to N. We can check $\delta(F^pC^*) \subset F^pC^*$ easily. This filtration defines a spectral sequence $E_r^{p,q}$, called the *Lyndon-Hochschild-Serre spectral sequence* for the group extension

$$1 \to N \to G \to G/N \to 1.$$

Definition 4.7 (Hochschild-Serre [9, Chapter II, 1]). Let G be a group, N be a normal subgroup of G and A be a left G-module. We set $Z_r^{p,q} = \{c \in F^pC^{p+q} \mid \delta c \in F^{p+r}C^{p+q+1}\}$. The *Lyndon-Hochschild-Serre* (LHS) spectral sequence is a spectral sequence defined as

$$E_r^{p,q} = Z_r^{p,q}/(Z_{r-1}^{p-1,q+1} + \delta Z_{r-1}^{p+1-r,q+r-2})$$

and differential operators $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ are induced by δ.

Proposition 4.8 (Hochschild-Serre [9, Chapter II, 4, Theorem 2]). *Assume that $1 \to N \to G \to G/N \to 1$ is a short exact sequence. The LHS spectral sequence $E_r^{p,q}$ for this exact sequence satisfies the follows.*

1. *There exists an isomorphism*

$$E_1^{p,q} \cong C^p(G/N; H^q_{\mathrm{grp}}(N; A)).$$

and the differential d_1 corresponds to the group coboundary map $\delta_{G/N}$: $C^p(G/N; H^q_{\mathrm{grp}}(N; A)) \to C^{p+1}(G/N; H^q_{\mathrm{grp}}(N; A))$.

2. *There exists an isomorphism*

$$E_2^{p,q} \cong H^p_{\mathrm{grp}}(G/N; H^q_{\mathrm{grp}}(N; A)).$$

Sketch of proof. Let σ be a (p, q)-shuffle, that is, σ is a permutation of $(1, \ldots, p+q)$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$. For a group 1-cochain $c \in C^{p+q}(G; A)$, we define $c_\sigma \in C^{p+q}(G; A)$ as

$$c_\sigma(h_1, \ldots, h_q, g_1, \ldots, g_p) = c(\gamma_1, \ldots, \gamma_{p+q}),$$

where

$$\gamma_{\sigma(i)} = \begin{cases} g_i & \text{if } 1 \leq i \leq p, \\ g_{\sigma(i)+p-i}^{-1} \cdots g_1^{-1} h_{i-p} g_1 \cdots g_{\sigma(i)+p-i}, & \text{if } p+1 \leq i \leq p+q. \end{cases}$$

Then we define $\varphi(c) \in C^p(G/N; C^q(N; A))$ as

$$\varphi(c)(g_1, \ldots, g_p)(n_1, \ldots, n_q) = \sum_\sigma c_\sigma(n_1, \ldots, n_q, g_1, \ldots, g_p),$$

where σ run over all (p, q)-shuffles.

This gives a homomorphism $\varphi : C^{p+q}(G; A) \to C^p(G/N; C^q(N; A))$ and this homomorphism φ induces isomorphisms $E_1^{p,q} \to C^p(G/N; H^q_{\mathrm{grp}}(N; A))$ and $E_2^{p,q} \to H^p_{\mathrm{grp}}(G/N; H^q_{\mathrm{grp}}(N; A))$. □

Proposition 4.9 (Hochschild–Serre [9, Chapter III, 1]). *The LHS spectral sequence $E_r^{p,q}$ for a short exact sequence $1 \to N \to G \to G/N \to 1$ converges to $H^{p+q}_{\mathrm{grp}}(G;A)$, that is, there is an isomorphism*

$$E_\infty^{p,q} \cong \mathrm{Im}(H^{p+q}_{\mathrm{grp}}(F^pC^*) \to H^{p+q}_{\mathrm{grp}}(G;A))/\mathrm{Im}(H^{p+q}_{\mathrm{grp}}(F^{p+1}C^*) \to H^{p+q}_{\mathrm{grp}}(G;A)).$$

Remark 4.10. Assume that $1 \to N \to G \to G/N \to 1$ is a short exact sequence and let A be a left G-module on which N acts trivially. Recall $H^1_{\mathrm{grp}}(N;A) = \mathrm{Hom}(N,A)$ in Remark 3.7.

Let $f : N \to A$ be a G-equivariant homomorphism and let $\tau : G \to A$ be a connection cochain over f. According to Proposition 4.8, the homomorphism $f \in H^1_{\mathrm{grp}}(N;A)^G$ can be regarded as an element in $E_2^{0,1}$ in the LHS spectral sequence and $\tau \in Z_2^{0,1} \subset C^1(G;A)$ is a representative group 1-cocycle of f as an element in $E_2^{0,1}$.

By Definition 4.7, the differential $d_2 f \in H^2_{\mathrm{grp}}(G/N;A) \cong E_2^{2,0}$ is given by

$$d_2 f = [\delta\tau] \in H^2_{\mathrm{grp}}(G/N;A).$$

Lemma 4.11. *Let $0 \to A \to \widehat{G} \xrightarrow{p} G \to 1$ be a central extension with its extension class $e(\widehat{G}) \in H^2_{\mathrm{grp}}(G;A)$. In the five-term exact sequence*

$$0 \to H^1_{\mathrm{grp}}(G;A) \to H^1_{\mathrm{grp}}(\widehat{G};A) \to H^1_{\mathrm{grp}}(A;A) \xrightarrow{d_2} H^2_{\mathrm{grp}}(G;A) \to H^2_{\mathrm{grp}}(\widehat{G};A),$$

The negative identity $-\mathrm{id}_A \in \mathrm{Hom}(A,A) = H^1_{\mathrm{grp}}(A;A)$ transgresses to the extension class $d_2(-\mathrm{id}_A) = e(\widehat{G}) \in H^2_{\mathrm{grp}}(G;A)$.

Proof. Let $s : G \to \widehat{G}$ be a set-theorical section of the surjection $p : \widehat{G} \to G$. We define $\tau_s : \widehat{G} \to A$ as

$$\tau_s(\hat{g}) = \hat{g}\{s \circ p(\hat{g})\}^{-1} \in A$$

since $p(\hat{g}\{s \circ p(\hat{g})\}^{-1}) = 1 \in G$.

Then τ_s is a connection cochain over id_A:

$$\tau_s(a) = a, \quad \tau_s(\hat{g}a) = \tau_s(\hat{g}) + a.$$

By Remark 4.10, we obtain $d_2\mathrm{id}_A = [\delta\tau_s] \in H^2_{\mathrm{grp}}(G;A)$. On the other hand, by applying Proposition 3.16, we have $e(\widehat{G}) = [-\delta\tau_s] \in H^2_{\mathrm{grp}}(G;A)$. Thus we conclude $d_2(-\mathrm{id}_A) = [-\delta\tau_s] = e(\widehat{G}) \in H^2_{\mathrm{grp}}(G;A)$. □

4.3 The Leray-Serre spectral sequence

Let B be a connected CW-complex and denote its p-skeleton by B_p:

$$B_0 \subset B_1 \subset B_2 \subset \cdots \subset B.$$

For a fibration $F \hookrightarrow E \xrightarrow{\pi} B$ with CW-complexes F and E, put $E_p := \pi^{-1}(B_p)$:

$$E_0 \subset E_1 \subset E_2 \subset \cdots \subset E.$$

Then we have a sequence of the singular cochain complexes:

$$C^*(E_0;\mathbb{Z}) \leftarrow C^*(E_1;\mathbb{Z}) \leftarrow C^*(E_2;\mathbb{Z}) \leftarrow \ldots \leftarrow C^*(E;\mathbb{Z}).$$

We define a filtraion $(F^pC^*)_{p\in\mathbb{Z}}$ of the singular complex $C^*(E;\mathbb{Z})$ as follows:

$$\begin{cases} F^pC^q = C^q(E;\mathbb{Z}), & \text{if } p \leq 0 \\ F^pC^q = 0, & \text{if } q < p \end{cases}$$

and for $0 < p \leq q$, we set

$$F^pC^q = \mathrm{Ker}(C^q(E;\mathbb{Z}) \to C^q(E_p;\mathbb{Z}))$$

the kernel of the restriction map.

Proposition 4.12 (see e.g. McCleary [12, 5, Theorem 5.2])**.** *A spectral sequence $\{(E_r^{*,*}, d_r)\}$ of the filtered cochain complex F^*C^* converges to $H^*(E;\mathbb{Z})$ with*

$$E_2^{p,q} \cong H^p(B;\mathcal{H}^q(F;\mathbb{Z})),$$

where $\mathcal{H}^p(F;\mathbb{Z})$ is a local system of coefficients on B of the cohomology group $H^p(F;\mathbb{Z})$.

This spectral sequence $E_2^{p,q} \cong H^p(B;\mathcal{H}^q(F;\mathbb{Z}))$ is called the *Leray-Serre spectral sequence* for the fibration $F \hookrightarrow E \to B$.

Assume that $F \hookrightarrow E \to B$ is a fiber bundle with connected CW-complexes B and F and with the structure group G. In general theory of spectral sequence, we have the five-term exact sequence of the Leray-Serre spectral sequence:

$$0 \to H^1(B;\mathbb{Z}) \to H^1(E;\mathbb{Z}) \to H^0(B;\mathcal{H}^1(F;\mathbb{Z})) \to H^2(B;\mathbb{Z}) \to H^2(E;\mathbb{Z}).$$

We will explain what $H^0(B;\mathcal{H}^1(F;\mathbb{Z}))$ is.

Proposition 4.13 (see e.g. Davis-Kirk [5, 5.3, Proposition 5.14])**.** *Let B be a connected CW-complex and let A be a left $\pi_1(B)$-module. We consider the cohomology group $H^p(B;\mathcal{A})$, where $\mathcal{A}$ is a local system of coefficients on B of A.*

Then

$$H^0(B;\mathcal{A}) \cong A^{\pi_1(B)} = \{a \in A \mid a = \gamma \cdot a, \forall \gamma \in \pi_1(B)\}.$$

Let B and F be connected CW-complexes. Assume that $\pi : E \to B$ is a fiber bundle with F the typical fiber and with G the structure group. Then the fundamental group $\pi_1(B)$ acts on F as elements of the structure group G, called the *monodromy action*. Therefore, we obtain the following lemma:

Lemma 4.14. *Let B and F be connected CW-complexes. Assume that $\pi : E \to B$ is a fiber bundle with F the typical fiber and with G the structure group.*

Then we have the five-term exact sequence

$$0 \to H^1(B;\mathbb{Z}) \to H^1(E;\mathbb{Z}) \to H^1(F;\mathbb{Z})^{\pi_1(B)} \xrightarrow{d_2} H^2(B;\mathbb{Z}) \to H^2(E;\mathbb{Z})$$

and there is a natural inclusion

$$H^1(F;\mathbb{Z})^{\pi_1(B)} \subset H^1(F;\mathbb{Z})^G.$$

Remark 4.15. We recall the relation between the Leray-Serre spectral sequence for a fibration and the Lyndon-Hochschild-Serre spectral sequence for a short exact sequence for groups.

Assume that $1 \to N \to G \to G/N \to 1$ is a short exact sequence for groups. We obtain a fibration $BN \hookrightarrow BG \to B(G/N)$ as

$$EG/N \hookrightarrow (EG \times E(G/N))/G \to B(G/N),$$

where $EN \to BN$, $EG \to BG$, and $E(G/N) \to B(G/N)$ are classifying spaces of N, G, and G/N, respectively.

Then the Leray-Serre spectral sequence $H^p(B(G/N);\mathcal{H}^q(BN;\mathbb{Z}))$ is naturally isomorphic to the LHS spectral sequence $H^p_{\mathrm{grp}}(G/N;H^q_{\mathrm{grp}}(N;\mathbb{Z}))$.

5 The Central Extension associated to the $\mathbb{Z}$-covering space

In this section, we introduce and investigate the central extension associated to a regular $\mathbb{Z}$-covering space mentioned in Introduction.

First, we shall introduce a new central extension, called the *central extension associated to* a given regular $\mathbb{Z}$-covering space. In the case of the universal covering $\mathbb{R} \to S^1$, the central extension associated to the universal covering space is just a central extension made up of the universal covering group of the orientation-preserving homeomorphism group $\mathrm{Homeo}_+(S^1)$ on S^1.

Second, we shall give the geometrical meaning to the central extension associated to a regular $\mathbb{Z}$-covering space. We define the notion of (X, α)-bundle, which is an analogy to that of oriented S^1-bundles. The (X, α)-bundle is a fiber bundle with X the typical fiber and with $\mathrm{Homeo}_\alpha(X)$ the structure group. Here $\mathrm{Homeo}_\alpha(X)$ is the group consisting of homeomorphisms on X that preserve the cohomology class $\alpha \in H^1(X; \mathbb{Z})$ (see Definition 5.7), which is regarded as the isomorphism class of the regular $\mathbb{Z}$-covering space over X. We show that such an (X, α)-bundle which is not necessarily flat has a characteristic class, so called the *Euler class*. Then we clarify the relation between the central extension associated to α and the Euler class for the universal flat (X, α)-bundle. It is known that the central extension of $\mathrm{Homeo}_+(S^1)$ given by the universal covering group corresponds to the Euler class for the universal oriented flat S^1-bundle. As in the case of oriented flat S^1-bundle, when the (X, α)-bundle is flat, we prove that the Euler class for the universal flat (X, α)-bundle corresponds to the extension class of the central extension associated to α.

Third, we shall give formulas of the representative cocycles for the extension class of the central extension associated to a given $\mathbb{Z}$-covering space. For the given covering space $\widehat{M} \to M$ over a smooth manifold M, if there exists a certain manifold W with M the boundary of W, then we prove a transgression formula connecting the flux homomorphism on the volume-preserving relative diffeomorphism group with the extension class of the *restricted smooth* central extension associated to α.

In this section, we assume that all topological spaces are CW-complexes and that the base space of a covering space is connected unless otherwise stated.

5.1 Central extension associated to $\mathbb{Z}$-covering space

In this subsection, we introduce the notion of the central extension associated to a regular $\mathbb{Z}$-covering space. In the following, we observe that the total space of covering spaces is not necessarily connected. For instance, a disjoint union $X \sqcup X$ of a connected CW-complex X with itself can be considered as a double covering space over X, which is not connected clearly.

Definition 5.1 (see e.g. Hatcher [8, Chapter I, section 1.3])**.** Let X be a connected CW-complex and let $\pi : \widehat{X} \to X$ be a covering space with Γ the deck transformation group. The covering space $\widehat{X} \to X$ is *regular* or *normal* if there exists a deck transformation $g \in \Gamma$ with $g \cdot \hat{x} = \hat{x}'$ for any $\hat{x}, \hat{x}' \in \widehat{X}$ with $\pi(\hat{x}) = \pi(\hat{x}')$. We call a covering space $\widehat{X} \to X$ a *regular* Γ*-covering space* if $\widehat{X} \to X$ is a regular covering space with Γ the deck transformation group.

Let $\pi : \widehat{X} \to X$ be a regular covering space with the deck transformation group Γ. Assume that $\widehat{X}$ is connected. Take and fix a base point $\hat{x}_0 \in \widehat{X}$ and put $x_0 = \pi(\hat{x}_0) \in X$. Since the covering space is regular, for each $\hat{x} \in \pi^{-1}(x_0) \subset \widehat{X}$, there is a unique element $g \in \Gamma$ such that $\hat{x} = g \cdot \hat{x}_0$. Thus we obtain a surjective homomorphism $\alpha_{x_0} : \pi_1(X, x_0) \to \pi_0(\pi^{-1}(\hat{x}_0), \hat{x}_0) \cong \Gamma$, called the *monodromy transformation*. Then we obtain a short exact sequence from the homotopy exact sequence for fiber bundles:

$$1 \to \pi_1(\widehat{X}, \hat{x}_0) \to \pi_1(X, x_0) \xrightarrow{\alpha_{x_0}} \pi_0(\pi^{-1}(x_0), \hat{x}_0) \to 1.$$

Lemma 5.2. *Let $\pi : \widehat{X} \to X$ be a regular covering space with the deck transformation group Γ. Suppose that $\widehat{X}$ is connected. Assume that a homomorphism $\phi : X \to X$ satisfies the following diagram:*

$$\begin{tikzcd}
\pi_1(X, x) \arrow[r, "\alpha_x"] \arrow[d, "\phi_*"'] \arrow[dr, phantom, "\circlearrowright"] & \Gamma \arrow[d, equal] \\
\pi_1(X, \phi(x)) \arrow[r, "\alpha_{\phi(x)}"'] & \Gamma
\end{tikzcd}$$

for any $x \in X$. Then there exists a homeomorphism $\hat{\phi} : \widehat{X} \to \widehat{X}$ such that

$$\hat{\phi}(g \cdot \hat{x}) = g \cdot \hat{\phi}(\hat{x})$$

for $\hat{x} \in \widehat{X}$ and $g \in \Gamma$.

