Complete Resource Theory of Quantum Incompatibility as Quantum Programmability

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(Received 1 November 2019; accepted 24 February 2020; published 24 March 2020)

Measurement incompatibility describes two or more quantum measurements whose expected joint outcome on a given system cannot be defined. This purely nonclassical phenomenon provides a necessary ingredient in many quantum information tasks such as violating a Bell inequality or nonlocally steering part of an entangled state. In this Letter, we characterize incompatibility in terms of programmable measurement devices and the general notion of quantum programmability. This refers to the temporal freedom a user has in issuing programs to a quantum device. For devices with a classical control and classical output, measurement incompatibility emerges as the essential quantum resource embodied in their functioning. Based on the processing of programmable measurement devices, we construct a quantum resource theory of incompatibility. A complete set of convertibility conditions for programmable devices is derived based on quantum state discrimination with postmeasurement information.

DOI: 10.1103/PhysRevLett.124.120401

The theory and practice of quantum measurement is a topic that sits at the foundation of quantum mechanics. Unlike its classical counterpart, quantum measurement offers a variety of ways to probe a system and extract classical information. A highly nonclassical feature that emerges in quantum mechanics is measurement incompatibility. The most general quantum measurements are described by positive-operator valued measures (POVMs), and the incompatibility of POVMs is typically defined in terms of joint measurability [1–3]. Roughly speaking, a family of POVMs is called jointly measurable if the outcomes of the constituent POVMs can be simulated through the measurement of a single "mother" POVM.

There has been much interest in measurement incompatibility and its relationship to various primitive tasks in quantum information theory [1]. For the demonstration of quantum nonlocality, it is not difficult to see that a Bell inequality can be violated only if incompatible measurements are employed by each of the parties involved in the experiment [4]. While for certain families of measurements the converse is true [5], only recently has it been found not to hold in general [6,7]. However, this asymmetry between measurement incompatibility and nonlocality vanishes when considering the more general task of quantum steering. That is, a family of POVMs is incompatible if and only if it can be used to steer some quantum state in a nonclassical way [8,9]. In recent works, it was also shown that a family of POVMs is incompatible if and only if it offers an advantage in some state discrimination game [10–14]. The main result of this Letter offers a generalization of these results.

Given the ability of incompatible measurements to generate nonclassical effects and enhance quantum state discrimination tasks, it becomes natural to view measurement incompatibility as a resource in quantum information processing. This interpretation can be formalized using the framework of a quantum resource theory (QRT) [15]. In general, a QRT isolates some particular feature of a quantum system, referred to as a resource, such as entanglement or coherence, and studies how this resource transforms under a restricted set of "free" operations; crucially, the free operations cannot generate the resource on their own. While entanglement and coherence represent static resources that are commonly studied in the literature, it is also possible to formulate resource theories for dynamic resources, such as certain families of quantum measurements [16–19].

In particular, resource theories of measurement incompatibility have been previously proposed in which the resources are incompatible families of POVMs [14,20,21]. However, a drawback to these approaches is that the free operations identified are not large enough to fully capture the notion of measurement incompatibility in an operational way. Reference [14] only considers measurement convertibility under quantum preprocessing, while Refs. [20,21] only consider conditional classical postprocessing as the free operations. Both of these on their own are too weak in that they do not allow for the free convertibility of one compatible POVM family to another. Moreover, there is no *a priori* reason why an experimenter should be restricted to performing either just quantum pre- or classical postprocessing when their combination is equally unable to generate incompatibility.

In this Letter, we construct a resource theory of measurement incompatibility that combines both quantum preprocessing and conditional classical postprocessing in the context of programmable measurement devices (PMDs).

PMDs are objects that emerge through the following consideration. In any experiment where different measurements are being employed, there are two relevant systems: the quantum system Q that is subjected to the particular measurement and the "program" system, whose state $x \in X$ represents the choice of measurement. The measurement apparatus in such an experiment thus exemplifies a PMD since the type of measurement it performs depends on the program it receives.

To formulate a resource theory in this setting, we shift the primary focus away from quantum measurement and place it on "programmability", which we consider broadly to be any sort of classical control over a device that can be implemented at the programmer's discretion. In other words, we envision programmability to mean that some device can be obtained at time t_0 and then controlled in whatever way the device allows at some later time t. This reflects the natural interplay between computing hardware and software: one first purchases or builds a computing device and then later programs it to perform whatever computational task is desired. However, adopting such a perspective then requires constraining the type of interaction between the program and quantum system described in the previous paragraph. Namely, the program system should not be allowed to affect the preparation of the quantum system, since the former is decided at time t, while the latter is set at time $t_0 < t$. In satisfying this restriction, we are thus led to a resource theory of programmability for which the free operations arise from very natural physical considerations.