Proof. Take and fix points $x \in X$ and $\hat{x}, \hat{x}' \in \widehat{X}$ such that $\pi(\hat{x}) = x$ and $\pi(\hat{x}') = \phi(x)$. By the assumption, we have

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\widehat{X}, \hat{x}) \arrow[r] & \pi_1(X, x) \arrow[r, "\alpha_x"] \arrow[d, "\phi_*"'] & \Gamma \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(\widehat{X}, \hat{x}') \arrow[r] & \pi_1(X, \phi(x)) \arrow[r, "\alpha_{\phi(x)}"'] & \Gamma \arrow[r] & 1,
\end{tikzcd}$$

and obtain

$$(\pi \circ \phi)_* \pi_1(\widehat{X}, \hat{x}) = \phi_* \mathrm{Ker}(\alpha_x) = \mathrm{Ker}(\alpha_{\phi(x)}) = \pi_* \pi_1(\widehat{X}, \hat{x}') \subset \pi_1(X, \phi(x)).$$

By using the Covering Lifting Property, there exist continuous maps $\hat{\phi}$ and $\widehat{\phi^{-1}}$ which make the following diagrams commutative:

$$\begin{tikzcd}
\widehat{X} \arrow[r, dashed, "\hat{\phi}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \widehat{X} \arrow[d, "\pi"] \\
X \arrow[r, "\phi"'] & X,
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{X} \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \widehat{X} \arrow[l, dashed, "\widehat{\phi^{-1}}"'] \arrow[d, "\pi"] \\
X & X. \arrow[l, "\phi^{-1}"]
\end{tikzcd}$$

Here, we may choose $\hat{\phi}$ and $\widehat{\phi^{-1}}$ such as $\hat{\phi}(\hat{x}) = \hat{x}'$ and $\widehat{\phi^{-1}}(\hat{x}') = \hat{x}$, respectively. By the uniqueness of lifts that agree at one point, the composition $\hat{\phi} \circ \widehat{\phi^{-1}}$ coincides with the identity on $\widehat{X}$ since $\hat{\phi} \circ \widehat{\phi^{-1}}(\hat{x}') = \hat{x}'$. Similarly, $\widehat{\phi^{-1}} \circ \hat{\phi}$ also coincides with the identity on $\widehat{X}$. Therefore $\hat{\phi} : \widehat{X} \to \widehat{X}$ is a homeomorphism which satisfies $\pi \circ \hat{\phi} = \hat{\phi}$ and $\hat{\phi}^{-1} = \widehat{\phi^{-1}}$.

The pair $(\hat{\phi}, \phi)$ induces the commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\widehat{X}, \hat{x}) \arrow[r] \arrow[d, "\hat{\phi}_*"'] & \pi_1(X, x) \arrow[r, "\alpha_x"] \arrow[d, "\phi_*"'] & \Gamma \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(\widehat{X}, \hat{\phi}(\hat{x})) \arrow[r] & \pi_1(X, \phi(x)) \arrow[r, "\alpha_{\phi(x)}"'] & \Gamma \arrow[r] & 1,
\end{tikzcd}$$

for each $x \in X$ and $\hat{x} \in \widehat{X}$ such that $\pi(\hat{x}) = x$. This implies that $\hat{\phi}$ commutes with the Γ-action. $\square$

Definition 5.3. Let $\pi : \widehat{X} \to X$ and $\pi' : \widehat{X} \to X$ be two different covering spaces over the same connected CW-complex X.

1. The covering spaces $\pi : \widehat{X} \to X$ and $\pi' : \widehat{X}' \to X$ are *isomorphic* if there exists a homeomorphism $\hat{\phi} : \widehat{X} \to \widehat{X}'$ which satisfies the following diagram:

$$\begin{tikzcd} \widehat{X} \arrow[rr, "\hat{\phi}"] \arrow[dr, "\pi"'] & \circlearrowleft & \widehat{X}' \arrow[dl, "\pi'"] \\ & X. & \end{tikzcd}$$

2. In addition, suppose that both of the covering spaces $\pi : \widehat{X} \to X$ and $\pi' : \widehat{X}' \to X$ are regular Γ-covering spaces. The regular Γ-covering spaces $\pi : \widehat{X} \to X$ and $\pi' : \widehat{X}' \to X$ are *isomorphic* if there exists a homeomorphism $\hat{\phi} : \widehat{X} \to \widehat{X}'$ that is an isomorphism between covering spaces and that satisfies

$$\hat{\phi}(g \cdot \hat{x}) = g \cdot \hat{\phi}(\hat{x})$$

for each $g \in \Gamma$ and $\hat{x} \in \widehat{X}$.

Remark 5.4. Let $\pi : \widehat{X} \to X$ and $\pi' : \widehat{X} \to X$ be two different regular Γ-covering spaces over the same connected CW-complex X. If they are isomorphic as regular Γ-covering spaces, then they are also isomorphic as covering spaces. However, the reverse is not always true.

Consider the universal covering space $\mathbb{R} \to S^1$ with the deck transformation group $\mathbb{Z}$. The integers $\mathbb{Z}$ acts on $\mathbb{R}$ as the deck transformations by two different ways: On the one hand, $\mathbb{Z}$ acts on $\mathbb{R}$ by

$$\mathbb{Z} \times \mathbb{R} \ni (n, x) \mapsto n + x \in \mathbb{R}.$$

On the other hand, $\mathbb{Z}$ acts on $\mathbb{R}$ by

$$\mathbb{Z} \times \mathbb{R} \ni (n, x) \mapsto -n + x \in \mathbb{R}.$$

The universal covering space $\mathbb{R} \to S^1$ with the former deck transformations is *not* isomorphic to the one with the latter deck transformations. In the following, we always assume that the universal covering space $\mathbb{R} \to S^1$ admits the former action of $\mathbb{Z}$ on $\mathbb{R}$ by fixing an orientation $u \in H^1(S^1; \mathbb{Z})$.

Lemma 5.5. *Let X be a connected CW-complex. The following notions are equivalent to each other.*

1. *An isomorphism class of regular $\mathbb{Z}$-covering spaces over X.*

2. *A cohomology class in $H^1(X;\mathbb{Z})$.*

3. *A homotopy class of continuous map from X to S^1.*

A sketch of proof. **1. $\Longrightarrow$ 2.** Choose and fix a point $\hat{x} \in \widehat{X}$ and put $x = \pi(\hat{x}) \in X$. For a regular $\mathbb{Z}$-covering space $\pi : \widehat{X} \to X$, we have the homotopy exact sequence:

$$1 \to \pi_1(\widehat{X}, \hat{x}) \to \pi_1(X, x) \xrightarrow{\alpha_x} \mathbb{Z} \to \pi_0(\widehat{X}, \hat{x}) \to 1.$$

Assume that $\pi' : \widehat{X}' \to X$ is a regular $\mathbb{Z}$-covering space which is isomorphic to $\pi : \widehat{X} \to X$ and take a point $\hat{x}' \in {\pi'}^{-1}(x) \subset \widehat{X}$ that corresponds to $\hat{x} \in \widehat{X}$. Then we obtain the homotopy exact sequence with the same homomorphism α_x:

$$1 \to \pi_1(\widehat{X}', \hat{x}') \to \pi_1(X, x) \xrightarrow{\alpha_x} \mathbb{Z} \to \pi_0(\widehat{X}', \hat{x}') \to 1.$$

Since the first homology group $H_1(X;\mathbb{Z})$ is known as the abelianization of $\pi_1(X, x)$, the homomorphism group $\alpha_x : \pi_1(X, x) \to \mathbb{Z}$ gives a homomorphism $H_1(X;\mathbb{Z}) \to \mathbb{Z}$. By the universal coefficient theorem for cohomology theory, the first cohomology group $H^1(X;\mathbb{Z})$ is isomorphic to $\mathrm{Hom}(H_1(X;\mathbb{Z}), \mathbb{Z})$. Hence we obtain the correspondence from an isomorphism class of regular $\mathbb{Z}$-covering space over X to a cohomology class in $H^1(X;\mathbb{Z})$.

2. $\Longrightarrow$ 3. Since Brown's representability theorem, the first cohomology group $H^1(X;\mathbb{Z})$ is isomorphic to the homotopy set $[X, B\mathbb{Z}]$, where $B\mathbb{Z}$ is the classifying space of $\mathbb{Z}$, which is homotopy equivalent to S^1.

3. $\Longrightarrow$ 2. As is explained above, we have the orientation class $u \in H^1(S^1;\mathbb{Z})$ that corresponds to the universal bundle $\mathbb{R} \to S^1$. Thus, given a continuous map $f : X \to S^1$, we obtain a cohomology class $\alpha = f^*u \in H^1(X;\mathbb{Z})$.

3. $\Longrightarrow$ 1. Given a continuous map $f : X \to S^1$, we set

$$\widehat{X}_f = \{(a, x) \in \mathbb{R} \times X \mid f(x) \equiv a \bmod \mathbb{Z}\}.$$

Then the natural projection $\widehat{X}_f \to X$ is a regular $\mathbb{Z}$-covering space.

□

Remark 5.6. Fix a regular $\mathbb{Z}$-covering space $\widehat{X} \to X$. Then the cohomology class $\alpha \in H^1(X;\mathbb{Z})$ corresponding to the isomorphism class of $\widehat{X} \to X$ is called the *classifying class*. A continuous map $X \to S^1$ in the homotopy class corresponding to the isomorphism class of $\widehat{X} \to X$ is also a *classifying map*.

Recall the central extension of the orientation-preserving homeomorphism group on the unit circle S^1 in Example 3.18:

$$0 \to \mathbb{Z} \to \mathrm{Homeo}(\mathbb{R})^{\mathbb{Z}} \to \mathrm{Homeo}_+(S^1) \to 1.$$

Here $\mathrm{Homeo}(\mathbb{R})^{\mathbb{Z}}$ consists of homeomorphisms on $\mathbb{R}$ that commute with the $\mathbb{Z}$-action on $\mathbb{R}$. It is at the same time isomorphic to the universal covering group of $\mathrm{Homeo}_+(S^1)$. Moreover, it is known that the extension class of this central extension above coincides with the Euler class for the universal oriented flat S^1-bundle (see Ghys [7, 6.2] for instance). This is the motivating example for our new central extensions of homeomorphism groups. Considering the above as a central extension of $\mathrm{Homeo}_+(S^1)$ by deck transformation group $\mathbb{Z}$, we shall extend the construction to arbitrary regular $\mathbb{Z}$-covering spaces. The existence of such a central extension of a homeomorphism group (to be precise, it is a certain subgroup of the homeomorphism group as is explained in the following) is one of the novelties of the present thesis.

Let $\pi : \widehat{X} \to X$ be a regular $\mathbb{Z}$-covering space over a connected CW-complex X. There is a cohomology class $\alpha \in H^1(X;\mathbb{Z})$ corresponding to the isomorphism class containing $\pi : \widehat{X} \to X$. We denote the $\mathbb{Z}$-action on $\widehat{X}$ by

$$n + \hat{x}$$

for $n \in \mathbb{Z}$ and $\hat{x} \in \widehat{X}$. Denote by $\widehat{G} = \mathrm{Homeo}(\widehat{X})^{\mathbb{Z}}$ the group of homeomorphisms on $\widehat{X}$ that commute with the deck transformations on $\widehat{X}$ by $\mathbb{Z}$:

$$\mathrm{Homeo}(\widehat{X})^{\mathbb{Z}} = \{\hat{\phi} \in \mathrm{Homeo}(\widehat{X}) \mid \hat{\phi}(1+\hat{x}) = 1 + \hat{\phi}(\hat{x}) \text{ for each } \hat{x} \in \widehat{X}\}.$$

Then we have the following exact sequence for groups:

$$0 \to \mathbb{Z} \to \mathrm{Homeo}(\widehat{X})^{\mathbb{Z}} \xrightarrow{p} \mathrm{Homeo}(X),$$

where the last homomorphism $p : \mathrm{Homeo}(\widehat{X})^{\mathbb{Z}} \to \mathrm{Homeo}(X)$ is not necessarily surjective.

Definition 5.7 (The homeomorphism group G_α [2])**.** We define a subgroup $\mathrm{Homeo}_\alpha(X) \subset \mathrm{Homeo}(X)$ to be

$$\mathrm{Homeo}_\alpha(X) = \{\phi \in \mathrm{Homeo}(X) \mid \phi^*\alpha = \alpha\},$$

which is called the *α-preserving homeomorphism group on X*. In the sequel we often denote $\mathrm{Homeo}_\alpha(X)$ by G_α.

Lemma 5.8. *The homomorphism $p : \widehat{G} \to \mathrm{Homeo}(X)$ factors through $G_\alpha = \mathrm{Homeo}_\alpha(X)$. Furthermore, $p : \widehat{G} \to G_\alpha$ is surjective.*

Proof. Take a homeomorphism $\hat{\phi} \in \widehat{G}$. Then the pair $(\hat{\phi}, \phi = p(\hat{\phi}))$ makes the following diagram commutative:

$$\begin{tikzcd}
\widehat{X} \arrow[r, "\hat{\phi}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \widehat{X} \arrow[d, "\pi"] \\
X \arrow[r, "\phi = p(\hat{\phi})"'] & X.
\end{tikzcd}$$

For each $\hat{x} \in \widehat{X}$ and $x = \pi(\hat{x}) \in X$, the pair $(\hat{\phi}, \phi)$ induces homomorphisms between two homotopy exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\widehat{X}, \hat{x}) \arrow[r] \arrow[d, "\hat{\phi}_*"'] & \pi_1(X, x) \arrow[r, "\alpha"] \arrow[d, "\phi_*"'] & \mathbb{Z} + \hat{x} \arrow[r] \arrow[d] & \pi_0(\widehat{X}, \hat{x}) \arrow[r] \arrow[d, "\hat{\phi}_*"] & 0, \\
1 \arrow[r] & \pi_1(\widehat{X}, \hat{\phi}(\hat{x})) \arrow[r] & \pi_1(X, \phi(x)) \arrow[r, "\alpha"] & \mathbb{Z} + \hat{\phi}(\hat{x}) \arrow[r] & \pi_0(\widehat{X}, \hat{\phi}(\hat{x})) \arrow[r] & 0.
\end{tikzcd}$$

Since $\hat{\phi}$ commutes with the $\mathbb{Z}$-action on $\widehat{X}$, we obtain the identity $\phi^*\alpha = \alpha$, so that the image of $p : \widehat{G} \to \mathrm{Homeo}(X)$ is contained in G_α.

Next, we prove that the homomorphism $p : \widehat{G} \to G_\alpha$ is surjective. We first assume that $\widehat{X}$ is connected. Let $\phi \in G_\alpha$. By the definition of G_α, the homeomorphism $\phi \in G_\alpha$ satisfies $\phi^*\alpha = \alpha$. This implies that the diagram

$$\begin{tikzcd}
\pi_1(X, x) \arrow[r, "\alpha_x"] \arrow[d, "\phi_*"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z} \arrow[d, equal] \\
\pi_1(X, \phi(x)) \arrow[r, "\alpha_{\phi(x)}"'] & \mathbb{Z}
\end{tikzcd}$$

[2] This definition of G_α was suggested by my superviser, Professor Moriyoshi.

is commutative for each $x \in X$. Applying Lemma 5.2, there exists a homeomorphism $\hat{\phi} \in \widehat{G} = \mathrm{Homeo}(\widehat{X})^{\mathbb{Z}}$ such that $p(\hat{\phi}) = \phi$.

Second, we assume that $\widehat{X}$ is not connected. Denote by $\widehat{X}_0 \subset \widehat{X}$ a connected component of $\widehat{X}$. Then $\widehat{X}_0 \to X$ is a connected regular covering space over X. Take and fix a point $\hat{x}_0 \in \widehat{X}_0$ and put $x = \pi(\hat{x}_0) \in X$. We also fix a point $\hat{x}'_0 \in \widehat{X}_0$ such that $\pi(\hat{x}'_0) = \phi(x)$. Then we have two exact sequences from homotopy exact sequences:

$$1 \to \pi_1(\widehat{X}, \hat{x}) \to \pi_1(X, x) \xrightarrow{\alpha_0} \mathbb{Z} \to \pi_0(\widehat{X}, \hat{x}_0) \to 0,$$
$$1 \to \pi_1(\widehat{X}, \hat{x}') \to \pi_1(X, \phi(x)) \xrightarrow{\alpha_1} \mathbb{Z} \to \pi_0(\widehat{X}, \hat{x}'_0) \to 0.$$

Now, we obtain the following commutative diagram with exact lows:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\widehat{X}, \hat{x}_0) \arrow[r] \arrow[d, equal] & \pi_1(X, x) \arrow[r, "\alpha_0"] \arrow[d, equal] & \mathbb{Z} \arrow[r] & \pi_0(\widehat{X}, \hat{x}_0) \arrow[r] & 0 \\
1 \arrow[r] & \pi_1(\widehat{X}, \hat{x}_0) \arrow[r] & \pi_1(X, x) \arrow[r, "\alpha_0"] & \mathrm{Im}(\alpha_0) \arrow[u, hook] \arrow[rr] & & 0.
\end{tikzcd}$$

Since $\alpha_0 \circ \phi_* = \alpha_1$, we obtain

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\widehat{X}_0, \hat{x}_0) \arrow[r] & \pi_1(X, x) \arrow[r, "\alpha_0"] \arrow[d, "\phi_*"'] & \mathrm{Im}(\alpha_0) \arrow[r] \arrow[d, equal] & 0 \\
1 \arrow[r] & \pi_1(\widehat{X}_0, \hat{x}'_0) \arrow[r] & \pi_1(X, \phi(x)) \arrow[r, "\alpha_1"'] & \mathrm{Im}(\alpha_1) \arrow[r] & 0.
\end{tikzcd}$$

Applying Lemma 5.2, there exists a homeomorphism $\hat{\phi}_0 : \widehat{X}_0 \to \widehat{X}_0$ which commutes with the $\mathrm{Im}(\alpha_0)$-action on $\widehat{X}_0$ and makes the following diagram commutative:

$$\begin{tikzcd}
\widehat{X}_0 \arrow[r, "\hat{\phi}_0"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \widehat{X}_0 \arrow[d, "\pi"] \\
X \arrow[r, "\phi"'] & X.
\end{tikzcd}$$

Put

$$\widehat{X}_n = n + \widehat{X}_0 = \{n + \hat{x} \in \widehat{X} \mid \hat{x} \in \widehat{X}_0\}.$$

We set $\hat{x}_n = n + \hat{x}_0 \in \widehat{X}_n$ and $\hat{x}'_n = n + \hat{x}'_0 \in \widehat{X}$. For each integer $n \in \mathbb{Z}$, we obtain a homeomorphism $\hat{\phi}_n : \widehat{X}_n \to \widehat{X}_n$ satisfying $\hat{\phi}_n(\hat{x}_n) = \hat{x}'_n$ and the following commutative diagram:

$$\begin{tikzcd}
\widehat{X}_n \arrow[r, "\hat{\phi}_n"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & \widehat{X}_n \arrow[d, "\pi"] \\
X \arrow[r, "\phi"'] & X.
\end{tikzcd}$$

If $\widehat{X}_n = \widehat{X}_m$, then we have

$$\hat{\phi}_n(\hat{x}_m) = \hat{\phi}_n((m-n) + \hat{x}_n) = (m-n) + \hat{\phi}_n(\hat{x}_n) = (m-n) + \hat{x}'_n = \hat{x}'_m.$$

This implies that $\hat{\phi}_n = \hat{\phi}_m$ according to the uniqueness of lifts that agree at one point. Hence the homeomorphism $\hat{\phi} : \widehat{X} \to \widehat{X}$ defined as

$$\hat{\phi}|_{\widehat{X}_n} = \hat{\phi}_n : \widehat{X}_n \to \widehat{X}_n$$

is well-defined.

For each $\hat{x} \in X$ and $x = \pi(\hat{x}) \in \widehat{X}$, the pair $(\hat{\phi}, \phi)$ induces the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\widehat{X}, \hat{x}) \arrow[r] \arrow[d, "\hat{\phi}_*"'] & \pi_1(X, x) \arrow[r, "\alpha_x"] \arrow[d, "\phi_*"'] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \pi_0(\widehat{X}, \hat{x}) \arrow[r] \arrow[d, "\hat{\phi}_*"] & 1 \\
1 \arrow[r] & \pi_1(\widehat{X}, \hat{\phi}(\hat{x})) \arrow[r] & \pi_1(X, \phi(x)) \arrow[r, "\alpha_{\phi(x)}"'] & \mathbb{Z} \arrow[r] & \pi_0(\widehat{X}, \hat{\phi}(\hat{x})) \arrow[r] & 1.
\end{tikzcd}$$

This implies that $\hat{\phi}$ commutes with the $\mathbb{Z}$-action on $\widehat{X}$. □

Definition 5.9 (The central extension associated to a regular $\mathbb{Z}$-covering space)**.** Let $\pi : \widehat{X} \to X$ be a regular $\mathbb{Z}$-covering space with the classifying class $\alpha \in H^1(X; \mathbb{Z})$. Take $G_\alpha = \mathrm{Homeo}_\alpha(X)$ and $\widehat{G} = \mathrm{Homeo}(\widehat{X})^{\mathbb{Z}}$. A central extension

$$0 \to \mathbb{Z} \to \widehat{G} \to G_\alpha \to 1$$

is called the *central extension associated to* $\pi : \widehat{X} \to X$ or the *central extension associated to* α.

Remark 5.10. We can define the notion of smooth central extension associated to a regular $\mathbb{Z}$-covering space over a smooth manifold. For a given regular $\mathbb{Z}$-covering space $\pi : \widehat{M} \to M$ over the smooth manifold M with the classifying class $\alpha \in H^1(M;\mathbb{Z})$, we obtain a central extension

$$0 \to \mathbb{Z} \to \widehat{G}^\infty \to G^\infty_\alpha \to 1.$$

Here, $G^\infty_\alpha = \mathrm{Diff}_\alpha(M)$ is the group consisting of diffeomorphisms on M that preserve $\alpha \in H^1(M;\mathbb{Z})$ and $\widehat{G}^\infty = \mathrm{Diff}(\widehat{M})^\mathbb{Z}$ is the group of diffeomorphisms on $\widehat{M}$ that commute with the $\mathbb{Z}$-action on $\widehat{M}$. We call the above the *smooth central extension associated to* the regular $\mathbb{Z}$-covering space $\pi : \widehat{M} \to M$.

Example 5.11. Take the universal covering space $\mathbb{R} \to S^1$ as a regular $\mathbb{Z}$-covering space. We have chosen the orientation class $u \in H^1(S^1;\mathbb{Z})$ as the classifying class of $\mathbb{R} \to S^1$. Then $G_u = \mathrm{Homeo}_u(S^1)$ is just the orientation-preserving homeomorphism group on S^1. Hence the central extension associated to the universal covering space $\mathbb{R} \to S^1$ is

$$0 \to \mathbb{Z} \to \mathrm{Homeo}(\mathbb{R})^\mathbb{Z} \to \mathrm{Homeo}_+(S^1) \to 1$$

and this extension turned out to be non-trivial (see Example 3.18).

Example 5.12. Let X be a topological space. The direct product $X \times S^1$ has a natural $\mathbb{Z}$-covering space $X \times \mathbb{R} \to X \times S^1$ and its classifying class $1 \otimes u \in H^1(X \times S^1;\mathbb{Z})$. Here, we consider $H^*(X \times S^1;\mathbb{Z})$ as the graded tensor product $H^*(X;\mathbb{Z}) \otimes H^*(S^1;\mathbb{Z})$ by the Künneth formula. Then we have the central extension

$$0 \to \mathbb{Z} \to \mathrm{Homeo}(X \times \mathbb{R})^\mathbb{Z} \to \mathrm{Homeo}_{1\otimes u}(X \times S^1) \to 1.$$

Now, we have natural inclusions and the following commuting diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathrm{Homeo}(X \times \mathbb{R})^\mathbb{Z} \arrow[r] & \mathrm{Homeo}_{1\otimes u}(X \times S^1) \arrow[r] & 1 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[u, equal] & \mathrm{Homeo}(\mathbb{R})^\mathbb{Z} \arrow[r] \arrow[u, hook] & \mathrm{Homeo}_+(S^1) \arrow[r] \arrow[u, hook] & 1.
\end{tikzcd}$$

Since the lower low is a non-trivial central extension, the upper central extension is also non-trivial.

Especially when $X = S^1$, we obtain a non-trivial central extension associated to the $\mathbb{Z}$-covering space $S^1 \times \mathbb{R} \to T^2$ over the torus.

Remark 5.13. The group $\mathrm{Homeo}_\alpha(X)$ always contains $\mathrm{Homeo}(X)_0$ the identity component of the group of homeomorphism on X. Especially if $X = S^1$, then $\mathrm{Homeo}_+(S^1)$ agrees with $\mathrm{Homeo}(S^1)_0$. However, $\mathrm{Homeo}_\alpha(X)$ does not always agree with $\mathrm{Homeo}(X)_0$. For instance, the map $\mathbb{R}^2 \to \mathbb{R}^2$, $(s,t) \mapsto (s,-t)$ induces a orientation-reversing homeomorphism ϕ on the torus T^2. This homeomorphism ϕ preserves the cohomology class $u \otimes 1 \in H^1(T^2;\mathbb{Z})$. Thus $\phi \notin \mathrm{Homeo}(T^2)_0$ and $\phi \in \mathrm{Homeo}_{u\otimes 1}(T^2)$.

Example 5.14. Let Σ_g be an oriented closed surface with genus g. We consider Σ_∞ which is a connected sum of a cylinder $S^1 \times \mathbb{R}$ and countably many toruses (see Figure 1). Then the $\mathbb{Z}$-covering space $S^1 \times \mathbb{R} \to T^2$ induces a $\mathbb{Z}$-covering space $\Sigma_\infty \to \Sigma_2$. Similarly, $n\mathbb{Z} \subset \mathbb{Z}$ acts on Σ_∞ and we obtain $n\mathbb{Z}$-covering space $\Sigma_\infty \to \Sigma_{n+1}$ since Σ_{n+1} is an n-fold covering space over Σ_2.

Fon $n \geq 2$, we have natural inclusions which satisfies the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathrm{Homeo}(\Sigma_\infty)^{\mathbb{Z}} \arrow[r] & \mathrm{Homeo}_\alpha(\Sigma_{n+1}) \arrow[r] & 1 \\
0 \arrow[r] & \mathbb{Z} \arrow[u, equal] \arrow[r, "\times n"] & \mathbb{Z} \arrow[u, hook] \arrow[r] & \mathbb{Z}/n\mathbb{Z} \arrow[u, hook] \arrow[r] & 1
\end{tikzcd}$$

The lower central extension is non-trivial since $\mathbb{Z}$ is torsion-free and $\mathbb{Z}/n\mathbb{Z}$ has torsion. Therefore the upper central extension is non-trivial.

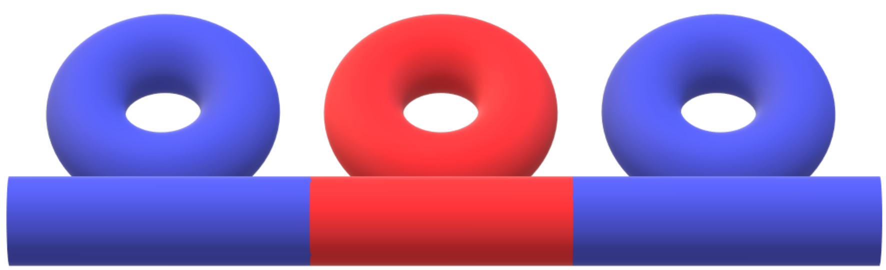

Figure 1: Imaginary drawing of Σ_∞

Remark 5.15. In the above Example 5.14, we don't know if the central extension

$$0 \to \mathbb{Z} \to \mathrm{Homeo}(\Sigma_\infty)^{\mathbb{Z}} \to \mathrm{Homeo}_\alpha(\Sigma_2) \to 1$$

is non-trivial.

5.2 The Euler class for oriented S^1-bundles

In this subsection, we recall a characteristic class for an oriented S^1-bundle, called the *Euler class*. The contents in this subsection are well known. We refer to Milnor [14], Wood [17], and Ghys [7], for instance.

Let $\pi : E \to B$ be an oriented S^1-bundle over a connected CW-complex B. Assume that $u \in H^1(S^1;\mathbb{Z})$ is the orientation class of S^1. The oriented S^1-bundle is a special example of (X,α)-bundle with $X = S^1$ and $\alpha = u \in H^1(S^1;\mathbb{Z})$ as is discussed in Definition 5.19. Recall the Leray-Serre spectral sequence and the five-term exact sequence from Proposition 4.12 and Lemma 4.14. For the Leray-Serre spectral sequence

$$E_2^{p,q} \cong H^p(B;\mathcal{H}^q(S^1;\mathbb{Z}))$$

for a bundle $E \to B$, the *Euler class* $e(E) \in H^2(B;\mathbb{Z})$ is defined by the five-term exact sequence as follows:

$$\begin{tikzcd}
0 \arrow[r] & H^1(B;\mathbb{Z}) \arrow[r] & H^1(E;\mathbb{Z}) \arrow[r] & H^1(S^1;\mathbb{Z}) \arrow[r, "d_2"] & H^2(B;\mathbb{Z}) \arrow[r] & H^2(E;\mathbb{Z}). \\
 & & & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \\
 & & & -u \arrow[r, mapsto] & e(E). &
\end{tikzcd}$$

Here, $\mathrm{Homeo}_+(S^1)$, the structure group of the oriented S^1-bundle $E \to B$, acts on $H^*(S^1;\mathbb{Z})$ trivially so that one has

$$E_2^{0,1} \cong H^1(S^1;\mathbb{Z})^{\pi_1(B)} = H^1(S^1;\mathbb{Z}).$$

This Euler class for an oriented S^1-bundle is a special case of the Euler class for a (S^1,u)-bundle $E \to B$ in Definition 5.20.

It is known that the universal Euler class $e \in H^2(B\mathrm{Homeo}_+(S^1);\mathbb{Z})$ is a generator of the polynomial ring $H^*(B\mathrm{Homeo}_+(S^1);\mathbb{Z})$. Therefore the Euler class for an oriented S^1-bundle is the unique characteristic class up to a constant multiple.

Proposition 5.16 (Thurston [16, Corollary (b)]). *Let M be a closed manifold. Denote by* $\mathrm{Homeo}(M)$ *the group of homeomorphisms on M with compact open topology. We also denote by* $\mathrm{Homeo}(M)^\delta$ *the group* $\mathrm{Homeo}(M)$ *with discrete topology. Then the map* $B\mathrm{Homeo}(M)^\delta \to B\mathrm{Homeo}(M)$ *induces an isomorphism on cohomology group.*

According to this proposition in the case of $M = S^1$, it follows that the Euler class $e(E) \in H^2(B;\mathbb{Z})$ is not always trivial when an oriented S^1-bundle $\pi : E \to B$ is flat. The Euler class for the universal flat S^1-bundle has a well-known representation under the isomorphism $H^2(B\mathrm{Homeo}_+(S^1)^\delta;\mathbb{Z}) \cong H^2_{\mathrm{grp}}(\mathrm{Homeo}_+(S^1);\mathbb{Z})$.