Let us now put the discussion in more formal terms.

Definition 1: Programmable measurement devices.—A (classically) PMD is a collection of POVMs on the same Hilbert space $\mathcal{H}^Q$, $\{M^Q(a|x)\colon a \in \mathcal{A}, x \in \mathcal{X}\}$, such that $M^Q(a|x) \geq 0$ and $\sum_a M^Q(a|x) = \mathbb{1}^Q$ for all x. The set $\mathcal{X}$ is interpreted as the program set (an element x being the program), while the set $\mathcal{A}$ is interpreted as the outcome set.

While PMDs are mathematically equivalent to $cq \to c$ channels, the two inputs of a PMD are always assumed to be *separate* systems. Crucially, we assume that it takes a finite amount of time for the program to be able to influence the measurement performed on the quantum system. This assumption immediately implies a necessary condition for a PMD to be able to implement an incompatible family of POVMs: the PMD must be able to effectively preserve the quantum system at least for the time it takes the program to influence the measurement process. This simple observation leads us to define the free objects in our QRT as those PMDs corresponding to compatible families of POVMs.

Definition 2: Simple PMDs, alias compatible POVMs.—A PMD $M^Q(a|x)$ is called "simple" if its constituting POVMs can be written as

$$M^Q(a|x) = \sum_{i \in I} p(a|i,x) \tilde{M}^Q(i), \qquad (1)$$

where the $\tilde{M}^Q(i)$ are elements of a single POVM (sometimes referred to as the mother POVM), and $p(a|i,x)$ is a conditional probability distribution.

Compatible POVMs are also often defined in terms of coarse graining over a single POVM, and Eq. (1) is equivalent to this characterization (see, e.g., Ref. [20]). From their definition, simple PMDs can be perfectly simulated without the need to store the quantum system, which can be immediately measured using the mother POVM, with the program influencing only the classical postprocessing of its outcome.

Let us then turn to the free operations in this resource theory, which will be a restricted set of maps converting $cq \to c$ channels to $cq \to c$. Such maps convert channel $\mathcal{M}$ into $\mathcal{F}_{\text{post}} \circ \mathcal{M} \circ \mathcal{E}_{\text{pre}}$, where $\mathcal{E}_{\text{pre}}$ and $\mathcal{F}_{\text{post}}$ are pre- and postprocessing maps, possibly connected by a memory side channel [22,23]. Every nonsimple PMD functions as a quantum memory, as it must preserve the quantum system until the program arrives. Quantum memory, then, is essential for a device to be programmable, and so it should not be something freely available in a resource theory of programmability. The memory connecting $\mathcal{E}_{\text{pre}}$ and $\mathcal{F}_{\text{post}}$ should therefore be classical, and the preprocessing map $\mathcal{E}_{\text{pre}}$ should be causally independent of the program, since at that time the program has not arrived yet. What remains are the free operations of this QRT, and they are described by Eq. (2) in the following definition (see Fig. 1 for a schematic representation).

Definition 3.—Given two PMDs $M^Q(a|x)$ and $N^{Q'}(b|y)$ on $\mathcal{H}^Q$ and $\mathcal{H}^{Q'}$, respectively, we write $M^Q(a|x) \succeq N^{Q'}(b|y)$ whenever

$$N^{Q'}(b|y) = \sum_r \mu(r) \sum_{i,x,a} q(b|a,x,i,y,r) \times p(x|i,y,r) (\mathcal{E}_{i|r}^{Q' \to Q})^\dagger [M^Q(a|x)], \qquad (2)$$

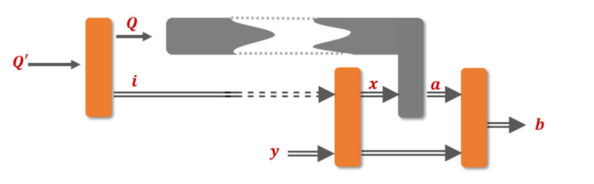

FIG. 1. PMDs processing, according to Definition 3. Time flows from left to right. The program y (i.e., the postinformation) arrives after the preprocessing has been performed. Since the only quantum memory resides in the PMD, the quantum input must be committed to the PMD until the program arrives. On the other hand, the classical output i of the preprocessing instrument can be stored in a classical memory and interact with the program before it reaches the PMD. Notice that, even though it is not explicitly depicted in the picture, classical randomness can be shared between all processing boxes (orange on-line), so that the set of possible processings is convex.