Put $G_\partial = \mathrm{Homeo}_+(S^1)$. Denote by $\widetilde{G}_\partial$ the universal covering group of G_∂. The group $\widetilde{G}_\partial$ is known as $\mathrm{Homeo}(\mathbb{R})^\mathbb{Z}$ the group of homeomorphisms on $\mathbb{R}$ which commutes with the $\mathbb{Z}$-action on $\mathbb{R}$:

$$\hat{\phi}(n+a) = n + \hat{\phi}(a), \quad \hat{\phi} \in \widetilde{G}_\partial, n \in \mathbb{Z}, a \in \mathbb{R}.$$

Then we have a central extension

$$0 \to \mathbb{Z} \hookrightarrow \widetilde{G}_\partial \to G_\partial \to 1$$

and put $e(\widetilde{G}_\partial) \in H^2_{\mathrm{grp}}(G_\partial;\mathbb{Z})$ its extension class.

Proposition 5.17 (see e.g. Ghys [7, 6.2])**.** *The Euler class $e \in H^2(BG_\partial^\delta;\mathbb{Z})$ for the universal oriented flat S^1-bundle coincides with the extension class $e(\widetilde{G}_\partial) \in H^2_{\mathrm{grp}}(G_\partial;\mathbb{Z})$.*

Proof. For the central extension $0 \to \mathbb{Z} \to \widetilde{G}_\partial \to G_\partial \to 1$, we have an induced fibration $B\mathbb{Z} \hookrightarrow B\widetilde{G}_\partial^\delta \to BG_\partial^\delta$:

$$E\widetilde{G}_\partial^\delta/\mathbb{Z} \hookrightarrow (E\widetilde{G}_\partial^\delta \times EG_\partial^\delta)/\widetilde{G}_\partial^\delta \to BG_\partial^\delta.$$

As we described in Remark 4.15, the Leray-Serre spectral sequence for this induced fibration agrees with the Lyndon-Hochschild-Serre spectral sequence for the central extension $0 \to \mathbb{Z} \to \widetilde{G}_\partial \to G_\partial \to 1$ and the orientation class $u \in H^1(S^1;\mathbb{Z}) \cong H^1(B\mathbb{Z};\mathbb{Z})$ corresponds to the identity $\mathrm{id} \in \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) = H^1_{\mathrm{grp}}(\mathbb{Z};\mathbb{Z})$.

Since $\mathbb{Z}$ acts on $E\widetilde{G}_\partial^\delta/\mathbb{Z} \times EG_\partial^\delta$ trivially, we have a homeomorphism

$$(E\widetilde{G}_\partial^\delta \times EG_\partial)/\widetilde{G}_\partial^\delta \cong (E\widetilde{G}_\partial^\delta/\mathbb{Z} \times EG_\partial^\delta)/G_\partial^\delta.$$

This implies that the induced fibration $B\widetilde{G}_\partial^\delta \to BG_\partial^\delta$ is the $B\mathbb{Z} \simeq E\widetilde{G}_\partial^\delta/\mathbb{Z}$-bundle associated with the universal flat G_∂-bundle $EG_\partial^\delta \to BG_\partial^\delta$. By the homotopy equivalence $B\mathbb{Z} \simeq S^1$, the Leray-Serre spectral sequence for the universal flat S^1-bundle agrees with the one of this $B\mathbb{Z}$-bundle.

Therefore, we proved the following:

$$
\begin{tikzcd}
H^1(S^1;\mathbb{Z}) \arrow[r, phantom, "\cong"] \arrow[d, "d_2"'] & H^1(B\mathbb{Z};\mathbb{Z}) \arrow[r, phantom, "\cong"] \arrow[d, "d_2"'] & H^1_{\mathrm{grp}}(\mathbb{Z};\mathbb{Z}) \arrow[d, "d_2"] \\
H^2(BG^\delta_\partial;\mathbb{Z}) \arrow[r, phantom, "="] & H^2(BG^\delta_\partial;\mathbb{Z}) \arrow[r, phantom, "\cong"] & H^2_{\mathrm{grp}}(G_\partial;\mathbb{Z}) \\
\Cup & & \Cup \\
e \arrow[rr, mapsto] & & e(\widetilde{G}_\partial).
\end{tikzcd}
$$

□

Proposition 5.17 is a special case of Theorem 5.23 discussed later:

Proposition 5.17	Theorem 5.23
S^1 with the orientation $u \in H^1(S^1;\mathbb{Z})$	CW-complex X with the classifying class $\alpha \in H^1(X;\mathbb{Z})$
an oriented S^1-bundle $E \to B$	an (X,α)-bundle $E \to B$
the Euler class $e(E) \in H^2(B;\mathbb{Z})$	the Euler class $e_\alpha(E) \in H^2(B;\mathbb{Z})$
the universal covering group $0 \to \mathbb{Z} \to \widetilde{G}_\partial \to G_\partial \to 1$	the central extension associated to α $0 \to \mathbb{Z} \to \widehat{G} \to G_\alpha \to 1$

Remark 5.18. We explain the correspondence between the cohomology group $H^1(S^1;\mathbb{Z})$ and the equivalence classes of regular $\mathbb{Z}$-covering spaces over S^1.

Let $u \in H^1(S^1;\mathbb{Z})$ be the orientation class of S^1, which is known as a generator of $H^1(S^1;\mathbb{Z})$. The universal covering space $\mathbb{R} \to S^1$ has u as the classifying class.

For $0 \in H^1(S^1;\mathbb{Z})$, the associated regular $\mathbb{Z}$-covering space over S^1 is just the trivial covering space $\mathbb{Z} \times S^1 \to S^1$.

For the $-u \in H^1(S^1;\mathbb{Z})$, the associated regular $\mathbb{Z}$-covering space is just the same universal covering space $\mathbb{R} \to S^1$ as a covering space. However, the deck transformation group $\mathbb{Z}$ acts on $\mathbb{R}$ as *orientation-reversing* homeomorphisms on $\mathbb{R}$.

For $n \in \mathbb{Z}_{\geq 0}$, the regular $\mathbb{Z}$-covering space $\widehat{S}^1 \to S^1$ with the classifying class $nu \in H^1(S^1; \mathbb{Z})$ is a disjoint union of n-copies of the universal covering space $\mathbb{R} \to S^1$:

$$\widehat{S}^1 = \mathbb{Z}/n\mathbb{Z} \times \mathbb{R} \to S^1.$$

Then the deck transformations $\mathbb{Z}$ acts on $\widehat{S}^1$ as

$$\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z} \times \mathbb{R}) \ni (m, ([n], a)) \mapsto ([n+m], \lfloor \frac{m}{n} \rfloor + a) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{R},$$

where $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ is the floor function.

5.3 The Euler class for (X, α)-bundles

In this subsection, we give a geometric meaning to the central extension associated to a regular $\mathbb{Z}$-covering space. We introduce the *Euler class* for (X, α)-bundles. The class does not necessarily vanish for *flat* bundles and it turned out that there exists a relation between the Euler class and the central extension associated to a regular $\mathbb{Z}$-covering space.

Definition 5.19 ((X, α)-bundle)**.** Let X be a connected CW-complex and α a cohomology class in $H^1(X; \mathbb{Z})$. Denote by $G_\alpha = \mathrm{Homeo}_\alpha(X)$ the α-preserving homeomorphism group on X. A fiber bundle over B is called an (X, α)-*bundle* if the typical fiber is X and the structure group is G_α.

We shall recall the Leray-Serre spectral sequence $E_2^{p,q}$ for a fibration (we described in more detail in subsection 4.3). For a fibration $F \hookrightarrow E \to B$, we have the Leray-Serre spectral sequence $E_2^{p,q} \cong H^p(B; \mathcal{H}^q(F; \mathbb{Z})) \Rightarrow H^{p+q}(E; \mathbb{Z})$, where $\mathcal{H}^q(F; \mathbb{Z})$ is the local system of coefficients on B by the cohomology group $H^q(F; \mathbb{Z})$.

Now, we consider an (X, α)-bundle $E \to B$, which is not necessarily flat. Recall from Lemma 4.14 that there exists the five-term exact sequence for the Leray-Serre spectral sequence

$$0 \to H^1(B; \mathbb{Z}) \to H^1(E; \mathbb{Z}) \to H^1(X; \mathbb{Z})^{\pi_1(B)} \xrightarrow{d_2} H^2(B; \mathbb{Z}) \to H^2(E; \mathbb{Z})$$

and that there is a natural inclusion

$$H^1(X; \mathbb{Z})^{G_\alpha} \subset H^1(X; \mathbb{Z})^{\pi_1(B)}.$$

By the definition of G_α, the cohomology class $\alpha \in H^1(X;\mathbb{Z})$ is G_α-invariant, so that

$$-\alpha \in H^1(X;\mathbb{Z})^{G_\alpha} \subset H^1(X;\mathbb{Z})^{\pi_1(B)} \cong H^0(B;\mathcal{H}^1(X;\mathbb{Z})).$$

Thus, in the five-term exact sequence for the Leray-Serre spectral sequence, we have a cohomology class $e_\alpha(E) \in H^2(B;\mathbb{Z})$ given by

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^1(B;\mathbb{Z}) & \longrightarrow & H^1(E;\mathbb{Z}) & \longrightarrow & H^1(X;\mathbb{Z})^{\pi_1(B)} & \xrightarrow{d_2} & H^2(B;\mathbb{Z}) & \longrightarrow H^2(E;\mathbb{Z}) \\ & & & & & \cup & & \cup & \\ & & & & & -\alpha & \longmapsto & e_\alpha(E). & \end{array} \tag{5.1}$$

Definition 5.20 (The Euler class for an (X,α)-bundle)**.** Let X be a connected CW-complex and α a cohomology class in $H^1(X;\mathbb{Z})$. Let $E \to B$ be an (X,α)-bundle. The *Euler class* for $E \to X$ is defined to be the cohomology class $e_\alpha(E) \in H^2(B;\mathbb{Z})$ in the above diagram (5.1).

Example 5.21. Let $u \in H^1(S^1;\mathbb{Z})$ be the orientation class of S^1. Then the (S^1,u)-bundle $E \to B$ is just an *oriented* S^1-bundle over B and the Euler class $e_u(E) \in H^2(B;\mathbb{Z})$ is the usual Euler class as an oriented S^1-bundle. We described in more detail in subsection 5.2

Remark 5.22. The Euler class for the (X,α)-bundle turned out to be the first Chern class in the following situation. Let $E \to B$ be a principal $U(n)$-bundle over a connected smooth manifold B. Let $u \in H^1(U(1);\mathbb{Z}) \cong \mathbb{Z}$ be the orientation class for $U(1) = S^1 \subset \mathbb{C}$. In the Leray-Serre spectral sequence for $E \to B$, it is known that the generator $-u \in H^1(U(1);\mathbb{Z})$ is mapped to $c_1(E) \in H^2(B;\mathbb{Z})$, the first Chern class of the principal $U(n)$-bundle:

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^1(B;\mathbb{Z}) & \longrightarrow & H^1(E;\mathbb{Z}) & \longrightarrow & H^1(U(1);\mathbb{Z}) & \xrightarrow{d_2} & H^2(B;\mathbb{Z}) & \longrightarrow H^2(E;\mathbb{Z}) \\ & & & & & \cup & & \cup & \\ & & & & & -u & \longmapsto & c_1(E). & \end{array}$$

Thus the Euler class for (X,α)-bundle can be considered as a generalization of the first Chern class also.

We recall the relation between the Leray-Serre spectral sequence for a fibration and the Lyndon–Hochschild–Serre spectral sequence for a short exact sequence for groups, described in Remark 4.15. For a short exact sequence

$1 \to N \to G \to G/N \to 1$, we have a induced fibration $BN \hookrightarrow BG \to B(G/N)$. Then the Leray-Serre spectral sequence for this fibration coincides with the LHS spectral sequence for the given short exact sequence. Especially, we have isomorphisms between the five-term exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & H^1_{\mathrm{grp}}(G/N;\mathbb{Z}) \arrow[r] \arrow[d, phantom, "\wr\|"] & H^1_{\mathrm{grp}}(G;\mathbb{Z}) \arrow[r] \arrow[d, phantom, "\wr\|"] & H^1_{\mathrm{grp}}(N;\mathbb{Z})^G \arrow[r, "d_2"] \arrow[d, phantom, "\wr\|"] & H^2_{\mathrm{grp}}(G/N;\mathbb{Z}) \arrow[r] \arrow[d, phantom, "\wr\|"] & H^2_{\mathrm{grp}}(G;\mathbb{Z}) \arrow[d, phantom, "\wr\|"] \\
0 \arrow[r] & H^1(B(G/N);\mathbb{Z}) \arrow[r] & H^1(BG;\mathbb{Z}) \arrow[r] & H^1(BN;\mathbb{Z})^G \arrow[r, "d_2"] & H^2(B(G/N);\mathbb{Z}) \arrow[r] & H^2(BG;\mathbb{Z}).
\end{tikzcd}$$

Let H be an arbitrary topological group and H^δ denote the group equipped with discrete topology. Given a central extension $0 \to \mathbb{Z} \to \widehat{H} \to H \to 1$, it is obviously an exact sequence of topological groups with discrete topology. Thus, we obtain the induced fibration $B\mathbb{Z} \hookrightarrow B\widehat{H}^\delta \to BH^\delta$.

Theorem 5.23. [3] *Let X be a connected CW-complex and α a cohomology class in $H^1(X;\mathbb{Z})$. Take and fix a regular $\mathbb{Z}$-covering space $\pi : \widehat{X} \to X$ with α the classifying class. Recall that $G_\alpha = \mathrm{Homeo}_\alpha(X)$ and $\widehat{G} = \mathrm{Homeo}(\widehat{X})^{\mathbb{Z}}$, the α-preserving homeomorphism group on X and the group of homeomorphisms on $\widehat{X}$ that commute with the $\mathbb{Z}$-action on $\widehat{X}$. We then have the central extension associated to α:*

$$0 \to \mathbb{Z} \to \widehat{G} \to G_\alpha \to 1.$$

Let $e(\widehat{G}) \in H^2_{\mathrm{grp}}(G_\alpha;\mathbb{Z})$ denote the extension class of the extension above. Denote also by E_α the universal flat (X,α)-bundle over BG_α^δ and let $e_\alpha(E_\alpha) \in H^2(BG_\alpha^\delta;\mathbb{Z})$ be the Euler class for E_α. It then follows that $e_\alpha(E_\alpha)$ corresponds to $e(\widehat{G})$ under the natural isomorphism $H^2(BG_\alpha^\delta;\mathbb{Z}) \cong H^2_{\mathrm{grp}}(G_\alpha;\mathbb{Z})$.

We shall prove Theorem 5.23 after providing the following:

Lemma 5.24. *Let $X \hookrightarrow E_\alpha \to BG_\alpha^\delta$ be the universal flat (X,α)-bundle. Then there exists a continuous map $\psi : B\widehat{G}^\delta \to E_\alpha$ which makes the following diagram commutative:*

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] & E_\alpha \arrow[r] \arrow[d, "\psi"'] & BG_\alpha^\delta \arrow[d, equal] \\
B\mathbb{Z} \arrow[r, hook] & B\widehat{G}^\delta \arrow[r] & BG_\alpha^\delta,
\end{tikzcd}$$

[3]This theorem is based on the joint work with Moriyoshi [6].

where the restriction $f = \psi|_X : X \to B\mathbb{Z} \simeq S^1$ is a classifying map of the regular $\mathbb{Z}$-covering space $\pi : \widehat{X} \to X$.

Proof. The universal flat (X, α)-bundle $E_\alpha \to BG_\alpha^\delta$ is an X-bundle associated the universal G_α^δ-bundle:

$$(X \times EG_\alpha^\delta)/G_\alpha^\delta \to BG_\alpha^\delta.$$

Now, we consider a projection $(\pi, \mathrm{id}) : \widehat{X} \times EG_\alpha^\delta \to X \times EG_\alpha^\delta$. Since $\mathbb{Z}$ acts on $X \times EG_\alpha^\delta$ trivially, we obtain an induced homeomorphism $(\widehat{X} \times EG_\alpha^\delta)/\widehat{G}^\delta \to (X \times EG_\alpha^\delta)/G_\alpha^\delta$. Hence we obtain the universal flat (X, α)-bundle

$$(\widehat{X} \times EG_\alpha^\delta)/\widehat{G}^\delta \to BG_\alpha^\delta.$$

Since $\widehat{G}^\delta$ acts on $\widehat{X}$ freely, $\widehat{G}^\delta$ acts on $\widehat{X} \times EG_\alpha^\delta$ freely, too. By assumption, G_α^δ acts on EG_α^δ properly. We have the $\widehat{G}$-action on EG_α^δ through $p : \widehat{G} \to G_\alpha$ and for each point of EG_α^δ, the stabilizer subgroup of $\widehat{G}^\delta$ which is isomorphic to $\mathbb{Z}$ acts on $\widehat{X}$ properly. Thus the diagonal action $\widehat{G} \times \widehat{X} \times EG_\alpha^\delta \to \widehat{X} \times EG_\alpha^\delta$,

$$(\hat{\phi}, \hat{x}, g) \mapsto (\hat{\phi}(\hat{x}), p(\hat{\phi})g)$$

is also proper.

Hence we have a principal $\widehat{G}^\delta$-bundle $\widehat{X} \times EG_\alpha^\delta \to (\widehat{X} \times EG_\alpha^\delta)/\widehat{G}^\delta$. Therefore, there exists a classifying map $\psi' : (\widehat{X} \times EG_\alpha^\delta)/\widehat{G}^\delta \to B\widehat{G}^\delta$ and its lift $\tilde{\phi}' : \widehat{X} \times EG_\alpha^\delta \to E\widehat{G}^\delta$:

$$\begin{tikzcd}
\widehat{X} \times EG_\alpha^\delta \arrow[r, "\tilde{\psi}'"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & E\widehat{G}^\delta \arrow[d] \\
(\widehat{X} \times EG_\alpha^\delta)/\widehat{G}^\delta \arrow[r, "\psi'"'] & B\widehat{G}^\delta.
\end{tikzcd}$$

We define a new continuous map $\psi : (\widehat{X} \times EG_\alpha^\delta)/\widehat{G}^\delta \to (E\widehat{G}^\delta \times EG_\alpha^\delta)/\widehat{G}^\delta$, $\psi([\hat{x}, g]) = [\tilde{\psi}'(\hat{x}, g), g]$:

$$\begin{tikzcd}
\widehat{X} \times EG_\alpha^\delta \arrow[r, "(\tilde{\psi}', \mathrm{id}_{EG_\alpha^\delta})"] \arrow[d] & E\widehat{G}^\delta \times EG_\alpha^\delta \arrow[r] \arrow[d] & E\widehat{G}^\delta \arrow[d] \\
(\widehat{X} \times EG_\alpha^\delta)/\widehat{G}^\delta \arrow[r, "\psi"'] & (E\widehat{G}^\delta \times EG_\alpha^\delta)/\widehat{G}^\delta \arrow[r] & B\widehat{G}^\delta.
\end{tikzcd}$$

By construction, the classifying map ψ induces the identity map on BG^δ_α:

$$
\begin{tikzcd}
(\widehat{X} \times EG^\delta_\alpha)/\widehat{G}^\delta \arrow[r] \arrow[d, "\psi"'] & BG^\delta_\alpha \arrow[d, equal] \\
(E\widehat{G} \times EG^\delta_\alpha)/\widehat{G}^\delta \arrow[r] & BG^\delta_\alpha .
\end{tikzcd}
$$

Take and fix a point $* \in EG^\delta_\alpha$. Then $(\widehat{X} \times G^\delta_\alpha *)/\widehat{G}^\delta$ is the fiber over $[*] \in BG^\delta_\alpha$ of the universal flat (X, α)-bundle. On the other hand, $(E\widehat{G}^\delta \times G^\delta_\alpha *)/\widehat{G}^\delta$ is the fiber over $[*] \in BG^\delta_\alpha$ of the fibration induced by the central extension. We obtain the following commutative diagram:

$$
\begin{tikzcd}[column sep=small]
\widehat{X} \times * \arrow[rr] \arrow[dr] \arrow[dd, "\tilde{\psi}|_{\mathrm{fiber}}"'] & & \widehat{X} \times EG^\delta_\alpha \arrow[rr] \arrow[dr] \arrow[dd, "{(\tilde{\psi}', \mathrm{id}_{EG^\delta_\alpha})}"'] & & EG^\delta_\alpha \arrow[dr] \arrow[dd, equal] & \\
& (\widehat{X} \times G^\delta_\alpha *)/\widehat{G}^\delta \arrow[rr] \arrow[dd, "\psi|_{\mathrm{fiber}}"'] & & (\widehat{X} \times EG^\delta_\alpha)/\widehat{G}^\delta \arrow[rr] \arrow[dd, "\psi"'] & & BG^\delta_\alpha \arrow[dd, equal] \\
E\widehat{G}^\delta \times * \arrow[rr] \arrow[dr] & & E\widehat{G}^\delta \times EG^\delta_\alpha \arrow[rr] \arrow[dr] & & EG^\delta_\alpha \arrow[dr] & \\
& (E\widehat{G}^\delta \times G^\delta_\alpha *)/\widehat{G}^\delta \arrow[rr] & & (E\widehat{G}^\delta \times EG^\delta_\alpha)/\widehat{G}^\delta \arrow[rr] & & BG^\delta_\alpha .
\end{tikzcd}
$$

The restricted map $f = \psi|_{\mathrm{fiber}} : X \cong \widehat{X}/\mathbb{Z} \to E\widehat{G}^\delta/\mathbb{Z}$ has a lift $\widehat{f} = \tilde{\psi}'|_{\mathrm{fiber}} : \widehat{X} \to E\widehat{G}^\delta$:

$$\hat{f}(\hat{x}) = \tilde{\psi}'(\hat{x}, *) \in E\widehat{G}^\delta.$$

Hence f is a classifying map of the regular $\mathbb{Z}$-covering space $\widehat{X} \to X$.

From the above, we obtain the natural fibration map $\psi : E_\alpha \to B\widehat{G}^\delta$ whose restriction to fiber is a classifying map of the $\mathbb{Z}$-covering space $\pi : \widehat{X} \to X$. □

Now we prove Theorem 5.23.

Proof of Theorem 5.23. [4] We denote by $X \hookrightarrow E_\alpha \to BG^\delta_\alpha$ the universal flat

[4] Theorem 5.23 was originally proved by employing homotopy commutativity of certain cohomology diagrams. Here, the author provides an improved proof, which avoids homotopy commutativity in the first proof. It makes a clearer understanding of Theorem 5.23.

(X,α)-bundle. We also denote by $B\mathbb{Z} \hookrightarrow B\widehat{G}^\delta \to BG_\alpha^\delta$ the induced fibration by the short exact sequence $0 \to \mathbb{Z} \to \widehat{G} \to G_\alpha \to 1$.

By Lemma 5.24, there exists a continuous map $\psi : E_\alpha \to B\widehat{G}^\delta$ which satisfies the commutative diagram

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "f"'] & E_\alpha \arrow[r] \arrow[d, "\psi"'] & BG_\alpha^\delta \arrow[d, equal] \\
B\mathbb{Z} \arrow[r, hook] & B\widetilde{G}^\delta \arrow[r] & BG_\alpha^\delta
\end{tikzcd}$$

and the restriction $f = \psi|_X : X \to B\mathbb{Z} \simeq S^1$ is a classifying map of the regular $\mathbb{Z}$-covering space $\pi : \widehat{X} \to X$.

The fibration map $\psi : E_\alpha \to B\widehat{G}$ induces a morphism between the two Leray-Serre spectral sequences; $\psi^* : H^p(BG_\alpha^\delta; \mathcal{H}^q(B\mathbb{Z};\mathbb{Z})) \to H^p(E_\alpha; \mathcal{H}^q(X;\mathbb{Z}))$. Especially, we obtain the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^1(BG_\alpha^\delta;\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^1(B\widehat{G}^\delta;\mathbb{Z}) \arrow[r] \arrow[d, "\psi^*"'] & H^1(B\mathbb{Z};\mathbb{Z})^{G_\alpha^\delta} \arrow[r, "d_2"] \arrow[d, "f^*"'] & H^2(BG_\alpha^\delta;\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^2(B\widehat{G}^\delta;\mathbb{Z}) \arrow[d, "\psi^*"] \\
0 \arrow[r] & H^1(BG_\alpha^\delta;\mathbb{Z}) \arrow[r] & H^1(E_\alpha;\mathbb{Z}) \arrow[r] & H^1(X;\mathbb{Z})^{G_\alpha^\delta} \arrow[r, "d_2"] & H^2(BG_\alpha^\delta;\mathbb{Z}) \arrow[r] & H^2(E_\alpha;\mathbb{Z}).
\end{tikzcd}$$

By Remark 4.15, the Leray-Serre spectral sequence $H^p(BG_\alpha^\delta; \mathcal{H}^q(B\mathbb{Z};\mathbb{Z}))$ is naturally isomorphic to the LHS spectral sequence $H^p_{\mathrm{grp}}(G_\alpha; H^q_{\mathrm{grp}}(\mathbb{Z};\mathbb{Z}))$. We can regard the upper five-term exact sequence for $B\mathbb{Z} \hookrightarrow B\widehat{G}^\delta \to BG_\alpha^\delta$ as the five-term exact sequence for group cohomology. The generator $u \in H^1(B\mathbb{Z};\mathbb{Z})$ corresponds to the identity id $\in \mathrm{Hom}(\mathbb{Z},\mathbb{Z})$ under the isomorphism $H^1(B\mathbb{Z};\mathbb{Z}) \cong H^1_{\mathrm{grp}}(\mathbb{Z};\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z},\mathbb{Z})$. Thus the transgression image of id by d_2 of the LHS spectral sequence is the extension class $-e(\widehat{G}) \in H^2_{\mathrm{grp}}(G_\alpha;\mathbb{Z})$ by Lemma 4.11. On the other hand, we recall that $\alpha = f^*u \in H^1(X;\mathbb{Z})$ and $d_2\alpha = -e_\alpha(E_\alpha) \in H^2(BG_\alpha^\delta;\mathbb{Z})$. By the above commutative diagram, we have proven $e_\alpha(E_\alpha) = e(\widehat{G})$. $\square$

5.4 Construction of representative group 2-cocyles

In this subsection, we give two representative 2-cocycles corresponding to the central extension for a regular $\mathbb{Z}$-covering space.

Proposition 5.25. *Let X be a connected CW-complex and let $\pi : \widehat{X} \to X$ be a regular $\mathbb{Z}$-covering space. Choose and fix a classifying map $f : X \to S^1$ and its lift $\hat{f} : \widehat{X} \to \mathbb{R}$ of the given regular $\mathbb{Z}$-covering space $\pi : \widehat{X} \to X$. We have $\alpha \in H^1(X;\mathbb{Z})$ the classifying class of $\pi : \widehat{X} \to X$. We have the central extension*

$$0 \to \mathbb{Z} \to \widehat{G} \to G_\alpha \to 1$$

and denote its extension class by $e(\widehat{G}) \in H^2_{\mathrm{grp}}(G_\alpha;\mathbb{Z})$. Now, take and fix a base point $\hat{x}_0 \in \widehat{X}$ such that $\hat{f}(\hat{x}_0) = 0$ and put $x_0 = p(\hat{x}_0) \in X$. Define a group 1-cochain $\tau_0 : \widehat{G} \to \mathbb{Z}$ as

$$\tau_0(\hat{\phi}) = \lfloor \hat{f} \circ \hat{\phi}(\hat{x}_0) \rfloor,$$

where $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ is the floor function. Then the followings follow:

1. *The group 1-cochain $\tau_0 \in C^1(\widehat{G};\mathbb{Z})$ is a connection cochain over the identity on $\mathbb{Z}$.*

2. *The negative coboundary $-\delta\tau_0 \in C^2(\widehat{G};\mathbb{Z})$ represents the extension class of the central extension associated to α.*

Proof. Remark that both of $\hat{f} : \widehat{X} \to \mathbb{R}$ and $\hat{\phi} \in \widehat{G}$ commute with the $\mathbb{Z}$-actions:

$$\begin{tikzcd} \widehat{X} \arrow[r, "\hat{f}"] \arrow[d, "n+"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{R} \arrow[d, "n+"] \\ \widehat{X} \arrow[r, "\hat{f}"'] & \mathbb{R}, \end{tikzcd} \quad \begin{tikzcd} \widehat{X} \arrow[r, "\hat{\phi}"] \arrow[d, "n+"'] \arrow[dr, phantom, "\circlearrowright"] & \widehat{X} \arrow[d, "n+"] \\ \widehat{X} \arrow[r, "\hat{\phi}"'] & \widehat{X}. \end{tikzcd}$$

For each $n \in \mathbb{Z}$, we have

$$\tau_0(n) = \lfloor \hat{f}(n + \hat{x}_0) \rfloor = \lfloor n + \hat{f}(\hat{x}_0) \rfloor = \lfloor n \rfloor = n.$$

We also have

$$\tau_0(n + \hat{\phi}) = \lfloor \hat{f}(n + \hat{\phi}(\hat{x}_0)) \rfloor = n + \lfloor \hat{f}(\hat{\phi}(\hat{x}_0)) \rfloor = n + \tau_0(\hat{\phi}).$$

Thus τ_0 is a connection cochain over the identity on $\mathbb{Z}$.

According to Proposition 3.16, we have the transgression formula $[-\delta\tau_0] = e(\widehat{G}) \in H^2_{\mathrm{grp}}(G_\alpha;\mathbb{Z})$. □

Remark 5.26. In the case of $M = S^1$, the connection cochain $\tau_0 \in C^1(\mathrm{Homeo}(\mathbb{R})^{\mathbb{Z}}; \mathbb{Z})$ is already known; See Moriyoshi [15, Example 1].

We have constructed a group 2-cocycle $-\delta\tau_0$ representing the central extension associated to α. Next, we construct another group 2-cocycle representing the smooth central extension associated to α by using the de Rham complex.

Recall the construction of a representative cocycle of the central extension associated to the universal covering space $\mathbb{R} \to S^1$ in Example 3.19. Put $\omega = \frac{d\theta}{2\pi} \in \Omega^1(S^1)$, where θ is an angular coordinate on S^1. It is well-known that the de Rham cohomology class in $H^1(S^1; \mathbb{R})$ containing ω is the generator of the cohomology ring $H^*(S^1; \mathbb{R})$ and that the induced homomorphism $i_* : H^1(S^1; \mathbb{Z}) \to H^1(S^1; \mathbb{R})$ sends the generator $u \in H^1(S^1; \mathbb{Z})$ to the cohomology class of ω. In Example 3.19, we construct a group 2-cocycle $-\delta\sigma \in C^2(\mathrm{Diff}_+(S^1); \mathbb{R})$ which represents the cohomology class $i_* e(\mathrm{Diff}(\mathbb{R})^{\mathbb{Z}}) \in H^2_{\mathrm{grp}}(\mathrm{Diff}_+(S^1); \mathbb{R})$, where $e(\mathrm{Diff}(\mathbb{R})^{\mathbb{Z}})$ is the extension class corresponding to the central extension

$$0 \to \mathbb{Z} \to \mathrm{Diff}(\mathbb{R})^{\mathbb{Z}} \to \mathrm{Diff}_+(S^1) \to 1.$$

Here, $\sigma \in C^1(\mathrm{Diff}_+(S^1); \Omega^0(S^1))$ is a group 1-cochain satisfying $d\sigma = \delta\omega$. Put $C^{p,q} = C^p(\mathrm{Diff}_+(S^1); \Omega^q(S^1))$.

$$\begin{tikzcd}
\Omega^1(S^1) \arrow[r, "\delta"] & C^{1,1} & \\
& C^{1,0} \arrow[u, "d"] \arrow[r, "\delta"] & C^{2,0}
\end{tikzcd}
\qquad
\begin{tikzcd}
\omega \arrow[r, mapsto] & \delta\omega = d\sigma & \\
& \sigma \arrow[u, mapsto] \arrow[r, mapsto] & \delta\sigma.
\end{tikzcd}$$

Let M be a connected smooth manifold and let $\pi : \widehat{M} \to M$ be a regular $\mathbb{Z}$-covering space. We also denote by $\alpha \in H^1(M; \mathbb{Z})$ the classifying class of $\pi : \widehat{M} \to M$. Then we have the smooth central extension associated to α:

$$0 \to \mathbb{Z} \to \widehat{G}^\infty \to G_\alpha^\infty \to 1, \tag{5.2}$$

where $\widehat{G}^\infty = \mathrm{Diff}(\widehat{M})^{\mathbb{Z}}$ and $G_\alpha^\infty = \mathrm{Diff}_\alpha(M)$. Set $i : \mathbb{Z} \to \mathbb{R}$ the inclusion.