where (i) $\mu(r)$ is a probability distribution modeling a shared source of classical randomness, (ii) $\{\mathcal{E}_{i|r}^{Q'\to Q}\}$ is a family of quantum instruments labeled by r, with classical outcome i, and $\mathcal{E}^\dagger$ denotes the adjoint (i.e., trace-dual) map of $\mathcal{E}$, and (iii) $p(x|i,y,r)$ and $q(b|a,x,i,y,r)$ are classical noisy channels (conditional probability distributions). The relation $M^Q(a|x)\succeq N^{Q'}(b|y)$ expresses convertibility of PMDs by free operations in this QRT.

Before proceeding further, we stress that the free operations considered here need not constitute the *only* meaningful operational framework to study the properties of programmability and compatibility. However, as shown in the Supplemental Material [24], they do satisfy the important property that any two simple PMDs can always be freely interconverted.

We refer to Fig. 1 as the temporal model of PMD processing, and there is an alternative spatial model that characterizes PMDs in terms of bipartite channels shared between two spatially separated parties (Alice and Bob). As shown in Fig. 2, the programmability of a PMD is then translated into a no-signaling constraint from Bob to Alice. Hence, the correct operational setting for PMD processing in the spatial model is one-way local quantum operations and classical communication (LOCC), and the following proposition makes this connection precise.

Proposition 1.—$M^Q(a|x)\succeq N^{Q'}(b|y)$ if and only if $M^Q(a|x)$ can be converted to $N^{Q'}(b|y)$ by a one-way LOCC from Alice to Bob.

We stress that, although the bipartite processing of PMDs by one-way LOCC is intuitively simple, without the temporal model in mind, the physical motivation for studying the QRT of $cq\to c$ channels under one-way LOCC is less clear. Why is one-way LOCC the free set of operations in such a QRT, and why must it only be from Alice to Bob? The answers come from the allowed operations in the temporal model, which do have clear physical motivation in terms of programmability. It just so happens that these free operations correspond to Alice-to-Bob one-way LOCC in the spatial model.

PMDs and postinformation guessing games.—The main result of this Letter is a characterization of free PMD convertibility in terms of quantum state guessing games with side information [12,13,21,25]. These games involve a

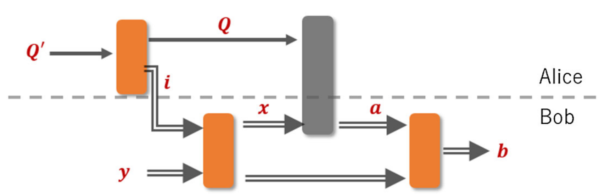

FIG. 2. The spatial model of PMD processing. The quantum and program inputs are separated between Alice and Bob, and the free operations depicted in Fig. 1 translate into one-way LOCC maps from Alice to Bob.

referee who distributes to the player a quantum state and some classical side information, information that we will henceforth refer to as "postinformation," since it more appropriately fits our temporal model. More formally, let $\{\rho_{w,z}^R : w\in\mathcal{W}, z\in\mathcal{Z}\}$ be a two-index quantum ensemble such that $p(w,z) \coloneqq \mathrm{Tr}[\rho_{w,z}^R]$ is a normalized joint probability distribution. A postinformation guessing game consists of the following components: (i) the referee picks one pair $(w,z)\in\mathcal{W}\times\mathcal{Z}$ at random according to the distribution $p(w,z)$, (ii) the normalized quantum state $p(w,z)^{-1}\rho_{w,z}^R$ is sent to the player, followed, after some finite time, by the index w, and (iii) the player attempts to maximize the probability of correctly guessing the value z using the given PMD $M^Q(a|x)$ and any free processing described in Definition 3. In this game, the label w is interpreted as the postinformation, since it is imported into the program register of the PMD *after* the quantum state, and it cannot be used in any preprocessing of the PMD.

When playing guessing games with postinformation, certain processing strategies will lead to greater success probabilities in guessing z. In particular, if the referee's questions $\rho_{w,z}^R$ are encoded on a quantum system that is different from the quantum input of the PMD $M^Q(a|x)$, then the player must do some sort of quantum preprocessing of R into Q, represented without loss of generality by a quantum instrument $\{\mathcal{E}_i^{R\to Q}\}$. The optimum success probability over all strategies is thus given by

$$P_{\mathrm{guess}}(M^Q(a|x);\rho_{w,z}^R) \coloneqq \max_{\mu,q,p,\mathcal{E}} \sum_{w,z}\sum_r\sum_{i,x,a}\mu(r)q(z|a,w,i,r)p(x|w,i,r) \times \mathrm{Tr}[\mathcal{E}_{i|r}^{R\to Q}(\rho_{w,z}^R)M^Q(a|x)], \tag{3}$$

where the probability distribution $\mu(r)$ is included to describe mixed strategies, i.e., those in which a different strategy, labeled by r, is chosen at random. [The optimum guessing probability will then be achieved on pure strategies, but it is convenient to explicitly include this in Eq. (3).]

We are now ready to state the main result, whose proof, which closely follows those in [26,27], is given in the Supplemental Material [24].

Theorem 1.—Given two PMDs $M^Q(a|x)$ and $N^{Q'}(b|y)$, the following are equivalent: (a) $M^Q(a|x)\succeq N^{Q'}(b|y)$; (b) for all guessing games with postinformation $\{\rho_{w,z}^R : w\in\mathcal{W}, z\in\mathcal{Z}\}$,

$$P_{\mathrm{guess}}(M^Q(a|x);\rho_{w,z}^R) \geq P_{\mathrm{guess}}(N^{Q'}(b|y);\rho_{w,z}^R).$$

In (b), it is possible to consider only guessing games with $\mathcal{H}^R=\mathcal{H}^{Q'}$, $\mathcal{W}=\mathcal{Y}$, and $\mathcal{Z}=\mathcal{B}$.

Simply by noticing that it is impossible to turn a simple PMD into an incompatible one by means of free operations, we obtain as a corollary that quantum incompatibility can always be witnessed by means of a suitable guessing game with postinformation.

Corollary.—A PMD $M^Q(a|x)$ is incompatible, if and only if there exists an ensemble $\{\rho^Q_{x,a}: x \in \mathcal{X}, a \in \mathcal{A}\}$, such that

$$\sum_{a,x} \mathrm{Tr}[M^Q(a|x)\rho^Q_{x,a}] > P^{\mathrm{simple}}_{\mathrm{guess}}(\rho^Q_{x,a}),$$

where $P^{\mathrm{simple}}_{\mathrm{guess}}(\rho^Q_{x,a})$ is defined as the optimum guessing probability achievable with simple PMDs.

As a special case, Theorem 1 provides necessary and sufficient conditions of a single POVM under quantum preprocessing and conditional postprocessing. That is, we have $M^Q(a) \succ N^{Q'}(b)$ in the sense of Eq. (2) if and only if the POVM $M^Q(a)$ is always more useful than $N^{Q'}(b)$ for the task of minimum-error state discrimination; i.e., for every ensemble $\{\rho_z\}_z$ we have $P_{\mathrm{guess}}(M^Q(b);\rho^R_z) \geq P_{\mathrm{guess}}(N^{Q'}(b);\rho^R_z)$.

Robustness of nonsimple PMDs.—In any QRT with minimal structure, it is possible to define a (generalized) robustness measure of resource [28]. Roughly speaking, the robustness captures how tolerant an object is to mixing before it loses all its resource. A PMD robustness measure $\mathfrak{R}(\{M(a|x)\}_{a,x})$ can also be defined in this QRT directly analogous to incompatibility robustness measures previously studied [12,21]. Specifically, we have

$$\mathfrak{R}(\{M(a|x)\}_{a,x}) = \min\left\{r \geq 0: \frac{M(a|x) + rN(a|x)}{1+r} \in \mathcal{F}\right\},$$

where $\mathcal{F}$ is the convex, compact set of simple PMDs matching input and output spaces of $M(a|x)$. In the above corollary, we showed that every incompatible PMD has an advantage over incompatible ones in some guessing game with postinformation. This advantage can also be quantified as the maximum ratio between the optimum guessing probability of the given incompatible PMD versus the optimal guess probability of any simple PMD,

$$\max_{\{\rho^Q_{x,a}\}} \frac{P_{\mathrm{guess}}(M^Q(a|x);\rho^Q_{x,a})}{P^{\mathrm{simple}}_{\mathrm{guess}}(\rho^Q_{x,a})}.$$