Take a 1-form $\omega \in \Omega^1(M)$ such that

$$[\omega] = i_*\alpha \in H^1(M; \mathbb{R}). \tag{5.3}$$

Since $\phi \in G_\alpha^\infty = \mathrm{Diff}_\alpha(M)$ preserves $\alpha \in H^1(M;\mathbb{Z})$ and the following diagram is commutative,

$$\begin{tikzcd}
H^1(S^1;\mathbb{Z}) \arrow[r, "f^*"] \arrow[d] & H^1(M;\mathbb{Z}) \arrow[r, "\phi^*"] \arrow[d] & H^1(M;\mathbb{Z}) \arrow[d] \\
H^1(S^1;\mathbb{R}) \arrow[r, "f^*"] & H^1(M;\mathbb{R}) \arrow[r, "\phi^*"] & H^1(M;\mathbb{R}),
\end{tikzcd}$$

the pullback $\phi^*\omega$ is cohomologous to ω on M. Thus there exists a $\Omega^0(M)$-valued group 1-cochain $\sigma \in C^1(G_\alpha^\infty;\Omega^0(M))$ such that

$$d(\sigma(\phi)) = \omega - \phi^*\omega = \delta\omega(\phi) \tag{5.4}$$

for each $\phi \in G_\alpha^\infty$.

Proposition 5.27. *Let $\widehat{M} \to M$ be a regular $\mathbb{Z}$-covering over a connected smooth manifold M. Suppose that $\alpha \in H^1(M;\mathbb{Z})$ is the classifying class of the regular $\mathbb{Z}$-covering space $\widehat{M} \to M$. Set $i : \mathbb{Z} \hookrightarrow \mathbb{R}$ the natural inclusion. Let a 1-form $\omega \in \Omega^1(M)$ which satisfies* (5.3). *Take a group 1-cochain $\sigma \in C^1(G_\alpha^\infty;\Omega^0(M))$ defined by* (5.4).

Then the followings follow:

1. *The coboundary $\delta\sigma \in C^2(G_\alpha^\infty;\Omega^0(M))$ takes values in the constant functions on M, that is*

$$\delta\sigma \in \mathrm{Im}(C^2(G_\alpha^\infty;\mathbb{R}) \to C^2(G_\alpha^\infty;\Omega^0(M))).$$

2. *The cohomology class $[\delta\sigma] \in H^2_{\mathrm{grp}}(G_\alpha^\infty;\mathbb{R})$ is independent of the choice of $\omega \in \Omega^1(M)$ and $\sigma \in C^1(G_\alpha^\infty;\Omega^0(M))$which satisfy* (5.3) *and* (5.4).

3. *The group 2-cocycle $\delta\sigma \in C^2(G_\alpha^\infty;\mathbb{R})$ represents the cohomology class $i_*e(\widehat{G}^\infty) \in H^2_{\mathrm{grp}}(G_\alpha^\infty;\mathbb{R})$, where $e(\widehat{G}^\infty)$ is the extension class of the central extension* (5.2).

Proof. 1. Since the differential d commutes with the coboundary operator δ, we obtain

$$d(\delta\sigma) = \delta(d\sigma) = \delta(\delta\omega) = 0.$$

Thus for each $\phi \in G_\alpha^\infty$, $\delta\sigma(\phi)$ is a constant function on M.

2. Fix a 1-form $\omega \in \Omega^1(M)$ and a group 1-cochain $\sigma \in C^1(G_\alpha^\infty; \Omega^1(M))$ which satisfy (5.3) and (5.4). We first suppose that we have another group 1-cochain $\sigma' \in C^1(G_\alpha^\infty; \Omega^0(M))$ such that $d\sigma' = \delta\omega$. We put $c = \sigma - \sigma' \in C^1(G_\alpha^\infty; \Omega^0(M))$. Since
$$dc = d\sigma - d\sigma' = \omega - \omega' = 0,$$
for each $\phi \in G_\alpha^\infty$, $c(\phi) \in C^\infty(M; \mathbb{R}) = \Omega^0(M)$ is constant. This implies that $\delta\sigma' \in C^2(G_\alpha^\infty; \mathbb{R})$ is cohomologous to $\delta\sigma \in C^2(G_\alpha^\infty; \mathbb{R})$.

   Next, Suppose that we have another 1-form $\omega' \in \Omega^1(M)$ representing $i_*\alpha \in H^1(M; \mathbb{R})$. Since ω' is cohomologous to ω, there exists a smooth function $F \in C^\infty(M; \mathbb{R}) = \Omega^0(M)$ such that $\omega - \omega' = dF$. Thus we can take $\sigma + F$ as the group 1-cochain $\sigma' \in C^1(G_\alpha^\infty; \Omega^0(M))$ such that $d\sigma' = \delta\omega'$. Then the group 2-cocycle $\delta\sigma' = \delta\sigma + \delta F$ in $C^2(G_\alpha^\infty; \mathbb{R})$ is cohomologous to the group 2-cocycle $\delta\sigma \in C^2(G_\alpha^\infty; \mathbb{R})$.

3. Choose and fix a classifying map $f : M \to S^1$ with a lift $\hat{f} : \widehat{M} \to \mathbb{R}$ of the given regular $\mathbb{Z}$-covering space $\pi : \widehat{M} \to M$. Let $\tilde{\pi} : \mathbb{R} \to S^1$ be the universal covering space over S^1 and let θ be the angular coordinate on S^1. Since the cohomology class $[\delta\sigma] \in H^2_{\mathrm{grp}}(G_\alpha^\infty; \mathbb{R})$ is independent of the choice of $\omega \in \Omega^1(M)$ and $\sigma \in C^1(G_\alpha^\infty; \Omega^0(M))$ satisfying (5.3) and (5.4), we may take the pullback $f^* \frac{d\theta}{2\pi}$ as $\omega \in \Omega^1(M)$.

   Let $s : G_\alpha^\infty \to \widehat{G}^\infty$ be a set-theorical section. For each $\phi \in G_\alpha^\infty$, we consider $f_s(\phi) = \hat{f} - \hat{f} \circ s(\phi) \in C^\infty(\widehat{M}; \mathbb{R})$. Since both of $\hat{f}$ and $s(\phi) \in \widehat{G}^\infty$ commute with the $\mathbb{Z}$-actions, we obtain
$$\begin{aligned} f_s(\phi)(n + \hat{x}) =& \hat{f}(n + \hat{x}) - \hat{f} \circ s(\phi)(n + \hat{x}) \\ =& \{n + \hat{f}(\hat{x})\} - \{n + \hat{f} \circ s(\phi)(\hat{x})\} = f_s(\phi)(\hat{x}). \end{aligned}$$
Thus we can consider $f_s(\phi)$ as a smooth function $M \to \mathbb{R}$. We define a group 1-cochain $\sigma_s \in C^1(G_\alpha^\infty; \Omega^0(M))$ as
$$\sigma_s(\phi) = f_s(\phi) = \hat{f} - \hat{f} \circ s(\phi) \in \Omega^0(M). \tag{5.5}$$
Then we obtain
$$\begin{aligned} d\sigma_s(\phi) =& d\hat{f} - d\hat{f} \circ s(\phi) \\ =& \hat{f}^* \tilde{\pi}^* \frac{d\theta}{2\pi} - s(\phi)^* \hat{f}^* \tilde{\pi}^* \frac{d\theta}{2\pi} \\ =& f^* \frac{d\theta}{2\pi} - \phi^* f^* \frac{d\theta}{2\pi} = \delta\omega(\phi). \end{aligned}$$

Then we calculate $\delta\sigma_s$:

$$\begin{aligned}\delta\sigma_s(\phi_1,\phi_2) =&\sigma_s(\phi_2) - \sigma_s(\phi_1\phi_2) + \phi_2^*\sigma_s(\phi_1)\\ =&(\hat{f} - \hat{f}\circ s(\phi_2)) - (\hat{f} - \hat{f}\circ s(\phi_1\phi_2)) + (\hat{f}\circ s(\phi_2) - \hat{f}\circ s(\phi_1)\circ s(\phi_2))\\ =&(\hat{f}\circ s(\phi_1\phi_2) - \hat{f}\circ s(\phi_1)\circ s(\phi_2))\\ =&\hat{f}\circ(\phi_1\phi_2) - \hat{f}\circ(s(\phi_1)s(\phi_2)s(\phi_1\phi_2)^{-1})\circ(\phi_1\phi_2).\end{aligned}$$

for $\phi_1,\phi_2 \in G_\alpha^\infty$. Recall that the extension class $i_*e(\widehat{G}^\infty) \in H^2_{\mathrm{grp}}(G_\alpha^\infty;\mathbb{R})$ has a representative group 2-cocycle $\chi \in C^2(G_\alpha^\infty;\mathbb{R})$ defined as

$$\chi(\phi_1,\phi_2) = s(\phi_1)s(\phi_2)s(\phi_1\phi_2)^{-1} \in \mathbb{Z} \subset \mathbb{R}.$$

Since $\hat{f}$ commutes with the $\mathbb{Z}$-action on $\widehat{M}$, we obtain that

$$\begin{aligned}\delta\sigma_s(\phi_1,\phi_2) =&\hat{f}\circ(\phi_1\phi_2) - \hat{f}\circ(s(\phi_1)s(\phi_2)s(\phi_1\phi_2)^{-1})\circ(\phi_1\phi_2)\\ =&\hat{f}\circ(\phi_1\phi_2) - \{s(\phi_1)s(\phi_2)s(\phi_1\phi_2)^{-1} + \hat{f}\circ(\phi_1\phi_2)\}\\ =&-\chi(\phi_1,\phi_2)\end{aligned}$$

Therefore we have $\chi = -\delta\sigma_s \in C^2(G_\alpha^\infty;\mathbb{R})$, so that $[-\delta\sigma_s] = i_*e(\widehat{G}^\infty) \in H^2_{\mathrm{grp}}(G_\alpha^\infty;\mathbb{R})$.

□

Corollary 5.28. *Let $\widehat{M} \to M$ be a regular $\mathbb{Z}$-covering over a connected smooth manifold M. Suppose that $\alpha \in H^1(M;\mathbb{Z})$ is the classifying class of the regular $\mathbb{Z}$-covering space $\widehat{M} \to M$. We have the smooth central extension associated to α:*

$$0 \to \mathbb{Z} \to \widehat{G}^\infty \to G_\alpha^\infty \to 1$$

with its extension class $e(\widehat{G}^\infty) \in H^2_{\mathrm{grp}}(G_\alpha^\infty;\mathbb{Z})$. Set $i : \mathbb{Z} \to \mathbb{R}$ the inclusion. Choose and fix a classifying map $f : M \to S^1$ with a lift $\hat{f} : \widehat{M} \to \mathbb{R}$ of the given regular $\mathbb{Z}$-covering space $\pi : \widehat{M} \to M$. Take a fixed point $\hat{x}_0 \in \widehat{M}$ satisfying $\hat{f}(\hat{x}_0) = 0$. Let $s : G_\alpha^\infty \to \widehat{G}^\infty$ be a set-theorical section. We define a group 1-cochain $\sigma_\infty \in C^1(G_\alpha^\infty;\Omega^0(M))$ as

$$\sigma_\infty(\phi)(x) = \hat{f}(\hat{x}) - \hat{f}\circ s(\phi)(\hat{x}) + \hat{f}\circ s(\phi)(\hat{x}_0),$$

where $\pi(\hat{x}) = x$.

Then the followings follows:

1. *The group 1-cochain $\sigma_\infty \in C^1(G_\alpha^\infty; \Omega^0(M))$ is well-defined, that is*

$$\sigma_\infty(\phi)(x) = \hat{f}(\hat{x}) - \hat{f} \circ s(\phi)(\hat{x}) + \hat{f} \circ s(\phi)(\hat{x}_0),$$

is independent of the choice of $\hat{x} \in \widehat{X}$ such that $\pi(\hat{x}) = x$.

2. *The group 1-cochain σ_∞ is independent of the choice of section $G_\alpha^\infty \to \widehat{G}^\infty$.*

3. *The negative coboundary $-\delta\sigma_\infty$ represents the extension class $i_*e(\widehat{G}^\infty) \in H^2_{\mathrm{grp}}(G_\alpha^\infty; \mathbb{R})$.*

Proof. 1. For a section $s : G_\alpha^\infty \to \widehat{G}^\infty$, we recall $\sigma_s \in C^1(G_\alpha^\infty; \Omega^0(M))$ defined as

$$\sigma_s(x) = \hat{f}(\hat{x}) - \hat{f} \circ s(\phi)(\hat{x}).$$

at equation (5.5) in the proof of Proposition 5.27. The group 1-cochain σ_s is well-defined and for each $\phi \in G_\alpha^\infty$, $\hat{f} \circ s(\phi)(\hat{x}_0) \in \Omega^0(M)$ is constant. Thus $\sigma_\infty \in C^1(G_\alpha^\infty; \Omega^0(M))$ defined as

$$\sigma_\infty(\phi) = \sigma_s(\phi) + \hat{f} \circ s(\phi)(\hat{x}_0)$$

is well-defined.

2. Take different sections $s, s' : G_\alpha^\infty \to \widehat{G}^\infty$ of the surjection $p : \widehat{G}^\infty \to G_\alpha^\infty$. For each $\phi \in G_\alpha^\infty$, there is an integer $n \in \mathbb{Z}$ such that $s'(\phi) = n + s(\phi) \in \widehat{G}^\infty$. Then we obtain

$$\begin{aligned}
&\hat{f}(\hat{x}) - \hat{f} \circ s'(\phi)(\hat{x}) + \hat{f} \circ s'(\phi)(\hat{x}_0) \\
&= \hat{f}(\hat{x}) - \hat{f} \circ (n + s(\phi))(\hat{x}) + \hat{f} \circ (n + s(\phi))(\hat{x}_0) \\
&= \hat{f}(\hat{x}) - (n + \hat{f} \circ s(\phi)(\hat{x})) + (n + \hat{f} \circ s(\phi)(\hat{x}_0)) \\
&= \hat{f}(\hat{x}) - \hat{f} \circ s(\phi)(\hat{x}) + \hat{f} \circ s(\phi)(\hat{x}_0)
\end{aligned}$$

since $\hat{f} : \widehat{M} \to \mathbb{R}$ commutes with the $\mathbb{Z}$-actions. Thus the group 1-cochain σ_∞ is independent of the choice of section $G_\alpha^\infty \to \widehat{G}^\infty$.