It is possible to show an equivalence between the advantage and the robustness; namely,

$$1 + \mathfrak{R}(\{M(a|x)\}_{a,x}) = \max_{\{\rho^Q_{x,a}\}} \frac{P_{\mathrm{guess}}(M^Q(a|x);\rho^Q_{x,a})}{P^{\mathrm{simple}}_{\mathrm{guess}}(\rho^Q_{x,a})},$$

where the maximization is over all possible guessing games with postinformation, mathematically represented by a double-index ensemble $\{\rho^Q_{x,a}: x \in \mathcal{X}, a \in \mathcal{A}\}$. The proof follows in the same way as the proof of Theorem 2 in [29], and one can check for details in the Supplemental Material [24]. This establishes an operational interpretation of $\mathfrak{R}(\{M(a|x)\}_{a,x})$ in terms of guessing games with postinformation.

Conclusion.—In this Letter, we have shown that a resource theory of quantum incompatibility can be naturally formulated as a resource theory of programmability, and that new insights can be gained by doing so. In particular, this resource theory is complete in the sense that all free devices are naturally equivalent to each other. This was accomplished by identifying programmability as a key resource that requires quantum memory for its realization. From this perspective, both quantum preprocessing and classical conditional postprocessing can be integrated into the picture, while remaining, however, within the operational scenario provided by postinformation guessing games [10].

The approach that we followed here in order to formulate a resource theory of quantum incompatibility is very much inspired by the concept of statistical comparison, introduced in mathematical statistics chiefly by Blackwell [30] and extended to the quantum case by one of the present authors [31]. Indeed, the aim of statistical comparison, as originally envisaged by Blackwell, is that of expressing the possibility of transforming an initial statistical model into another one, in terms of the utility that the two statistical models provide in operationally motivated scenarios (that is, statistical decision problems in Blackwell's original paper). *Mutatis mutandis*, this is exactly the scope of any resource theory, where the aim is to identify a set of operationally motivated monotones that dictate when an allowed transformation between resources exists or not. Among the numerous examples of such an approach, which at present ranges from quantum nonlocality [31] to quantum thermodynamics [32], this Letter bears some similarities with the resource theory of quantum memories, viz. nonentanglement-breaking channels, recently put forth in Ref. [33]. Even though no program register is considered in [33], there, as it happens here, the quantum memory is probed by means of "timed" decision problems, in which two tokens of the problem (there, two quantum tokens; here, one token is classical) are given to the player at subsequent times, who is then asked to formulate an educated guess so to maximize the expected payoff. Further relations between the two frameworks are left for future research.

We thank Teiko Heinosaari, Gilad Gour, and Otfried Gühne for helpful discussions on measurement incompatibility. W. Z. thanks Ryuji Takagi and Bartosz Regula for pointing out some mistakes while writing the proof of robustness. F. B. acknowledges partial support from the Japan Society for the Promotion of Science (JSPS)

KAKENHI, Grant No. 19H04066. E. C. acknowledges the support of NSF Grant No. 1820871. W. Z. is supported by the Program (RWDC) for Leading Graduate Schools of Nagoya University.

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[1] T. Heinosaari, T. Miyadera, and M. Ziman, An invitation to quantum incompatibility, J. Phys. A **49**, 123001 (2016).

[2] T. Heinosaari, D. Reitzner, and P. Stano, Notes on joint measurability of quantum observables, Found. Phys. **38**, 1133 (2008).

[3] P. Lahti, Coexistence and joint measurability in quantum mechanics, Int. J. Theor. Phys. **42**, 893 (2003).

[4] A. Fine, Joint distributions, quantum correlations, and commuting observables, J. Math. Phys. (N.Y.) **23**, 1306 (1982).

[5] M. M. Wolf, D. Perez-Garcia, and C. Fernandez, Measurements Incompatible in Quantum Theory Cannot be Measured Jointly in Any Other No-Signaling Theory, Phys. Rev. Lett. **103**, 230402 (2009).

[6] E. Bene and T. Vértesi, Measurement incompatibility does not give rise to bell violation in general, New J. Phys. **20**, 013021 (2018).

[7] M. T. Quintino, J. Bowles, F. Hirsch, and N. Brunner, Incompatible quantum measurements admitting a local-hidden-variable model, Phys. Rev. A **93**, 052115 (2016).

[8] M. T. Quintino, T. Vértesi, and N. Brunner, Joint Measurability, Einstein-Podolsky-Rosen Steering, and Bell Nonlocality, Phys. Rev. Lett. **113**, 160402 (2014).