3. For a section $s : G_\alpha^\infty \to \widehat{G}^\infty$, we recall $\sigma_s \in C^1(G_\alpha^\infty; \Omega^0(M))$. We have

$$\sigma_\infty(\phi) = \sigma_s(\phi) + \hat{f} \circ s(\phi)(\hat{x}_0) \in \Omega^0(M)$$

and $\hat{f} \circ s(\phi)(\hat{x}_0)$ is constant on M. Applying Proposition 5.27, the negative coboundary $-\delta\sigma_\infty$ represents the extension class $i_* e(\widehat{G}^\infty) \in H^2_{\mathrm{grp}}(G^\infty_\alpha; \mathbb{R})$.

$\square$

Remark 5.29. We have the central extension associated to the universal covering space $\mathbb{R} \to S^1$:

$$0 \to \mathbb{Z} \to \mathrm{Homeo}(\mathbb{R})^{\mathbb{Z}} \xrightarrow{p} \mathrm{Homeo}_+(S^1) \to 1.$$

It is known that $-\delta\sigma \in C^2(\mathrm{Homeo}_+(S^1); \mathbb{R})$ represents the extension class $i_* e(\mathrm{Homeo}(\mathbb{R})^{\mathbb{Z}}) \in H^2_{\mathrm{grp}}(\mathrm{Homeo}_+(S^1); \mathbb{R})$, where $\sigma \in C^1(\mathrm{Homeo}_+(S^1); C(X; \mathbb{R}))$ is defined as

$$\sigma(\phi)(x) = x - \hat{\phi}(x) + \hat{\phi}(0)$$

Here $x \in \mathbb{R}/2\pi\mathbb{Z} \cong S^1$ and $p(\hat{\phi}) = \phi$ (see Moriyoshi [15] for instance). Corollary 5.28 is an analogy to this construction.

5.5 Central extension of diffeomorphism group and the flux homomorphism

In this subsection, we will show a relation between the smooth central extension associated to a smooth $\mathbb{Z}$-covering space and the flux homomorphism on the relative diffeomorphism group on a compact manifold with boundary.

Let M be a closed, oriented, smooth manifold. We will consider the identity component of the orientation-preserving diffeomorphism group on M, denote $\mathrm{Diff}_+(M)_0$. Since every diffeomorphism in $\mathrm{Diff}_+(M)_0$ is isotopic to the identity, $G^\infty_\alpha = \mathrm{Diff}_\alpha(M)$ includes $\mathrm{Diff}_+(M)_0$ for any $\alpha \in H^1(M; \mathbb{Z})$. Then this inclusion induces a central extension, called *the restricted* smooth central extension associated to α:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathrm{Diff}(\widehat{M})^{\mathbb{Z}} \arrow[r] & \mathrm{Diff}_\alpha(M) \arrow[r] & 1 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[u, equal] & p^{-1}(\mathrm{Diff}_+(M)_0) \arrow[r] \arrow[u, hook] & \mathrm{Diff}_+(M)_0 \arrow[r] \arrow[u, hook] & 1.
\end{tikzcd} \tag{5.6}$$

Put $G_\partial = \mathrm{Diff}_+(M)_0$ and $\widehat{G}_\partial = p^{-1}(\mathrm{Diff}_+(M)_0)$.

Let W be an $(n+1)$-dimensional compact, oriented, connected, smooth manifold with connected smooth boundary ∂W and $\Omega \in \Omega^{n+1}(W)$ be a normalized volume form on W, that is $\int_W \Omega = 1$. Since W is compact oriented with boundary, there is an n-form $\eta \in \Omega^n(W)$ such that $-d\eta = \Omega$.

Denote by $\mathrm{Diff}_\Omega(W)$ the group of the volume-preserving diffeomorphisms on W and by $\mathrm{Diff}_\Omega(W, \partial W)$ the subgroup of $\mathrm{Diff}_\Omega(W)$ consisting of ones which coincide with the identity on the boundary ∂W. Put $G = \mathrm{Diff}_\Omega(W)_0$ the identity component of the group $\mathrm{Diff}_\Omega(W)$.

Lemma 5.30 (Krygin [10, Theorem 1])**.** *Let W be a compact oriented manifold with boundaty ∂W. The restriction map $\mathrm{Diff}_\Omega(W)_0 \to \mathrm{Diff}_+(\partial W)_0$ is a surjective homomorphism.* □

By this lemma, we have a short exact sequence

$$1 \to G_{\mathrm{rel}} \to \mathrm{Diff}_\Omega(W)_0 \to \mathrm{Diff}_+(\partial W)_0 \to 1, \tag{5.7}$$

where $G_{\mathrm{rel}} \subset \mathrm{Diff}_\Omega(W, \partial W)$ is the kernel of the surjective $\mathrm{Diff}_\Omega(W)_0 \to \mathrm{Diff}_+(M)_0$.

We set the subcomplex

$$\Omega^p(W, \partial W) = \mathrm{Ker}(\Omega^p(W) \to \Omega^p(\partial W)).$$

It is known that the cohomology $H^p(W, \partial W; \mathbb{R}) = H^p(\Omega^p(W, \partial W), d)$ is the *relative cohomology* of the pair $(W, \partial W)$.

Let $\mathrm{pr} : W \times [0,1] \to W$ be the first projection. Every p-form on $W \times [0,1]$ is a linear combination of the following two types of forms:

$$\begin{cases} \mathrm{pr}^*\mu & \mu \in \Omega^p(W), \\ \mathrm{pr}^*\nu \wedge dt & \nu \in \Omega^{p-1}(W). \end{cases}$$

Here, t is the coordinate system on $[0,1]$. Define a linear map $K : \Omega^p(W \times [0,1]) \to \Omega^{p-1}(W)$ as follows:

$$\begin{cases} K(\mathrm{pr}^*\mu) = 0 & \mu \in \Omega^p(W), \\ K(\mathrm{pr}^*\nu \wedge dt) = \nu & \nu \in \Omega^{p-1}(W). \end{cases}$$

Lemma 5.31 (see e.g. Bott-Tu [1, I, §4])**.** *Let $\lambda \in \Omega^p(W)$ be a p-form on W and let $h : W \times [0,1] \to W$ be a smooth map.*

Then we obtain the identity

$$h_1^*\lambda - h_0^*\lambda = dK(h^*\lambda) - K(d(h^*\lambda)).$$

Here, h_0 and h_1 are smooth maps from W to W defined by

$$h_0(\bar{x}) = h(\bar{x}, 0), \quad h_1(\bar{x}) = h(\bar{x}, 1)$$

for $\bar{x} \in W$.

By the definition, for each $\bar{\phi} \in \mathrm{Diff}_\Omega(W, \partial W)$, $\delta\eta(\bar{\phi}) = \eta - \bar{\phi}^*\eta$ is closed and vanishes on the boundary ∂W. Since any element in G_{rel} is isotopic to the identity, G_{rel} acts on $H^n(W, \partial W; \mathbb{R})$ trivially. The following is well-known:

Definition 5.32 (see e.g. McDuff–Salamon [13, III, 10.2])**.** The *flux homomorphism* $\mathrm{Flux} : G_{\mathrm{rel}} \to H^n(W, \partial W; \mathbb{R})$ is defined by

$$\mathrm{Flux}(\bar{\phi}) = -[\delta\eta(\bar{\phi})] = [\bar{\phi}^*\eta - \eta] \in H^n(W, \partial W; \mathbb{R}).$$

Remark that the flux homomorphism $\mathrm{Flux} : G_{\mathrm{rel}} \to H^n(W, \partial W; \mathbb{R})$ is independent of the choice of n-form η such that $\Omega = -d\eta \in \Omega^{n+1}(W)$. We extend the flux homomorphism on G_{rel} to a map $F : \mathrm{Diff}_\Omega(W)_0 \to \Omega^n(W)$ by $F(\bar{\phi}) = -\delta\eta(\bar{\phi}) = \bar{\phi}^*\eta - \eta$: $\mathrm{Flux}(\bar{\psi}) = [F(\bar{\psi})] \in H^n(W, \partial W; \mathbb{R})$ for $\bar{\psi} \in G_{\mathrm{rel}}$. This extended map factors through closed n-forms on W. However, this map is not a homomorphism and the restriction of an n-form $F(\bar{\phi}) \in \Omega^n(W)$ to ∂W does not vanish. Remark that this map F depends on the choice of n-form η such that $\Omega = -d\eta \in \Omega^{n+1}(W)$.

Lemma 5.33. *1. For each $\bar{\phi} \in G$ and $\bar{\psi} \in G_{\mathrm{rel}}$, we have the identity:*

$$F(\bar{\phi}\bar{\psi}\bar{\phi}^{-1}) = (\bar{\phi}^{-1})^* F(\bar{\psi}) + (\bar{\phi}^{-1})^* \{\bar{\psi}^* F(\bar{\phi}) - F(\bar{\phi})\}.$$

2. The flux homomorphism $\mathrm{Flux} : G_{\mathrm{rel}} \to H^n(W, \partial W; \mathbb{R})$ is G-invariant.

Proof. 1. Recall a map $F : G \to \Omega^n(W)$ defined as $F(\bar{\phi}) = \bar{\phi}^*\eta - \eta = -\delta\eta(\bar{\phi})$. By the definition, $0 = \delta F \in C^1(G; \Omega^n(W))$ and $0 = F(\mathrm{id}_W) \in \Omega^n(W)$. Especially, we have

$$\begin{aligned}
0 =& \delta F(\bar{\phi}\bar{\psi}, \bar{\phi}^{-1}) = F(\bar{\phi}^{-1}) - F(\bar{\phi}\bar{\psi}\bar{\phi}^{-1}) + (\bar{\phi}^{-1})^* F(\bar{\phi}\bar{\psi}),\\
0 =& \delta F(\bar{\phi}, \bar{\psi}) = F(\bar{\psi}) - F(\bar{\phi}\bar{\psi}) + \bar{\psi}^* F(\bar{\phi}),\\
0 =& \delta F(\bar{\phi}, \bar{\phi}^{-1}) = F(\bar{\phi}^{-1}) - F(\mathrm{id}_W) + (\bar{\phi}^{-1})^* F(\bar{\phi}) = F(\bar{\phi}^{-1}) + (\bar{\phi}^{-1})^* F(\bar{\phi})
\end{aligned}$$

for $\bar{\psi} \in G_{\mathrm{rel}}$ and $\bar{\phi} \in G$. Thus we obtain

$$F(\bar{\phi}\bar{\psi}\bar{\phi}^{-1}) = (\bar{\phi}^{-1})^* F(\bar{\psi}) + (\bar{\phi}^{-1})^* \{\bar{\psi}^* F(\bar{\phi}) - F(\bar{\phi})\}.$$

2. Since the diffeomorphism $\bar{\psi}$ is isotopic to the identity on W in $\mathrm{Diff}_\Omega(W, \partial W)$, there exists a smooth homotopy map $\bar{\Psi} : W \times [0,1] \to W$ which satisfies the following:

$$\begin{cases} \bar{\Psi}(\bar{x}, 0) = \bar{x} & \text{for } \bar{x} \in W. \\ \bar{\Psi}(\bar{x}, 1) = \bar{\psi}(\bar{x}) & \text{for } \bar{x} \in W. \\ \bar{\Psi}(\bar{x}, t) = \bar{x} & \text{for } \bar{x} \in \partial W \text{ and } t \in [0,1]. \end{cases}$$

Applying Lemma 5.31, we obtain

$$\bar{\psi}^* F(\bar{\phi}) - F(\bar{\phi}) = dK(\bar{\Psi}^* F(\bar{\phi})) - K(d(\bar{\Psi}^* F(\bar{\phi}))).$$

Since $F(\bar{\phi}) \in \Omega^n(W)$ is closed, we have

$$\bar{\psi}^* F(\bar{\phi}) - F(\bar{\phi}) = dK(\bar{\Psi}^* F(\bar{\phi})).$$

Since $\bar{\Psi}|_{\partial W \times [0,1]} = \mathrm{pr}$, we have

$$K(\bar{\Psi}^* F(\bar{\phi}))|_{\partial W} = 0,$$

so that $K(\bar{\Psi}^* F(\bar{\phi})) \in \Omega^{n-1}(W, \partial W)$. Thus $\bar{\psi}^* F(\bar{\phi}) - F(\bar{\phi})$ is an exact n-form in $\Omega^n(W, \partial W)$.

Since the diffeomorphism $\bar{\phi}$ is homotopic to the identity, it preserves closed forms on W up to exact forms. Thus $(\bar{\phi}^{-1})^* F(\bar{\psi}) - F(\bar{\psi})$ is exact in $\Omega^n(W, \partial W)$.

These implies that $F(\bar{\phi}\bar{\psi}\bar{\phi}^{-1})$ is cohomologous to $F(\bar{\psi})$ in $\Omega^n(W, \partial W)$.

□

Let $\widehat{W} \to W$ be a regular $\mathbb{Z}$-covering space over a compact, oriented, connected, smooth manifold W with the boundary $\partial W = M$. Then we have a regular $\mathbb{Z}$-covering space $\partial \widehat{W} = \widehat{M} \to M$ as the boundary. There are classifying classes $\bar{\alpha} \in H^1(W; \mathbb{Z})$ and $\alpha \in H^1(M; \mathbb{Z})$ of $\widehat{W} \to W$ and $\widehat{M} \to M$, respectively. They satisfy that $\bar{\alpha}|_{\partial W} = \alpha$. Take a closed 1-form $\bar{\omega} \in \Omega^1(W)$ which represents $i_* \bar{\alpha} \in H^1(W; \mathbb{R})$ with the natural inclusion $i : \mathbb{Z} \hookrightarrow \mathbb{R}$. Remark that $G = \mathrm{Diff}_\Omega(W)_0$ preserves $\bar{\omega}$ up to exact forms since every diffeomorphisms in G are the identity homotopic. We regard $\bar{\omega} \in \Omega^1(W)$ as a cochain in the group-de Rham double-complex $C^0(G; \Omega^1(W))$

and take the group coboundary $\delta\bar{\omega} \in C^1(G;\Omega^1(W))$. For $\bar{\phi}_1, \bar{\phi}_2 \in G$, define a group 2-cochain $L \in C^2(G;\mathbb{R})$ as

$$L(\bar{\phi}_1, \bar{\phi}_2) := \int_W F \cup \delta\bar{\omega}(\bar{\phi}_1, \bar{\phi}_2) = \int_W \bar{\phi}_2^* F(\bar{\phi}_1) \wedge \delta\bar{\omega}(\bar{\phi}_2), \tag{5.8}$$

where $\cup : C^1(G;\Omega^1(W)) \times C^1(G;\Omega^1(W)) \to C^2(G;\Omega^2(W))$ is the cup product of group cochains referred in Example 3.6.

Theorem 5.34. *Let W be a $(n+1)$-dimensional, compact, oriented, connected, smooth manifold W with the boundary M and let $\widehat{W} \to W$ be a regular $\mathbb{Z}$-covering space with the classifying class $\bar{\alpha} \in H^1(W;\mathbb{Z})$. The boundary of $\widehat{W} \to W$ is a regular $\mathbb{Z}$-covering space $\partial\widehat{W} = \widehat{M} \to M$ with the classifying class $\alpha = \bar{\alpha}|_{\partial W} \in H^1(M;\mathbb{Z})$. Choose and fix a closed 1-form $\bar{\omega} \in \Omega^1(W)$ which represents $i_*\bar{\alpha} \in H^1(W;\mathbb{R})$ induced by the inclusion $i : \mathbb{Z} \hookrightarrow \mathbb{R}$.*

Take and fix a 1-form $\eta \in \Omega^n(W)$ whose negative differential $\Omega = -d\eta \in \Omega^{n+1}(W)$ is the normalized volume form on W. Let $1 \to G_{\mathrm{rel}} \to G \xrightarrow{\bar{p}} G_\partial \to 1$ be a short exact sequence defined by (5.7)*.*

Put $F = -\delta\eta \in C^1(G;\Omega^n(W))$. Let $L \in C^2(G;\mathbb{R})$ be a group 2-cochain defined by (5.8)*.*

Then the following holds.