[9] R. Uola, T. Moroder, and O. Gühne, Joint Measurability of Generalized Measurements Implies Classicality, Phys. Rev. Lett. **113**, 160403 (2014).

[10] C. Carmeli, T. Heinosaari, and A. Toigo, State discrimination with postmeasurement information and incompatibility of quantum measurements, Phys. Rev. A **98**, 012126 (2018).

[11] J. Mori, Operational characterization of incompatibility of quantum channels with quantum state discrimination, arXiv:1906.09859.

[12] R. Uola, T. Kraft, J. Shang, X.-D. Yu, and O. Gühne, Quantifying Quantum Resources with Conic Programming, Phys. Rev. Lett. **122**, 130404 (2019).

[13] C. Carmeli, T. Heinosaari, and A. Toigo, Quantum Incompatibility Witnesses, Phys. Rev. Lett. **122**, 130402 (2019).

[14] T. Heinosaari, J. Kiukas, and D. Reitzner, Noise robustness of the incompatibility of quantum measurements, Phys. Rev. A **92**, 022115 (2015).

[15] E. Chitambar and G. Gour, Quantum resource theories, Rev. Mod. Phys. **91**, 025001 (2019).

[16] S. Designolle, M. Farkas, and J. Kaniewski, Incompatibility robustness of quantum measurements: A unified framework, New J. Phys. **21**, 113053 (2019).

[17] M. Oszmaniec and T. Biswas, Operational relevance of resource theories of quantum measurements, Quantum **3**, 133 (2019).

[18] M. Oszmaniec, L. Guerini, P. Wittek, and A. Acín, Simulating Positive-Operator-Valued Measures with Projective Measurements, Phys. Rev. Lett. **119**, 190501 (2017).

[19] R. Takagi and B. Regula, General Resource Theories in Quantum Mechanics and Beyond: Operational Characterization via Discrimination Tasks, Phys. Rev. X **9**, 031053 (2019).

[20] L. Guerini, J. Bavaresco, M. T. Cunha, and A. Acín, Operational framework for quantum measurement simulability, J. Math. Phys. (N.Y.) **58**, 092102 (2017).

[21] P. Skrzypczyk, I. Šupić, and D. Cavalcanti, All Sets of Incompatible Measurements Give an Advantage in Quantum State Discrimination, Phys. Rev. Lett. **122**, 130403 (2019).

[22] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Transforming quantum operations: Quantum supermaps, Europhys. Lett. **83**, 30004 (2008).

[23] G. Gour, Comparison of quantum channels by superchannels, IEEE Trans. Inf. Theory **65**, 5880 (2019).

[24] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.124.120401 for the proof of Theorem 1 and a detailed calculation of the robustness measure in terms of postinformation guessing games.

[25] M. A. Ballester, S. Wehner, and A. Winter, State discrimination with post-measurement information, IEEE Trans. Inf. Theory **54**, 4183 (2008).

[26] F. Buscemi, Degradable channels, less noisy channels, and quantum statistical morphisms: An equivalence relation, Probl. Inf. Transm. **52**, 201 (2016).

[27] F. Buscemi, Comparison of noisy channels and reverse data-processing theorems, in *Proceedings of 2017 IEEE Information Theory Workshop (ITW), Kaohsiung, Taiwan* (IEEE, New York, 2017).

[28] F. G. S. L. Brandão and G. Gour, Reversible Framework for Quantum Resource Theories, Phys. Rev. Lett. **115**, 070503 (2015).

[29] R. Takagi and B. Regula, General Resource Theories in Quantum Mechanics and Beyond: Operational Characterization via Discrimination Tasks, Phys. Rev. X **9**, 031053 (2019).

[30] D. Blackwell, Equivalent comparisons of experiments, Ann. Math. Stat. **24**, 265 (1953).

[31] F. Buscemi, Comparison of quantum statistical models: Equivalent conditions for sufficiency, Commun. Math. Phys. **310**, 625 (2012).

[32] G. Gour, D. Jennings, F. Buscemi, R. Duan, and I. Marvian, Quantum majorization and a complete set of entropic conditions for quantum thermodynamics, Nat. Commun. **9**, 5352 (2018).

[33] D. Rosset, F. Buscemi, and Y.-C. Liang, Resource Theory of Quantum Memories and Their Faithful Verification with Minimal Assumptions, Phys. Rev. X **8**, 021033 (2018).