1. *The group 2-cochain $L \in C^2(G;\mathbb{R})$ is contained in the image of $\bar{p}^* : C^2(G_\partial;\mathbb{R}) \to C^2(G;\mathbb{R})$.*

2. *This group 2-cocycle $(-1)^{n+1}L \in C^2(G_\partial;\mathbb{R})$ represents $i_*e(\widehat{G}_\partial) \in H^2_{\mathrm{grp}}(G_\partial;\mathbb{R})$, where $e(\widehat{G}_\partial)$ is the extension class of the central extension* (5.6)*.*

Proof. 1. Since G preserves $\bar{\omega}$ up to exact forms, there is a group 1-cochain $\bar{\sigma} \in C^1(G;\Omega^1(W))$ such that $d\bar{\sigma} = \delta\bar{\omega}$. Remark that for $\lambda \in \Omega^p(W)$ and $\bar{\phi} \in G$,

$$(\lambda - \bar{\phi}^*\lambda)|_{\partial W} = \lambda|_{\partial W} - \bar{p}(\bar{\phi})^*(\lambda|_{\partial W}). \tag{5.9}$$

Put $\omega = \bar{\omega}|_{\partial W} \in \Omega^1(M)$, which represents $i_*\alpha \in H^1(M;\mathbb{R})$. Recall that $\sigma_\infty \in C^1(G_\partial;\Omega^0(M))$ defined in Corollary 5.28. Then we have

$$d\bar{\sigma}|_{\partial W} = \delta\bar{\omega}|_{\partial W} = \delta\omega = d\bar{p}^*\sigma_\infty$$

by Equation (5.9). Thus $\bar{\sigma} = \bar{p}^*\sigma_\infty + C$, where $C \in \mathrm{Im}(C^1(G;\mathbb{R}) \to C^1(G;\Omega^0(M)))$. Similarly, $F|_{\partial W} = -\delta\eta|_{\partial W} \in C^1(G_\partial;\Omega^1(M))$ by Equation (5.9).

By Stokes' theorem, we obtain

$$\begin{aligned}
L = \int_W F \cup \delta\bar{\omega} = (-1)^n \int_W d(F \cup \bar{\sigma}) \\
=&(-1)^n \int_{\partial W} F|_{\partial W} \cup \bar{\sigma}|_{\partial W} \\
=&(-1)^n \int_{\partial W} F|_{\partial W} \cup \bar{p}^* \sigma_\infty|_{\partial W} + (-1)^n \int_{\partial W} F|_{\partial W} \cup C
\end{aligned}$$

Here,

$$\int_{\partial W} (F \cup C)(\bar{\phi}_1, \bar{\phi}_2) = \int_{\partial W} \bar{\phi}_2^* F(\bar{\phi}_1) \wedge C(\bar{\phi}_2) = C(\bar{\phi}_2) \int_{\partial W} \bar{\phi}_2^* F(\bar{\phi}_1) = 0.$$

Thus the group 2-cochain

$$L = (-1)^n \int_{\partial W} F|_{\partial W} \cup \sigma_\infty|_{\partial W}$$

is contained in the image of $\bar{p}^* : C^2(G_\partial; \mathbb{R}) \to C^2(G; \mathbb{R})$.

2. According to Corollary 5.28, $-\delta\sigma_\infty$ represents the extension class $i_* e(\widehat{G}_\partial) \in H^2_{\mathrm{grp}}(G_\partial; \mathbb{R})$. We can calculate:

$$\begin{aligned}
L =&(-1)^n \int_{\partial W} F|_{\partial W} \cup \sigma_\infty|_{\partial W} \\
=&(-1)^{n+1} \int_{\partial W} \eta \cup \delta\sigma_\infty + (-1)^n \int_{\partial W} \delta(\eta \cup \sigma_\infty) \\
=&(-1)^{n+1} \left(\int_W d\eta \right) \delta\sigma_\infty + (-1)^n \delta \int_{\partial W} \eta \cup \sigma_\infty \\
=&(-1)^n \delta\sigma_\infty + \delta c,
\end{aligned}$$

where $c = (-1)^n \int_{\partial W} \eta \cup \sigma_\infty \in C^1(G_\partial; \mathbb{R})$. Thus $\delta\sigma_\infty$ and $(-1)^n L$ represents the same cohomology class $i_* e(\widehat{G}_\partial) \in H^2_{\mathrm{grp}}(G_\partial; \mathbb{R})$.

□

Consider the pairing $\langle \mathrm{Flux}, i_* \bar{\alpha} \rangle : G_{\mathrm{rel}} \to \mathbb{R}$,

$$\langle \mathrm{Flux}, i_* \bar{\alpha} \rangle (\bar{\phi}) = \left\langle \mathrm{Flux}(\bar{\phi}) \cup i_* \bar{\alpha}, [W, \partial W] \right\rangle.$$

Here, $\cup : H^n(W, \partial W; \mathbb{R}) \otimes H^1(W; \mathbb{R}) \to H^{n+1}(W, \partial W; \mathbb{R})$ is the cup product, $\langle \cdot, \cdot \rangle : H^{n+1}(W, \partial W; \mathbb{R}) \times H_{n+1}(W, \partial W; \mathbb{R}) \to \mathbb{R}$ is the evaluation map, and $[W, \partial W] \in H_{n+1}(W, \partial W; \mathbb{R})$ denotes the relative fundamental class of the pair $(W, \partial W)$. Therefore $\langle \mathrm{Flux}, i_* \bar{\alpha} \rangle$ is a homomorphism.

Corollary 5.35. *Consider the five-term exact sequence of the LHS spectral sequence*

$$0 \to H^1_{\mathrm{grp}}(G_\partial;\mathbb{R}) \to H^1_{\mathrm{grp}}(G;\mathbb{R}) \to H^1_{\mathrm{grp}}(G_{\mathrm{rel}};\mathbb{R})^G \xrightarrow{d_2} H^2_{\mathrm{grp}}(G_\partial;\mathbb{R}) \to H^2_{\mathrm{grp}}(G;\mathbb{R})$$

for the short exact sequence $1 \to G_{\mathrm{rel}} \to G \to G_\partial \to 1$ *defined in Definition 4.7.*

Then the following holds:

1. *The pairing* $\langle \mathrm{Flux}, i_*\bar{\alpha}\rangle$ *is in the third term.*

2. *The transgression* $(-1)^n d_2 \langle \mathrm{Flux}, i_*\bar{\alpha}\rangle$ *is the extension class of the central extension* (5.6).

Proof. 1. The third term $H^1_{\mathrm{grp}}(G_{\mathrm{rel}};\mathbb{R})^G$ is the group of G-invariant homomorphisms from G_{rel} to $\mathbb{R}$.

We first prove that $\langle \mathrm{Flux}, i_*\bar{\alpha}\rangle : G_{\mathrm{rel}} \to \mathbb{R}$ is a homomorphism. Since $\mathrm{Flux} : G_\partial \to H^1(W;\partial W;\mathbb{R})$ is a homomorphism, we have $0 = \delta\mathrm{Flux} \in C^2(G_{\mathrm{rel}}; H^1(W;\partial W;\mathbb{R}))$. Since all of the elements in G_{rel} are homotopic to the identity on W, we also have $0 = \delta(i_*\alpha) \in C^1(G_{\mathrm{rel}}; H^1(W;\mathbb{R}))$. Therefore we obtain

$$\begin{aligned}\delta \langle \mathrm{Flux}, i_*\alpha\rangle =& \delta \langle \mathrm{Flux} \cup i_*\bar{\alpha}, [W,\partial W]\rangle \\ &= \langle \delta\mathrm{Flux} \cup i_*\bar{\alpha}, [W,\partial W]\rangle - \langle \mathrm{Flux} \cup \delta(i_*\bar{\alpha}), [W,\partial W]\rangle = 0.\end{aligned}$$

This means that $\langle \mathrm{Flux}, i_*\bar{\alpha}\rangle : G_{\mathrm{rel}} \to \mathbb{R}$ is a homomorphism.

Next, we prove that $\langle \mathrm{Flux}, i_*\bar{\alpha}\rangle : G_{\mathrm{rel}} \to \mathbb{R}$ is G-invariant. Since $\mathrm{Flux} : G_{\mathrm{rel}} \to H^n(W,\partial W;\mathbb{R})$ is G-invariant by Lemma 5.33, $\langle \mathrm{Flux}, i_*\bar{\alpha}\rangle$ is also G-invariant.

2. We prove the transgression formula. Define $\tau : G \to \mathbb{R}$ as

$$\tau(\bar{\phi}) = \int_W F(\bar{\phi}) \wedge \bar{\omega}.$$

By the definition, we have

$$\delta\tau = \delta \int_W F \cup \bar{\omega} = (-1)^{n+1} \int_{\partial W} F \cup \sigma.$$

Since $0 = F(\bar{\psi})|_{\partial W} \in \Omega^n(M)$ for $\bar{\psi} \in G_{\mathrm{rel}}$, we obtain

$$\delta\tau(\bar{\psi}, \bar{\phi}) = (-1)^{n+1} \int_{\partial W} \bar{\phi}^* F(\bar{\psi}) \cup \sigma(\bar{\phi}) = 0$$

for $\bar{\phi} \in G$ and $\bar{\psi} \in G_{\mathrm{rel}}$, so that

$$\tau(\bar{\psi}\bar{\phi}) = \langle \mathrm{Flux}, i_*\bar{\alpha} \rangle (\bar{\psi}) + \tau(\bar{\phi}).$$

Thus τ is a connection cochain over $\langle \mathrm{Flux}, i_*\bar{\alpha}\rangle$. Furthermore, the identity

$$L = \int_W F \cup \delta\bar{\omega} = -\delta \int_W F \cup \bar{\omega} = -\delta\tau$$

implies that $d_2 \langle \mathrm{Flux}, i_*\bar{\alpha}\rangle$ is represented by $-L \in C^2(G; \mathbb{R})$, so that $(-1)^n i_* e(\widehat{G}_\partial) \in H^2_{\mathrm{grp}}(G_\partial; \mathbb{R})$.

□

Example 5.36. We consider a solid torus $W = D \times S^1$. We have a smooth regular $\mathbb{Z}$-covering space $\widehat{W} = D \times \mathbb{R} \to W$. Define a 2-form $\eta \in \Omega^2(D \times S^1)$ as

$$\eta = -\frac{1}{8\pi^2} r^2 d\theta \wedge d\varphi, \quad ((r, \theta), \varphi) \in D \times S^1.$$

Then $\Omega = -d\eta \in \Omega^3(W)$ is a normalized volume form on W. Thus, we have the flux homomorphism $\mathrm{Flux} : G_{\mathrm{rel}} \to H^2(W, \partial W; \mathbb{R})$:

$$\mathrm{Flux}(\bar{\phi}) = [\bar{\phi}^*\eta - \eta] \in H^2(W, \partial W; \mathbb{R}).$$

On the other hand, the covering space over the solid torus has the boundary $\widehat{T}^2 = S^1 \times \mathbb{R} \to T^2$. The restricted smooth central extension associated $\widehat{T}^2 \to T^2$ is non-trivial (see Example 5.12):

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathrm{Diff}(S^1 \times \mathbb{R})^{\mathbb{Z}} \arrow[r, "p"] & \mathrm{Diff}_\alpha(T^2) \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & p^{-1}(G_\partial) \arrow[r] \arrow[u, hook] & G_\partial = \mathrm{Diff}_+(T^2)_0 \arrow[r] \arrow[u, hook] & 0.
\end{tikzcd}$$

Define $\bar{\omega} \in \Omega^1(W)$ as $\bar{\omega} = \frac{d\varphi}{2\pi}$. This form $\bar{\omega}$ represents the classifying class $i_*\bar{\alpha} \in H^1(W; \mathbb{R})$ of $\widehat{W} \to W$ with coefficients in $\mathbb{R}$

Thus the G-invariant homomorphism $\langle \mathrm{Flux}, i_*\bar{\alpha} \rangle : G_{\mathrm{rel}} \to \mathbb{R}$ defined by the evaluation

$$\langle \mathrm{Flux}, i_*\bar{\alpha} \rangle (\phi) = \langle \mathrm{Flux}(\phi) \cup i_*\bar{\alpha}, [W, \partial W] \rangle$$

transgresses to the extension class of the restricted smooth central extension associated to $\widehat{T}^2 \to T^2$ by the LHS spectral sequence for the short exact sequence $1 \to G_{\mathrm{rel}} \to G \to G_\partial \to 1$.

5.6 Future developments

We will introduce two future developments.

Let X be a connected CW-complex and let Γ be an arbitrary discrete group. For each regular Γ-covering space $\widehat{X} \to X$, we have an exact sequence

$$0 \to Z(\Gamma) \to \mathrm{Homeo}(\widehat{X})^\Gamma \xrightarrow{p} \mathrm{Homeo}(X),$$

where $Z(\Gamma)$ is the center of Γ. Thus we obtain a central extension

$$0 \to Z(\Gamma) \to \mathrm{Homeo}(\widehat{X})^\Gamma \to \mathrm{Im}(p) \to 1.$$

We also call this extension the *central extension associated to the regular covering space* $\pi : \widehat{X} \to X$. Especially if Γ is Abelian, then we obtain a central extension

$$0 \to \Gamma \to \mathrm{Homeo}(\widehat{X})^\Gamma \to \mathrm{Im}(p) \to 1.$$

By Theorem 5.23, the central extension associated to the regular $\mathbb{Z}$-covering space means the Euler class for the universal flat (X, α)-bundle.

Problem 5.37. *What does the central extension associated to the regular Γ-covering space mean geometrically?*

Let $\mathcal{H}$ be an infinite-dimensional, separable, Hilbert space. Denote by $U(\mathcal{H})$ the group of unitary operators on $\mathcal{H}$. It is known that $U(\mathcal{H})$ is weakly contractible due to Kuiper [11, Theorem (3)] and the center of $U(\mathcal{H})$ is the unitary group $U(1)$. Thus the projective unitary group $PU(\mathcal{H}) = U(\mathcal{H})/U(1)$ is homotopic to the classifying space $BU(1)$. Take the generator $u \in H^2(PU(\mathcal{H}); \mathbb{Z}) \cong H^1(U(1); \mathbb{Z})$.

For a principal $PU(\mathcal{H})$-bundle $E \to B$, we consider the Leray-Serre spectral sequence $E_2^{p,q} \cong H^p(B; \mathcal{H}^q(PU(\mathcal{H}); \mathbb{Z})$. Then, under the differential

$d_3 : H^2(PU(\mathcal{H});\mathbb{Z}) \to H^3(B;\mathbb{Z})$, the cohomology class $d_3u \in H^3(B;\mathbb{Z})$ is known as the *Dixmier-Douady class* (see Bouwknegt-Mathai [2, 2], Brylinski [4, Chapter 4], for instance).

Let X be a connected smooth manifold with a *second* cohomology class $\alpha \in H^2(X;\mathbb{Z})$. Assume that the first cohomology group $H^1(X;\mathbb{Z})$ is trivial. We denote by $G_\alpha = \mathrm{Homeo}_\alpha(X)$ the group of homeomorphisms on X which preserves the class $\alpha \in H^2(X;\mathbb{Z})$. We can define the (X,α)-bundle $E \to B$ which is a fiber bundle $E \to B$ with X the typical fiber and with G_α the structure group. In the Leray-Serre spectral sequence $E_2^{p,q} \cong H^p(B;\mathcal{H}^q(X;\mathbb{Z}))$, there is the third differential

$$H^2(X;\mathbb{Z})^{\pi_1(B)} \cong E_2^{0,2} \to E_3^{0,2} \xrightarrow{d_3} E_3^{3,0} \to E_2^{3,0} \cong H^3(B;\mathbb{Z}).$$

We call $d_3u \in H^3(B;\mathbb{Z})$ the *Dixmier-Douady class* for the (X,α)-bundle $E \to B$.

On the other hand, it is known that there is the one-to-one correspondence between the second cohomology group $H^2(X;\mathbb{Z})$ and the equivalence class of principal $U(1)$-bundles over X.

Problem 5.38. *Can we define a central $U(1)$-extension for a principal $U(1)$-bundle whose extension class agree with the Dixmier-Douady class for universal flat (X,α)-bundle?*

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